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The signed (k,k)-domatic number of digraphs

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Abstract. Let D be a finite and simple digraph with vertex set $V(D)$, and let $f : V(D) \to \{-1, 1\}$ be a two-valued function. If $k \geq 1$ is an integer and $\sum_{x \in N^-[v]} f(x) \geq k$ for each $v \in V(D)$, where $N^-[v]$ consists of v and all vertices of D from which arcs go into v, then f is a signed k-dominating function on D. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed k-dominating functions on D with the property that $\sum_{i=1}^{d} f_i(x) \leq k$ for each $x \in V(D)$, is called a signed (k,k)-dominating family (of functions) on D. The maximum number of functions in a signed (k,k)-dominating family on D is the signed (k,k)-domatic number on D, denoted by $d_S^k(D)$.
In this paper, we initiate the study of the signed (k,k)-domatic number of digraphs, and we present different bounds on $d_S^k(D)$. Some of our results are extensions of well-known properties of the signed domatic number $d_S(D) = d_S^1(D)$ of digraphs D as well as the signed (k,k)-domatic number $d_S^k(G)$ of graphs G.

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1. Terminology and introduction

In this paper, D is a finite and simple digraph with vertex set $V(D)$ and arc set $A(D)$. The integers $n(D) = |V(D)|$ and $m(D) = |A(D)|$ are the *order* and the *size* of the digraph D, respectively. We write $d_D^+(v) = d^+(v)$ for the *outdegree* of a vertex v and $d_D^-(v) = d^-(v)$ for its *indegree*. The *minimum* and *maximum indegree* are $\delta^-(D)$ and $\Delta^-(D)$. The sets $N^+(v) = \{x \mid (v,x) \in A(D)\}$ and $N^-(v) = \{x \mid (x,v) \in A(D)\}$ are called the *outset* and *inset* of the vertex v. Likewise, $N^+[v] = N^+(v) \cup \{v\}$ and $N^-[v] = N^-(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. For an arc $(x,y) \in A(D)$, the vertex y is an *outer neighbor* of x and x is an *inner neighbor* of y. For a real-valued function $f : V(D) \longrightarrow \mathbf{R}$ the weight of f is $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$. Consult [3] and [4] for notation and terminology which are not defined here.

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If $k \geq 1$ is an integer, then the *signed k-dominating function* is defined as a function $f : V(D) \longrightarrow \{-1, 1\}$ such that $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq k$ for every $v \in V(D)$. The *signed k-domination number* for a digraph D is

$$\gamma_{kS}(D) = \min\{w(f) \mid f \text{ is a signed } k\text{-dominating function of } D\}.$$

A $\gamma_{kS}(D)$-function is a signed k-dominating function on D of weight $\gamma_{kS}(D)$. As the assumption $\delta^-(D) \geq k-1$ is necessary, we always assume that when we discuss $\gamma_{kS}(D)$, all digraphs involved satisfy $\delta^-(D) \geq k-1$ and thus $n(D) \geq k$.

The signed k-domination number of digraphs was introduced by Atapour, Hajypory, Sheikholeslami and Volkmann [1]. When $k = 1$, the signed k-domination number $\gamma_{kS}(D)$ is the usual *signed domination number* $\gamma_S(D)$, which was introduced by Zelinka in [13] and has been studied by several authors (see for instance [5] and [10]).

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed k-dominating functions on D with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(D)$, is called a *signed (k,k)-dominating family* on D. The maximum number of functions in a signed (k,k)-dominating family on D is the *signed (k,k)-domatic number* of D, denoted by $d_S^k(D)$. When $k = 1$, the signed (k,k)-domatic number of a digraph D is the usual *signed domatic number* $d_S(D)$, which was introduced by Sheikholeslami and Volkmann [7] and has also been studied in [10].

In this paper, we initiate the study of the signed (k,k)-domatic number of digraphs, and we present different bounds on $d_S^k(D)$. Some of our results are extensions of well-known properties of the signed domatic number $d_S(D) = d_S^1(D)$ of digraphs (see for example [7]) as well as the signed (k,k)-domatic number $d_S(G)$ of graphs G (see for example [6, 8, 9, 11]).

Our first proposition shows that the signed (k,k)-domatic number $d_S^k(D)$ is well-defined for every digraph D with $\delta^-(D) \geq k-1$.

