Queen's University Belfast

Doctoral Thesis

Algebraic Finite Domination

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ABSTRACT

This thesis focuses on proving a finite domination condition on bounded chain complexes of finitely generated free R-modules where R is a strongly $\mathbb{Z}^n$-graded ring.

CONTENTS

0. INTRODUCTION

Acknowledgements

I would like to acknowledge the support and assistance of my supervisor Thomas Hüttemann and the financial support of DEL.

0.1 Topological and algebraic finite domination

This thesis focuses on an interesting generalisation of the characterisation of algebraic finite domination of chain complexes. To outline the background of the result, it seems best to begin by referring to topological finite domination.

Definition 0.1.1. A topological space X is *finitely dominated* if it is a homotopy retract of a finite CW complex, that is if there exists a finite CW complex K, maps $f\colon X \to K, g\colon K \to X$ and a homotopy $gf \simeq 1\colon X \to X$.

In the paper of Ranicki, [Ran95], the author discusses a number of results pertaining to the finite domination properties of infinite cyclic covers of finite CW complexes. In particular, by making use of Novikov rings, he produces the following characterisation of finite domination for an infinite cyclic cover of a finite CW complex:

Theorem 0.1.2 (Theorem 1 [Ran95])**.** *Let X be a CW complex with universal cover $\tilde{X}$, and fundamental group $\pi_1(X) = \pi \times \mathbb{Z}$ with π a group, so that $\mathbb{Z}[\pi_1(X)] = \mathbb{Z}[\pi][z, z^{-1}]$. Let $C(Y)$ be the cellular chain complex of the CW complex Y. The infinite cyclic cover $\overline{X} = \tilde{X}/\pi$ is finitely dominated if and only if X is $\mathbb{Z}[\pi]((z))$-acyclic and $\mathbb{Z}[\pi]((z^{-1}))$-acyclic:*

$$H_*(\mathbb{Z}[\pi]((z)) \otimes_{\mathbb{Z}[\pi][z,z^{-1}]} C(\tilde{X})) = 0 = H_*(\mathbb{Z}[\pi]((z^{-1})) \otimes_{\mathbb{Z}[\pi][z,z^{-1}]} C(\tilde{X})).$$

The name Novikov ring stems from their use by Sergei Novikov in his study of Morse theory. The above result of Ranicki is the natural place to begin the discussion of Novikov rings in this work as it is the topological equivalent of the algebraic result studied in this thesis. Helpfully, in the same paper, a related key result of algebraic finite domination is presented. The key definition of algebraic finite domination, that is investigated in this work, is the following:

Definition 0.1.3. Let L be a unital ring, and let K be a subring of L. A chain complex C of (right) L-modules is *K-finitely dominated* if C, considered as a complex of K-modules, is a retract up to homotopy of a bounded complex of finitely generated free K-modules.

Theorem 0.1.4. *From [Ran85, Proposition 3.2. (ii)], we know that an L complex C is K-finitely dominated if and only if it is homotopy equivalent to a bounded complex of finitely generated projective K-modules.*

The relevant result, that transfers [Ran95, Theorem 1] to the algebraic setting, is the following:

Theorem 0.1.5 (Ranicki [Ran95, Theorem 2])**.** *Let R be a unital ring, and let $R[t, t^{-1}]$ denote the Laurent polynomial ring in the indeterminate t. Let C be a bounded chain complex of finitely generated free $R[t, t^{-1}]$-modules. The complex C is R-finitely dominated if and only if both*

$$C \underset{R[t,t^{-1}]}{\otimes} R(\!(t^{-1})\!) \qquad \text{and} \qquad C \underset{R[t,t^{-1}]}{\otimes} R(\!(t)\!)$$

have vanishing homology in all degrees. Here we write $R(\!(t)\!) = R[\![t]\!][t^{-1}]$ for the ring of formal Laurent series in t, and similarly $R(\!(t^{-1})\!) = R[\![t^{-1}]\!][t]$ stands for the ring of formal Laurent series in t^{-1}.

The rings $R(\!(t)\!)$ and $R(\!(t^{-1})\!)$ are called Novikov rings. We state that a complex of $R[t, t^{-1}]$-modules C has *trivial Novikov homology* whenever the complexes $C \otimes_{R[t,t^{-1}]} R(\!(t^{-1})\!)$ and $C \otimes_{R[t,t^{-1}]} R(\!(t)\!)$ are acyclic. Hence, we can summarise the result thus: *A bounded chain complex of finitely generated free $R[t, t^{-1}]$-modules C is R-finitely dominated if and only if it has trivial Novikov homology.* The main result of this thesis generalises Theorem 0.1.5

and as we progress towards generality we look at more general concepts of trivial Novikov homology.

It therefore is necessary to spend some time discussing what Novikov rings actually are in these higher generalities, beginning with the simplest case of polynomial rings. The above rings $R(\!(t)\!)$ and $R(\!(t^{-1})\!)$ are the versions built upon a Laurent polynomial ring in one indeterminate. These can be seen as localisations of power series $R[\![t]\!], R[\![t^{-1}]\!]$ where R is a division ring.

When extending to Novikov rings in two indeterminates we have more choices than just orientation. Starting with a Laurent polynomial ring $R[x, x^{-1}, y, y^{-1}]$ we have a collection of power series

$$R[\![x, y]\!], \qquad R[y, y^{-1}][\![x]\!], \qquad R[x, x^{-1}][\![y]\!].$$

Let $R_*(\!(x, y)\!)$ be the ring with elements in the set

$$\{\sum_{a,b\geq t}^{\infty} r_{a,b}x^a y^b \colon t \in \mathbb{Z},\, r_{a,b} \in R\}.$$

The following rings constitute the Novikov rings that correspond directly to the Novikov homology of a chain complex of $R[x, x^{-1}, y, y^{-1}]$:

$$\begin{array}{ll} R[x,\, x^{-1}](\!(y)\!) & R[x,\, x^{-1}](\!(y^{-1})\!) \\ R[y,\, y^{-1}](\!(x)\!) & R[y,\, y^{-1}](\!(x^{-1})\!) \\ R(\!(x,\, y)\!) & R(\!(x^{-1},\, y^{-1})\!) \\ R(\!(x,\, y^{-1})\!) & R(\!(x^{-1},\, y)\!). \end{array}$$

There are also similar rings, such as $R(\!(x)\!)(\!(y)\!)$ which are not directly associated to the definition of Novikov homology but feature in the proof given in [HQ15]. In fact, trivial Novikov homology for a chain complex with satisfactory conditions on it will imply that the tensor product of the complex with any of these rings are acyclic as an implication. Specifically, these rings are:

$$\begin{array}{ll} R(\!(x)\!)(\!(y)\!) & R(\!(x)\!)(\!(y^{-1})\!) \\ R(\!(x^{-1})\!)(\!(y)\!) & R(\!(x^{-1})\!)(\!(y^{-1})\!) \end{array}$$

and similar ones with x and y swapped. The first extension of [Ran95, Theorem 2] by my supervisor Thomas Hüttemann and David Quinn was to the two dimensional case:

Theorem 0.1.6 (Hüttemann and Quinn[HQ15, Main Theorem 1.1.2])**.** *Write $L = R[x, x^{-1}, y, y^{-1}]$. Let C be a bounded chain complex of finitely generated free L-modules. Then the following two statements are equivalent:*

1. *The complex C is R-finitely dominated, i.e., C is homotopy equivalent, as an R-module chain complex, to a bounded chain complex of finitely generated projective R-modules.*

2. *The eight chain complexes listed below are acyclic (all tensor products are taken over L):*

$$\left.\begin{array}{ll} C \otimes R[x,\, x^{-1}](\!(y)\!) & C \otimes R[x,\, x^{-1}](\!(y^{-1})\!) \\ C \otimes R[y,\, y^{-1}](\!(x)\!) & C \otimes R[y,\, y^{-1}](\!(x^{-1})\!) \end{array}\right\} \tag{0.1.6.1a}$$

$$\left.\begin{array}{ll} C \otimes R(\!(x,\, y)\!) & C \otimes R(\!(x^{-1},\, y^{-1})\!) \\ C \otimes R(\!(x,\, y^{-1})\!) & C \otimes R(\!(x^{-1},\, y)\!) \end{array}\right\} \tag{0.1.6.1b}$$

Condition 2 outlines the concept of Novikov homology for rings with two indeterminates.

In a different direction, Thomas and myself extended Theorem 0.1.5 to bounded chain complexes of strongly $\mathbb{Z}$-graded ring modules. To even state the result in this context, we need to quickly note the definition of a $\mathbb{Z}$-graded ring.

Definition 0.1.7. A $\mathbb{Z}$*-graded ring* is a (unital) ring R equipped with a direct sum decomposition into additive subgroups $R = \bigoplus_{k\in\mathbb{Z}} R_k$ such that $R_k R_\ell \subseteq R_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$, where $R_k R_\ell$ consists of the finite sums of ring products xy with $x \in R_k$ and $y \in R_\ell$. The summands R_k are called the *(homogeneous) components* of R; elements of R_k are called *homogeneous of degree k*. — Following Dade [Dad80] we call R a *strongly $\mathbb{Z}$-graded ring* if $R_k R_\ell = R_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$.

The $\mathbb{Z}$-graded ring is a generalisation of a Laurent polynomial ring in one indeterminate. For a Laurent polynomial ring $T[x, x^{-1}]$, T is the ground ring and hence the ring that the relevant chain complex will be finitely dominated over is the ring T itself. For the $\mathbb{Z}$-graded ring R the subring that plays this role is R_0, the component of R indexed by 0, which is known to be a unital ring when R is itself unital [Dad80, Proposition 1.4].

Next, we need to discuss the analogue of Novikov rings. For one dimension, this is fairly simple — we have two, $R_*((t)) = \bigoplus_{t<0} R_t \oplus \prod_{t\geq 0} R_t$ and $R_*((t^{-1})) = \prod_{t<0} R_t \oplus \bigoplus_{t\geq 0} R_t$. Trivial Novikov homology in this case is related to tensoring with these rings over R.

Theorem 0.1.8 (Hüttemann and Steers [HS16, Theorem 1.3])**.** *Let $R = \bigoplus_{k\in\mathbb{Z}} R_k$ be a strongly $\mathbb{Z}$-graded ring, and let C be a bounded chain complex of finitely generated free R-modules. The complex C is R_0-finitely dominated if and only if both*

$$C \underset{R}{\otimes} R_*((t^{-1})) \qquad \textit{and} \qquad C \underset{R}{\otimes} R_*((t))$$

have vanishing homology in all degrees.

Thomas and myself have also worked on the case where the ring is a strongly $\mathbb{Z}^2$-graded ring, which has produced a paper that is at the time of writing being finalised. The eight Novikov rings associated with a $\mathbb{Z}^2$-graded ring correspond with the eight Novikov rings seen in condition 2 of Theorem 0.1.6.

This paper deals with the analogous case for when R is a strongly $\mathbb{Z}^n$ graded ring. As the number of indeterminates grows, we will make sense of the collection of Novikov rings for a Laurent polynomial of n indeterminates by associating each ring with a flag of faces of a cube $S = [-1, 1]^n$. Thomas and David worked on the case with a Laurent polynomial in n indeterminates. The result can be seen in [HQ16, Theorem III.6.4], broadly speaking it deals with a more general collection of homological conditions but with a class of rings more specific than that looked at in this thesis.

Another result that looks at a similar case is from Schütz.

Theorem 0.1.9 (Schütz [Sch06, Theorem 4.7])**.** *Let G be a group and C be a bounded chain complex of finitely generated free $R[G]$-modules. Let*

$$\widehat{RG_\chi} = \{f\colon G \to R | \forall t \in R\colon \#\Big(\mathrm{supp}(f) \cap \chi^{-1}\big([t,\infty)\big)\Big) < \infty\}.$$

Suppose that N is a normal subgroup of G with quotient $G/N \cong \mathbb{Z}^n$ a free Abelian group of finite rank. The complex C is $R[N]$-finitely dominated if *and only if for every character $\chi\colon G \to \mathbb{R}$ which is trivial on N the complex $C \underset{R[G]}{\otimes} \widehat{RG_\chi}$ is acyclic.*

This result effectively uses infinitely many conditions, while this paper has managed to express a similar result in finitely many conditions. Precisely, the ring $R[G/N]$ is a specific strongly $\mathbb{Z}^n$-graded ring.

0.2 Structure of the thesis

The first chapter introduces the two key definitions required for the main result. Firstly, I define strongly $\mathbb{Z}^n$-graded rings and their properties, including a number of key characterisations and the partition of unity. Secondly, the definition of the generalised Novikov ring. These are the analogues of Novikov rings and power series for the polynomial case in the graded ring case.

The second chapter involves proving one direction of the actual result, that trivial Novikov homology implies finite domination, introducing a number of concepts to better explain Novikov rings to this end. This section will feature a canonical resolution of R_0, for a strongly $\mathbb{Z}^n$-graded ring R.

The third chapter deals with setting conventions and forming a category whose objects can be totalised as iterated mapping cones of maps in a category of chain complexes of R-modules, this will have left and right adjoints to the category of chain complexes of R-modules. The objects of these mapping cone categories, N-cubes, will be of central use in the final section, when proving that finite domination implies that the Novikov homology is trivial.

The fourth chapter goes through the other direction, making use of the N-cube definition. This section will feature a canonical resolution of R for a given strongly $\mathbb{Z}^n$-graded ring.

Finally, the Appendix consists of a reproduction of the paper worked on by my supervisor and myself.

0.3 Setting conventions

We take a moment to set a few conventions. Firstly, given a map between two direct sums $f\colon A \oplus B \to C \oplus D$ we consider f as a matrix

$$f = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$$

with

$$\begin{aligned} w &\colon A \to C & x &\colon B \to C \\ y &\colon A \to D & z &\colon B \to D. \end{aligned}$$

That is we view elements of $A \oplus B$ as a column vector with the matrix f acting on the left.

It is worth taking a moment to explain, for the total avoidance of doubt, what a chain complex is. For a ring R, given a collection of R-modules $(C_k)_{k\in\mathbb{Z}}$ and maps $d_k\colon C_k \to C_{k-1}$ satisfying $d_k d_{k-1} = 0$, a chain complex C is a pair:

$$C = (C_k, d)$$

where C_k is the module at position k. In particular, due to potential confusion later on, we avoid using any suspension definition, rather I will express chain complexes where the index of the modules have changed explicitely.

