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Ubiquitous Archimedean Circles of the Collinear Arbelos

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ABSTRACT

We generalize the arbelos and its Archimedean circles, and show the existence of the generalized Archimedean circles which cover the plane.

Key words: arbelos, collinear arbelos, ubiquitous Archimedean circles

MSC 2000: 51M04, 51M15, 51N10

Sveprisutne Arhimedove kružnice kolinearnog arbelosa

SAŽETAK

Generaliziramo arbelos i njegove Arhimedove kružnice te pokazujemo postojanje generaliziranih Arhimedovih kružnica koje pokrivaju ravninu.

Ključne riječi: arbelos, kolinearni arbelos, sveprisutne Arhimedove kružnice

1 Introduction

For a point O on the segment AB in the plane, the area surrounded by the three semicircles with diameters AO, BO and AB erected on the same side is called an arbelos. It has lots of unexpected but interesting properties (for an extensive reference see [1]). The radical axis of the inner semicircles divides the arbelos into two curvilinear triangles with congruent incircles called the twin circles of Archimedes. Circles congruent to those circles are said to be Archimedean. In this paper we generalize the arbelos and the Archimedean circles, and show the existence of the generalized Archimedean circles covering the plane, which is a generalization of the ubiquitous Archimedean circles of the arbelos in [4].

The arbelos is generalized in several ways, the generalized arbelos of intersecting type [7], the generalized arbelos of non-intersecting type [6] and the skewed arbelos [5], [8]. For the generalized arbelos of intersecting type and non-intersecting type, the twin circles of Archimedes are considered in a general way as Archimedean circles in aliquot parts. But Archimedean circles are still not given except them. In this paper we unify the two generalized arbeloi with one more additional generalized arbelos.

2 The collinear arbelos

In this section we generalize the arbelos and the twin circles of Archimedes to a generalized arbelos. For two points P and Q in the plane, (PQ) denotes the circle with diameter PQ. Let P and Q be point on the line AB, and let $\alpha = (AP)$, $\beta = (BQ)$ and $\gamma = (AB)$. Let O be the point of intersection of AB and the radical axis of the circles α and β and let $u = |AB|$, $s = |AQ|/2$ and $t = |BP|/2$. We use a rectangular coordinate system with origin O such that the points A, B have coordinate $(a,0)$, $(b,0)$ respectively with $a-b=u$. The configuration (α,β,γ) is called a collinear arbelos if the four points lie in the order (i) B, Q, P, A or (ii) B, P, Q, A, or (iii) P, B, A, Q. In each of the cases the configuration is explicitly denoted by $(BQPA)$, $(BPQA)$ and $(PBAQ)$ respectively. $(BQPA)$ and $(BPQA)$ are the generalized arbelos of non-intersecting type and the generalized arbelos of intersecting type respectively.

Let $(p,0)$ and $(q,0)$ be the coordinates of P and Q respectively. Since the point O lies on the radical axis of α and β, the powers of O with respect to α and β are equal, i.e., $ap = bq$ holds. Hence there is a real number $k < 0$ such that $b = ka$ and $p = kq$. Therefore we get

$$ta + sb = tq + sp = 0. \quad (1)$$

For points V and W on the line AB with x-coordinates v and w respectively, $V \le W$ describes $v \le w$, and $\mathcal{P}_W$ denotes the perpendicular to AB passing through W. The part (i) of the following lemma is proved in [3]. The proof of (ii) is similar and is omitted (see Figure 1).

Lemma 1 *The following circles have radii* $|AW||BV|/(2u)$ *for points* V *and* W *on the line* AB*.*

(i) *The circles touching the circles* γ *internally,* (AV) *externally and the line* $\mathcal{P}_W$ *from the side opposite to the point* B *in the case* $B \le V \le A$ *and* $B \le W \le A$*.*

(ii) *The circles touching* γ *externally,* (AV) *internally and* $\mathcal{P}_W$ *from the side opposite to the point* A *in the case* $V \le B \le W \le A$*.*

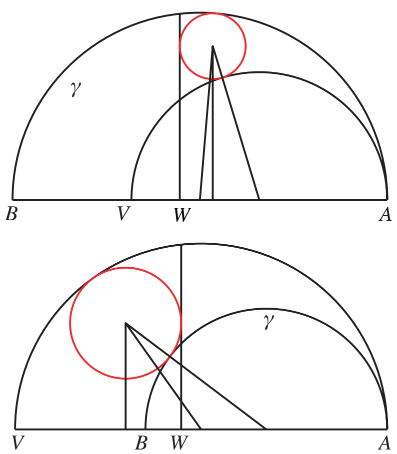

Figure 1: *Circles of radii* $|AW||BV|/(2u)$.

