ON THE SET-SEMIDEFINITE REPRESENTATION OF NONCONVEX QUADRATIC PROGRAMS WITH CONE CONSTRAINTS

Gabriele Eichfelder
University of Erlangen-Nürnberg, Department of Mathematics
Martensstr. 3, D-91058 Erlangen, Germany, E-mail: Gabriele.Eichfelder@am.uni-erlangen.de

Janez Povh[1]
Faculty of Information Studies in Novo Mesto
Novi trg 5, 8000 Novo mesto, Slovenia, E-mail: janez.povh@fis.unm.si

Abstract: The well-known result stating that any non-convex quadratic problem over the non-negative orthant with some additional linear and binary constraints can be rewritten as linear problem over the cone of completely positive matrices (Burer, 2009) is generalizes by replacing the non-negative orthant with arbitrary closed convex and pointed cone. This set-semidefinite representation result implies new semidefinite lower bounds for quadratic problems over the Bishop-Phelps cones, based on the Euclidian norm.

Key words: *set positivity, Bishop-Phelps cones*

1. Introduction

In [2, 10, 14, 15] several hard problems from combinatorial optimization have been reformulated as linear programs over the cone of copositive or completely positive matrices. In [3] these results are generalized as follows: the optimal value of the nonconvex quadratic problem

$$\begin{array}{ll} \inf & x^TQx + c^Tx \\ \text{s. t.} & a_i^Tx = b_i, \\ & x_j \in \{0, 1\}, \forall j \in \mathcal{J}, \\ & x \geq 0 \end{array}$$

is under some mild assumptions equal to the optimal value of the following program:

$$\begin{array}{ll} \inf & \langle Q, X\rangle + c^Tx \\ \text{s. t.} & a_i^Tx \ = \ b_i, \ \langle a_ia_i^T, X\rangle = b_i^2 \\ & X_{jj} \ = \ x_j, \forall j \in \mathcal{J}, \ Y \ = \ \begin{bmatrix} 1 & x^T \\ x & X \end{bmatrix} \in \mathbb{C}_{n+1}^*. \end{array}$$

This is a linear optimization problem over the cone of completely positive matrices (we call such problem a copositive programming problem)

In this paper we generalize this result further. We prove that we can replace in the Burer's result the nonnegativity constraint $x \in \mathbb{R}_+^n$ by the more general constraint $x \in K$ for some arbitrary nonempty closed convex cone $K \subseteq \mathbb{R}^n$. This is the main contribution of this paper.

In the second part of the paper we present how to use this result to obtain tractable relaxations for optimization problems over Bishop-Phelps cones. We shall mention that at the very last stage of preparation of this paper we realized that a very similar generalization was obtained independently by Burer [4]. However, there remains a substantial difference between the papers since we focus in the sequel on the Bishop-Phelps cone while the rest of [4] is a review of the existing results.

The nonnegativity constraint $x \in \mathbb{R}_+^n$ can be interpreted by assuming that the space $\mathbb{R}^n$ is partially ordered by the natural (or componentwise) ordering. Any partial ordering $\leq$, i.e. any

[1] Supported by the Slovenian Research Agency (project no. 1000-08-210518).

reflexive and transitive binary relation which is compatible with the linear structure of the space, can be represented by a convex cone defined by $\{x \in \mathbb{R}^n : x \geq 0\}$. On the other hand, any convex cone $K \subseteq \mathbb{R}^n$ defines a partial ordering in $\mathbb{R}^n$ by $x \leq_K y$ if and only if $y - x \in K$, i.e. $x \in K$ corresponds to $x \geq_K 0$ w.r.t. the partial ordering given by the cone K. Hence $x \in K$ is just a more general nonnegativity constraint inducing a wider class of optimization problems as e.g. second order cone programming.

One of the motivating problems for our research comes from vector optimization problems, i.e. from optimization problems with a vector valued objective function $f : \mathbb{R}^m \to \mathbb{R}^n$. Assuming the space $\mathbb{R}^n$ to be ordered by some pointed convex cone K, a point $\bar{x} \in \mathbb{R}^m$ is called a *minimal solution* of $\inf_{x \in S} f(x)$ with S a nonempty subset of $\mathbb{R}^m$ if $(f(\bar{x}) - K) \cap f(S) = \{f(\bar{x})\}$. Recall that a cone K is called pointed if $K \cap (-K) = \{0\}$. Such minimal solutions can be determined by scalarization techniques which may result in an optimization problem with a cone constraint. For instance using the scalarization result introduced in [12] for two parameters $a, r \in \mathbb{R}^n$ yields the problem

$$\begin{gathered} \inf t \\ \text{such that} \\ t\,r - f(x) - y = a, \\ y \in K, \\ t \in \mathbb{R}, \\ x \in S. \end{gathered}$$

If f is a linear objective function and S is defined by linear equations and binary constraints, then the technique presented in this paper yields a reformulation of this problem as a linear problem over a special cone eliminating the binary constraints.

The remaining of this paper is structured as follows: in Section 2 we recall some basic definitions and results for the cone of set-semidefinite matrices and we give the basic idea of our main result, which is then given in Section 3. Finally, in Section 4, we discuss nonconvex quadratic optimization problems over Bishop-Phelps cones in Euclidean spaces and give relaxations of the reformulated problems as semidefinite programs.

2. Definitions and Preliminaries

In [3] the reformulation of the nonconvex quadratic optimization problem is done over the cone of *completely positive* matrices

$$C^*_{\mathbb{R}^n_+} := \left\{ A \in \mathcal{S}^n : A = \sum_{i=1}^{n} a_j a_j^\top,\ a_j \in \mathbb{R}^n_+,\ j = 1, \ldots, m \right\} ,$$

which is the dual cone of the cone of *copositive* matrices defined by

$$C_{\mathbb{R}^n_+} := \left\{ A \in \mathcal{S}^n : x^\top A x \geq 0 \ \text{ for all } \ x \in \mathbb{R}^n_+ \right\} . \tag{1}$$

Here, $\mathcal{S}^n$ denotes the space of real symmetric $n \times n$ matrices equipped with the inner product defined by $\langle A, B \rangle := \operatorname{trace}(AB)$ for all $A, B \in \mathcal{S}^n$. Recall that the dual cone of a cone C in a topological linear space X is in general defined by

$$C^* := \{x^* \in X^* : x^*(x) \geq 0 \text{ for all } x \in C\} .$$

with X^* denoting the topological dual space, i.e. the space of all continuous linear maps from X to $\mathbb{R}$.