Mapping cones feature heavily in this paper. I settle on the following sign convention.

Definition 0.3.1 (Mapping cone)**.** Let $f\colon C \to D$ be a chain map between two chain complexes. The *mapping cone of* f, written $\mathrm{cone}(f)$, is a chain complex

$$\left(C_{k-1} \oplus D_k, \begin{pmatrix} d^C_{k-1} & 0 \\ f_{k-1} & -d^D_k \end{pmatrix} \right).$$

Note that this is a non-standard convention, but a mapping cone using this convention is isomorphic to the more standard convention (i.e., $\begin{pmatrix} -d^C_{k-1} & 0 \\ f_{k-1} & d^D_k \end{pmatrix}$).

Definition 0.3.2. Let C be a right R-module and D a left R-module. Define the tensor product of the chain complexes over R, $C \underset{R}{\otimes} D$, as the complex with modules

$$(C \underset{R}{\otimes} D)_n = \sum_{k+\ell=n} C_k \otimes D_\ell$$

and boundary

$$d(x \otimes y) = d(x) \otimes y + (-1)^{\deg(x)} x \otimes d(y).$$

We define double complexes and their totalisations.

Definition 0.3.3. Define a *double complex* as a collection of R-modules $C_{a,b}$, indexed over $\mathbb{Z}^2$, and maps

$$d_h \colon C_{a,b} \to C_{a-1,b},\ d_v \colon C_{a,b} \to C_{a,b-1}$$

that satisfy

$$d_h d_h = 0 = d_v d_v$$

and

$$d_h d_v = -d_v d_h.$$

This convention provides that the boundaries anti-commute. This ensures that the obvious candidate for totalisation is a chain complex.

Definition 0.3.4. Given a double complex $C = \{C_{a,b},\, d_h,\, d_v\}$, define the *totalisation of the double complex*, $\mathrm{Tt}(C)$, as the complex with module at degree k

$$\bigoplus_{a+b=k} C_{a,b}$$

and boundary consisting of

$$d_h + d_v$$

applied to each summand $C_{a,b}$.

Remark 0.3.5. For a double complex C, the boundary map of $\mathrm{Tt}(C)$ satisfies $(d_h + d_v)(d_h + d_v) = d_h d_h + d_h d_v + d_v d_h + d_v d_v = 0$, hence $\mathrm{Tt}(C)$ is a chain complex.

Both of these definitions follow [HQ15], so that we can lift the following result from it for later use.

Lemma 0.3.6. *Let $f \colon C \to D$ be a map of double complexes which are concentrated in finitely many columns. If f is a quasi-isomorphism on each column or on each row, then the induced map*

$$\mathrm{Tt}(f) \colon \mathrm{Tt}(C) \to \mathrm{Tt}(D)$$

is a quasi-isomorphism.

Proof. Seen in [HQ15, Lemma 2.2.2]. □

1. DEFINITION OF GENERALISED NOVIKOV RINGS

As noted in the introduction, we are interested in a generalised case of polynomial rings in n indeterminates, strongly $\mathbb{Z}^n$-graded rings. We now need to define and investigate the Novikov rings that are used to encode the condition of trivial Novikov homology in the strongly $\mathbb{Z}^n$-graded case. After defining strongly $\mathbb{Z}^n$-graded rings we move onto defining Novikov rings in this case.

1.1 Strongly $\mathbb{Z}^n$-graded rings

This section's purpose is to introduce the $\mathbb{Z}^n$-graded rings. We can define a broad collection of rings, G-graded rings, where G is a general group. However we are mainly interested in $\mathbb{Z}^n$-graded rings.

Definition 1.1.1. For $n \in \mathbb{N}$, a $\mathbb{Z}^n$*-graded ring* is a (unital) ring R equipped with a direct sum decomposition into additive subgroups $R = \bigoplus_{k \in \mathbb{Z}^n} R_k$ such that $R_k R_\ell \subseteq R_{k+\ell}$ for all $k, \ell \in \mathbb{Z}^n$, where $R_k R_\ell$ consists of the finite sums of ring products xy with $x \in R_k$ and $y \in R_\ell$. The summands R_k are called the *(homogeneous) components* of R; elements of R_k are called *homogeneous of degree* k. If R satisfies $R_k R_\ell = R_{k+\ell}$ for all $k, \ell \in \mathbb{Z}^n$ then we call R a *strongly* $\mathbb{Z}^n$*-graded ring.*

The paramount case is the Laurent polynomial ring $V[x_1, x_1^{-1}, ..., x_n, x_n^{-1}]$ in n variables over a ring V, which for $\rho = \sum_{1=j}^{n} m_j e_j \in \mathbb{Z}^n$ has homogenous component

$$V[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]_\rho = \{v x_1^{m_1} x_2^{m_2}, \dots, x_n^{m_n} : v \in V\}$$

for $m_j \in \mathbb{Z}$. This is, in particular, a strongly $\mathbb{Z}^n$-graded ring.

This definition can be extended from $\mathbb{Z}^n$ to general groups, but generally we will focus only on $\mathbb{Z}^n$-graded rings in this work. All rings used in this thesis will be unital, on which note the following ought to be stated.

Remark 1.1.2. Whenever R is unital, $R_0 = R_{0_{\mathbb{Z}^n}}$ is a unital ring [Dad80, Proposition 1.4].

Definition 1.1.3. Given two $\mathbb{Z}^n$-graded rings R, R', we call a ring homomorphism $f\colon R \to R'$ a *$\mathbb{Z}^n$-graded map* if f respects the grading, that is for all $k \in \mathbb{Z}^n$, $f(R_k) \subseteq R'_k$.

A characterisation of strongly $\mathbb{Z}^n$-graded is needed next.

Definition 1.1.4 (Partition of unity of type $(-\rho, \rho)$)**.** Given a $\mathbb{Z}^n$-graded unital ring R and some $\rho \in \mathbb{Z}^n$, an expression of the form

$$1 = \sum_{j=1}^{q} u_j v_j,$$

where $u_j \in R_{-\rho}$, $v_j \in R_\rho$, is called a *partition of unity of type* $(-\rho, \rho)$.

Proposition 1.1.5 (Characterisation of strongly graded rings)**.** *The following statements are equivalent:*

1. *The ring R is strongly graded.*
2. *For every $\rho \in \mathbb{Z}^n$ there is at least one partition of unity of type $(\rho, -\rho)$.*
3. *There is at least one partition of unity of each of the types $(e_k, -e_k)$ and $(-e_k, e_k)$ for all e_k in a basis $e_k\colon 1 \leq k \leq n$ of $\mathbb{Z}^n$.*

Proof. For the equivalence of statements (1) and (2) see Proposition 1.6 of [Dad80]. That (2) implies (3) is trivial.

For the converse, suppose that $1 = \sum_j {}_{\pm k}u_j \, {}_{\pm k}v_j$ is a partition of unity of type $(\pm e_k, \mp e_k)$ for all $1 \leq k \leq n$.

For all $1 \leq k \leq n$ let ${}_{\pm k}\overline{u}$ be the association

$${}_{\pm k}\overline{u}\colon a \mapsto \sum_j {}_{\pm k}u_j a_{\pm k} v_j.$$

In particular ${}_{\pm k}\overline{u}(1) = 1$, making it a partition of unity of type $(\pm e_k, \mp e_k)$. It follows that ${}_{\pm k}\overline{u}^p(1)$ is a partition of unity of type $(\pm p e_k, \mp p e_k)$.

For a general $\rho = \sum_{k=1}^{n} m_k e_k \in \mathbb{Z}^n$, where $\delta_k = |m_k|/m_k$,

$$_{\delta_1 1}\overline{u}^{m_1}{}_{\delta_2 2}\overline{u}^{m_2} \cdots {}_{\delta_n n}\overline{u}^{m_n}(1)$$

is a partition of unity of type $(\rho, -\rho)$. □

Proposition 1.1.6. *If R is strongly graded, then each R_k is finitely generated projective as both a right and left R_0-module.*

Proof. See Proposition 1.6 of [HS16]. □

Now I want to add the definition of a $\mathbb{Z}^n$-graded module. We will need to use these later on in the proof.

Definition 1.1.7 ($\mathbb{Z}^n$-Graded R-module)**.** For a unital $\mathbb{Z}^n$-graded ring R, a *$\mathbb{Z}^n$-graded right R-module* is an R-module M with a direct sum decomposition into right R_0-modules

$$M = \bigoplus_{\rho \in \mathbb{Z}^n} M_\rho$$

such that

$$\forall \rho, \ell \in \mathbb{Z}^n : M_\rho R_\ell \subseteq M_{\rho+\ell}.$$

Call $\bigoplus_{\rho \in \mathbb{Z}^n} M_\rho$ the $\mathbb{Z}^n$-grading of M, and M_ρ the ρ-component of M. When $M_\rho R_\ell = M_{\rho+\ell}$ for all $\rho, \ell \in \mathbb{Z}^n$, we call M a *strongly $\mathbb{Z}^n$-graded R-module.*

We want to give some examples of strongly $\mathbb{Z}^n$-graded rings that are not Laurent polynomial rings.

Example 1.1.8. Let K be a field. Consider the $\mathbb{Z}$-graded ring

$$\overline{K} = K[A, B, C, D]/AB + CD - 1$$

with $\mathbb{Z}$-grading resulting from letting A, C have degree 1 and B, D have degree -1. That the ring is strongly $\mathbb{Z}$-graded follows immediately by the fact that the relation $AB + CD - 1 = 0$ makes $AB + CD$ a partition of unity of type $(1, -1)$ and, due to commutativity of indeterminates, there is also a partition of unity $BA + DC$ of type $(-1, 1)$. It can be shown via ideas from Gröbner basis theory that $\overline{K}$ is not a Laurent polynomial ring in one indeterminate as it does not contain a unit of degree 1.

Consider the strongly $\mathbb{Z}^n$-graded ring

$$\hat{K} = \overbrace{\overline{K} \underset{\overline{K}_0}{\otimes} \overline{K} \underset{\overline{K}_0}{\otimes} \dots \underset{\overline{K}_0}{\otimes} \overline{K}}^{n-\text{times}}$$

which can be seen to have a strongly $\mathbb{Z}^n$-graded structure as the strong grading of $\overline{K}$ provides that the necessary partitions of unity are present as in point (3) of Proposition 1.1.5. It also follows that $\hat{K}$ is not a Laurent polynomial ring in n indeterminates.

Example 1.1.9. For a group G and a ring R let $R[G]$ be the group ring of G over R where for $g \in G$,

$$R[G] = \bigoplus_{g \in G} R \cdot g.$$

Taking the same situation as Theorem 0.1.9, that is let there be a normal subgroup N of G such that $G/N = \mathbb{Z}^n$, we see that the ring $R[G/N]$ is a $\mathbb{Z}^n$-graded ring with grading where for $[g] = a \in \mathbb{Z}^n$,

$$R[G]_a = \bigoplus_{n \in gN} R \cdot n$$

and in particular

$$R[G]_0 = R[N].$$

1.2 Generalised Novikov Rings

Now we define the collection of generalised Novikov rings. The main definition of this section will cover a larger collection of rings than those pertaining directly to what we will come to refer to as Novikov homology. Once we have set the definition we can state the result that is the main concern of this thesis.

We begin by setting some notation, all of which will feature in the definition of a generalised Novikov ring. Let S be the n-dimensional cube

$[-1,1]^n \subset \mathbb{R}^n$ and let $\{e_k : 1 \leq k \leq n\}$ be the standard basis of $\mathbb{Z}^n$. We are concerned with the collection of non-trivial subfaces F of S.

Let η_F be the sum of inward pointing normal vectors of length 1 of the $(n-1)$-dimensional faces that contain F. Also set $\eta_S = 0$.

Example 1.2.1. Let $S = [-1,1]^2$ and label the faces like so:

$$\begin{array}{ccc} v_{tl} & \overset{\epsilon_t}{\rule{2em}{0.4pt}} & v_{tr} \\ \epsilon_l \Big| & S & \epsilon_r \Big| \\ v_{bl} & \overset{\epsilon_b}{\rule{2em}{0.4pt}} & v_{br} \end{array}$$

where v stands for vertex, ϵ stands for edge and S stands for square. Then $\eta_{v_{bl}} = (1,1)$, $\eta_{\epsilon_b} = (0,1)$ and $\eta_S = 0$.

Remark 1.2.2. For general n, the polytope $S = [-1,1]^n$ is a simple polytope and any given face F is contained in precisely $\operatorname{codim}(F)$ many faces of dimension $n-1$.

Let R be a $\mathbb{Z}^n$-graded ring for $n \in \mathbb{N}$. Consider the collection $\mathfrak{A}$ of maps

$$f\colon \mathbb{Z}^n \to R = \bigoplus_{a \in \mathbb{Z}^n} R_a$$

such that $f(a) \in R_a$ for all $a \in \mathbb{Z}^n$. The ring R can be thought of as the subcollection of $\mathfrak{A}$ where for all $f \in R$, the support $\operatorname{supp}(f)$ of f is a finite subset of $\mathbb{Z}^n$. By considering subcollections of $\mathfrak{A}$ whose maps satisfy other conditions, we can form rings that are the analogue of the Novikov rings and power series rings associated with polynomial rings.

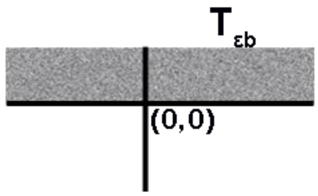
Before we can actually present the definition of Novikov rings we still have a few more definitions to go through.

Definition 1.2.3. Let F be a non-empty face of S. For a set $A \subseteq \mathbb{R}^n$, define $\operatorname{pos}(A)$ as the set of linear combinations of elements of A with non-negative coefficients, that is $\operatorname{pos}(A) = \{\sum_i \lambda_i a_i \colon a_i \in A,\ \lambda_i \geq 0\}$. Let

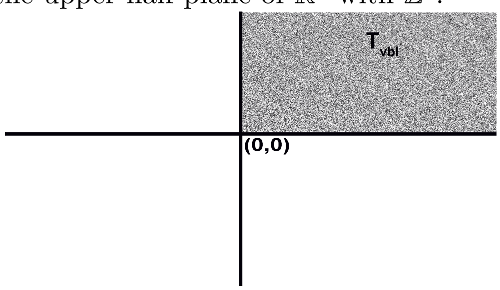
$$B_F = \operatorname{pos}\big(\{(s-f) : s \in S, f \in F\}\big) \subseteq \mathbb{R}^n$$

which we call the *barrier cone of F in relation to S*, and $T_F = B_F \cap \mathbb{Z}^n$ the *barrier lattice of F in relation to S* for all faces $F \subseteq S$.