Proposition 1. *The signed domatic number $d_S^k(D)$ is well-defined for each digraph D with $\delta^-(D) \geq k-1$.*

Proof. Let $k \geq 1$ be an integer, and let $\delta^-(D) \geq k-1$. Since the function $f : V(D) \to \{-1, 1\}$ with $f(v) = 1$ for each $v \in V(D)$ is a signed k-dominating function on D, the family $\{f\}$ is a signed (k,k)-dominating family on D. Therefore, the set of signed k-dominating functions on D is non-empty and there exists the maximum of their cardinalities, which is the signed (k,k)-domatic number of D. □

2. Properties of the signed (k,k)-domatic number

In this section we present basic properties of the signed (k,k)-domatic number and find some sharp bounds for this parameter.

Theorem 1. *If D is a digraph with $\delta^-(D) \geq k-1$, then*

$$d_S^k(D) \leq \delta^-(D) + 1.$$

Moreover, if $d_S^k(D) = \delta^-(D) + 1$, then for each function of any signed (k,k)-dominating family $\{f_1, f_2, \ldots, f_d\}$ on D and for all vertices v of indegree $\delta^-(D)$, $\sum_{x \in N^-[v]} f_i(x) = k$ and $\sum_{i=1}^{d} f_i(x) = k$ for every $x \in N^-[v]$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on D such that $d = d_S^k(D)$. If $v \in V(G)$ is a vertex of minimum indegree $\delta^-(D)$, then it follows that

$$d \cdot k = \sum_{i=1}^{d} k \le \sum_{i=1}^{d} \sum_{x \in N^-[v]} f_i(x)$$
$$= \sum_{x \in N^-[v]} \sum_{i=1}^{d} f_i(x)$$
$$\le \sum_{x \in N^-[v]} k = k(\delta^-(D) + 1),$$

and this implies the desired upper bound on the signed (k,k)-domatic number.

If $d_S^k(D) = \delta^-(D) + 1$, then the two inequalities occurring in the inequality chain above become equalities. Therefore, for all vertices v of indegree $\delta^-(D)$, we observe that $\sum_{x \in N^-[v]} f_i(x) = k$ for $1 \le i \le d$ and $\sum_{i=1}^{d} f_i(x) = k$ for every $x \in N^-[v]$. $\square$

Theorem 2. *Let D be an r-regular digraph of order n such that $r \ge 1$ and $\gcd(n, r+1) = 1$, and let k be a positive integer. Then $d_S^k(D) \le \delta^-(D) = r$.*

Proof. Suppose to the contrary that $d_S^k(D) > \delta^-(D)$. Then by Theorem 1, $d_S^k(D) = \delta^-(D) + 1$. Let f belong to a signed (k,k)-dominating family on D of order $\delta^-(D) + 1$. By Theorem 1, we have $\sum_{x \in N^-[v]} f(x) = k$ for every $v \in V(D)$. This implies that

$$nk = \sum_{v \in V(D)} \sum_{x \in N^-[v]} f(x) = \sum_{x \in V(D)} (r+1) f(x) = (r+1) \sum_{x \in V(D)} f(x) = (r+1) w(f).$$

Since $w(f)$ is an integer and $\gcd(n, r+1) = 1$, the number $r+1$ is a divisor of k. It follows from $k \le \delta^-(D) + 1 = r + 1$ that $k = r + 1$. Thus $\sum_{x \in N^-[v]} f(x) = r + 1$ for every $v \in V(D)$. Since $f(x) \le 1$ for each $x \in V(D)$, we deduce that $f(v) = 1$ for each $v \in V(D)$. Hence f is the only element of the signed (k,k)-dominating family on D which is a contradiction. This completes the proof. $\square$

Theorem 3. *If D is a digraph of order n with $\delta^-(D) \ge k - 1$, then*

$$\gamma_{kS}(D) \cdot d_S^k(D) \le k \cdot n.$$

Moreover, if $\gamma_{kS}(D) \cdot d_S^k(D) = k \cdot n$, then for each signed (k,k)-dominating family $\{f_1, f_2, \ldots, f_d\}$ on D with $d = d_S^k(D)$, each function f_i is a $\gamma_{kS}(D)$-function and $\sum_{i=1}^{d} f_i(x) = k$ for each $x \in V(D)$.