Replacing $\mathbb{R}^n_+$ in (1) by an arbitrary nonempty set $K \subseteq \mathbb{R}^n$ (later we assume K to be a nonempty closed convex cone) we get the cone

$$C_K := \left\{A \in \mathcal{S}^n : x^\top Ax \geq 0 \text{ for all } x \in K\right\}$$

which is called K-*semidefinite* (or *set-semidefinite*) cone. In opposition to [5, 6] we define here the K-semidefinite cone in the subspace of symmetric matrices instead of in the whole space of linear maps mapping from the finite dimensional Euclidean space $\mathbb{R}^n$ to $\mathbb{R}^n$. The K-semidefinite cone is a convex cone and hence defines itself a partial ordering in the space of symmetric matrices.

Under the assumptions here, i.e. $K \subseteq \mathbb{R}^n$, the dual cone of the K-semidefinite cone was given in [16] and in [8, Lemma 7.5]:

Lemma 2.1. (a) *Let $K \subseteq \mathbb{R}^n$ be a nonempty given set, then*

$$C_K^* = \text{cl cone } \{xx^\top : x \in K\}.$$

(b) *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone, then*

$$C_K^* = \text{conv}\{xx^\top : x \in K\}$$

and C_K^* is closed.

Here cone(Ω) for some set Ω denotes the convex cone generated by the set Ω, conv(Ω) is the convex hull and cl(Ω) is the closure of the set Ω. In the following we additionally assume the set $K \subseteq \mathbb{R}^n$ to be a nonempty closed convex cone. We start by summing up some basic properties, compare [7, 5].

Propositio 2.2. *Let $K_1, K_2, K \subseteq \mathbb{R}^n$ be closed convex nontrivial cones in $\mathbb{R}^n$.*

(i) $C_{K_1}^* + C_{K_2}^* \subseteq (C_{K_1} \cap C_{K_2})^* = (C_{K_1 \cup K_2})^*$

(ii) *$K_1 \subseteq K_2$ implies $C_{K_2} \subseteq C_{K_1}$ and $C_{K_1}^* \subseteq C_{K_2}^*$.*

(iii) $(C_{K_1} \cup C_{K_2})^* = C_{K_1}^* \cap C_{K_2}^*$

(iv) *For the interior of the cone C_K it holds*

$$int(C_K) = \{A \in \mathcal{S}^n : x^\top Ax > 0 \text{ for all } x \in K \setminus \{0\}\} \neq \varnothing$$

and thus the dual cone C_K^* is pointed.

Since K is a cone in $\mathbb{R}^n$, we can use the Carathéodory theorem and represent the dual cone by

$$C_K^* = \left\{ \sum_{i=1}^{(n(n+1)/2)+1} x^i(x^i)^\top : x^i \in K,\ \forall i = 1, \ldots, \frac{n(n+1)}{2} + 1 \right\}.$$

For shortness of the representation we omit the upper limit $p := (n(n+1)/2) + 1$ in the sum above and write instead in the following $C_K^* = \left\{\sum_i x^i(x^i)^\top : x^i \in K\right\}$. The following lemma is the base for our main result and states that the minimal value of a linear function over this dual cone is always attained in a matrix which can be written as $xx^\top$ for some $x \in K$.

Lemma 2.3. *Let a matrix $Q \in \mathcal{S}^n$ and a nonempty set $S \subseteq \mathbb{R}^n$ be given. If the matrix $\overline{Y}$ is a minimal solution of*

$$\begin{array}{c} \inf \langle Q, Y\rangle \\ \text{such that} \\ Y \in \text{conv}\{xx^\top : x \in S\}, \end{array} \qquad (P')$$

then there exists some $\bar{x} \in S$ such that $\bar{x}\bar{x}^\top$ is also a minimal solution of (P'), i.e.

$$\langle Q, \bar{x}\bar{x}^\top\rangle = \langle Q, \overline{Y}\rangle,$$

and $\bar{x}$ is also a minimal solution of

$$\begin{array}{c} \textit{inf } \langle Q, Y \rangle \\ \textit{such that} \\ Y = xx^\top, \\ x \in S. \end{array} \tag{P_C}$$

Hence, the optimization problems (P') and (P_C) are equivalent regarding the minimal value.

Proof. Let $\overline{Y}$ be a minimal solution of (P'). Then there exists some $k \in \mathbb{N}$, some $x^i \in S$ and $\lambda_i \geq 0$ for $i = 1, \ldots, k$ with $\sum_i \lambda_i = 1$ and $\overline{Y} = \sum_i \lambda_i x^i (x^i)^\top$. Let $j \in \{1, \ldots, k\}$ such that $(x^j)^\top Q x^j = \min_i \{(x^i)^\top Q x^i\}$, then

$$\langle Q, \overline{Y} \rangle = \sum_i \lambda_i (x^i)^\top Q x^i \geq \left(\sum_i \lambda_i \right) (x^j)^\top Q x^j = (x^j)^\top Q x^j = \langle Q, x^j (x^j)^\top \rangle.$$

As $\overline{Y}$ is minimal for (P') and $x^j (x^j)^\top$ is also feasible for (P') we get $\langle Q, \overline{Y} \rangle = \langle Q, x^j (x^j)^\top \rangle$. Of course, x^j is then also a minimal solution of (P_C). ■

3. Set-semidefinite reformulation of quadratic programs

In this section we examine the equivalence between a quadratic optimization problem with linear constraints, a cone constraint and binary variables, and the relaxed problem over the dual cone of set-semidefinite matrices. Let $Q \in \mathcal{S}^n$ be a symmetric matrix, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $K \subseteq \mathbb{R}^n$ a nonempty closed and convex cone and $B \subseteq \{1, \ldots, n\}$ an index set. We study the following quadratic optimization problem

$$\begin{array}{c} \inf x^\top Q x + 2c^\top x \\ \text{such that} \\ Ax = b, \\ x_j \in \{0,1\} \text{ for all } j \in B, \\ x \in K. \end{array} \tag{QP}$$