Example 1.2.4. Again let $S = [-1, 1]^2$, then T_S is $\mathbb{Z}^2$.



For the bottom edge $\epsilon_b = \{v_{bl}, v_{br}\}$ of the square S, T_{ϵ_b} is the intersection of the upper half plane of $\mathbb{R}^2$ with $\mathbb{Z}^2$.



For the bottom left vertex v_{bl} of S, $T_{v_{bl}}$ is the top right quadrant of $\mathbb{R}^2$ intersected with $\mathbb{Z}^2$.

Remark 1.2.5. For any $n-1$ dimensional face G where $\eta_G = e_G$, it can be shown that

$$T_G = \{\rho = \sum_{i=1}^{n} \rho_i e_i \in \mathbb{Z} \colon \rho_G \geq 0\},$$

i.e., the half space of $\mathbb{Z}^n$ with positive coefficients of e_G. Similarly for $-G$ which will satisfy $\eta_{-G} = -e_G$,

$$T_{-G} = \{\rho = \sum_{i=1}^{n} \rho_i e_i \in \mathbb{Z} \colon \rho_G \leq 0\}.$$

Remark 1.2.6. If $F \subseteq F'$, then the definition of the barrier cone immediately informs us that $B_F \subseteq B_{F'}$ and hence $T_F \subseteq T_{F'}$. In fact, if $F = \cap_i F'_i$, then one can show that $T_F = \cap_i T_{F'_i}$.

Definition 1.2.7. We call a collection $\mathcal{F}$ of faces of S with the following property $\mathcal{F} = \{F_0 \subset F_1 \subset ... \subset F_\ell\}$, $F_0 \neq \emptyset$, $0 \leq m \leq n$ a *flag of faces of S*.

Definition 1.2.8. For a given face $F \subseteq S$, define the *caterpillar of* $\mathcal{F}$ as

$$\mathrm{CP}(\mathcal{F}) = \{\text{non-empty faces } F \text{ of } S \mid F_{i-1} \subseteq F \subseteq F_i, \text{ for some } i, \ 0 \leq i \leq \ell\}$$

with $F_{-1} = \emptyset$.

Example 1.2.9. Firstly note that for any n where $S = [-1,1]^n$ $\mathrm{CP}(\{S\})$ is the collection of every non-empty subface of S including itself. Now let $S = [-1,1]^2$ be labelled as above. Then:

- $\mathrm{CP}(\{S\}) = \{v_{tl}, v_{tr}, v_{br}, v_{bl}, \epsilon_t, \epsilon_r, \epsilon_b, \epsilon_l, S\}$.
- $\mathrm{CP}(\{v_{tl}, \epsilon_t, S\}) = \{v_{tl}, \epsilon_t, S\}$.
- $\mathrm{CP}(\{\epsilon_t, S\}) = \{v_{tl}, v_{tr}, \epsilon_t, S\}$.
- $\mathrm{CP}(\{\epsilon_t\}) = \{v_{tl}, v_{tr}, \epsilon_t\}$.
- $\mathrm{CP}(\{v_{tl}, S\}) = \{v_{tl}, \epsilon_t, \epsilon_l, S\}$.
- $\mathrm{CP}(\{v_{tl}\}) = \{v_{tl}\}$.

The caterpillar has a largest face that is precisely the largest face of the flag. Generally speaking, the more faces in the flag and the lower the dimension of the largest face, the fewer faces are in the caterpillar. At the extremes, for a vertex v, $\mathrm{CP}(\{v\}) = \{v\}$ while $\mathrm{CP}(\{S\})$ is the set of all non-trivial subfaces of S. If a flag contains a face at every dimension $0 \leq i \leq n$ then the caterpillar contains only the faces of the flag. We see the same picture if a flag contains a face of every dimension smaller than its largest face.

Definition 1.2.10. Call a flag $\mathcal{F} = \{F_0 \subset F_1 \subset \cdots \subset F_\ell\}$ *maximal* if $\ell = n$. In particular $\mathcal{F}$ is maximal if and only if there is a face of each dimension within the flag, in particular $S \in \mathcal{F}$.

It follows that for $\mathcal{F}$ maximal, $\mathrm{CP}(\mathcal{F}) = \mathcal{F}$.

Now we can write down the key definition of this section, based on private communication with David Quinn.

Definition 1.2.11 (Generalised Novikov Group)**.** Let $S \subseteq \mathbb{R}^n$ be the n-dimensional cube $[-1,1]^n$ and $\mathcal{F} = \{F_0 \subset F_1 ... \subset F_\ell\}$, $1 \leq \ell \leq n$ a flag of faces of S with $\emptyset \neq F_0$, $0 \leq m \leq n$. The *generalised Novikov group associated with* $\mathcal{F}$, denoted $R_*(\!(\mathcal{F})\!)$, is the set of maps

$$f \colon \mathbb{Z}^n \to R$$

such that $f(k) \in R_k$ for all $k \in \mathbb{Z}^n$ satisfying the following condition in two parts:

(1) The support of f is in the barrier lattice of the largest face of $\mathcal{F}$, i.e. $\operatorname{supp}(f) \subseteq T_{F_\ell}$.

For each $F \in \mathrm{CP}(\mathcal{F})$:

(2-F) For all $q \in \mathbb{Z}^n$ there exists $k \gg 0$ such that

$$k\eta_F + \Big(\big(q + (-T_F)\big) \cap \operatorname{supp}(f)\Big) \subset T_F.$$

Example 1.2.12. For $n = 1$, the three faces of S, $\{v_l, v_r, S\}$ form the following diagram

$$v_l \overset{S}{\text{———}} v_r$$

and there are five flags of faces $\mathcal{F}$ that can be formed from the three faces. Note that $\eta_{v_l} = 1$, $\eta_{v_r} = -1$ and $T_{v_l} = \mathbb{Z}_{\geq 0} = -T_{v_r}$, $T_{v_r} = \mathbb{Z}_{\leq 0} = -T_{v_l}$. Hence given a general $\mathbb{Z}$-graded ring R there are five generalised Novikov groups $R_*(\!(\mathcal{F})\!)$ that can be formed.

- Firstly for the flag $\{S\}$ all three faces of S are contained within the caterpillar. Condition (1) is trivial in this case as $T_S = \mathbb{Z}^2$ so there is no restriction on the support. Similarly, (2-S) has no effect. Condition (2-v_l) means that a element f of $R_*(\!(\{S\})\!)$ must have a lower bound on its support within $\mathbb{Z}$. If not, then there will be no k such that the $k + \big(q + (-T_{v_l})\big) \cap \operatorname{supp}(f)$ is contained within T_{v_l}, the non-negative values of $\mathbb{Z}$. Dually, (2-v_r) ensures that the support of every element of $R_*(\!(\{S\})\!)$ has an upper bound. Hence $R_*(\!(\{S\})\!) = R$ and this will also follow for any $\mathbb{Z}^n$ graded ring and any n.

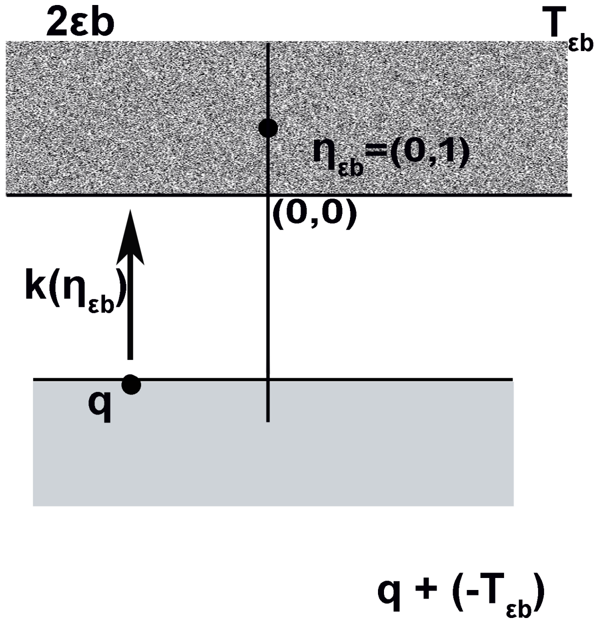
- For the two flags $\{v_l\}$ and $\{v_r\}$ we only have a single face within the caterpillar, the one face within the flag. In both cases the condition (1) is the strongest. For example, for an element of $R_*(\!(\{v_l\})\!)$, (1) tells us that the support must be contained within the non-negative part of $\mathbb{Z}$. We have already discussed that condition (2-$\{v_l\}$) puts a lower bound on the support at some point hence fixing it at 0 is clearly stronger so there is nothing left to think about. Hence $R_*(\!(\{v_l\})\!) = \prod_{t\geq 0} R_t$. If we let $R = V[x, x^{-1}]$, then in this case $R_*(\!(\{v_l\})\!) = V[\![x]\!]$. Dually, $R_*(\!(\{v_r\})\!) = \prod_{t\leq 0} R_t$ and for $R = V[x, x^{-1}]$ then $R_*(\!(\{v_r\})\!) = V[\![x^{-1}]\!]$. These are precisely the power series.

- Finally for the flags $\{v_l, S\}$ and $\{v_r, S\}$ there are in both cases two faces in the caterpillar. Considering $R_*(\!(\{v_l, S\})\!)$, note that (1) and (2-S) are trivial. The only condition that implies there is a boundary on the support of an element of the ring is (2-v_l). We know this puts a lower bound on the support of an element. Hence, we can write $R_*(\!(\{v_l, S\})\!) = \bigoplus_{t<0} R_t \oplus \prod_{t\geq 0} R_t$. If we let $R = V[x, x^{-1}]$, then $R_*(\!(\{v_r, S\})\!) = V(\!(x)\!)$, Dually, for the flag $\{v_r, S\}$ then $R_*(\!(\{v_l, S\})\!) = \prod_{t\leq 0} R_t \oplus \bigoplus_{t>0} R_t$ and for $R = V[x, x^{-1}]$, $R_*(\!(\{v_r, S\})\!) = V(\!(x^{-1})\!)$. These two cases are precisely the Novikov rings that encode Novikov homology in the case where R is a strongly-$\mathbb{Z}$-graded ring.

We now will discuss the collection of generalised Novikov groups for $R = V[x, x^{-1}, y, y^{-1}]$, a $\mathbb{Z}^2$-graded ring with $R_{a,b} = \{vx^a y^b : v \in V\}$. Let $S = [-1, 1]^2$ be labelled as before.

Let's consider the generalised Novikov groups associated to the flag consisting of $\mathcal{F} = \{\epsilon_b, S\}$ where ϵ_b is the bottom face of S. The caterpillar has precisely 4 faces (meaning condition 2 has to be considered for 4 different faces) — ϵ_b, S and the two vertices contained within ϵ_b, namely v_{bl} and v_{br}.

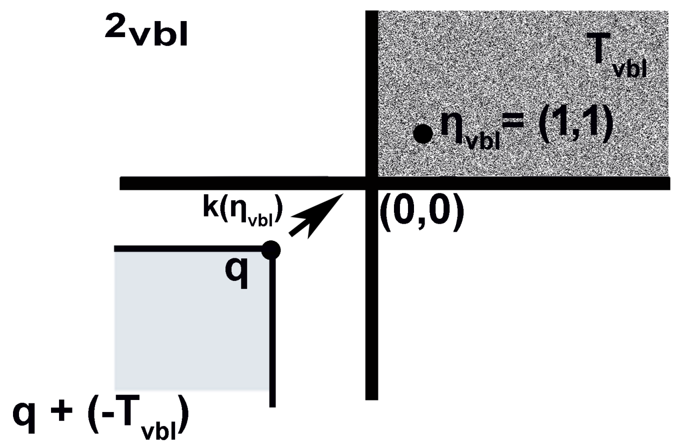
Condition (1) is trivial in this case as $S \in \mathcal{F}$. Condition (2-S) is also trivial.



Condition (2-ϵ_b) tells us that for any $q = (q_x, q_y)$, we can find k such that the subspace $q + (-T_{\epsilon_b}) \cap \operatorname{supp}(f)$, which is contained within the bottom half of the plane $\mathbb{Z}^2$ below the line $y = q_y$ inclusive, can be shifted by $(0, k)$ until being entirely contained within ϵ_b, the half of the plane above $y = 0$ inclusive. As an implication, for any given element $r \in R_*((\mathcal{F}))$, for the collection of subspaces of $\mathbb{Z}^2$ over $y \in \mathbb{Z}$

$$\{(x, y) \colon x \in \mathbb{Z}\}$$

there is some $p \in \mathbb{Z}$ such that for $y \leq p$ the intersection of the support of r and this subspace is empty. This is a global bound on the support of r.



Condition (2-v_{bl}) tells us that for any given element $r \in R_*((\mathcal{F}))$ and any fixed point $q = (q_x, q_y)$ we can generate some k such that the subspace $q + (-T_{v_{bl}})$, which is precisely the set

$$\{(x, y) \colon x \leq q_x, y \leq q_y\},$$

intersected with the support of an element r can be shifted by $k\eta_{v_{bl}} = k(1, 1)$ and it will be completely contained within the top right quadrant

$$\{(x, y) \colon x, y \geq 0\}.$$

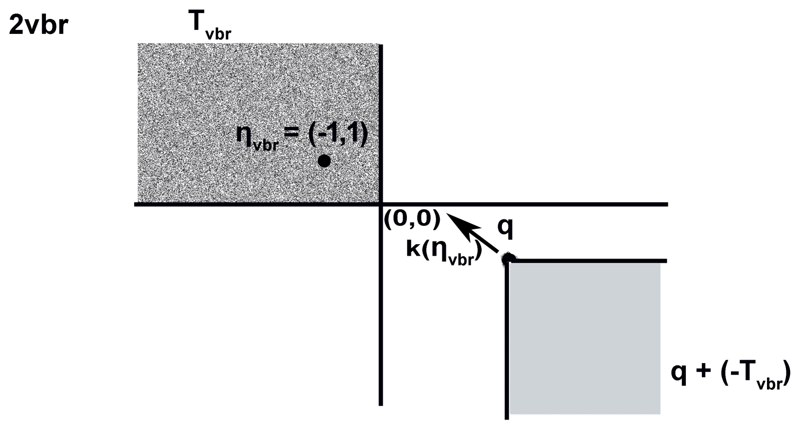
In particular, for a given element r and q, we can find $\gamma_x, \gamma_y \in \mathbb{Z}$ such that the sets

$$\{(q_x, y) \in \mathbb{Z}^2 \colon y \leq \gamma_y\}$$

and

$$\{(x, q_y) \in \mathbb{Z}^2 \colon x \leq \gamma_x\}$$

have empty intersection with the support of r, that is, the points can only be shifted finitely often in the directions of x or y before they are no longer contained within the support of r. In the y direction, this condition is strictly weaker than that of (2-ϵ_b). However, in the x direction this is a new condition. Together, (2-ϵ_b) and (2-v_{bl}) tell us that for a fixed $r \in R_*((\mathcal{F}))$ and fixed y, there is a lower bound on the possible values of x such that (x, y) is contained within the support of r. However there need not be a global lower bound on the possible values of x across the entire ring.



