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Upper bound for functions of bounded turning

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Abstract. For normalized analytic functions in the unit disk, we consider subclasses of bounded turning. The geometric representation is introduced, the radii of convexity (close to convex) are calculated and some subordination relations are suggested. Moreover, the upper bound of the pre-Schwarzian norm for these functions is computed.

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1. Introduction

Let $U := \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\mathcal{H}$ denote the space of all analytic functions on U. Here we suppose that $\mathcal{H}$ is a topological vector space endowed with the topology of uniform convergence over compact subsets of U. Also for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subspace of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Further, let $\mathcal{A} := \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}$ and $\mathcal{S}$ denote the class of univalent functions in $\mathcal{A}$. A function $f \in \mathcal{A}$ is called starlike if $f(U)$ is a starlike domain with respect to the origin, and the class of univalent starlike functions is denoted by S^*. It is called convex $\mathcal{C}$, if $f(U)$ is a convex domain. Each univalent starlike function f is characterized by the analytic condition $\Re(\frac{zf'(z)}{f(z)}) > 0$, $f(z) \neq 0$ in U. Also, it is known that $zf'(z)$ is starlike if and only if f is convex which is characterized by the analytic condition $\Re(1 + \frac{zf''(z)}{f'(z)}) > 0$, $f'(z) \neq 0$ in U. Let $f \in \mathcal{H}$, and g be a univalent function in U, with $f(0) = g(0)$. Then, f is said to be subordinated to g (or g is superordinated to f), denoted by $f(z) \prec g(z)$, if $f(\mathrm{U}) \subset g(\mathrm{U})$. For two functions $f, g \in \mathcal{A}$, the Hadamard product is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,$$

where a_n and b_n are the coefficients of f and g, respectively.

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The pre-Schwarzian derivative T_f of f is defined by

$$T_f(z) = \frac{f''(z)}{f'(z)}$$

with the norm

$$\|f\| = \sup_{z\in U}|T_f|(1-|z|^2).$$

It is known that $\|T_f\| < \infty$ if and only if f is uniformly locally univalent. It is also known that $\|T_f\| \leq 6$ for $f \in \mathcal{S}$ and that $\|T_f\| \leq 4$ for $f \in \mathcal{K}$. Moreover, it is shown that when $|T_f| \leq 3.05$, then f is univalent in U. And when $|T_f| \leq 2.28329.$, then f is starlike in U (see [7]). Recently, the sharp norm estimate for well-known integral operators are determined (see [1, 5, 9]).

For $0 \leq \nu < 1$, let $B(\nu)$ denote the class of functions f of the form (1) so that $\Re\{f'\} > \nu$ in U. The functions in $B(\nu)$ are called functions of bounded turning. By the Nashiro-Warschowski Theorem, the functions in $B(\nu)$ are univalent and also close-to-convex in U. It is well-known that $B(\nu) \nsubseteq S^*$ and $S^* \nsubseteq B(\nu)$. In [8], Mocanu obtained a subclass of S^* which is contained in $B(\nu)$. Recently, Tuneski generalized the class of convex functions with bounded turning (see [11])

$$\sqrt[k]{f'(z)} \prec \frac{1+Cz}{1+Dz}, \quad k \geq 1.$$

Different studies of the class of bounded turning functions can be found in [2 -4 , 10].

In this note we pose the following subclass of bounded turning functions in the unit disk: For given numbers $\epsilon \in (0, 1]$ and $\alpha > 1$, let us consider the class $B(p_\epsilon)$

$$B(p_\epsilon) = \{f \in \mathcal{A} : \left|\Big(f'(z)\Big)^{\alpha} - 1\right| < \epsilon,\, z \in U\}.$$

It is easy to see that $f \in B(p_\epsilon)$ if and only if

$$f'(z) \prec p_\epsilon := (1+\epsilon z)^{1/\alpha}, \quad z \in U.$$

Let us denote by $\mathcal{Q}$ a set of all points on the right half-plane such that the product of the distances from each point to the end points -1 and 1 is less than ϵ

$$\mathcal{Q} = \{w \in \mathbb{C} : 0 < \Re(w),\, |w^\alpha - 1| < \epsilon, \quad \alpha > 1\}.$$

Therefore its boundary satisfies the equation

$$w^2 = [\epsilon^2 + 1]^{2/\alpha}$$

and hence in Euler formula we have

$$\cos 2\theta + i \sin 2\theta = [\epsilon^2 + 1]^{2/\alpha},$$

where $r = 1$.