For collinear arbeloi $(BQPA)$ and $(BPQA)$, δ_α is the circle in the region $y > 0$ touching the circles γ internally, α externally and the line $\mathcal{P}_O$ from the side opposite to B. For a collinear arbelos $(PBAQ)$, δ_α is the circle in the region $y > 0$ touching γ externally, α internally and $\mathcal{P}_O$ from the side opposite to A. The circle δ_β is defined similarly (see Figure 2).

Theorem 1 *For a collinear arbelos* (α, β, γ)*, the circles* δ_α *and* δ_β *are congruent with common radii* $st/(s+t)$*.*

Proof. If $(\alpha, \beta, \gamma) = (PBAQ)$, by (ii) of Lemma 1 and (1) the radius of δ_α is

$$\frac{|AO||BP|}{2u} = \frac{a(b-p)}{2(a-b)} = \frac{a(-ta/s + tq/s)}{2(a + ta/s)} = \frac{st}{s+t}.$$

Similarly the radius of δ_β is equal to $st/(s+t)$. The other cases are proved similarly. □

We now call the circles δ_α and δ_β the twin circles of Archimedes of the collinear arbelos. Circles congruent to the twin circles are called Archimedean circles of the collinear arbelos.

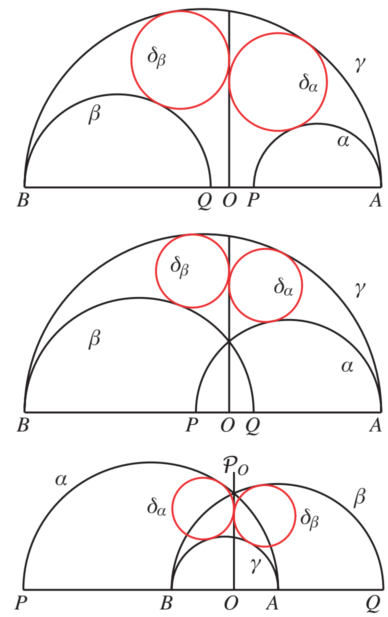

Figure 2: *The circles* δ_α *and* δ_β.

3 A pair of Archimedean circles generated by a point

We use the following lemma [4], which is easily proved by the properties of similar triangles.

Lemma 2 *For a triangle* RGH *with a point* S *on the segment* GH*, let* E *and* F *be points on the lines* RG *and* RH *respectively such that* $SERF$ *is a parallelogram. If* T *and* U *are points of intersection of* RS *with the lines parallel to* GH *passing through* E *and* F *respectively and* $g = |GS|$*,* $h = |HS|$*, then* $|ET| = |FU| = gh/(g+h)$*.*

Theorem 2 *For a collinear arbelos* (α, β, γ)*, let* R *be a point which does not lie on the line* AB*, and let* E *and* F *be points on the line* AR *and* BR *respectively such that* EP *and* FQ *are parallel to* BR *and* AR *respectively. If the lines passing through* E *and* F *parallel to* AB *intersect the line* OR *at points* T *and* U *respectively, the circles* (ET) *and* (FU) *are Archimedean.*

Proof. Let EP and FQ intersect the line OR at points S and S' respectively (see Figures 3, 4, 5). The triangles RAO and $S'QO$ are similar. Also the triangles RBO and SPO are similar. While $|QO|/|AO| = |PO|/|BO|$ for O lies on the radical axis of the circles α and β. Therefore the ratios of the similarity of the two pairs of the similar triangles are the same. Hence $|S'O|/|RO| = |SO|/|RO|$, i.e.,

$S = S'$. Let A' and B' be the points of intersection of the line passing through S parallel to AB with the lines AR and BR respectively. Then $|A'S| = |AQ| = 2s$ and $|B'S| = |BP| = 2t$. Therefore $|ET| = |FU| = 2st/(s+t)$ by Lemma 2, i.e., the circles (ET) and (FU) are Archimedean. □

