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On n-absorbing submodules

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Abstract. All rings are commutative with identity, and all modules are unital. The purpose of this article is to investigate n-absorbing submodules. For this reason we introduce the concept of n-absorbing submodules generalizing n-absorbing ideals of rings. Let M be an R-module. A proper submodule N of M is called an n-absorbing submodule if whenever $a_1 \cdots a_n m \in N$ for $a_1, \ldots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in (N :_R M)$ or there are $n-1$ of a_i's whose product with m is in N. We study the basic properties of n-absorbing submodules and then we study n-absorbing submodules of some classes of modules (e.g. Dedekind modules, Prüfer modules, etc.) over commutative rings.

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1. Introduction

In this paper, all rings are commutative with non-zero identity and all modules are unital. Let R be a ring, M an R-module and N a submodule of M. We will denote by $(N :_R M)$ the residual of N by M, that is, the set of all $r \in R$ such that $rM \subseteq N$. The annihilator of M which is denoted by $ann_R(M)$ is $(0 :_R M)$. An R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R. Note that, since $I \subseteq (N :_R M)$ then $N = IM \subseteq (N :_R M)M \subseteq N$. So that $N = (N :_R M)M$ [21]. Finitely generated faithful multiplication modules are cancellation modules [20, Corollary to Theorem 9], where an R-module M is defined to be a cancellation module if $IM = JM$ for ideals I and J of R implies $I = J$. It is well-known that if R is a commutative ring and M a non-zero multiplication R-module then every proper submodule of M is contained in a maximal submodule of M and K is a maximal submodule of M if and only if there exists a maximal ideal p of R such that $K = pM$ [21, Theorem 2.5]. For a submodule N of M, if $N = IM$ for some ideal I of R, then we say that I is a presentation ideal of N. Note that it is possible that for a submodule N, no such presentation ideal exists. For example, assume that M is a vector space over an arbitrary field F with $dim_F M \geq 2$ and let N be a proper subspace of M such that $N \neq 0$. Then M has finite length (so M is Noetherian, Artinian and injective), but M is not multiplication and N

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does not have any presentation. Clearly, every submodule of M has a presentation ideal if and only if M is a multiplication module. Let N and K be submodules of a multiplication R-module M with $N = I_1M$ and $K = I_2M$ for some ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1I_2M$. Then by [4, Theorem 3.4], the product of N and K is independent of presentations of N and K. Moreover, for $a, b \in M$, by ab, we mean the product of Ra and Rb. Clearly, NK is a submodule of M and $NK \subseteq N \cap K$ (see [4]).

A submodule N of M is called idempotent if $N = (N :_R M)N$, [2]. It is shown [2, Theorem 3] that if M is multiplication and $(N :_R M)$ is an idempotent ideal of R then N is idempotent in M. The converse is true if we assume further that M is finitely generated and faithful. A submodule N of the R-module M is called a nilpotent submodule if $(N :_R M)^n N = 0$ for some positive integer n, and $m \in M$ is said to be nilpotent if Rm is a nilpotent submodule of M, [2]. Assume that $Nil(M)$ is the set of all nilpotent elements of M; then $Nil(M)$ is a submodule of M provided that M is faithful module, and if in addition M is multiplication, then $Nil(M) = Nil(R)M = \bigcap P$, where the intersection runs over all prime submodules of M, [2, Theorem 6]. We recall that a submodule N of M is prime (resp., primary) if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then either $m \in N$ or $rM \subseteq N$ (resp., $r^nM \subseteq N$ for some positive integer n). If N is a prime (resp. primary) submodule of M, then $p := (N :_R M)$ (resp. $p := \sqrt{(N :_R M)}$) is a prime ideal of R. In this case we say that N is a p-prime (resp. p-primary) submodule of M.

Let S be the set of all non-zero divisors of R and R_S be the total quotient ring of R. For a non-zero ideal I of R, Let

$$I^{-1} = \{x \in R_S : xI \subseteq R\}.$$

I is called an invertible ideal of R if $II^{-1} = R$. Let M be an R-module and

$$T = \{t \in S : tm = 0 \text{ for m } \in M \text{ implies m} = 0\}.$$

T is a multiplicatively closed subset of S, and if M is torsion free then $T = S$. In particular, if M is a faithful multiplication R-module then $T = S$ [21, Lemma 4.1]. Let N be a non-zero submodule of the R-module M, and

$$N^{-1} = \{x \in R_T : xN \subseteq M\}.$$

N^{-1} is an R-submodule of R_T, $R \subseteq N^{-1}$ and $N^{-1}N \subseteq M$. N is said to be an invertible submodule if $N^{-1}N = M$, [18].