2. Main results

First, our result is in the following form:

Theorem 1. *A function $f \in B(p_\epsilon)$; $\epsilon \in (0,1]$ if and only if there exists an analytic function p, $p(z) \prec p_\epsilon(z) := (1+\epsilon z)^{1/\alpha}$ such that*

$$f(z) = \int_0^z p(t)\,dt, \quad p(0) = 1. \tag{1}$$

Moreover, if for function $f_\epsilon \in B(p_\epsilon)$, it takes the form

$$f_\epsilon(z) = \frac{[1 + (\sqrt{\epsilon+1}-1)z]^{\frac{2+\alpha}{\alpha}} - 1}{(\sqrt{\epsilon+1}-1)(\frac{2+\alpha}{\alpha})}, \tag{2}$$

then the subordination relation

$$\frac{f(z)}{z} \prec \frac{f_\epsilon(z)}{z}, \quad z \in U \tag{3}$$

holds.

Proof. Let $f \in B(p_\epsilon)$ and let $p(z) = f'(z) \prec (1+\epsilon z)^{1/\alpha}$. Integration of this equation yields (1). If f is given in (1) with an analytic function $p(z) \prec p_\epsilon(z)$, then by a differentiation of (1) we obtain that $f'(z) = p(z)$. Therefore $f'(z) \prec (1+\epsilon z)^{1/\alpha}$ and consequently $f \in B(p_\epsilon)$.

Now we proceed to prove that $f_\epsilon \in B(p_\epsilon)$. For this purpose we will show that the set

$$\mathcal{Q}_\epsilon := \{w \in \mathbb{C} : 0 < \Re(w),\, |w^{\frac{\alpha}{2}} - 1| < \sqrt{1+\epsilon} - 1, \quad \alpha > 1\} \subset \mathcal{Q}.$$

Let $w \in \mathcal{Q}_\epsilon$, then

$$|w^{\frac{\alpha}{2}} - 1| < \sqrt{1+\epsilon} - 1 \Rightarrow |w^{\frac{\alpha}{2}} + 1| < \sqrt{1+\epsilon} + 1.$$

By multiplying these inequalities we obtain

$$|w^\alpha - 1| < \epsilon \Rightarrow w \in \mathcal{Q}.$$

Denote $q_\epsilon(z) = [1 + (\sqrt{1+\epsilon} - 1)z]^{\frac{2}{\alpha}}$, we pose that

$$w^{\alpha/2} := [q_\epsilon(z)]^{\frac{\alpha}{2}} = 1 + (\sqrt{1+\epsilon} - 1)z,$$

thus

$$q_\epsilon(U) = \{w \in \mathbb{C} : 0 < \Re(w),\, |w^{\frac{\alpha}{2}} - 1| < \sqrt{1+\epsilon} - 1, \quad \alpha > 1\} \subset \mathcal{Q}.$$

Hence $q_\epsilon(z) \prec p_\epsilon(z)$, by putting $q_\epsilon(z)$ in (1) implies (2). To prove the subordination relation (3), first we show that $\frac{f_\epsilon(z)}{z}$ is a convex function. We observe that

$$\begin{aligned} f_\epsilon(z) &= \frac{[1+(\sqrt{\epsilon+1}-1)z]^{\frac{2+\alpha}{\alpha}}-1}{(\sqrt{\epsilon+1}-1)(\frac{2+\alpha}{\alpha})} \\ &= \frac{[1+(\frac{2+\alpha}{\alpha})(\sqrt{\epsilon+1}-1)z+\frac{(\frac{2+\alpha}{\alpha})(\sqrt{\epsilon+1}-1)^2}{\alpha}z^2+\cdots]-1}{(\sqrt{\epsilon+1}-1)(\frac{2+\alpha}{\alpha})} \\ &= z+\frac{(\sqrt{\epsilon+1}-1)}{\alpha}z^2+\cdots \\ &= z+\sum_{n=2}^{\infty}\lambda(\alpha,\epsilon)z^n \in \mathcal{A}. \end{aligned}$$