Proof. If $\{f_1, f_2, \ldots, f_d\}$ is a signed (k,k)-dominating family on D such that $d = d_S^k(D)$, then the definitions imply

$$d \cdot \gamma_{kS}(D) = \sum_{i=1}^{d} \gamma_{kS}(D) \le \sum_{i=1}^{d} \sum_{x \in V(D)} f_i(x)$$
$$= \sum_{x \in V(D)} \sum_{i=1}^{d} f_i(x) \le \sum_{x \in V(D)} k = k \cdot n.$$

If $\gamma_{kS}(D) \cdot d_S^k(D) = k \cdot n$, then the two inequalities occurring in the inequality chain above become equalities. Hence $\gamma_{kS}(D) = \sum_{x\in V(D)} f_i(x)$ for each $i \in \{1,2,\ldots,d\}$, and thus each function f_i is a $\gamma_{kS}(D)$-function. In addition, we see that $\sum_{i=1}^d f_i(x) = k$ for each $x \in V(D)$. $\square$

The special case $k = 1$ in Theorems 1 and 3 can be found in [7].

Theorem 4. *If v is a vertex of a digraph D such that $d^-(v)$ is odd and k is odd or $d^-(v)$ is even and k is even, then*

$$d_S^k(D) \le \frac{k}{k+1}(d^-(v)+1).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on D such that $d = d_S^k(D)$. Assume first that $d^-(v)$ and k are odd. The definition yields to $\sum_{x\in N^-[v]} f_i(x) \ge k$ for each $i \in \{1,2,\ldots,d\}$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore, it is an even number, and as k is odd, we obtain $\sum_{x\in N^-[v]} f_i(x) \ge k+1$ for each $i \in \{1,2,\ldots,d\}$. It follows that

$$\begin{aligned} k(d^-(v)+1) &= \sum_{x\in N^-[v]} k \ge \sum_{x\in N^-[v]} \sum_{i=1}^{d} f_i(x) \\ &= \sum_{i=1}^{d} \sum_{x\in N^-[v]} f_i(x) \\ &\ge \sum_{i=1}^{d} (k+1) = d(k+1), \end{aligned}$$

and this leads to the desired bound. Assume next that $d^-(v)$ and k are even integers. Note that $\sum_{x\in N^-[v]} f_i(x) \ge k$ for each $i \in \{1,2,\ldots,d\}$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore, it is an odd number, and as k is even, we obtain $\sum_{x\in N^-[v]} f_i(x) \ge k+1$ for each $i \in \{1,2,\ldots,d\}$. Now the desired bound follows as above, and the proof is complete. $\square$

The next result is an immediate consequence of Theorem 4.

Corollary 1. *If D is a digraph such that $\delta^-(D)$ and k are odd or $\delta^-(D)$ and k are even, then*

$$d_S^k(D) \le \frac{k}{k+1}(\delta^-(D)+1).$$

For special digraphs D we will improve the upper bound on $d_S^k(D)$ given in Theorem 1.

Corollary 2. *Let $k \ge 1$ be an integer. If D is a digraph such that $\delta^-(D) = k + 2t$ for an integer $t \ge 1$, then*

$$d_S^k(D) \le \delta^-(D) - 1.$$

Proof. Since k and $\delta^-(D)$ are of the same parity, Corollary 1 implies that

$$d_S^k(D) \le \frac{k}{k+1}(\delta^-(D)+1) = \frac{k}{k+1}(k+2t+1) < k+2t$$

and therefore $d_S^k(D) \le k+2t-1 = \delta^-(D)-1$. □

Theorem 5. *If D is a digraph such that k is odd and $d_S^k(D)$ is even or k is even and $d_S^k(D)$ is odd, then*

$$d_S^k(D) \le \frac{k-1}{k}(\delta^-(D)+1).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on D such that $d = d_S^k(D)$. Assume first that k is odd and d is even. If $x \in V(D)$ is an arbitrary vertex, then $\sum_{i=1}^d f_i(x) \le k$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore, it is an even number, and as k is odd, we obtain $\sum_{i=1}^d f_i(x) \le k-1$ for each $x \in V(G)$. If v is a vertex with $d^-(v) = \delta^-(D)$, then it follows that

$$\begin{aligned} d \cdot k = \sum_{i=1}^d k &\le \sum_{i=1}^d \sum_{x \in N^-[v]} f_i(x) \\ &= \sum_{x \in N^-[v]} \sum_{i=1}^d f_i(x) \\ &\le \sum_{x \in N^-[v]} (k-1) \\ &= (\delta^-(D)+1)(k-1), \end{aligned}$$

and this yields to the desired bound. Assume secondly that k is even and d is odd. If $x \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^d f_i(x) \le k$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore, it is an odd number, and as k is even, we obtain $\sum_{i=1}^d f_i(x) \le k-1$ for each $x \in V(G)$. Now the desired bound follows as above, and the proof is complete. □

According to Proposition 1, $d_S^k(D)$ is a positive integer. If we suppose in the case $k=1$ that $d_S(D) = d_S^1(D)$ is an even integer, then Theorem 5 leads to the contradiction $d_S(D) \le 0$. Consequently, we obtain the next known result.