In [18], Naoum and Al-Alwan generalized the concept of Dedekind domains to that of modules. An R-module M is a Dedekind module or D-module, if every non-zero submodule M is invertible and M is said to be a D_1-module if every non-zero cyclic submodule of M is invertible. It is clear that every D-module is a D_1-module. Let M be a faithful multiplication R-module. If M is a Dedekind module then R is a Dedekind domain, [18, Theorem 3.5]. Let M be a faithful multiplication R-module over the Dedekind domain R. Then M is a finitely generated Dedekind R-module, [18, Theorem 3.4]. Let R be an integral domain and M an R-module. M is called a valuation module if for all nonzero elements m and n of M, either $Rm \subseteq Rn$ or $Rn \subseteq Rm$. Equivalently, for any submodules N and K of M, either $N \subseteq K$

or $K \subseteq N$. A valuation module M such that every non-zero prime submodule P of M is not idempotent, that is, $P \neq (P :_R M)P$, is a discrete valuation module, [3]. An R-module M is called a Prüfer module, if every non-zero finitely generated submodule of M is invertible. An R-module M is said to be a Bézout module, if every finitely generated submodule is a principal submodule of M. Several properties of these classes of modules can be found in [1, 3] and [18].

In [7], Badawi introduced a new generalization of prime ideals in a commutative ring R. He defined a nonzero proper ideal I of R to be a 2-absorbing ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. This definition can obviously be made for any ideal of R. This concept has a generalization, called weakly 2-absorbing ideals, which has been studied in [8]. A proper ideal I of R to be a weakly 2-absorbing ideal of R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Later, Anderson and Badawi [5], introduced the concept of n-absorbing ideals of R for a positive integer n. A proper ideal I of R is called an n-absorbing (resp., strongly n-absorbing) ideal if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$ (resp, $I_1, \ldots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R), then there are n of the a_i's (resp., n of the I_i's) whose product is in I. It was shown that these two concepts agree when $n = 2$ in [7]. In [5, Corollary 6.9] it is shown that they agree for Prüfer domains, and it is conjectured that these two concepts agree for all positive integers n.

The concept of 2-absorbing (resp., weakly 2-absorbing) submodules was introduced and investigated in [22]. Let M be an R-module and N a proper submodule of M. N is said to be a 2-absorbing submodule (resp. weakly 2-absorbing submodule) of M if whenever $a, b \in R$ and $m \in M$ with $abm \in N$ (resp. $0 \neq abm \in N$), then $ab \in (N :_R M)$ or $am \in N$ or $bm \in N$. In this paper, we generalize the concepts of n-absorbing and strongly n-absorbing ideals of the ring R to that of submodules of an R-module M. Most of results are related to the reference [5] which have been proved for n-absorbing submodules. Let n be a positive integer. A proper submodule N of M is called an n-absorbing (resp., strongly n-absorbing) submodule if whenever $a_1 \cdots a_n m \in N$ for $a_1, \ldots, a_n \in R$ and $m \in M$ (resp, $I_1, \cdots I_n L \subseteq N$ for ideals $I_1, \ldots, I_n$ of R and submodule L of M), then either $a_1 \cdots a_n \in (N :_R M)$ (resp. $I_1 \cdots I_n \subseteq (N :_R M)$) or there are $n - 1$ of a_i's (resp. I_i's) whose product with m (resp. with L) is in N.

In this note, we study the concept of n-absorbing submodule, for a positive integer n. In fact, among the other things we prove that if R is a commutative ring and N is a 2-absorbing submodule of a faithful multiplication R-module M, then M-rad$N$ is a 2-absorbing submodule of M (see Theorem 1). We show (Theorem 2) that if N_j is an n_j-absorbing submodule of M for every $1 \leq j \leq k$, then $N_1 \cap \cdots \cap N_k$ is an n-absorbing submodule of M for $n = n_1 + \cdots + n_k$. In particular, if $N_1, \ldots, N_n$ are prime submodules of M, then $N_1 \cap \cdots \cap N_n$ is an n-absorbing submodule of M. In Theorem 3, we prove that if N is a p-primary submodule of M such that $p^n M \subseteq N$, then N is an n-absorbing submodule of M. In particular, if M is a multiplication module and $p^n M$ is a p-primary submodule of M, then $p^n M$ is an n-absorbing submodule of M. Theorem 7 implies that if R is a Noetherian ring and M a finitely generated R-module, then every non-zero proper submodule of M is an n-absorbing submodule of M for some positive integer n. In Section 3, we study 2-absorbing submodules of multiplication modules. Indeed, if we could give a positive answer to the Conjecture 1, then many of the results in Section 3 could be impled for n-absorbing submodules for every positive integer n.