Let us consider the function

$$F_\epsilon(z) = \frac{\alpha}{\sqrt{1+\epsilon}-1}[\frac{f_\epsilon(z)}{z}-1] \in \mathcal{A}.$$

Computations give

$$F'_\epsilon(z) = \frac{\alpha}{\sqrt{1+\epsilon}-1}[\frac{f'_\epsilon(z)}{z}-\frac{f_\epsilon(z)}{z^2}]$$

and

$$F''_\epsilon(z) = \frac{\alpha}{\sqrt{1+\epsilon}-1}[\frac{zf''_\epsilon(z)-f'_\epsilon(z)}{z^2}-\frac{z^2f'_\epsilon(z)-2zf_\epsilon(z)}{z^4}].$$

A calculation also implies that

$$\begin{aligned} f_\epsilon(z) &= \frac{[1+(\sqrt{\epsilon+1}-1)z]^{\frac{2+\alpha}{\alpha}}-1}{(\sqrt{\epsilon+1}-1)(\frac{2+\alpha}{\alpha})} \\ f'_\epsilon(z) &= [1+(\sqrt{\epsilon+1}-1)z]^{2/\alpha} \\ f''_\epsilon(z) &= \frac{2(\sqrt{\epsilon+1}-1)}{\alpha}[1+(\sqrt{\epsilon+1}-1)z]^{2/\alpha-1}. \end{aligned} \tag{4}$$

The aim is to show that $1+\frac{zF''_\epsilon(z)}{F'_\epsilon(z)}$ has a positive real part in the unit disk. Let $z \in \mathcal{Q}_\epsilon$ *i.e.* $\Re(z) > 0$. Since $0 < (\sqrt{\epsilon+1}-1) < 1$, then by using (4), we have

$$\begin{aligned} \Re\Big(1+\frac{zF''_\epsilon(z)}{F'_\epsilon(z)}\Big) &= \Re\Big(\frac{z^2f''_\epsilon(z)}{zf'_\epsilon(z)-f_\epsilon(z)}-1\Big) \\ &= \Re\Big(\frac{\frac{2z^2(\sqrt{\epsilon+1}-1)^2(\frac{2+\alpha}{\alpha})}{\alpha}}{[1+(\sqrt{\epsilon+1}-1)z]\{z(\sqrt{\epsilon+1}-1)(\frac{2+\alpha}{\alpha})-[1+(\sqrt{\epsilon+1}-1)z]-1\}}-1\Big). \end{aligned}$$

Hence for choosing suitable $\alpha > 1$ such that $\Re(z) > \frac{\alpha}{2(\sqrt{\epsilon+1}-1)} > 0$, we impose that

$$\Re\Big(1+\frac{zF''_\epsilon(z)}{F'_\epsilon(z)}\Big) > 0, \quad z \in U.$$

Consequently, we obtain that $F_\epsilon \in \mathcal{K}$; therefore $\frac{f_\epsilon(z)}{z}$ is a convex function.

Now by using the fact that if for $F, G \in \mathcal{K}$, satisfy $f \prec F$ and $g \prec G$, then $f * g \prec F * G$ and $k(z) = \frac{z}{1-z}$ is a convex function then we immediately establish (3). This completes the proof. □

Next we consider another class of functions of bounded turning. We will estimate the upper bound of these functions by using the pre-Schwarzian norm.