We can reformulate (QP) by introducing

$$Y = \begin{pmatrix} 1 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^\top \in \mathcal{S}^{n+1} \tag{2}$$

to obtain

$$\begin{array}{rcl} OPT_P & := & \inf \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\ & & \text{such that} \\ Y & = & \begin{pmatrix} 1 \\ x \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^\top \\ Ax & = & b, \\ x_j & = & x_j^2 \text{ for all } j \in B, \\ x & \in & K. \end{array} \tag{QP'}$$

A natural linearization and lifting of the problem (QP') into the dual cone of $C_{\mathbb{R}_+ \times K}$ generated by dyadic products of the type (2):

$$C^*_{\mathbb{R}_+ \times K} = \left\{ \sum_i \begin{pmatrix} \alpha_i \\ v^i \end{pmatrix} \begin{pmatrix} \alpha_i \\ v^i \end{pmatrix}^\top : \alpha_i \in \mathbb{R}_+,\ v^i \in K \right\}$$

yields the following linear problem:

$$
\begin{aligned}
OPT_C &:= \inf \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle \\
&\text{such that} \\
Y &= \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}, \\
Y &\in \mathcal{C}^*_{\mathbb{R}_+ \times K}, \\
Ax &= b, \\
x_j &= X_{jj} \text{ for all } j \in B, \\
\operatorname{Diag}(AXA^\top) &= b \circ b := \begin{pmatrix} b_1^2 \\ \vdots \\ b_n^2 \end{pmatrix}, \\
& x \in \mathbb{R},\ X \in \mathcal{S}^n.
\end{aligned}
\tag{QP$_C$}
$$

The main difference to the problems considered in [3] is that here we replace the inequalities $x \in \mathbb{R}^n_+$ by the constraint $x \in K$ for an arbitrary closed convex cone K.

We denote the feasible set of the problem (QP), which will be assumed to be nonempty, by Feas(P), and the feasible set of (QP$_C$) by Feas(C).

Let $L := \{x \in K\colon Ax = b\}$. Then we follow the line of [3] and assume in the following:

Assumption 3.1. If $x \in L$ then $x_j \in [0, 1]$ for all $j \in B$.

Remark 3.2. Assumption 3.1 is not very restrictive. Suppose that it does not hold for some x_j, $j \in B$, e.g. $x_j \in L$ does not imply $x_j \in [0, 1]$. Then we can add two more equations $x_j + y_j = 1$, $x_j - z_j = 0$ and two sign constraints: $y_j, z_j \geq 0$. Hence using

$$L' = \{(x, y_j, z_j) \in K \times \mathbb{R}^2_+ \colon Ax = b,\ x_j + y_j = 1,\ x_j - z_j = 0\}$$

the assumption holds for x_j and $K' := K \times \mathbb{R}^2_+$ is still a closed convex cone.

We denote the feasible set of the problem (QP), which will be assumed to be nonempty, by Feas(P), and the feasible set of (QP$_C$) by Feas(C).

Let $L := \{x \in K\colon Ax = b\}$. Then we follow the line of [3] and assume in the following:

Assumption 3.1. If $x \in L$ then $x_j \in [0, 1]$ for all $j \in B$.

Remark 3.2. Assumption 3.1 is not very restrictive. Suppose that it does not hold for some x_j, $j \in B$, e.g. $x_j \in L$ does not imply $x_j \in [0, 1]$. Then we can add two more equations $x_j + y_j = 1$, $x_j - z_j = 0$ and two sign constraints: $y_j, z_j \geq 0$. Hence using

$$L' = \{(x, y_j, z_j) \in K \times \mathbb{R}^2_+ \colon Ax = b,\ x_j + y_j = 1,\ x_j - z_j = 0\}$$

the assumption holds for x_j and $K' := K \times \mathbb{R}^2_+$ is still a closed convex cone.

If the set B is empty this assumption is dispensable. Additionally we define

$$
\begin{aligned}
L_\infty &:= \{d \in K \colon Ad = 0\}, \\
L_\infty^+ &:= \operatorname{cone}\left\{ \begin{pmatrix} 0 \\ d \end{pmatrix} \begin{pmatrix} 0 \\ d \end{pmatrix}^\top : d \in L_\infty \right\}, \\
\operatorname{Feas}^+(C) &:= \left\{ \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} : (x, X) \in \operatorname{Feas}(C) \right\}, \\
\operatorname{Feas}^+(P) &:= \operatorname{conv}\left\{ \begin{pmatrix} 1 \\ x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix}^\top : x \in \operatorname{Feas}(P) \right\}.
\end{aligned}
$$

Lemma 3.3. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and let Assumption 3.1 be satisfied. Then*

$$\mathit{Feas}^+(P) \subseteq \mathit{Feas}^+(C) \quad \text{and} \quad \mathit{Feas}^+(P) + L_\infty^+ \subseteq \mathit{Feas}^+(C).$$

Proof. The first part of the assertion is obvious. For the second part, following the ideas given in the proof of Prop. 2.1 in [3], we consider the convex cone

$$\begin{aligned} L(C)_\infty := & \left\{ \begin{pmatrix} 0 & x^\top \\ x & X \end{pmatrix} \in \mathcal{C}^*_{\mathbb{R}_+\times K} : Ax = 0,\ \mathrm{Diag}(AXA^\top) = 0,\ x_j = X_{jj} \text{ for all } j \in B \right\} \\ = & \left\{ \begin{pmatrix} 0 & 0^\top \\ 0 & X \end{pmatrix} \in \mathcal{C}^*_{\mathbb{R}_+\times K} : \mathrm{Diag}(AXA^\top) = 0,\ X_{jj} = 0 \text{ for all } j \in B \right\} \end{aligned}$$

for which it holds $\mathrm{Feas}^+(C) + L(C)_\infty \subseteq \mathrm{Feas}^+(C)$. Noting that by Assumption 3.1 $d \in L_\infty$ implies $d_j = 0$ for all $j \in B$, it can easily be seen that $L_\infty^+ \subseteq L(C)_\infty$ and thus $\mathrm{Feas}^+(P) + L_\infty^+ \subseteq \mathrm{Feas}^+(C)$. ■