2. Basic results

In this section, we study some basic properties of n-absorbing submodules of the R-module M. Let n be a positive integer. We recall that a proper submodule N of M is called an n-absorbing submodule if whenever $a_1 \cdots a_n m \in N$ for $a_1, \dots, a_n \in R$ and $m \in M$, then either $a_1 \cdots a_n \in (N :_R M)$ or there are $n-1$ of a_i's whose product with m is in N. A natural question is that if N is an n-absorbing submodule of M, whether the ideal $(N :_R M)$ is an n-absorbing ideal of R? For the cases where $n = 2$ or M is cyclic, we have the following results (compare Proposition 1 with [22, Proposition 2.9]).

Proposition 1. *Let R be a commutative ring and let M be an R-module. Assume that N is a 2-absorbing submodule of M. Then*

(1) *For every element $a, b \in R$ and every submodule K of M, $abK \subseteq N$ implies that $ab \in (N :_R M)$ or $aK \subseteq N$ or $bK \subseteq N$.*

(2) *$(N :_R M)$ is a 2-absorbing ideal of R.*

Proof. (1) Assume that $ab \notin (N :_R M)$, $aK \nsubseteq N$ and $bK \nsubseteq N$. Then $ax \notin N$ and $by \notin N$ for some $x, y \in K$. As $abx, aby \in N$ we have $ay \in N$ and $bx \in N$. Now it follows from $ab(x+y) \in N$ that either $a(x+y) \in N$ or $b(x+y) \in N$. Consequently, either $by \in N$ or $ax \in N$ which are contradictions.

(2) Suppose that $abc \in (N :_R M)$. Then setting $K = cM$ we have $abK \subseteq N$. As N is 2-absorbing, it follows from (1) that $ab \in (N :_R M)$ or $aK \subseteq N$ or $bK \subseteq N$. Hence $ab \in (N :_R M)$ or $ac \in (N :_R M)$ or $bc \in (N :_R M)$. □

Proposition 2. *Let R be a commutative ring and M a cyclic multiplication R-module. Then N is an n-absorbing submodule of M if and only if $(N :_R M)$ is an n-absorbing ideal of R.*

Proof. Let M be a cyclic R-module generated by $m \in M$. Let N be an n-absorbing submodule of M. Assume that $a_1, \dots, a_{n+1} \in R$ with $a_1 \cdots a_{n+1} \in (N :_R M)$. For every $1 \leq i \leq n$, let $\widehat{a_i}$ be the element of R which is obtained by eliminating a_i from $a_1 \cdots a_n$. Assume that $\widehat{a_i} a_{n+1} \notin (N :_R M)$ for every $1 \leq i \leq n$. Then $\widehat{a_i} a_{n+1} m \notin N$. So it follows from $(a_1 \cdots a_n)(a_{n+1} m) \in N$ and the fact that N is n-absorbing that $a_1 \cdots a_n \in (N :_R M)$, that is, $(N :_R M)$ is n-absorbing.

Conversely, assume that $(N :_R M)$ an n-absorbing ideal of R. Let $a_1, \dots, a_n \in R$ and $x \in M$ be such that $a_1 \cdots a_n x \in N$. There exists $a_{n+1} \in R$ such that $x = a_{n+1} m$. Thus $a_1 \cdots a_n a_{n+1} m \in N$. So $a_1 \cdots a_n a_{n+1} \in (N :_R m) = (N :_R M)$. But $(N :_R M)$ is an n-absorbing ideal of R, so there are n of the a_i's whose product is in $(N :_R M)$. This implies that either $a_1 \cdots a_n \in (N :_R M)$ or there are $n-1$ of a_i's whose product with x is in N, that is, N is n-absorbing. □

Conjecture 1. *Let R be a commutative ring and let M be an R-module. If N is an n-absorbing submodule of M, then $(N :_R M)$ is an n-absorbing ideal of R.*

Let N be a proper submodule of a nonzero R-module M. Then the M-radical of N, denoted by M-rad N, is defined to be the intersection of all prime submodules of M containing N. It is shown in [21, Theorem 2.12] that if N is a proper submodule of a multiplication R-module M, then M-rad $N = \sqrt{(N :_R M)}M$.