Theorem 2. *Consider the class $B(q_\epsilon), \epsilon \in (0,1]$ of functions $f \in \mathcal{A}$ of bounded turning which satisfies the relation*

$$f'(z) \prec q_\epsilon := (1-\epsilon z)^{1/\alpha}, \quad \alpha > 1.$$

Then

$$\|f\| \leq \frac{\epsilon(\epsilon+1)}{\alpha}. \tag{5}$$

Proof. Let $f \in B(q_\epsilon)$ and $P_f := f'(z)$. Then there exists an analytic function $w : U \to U$ with $w(0) = 0$ and

$$P_f = q_\epsilon \circ w = (1-\epsilon w)^{1/\alpha}. \tag{6}$$

Define a function $F \in \mathcal{A}$ such that $P_F = -q_\epsilon, i.e.$

$$F'(z) = -(1-\epsilon z)^{1/\alpha} \tag{7}$$

and thus

$$F(z) = -\int_0^z q_\epsilon(t)\, dt = \frac{\alpha}{\epsilon(\alpha+1)}(1-\epsilon z)^{\frac{1+\alpha}{\alpha}} - \frac{\alpha}{\epsilon(\alpha+1)}.$$

We proceed to determine the quantities $T_F(|z|)$ and $T_f(z)$. Logarithmic differentiation of (7) yields

$$T_F(|z|) = \frac{F''(|z|)}{F'(|z|)} = \frac{\epsilon}{\alpha(1-\epsilon|z|)}$$

And the logarithmic differentiation of (6) gives

$$T_f(z) = \frac{f''(z)}{f'(z)} = -\frac{\epsilon}{\alpha}\frac{w'(z)}{1-\epsilon w(z)}, \quad z \neq 0.$$

Thus by triangle inequality and Schwarz-Pick lemma, we obtain

$$\begin{aligned}
|T_f(z)| &= \frac{\epsilon}{\alpha}\left|\frac{w'(z)}{1-\epsilon w(z)}\right| \\
&\leq \frac{\epsilon}{\alpha}\frac{1-|w(z)|^2}{(1-|z|^2)(|1-\epsilon w(z)|)} \\
&\leq \frac{\epsilon}{\alpha}\frac{1-|w(z)|}{(1-|z|)(|1-\epsilon w(z)|)} \\
&\leq \frac{\epsilon}{\alpha}\frac{1}{(1-|z|)} \leq \frac{\epsilon}{\alpha}\frac{1}{(1-\epsilon|z|)} = T_F(|z|).
\end{aligned}$$

Consequently we have

$$(1-|z|^2)|T_f(z)| = (1-|z|^2)T_F(|z|).$$

Therefore, $\|f\| \leq \|F\|$ and this inequality is sharp. Thus to determine the upper estimate of $f \in B(q_\epsilon)$ it is enough to compute $\|F\|$. Let $t = |z|$, we have

$$\begin{aligned} sup(1-t^2)T_F(|z|) &= \sup(1-t^2)\frac{\epsilon}{\alpha(1-\epsilon t)} \\ &\leq \sup(1-\epsilon^2 t^2)\frac{\epsilon}{\alpha(1-\epsilon t)} \\ &= \sup\frac{\epsilon(1+\epsilon t)}{\alpha}. \end{aligned}$$

Hence we obtain (5). □

By applying Jack's Lemma, we pose sufficient conditions for convex functions f to belong to subclasses $B(p_\epsilon)$ and $B(q_\epsilon)$.

Lemma 1 (See [6]). *Let w be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point z_0, then*

$$z_0 w'(z_0) = k w(z_0),$$

where k is a real number and $k \geq 1$.

Theorem 3. *Assume that $\epsilon \in [\frac{1}{2}, 1)$ and $\alpha \geq \frac{\epsilon}{1-\epsilon}$. If $f \in \mathcal{A}$ is a convex function in U of order $0 \leq \frac{\alpha-\epsilon(1+\alpha)}{\alpha(1-\epsilon)} < 1$, then $f \in B(q_\epsilon)$.*

Proof. Let $f \in \mathcal{K}(\frac{\alpha-\epsilon(1+\alpha)}{\alpha(1-\epsilon)}), i.e.$

$$\Re\Big\{1 + \frac{zf''(z)}{f'(z)}\Big\} > \frac{\alpha-\epsilon(1+\alpha)}{\alpha(1-\epsilon)}, \quad z \in U. \tag{8}$$