Corollary 3 (Sheikholeslami, Volkmann [7])**.** *The signed domatic number $d_S(D)$ is an odd integer.*

Theorem 6. *Let $k \ge 2$ be an integer, and let D be a digraph with $\delta^-(D) \ge k-1$. Then $d_S^k(D) = 1$ if and only if for every vertex $v \in V(D)$ the set $N^+[v]$ contains a vertex x such that $d^-(x) \le k$.*

Proof. Assume that $N^+[v]$ contains a vertex x such that $d^-(x) \leq k$ for every vertex $v \in V(D)$, and let f be a signed k-dominating function on D. If $d^-(v) \leq k$, then it follows that $f(v) = 1$. If $d^-(x) \leq k$ for a vertex $x \in N^+(v)$, then we observe $f(v) = 1$ too. Hence $f(v) = 1$ for each $v \in V(D)$ and thus $d_S^k(D) = 1$.

Conversely, assume that $d_S^k(D) = 1$. If D contains a vertex w such that $d^-(x) \geq k+1$ for each $x \in N^+[w]$, then the functions $f_i : V(D) \to \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in V(D)$ and $f_2(w) = -1$ and $f_2(x) = 1$ for each $x \in V(D) \setminus \{w\}$ are signed k-dominating functions on D such that $f_1(x) + f_2(x) \leq 2 \leq k$ for each vertex $x \in V(D)$. Thus $\{f_1, f_2\}$ is a signed (k, k)-dominating family on D, a contradiction to $d_S^k(D) = 1$. This completes the proof. □

Theorem 7. *If D is a digraph with $\delta^-(D) \geq k+1$, then $d_S^k(D) \geq k$.*

Proof. Let $\{u_1, u_2, \ldots, u_k\} \subset V(D)$ be a subset of k vertices. The hypothesis $\delta^-(D) \geq k+1$ implies that the functions $f_i : V(D) \to \{-1, 1\}$ such that $f_i(u_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(D) \setminus \{u_i\}$ are signed k-dominating functions on D for $i \in \{1, 2, \ldots, k\}$. Since $f_1(x) + f_2(x) + \ldots + f_k(x) \leq k$ for each vertex $x \in V(D)$, we observe that $\{f_1, f_2, \ldots, f_k\}$ is a signed (k, k)-dominating family on D, and Theorem 7 is proved. □

Theorem 8. *Let $k \geq 1$ be an integer, and let D be a $(k+1)$-regular digraph of order n. If $n \not\equiv 0 \pmod{k+2}$, then $d_S^k(D) = k$.*

Proof. Since D is $(k+1)$-regular, we have $d^+(x) = d^-(x) = k+1$ for each vertex $x \in V(D)$. Let f be an arbitrary signed k-dominating function on D. If we define the sets $P = \{v \in V(D) \mid f(v) = 1\}$ and $M = \{v \in V(D) \mid f(v) = -1\}$, then we firstly show that

$$|P| \geq \left\lceil \frac{n(k+1)}{k+2} \right\rceil. \tag{1}$$

Because of $\sum_{x \in N^-[y]} f(x) \geq k$ for each vertex $y \in V(D)$, the $(k+1)$-regularity of D implies that each vertex $u \in P$ has at most one inner neighbor in M and each vertex $v \in M$ has exactly $k+1$ inner neighbors in P. Therefore, the subdigraph $D[M]$ contains no arc, and since $d^+(v) = k+1$, each vertex $v \in M$ has exactly $k+1$ outer neighbors in P. Altogether, we obtain

$$|P| \geq |M|(k+1) = (n - |P|)(k+1),$$

and immediately this leads to (1).