Theorem 1. *Let R be a commutative ring and M a faithful multiplication R-module. If N is a 2-absorbing submodule of M, then M-rad N is a 2-absorbing submodule of M.*

Proof. Since N is a 2-absorbing submodule of M then the ideal $(N :_R M)$ is a 2-absorbing ideal of R by Proposition 1. Then by [7, Theorem 2.4] we have the two following cases.

Case (1). $\sqrt{(N :_R M)} = p$ is a prime ideal of R. Since M is a multiplication module, then M-rad $N = \sqrt{(N :_R M)}M = pM$, where pM is a prime submodule of M by [21, Corollary 2.11]. Hence in this case M-rad N is a 2-absorbing submodule of M.

Case (2). $\sqrt{(N :_R M)} = p_1 \cap p_2$, where p_1, p_2 are distinct prime ideals of R that are minimal over $(N :_R M)$. In this case, we have M-rad $N = \sqrt{(N :_R M)}M = (p_1 + annM)M \cap (p_2 + annM)M = p_1M \cap p_2M$, where p_1M, p_2M are prime submodules of M by [21, Corollary 2.11, 1.7]. Consequently, M-rad N is a 2-absorbing submodule of M by [22, Theorem 2.3]. □

Theorem 2. *Let R be a ring and M an R-module. If N_j is an n_j-absorbing submodule of M for every $1 \leq j \leq k$, then $N_1 \cap \cdots \cap N_k$ is an n-absorbing submodule of M for $n = n_1 + \cdots + n_k$. In particular, if $N_1, \ldots, N_n$ are prime submodules of M, then $N_1 \cap \cdots \cap N_n$ is an n-absorbing submodule of M.*

Proof. Let $a_1, \ldots a_n \in R$ and $m \in M$ with $a_1 \cdots a_n m \in N_1 \cap \cdots \cap N_k := N$ such that there are not $n-1$ of the a_i's whose product with m lies in N. As $a_1 \cdots a_n m \in N_1 \cap \cdots \cap N_k$, so $a_1 \cdots a_m m \in N_j$ for every $1 \leq j \leq k$. Therefore $a_1 \cdots a_n \in (N_j :_R M)$ for every $1 \leq j \leq k$ since N_j is assumed to be an n_j-absorbing submodule of M and $n_j \leq n$. Therefore $a_1 \cdots a_n \in \bigcap_{j=1}^{k} (N_j :_R M) = (N :_R M)$, that is, N is n-absorbing. The " In particular" statement is clear. □

Let N be a proper submodule of an R-module M. It is clear that if N is an n-absorbing submodule, then it is an m-absorbing submodule of M for every integer $m \geq n$. If N is an n-absorbing submodule of M for some positive integer n, then define $\omega_M(N) = min\{n \mid \text{N is an n-absorbing submodule of M}\}$; otherwise, set $\omega_M(N) = \infty$ (we will just write $\omega(N)$ when the context is clear). Moreover, we define $\omega(M) = 0$. Therefore, for any submodule N of M, we have $\omega_M(N) \in \mathbb{N} \cup \{0, \infty\}$, with $\omega(N) = 1$ if and only if N is a prime submodule of M and $\omega(N) = 0$ if and only if $M = N$. Then $\omega(N)$ measures, in some sense, how far N is from being a prime submodule of M. On can ask how $\omega_M(N)$ and $\omega_R((N :_R M))$ compare.

Corollary 1. *Let M be an R-module.*

(1) *If* $N_1, \ldots, N_k$ *are submodules of* M*, then* $\omega(N_1 \cap \cdots \cap N_k) \leq \omega(N_1) + \cdots + \omega(N_k)$.

(2) $\omega(N_1 \cap \cdots \cap N_n) \leq n$*, where* $N_1, \ldots, N_n$ *are prime submodules of* M.

Notation. Let R be a commutative ring and $a_1, a_2, ..., a_n \in R$. We denote by $\widehat{a_i}$ the element $a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$. In this case the definition of an n-absorbing submodule can be reformulated as: the submodule N of the R-module M is called n-absorbing if whenever $a_1, \ldots, a_n \in R$ and $m \in M$ with $a_1 \cdots a_n m \in N$, then either $a_1 \cdots a_n \in (N :_R M)$ or $\widehat{a_i} m \in N$ for some $1 \leq i \leq n$.