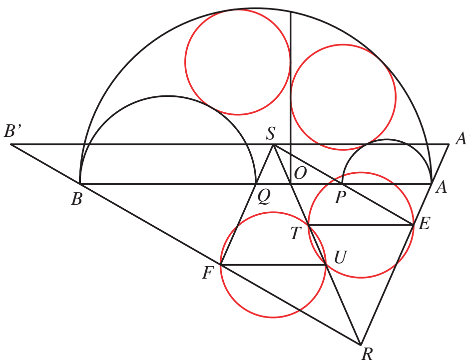

Figure 3

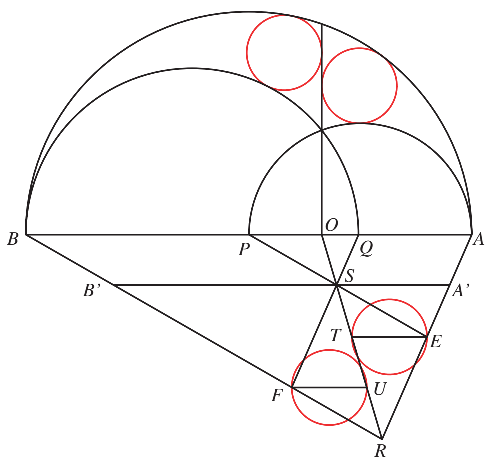

Figure 4

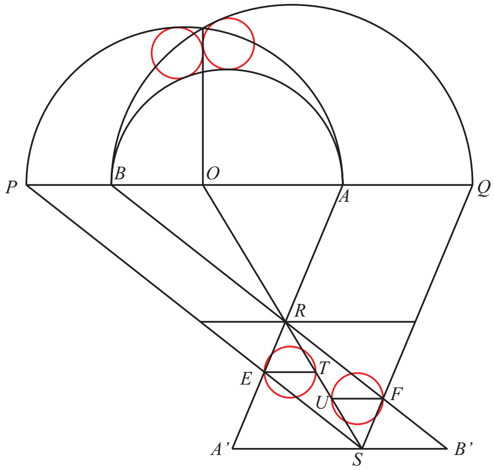

Figure 5

4 Ubiquitousness and parallelograms

However the word "ubiquitous" is used in the title of the paper [2], their Archimedean circles do not cover the plane. In this section, we show that our generalized Archimedean circles of the collinear arbelos cover the plane.

Let (α, β, γ) be a collinear arbelos. If a point X does not lie on the line AB, let Y and Z be points such that the midpoint of YZ is X, $\overrightarrow{YZ}$ and $\overrightarrow{AQ}$ are parallel with the same direction, and $|YZ| = 2r_{\mathrm{A}}$. Let R be the point of intersection of the lines AY and OZ. Using the point R, let us construct a parallelogram $SERF$ and a point T as in Figures 3, 4, 5. Then (ET) is an Archimedean circle with center X.

If a point X lies on the line AB, we choose a point X' lying inside of the Archimedean circle with center X, so that X' does not lie on AB. If we use the point X' instead of X, and construct the parallelogram $SERF$ and the point T as in Figures 3, 4, 5 just as mentioned above, the Archimedean circle (ET) with center X' contains X. Therefore there is an Archimedean circle containing the point X in any case, i.e., the Archimedean circles cover the plane. In this sense our Archimedean circles are Ubiquitous.

However the five points A, B, P, Q and O are involved in the construction of the Archimedean circles, the three circles α, β and γ are not. Therefore it seems that the Archimedean circles are not so closely related to the collinear arbelos. But we can show that for a point R, which does not lie on the line AB, the parallelogram $SERF$ in Figures 3, 4, 5 are constructed by the circles α, β and γ (see Figure 6). Let the circle γ intersect the lines AR and BR at points I and J respectively, and let AJ intersect α at a point K and BI intersect β at a point L. Then KP and JB are parallel, also LQ and IA are parallel. Therefore if E, F, S are the points of intersection of the lines AR and KP, BR and LQ, KP and LQ respectively, then $SERF$ is a parallelogram.

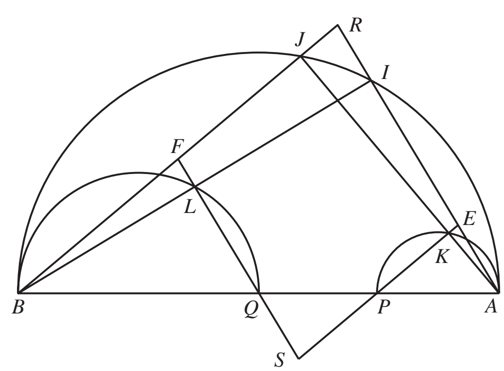

Figure 6

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