Similarly, (2-v_{br}) tells us that for any given element $r \in R_*(\!(\mathcal{F})\!)$ and any fixed point $q = (x, y)$ we can generate some k such that the subspace $q + (-T_{v_{br}})$, which is precisely the set

$$\{(x, y) \colon x \geq q_x, y \leq q_y\},$$

intersected with the support of an element r can be shifted untill it is completely contained within the top left quadrant

$$\{(x, y) \colon x \leq 0, y \geq 0\}.$$

It also puts a local upper bound on values of x for fixed y. Taken together, (2-v_{bl}) and (2-v_{br}) show that for all $b \in \mathbb{Z}$, the horizontal lines

$$\{(x, b) \colon x \in \mathbb{Z}\}$$

have finite intersection with the support of any element of $R_*(\!(\mathcal{F})\!)$. The ring $R_*(\!(\mathcal{F})\!)$ can therefore be seen to be a generalised analogue of a power series in y over a Laurent polynomial in x over a ring. For example if $R = V[x, x^{-1}, y, y^{-1}]$ then $R_*(\!(\{\epsilon_b, S\})\!)$ would be the ring $V[x, x^{-1}](\!(y)\!)$. To see an example of the distinction between global bounds and local bounds, observe that $\sum_{\ell \geq 0} x^\ell y^\ell \in V[x, x^{-1}](\!(y)\!)$ while $\sum_{\ell \geq 0} x^\ell \notin V[x, x^{-1}](\!(y)\!)$.

In general, the collection of generalised Novikov groups for $R = V[x, x^{-1}, y, y^{-1}]$, up to orientation, are the following:

- $R_*(\!(\{S\})\!) = V[x, x^{-1}, y, y^{-1}]$.

- $R_*((\{\epsilon_b, S\})) = V[x, x^{-1}]((y))$.
- $R_*((\{\epsilon_b\})) = V[x, x^{-1}][[y]]$.
- $R_*((\{v_{bl}, S\})) = V((x, y))$.
- $R_*((\{v_{bl}\})) = V[[x, y]]$.
- $R_*((\{v_{bl}, \epsilon_b, S\})) = V((x))((y))$.
- $R_*((\{v_{bl}, \epsilon_b\})) = V((x))[[y]]$.

It is clear that $R_*((\mathcal{F}))$ will form an abelian group under addition of maps. However we want the generalised Novikov groups to be rings. The next step is to show the obvious candidate of ring action is satisfactory for general $\mathbb{Z}^n$-graded rings.

Lemma 1.2.13. *Let $\mathcal{F}$ be a maximal flag. For all $f, g \in R_*((\mathcal{F}))$, the sum*

$$(f \cdot g)(t) = \sum_{t=a+b} f(a) \cdot g(b)$$

involves only finitely many non-zero terms for every $t \in \mathbb{Z}^n$.

Proof. Begin, for clarity, by considering the case for $\mathbb{Z}^2$. Specifically we look at the associated group to the flag consisting of the whole square, the lower edge ϵ_b and the lower left vertex v_{bl}, that is

$$R_*((\{v_{bl}, \epsilon_b, S\})).$$

Let $f, g \in R_*((\{v_{bl}, \epsilon_b, S\}))$. I argue that for all $t \in \mathbb{Z}^2$, $(f \cdot g)(t)$ is a finite sum. This is done by observing that there are only finitely many possible values of $b = t - a$ such that $f(a) \cdot g(t - a)$ is non-zero. Condition (2-ϵ_b) tells us that for every individual element there is a global lower bound f_y on the y-coordinates of elements in the support of f. Similarly (2-ϵ_b) tells us there is a lower bound g_y for g. Let $u_y = f_y + g_y$. If $t \in \mathbb{Z}^2$ is such that the y-coordinate t_y of t satisfies $t_y < u_y$, then $(f \cdot g)(t) = 0$ as either the y-coordinate of a is below f_y or otherwise the y-coordinate of $t - a$ is below $u_y - f_y = g_y$ since the sum of the y-coordinates of a and $t - a$ must equal the y-coordinate of t_y, which is strictly smaller than u_y. Therefore we can restrict attention to $t_y \geq u_y$.

So assume that $t_y \geq u_y$. We argue that for elements $a = (a_x, a_y)$, there is finitely many a_y such that the product $f(a) \cdot g(t-a)$ is non-zero. Let $p_y < \min(f_y, g_y)$. Note that $t_y - p_y \geq u_y - p_y > u_y - g_y > f_y$. Whenever $a_y < p_y$, $f(a) = 0$ and whenever $a_y > t_y - p_y$, it follows that $g(t-a) = 0$ as $t_y - a_y < t_y - (t_y - p_y) = p_y < g_y$. Hence a_y must be contained between p_y and $t_y - p_y$.

For $q = (t_x, t_y - p_y)$ find large enough k_f, k_g that satisfy the condition (2-v_{bl}) for f and g respectively. Recall that $\eta_{v_{bl}} = (1,1)$. Let $k = \max(k_f, k_g)$. Then, for all $r = (r_x, r_y) \in \mathbb{Z}^2$ where r_y is bounded by p_y and $t_y - p_y$, if $r_x \leq -k$ then $f(r) = 0$ and if $r_x \geq t_x + k$ then $g(t-r) = 0$, as the condition (2-v_{bl}) tells us that, beneath $y = t_y - p_y$, elements of the supports of both f and g have x values bounded below by $-k$. This means that the x-coordinates of a and b must be between $-k$ and $t+k$ to make $f(a) \cdot g(b)$ non-zero. Hence, for a given t, there are only finitely many pairs $a, t-a$ such that $f(a) \cdot g(t-a)$ providing the result. Other two dimensional, maximal flag cases follow with similar arguments.

Now, to show the result in the Lemma for any n, we argue using induction. Let $\mathcal{F}$ be a maximal flag, and let σ be the ordering of n such that σ_i is the index of the unique member (unique, as the flag is maximal) of the set

$$\{\pm e_j : 1 \leq j \leq n\}$$

such that only one of e_{σ_i} or $-e_{\sigma_i}$ are contained in T_F but both $\pm e_{\sigma_i}$ are contained within the barrier lattice of any higher dimensional face in the flag. We can follow the argument for $\mathbb{Z}^2$ above to show that for a given t the set of choices of a such that $f(a) \cdot g(t-a)$ is non-zero has bounds in the directions $\pm e_{\sigma_n}, \pm e_{\sigma_{n-1}}$. If we know that the possible a satisfying $f(a) \cdot g(t-a) \neq 0$ are bounded in the directions $\pm e_{\sigma_n}, ..., \pm e_{\sigma_z}$ for a fixed $2 \leq z \leq n-1$, then an similar argument to before will show that there are bounds in the $\pm e_{\sigma_{z-1}}$ directions as well (i.e., for the face F of dimension

$z-1$ contained within the maximal flag, generate k' for a satisfactory q' with (2-F)).

Hence we know from induction that there is a finite number of a satisfying the non-zero condition, making the sum $\sum\limits_{a+b=t} f(a)\cdot g(b)$ finite for all $t\in\mathbb{Z}^n$. $\square$

Lemma 1.2.14. *When $\mathcal{F}$ is maximal, $R_*(\!(\mathcal{F})\!)$ is a ring under the operation $R_*(\!(\mathcal{F})\!)\times R_*(\!(\mathcal{F})\!)\to R_*(\!(\mathcal{F})\!)$,*

$$(f\cdot g)(k)=\sum_{k=a+b} f(a)\cdot g(b).$$

Proof. The only non-trivial thing we need to show is that $f\cdot g\in R_*(\!(\mathcal{F})\!)$ for $f,g\in R_*(\!(\mathcal{F})\!)$, distributivity and associativity will follow immediately from the structure of R. Let $f,g\in R_*(\!(\mathcal{F})\!)$ for a maximal flag $\mathcal{F}$. To see that closure is satisfied, begin by noting that condition (1) is trivially satisfied for any map $\mathbb{Z}^n\to R$ including $f\cdot g$ as $S\in\mathcal{F}$, as is the case for (2-S). As we have done for Lemma 1.2.13, we will look at the case of $n=2$ for explanatory purposes, again specifically for the maximal flag

$$\{v_{bl},\epsilon_b,S\}$$

(the whole square, the lower edge and the lower left vertex). Given two elements $f,g\in R_*(\!(\mathcal{F})\!)$, we argue that $f\cdot g$ satisfies the conditions (2-S), (2-ϵ_b) and (2-v_{bl}). Firstly, we look at condition (2-ϵ_b). The cone T_{ϵ_b} is the entire upper half plane. Note that $\eta_{\epsilon_b}=e_y$. We can find $k_f,k_g\geq 0$ such that the supports of f and g have no support within the sets

$$\{(a,j),a,j\in\mathbb{Z},\ j\leq -k_f\}$$

and

$$\{(a,j),a,j\in\mathbb{Z},\ j\leq -k_g\}$$

respectively. Let $k=k_f+k_g$. I argue that the support of $f\cdot g$ has $-k$ as a global lower bound on the second index, i.e., $(a,j)\in\operatorname{supp}(f\cdot g)$ implies $j\geq -k$.

Let $(a, b) \in \operatorname{supp}(f \cdot g)$ such that $b < -k$. Then, owing to the definition of composition, there is some b_f, b_g such that $b_f + b_g = b$ and $b_f \in \operatorname{supp}(f)$, $b_g \in \operatorname{supp}(g)$. So $b = b_f + b_g > -k_f - k_g = -k$ which is a contradiction.

Now we look at the condition (2-v_{bl}). Note $\eta_{v_{bl}} = e_x + e_y$. Fix $q = (q_x, q_y)$, and note that we can find $\overline{k}_{f,q}, \overline{k}_{g,q} \geq 0$ such that the condition is met for f and g respectively and the y co-ordinate of $\overline{k}_{f,q}\eta_{v_{bl}}$ (respectively $\overline{k}_{g,q}\eta_{v_{bl}}$) is greater than k_f (respectively k_g) that satisfy condition (2-ϵ_{bl}) for f and g respectively (simply increase $\overline{k}_{f,q}$ and $\overline{k}_{g,q}$ until satisfied). Let $\overline{k}_q = \overline{k}_{f,q} + \overline{k}_{g,q} \geq k$. Then using the same argument as used to show that $-k = -k_f - k_g$ was a lower bound for the y coordinates, $-\overline{k}_q$ is a lower bound of the x coordinates of elements in the support of $f \cdot g$ below the line $y = q_y$. That is, by shifting $(q + (-T_{v_{bl}})) \cap \operatorname{supp}(f \cdot g)$ by $k'_q\eta_{v_{bl}}$ such that k'_q is larger than $\overline{k}_q$ we find it completely contained within $T_{v_{bl}}$ as $-\overline{k}_q$ is a lower bound in both the x and y coordinates respectively for the support of $f \cdot g$ within $q + (-T_{v_{bl}})$. Hence $f \cdot g$ is also contained within

$$R_*((\{v_{bl}, \epsilon_b, S\}))$$

making it a ring.

Now, to show for any n, we argue using induction. Let $\mathcal{F}$ be a maximal flag, and let σ be the ordering of n such that σ_i is the index of the unique member (unique, as the flag is maximal) of the set

$$\{\pm e_j : 1 \leq j \leq n\}$$

such that only one of e_{σ_i} or $-e_{\sigma_i}$ are contained in T_F but both $\pm e_{\sigma_i}$ are contained within the barrier cone of any higher dimensional face in the flag. We know that the conditions for F_{n-1} and F_{n-2} satisfy closure as we can repeat the argument for the two dimensional case but, for example, rather than the condition for F_{n-1} being that each element's support can be shifted into the upper half plane, we have that the support can be shifted into the space bounded below by the $n-1$-dimensional subspace where the coefficient of e_{σ_n} is 0.

Recall that $F_n = S$ for a maximal flag hence (2-F_n) is trivial. Fix $2 \leq z \leq n-2$. Assume that there is closure for the conditions taken together for the faces $F_z \subseteq F \subseteq F_{n-1}$ so that for a fixed q each condition provides some $(k_F)_q \geq 0$ that satisfies (2-F). Let $k_q = \max(\{(k_F)_q \colon F_z \subseteq F \subseteq F_{n-1},\, F \in \mathcal{F}\})$. We want to show that adding the condition for F_{z-1} also keeps closure. For a given $q \in \mathbb{Z}^n$, $-k_q\eta_{F_z}$ is a bound (upper or lower, depending on the orientation of F_z) in the support of $f \cdot g$ parallel to the axes $e_{\sigma_n}, ..., e_{\sigma_z}$ within the cone $q+(-T_{F_{z-1}})$ via the inductive assumption as $T_{F_{z-1}} \subseteq T_{F_j}$ for $z \leq j \leq n$. A similar argument for the two dimensional case gives the result, firstly find a satisfactory $\overline{k}_q = \overline{k}_{f,q} + \overline{k}_{f,q} \geq k_q$ for a (lower or upper) bound in the supports of f and g in the $e_{\sigma_{z-1}}$ axis within the cone $q + (-T_{F_{z-1}})$, by using the condition (2-F_{z-1}) for f and g. Then choosing k'_q larger than $\overline{k}_q$ will satisfy the condition (2-F_{z-1}) as required. □

For a flag $\mathcal{F}$ containing S, let $\mathrm{MF}(\mathcal{F})$ be the subset of the set of flags of faces in $\mathrm{CP}(\mathcal{F})$ consisting of maximal flags.