Lemma 3.4. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and let Assumption 3.1 be satisfied. Then*

$$\mathit{Feas}^+(C) = \mathit{Feas}^+(P) + L_\infty^+.$$

Proof. Also this proof follows the ideas given in the proof of Prop. 2.1 in [3]. By Lemma 3.3, it remains to show $\mathrm{Feas}^+(C) \subseteq \mathrm{Feas}^+(P) + L_\infty^+$. Let $Y \in \mathrm{Feas}^+(C)$. Then because of $Y \in \mathcal{C}^*_{\mathbb{R}_+\times K}$ there exists some $k \in \mathbb{N}$ and $\alpha_i \in \mathbb{R}_+$, $v^i \in K$ for $i = 1, \ldots, k$ with

$$Y = \sum_i \begin{pmatrix} \alpha_i \\ v^i \end{pmatrix} \begin{pmatrix} \alpha_i \\ v^i \end{pmatrix}^\top = \sum_i \begin{pmatrix} \alpha_i^2 & \alpha_i v^{i\top} \\ \alpha_i v^i & v^i v^{i\top} \end{pmatrix}$$

and $(\alpha_i, v^{i\top}) \neq 0$ for all $i = 1, \ldots, k$. As Y satisfies the constraints in $(\mathrm{QP_C})$, it holds that

$$\sum_i \alpha_i^2 = 1, \tag{3}$$

$$\sum_i \alpha_i a^j v^i = b_j \text{ for all } j = 1, \ldots, m, \tag{4}$$

with a^j the j-th row of the matrix A, and due to $\mathrm{Diag}(AXA^\top) = b \circ b$ also

$$\sum_{i=1}^k (a^j v^i)^2 = b_j^2 \text{ for all } j = 1, \ldots, m. \tag{5}$$

Then (3), (5) and (4) yield

$$\left(\sum_{i=1}^k \alpha_i^2\right) \cdot \sum_{i=1}^k (a^j v^i)^2 = b_j^2 = \left(\sum_{i=1}^k \alpha_i a^j v^i\right)^2$$

and we get

$$a^j v^i = \tau_j \alpha_i \text{ for } i = 1, \ldots, k$$

for some $\tau_j \in \mathbb{R}$ and with (4) and (3) we conclude $\tau_j = b_j$, $j = 1, \ldots, m$, i.e.

$$a^j v^i = b_j \alpha_i \text{ for } i = 1, \ldots, k,\ j = 1, \ldots, m. \tag{6}$$

Additionally for $j \in B$ it holds

$$\sum_{i=1}^{k} \alpha_i v_j^i = \sum_{i=1}^{k} (v_j^i)^2. \tag{7}$$

We define to $I := \{1, \ldots, k\}$ the index sets $I^+ := \{i \in I : \alpha_i \neq 0\}$ and $I^0 := \{i \in I : \alpha_i = 0\} = I \setminus I^+$. Then we have

$$Y = \underbrace{\sum_{i \in I^+} \alpha_i^2 \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} v^i \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{\alpha_i} v^i \end{pmatrix}^\top}_{:=Y^1} + \underbrace{\sum_{i \in I^0} \begin{pmatrix} 0 \\ v^i \end{pmatrix} \begin{pmatrix} 0 \\ v^i \end{pmatrix}^\top}_{:=Y^2} \tag{8}$$

with $\sum_{i \in I^+} \alpha_i^2 = 1$, see (3). Next we show $Y^1 \in \mathrm{Feas}^+(P)$ and $Y_2 \in L_\infty^+$.

Let $i \in I^+$. For showing $\frac{1}{\alpha_i} v^i \in \mathrm{Feas}(P)$ we use that $v^i \in K$, K is a cone and $\frac{1}{\alpha_i} > 0$ and thus $\frac{1}{\alpha_i} v^i \in K$. With (6) we have $a^j \left(\frac{1}{\alpha_i} v^i \right) = b_j$ for $j = 1, \ldots, m$. It remains to show

$$\frac{1}{\alpha_i} v_j^i \in \{0, 1\} \text{ for } j \in B.$$

Let $j \in B$. Based on the Assumption 3.1 $\frac{1}{\alpha_i} v_j^i \in [0, 1]$. From (7) and by setting $z_j^i := v_j^i / \alpha_i \in [0, 1]$ we get the equation

$$\sum_{i \in I^+} \alpha_i^2 \left(\frac{v_j^i}{\alpha_i} - \left(\frac{v_j^i}{\alpha_i} \right)^2 \right) = \sum_{i \in I^+} \underbrace{\alpha_i^2}_{>0} \underbrace{(z_j^i - (z_j^i)^2)}_{\geq 0} = 0$$

which implies $z_j^i - (z_j^i)^2 = 0$, i.e. $z_j^i = v_j^i / \alpha_i \in \{0, 1\}$.

Thus we have $Y^1 \in \mathrm{Feas}^+(P)$. Using (6) and $\alpha_i = 0$ for $i \in I^0$ we conclude that $Y_2 \in L_\infty^+$. ∎

Hence for $K \subseteq \mathbb{R}^n$ a nonempty closed convex cone we have

$$\mathrm{Feas}^+(P) \subseteq \mathrm{Feas}^+(C) = \mathrm{Feas}^+(P) + L_\infty^+.$$

We define for the objective functions of the problems (QP) and (QP$_\mathrm{C}$)

$$\begin{aligned} v_P(x) &:= x^\top Q x + 2c^\top x, \\ v_C(Y) &:= \left\langle \underbrace{\begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}}_{:=\widetilde{Q}}, Y \right\rangle. \end{aligned}$$

Instead of (x, X) a feasible element or a minimal solution of (QP$_\mathrm{C}$) we use the sometimes shorter notation that Y with

$$Y = \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}$$

is a feasible element or a minimal solution of (QP$_\mathrm{C}$). Thus we identify the problem

$$\inf_{Y \in \mathrm{Feas}^+(C)} \langle \widetilde{Q}, Y \rangle$$

with the problem (QP$_\mathrm{C}$).