Theorem 3. *Let* M *be an* R*-module and* N *a* p*-primary submodule of* M *such that* $p^n M \subseteq N$*. Then* N *is an* n*-absorbing submodule of* M*. Moreover,* $\omega(N) \leq n$*. In particular, if* M *is a multiplication module and* $p^n M$ *is a* p*-primary submodule of* M*, then* $p^n M$ *is an* n*-absorbing submodule of* M*. Moreover,* $\omega(p^n M) \leq n$.

Proof. Assume that $a_1, \ldots, a_n \in R$ and $m \in M$ with $a_1 \cdots a_n m \in N$ such that $\widehat{a_i} m \notin N$ for every $1 \leq i \leq n$. For every $1 \leq i \leq n$, as $a_i \widehat{a_i} m \in N$ with $\widehat{a_i} m \notin N$ and N is a p-primary submodule of M, we have $a_i \in p$. Consequently, $a_1 \cdots a_n \in p^n \subseteq (N :_R M)$, that is, N is an n-absorbing submodule of M. □

Let R be a ring with identity and M an R-module. Then $R(M) = R(+)M$ with multiplication $(a, m)(b, n) = (ab, an + bm)$ and with additition $(a, m) + (b, n) = (a + b, m + n)$ is a commutative ring with identity and $0(+)M$ is a nilpotent ideal of index 2. The ring $R(+)M$ is said to be the *idealization* of M or trivial extension of R by M. We view R as a subring of $R(+)M$ via $r \to (r, 0)$. An ideal H is said to be *homogeneous* if $H = I(+)N$ for some ideal I of R and some submodule N of M; whence $IM \subseteq N$ [14].

Theorem 4. *Let* I *be an ideal of* R *and* N *a submodule of* M*. Let* $I(+)N$ *be an* n*-absorbing ideal of* $R(M)$ *such that* $I(+)N$ *is a homogeneous ideal of* $R(M)$*. Then* I *is an* n*-absorbing ideal of* R *and* N *is an* n*-absorbing submodule of* M.

Proof. Assume that $I(+)N$ is an n-absorbing ideal of R. Let $a_1, \ldots, a_{n+1} \in R$ such that $a_1 \cdots a_{n+1} \in I$, then $(a_1, 0)(a_2, 0) \cdots (a_{n+1}, 0) \in I(+)N$. Since $I(+)N$ is an n-absorbing ideal, then $\widehat{(a_i, 0)} \in I(+)N$ for some $1 \leq i \leq n$. So $\widehat{a_i} \in I$ for some $1 \leq i \leq n$, that is, I is an n-absorbing ideal of R. Now, let $a_1, \ldots, a_n \in R$ and $m \in M$ be such that $a_1 \cdots a_n m \in M$. Since $I(+)N$ is a homogenous ideal of $R(M)$, we have $(a_1, 0)(a_2, 0) \cdots (a_n, 0)(0, m) \in I(+)N$. Since $I(+)N$ is an n-absorbing ideal of $R(+)M$, either $(a_1, 0) \cdots (a_n, 0) \in I(+)N$ or there exist $n - 1$ of $(a_i, 0)'s$ whose product with $(0, m)$ is in $I(+)N$. Then $a_1 \cdots a_n \in I \subseteq (N :_R M)$ or there are $n - 1$ of a_i's whose product with m is in N. Hence N is an n-absorbing submodule of M. □

Recall that a proper ideal I of an integral domain R is said to be divided if $I \subset Rc$ for every $c \in R \setminus I$, [11] and [6]. Generalizing this idea to modules we say that a proper submodule N of an R-module M is divided if $N \subset Rm$ for all $m \in M \setminus N$, [3].

Lemma 1. *Let R be a commutative ring and let M be a finitely generated faithful multiplication R-module. If P is a divided prime submodule of M, then $(P :_R M)$ is a divided prime ideal of R.*

Proof. [3, Proposition 6]. □

Theorem 5. *Let R be a commutative ring, M a finitely generated faithful multiplication R-module, and $P = pM$ a divided prime submodule of M, where $p = (P :_R M)$ is a prime ideal of R. If M-rad $N = P$ and N is an n-absorbing submodule of M for some positive integer n, then N is p-primary.*