Example 1.2.15. For a flag $\mathcal{F} = \{S\}$ where $S = [-1, 1]$,

$$\mathrm{MF}(\{S\}) = \{\{v_l, S\}, \{v_r, S\}\}.$$

Lemma 1.2.16. *For all flags $\mathcal{F}$ where $S \in \mathcal{F}$,*

$$R_*(\!(\mathcal{F})\!) = \bigcap_{\mathcal{G} \in \mathrm{MF}(\mathcal{F})} R_*(\!(\mathcal{G})\!).$$

Proof. That the condition (1) for $\mathcal{F}$ is equivalent on both sides follows trivially as $S \in \mathcal{F}$ and $S \in \mathcal{G}$ for all $\mathcal{G} \in \mathrm{MF}(\mathcal{F})$. The second collection of conditions are equivalent, by observing that a face is within a flag of $\mathrm{MF}(\mathcal{F})$ if and only if it is contained within $\mathrm{CP}(\mathcal{F})$, so the collection of conditions (2-F) are the same on both sides of the equation. □

Proposition 1.2.17. *Let $\mathcal{F}$ be a flag that does not contain S but does have faces of dimension 0 to $n-1$. Then $R_*(\!(\mathcal{F})\!)$ is also a ring.*

Proof. We know that $\mathcal{F} \cup \{S\}$ is a maximal flag hence $R_*((\mathcal{F} \cup \{S\}))$ is a ring. It follows that, since the only difference between the conditions on $R_*((\mathcal{F} \cup \{S\}))$ and $R_*((\mathcal{F}))$ is that (1) is no longer non-trivial and (2-F) for the case $F = S$ is lost, which is itself trivial, the only thing that needs to be checked is the first condition for the new case, that $\mathrm{supp}(f \cdot g) \in T_{F_{n-1}}$. However, it is clear that this satisfies closure hence there is nothing else to show. □

Proposition 1.2.18. *Let $\mathcal{F}$ be a flag with a face of dimensions 0 to m for $m < n - 1$. Then $R_*((\mathcal{F}))$ is a ring.*

Proof. Beginning with a flag $\mathcal{F} \cup \mathcal{G}$, where $\mathcal{G}$ has a face at every dimension from $m + 1$ to n. Then iterate the argument of Proposition 1.2.17 to arrive at the result. □

When the largest dimension of face a flag $\mathcal{F}$ is some m lower than n, this face is the largest dimension of face contained within $\mathrm{CP}(\mathcal{F})$ also. Let $\mathrm{MF}_m(\mathcal{F})$ be the collection of flags of faces of $\mathrm{CP}(\mathcal{F})$ such that there is a face at every dimension 1 to m. Proposition 1.2.18 tells us that $R_*((\mathcal{G}))$ is a ring for all $\mathcal{G} \in \mathrm{MF}_m(\mathcal{F})$.

Proposition 1.2.19. *Let $\mathcal{F}$ be a flag with maximal face of dimension m. Then $R_*((\mathcal{F}))$ is a ring.*

Proof. For every flag $\mathcal{G}$ inside $\mathrm{MF}_m(\mathcal{F})$, $R_*((\mathcal{G}))$ is a ring. Now apply the same argument as Proposition 1.2.16 to show that

$$R_*((\mathcal{F})) = \bigcap_{\mathcal{G} \in \mathrm{MF}_m(\mathcal{F})} R_*((\mathcal{G})).$$

□

Finally, Proposition 1.2.19 tells us that for all possible flags $\mathcal{F}$, $R_*((\mathcal{F}))$ is a ring. Henceforth, we use the term *generalised Novikov* ***rings*** to refer to these objects.

Remark 1.2.20. We can form generalised Novikov $R_*(\!(\mathcal{F})\!)$-modules $M_*(\!(\mathcal{F})\!)$ from an R-module M, analogous to the ring versions. That the $R_*(\!(\mathcal{F})\!)$-action is well defined and satisfies the conditions follows a similar argument to what has been seen in this section for proving that the ring $R_*(\!(\mathcal{F})\!)$ has a valid ring action.

We make another important note of the properties of these rings.

Lemma 1.2.21. *Let $\mathcal{F} \subseteq \mathcal{F}'$. Then there is an inclusion map $R_*(\!(\mathcal{F})\!) \to R_*(\!(\mathcal{F}')\!)$.*

Proof. Whenever the largest face of $\mathcal{F}$ is precisely the same face as the largest face of $\mathcal{F}'$, the result is clear since condition (1) is the same for both and the caterpillar of the latter is strictly contained within the caterpillar of the former making conditions (2-F) for $F \in \mathcal{F}'$ a sub-collection of the total collection of conditions for $\mathcal{F}$.

Whenever the largest face of $\mathcal{F}$, say F, is strictly smaller than the largest face of $\mathcal{F}'$, F' the argument follows with more effort. Firstly, from the difference between condition 1 for both flags, namely that since $T_F \subseteq T_{F'}$, (1) is strictly stronger for $\mathcal{F}$ than $\mathcal{F}'$. The face F is contained in both flags. For faces of dimension $\dim(F)$ and below that are contained within F, we see in a similar fashion to the first case that those within the caterpillar of $\mathcal{F}'$ are also contained within the caterpillar of $\mathcal{F}$.

However for faces G within the caterpillar of $\mathcal{F}'$ that are of dimension $\dim(F)+1$ and above, we argue that each possible condition (2-G) is satisfied by the condition (1) for the flag $\mathcal{F}$. This is immediately obvious, in fact since the support of an element $f \in R_*(\!(\mathcal{F})\!)$ is contained within T_F from condition (1) for $\mathcal{F}$, then for any strictly larger face $F \subseteq G$, it is immediate that (2-G) is satisfied by a choice of $k = 0$ as for all q, $(q + (-T_G)) \cap T_F$ is either empty or a fully bounded subspace of T_G and by (1) of $\mathcal{F}$, $\mathrm{supp}(f) \subseteq T_F$. $\square$

1.3 Novikov homology and the main theorem

We will now specify the Novikov Rings that will be the focus of the main result.

Definition 1.3.1. Let $\mathcal{F}$ be a flag of faces of $S = [-1,1]^n$. Whenever $\mathcal{F} = \{F, S\}$, where F is a face distinct from S, we call $R_*((\mathcal{F}))$ a *Novikov Ring.*

The main theorem can now be stated.

Theorem 1.3.2. *Let R be a strongly $\mathbb{Z}^n$-graded ring and write $R_{0_{\mathbb{Z}^n}} = R_0$. Let $S = [-1,1]^n$ and C be a bounded complex of finitely generated free R-modules. The complex C is R_0-finitely dominated if and only if for every flag $\mathcal{F}$ of the form $\mathcal{F} = \{F \subset S\}$, the complexes*

$$C \underset{R}{\otimes} R_*((\mathcal{F}))$$

are acyclic.

Whenever we say a chain complex C has *trivial Novikov homology*, we are referring to the above acyclicity condition on the complexes $C \underset{R}{\otimes} R_*((\mathcal{F}))$ where $\mathcal{F} = \{F \subset S\}$.

Remark 1.3.3. Note that this is specifically an acyclicity condition, not a contractibility condition. However, in our case, since we are focusing on a bonded complex of finitely generated free R-modules, assuming $C \underset{R}{\otimes} R_*((\mathcal{F}))$ is acyclic as abelian groups and observing that $C \underset{R}{\otimes} R_*((\mathcal{F}))$ is a bounded finitely generated free complex of $R_*((\mathcal{F}))$-modules provides us with a contraction on $C \underset{R}{\otimes} R_*((\mathcal{F}))$.

This is the explicit result, but the acyclicity condition listed here has a key implication that can be stated immediately.

Lemma 1.3.4. *Let R be a strongly $\mathbb{Z}^n$-graded ring. Let $S = [-1,1]^n$ and C be a bounded complex of finitely generated free R-modules. If for every flag $\mathcal{F}$ of the form $\mathcal{F} = \{F \subset S\}$, the complexes*

$$C \underset{R}{\otimes} R_*((\mathcal{F}))$$

are acyclic, then for all flags $\mathcal{F}'$ that contain S and at least one other face the complexes

$$C \underset{R}{\otimes} R_*(\!(\mathcal{F}')\!)$$

are acyclic also.

Proof. As per Remark 1.3.3, let c be the contraction of $C \underset{R}{\otimes} R_*(\!(\mathcal{F})\!)$. By noting from Lemma 1.2.21, that since $\mathcal{F} \subseteq \mathcal{F}'$, that $R_*(\!(\mathcal{F})\!) \subseteq R_*(\!(\mathcal{F}')\!)$ and hence $R_*(\!(\mathcal{F}')\!)$ is a $R_*(\!(\mathcal{F})\!)$-$R_*(\!(\mathcal{F})\!)$-bimodule. It follows that

$$C \underset{R}{\otimes} R_*(\!(\mathcal{F}')\!) = C \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) \underset{R_*(\!(\mathcal{F})\!)}{\otimes} R_*(\!(\mathcal{F}')\!),$$

the latter of which has contraction $c \otimes \mathrm{id}$, as if $c\colon \mathrm{id} \simeq 0$, then $c \otimes \mathrm{id} \simeq 0 \otimes \mathrm{id} = 0$. □

1.4 Skeleton of a flag

The definition of a generalised Novikov Ring deserves a detailed discussion. Perhaps the easiest way to consider this is as the natural evolution of the one dimensional Novikov ring, which can be seen by taking $S = [-1, 1]$.

The way to discern what happens is to work out what elements $\pm e_k$ for a standard basis of $\mathbb{Z}^n$ $\{e_k : 1 \leq k \leq n\}$ are contained within T_{F_i} for $F_i \in \mathcal{F}$, which is ultimately related to what faces of dimension $n-1$ the face F_i is contained in. Let E be the collection of elements $\pm e_k$.

Remark 1.4.1. From Remarks 1.2.5 and 1.2.6, we note that for a given flag $\mathcal{F} = \{F_0, \dots F_\ell\}$, for all k one of the three possibilities must be true:

1. $\pm e_k \in T_{F_0}$.
2. $+e_k \in T_{F_0}$ while $-e_k \notin T_{F_0}$.
3. $-e_k \in T_{F_0}$ while $+e_k \notin T_{F_0}$.

Lemma 1.4.2. *If, for a given k,* 2. *is true, then:*

- *There is a maximal a in the flag such that $+e_k \in T_{F_a}$ while $-e_k \notin T_{F_a}$.*

- *For any larger face in the flag $F_a \subseteq F$, both $\pm e_k \in T_F$.*
- *For all smaller faces in the flag $G \subseteq F_a$, we have $+e_k \in T_G$ while $-e_k \notin T_G$.*
- *The face F_a itself is contained in the unique $n-1$ dimensional face H such that $\eta_H = e_k$.*
- *For any larger face in the flag $F_a \subseteq F$, the face F is not contained within H.*

Proof. Using Remark 1.2.5 to see which half plane T_H corresponds to each $n-1$ dimensional face H and Remark 1.2.6 to represent each face as an intersection of $n-1$ dimensional faces, one simply notes the inclusion property of the faces of a flag to see this. □

A dual Lemma holds for when 3. is true for a given k.

Now we need to consider a few facts about these faces that are invaluable when proving anything with their flags.

Remark 1.4.3. Note firstly that each subface F of $S = [-1,1]^n$ is an unique intersection of faces of dimension $n-1$. In particular, if T_F contains e_k but not $-e_k$ then $F \subset G_k$ where G_k is the unique $n-1$ dimensional face such that $\eta_{G_k} = e_k$ (there is a dual result when $-e_k \in T_F$ and $e_k \notin T_F$). If we let G'_k be the $n-1$ dimensional face such that $\eta_{G'_k} = -e_k$ and $\mathrm{sgn}(F)_k$ be the sign of e_k that is contained within T_F, then we can write

$$F = \Big(\bigcap_{\mathrm{sgn}(F)_{k_i}=1} G_{k_i} \Big) \cap \Big(\bigcap_{\mathrm{sgn}(F)_{k_j}=-1} G'_{k_j} \Big)$$

noting that the total number of G_{k_i}, G'_{k_j} is precisely $\mathrm{codim}(F)$. We will very often intersect faces with certain $n-1$ dimensional faces based on what pairs $\pm e_k$ are contained within their barrier lattices, to form useful faces in future proofs. Also we will represent smaller faces as larger faces intersected by the correctly orientated faces of dimension $n-1$. These ideas will be fundamental in tackling a number of the proofs we see later on in the work.

We now wish to set a bit of notation.

Remark 1.4.4. Given a flag $\mathcal{F}$, let $\operatorname{sgn}(\mathcal{F})_k$ be the unique sign on e_k such that $\operatorname{sgn}(\mathcal{F})_k e_k \in T_{F_0}$ and $-\operatorname{sgn}(\mathcal{F})_k e_k \notin T_{F_0}$ if it exists (i.e., both $\pm e_k$ are not in T_{F_0}. If $\mathcal{F}$ does not contain a vertex, then not all $\operatorname{sgn}(\mathcal{F})_k$ will be defined, specifically those such that both $\pm e_k \in T_{F_0}$. That is:

$$\operatorname{sgn}(\mathcal{F})_k = \begin{cases} 1 \text{ when } e_k \in T_{F_0},\ -e_k \notin T_{F_0}. \\ -1 \text{ when } -e_k \in T_{F_0},\ e_k \notin T_{F_0}. \\ \text{undefined otherwise, that is when} \pm e_k \in T_{F_0}. \end{cases}$$

Evidentally when F_0 is a vertex, $\operatorname{sgn}(\mathcal{F})_k$ is defined for all $1 \leq k \leq n$. Hence. we can also define $\operatorname{sgn}(v)_k$ for every vertex v.

These ultimately provide the information that encodes what the Novikov rings actually are. We will always be interested in the collection of signs $\operatorname{sgn}(\mathcal{F})_k$ and the smallest face F_a such that $\pm e_k \in F$ for $F_a \subset F$. We now claim that we can discard a considerable number of the faces of $\mathrm{CP}(\mathcal{F})$ without weakening the definition of a Novikov ring.