Now let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k, k)-dominating family on D with $d = d_S^k(D)$. Since $\sum_{i=1}^{d} f_i(u) \leq k$ for every vertex $u \in V(D)$, each of these sums contains at least $\lceil (d-k)/2 \rceil$ summands of value -1 (note that Theorem 7 implies that $d \geq k$). Using this and inequality (1), we see that the sum

$$\sum_{x \in V(D)} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in V(D)} f_i(x) \tag{2}$$

contains at least $n\lceil (d-k)/2\rceil$ summands of value -1 and at least $d\lceil n(k+1)/(k+2)\rceil$ summands of value 1. As the sum (2) consists of exactly dn summands, we deduce that

$$n\left\lceil\frac{d-k}{2}\right\rceil + d\left\lceil\frac{n(k+1)}{k+2}\right\rceil \le dn. \tag{3}$$

It follows from the hypothesis $n \not\equiv 0 \pmod{(k+2)}$ that

$$\left\lceil\frac{n(k+1)}{k+2}\right\rceil > \frac{n(k+1)}{k+2},$$

and thus (3) leads to

$$\frac{n(d-k)}{2} + \frac{dn(k+1)}{k+2} < dn.$$

A simple calculation shows that this inequality implies $d < k+2$ and so $d \le k+1$. If we suppose that $d = k+1$, then we observe that d and k are of different parity. Applying Theorem 5, we obtain the contradiction

$$k+1 = d \le \frac{k-1}{k}(k+2) < k+1.$$

Therefore, $d \le k$, and Theorem 7 yields to the desired result $d = k$. □

On the one hand, Theorem 8 demonstrates that the bound in Theorem 7 is sharp, on the other hand, the following example shows that Theorem 8 is not valid in general when $n \equiv 0 \pmod{(k+2)}$.

Let $v_1, v_2, \ldots, v_{k+2}$ be the vertex set of the complete digraph $D = K^*_{k+2}$. We define the functions $f_i : V(D) \to \{-1, 1\}$ such that $f_i(v_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(D) \setminus \{v_i\}$ and each $i \in \{1, 2, \ldots, k+2\}$. Then we observe that f_i is a signed k-dominating function on K^*_{k+2} for each $i \in \{1, 2, \ldots, k+2\}$ and $\sum_{i=1}^{k+2} f_i(x) = k$ for each vertex $x \in V(K^*_{k+2})$. Therefore, $\{f_1, f_2, \ldots, f_{k+2}\}$ is a signed (k,k)-dominating family on D and thus $d^k_S(K^*_{k+2}) \ge k+2$. Using Theorem 1, we obtain $d^k_S(K^*_{k+2}) = k+2$.

Theorem 9. *Let $k \ge 1$ be an integer. If D is a $(k+2)$-regular digraph, then $d^k_S(D) = k$.*

Proof. Let f be an arbitrary signed k-dominating function on D. If we define the sets $P = \{v \in V(D) \mid f(v) = 1\}$ and $M = \{v \in V(D) \mid f(v) = -1\}$, then we obtain analogously to the proof of Theorem 8 the inequality

$$|P| \ge \left\lceil\frac{n(k+2)}{k+3}\right\rceil. \tag{4}$$

Now let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on D such that $d = d^k_S(D)$. Since $\sum_{i=1}^{d} f_i(u) \le k$ for every vertex $u \in V(D)$, each of these sums contains at least $\lceil (d-k)/2\rceil$ summands of value -1. Using this and inequality (4), we see that the sum

$$\sum_{x\in V(D)}\sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d}\sum_{x\in V(D)} f_i(x) \tag{5}$$

contains at least $n\lceil (d-k)/2\rceil$ summands of value -1 and at least $d\lceil n(k+2)/(k+3)\rceil$ summands of value 1. As the sum (5) consists of exactly dn summands, we deduce that

$$n\left\lceil \frac{d-k}{2}\right\rceil + d\left\lceil \frac{n(k+2)}{k+3}\right\rceil \le dn. \tag{6}$$

In view of Corollary 2, we deduce that $d \le k+1$. If we suppose that $d = k+1$, then inequality (6) leads to

$$n + \frac{n(k+1)(k+2)}{k+3} \le (k+1)n,$$

and we obtain the contradiction

$$\frac{(k+1)(k+2)}{k+3} \le k.$$

Therefore, $d \le k$, and Theorem 7 yields to the desired result $d = d_S^k(D) = k$. □

Theorem 9 also demonstrates that the bound in Theorem 7 is sharp.