From the proof of Theorem 2, we have

$$\Re\{\frac{zf''(z)}{f'(z)} + 1\} = \Re\{1 - \frac{\epsilon}{\alpha}\frac{zw'(z)}{1-\epsilon w(z)}\} > \frac{\alpha-\epsilon(1+\alpha)}{\alpha(1-\epsilon)}, \quad z \in U,$$

where w ia analytic in U and satisfies $w(0) = 0$ and

$$f'(z) = (1-\epsilon w(z))^{1/\alpha}.$$

Suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = 1.$$

Then, using Lemma 1 and letting $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = k e^{i\theta},\ k \geq 1$ yields

$$\begin{aligned} \Re\{\frac{z_0 f''(z_0)}{f'(z_0)} + 1\} &= \Re\{1 - \frac{\epsilon}{\alpha}\frac{z_0 w'(z_0)}{(1-\epsilon w(z_0))}\} \\ &= \Re\{1 - \frac{\epsilon}{\alpha}\frac{k e^{i\theta}}{(1-\epsilon e^{i\theta})}\} \\ &\leq \Re\{1 - \frac{\epsilon}{\alpha}\frac{e^{i\theta}}{(1-\epsilon e^{i\theta})}\}, \end{aligned}$$

which contradicts hypothesis (8). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$ that is $f \in B(q_\epsilon)$.

□

Theorem 4. *Assume that $\epsilon \in (0, 1]$ and $\alpha > 1$. If $f \in \mathcal{A}$ satisfies*

$$\Re\{\frac{zf''(z)}{f'(z)} + 1\} < \frac{\epsilon(\epsilon+2)}{\alpha(\epsilon+1)}, \quad z \in U, \tag{9}$$

then $f \in B(p_\epsilon)$.

Proof. Define a function $\omega(z)$ by

$$f'(z) = (1 + \epsilon\omega(z))^{1/\alpha}.$$

Then ω is analytic in U and satisfies $\omega(0) = 0$. It follows that

$$\Re\{\frac{zf''(z)}{f'(z)} + 1\} = \frac{\epsilon}{\alpha}\Re\{\frac{z\omega'(z) + \epsilon\omega(z) + 1}{1 + \epsilon\omega(z)}\} < \frac{\epsilon(\epsilon+2)}{\alpha(\epsilon+1)}.$$

In the same manner of Theorem 3, we find that

$$\begin{aligned}\Re\{\frac{z_0f''(z_0)}{f'(z_0)} + 1\} &= \frac{\epsilon}{\alpha}\Re\{\frac{z_0\omega'(z_0) + \epsilon\omega(z_0) + 1}{1 + \epsilon\omega(z_0)}\} \\ &= \frac{\epsilon}{\alpha}\Re\{\frac{ke^{i\theta} + \epsilon e^{i\theta} + 1}{1 + \epsilon e^{i\theta}}\} \\ &\geq \frac{\epsilon(\epsilon+2)}{\alpha(\epsilon+1)},\end{aligned}$$

which contradicts hypothesis (9). Therefore, we conclude that $|w(z)| < 1$ for all $z \in U$, that is $f \in B(p_\epsilon)$. □

Corollary 1. *Let the assumptions of Theorem 4 hold. Then f is strongly close to convex of order $\frac{1}{\alpha}$.*

Proof. Since $f \in B(p_\epsilon)$, then there exists an analytic function $\psi \in U$ such that $\psi(0) = 0, |\psi(z)| < 1$ and

$$(f'(z))^\alpha = 1 + \epsilon\psi(z).$$

But

$$|\psi(z)| = \Big|\frac{(f'(z))^\alpha - 1}{\epsilon}\Big| < 1,$$

which implies

$$\Big|(f'(z))^\alpha - 1\Big| < 1$$

and thus f is strongly close to convex of order $\frac{1}{\alpha}$. □

3. Conclusion

It is well-known that the class of bounded turning functions is not included in the class of starlike functions also starlike functions cannot be embedded in the class of bounded turning functions. From the above, we conclude that for some classes of bounded turning functions we can include the class of convex functions ($\mathcal{K} \subset B(q_\epsilon)$; Theorem 3). Moreover, some classes of bounded turning functions can be embedded in the class of close to convex functions ($B(p_\epsilon) \subset \mathcal{KL}$; Corollary 1). Hence we have the following inclusion relation:

$$\mathcal{K} \subset B \subset \mathcal{KL}.$$

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