Proposition 1.4.5. *Let $\mathcal{F} = \{F_0 \subset \cdots \subset F_\ell\}$ be a flag. The collection of conditions*

$$\{(1),\ (2\text{-}F)\colon F \in \mathrm{CP}(\mathcal{F})\}$$

is implied by the subcollection of conditions

$$\{(1),\ (2\text{-}F)\colon F \in \mathrm{CP}(\mathcal{F}),\, \dim F = \dim F_i - 1, F_i \in \mathcal{F}\}.$$

Proof. Begin by letting $S \in \mathcal{F}$ (making (1) trivial) and take a face $F_{\ell-1} \subseteq F \subset S$ with $\operatorname{codim}(F) = p$. Note by Remark 1.2.6 $T_F = \bigcap_{m_j} T_{G_{m_j}}$, $1 \leq j \leq p$, $1 \leq m \leq n$ for faces of dimension $n-1$. Since $F \subseteq G_{m_j} \subset S$, it follows that $G_{m_j} \in \mathrm{CP}(\mathcal{F})$. It can be seen that for all faces G within $\mathrm{CP}(\mathcal{F})$ of dimension $n-1$, $\eta_G = \operatorname{sgn}(\mathcal{F})_r e_r$ for some r. For a fixed $q \in \mathbb{Z}^n$, the conditions (2-G_{m_j}) generate a selection of $k_{G_{m_j}}$ that satisfy them. If we let $k = \max\{k_{G_{m_j}},\, 1 \leq j \leq p\}$, we can write

$$k\eta_{G_{m_j}} + \Big(\big(q + (-T_{G_{m_j}})\big) \cap \operatorname{supp}(f)\Big) \subset T_{G_{m_j}}.$$

Hence

$$k\sum_j \eta_{G_{m_j}} + \Big(\big(q + (-\bigcap_j T_{G_{m_j}})\big) \cap \operatorname{supp}(f) \Big) \subset \bigcap_j T_{G_{m_j}}$$

which is precisely

$$k\eta_F + \Big(\big(q + (-T_F)\big) \cap \operatorname{supp}(f) \Big) \subset T_F$$

i.e., (2-F) is satisfied.

Now let $F_{\ell-2} \subseteq F \subset F_{\ell-1}$ with $F_{\ell-1} = \bigcap_t G'_{m_t}$ and

$$F = (\bigcap_t G'_{m_t}) \cap (\bigcap_j G_{m_j})$$

for some distinct faces G'_{m_t}, G_{m_j} of dimension $n-1$. Clearly, $F_{\ell-1}$ is not contained within G_{m_j} while $F \subseteq G_{m_j}$ for all j (recall that a face F is contained in $\operatorname{codim}(F)$ many faces of dimension $n-1$). It follows that while each of the faces G'_{m_t} are contained within $\mathrm{CP}(\mathcal{F})$, each of the faces G_{m_j} are not contained within $\mathrm{CP}(\mathcal{F})$.

However, observe that for each j there is a face $F \subseteq L_j \subset F_{\ell-1}$ of dimension $\dim(F_{\ell-1}) - 1$ such that $L_j = (\bigcap_t G'_{m_t}) \cap G_{m_j}$. Precisely this is the face L such that $\eta_{L_j} = \eta_{G_{m_j}} + \sum_t \eta_{G'_{m_t}}$, which is a face of dimension $\dim(F_{\ell-1}) - 1$. If we combine the conditions (2-L_j) of faces L_j for each G_{m_j} with the conditions (2-G'_{m_t}) for each G'_{m_t}, we see the following is satisfied

$$\begin{aligned} k\sum_j \eta_{L_j} + k\sum_t \eta_{G'_{m_t}} + \Big(\big(q + \big(- (\bigcap_j T_{L_j}) \cap (\bigcap_t T_{G'_{m_t}}) \big) \big) \cap \operatorname{supp}(f) \Big) \\ \subset (\bigcap_j T_{L_j}) \cap (\bigcap_t T_{G'_{m_t}}). \end{aligned}$$

Firstly note that

$$\begin{aligned} (\bigcap_j T_{L_j}) \cap (\bigcap_t T_{G'_{m_t}}) &= \bigcap_j \Big((\bigcap_t T_{G'_{m_t}}) \cap T_{G_{m_j}} \Big) \cap (\bigcap_t T_{G'_{m_t}}) \\ &= T_F \cap T_{F_{i+1}} = T_F. \end{aligned}$$

Since $k\sum_j \eta_{L_j} + k\sum_t \eta_{G'_{m_t}}$ is just a linear combination of $\eta_{G'_{m_t}}$ and $\eta_{G_{m_j}}$, it follows that given $k\sum_j \eta_{L_j} + k\sum_t \eta_{G'_{m_t}}$ we can find k' such that

$$k'(\sum_t \eta_{G'_{m_t}} + \sum_j \eta_{G_{m_j}}) = k'\eta_F > k\sum_j \eta_{L_j} + k\sum_{m_t} \eta_{G'_{m_t}}$$

hence satisfying the below condition

$$k'\eta_F + \Big(\big(q + (-T_F)\big) \cap \operatorname{supp}(f)\Big) \subset T_F$$

which is precisely (2-F) as required.

We continue in the same manner, for faces $F_{i-1} \subseteq F \subset F_i$, we see that the conditions (2-G) where $G \in \mathrm{CP}(\mathcal{F})$ are of dimension $\dim(F_x) - 1$ for $x \geq i$ imply the conditions (2-F). Eventually, we see that for a flag $S \in \mathcal{F}$ and any face F within the caterpillar $\mathrm{CP}(\mathcal{F})$, the collection of conditions (2-G) for faces G of dimension $\dim(F_x) - 1$ for certain $0 \leq x \leq \ell$ will imply (2-F) as required. At this point, we have shown the required result for whenever $S \in \mathcal{F}$. If $S \notin \mathcal{F}$ so that $F_\ell \neq S$, let $F_\ell = \bigcap_w G_{m_w}$ for faces of dimension $n-1$. Now simply note that (1) in this case will be stronger than (2-G_{m_w}) and therefore a similar method to the above will provide the required result. □

We will now adapt the definition of the Novikov ring as we can limit our concern to faces within $\mathrm{CP}(\mathcal{F})$ of certain dimensions.

Definition 1.4.6. Given a flag $\mathcal{F} = \{F_0 \subset \cdots \subset F_\ell\}$ of faces of $S = [-1,1]^n$, define the *Christmas tree of* $\mathcal{F}$ as the following:

$$\mathrm{CT}(\mathcal{F}) = \{F \in \mathrm{CP}(\mathcal{F}),\, \dim(F) = \dim(F_i) - 1,\, 0 \leq i \leq \ell\}.$$

The new definition of the Generalised Novikov rings therefore only covers conditions (2-F) for $F \in \mathrm{CT}(\mathcal{F})$.

As a result of Proposition 1.4.5, given a flag $\mathcal{F} = \{F_0 \subset, \cdots \subset F_\ell\}$ and a $\mathbb{Z}^n$-graded ring R, to understand the structure of $R_*(\!(\mathcal{F})\!)$ we are only interested in those faces in $\mathrm{CP}(\mathcal{F})$ that are precisely one dimension lower than a face in the flag itself. The nature of the faces of dimension $\dim(F_i)-1$ for some $F_i \in \mathcal{F}$ in $\mathrm{CT}(\mathcal{F})$ depend on what elements of the form $\operatorname{sgn}(\mathcal{F})_k e_k$

have F_{i-1} as the maximal face as per Lemma 1.4.2 (that is, what k satisfies $\mathrm{sgn}(\mathcal{F})_k e_k \in T_{F_{i-1}}$ and for $F_j \supset F_{i-1}$, $\pm e_k \in T_{F_j}$). Precisely this condition dictates what faces of dimension $n-1$ can be intersected with F_i to form the faces of dimension $\dim(F_i)-1$ that are within the caterpillar, and hence the Christmas tree of $\mathcal{F}$.

Not only are we interested in the orientation, we are also interested in the dimension of the maximal face for a given basis element e_k. For a given k, the bound on the support is stronger in either the e_k or $-e_k$ direction the larger the maximum face that does not contain both $\pm e_k$ in its barrier cone is, simply because the larger the face, the stronger the condition. Recall, for example, that faces of $n-1$ dimension give global bounds on the support, but only in the relevant direction (opposite to the inward normal vector).

We now use the above simplification to encode the information, both pertaining to orientation and the strength of the condition in a given direction, in an enlightening way.

Definition 1.4.7 (Skeleton of a flag)**.** Given a flag $\mathcal{F} = \{F_0, \dots, F_\ell\}$, of length ℓ where $0 \leq \ell \leq n$ let E be the set of elements $E = \{\pm e_k,\ 1 \leq k \leq n\}$ (where e_k form a standard basis of $\mathbb{Z}^n$) and $\mathrm{sgn}(\mathcal{F})_k$ be as in Remark 1.4.4. Define the following collection of sets of elements of E:

- $P = \{\pm e_k \colon \pm e_k \in T_{F_0}\}$.
- $A_i = \{\mathrm{sgn}(\mathcal{F})_k e_k \colon \mathrm{sgn}(\mathcal{F})_k e_k \in T_{F_i},\ -\mathrm{sgn}(\mathcal{F})_k e_k \notin T_{F_i},\ \pm e_k \in T_{F_{i+1}}\}$.
- $W = \{\mathrm{sgn}(\mathcal{F})_k e_k \colon \mathrm{sgn}(\mathcal{F})_k e_k \in T_{F_\ell},\ -\mathrm{sgn}(\mathcal{F})_k e_k \notin T_{F_\ell}\}$.

The *skeleton of* $\mathcal{F}$, $\mathrm{SK}(\mathcal{F})$ is the following collection of sets:

$$\{P, A_0, ..., A_{\ell-1}, W\}.$$

Each skeleton corresponds to a unique flag, and hence a unique generalised Novikov ring. Note that when $F_\ell = S$, $W = \emptyset$ and when F_0 is a vertex, $P = \emptyset$.

Remark 1.4.8. The skeleton can be used to quickly state the nature of the faces of a flag in the following manner. For a flag $\mathcal{F}$, the face F_i is precisely the intersection of F_{i+1} with the $n-1$ dimensional faces G_{j_u},

$$F_i = F_{i+1} \cap (\bigcap_u G_{j_u})$$

such that $\eta_{G_{j_u}} = \mathrm{sgn}(\mathcal{F})_{j_u} e_{j_u} \in A_i$. The set W will tell us the nature of the largest face, precisely:

$$F_\ell = (\bigcap_b G_{j_b})$$

for $n-1$-dimensional faces G_{j_b} such that $\eta_{G_{j_b}} = \mathrm{sgn}(\mathcal{F})_{j_b} e_{j_b} \in W$ in particular, $W = \emptyset$ implies $F_\ell = S$. We see that no face within $\mathcal{F}$ is contained within G or $-G$ such that $\pm\eta_G$ are in P. Again, these ideas will be repeatedly used in later proofs.

The skeleton attempts to boil down the flag to only the important information pertaining to Novikov rings. Let $\mathrm{sgn}(\mathcal{F})_k$ be as in Remark 1.4.4, that is the unique sign of e_k such that $\mathrm{sgn}(\mathcal{F})_k e_k \in T_{F_0}$ if one exists, otherwise it is undefined. For k where $\mathrm{sgn}(\mathcal{F})_k$ is defined there will be some A_i, or W, that contains $\mathrm{sgn}(\mathcal{F})_k e_k$. The bound conditions on the support for a given direction, always $-\mathrm{sgn}(\mathcal{F})_k e_k$, is stronger whenever $\mathrm{sgn}(\mathcal{F})_k e_k \in A_i$ for larger i. There are always as many faces in $\mathrm{CP}(\mathcal{F})$ of dimension $F_{i+1} - 1$ as there are elements in A_i. In the case that the element is in A_i, the faces of $\dim(F_{i+1}) - 1$ are key. For elements in W condition (1) is key. For elements in P, it is the faces of dimension $\dim(F_0) - 1$ that are important. Here, there is again a face within $\mathrm{CP}(\mathcal{F})$ of dimension $\dim(F_0) - 1$ for each element of P, specifically there are two faces for each pair of elements $\pm e_j \in P$.

We use the following results to attempt to visualise what structure the generalised Novikov rings actually have.

Proposition 1.4.9. *Let $\mathcal{F}$ be a flag and $F \subset [-1,1]^n$ a face of the flag. Let $F = \bigcap_t G_{j_t}$ for faces G_{j_t} of dimension $n-1$ with $\eta_{G_{j_t}} = \mathrm{sgn}(\mathcal{F})_{j_t} e_{j_t}$. The condition (2-F) implies that for an element $f \in R_*(\!(\mathcal{F})\!)$ and fixed $q \in \mathbb{Z}^n$, the subspace*

$$F_q^* = \{q + \sum_{j=1}^{n} m_j e_j \colon m_j \in \mathbb{Z},\, m_j = 0 \text{ if } j = j_t \text{ for some } t\}$$

of $\mathbb{Z}^n$ can be shifted only finitely often in the directions $-\mathrm{sgn}(\mathcal{F})_{j_t} e_{j_t}$ before having no intersection with $\mathrm{supp}(f)$.

Proof. Begin by observing that, for $q = 0$, $F_0^* \subseteq T_F$ and $F_0^* \subseteq -T_F$ as

$$\{\sum_{j=1}^{n} m_j e_j \colon m_j \in \mathbb{Z},\, m_j = 0 \text{ if } j = j_t \text{ for some } t\} \subset T_{G_{j_t}}$$

for all t where $T_F = \bigcap_t T_{G_{j_t}}$. Furthermore, $F_q^* \subset q + (-T_F)$. Immediately we can see that for all t,

$$\bigoplus_{k\geq 0} F^*_{q-k\,\mathrm{sgn}(\mathcal{F})_{j_t} e_{j_t}} \subseteq q + (-T_F).$$

Now argue by contradiction. If there was no k such that for a fixed t' and all $k' \geq k$, the plane

$$\begin{aligned} &F^*_{q-k'\mathrm{sgn}(\mathcal{F})_{j_{t'}} e_{j_{t'}}} = \\ &\quad \{q + \sum_{j=1}^{n} m_j e_j - k'\mathrm{sgn}(\mathcal{F})_{j_{t'}} e_{j_{t'}} \colon m_j \in \mathbb{Z},\, m_j = 0 \text{ if } j = j_t \text{ for some } t\} \end{aligned}$$

would have empty intersection with $\mathrm{supp}(f)$, then (2-F) could never be satisfied as $k'\eta_F + \big(\bigoplus_{k\geq 0} F^*_{q-k\,\mathrm{sgn}(\mathcal{F})_{j_{t'}} e_{j_{t'}}} \cap \mathrm{supp}(f)\big)$ would never be within T_F for any k'. Hence there must be some k that will ensure that for large enough shifts the intersection of the shifted planes with the support is empty. $\square$

This hopefully gives a little more visual representation onto the abstract bones of the Novikov ring definition. We already know that we are not particularly interested in the flags within $\mathrm{CP}(\mathcal{F})$ not of dimension $\dim F_i - 1$ for $F_i \in \mathcal{F}$. For those faces of dimension $\dim F_i - 1$, it is precisely the skeleton of the flag that tells us what planes have bounds and in what direction.