Corollary 3.5. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and let Assumption 3.1 be satisfied. Then the following holds*

$$OPT_P \geq OPT_C. \tag{9}$$

Proof. Let $\bar{x} \in \text{Feas}(P)$. From Lemma 3.3 we conclude that the matrix $\bar{Y} = (1, \bar{x}^\top)^\top(1, \bar{x}^\top)$ is feasible for (QP_C) and $v_P(\bar{x}) = v_C(\bar{Y})$, hence for every feasible point of (QP) we have a feasible matrix of (QP_C) with the same objective value. Since we are looking for minimum value, the assertion follows. ∎

Theorem 3.6. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone and let Assumption 3.1 be satisfied. The the following is true:*

$$OPT_P = OPT_C$$

Proof. The proof follows the ideas of the proof of Lemma 2.3. Due to Corollary 3.5 we need to prove only $OPT_P \leq OPT_C$.

Suppose that $Y \in \text{Feas}^+(C)$. Lemma 3.4 implies that $Y = Y_1 + Y_2$ with $Y_1 \in \text{Feas}^+(P)$ and $Y_2 \in L_\infty^+$. By definition we can write

$$Y_1 = \sum_{i \in I} \lambda_i \begin{pmatrix} 1 \\ z^i \end{pmatrix} \begin{pmatrix} 1 \\ z^i \end{pmatrix}^\top$$

with $z^i \in \text{Feas}(P)$ for $i \in I$ and $\sum_{i \in I} \lambda_i = 1$, where I is some finite index set. Similarly we have

$$Y_2 = \sum_{i \in J} \begin{pmatrix} 0 \\ d^i \end{pmatrix} \begin{pmatrix} 0 \\ d^i \end{pmatrix}^\top,$$

with $d^i \in L_\infty$, and J another finite index set.

Set

$$\bar{z} \in \text{argmin}\{(1, z^{i\top})\widetilde{Q}(1, z^{i\top})^\top : i \in I\}.$$

Then

$$\begin{aligned} \left\langle \widetilde{Q}, Y_1 \right\rangle &= \sum_{i \in I} \lambda_i \left\langle \widetilde{Q}, \begin{pmatrix} 1 \\ z^i \end{pmatrix} \begin{pmatrix} 1 \\ z^i \end{pmatrix}^\top \right\rangle \\ &\geq \left\langle \widetilde{Q}, \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{z} \end{pmatrix}^\top \right\rangle. \end{aligned}$$

To finish the proof we need to consider $\langle \widetilde{Q}, Y_2 \rangle$:

- If $\langle \widetilde{Q}, Y_2 \rangle < 0$ then there exists $\bar{d} \in L_\infty$ such that $\bar{d}^\top Q \bar{d} < 0$ and for any feasible $x \in \text{Feas}(P)$ there exists $\bar{\tau} > 0$ such that $(x + \tau\bar{d})^\top Q(x + \tau\bar{d}) = x^\top Qx + 2\tau x^\top Q\bar{b} + \tau^2\bar{d}^\top Q\bar{d} < 0$ for every $\tau > \bar{\tau}$. Since $x + \tau\bar{d} \in \text{Feas}(P)$ for every $\tau > 0$ sending τ to infinity implies that $OPT_P = -\infty$, hence by Corollary 3.5 we have $OPT_P = OPT_C = -\infty$.
- If $\langle \widetilde{Q}, Y_2 \rangle \geq 0$, then $\langle \widetilde{Q}, Y \rangle \geq \langle \widetilde{Q}, Y_1 \rangle \geq \bar{z}^\top Q\bar{z} + 2c^\top \bar{z}$. Hence we found for this particular Y a feasible point ($\bar{z}$) for (QP) with smaller objective value.

Since Y was chosen arbitrary, the theorem is proven. ∎

Lemma 3.7. *Let $K \subseteq \mathbb{R}^n$ be a nonempty closed convex cone, let Assumption 3.1 be satisfied and let Feas(P) be bounded, then Feas$^+(P) =$ Feas(C).*

Proof. If Feas(P) is bounded then $L_\infty = \varnothing$ and the assertion follows by Lemma 3.4. ∎

4. Optimization over the Bishop-Phelps Cone

In 1962, Bishop and Phelps [1] introduced a class of ordering cones which have a rich mathematical structure and which have proven to be useful for instance in functional analysis and vector optimization. Well known cones as the nonnegative orthant or the Lorentz cone are special Bishop-Phelps cones.

Definition 4.1. For an arbitrary continuous linear functional $\phi\colon Y \to \mathbb{R}$ on the normed space $(Y, \|\cdot\|)$ the cone

$$\mathrm{K}_\phi := \{y \in Y\colon \|y\| \leq \phi(y)\}$$

is called *Bishop-Phelps cone.*

Note that the definition of Bishop-Phelps cone (BP cone) introduced in [1] is slightly different from the above one: Bishop and Phelps required that $\|\phi\| = 1$ and $t\|y\| \leq \phi(y)$ for some scalar $t \in (0, 1)$. Nowadays, several authors, see for instance [9], do not use the constant t and the assumption $\|\phi\| = 1$ and the Definition 4.1 follows this line.

We first collect some properties of BP cones [9]. Recall that a base B_K of a nontrivial convex cone K is a nonempty convex subset such that each element $x \in K \setminus \{0\}$ is uniquely representable as $x = \lambda b$ for some $\lambda > 0$ and some $b \in B_K$. The norm $\|\cdot\|$ on Y induces also a norm on the topological dual space Y^* by

$$\|y^*\| := \sup_{y \neq 0} \frac{|y^*(y)|}{\|y\|} \quad \text{for all} \quad y^* \in Y^*.$$

Propositio 4.2. *Let $(Y, \|\cdot\|)$ be a normed space and let $\phi \in Y^*$ be given.*

(i) *K_ϕ is a closed, pointed and convex cone.*
(ii) *If $\|\phi\| > 1$ then K_ϕ is nontrivial and*

$$\mathrm{int}(\mathrm{K}_\phi) = \{y \in Y\colon \|y\| < \phi(y)\}.$$

If $\|\phi\| < 1$ then $\mathrm{K}_\phi = \{0\}$.
(iii) *If $\|\phi\| > 1$ then $B_K := \{y \in \mathrm{K}_\phi\colon \phi(y) = 1\}$ is a closed and bounded base for the cone K_ϕ.*
(iv) *The dual cone is $\mathrm{K}_\phi{}^* = cl\{\lambda z \in Y^*\colon \lambda \geq 0,\ z \in B(\phi, 1)\} \subseteq Y^*$ with $B(\phi, 1) := \{y^* \in Y^*\colon \|y^* - \phi\| \leq 1\}$.*
(iv) *$x \in \mathrm{K}_\phi$ if and only if the matrix*

$$M(x) := \begin{pmatrix} \phi(x) & \|x\| \\ \|x\| & \phi(x) \end{pmatrix}$$

is positive semidefinite

According to [13] every nontrivial convex cone in $\mathbb{R}^n$ is representable as a BP cone if and only if it is closed and pointed. But note that in $\mathbb{R}^n$ one might need different equivalent norms to present different nontrivial convex closed pointed cones as BP cones.