Proof. Note first that by [21, Theorem 2.12], M-rad $N = \sqrt{(N :_R M)}M$. On the other hand, M-rad $N = P = pM$ by [21, Corollary 2.11]. Moreover, every finitely generated faithful multiplication module is cancellation. So that $p = (P :_R M) = \sqrt{(N :_R M)}$. Assume that $am \in N$ but $a \notin p$. Then from $am \in P$, $a \notin (P :_R M)$ and P prime we get $m \in P$. By Lemma 1, p is a divided prime ideal of R. So $p \subset Ra^{n-1}$ since $a \notin p$. Therefore $P = pM \subset Ma^{n-1}$, and hence $m = a^{n-1}z$ for some $z \in M$. Now it follows from $a^n z = am \in N$ and $a^n \notin (N :_R M)$ that $m = a^{n-1}z \in N$ since N is assumed to be n-absorbing. This shows that N is a p-primary submodule of M. □

Theorem 6. *Let R be a ring and let M be a finitely generated faithful multiplication R-module. Let $Nil(M) \subset P$ be divided prime submodules of M. Then P^n is a $(P :_R M)$-primary submodule of M, and thus P^n is an n-absorbing submodule of M with $\omega(P^n) \leq n$, for every positive integer n.*

Proof. Since M is a faithful multiplication module, we have $Nil(M) = Nil(R)M$ by [2, Theorem 6]. On the other hand, M is a cancellation module by [21, Theorem 3.1]. Therefore $Nil(R) \subset (P :_R M)$ are divided prime ideals of R by Lemma 1. It follows now from [5, Theorem 3.3] that $(P :_R M)^n$ is a $(P :_R M)$-primary ideal of R. Hence $P^n = (P :_R M)^n M$ is a $(P :_R M)$-primary submodule of M by [12, Corollary 2]. Therefore P^n is n-absorbing by Theorem 3. □

Corollary 2. *Let R be an integral domain and let M be a faithful multiplication prime R-module. Assume that P is a nonzero divided prime submodule of M. Then P^n is an n-absorbing submodule of M for every positive integer n.*

Proof. Since R is an integral domain and M is a prime module, then $Nil(M) = 0$ is a divided prime submodule of M by [2, Theorem 6]. □

Theorem 7. *Let R be a Noetherian ring and let M be a finitely generated R-module. Then every non-zero proper submodule of M is an n-absorbing submodule of M for some positive integer n.*

Proof. Let N be a p-primary submodule of M. So $(N :_R M)$ is a p-primary ideal of R. Since R is a Noetherian ring, there exists a positive integer m for which $p^m \subseteq (N :_R M)$. Thus N is an m-absorbing submodule of M by Theorem 3. Now assume that K is a proper submodule of M. Since M is a Noetherian module, K is representable. Assume that $K = N_1 \cap \cdots \cap N_k$ is a primary decomposition of K, where N_i is a p_i-primary submodule of M for any $1 \leq i \leq n$. By the first part, each N_i $(1 \leq i \leq n)$ is an m_i-absorbing submodule of M for some positive integer m_i. Now K is an n-absorbing submodule in which $n = m_1 + \cdots + m_k$. Therefore the result follows. □

Let R be a commutative ring. The concept of strongly n-absorbing ideals of R was introduced and studied in [5]. A proper ideal I of R is said to be a strongly n-absorbing ideal of R if whenever $I_1 \cdots I_{n+1} \subseteq I$ for ideals $I_1, \ldots, I_{n+1}$ of R, then the product of some n of the I_i's is in I. It is clear that a strongly n-absorbing ideal of R is also an n-absorbing ideal of R, and in [7, Theorem 2.13], it was shown that these two concepts agree when $n = 2$. In [5, Corollary 6.9] it is shown that they agree for Prüfer domains, and it is conjectured that these two concepts agree for all positive integers n. Now let M be an R-module. It is easy to show that a proper submodule N of M is prime if and only if whenever $IL \subseteq N$ for an ideal I of R and a submodule L of M, then either $L \subseteq N$ or $I \subseteq (N :_R M)$. Let n be a positive integer. We say that a proper submodule N of an R-module M is a *strongly n-absorbing* submodule, if whenever $I_1 I_2 \cdots I_n L \subseteq N$ for ideals $I_1, I_2, \ldots, I_n$ of R and submodule K of M, then either $I_1 I_2 \cdots I_n \subseteq (N :_R M)$ or there are $n-1$ of the I_j's whose product with L is contained in N. Thus a strongly 1-absorbing submodule is just a prime submodule, and the intersection of n prime submodules of M is a strongly n-absorbing submodule of M. It is also clear that every strongly n-absorbing submodule of M is an n-absorbing submodule of M.