Theorem 10. *If D is a digraph of order n with $\delta^-(D) \ge k-1$, then*

$$d_S^k(D) + \gamma_{kS}(D) \le kn + 1.$$

Proof. According to Theorem 3, we deduce that

$$d_S^k(D) + \gamma_{kS}(D) \le d_S^k(D) + \frac{kn}{d_S^k(D)}. \tag{7}$$

By Proposition 1 and Theorem 1, we have $1 \le d_S^k(D) \le n$. Using the fact that the function $g(x) = x + kn/x$ is decreasing for $1 \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le n$, inequality (7) leads to

$$d_S^k(D) + \gamma_{kS}(D) \le \max\left\{1 + kn, n + \frac{kn}{n}\right\} = kn + 1.$$

□

Corollary 4 (Sheikholeslami, Volkmann [7]). *If D is a digraph of order n, then $d_S(D) + \gamma_S(D) \le n + 1$.*

If $k \ge 2$ and $\delta^-(D) \ge k+1$, then we can improve Theorem 10 considerably.

Theorem 11. *If D is a digraph of order n with $\delta^-(D) \ge k+1$, then*

$$d_S^k(D) + \gamma_{kS}(D) \le k + n.$$

Proof. By Theorems 1 and 7, we have $k \le d_S^k(D) \le n$. Using inequality (7) and the fact that the function $g(x) = x + kn/x$ is decreasing for $k \le x \le \sqrt{kn}$ and increasing for $\sqrt{kn} \le x \le n$, we obtain

$$d_S^k(D) + \gamma_{kS}(D) \le \max\left\{k + \frac{kn}{k}, n + \frac{kn}{n}\right\} = k + n.$$

□

3. Signed (k,k)-domatic number of graphs

The *signed k-dominating function* of a graph G is defined in [12] as a function $f: V(G) \longrightarrow \{-1,1\}$ such that $\sum_{x \in N_G[v]} f(x) \geq k$ for all $v \in V(G)$. The sum $\sum_{x \in V(G)} f(x)$ is the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed k-dominating functions f on G is called the *signed k-domination number* of G, denoted by $\gamma_{kS}(G)$. The special case $k=1$ was defined and investigated in [2].

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *signed* (k,k)*-dominating family* on G. The maximum number of functions in a signed (k,k)-dominating family on G is the *signed* (k,k)*-domatic number* of G, denoted by $d_S^k(G)$. This parameter was introduced by Sheikholeslami and Volkmann in [6]. In the case $k=1$, we write $d_S(G)$ instead of $d_S^1(G)$.

The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}[v] = N_G[v]$ for each vertex $v \in V(G) = V(D(G))$, the following useful Proposition is valid.

Proposition 2. *If $D(G)$ is the associated digraph of a graph G, then $\gamma_{kS}(D(G)) = \gamma_{kS}(G)$ and $d_S^k(D(G)) = d_S^k(G)$.*

There are a lot of interesting applications of Proposition 2, as for example the following results. Using Corollary 3, we obtain the first one.

Corollary 5 (Volkmann, Zelinka [11] 2005)**.** *The signed domatic number $d_S(G)$ of a graph G is an odd integer.*

Since $\delta^-(D(G)) = \delta(G)$, the next result follows from Proposition 2 and Theorem 1.

Corollary 6 (Sheikholeslami, Volkmann [6] 2010)**.** *If G is a graph with minimum degree $\delta(G) \geq k-1$, then*

$$d_S^k(G) \leq \delta(G) + 1.$$

The case $k=1$ in Corollary 6 can be found in [11].

Corollary 7 (Volkmann [8] 2009)**.** *Let G be a graph, and let v be a vertex of odd degree $d_G(v) = 2t+1$ with an integer $t \geq 0$. Then $d_S(G) \leq t+1$ when t is even and $d_S(G) \leq t$ when t is odd.*

Proof. Since $d^-_{D(G)}(v) = d_G(v) = 2t+1$, it follows from Proposition 2 and Theorem 4 that

$$d_S(G) = d_S(D(G)) \leq \frac{d^-_{D(G)}(v)+1}{2} = \frac{d_G(v)+1}{2} = t+1.$$

Applying Corollary 5, we obtain the desired result. □

In view of Proposition 2 and Theorem 10, we immediately obtain the next result.

Corollary 8 (Volkmann [9] 2011)**.** *If G is a graph of order n, then*

$$\gamma_S(G) + d_S(G) \leq n + 1.$$

Theorem 9 and Proposition 2 lead to our last corollary.

Corollary 9. *If G is a $(k+2)$-regular graph, then $d_S^k(G) = k$.*

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