4.1. **BP cones with Euclidean norm.** Let us consider the case where $Y = \mathbb{R}^n$ with the Euclidean norm $\|\cdot\|_2$. Then $\phi \in \mathbb{R}^n$. We have

$$\mathrm{K}^2_\phi = \{x \in \mathbb{R}^n \colon \|x\|_2 \le \phi^\top x\} = \{x \in \mathbb{R}^n \colon x^\top(\phi\phi^\top - I)x \ge 0,\ \phi^\top x \ge 0\}. \tag{10}$$

Example 4.3. (a) The Lorentz cone (or second order cone or ice-cream cone - see e.g. [11] for definitions and applications of second order cone programming)

$$K_L := \{y \in \mathbb{R}^n \colon \|(y_1,\ldots,y_{n-1})\|_2 \le y_n\}$$

is representable as a BP cone using the Euclidean norm and choosing $\phi := \sqrt{2}e_n$ with e_n denoting the nth unit vector [9, Lemma 2.4(a)], i.e.

$$K_L = \{y \in \mathbb{R}^n \colon \|y\|_2 \le \sqrt{2}e_n^\top y\}.$$

(b) In the Euclidean space $\mathbb{R}^2$ Figure 1 illustrates the relation between $\phi = (\phi_1, \phi_2) \in \mathbb{R}^2$, to be more concrete between $1/\phi_1$ and $1/\phi_2$, and the represented BP cone $\mathrm{K}^2_\phi = \{y \in \mathbb{R}^2 \colon \|y\|_2 \le \phi^\top y\}$.

Next we illustrate how a representation as a BP cone (using the Euclidean norm) of an arbitrary closed convex pointed cone in $\mathbb{R}^2$ can be constructed. Let $a \in \mathbb{R}^2$ and $b \in \mathbb{R}^2$ denote the intersection points of the boundary of the cone and the unit ball w.r.t. the

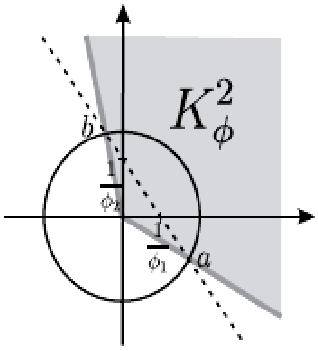

FIGURE 1. BP cone K^2_ϕ of Example 4.3(b) as well as the unit ball w.r.t. the Euclidean norm and (in dashed line) the line connecting the points $(1/\phi_1, 0)$ and $(0, 1/\phi_2)$.

Euclidean norm. Assume $a_1 \neq b_1$ and $a_2 \neq b_2$. Then the line connecting a and b is given by all points (x_1, x_2) with

$$x_2 = \frac{a_2 - b_2}{a_1 - b_1}x_1 + \frac{a_1b_2 - a_2b_1}{a_1 - b_1}.$$

This line intersects the coordinate axes in the points

$$\left(\frac{a_2b_1 - a_1b_2}{a_2 - b_2}, 0\right) \quad \text{and} \quad \left(0, \frac{a_1b_2 - a_2b_1}{a_1 - b_1}\right).$$

Setting

$$\phi_1 := \frac{a_2 - b_2}{a_2b_1 - a_1b_2} \quad \text{and} \quad \phi_2 := \frac{a_1 - b_1}{a_1b_2 - a_2b_1}$$

the linear functional ϕ describes by $\mathrm{K}^2_\phi = \{y \in \mathbb{R}^2 \colon \|y\|_2 \le \phi^\top y\}$ the given cone.

We can say more on the relation between BP cones with Euclidean norm and the second order cones.

Lemma 4.4. *Let $K_L \subseteq \mathbb{R}^n$ be the second order cone and $\mathrm{K}_\phi^2 \subseteq \mathbb{R}^n$ a BP cone with respect to the Euclidian norm and the linear operator $\phi \in \mathbb{R}^n$, $\|\phi\|_2 > 1$. There exists a nonsingular linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathrm{K}_\phi^2 = T(K_L)$.*

Proof. Note that K_L and K_ϕ^2 are determined by quadratic forms

$$f_L(x) = \sum_{i=1}^{n-1} x_i^2 - x_n^2 = x^\top \underbrace{(I_n - 2E_{nn})}_{:=A_L} x = x^\top A_L x, \tag{11}$$

$$f_K(x) = \sum_{i=1}^{n} x_i^2 - x^\top \phi\phi^\top x = x^\top \underbrace{(I - \phi\phi^\top)}_{:=A_\phi} x = x^\top A_\phi x, \tag{12}$$

i.e. $K_L = \{x \in \mathbb{R}^n : x^\top A_L x \leq 0, x_n \geq 0\}$ and $\mathrm{K}_\phi^2 = \{x \in \mathbb{R}^n : x^\top A_\phi x \leq 0, \phi^\top x \geq 0\}$.

The rank of the matrix A_L is n and its eigenvalues are 1 (with order $n-1$) and -1 (with order 1). Similarly we find out that A_K has eigenvalues 1 (with order $n-1$) and $1 - \|\phi\|_2^2 < 1$ (with order 1) to the eigenvector ϕ.