If N is a strongly n-absorbing submodule of M for some positive integer n, then we define $\omega^*_M(N) = min\{n|$ N is a strongly n-absorbing submodule$\}$; otherwise set $\omega^*_M(N) = \infty$ and $\omega^*_M(M) = 0$. Then $\omega^*_M(N) = 1$ if and only if N is a prime submodule of M, and $\omega_M(N) \leq \omega^*_M(N)$. Then $\omega^*_M(N) \in \mathbb{N} \cup \{0, \infty\}$. Also, we define $\Omega^*(M) = \{\omega^*_M(N)|$ N is a proper submodule$\}$; so $\{1\} \subseteq \Omega^*(M) \subseteq \mathbb{N} \cup \{\infty\}$. Always $\omega^*(N_1 \cap \cdots \cap N_m) \leq \omega^*(N_1) + \cdots + \omega^*(N_m)$.

3. 2-absorbing submodules in multiplication modules

In this section we study 2-absorbing submodules of some specific modules M(e.g. Dedekind module, Prüfer module, etc.), where M is a multiplication module.

Lemma 2. *Let R be an integral domain and M a Bézout finitely generated faithful multiplication R-module. If N is a 2-absorbing submodule and P a prime submodule of M such that M-rad $N = P$, then $P^2 \subseteq N$. In particular, this holds if M is a valuation module.*

Proof. Since R is an integral domain and M is a Bézout faithful multiplication R-module, then R is a Bézout ring by [1, Proposition 2.2]. On the other hand, by Proposition 1, $(N :_R M)$ is a 2-absorbing ideal of R since N is assumed to be a 2-absorbing submodule of M. As M-rad $N = P$, there exists a prime ideal p or R with $P = pM$. As M is a finitely generated faithful multiplication module, we have $\sqrt{(N :_R M)} = p$ by [21, Theorem 2.12, Theorem 3.1]. Consequently, $p^2 \subseteq (N :_R M)$ by [5, Lemma 5.1]. Now we have $P^2 = p^2 M \subseteq (N :_R M)M = N$. The "In particular" statement is clear. □

The next result shows that 2-absorbing submodules of a valuation module M are of the form P^m, where P is a prime submodule of M and $m = 1$ or 2.

Theorem 8. *Let R be a an integral domain, and M a finitely generated faithful multiplication R-module. In addition, if M is a valuation module, then the following statements are equivalent for a submodule N of M:*

(1) *N is a 2-absorbing submodule of M.*

(2) *N is a p-primary submodule of M for some prime ideal p of R with $p^2M \subseteq N$.*

(3) *$N = P$ or P^2 for some prime submodule P (= M-rad N) of M.*

Proof. (1) $\Rightarrow$ (2) Assume that N is a 2-absorbing submodule of M. Then $(N :_R M)$ is an n-absorbing ideal of R by Proposition 1. Moreover, M is a valuation module, so R is a valuation domain by [1, Proposition 2.2]. It follows that $\sqrt{(N :_R M)} = p$ is a prime ideal of R, and $(N :_R M)$ is a p-primary ideal of R with $p^2 \subseteq (N :_R M)$ by [5, Lemma 5.5]. Thus N is a p-primary submodule of M with $p^2M \subseteq (N :_R M)M = N$.

(2) $\Rightarrow$ (3) Assume that N is a p-primary submodule of M wit$p^2M \subseteq N$. In this case $(N :_R M)$ is a p-primary ideal of R. Moreover, it follows from $p^2M \subseteq (N :_R M)M$ that $p^2 \subseteq (N :_R M)$ by [21, Theorem 3.1]. Now, by [13, Theorem 17.3], $(N :_R M) = p$ or p^2 with $p = \sqrt{(N :_R M)}$. In this case $N = (N :_R M)M = pM$or $(pM)^2$, where $P := pM$ is a prime submodule of M with $P = pM = \sqrt{(N :_R M)}M$ by [21, Theorem 2.12].

(3) $\Rightarrow$ (1) Assume that $N = P$ or P^2 for some prime submodule P (= M-rad N) of M. If $N = 0$, then it is 2-absorbing as M is assumed to be faithful. Moreover, there will be nothing to prove if $N = P$. So we may assume that $0 \neq N \neq P^2$. Since M is a valuation module, $Nil(M) \subset P$ are divided prime submodules of M. In this case, $N = P^2$ is a 2-absorbing submodule of M by Theorem 6. □

Theorem 9. *Let R be a commutative ring and M a faithful multiplication R-module.*

(1) *If M is a Dedekind module and if N is a 2-absorbing submodule of M, then either N is a maximal submodule of M or $N = N_1N_2$ for maximal submodules N_1, N_2 of M.*