Remark 1.4.10. Let

$$\mathrm{SK}(\mathcal{F}) = \{P, A_0, ..., A_{\ell-1}, W\}$$

and fix F_i such that for $1 \leq k \leq n$, $\pm e_{k_p} \in P$, $\mathrm{sgn}(w)_{k_a} e_{k_a} \in \bigcup_{0\leq x\leq i-1} A_x$. We can rewrite $(F_i)_q^*$ as the following:

$$(F_i)_q^* = \{q + \sum_{k=k_p,k_a} m_k e_k \colon m_k \in \mathbb{Z}\}$$

and can observe that for a given element $r \in R_*((\mathcal{F}))$ the plane can be shifted only finitely often in the directions $-\mathrm{sgn}(\mathcal{F})_k e_k \in (\bigcup_{x \geq j} A_x) \cup W$ until the intersection with the support is trivial. However, we only care about faces of dimension $\dim F_i - 1$. Let $F = \dim F_i - 1$ and $F \in \mathrm{CP}(\mathcal{F})$, then there is some G_t' such that $F = F_i \cap G_t'$, $\eta_{G_t'} = \mathrm{sgn}(\mathcal{F})_t e_t \in A_{i-1}$. We can write F_q^* as the following:

$$(F)_q^* = \{q + \sum_{k=k_p,k_a} m_k e_k \colon m_k \in \mathbb{Z},\, m_t = 0\}$$

and note that this subspace can be shifted only finitely often in the directions $-\mathrm{sgn}(\mathcal{F})_k e_k \in (\bigcup_{x \geq j} A_x) \cup W$ and $-\mathrm{sgn}(\mathcal{F})_t e_t$ before having trivial intersection with the support of an element.

We will refer to the condition that a subspace can be shifted only finitely often in a certain direction, say $-\mathrm{sgn}(\mathcal{F})_k e_k$, as having *the subspace bound condition in* $-\mathrm{sgn}(\mathcal{F})_k e_k$.

Note that there is a face of dimension $\dim F_i - 1$ for every element within A_{i-1}. We will use this shifted plane argument to help understand what precisely these Novikov rings are. The above gives a good description of what happens for A_i, but W and P need to be discussed too.

What W tells us. For the elements of W the support of a given $r \in R_*((\mathcal{F}))$ is infinite and bounded into either the non-negative or non-positive hyperplane, depending on $\mathrm{sgn}(\mathcal{F})_k$ for $\mathrm{sgn}(\mathcal{F})_k e_k \in W$. This is from condition (1). From observing the definition of the Novikov ring, we see that this translates to global lower or upper bounds by zero on the supports of elements in the ring for these directions within $\mathbb{Z}^n$. When $W = \emptyset$ it follows that $S \in \mathcal{F}$ and hence condition (1) of the Novikov ring definition is trivial.

What A_j tells us. These are the 'typical' Novikov conditions. The element $\mathrm{sgn}(\mathcal{F})_k e_k \in A_j$ gives the orientation, there is no bound in the direction of $\mathrm{sgn}(\mathcal{F})_k e_k$ but some kind of local bound condition in the direction of $-\mathrm{sgn}(\mathcal{F})_k e_k$. If there is a bound the larger j is the stronger the bound on the support is in that direction. Namely, letting $\pm e_{k_p} \in P$ and $\mathrm{sgn}(\mathcal{F})_{k_a} e_{k_a} \in \bigcup_{0 \leq x \leq j-1} A_x$, for each $\mathrm{sgn}(\mathcal{F})_t e_t \in A_j$, the plane

$$(F)_q^* = \{q + \sum_{k=k_p,k_a} m_k e_k \colon m_k \in \mathbb{Z},\, m_t = 0\}$$

can be shifted only finitely often in the directions $-\mathrm{sgn}(\mathcal{F})_k e_k \in (\bigcup_{x\geq j} A_x) \cup W$ and $-\mathrm{sgn}(\mathcal{F})_t e_t$. Note that we do not take all the elements of A_j at once when we assess our plane conditions, in fact we do not take more than one at once. This is because it is the faces of dimension $\dim F_{j+1} - 1$ that provide the conditions here, and each one is associated with a different element of A_j.

What P tells us. In a similar manner as A_i, there is again one face of dimension $\dim F_0 - 1$ for every element of P, this time however we have pairs of elements $\pm e_k$. Letting $\pm e_{k_p} \in P$, begin by considering the condition on the subspace

$$(F_0)^*_q = \{q + \sum_{k=k_p} m_k e_k \colon m_k \in \mathbb{Z}\}$$

from (2-F_0), which implies there is a plane bound condition in the directions of $-\mathrm{sgn}(\mathcal{F})_k e_k \in (\bigcup_{x\geq 0} A_x) \cup W$.

Now, we consider the effect of the faces of dimension $\dim F_0 - 1$. Consider that for two faces $F = F_0 \cap G_t$ and $F' = F_0 \cap G_{-t}$ where $\eta_{G_t} = e_t = -\eta_{G_{-t}}$,

$$F^*_q = (F')^*_q = \{q + \sum_{k=k_p} m_k e_k \colon m_k \in \mathbb{Z},\, m_t = 0\}.$$

It is immediately apparent that $F^*_q = (F')^*_q$ has the plane bound condition in the both of the directions $+e_t$ and $-e_t$ by considering the plane bound conditions derived from (2-F) and (2-F'). Taking across all the elements of P, we see that for any fixed q the following subspace will always have finite intersection with the support of an element of $R_*(\!(\mathcal{F})\!)$

$$\{q + \sum_{k=k_p} m_k e_k \colon m_k \in \mathbb{Z}\}.$$

As a final observation for this section, whenever P is non-empty and has k many pairs of elements $\pm e_t$, we in actuality see a ring that is precisely a Novikov ring over an $n-k$-graded ring.

Remark 1.4.11. For F a k-dimensional face let $X = \mathrm{span}\{x - y \colon x, y \in F\}$. There is a projection map

$$\gamma \colon \mathbb{R}^n \to \mathbb{R}^n / X \cong \mathbb{R}^{n-k}$$

that has an obvious splitting as X is a coordinate subspace, i.e., there is δ such that $\gamma\delta = \mathrm{id}$. There is a similar map $\mathbb{Z}^n \to \mathbb{Z}^{n-k}$. The ring R can be, therefore, understood as a $\mathbb{Z}^{n-k}$ ring U where $U_\rho = \bigoplus_{z \in \gamma^{-1}(\rho)} R_z$. Consider the image of $S = [-1,1]^n$ by γ, this is the cube $[-1,1]^{n-k}$. For any flag G within S containing F, we can define $\gamma(G)$ where $\delta\gamma(G) = G$. Given a flag $\mathcal{F} = \{F_0 \subset \cdots \subset F_\ell\}$ contained within $\mathcal{N}_F$, which are precisely those faces that contain F, we can define a flag $\gamma(\mathcal{F}) = \{\gamma(F_0) \subset \cdots \subset \gamma(F_\ell)\}$. We can also understand the ring $R_*(\!(\mathcal{F})\!)$ as the ring $U_*(\!(\gamma(\mathcal{F}))\!)$. Here outside $\gamma(\mathbb{Z}^n)$ the support of any element is locally finite, that is each component of $U_*(\!(\gamma(\mathcal{F}))\!)$ for $\rho \in \mathbb{Z}^{n-k}$ is still $\bigoplus_{z \in \gamma^{-1}(\rho)} R_z$, matching what occurs for those elements $\pm e_j \in P$ for the skeleton of $\mathcal{F}$ as discussed earlier.

2. CONTRACTIBILITY OF NOVIKOV HOMOLOGY IMPLIES FINITE DOMINATION

Having set out what trivial Novikov homology actually means, we can begin to tackle the actual main result. We begin by assuming a bounded chain complex of finitely generated free R-modules C, for a unital, graded module R, has trivial Novikov homology and we will now work towards a proof that C is R_0-finitely dominated.

2.1 Čech complexes of diagrams

We can define diagrams with entries in the category of R_0-modules indexed by given posets P (that is, functors $P \to R_0$-mod). Genrally speaking we can equip P with a strictly order reversing rank map

$$\mathrm{rk}\colon P \to \mathbb{Z}_{\leq 0}$$

and a (potentially trivial) incidence function

$$[-,-]\colon P \times P \to \mathbb{Z}$$

such that $[x\colon y]$ for $x, y \in P$ satisfies the following conditions:

- (DI1) $[x\colon y] = 0$ unless $x < y$ and $\mathrm{rk}(x) - 1 = \mathrm{rk}(y)$.
- (DI2) for all $x < z$ with $\mathrm{rk}(x) - 2 = \mathrm{rk}(z)$, the open interval $I(x\colon z) = \{y \in P | x < y < z\}$ is finite, and we have

$$\sum_{y \in I(x\colon z)} [y\colon z] \cdot [x\colon y] = 0.$$

- (DI3) for $z \in P$ with $\mathrm{rk}(z) = -1$ the set $I(< z) = \{y \in P | y < z\}$ (noting that there is only one level below z) is finite and we have

$$\sum_{y \in I(<z)} [y\colon z] = 0.$$

In this paper we will affix specific P with known rank and incidence functions which will be described later.

We can use the poset, these two maps and an associated diagram to define a chain complex.

Definition 2.1.1. Let X be an additive category and P a poset equipped with a strictly order reversing rank map $\mathrm{rk}\colon P \to \mathbb{Z}_{\leq 0}$ and an incidence function $[-,-]\colon P \times P \to \mathbb{Z}$ as above. For a diagram $\Phi\colon P \to X$ with structure maps $\varphi_{p,q}\colon \Phi(p) \to \Phi(q)$, define the *Čech complex* $\Gamma(\Phi)$ as the collection of objects indexed by $i \in \mathbb{Z}_{\leq 0}$ where

$$\Gamma(\Phi)_i = \bigoplus_{\mathrm{rk}(p)=i} \Phi(p)$$

is the direct sum of the objects of Φ with rank i and structure maps are

$$d_i = \bigoplus_{\substack{\mathrm{rk}(p)=i \\ \mathrm{rk}(q)=i-1}} [p:q]\varphi_{p,q}.$$

It can be shown that the conditions (DI1) (only non trivial maps are of degree -1) and (DI2) (composition of d is trivial) ensure the Ĉech complex of Φ, $\Gamma(\Phi)$, is a chain complex. The condition (DI3) will ensure certain maps we will define later are chain complex maps. All diagrams in this paper will have objects that are either R_0-R_0-modules or chain complexes of R_0-R_0-modules. The act of forming a Čech complex 'preserves quasi-isomorphisms'.

Lemma 2.1.2. *For two diagrams* $\Phi, \Psi\colon P \to X$ *with* P *finite,* $\max(\mathrm{rk}(P)) = 0$ *and* $\min(\mathrm{rk}(P)) = k$*, and a map* $\chi\colon \Phi \to \Psi$ *such that for each* $p \in P$ *the component* $\chi(p)$ *is a quasi-isomorphism, the induced map* $\Gamma(\chi)\colon \Gamma(\Phi) \to \Gamma(\Psi)$ *is also a quasi-isomorphism.*

Proof. Let ${}_{Tr}\Gamma_j(\Phi)$ be the truncation of $\Gamma(\Phi)$ below and including j. Also let $\Gamma(\Phi)_j$ be the jth chain level of $\Gamma(\Phi)$, thought of as a chain complex concentrated at the jth level. Note that $\Gamma(\Phi)_j = \bigoplus_{\mathrm{rk}(p)=j} \Phi(p)$. For all j there is a quasi-isomorphism

$$\bigoplus_{\mathrm{rk}(p)=j} \chi(p)\colon \Gamma(\Phi)_j \to \Gamma(\Psi)_j.$$

Noting that ${}_{Tr}\Gamma_k(\Phi) = \Gamma(\Phi)_k$, we have a short exact sequence of chain complexes

$$0 \to {}_{Tr}\Gamma_k(\Phi) \to {}_{Tr}\Gamma_{k+1}(\Phi) \to \Gamma(\Phi)_{k+1} \to 0$$

that combined via the 5-lemma tell us that there is a quasi-isomorphism

$${}_{Tr}\Gamma_{k+1}(\Phi) \to {}_{Tr}\Gamma_{k+1}(\Psi).$$

Next, note the following short exact sequence of chain complexes:

$$0 \to {}_{Tr}\Gamma_{k+1}(\Phi) \to {}_{Tr}\Gamma_{k+2}(\Phi) \to \Gamma(\Phi)_{k+2} \to 0$$

shows that there is a quasi-isomorphism

$${}_{Tr}\Gamma_{k+2}(\Phi) \to {}_{Tr}\Gamma_{k+2}(\Psi).$$

Iterate this argument to find that $\Gamma(\chi)\colon \Gamma(\Phi) \to \Gamma(\Psi)$ is a quasi-isomorphism as claimed. $\square$

Remark 2.1.3. For a diagram of chain complexes Ξ, where for each $p \in P$ $\Xi(p)$ is a chain complex we define $\Gamma(\Xi)$ as the double complex with objects

$$\Gamma(\Xi)_{j,n} = \bigoplus_{\mathrm{rk}(p)=j} \Xi(p)_n$$

where the 'horizontal' boundary is $\bigoplus_{\mathrm{rk}(p)=j}[p\colon \ell]\xi_{p,\ell}$ for the structure maps $\xi_{p,\ell}$, effectively the Čech complex of a chain level of Ξ, and the vertical is $\bigoplus_{\mathrm{rk}(k)=j}(-1)^j d_{\Xi_k}$. Hence, $\Gamma(\Xi)$ has anti-commutative differentials making its totalisation a chain complex by our convention. Write $\check{\Gamma}(\Xi)$ to denote the totalisation of the double complex $\Gamma(\Xi)$.

2.2 The quasi-coherent diagram $D(k)$

We wish to define a specific diagram with special properties that will be of use to us. Specifically, the Čech totalisation of this diagram will be a resolution of R_0. Before we can do so, we need to go through another short round of definitions first.