As A_ϕ is a symmetric matrix we ca find a spectral decomposition of A_ϕ by $A_\phi = \widetilde{V}\widetilde{\Lambda}\widetilde{V}^\top$, where $\widetilde{\Lambda} = \mathrm{Diag}(1, \dots, 1, 1 - \|\phi\|_2^2)$ and $\widetilde{V}$ contains an orthonormal basis of eigenvectors of A_ϕ, i.e. the last column of $\widetilde{V}$ is $(1/\|\phi\|_2)\phi$. By setting $\Lambda := \mathrm{Diag}(1, \dots, 1, -1)$ and $V := \widetilde{V}\,\mathrm{Diag}(1, \dots, 1, \sqrt{\|\phi\|_2^2 - 1})$ we get $A_\phi = V\Lambda V^\top$ with $V^\top V = \mathrm{Diag}(1, \dots, 1, \|\phi\|_2^2 - 1) =: D_\phi$. Then

$$V^{-1} = D_\phi^{-1} V^\top . \tag{13}$$

The last column of V is $(\sqrt{\|\phi\|_2^2 - 1}/\|\phi\|_2)\phi$. Let us take $T := V^{-\top}$. It is a nonsingular matrix, defining a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^n$. It follows

$$(\phi^\top T)^\top = T^\top \phi = V^{-1}\phi = D_\phi^{-1} V^\top \phi = D_\phi^{-1}\sqrt{(\|\phi\|_2^2 - 1)\|\phi\|_2^2}\, e_n = \sqrt{\frac{\|\phi\|_2^2}{\|\phi\|_2^2 - 1}}\, e_n .$$

By substitution $x = Tu$ and noting that $A_L = \Lambda$ it follows

$$\begin{aligned}
\mathrm{K}_\phi^2 &= \{x \in \mathbb{R}^n : x^\top A_\phi x \leq 0,\ \phi^\top x \geq 0\} \\
&= \{Tu \in \mathbb{R}^n : u^\top T^\top A_\phi T u \leq 0,\ \phi^\top T u \geq 0\} \\
&= T(\{u \in \mathbb{R}^n : u^\top A_L u \leq 0,\ e_n^\top u \geq 0\}) \\
&= T(K_L)
\end{aligned}$$

∎

The following example shows that the assumption $\|\phi\|_2 > 1$ is essential.

Example 4.5. Consider the BP cone $\mathrm{K}_\phi^2 \subseteq \mathbb{R}^3$ for $\phi := (0, 0, 1)^\top$. Then $\mathrm{K}_\phi^2 = \{y \in \mathbb{R}^3 : y_1 = y_2 = 0,\ y_3 \geq 0\}$ has an empty interior, but the Lorentz cone has a nonempty interior. So, there exists no nonsingular linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ with $\mathrm{K}_\phi^2 = T(K_L)$. But there exists a transformation map $t\colon \mathbb{R} \to \mathbb{R}^3$, $t(y) := (0, 0, y)^\top$ for all $y \in \mathbb{R}$ with $\mathrm{K}_\phi^2 = t(K_L)$ for the Lorentz cone $K_L = \{y \in \mathbb{R} : y \geq 0\}$ in $\mathbb{R}$.

We will keep considering the BP cone K^2_ϕ even though we could stick to the second order cone. The reason is that after the transformation described in Lemma 4.4 the structure of the problem gets less transparent.

Following the procedure from the previous section, we can show that the optimization problem:

$$\begin{array}{c} \inf x^\top Qx + 2c^\top x \\ \text{such that} \\ Ax = b, \\ x_j \in \{0,1\} \ \text{ for all } \ j \in B, \\ x \in \mathrm{K}^2_\phi \end{array} \tag{P_{BP}}$$

has according to Theorem 3.6, under the assumptions mentioned there, the same optimal value as

$$\begin{array}{rcl} & \inf \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle & \\ & \text{such that} & \\ Y & = & \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}, \\ Y & \in & \mathcal{C}^*_{\mathbb{R}_+ \times \mathrm{K}^2_\phi}, \\ Ax & = & b, \\ x_j & = & X_{jj} \ \text{ for all } j \in B, \\ \mathrm{Diag}(AXA^\top) & = & b \circ b, \\ & x \in \mathbb{R}, \ X \in \mathcal{S}^n. & \end{array} \tag{C_{BP}}$$

Note that

$$\mathcal{C}^*_{\mathbb{R}_+ \times \mathrm{K}^2_\phi} = \left\{ \sum_i \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix} \cdot \begin{pmatrix} \alpha_i \\ x^i \end{pmatrix}^\top : \|x^i\|_2 \le \phi^\top x^i, \ \alpha_i \ge 0 \right\}. \tag{14}$$

Lemma 4.6. *Let $x \in \mathbb{R}^n$, $X \in \mathcal{S}^n$, $\phi \in \mathbb{R}^n$ and*

$$Y = \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}.$$

If $Y \in \mathcal{C}^*_{\mathbb{R}_+ \times \mathrm{K}^2_\phi}$ then

$$\textit{trace}(X) \le \langle \phi\phi^\top, X \rangle, \ \ x \in \mathrm{K}^2_\phi \ \ \textit{and } Y \textit{ is positive semidefinite.}$$

Proof. Note that if $Y \in \mathcal{C}^*_{\mathbb{R}_+ \times \mathrm{K}^2_\phi}$ then using (14) it follows $X = \sum_i x^i (x^i)^\top$ and $x = \sum_i \alpha_i x^i$, where $\alpha_i \ge 0$, $\|x^i\|_2 \le \phi^\top x^i$ for all i. For $Y \in \mathcal{C}^*_{\mathbb{R}_+ \times \mathrm{K}^2_\phi}$ we conclude

$$\begin{aligned} \mathrm{trace}(X) &= \mathrm{trace}\left(\textstyle\sum_i x^i (x^i)^\top\right) = \textstyle\sum_i \|x^i\|_2^2 \\ &\le \textstyle\sum_i (x^i)^\top \phi\phi^\top x^i = \textstyle\sum_i \langle \phi\phi^\top, x^i (x^i)^\top \rangle = \langle \phi\phi^\top, X \rangle \,. \end{aligned}$$

Because K^2_ϕ is a cone, $x^i \in \mathrm{K}^2_\phi$ implies $\alpha_i x^i \in \mathrm{K}^2_\phi$ and as K^2_ϕ is also convex, we get $x = \sum_i \alpha_i x^i \in \mathrm{K}^2_\phi$. ■

The following example shows that the inequality sign in $\mathrm{trace}(X) \le \langle \phi\phi^\top, X \rangle$ in the above lemma cannot be replaced by an equality sign in general.