(2) *If M is a Prüfer module and N a nonzero 2-absorbing submodule of M, then N is a prime submodule of M or $N = p^2M$ is a p-primary submodule of M or $N = P_1 \cap P_2$, where P_1 and P_2 are nonzero prime submodules of M.*

Proof. (1) Assume that M is a Dedekind module. Then R is a Dedekind domain by [18, Theorem 3.5]. Now assume that N is a 2-absorbing submodule of M. Then $(N :_R M)$ is a 2-absorbing ideal of R by Proposition 1. Consequently, by [5, Theorem 5.1], either $(N :_R M)$ is a maximal ideal of R or $(N :_R M) = \underline{m}_1\underline{m}_2$ for maximal ideals $\underline{m}_1, \underline{m}_2$ of R. It follows from [21, Theorem 2.5] that either $N = (N :_R M)M$ is a maximal submodule of M or $N = N_1N_2$ for maximal submodules $N_1 = \underline{m}_1M$ and $N_2 = \underline{m}_2M$ of M.

(2) Since M is a Prüfer faithful multiplication module, R is a Prüfer domain by [10, Theorem 3.6]. Hence $(N :_R M)$ is a 2-absorbing ideal of R by Proposition

1. It follows now from [7, Theorem 3.14] that $(N :_R M)$ is a prime ideal of R or $(N :_R M) = p^2$ is a p-primary ideal of R or $(N :_R M) = p_1 \cap p_2$, where p_1 and p_2 are nonzero prime ideals of R. Hence, by [21, Theorem 2.11] and [12, Corollary 2], $N = (N :_R M)M$ is a prime submodule of M or $N = p^2M$ is a p-primary submodule of M or $N = P_1 \cap P_2$, where $P_1 = p_1M$ and $P_2 = p_2M$ are nonzero prime submodules of M. □

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References

[1] M. M. Ali, *Invertibility of multiplication modules*, New Zealand J. Math. **35**(2006), 17–29.
[2] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Comm. Algebra **36**(2008), 4620–4642.
[3] M. M. Ali, *Invertibility of multiplication modules III*, New Zealand J. Math. **39**(2009), 193–213.
[4] R. Ameri, *On the prime submodules of multiplication modules*, Int. J. Math. Math. Sci. **27**(2003), 1715–1724.
[5] D. F. Anderson, A. Badawi, *On n-absorbing ideals of commutative rings*, Comm. Algebra **39**(2011), 1646-1672.
[6] A. Badawi, *On divided commutative rings*, Comm. Algebra **27**(1999), 1465–1474.
[7] A. Badawi, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. **75**(2007), 417–429.
[8] A. Badawi, A. Yousefian Darani, *On weakly 2–absorbing ideals of commutative rings*, Houston J. Math., to appear.
[9] M. Baziar, M. Behboodi, *Classical primary submodules and decomposition theory of modules*, J. Algebra Appl. **8**(2009), 351–362.
[10] Y. Hwan Cho, *On multiplication modules (II)*, Comm. Korean Math. Soc. **13**(1998), 727–733.
[11] D. E. Dobbs, *Divided rings and going down*, Pacific J. Math. **67**(1976), 353–363.
[12] S. Ebrahimi Atani, F. Callalp, Ü. Tekir, *A short note on the primary submodules of multiplication modules*, Inter. J. Algebra, **8**(2007), 381–384.
[13] R. Gilmer, *Multiplicative ideal theory*, Marcel Dekker Inc., New York, 1972.
[14] J. A. Huckaba, *Commutative rings with zero–divisors*, Marcel Dekker Inc., New York, 1988.
[15] M. D. Larsen, P. J. MacCarthy, *Multiplicative theory of ideals*, Academic Press, New York, 1971.
[16] C. P. Lu, *Spectra of modules*, Comm. Algebra **10**(1995), 3741–3752.
[17] H. Matsumara, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.
[18] A. G. Naoum, F. H. Al-Alwan, *Dedekind modules*, Comm. Algebra. **24**(1996), 397–412.
[19] R. Y. Sharp, *Steps in commutative algebra*, London Mathematical Society, London, 1990.

[20] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math. **50**(1988), 223–235.

[21] P. F. Smith, Z. El-Bast, *Multiplication modules*, Comm. Algebra **16**(1988), 755–799.

[22] A. Yousefian Darani, F. Soheilnia, 2*–absorbing and weakly* 2*–absorbing submodules*, Thai J. Math. **9**(2011), 577–584.