Example 4.7. Consider

$$Y = \begin{pmatrix} 1 & \phi \\ \phi & \phi\phi^\top \end{pmatrix} = \begin{pmatrix} 1 \\ \phi \end{pmatrix} \begin{pmatrix} 1 \\ \phi \end{pmatrix}^\top$$

with $X = \phi\phi^\top$ and assume $\|\phi\|_2 \geq 1$ (otherwise $\mathrm{K}^2_\phi = \{0\}$). Because of $\|\phi\|_2 \leq \|\phi\|_2^2 = \phi^\top\phi$ it holds $Y \in C^*_{\mathbb{R}_+\times \mathrm{K}^2_\phi}$ and $\mathrm{trace}(X) = \mathrm{trace}(\phi\phi^\top) = \|\phi\|_2^2$ but $\langle \phi\phi^\top, X\rangle = \|\phi\|_2^4$.

The following corollary follows immediately from Corollary 3.5 and the relaxation given in Lemma 4.6:

Corollary 4.8. *Let Assumption 3.1 be satisfied. The optimal value of (P_{BP}) is bounded from below by the optimal value of the following semidefinite program:*

$$\begin{array}{rcll}
\multicolumn{3}{c}{\mathit{inf} \left\langle \begin{pmatrix} 0 & c^\top \\ c & Q \end{pmatrix}, Y \right\rangle} & \\
\multicolumn{3}{c}{\textit{such that}} & \\
Y & = & \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix} \textit{ positive semidefinite}, & \\
\langle I - \phi\phi^\top, X\rangle & \leq & 0, & (SDP^2_{BP}) \\
\|x\|_2 & \leq & \phi^\top x, & \\
Ax & = & b, & \\
x_j & = & X_{jj} \quad \textit{for all } j \in B, & \\
\mathit{Diag}(AXA^\top) & = & b \circ b, & \\
\multicolumn{3}{c}{x \in \mathbb{R},\ X \in \mathcal{S}^n.} &
\end{array}$$

Note that the constraint $\|x\|_2 \leq \phi^\top x$ is a semidefinite programming (SDP) constraint (see e.g. [11, Subsec. 1.4] for details).

4.2. **Semidefinite relaxation for the Lorentz cone.** Using the representation of the Lorentz cone K_L given in Example 4.3(a) we get especially for this cone the following relaxation, which was already given in [4, Prop. 7].

Corollary 4.9. *Let $x \in \mathbb{R}^n$, $X \in \mathcal{S}^n$ and*

$$Y = \begin{pmatrix} 1 & x^\top \\ x & X \end{pmatrix}.$$

If $Y \in C^*_{\mathbb{R}_+\times K_L}$ then

$$\sum_{i=1}^{n-1} X_{ii} \leq X_{nn},\ x \in K_L \quad \textit{and } Y \textit{ is positive semidefinite.}$$

5. Conclusions

In the paper we presented the extension of the results stating that any non-convex quadratic problem over the nonnegative orthant with some additional linear and binary constraints can be rewritten as linear problem over the cone of completely positive matrices (Burer, 2009). Our result covers all non-convex quadratic problems over a closed convex cone with some additional linear and binary constraints. In the second part we show the implication of this extended result to quadratic optimization over the Bishop-Phelps (BP) cones, especially to the BP cone with the Euclidean norm, and present a natural semidefinite relaxation.

References

[1] E. Bishop and R.R. Phelps, "The support functionals of a convex set", *Proc. Sympos. Pure Math.* 7 (1962) 27–35.

[2] I. Bomze, M. Duer, E. de Klerk, C. Roos, A.J. Quist, T. Terlaky, "On copositive programming and standard quadratic optimization problems", *J. Glob. Optim.* 18 (2000) 301–320.

[3] S. Burer, "On the copositive representation of binary and continuous nonconvex quadratic programs", *Math. Programming* 120 (2009) 479–495.

[4] S. Burer, "Copositive Programming", submitted to Handbook of Semidefinite, Cone and Polynomial Optimization: Theory, Algorithms, Software and Applications, edited by M. Anjos and J.B. Lasserre, June 2010.

[5] G. Eichfelder and J. Jahn, "Set-Semidefinite Optimization", *Journal of Convex Analysis* 15 (2008) 767–801.

[6] G. Eichfelder and J. Jahn, "Foundations of Set-Semidefinite Optimization", Chapter 18 in: *Nonlinear Analysis and Variational Problems*, Pardalos, P., Rassias, Th.M. and Khan, A.A. (Eds.), Springer (2009) 259–284.

[7] J. Jahn, *Vector Optimization - Theory, Applications, and Extensions* (Springer, Berlin, 2004).

[8] J. Jahn, *Introduction to the Theory of Nonlinear Optimization* (Springer, Berlin, 2007).

[9] J. Jahn, "Bishop-Phelps Cones in Optimization", *International Journal of Optimization: Theory, Methods and Applications* 1 (2009) 123-139.

[10] E. de Klerk, D.V. Pasechnik, "Approximation of the stability number of a graph via copositive programming", *SIAM J. Optim.* 12 (2002) 875–892.

[11] M. S. Lobo, L. Vandenberghe, S. Boyd and H. Lebret, "Applications of second-order cone programming". ILAS Symposium on Fast Algorithms for Control, Signals and Image Processing (Winnipeg, MB, 1997). Linear Algebra Appl. 284 (1998) no. 1-3, 193–228.

[12] A. Pascoletti, and P. Serafini, "Scalarizing vector optimization problems", *J. Optim. Theory Appl.* 42 (1984) 499–524.

[13] M. Petschke, "On a theorem of Arrow, Barankin, and Blackwell", *SIAM J. Control Optim.* 28 (1990) 395–401.

[14] J. Povh, F. Rendl, "A copositive programming approach to graph partitioning", *SIAM J. Optim.* 18 (2007) 223–241.

[15] J. Povh, F. Rendl, "Copositive and semidefinite relaxations of the quadratic assignment problem, *Discrete Optim.* 6 (2009) 231–241.

[16] J.F. Sturm and S. Zhang, "On cones of nonnegative quadratic functions", *Math. Oper. Res.* 28 (2003) 246–267.