Definition 2.2.1 (Shifted barrier lattices)**.** For a given $k \in \mathbb{Z}$, we define *shifted barrier lattices*

$$(kF + B_F) \cap \mathbb{Z}^n = \{kf + t | f \in F, t \in B_F\} \cap \mathbb{Z}^n$$

with $(0F + B_F) \cap \mathbb{Z}^n = T_F$. Write $k \circ T_F := (kF + B_F) \cap \mathbb{Z}^n$.

Example 2.2.2. Observing that $\epsilon_b = \{(x, y) \colon -1 \leq x \leq 1,\ y = -1\}$, note that $(\epsilon_b + B_{\epsilon_b})$ is the set

$$\{(x, y) \in \mathbb{R}^2,\ y \geq -1\},$$

the upper half plane shifted down by one in the y direction. Similarly, noting that $v_{bl} = (-1, 1)$, $(v_{bl} + B_{v_{bl}})$ is the set

$$\{(x, y) \in \mathbb{R}^2,\ x, y \geq -1\},$$

the upper right quadrant shifted diagonally by $(-1, -1)$.

Definition 2.2.3 (The shifted cone submodule of R)**.** Let $R_*[k \circ T_F]$ be the collection of elements of R that have finite support in $k \circ T_F$, called *the shifted cone submodule of* R. This is a left and right R_0-module and $R_*[0 \circ T_F] = R_*[T_F]$ is a ring. Also, $R_*[T_S] = R$. Given two faces $F \subseteq G$, note that $T_F \subseteq T_G$, and hence $R_*[k \circ T_F] \subseteq R_*[k \circ T_G]$ for $k \in T_F$.

Example 2.2.4. If we consider $R = V[x, x^{-1}, y, y^{-1}]$ for a ring V, then:

$$R_*[T_{v_{bl}}] = V[x, y]. \qquad R_*[T_{v_{br}}] = V[x^{-1}, y].$$
$$R_*[T_{\epsilon_b}] = V[x, x^{-1}, y].$$

also, for $k \in \mathbb{Z}$:

$$R_*[k \circ T_{v_{bl}}] = x^{-k} y^{-k} V[x, y]. \qquad R_*[k \circ T_{v_{br}}] = x^k y^{-k} V[x^{-1}, y].$$
$$R_*[k \circ T_{\epsilon_b}] = y^{-k} V[x, x^{-1}, y].$$

Remark 2.2.5. For any flag $\mathcal{F}$ such that F is the largest face, $R_*(\!(\mathcal{F})\!)$ has an $R_*[T_F]$-$R_*[T_F]$ bimodule structure, as the support of the elements of $R_*(\!(\mathcal{F})\!)$ are contained within T_F. In particular for every $\mathcal{F}$ that contains S the ring $R_*(\!(\mathcal{F})\!)$ has an R-R bimodule structure.

Definition 2.2.6 (Quasi-coherent diagram of modules)**.** For $S = [-1,1]^n$ let S be the category of faces of S with inclusions as maps. Let $\mathfrak{Q}\colon \mathsf{S} \to R_0$-mod be a diagram where $\mathfrak{Q}(S)$ is an R-module, for faces $F \subset S$, $\mathfrak{Q}(F)$ is a module over $R_*[T_F]$ and for $F \subset G \subseteq S$ the map $\gamma_{F,G}\colon \mathfrak{Q}(F) \to \mathfrak{Q}(G)$ is a $R_*[T_F]$-linear structure map for all $F \subset G$ with compositions $\gamma_{G,H}\gamma_{F,G} = \gamma_{F,H}$ for all $F \subset G \subset H$. If $\mathfrak{Q}$ satisfies the condition for all $F \subset G$ that the adjoint map of $\gamma_{F,G}$

$$\mathfrak{Q}(F) \underset{R_*[T_F]}{\otimes} R_*[T_G] \to \mathfrak{Q}(G), \quad r \otimes s \to rs$$

is an isomorphism of $R_*[T_G]$ modules we call it a *quasi-coherent diagram of modules.*

We have the definitions we need, we now ensure the diagram we will construct satisfies the adjoint map condition.

Proposition 2.2.7. *For all $k \in \mathbb{Z}, F \subseteq S$, the $R_*[T_G]$-linear maps*

$$\alpha_{k,F,G}\colon R_*[k \circ T_F] \underset{R_*[T_F]}{\otimes} R_*[T_G] \longrightarrow R_*[k \circ T_G], \quad r \otimes s \mapsto rs$$

are isomorphisms provided the ring R is strongly $\mathbb{Z}^n$-graded.

Proof. Suppose R is strongly graded. For j where only one of $+e_j, -e_j$ is contained within T_F, let $\mathrm{sgn}(F)_j e_j \in T_F$. Then $\eta_F = \sum_j \mathrm{sgn}(F)_j e_j$. Then we may choose a partition of unity of type $(-k\eta_F, k\eta_F)$, say $1 = \sum_\ell u_\ell v_\ell$ with $u_\ell \in R_{-k\eta_F}$ and $v_\ell \in R_{k\eta_F}$, so that $u_\ell \in R_*[k \circ T_F]$ and for $r \in R_*[k \circ T_G]$, $v_\ell r \in R_*[T_G]$. The $R_*[T_G]$-linear map

$$\beta_{k,F,G}\colon R_*[k \circ T_G] \longrightarrow R_*[k \circ T_F] \underset{R_*[T_F]}{\otimes} R_*[T_G], \quad r \mapsto \sum_\ell u_\ell \otimes v_\ell r$$

satisfies $\alpha\beta(r) = \sum_\ell u_\ell v_\ell r = r$ so that $\alpha\beta = \mathrm{id}$. Also

$$\beta\alpha(r \otimes s) = \beta(rs) = \sum_\ell u_\ell \otimes v_\ell rs \underset{(*)}{=} \sum_\ell u_\ell v_\ell r \otimes s = r \otimes s$$

(where the equality labelled $(*)$ is true since $v_\ell r \in R_*[T_F]$ for any $r \in R_*[k \circ T_F]$), hence $\beta\alpha = \mathrm{id}$. □

Definition 2.2.8 (The quasi-coherent diagram $D(k)$)**.** Given a $\mathbb{Z}^n$-graded ring R and $k \geq 0$, let $D(k)$ be the diagram with $D(k)(F) = R_*[k \circ T_F]$ and all maps inclusions ι. We see that this diagram is a quasi-coherent diagram of modules with the adjoint map assumptions satisfied via Proposition 2.2.7.

Consider a pair of faces F, F' such that $\dim(F) + 1 = \dim(F')$. Assume $F = F' \cap G_j$ for some $n-1$ dimensional face G_j. We know that there is precisely one j such that $\pm e_j \in T_{F'}$ but only one of $+e_j$ or $-e_j$ is contained within T_F. Let

$$y_{F,F'} = \begin{cases} 0 \text{ when } e_j \in T_F. \\ 1 \text{ when } -e_j \in T_F. \end{cases}$$

Let $L_{F,j}$ be the number of e_i, $0 \leq i < j$ such that both $\pm e_i \in T_F$. Equip S with the rank map $\mathrm{rk}(F) = -\dim(F)$.

We now need to apply satisfactory signs so that we can form the Čech complex of $D(k)$.

Lemma 2.2.9. *Let the map $[-,-]\colon \mathsf{S} \times \mathsf{S} \to \mathbb{Z}$ be such that $[F\colon F']$ is the following sign:*

$$[F\colon F'] = \begin{cases} (-1)^{y_{F,F'} + L_{F,j}} \ \textit{when } \dim(F) + 1 = \dim(F'). \\ 0 \qquad \textit{otherwise.} \end{cases}$$

This map satisfies conditions (DI1),(DI2) and (DI3).

Proof. The map $[-,-]\colon \mathsf{S} \times \mathsf{S} \to \mathbb{Z}$ satisfies condition (DI1) trivially. Let F, G satisfy $\dim(F) + 2 = \dim(G)$, so that $\mathrm{rk}(F) - 2 = \mathrm{rk}(G)$. Then there are precisely two faces $A, B \in I(F\colon G)$, and two indices a, b such that

- $\pm e_a, \pm e_b \in T_F$.

- $\mathrm{sgn}(a)e_a \in T_A$, $-\mathrm{sgn}(a)e_a \notin T_A$ for some $\mathrm{sgn}(a) \in \{-1, 1\}, \pm e_b \in T_A$.
- $\mathrm{sgn}(b)e_b \in T_B$, $-\mathrm{sgn}(b)e_b \notin T_B$ for some $\mathrm{sgn}(b) \in \{-1, 1\}, \pm e_a \in T_B$.
- $\mathrm{sgn}(a)e_a, \mathrm{sgn}(b)e_b \in T_G$, $-\mathrm{sgn}(a)e_a, -\mathrm{sgn}(b)e_b \notin T_G$.

Hence, there are only two summands $[A\colon G][F\colon A], [B\colon G][F\colon B]$ to check in relation to condition (DI2). Note that

$$y_{F,A} + y_{A,G} = y_{F,B} + y_{B,G}$$

as $y_{F,A} = y_{B,G}$, $y_{A,G} = y_{F,B}$. Without loss of generality, let $a \leq b$. Then we know that $L_{F,b} + 1 = L_{A,b}$, as $\pm e_a \in T_F$ but $-\mathrm{sgn}(a)e_a \notin T_A$, and $L_{F,a} = L_{B,a}$. The products $[A\colon G][F\colon A] = (-1)^{y_{F,A}+L_{F,a}+y_{A,G}+L_{A,b}}$ and $[B\colon G][F\colon B] = (-1)^{y_{F,B}+L_{F,b}+y_{B,G}+L_{B,a}}$ have trivial sum as

$$y_{F,A} + L_{F,a} + y_{A,G} + L_{A,b} = y_{F,B} + L_{F,b} + 1 + y_{B,G} + L_{B,a}$$

which makes the signs in this case differ so (DI2) is therefore satisfied.

Finally, note that for a given face F of dimension 1, there is only one j such that $\pm e_j \in T_F$ and two vertices such that $v, v' \subset F$. Note that $L_{v,i} = 0 = L_{v',i}$ for all $1 \leq i \leq n$. The signs of $[v\colon F] = (-1)^{y_{v,F}+L_{v,j}}$ and $[v'\colon F] = (-1)^{y_{v',F}+L_{v',j}}$ differ, as if $e_j \in T_v$ then $-e_j \in T_{v'}$ so $y_{v,F} = 0$ and $y_{v',F} = 1$. Similarly if $-e_j \in T_v$ then $e_j \in T_{v'}$ so $y_{v,F} = 1$ and $y_{v',F} = 0$. Hence (DI3) is satisfied also. □

Remark 2.2.10 (Čech complex of $D(k)$)**.** The complex $\Gamma(D(k))$ is the complex with entries indexed with faces of dimension i at level $-i$, that is the sum of

$$\Gamma(D(k))_{-i} = \bigoplus_{\dim F = i} R_*[T_F]$$

and maps

$$[F\colon F']\iota\colon R_*[T_F] \to R_*[T_{F'}]$$

where for $F \subset F'$, $\dim(F) + 1 = \dim(F')$, $[F\colon F']$ is the following sign:

$$[F\colon F'] = (-1)^{y_{F,F'}+L_{F,j}}$$

and is 0 otherwise. Since the objects indexed by F, A, B, F' form a commutative square consisting of inclusions, the sum of maps

$$[A\colon F']\alpha_{A,F'}[F\colon A]\alpha_{F,A} + [B\colon F']\alpha_{B,F'}[F\colon B]\alpha_{F,B}$$

must equal zero. Hence the satisfaction of (DI2) implies that the maps $\sum\limits_{\dim(F)=k}[F\colon G]\alpha_{F,G}$ are boundaries as the composition is the sum of pairs of maps $[A\colon F']\alpha_{A,F'}[F\colon A]\alpha_{F,A} + [B\colon F']\alpha_{B,F'}[F\colon B]\alpha_{F,B} = 0$.

2.3 The homology of the Čech complex of $D(k)$

We now wish to find the homology of the Čech complex of $D(k)$. We will see that $\Gamma(D(k))$ is homotopy equivalent to a complex that will be used as the components of the chain levels of a bounded complex of finitely generated projective R_0-modules, hence playing an important role in the finite domination result.

To begin we prove something in a greater generality than required, so that we can use special cases in a number of situations.

Note that for any group G and any set S of elements of G, and unital G-graded ring $\bigoplus\limits_{g\in G} R_g$, R_{id_G} is also an unital ring and the sum of homogenous components with index in S, $\bigoplus\limits_{s\in S} R_s$ is an R_{id_G}-R_{id_G}-bimodule. Note that whenever a unital G-graded ring is strongly graded, there are still partition of unities for all $g \in G$ (a consequence of Proposition 1.6 of [Dad80], which is more general than $\mathbb{Z}^n$-graded rings).

Proposition 2.3.1. *Let X be a strongly G-graded ring for a group G. Consider sets A, B contained within G that satisfy $A \cap B = \{\mathrm{id}_G\}$ and the condition for all $a, a' \in A$, $b, b' \in B$,*

$$ab = a'b' \text{ if and only if } a = a' \text{ and } b = b'. \tag{2.3.1.1}$$

Let $AB = \{ab\colon a \in A, b \in B\}$ and $X_I = \bigoplus\limits_{i\in I} X_i$ be the restriction of X to any subset I of G. The following maps

$$\pi_{a,b}\colon X_a \underset{X_{\mathrm{id}_G}}{\otimes} X_b \to X_{ab},\ x_a \otimes x_b \mapsto x_a x_b$$

form an isomorphism as X_{id_G}-X_{id_G} bimodules:

$$\pi\colon X_A \underset{X_{\mathrm{id}_G}}{\otimes} X_B \cong X_{AB}.$$

Proof. Given $x_i \in X_i$ and $a \in A, b \in B$ the maps

$$\pi\colon X_A \underset{X_{\mathrm{id}_G}}{\otimes} X_B \to X_{AB}$$

where

$$\pi_{a,b}\colon X_a \underset{X_{\mathrm{id}_G}}{\otimes} X_b \cong X_{ab},\, x_a \otimes x_b \mapsto x_a x_b$$

and

$$\beta\colon X_{AB} \to X_A \underset{R_{\mathrm{id}_G}}{\otimes} X_B$$

where

$$\beta_{a,b}\colon X_{ab} \to X_a \underset{X_{\mathrm{id}_G}}{\otimes} X_b,\, x_{ab} \mapsto \sum_j x_{ab} u_j \otimes v_j$$

for a partition of unity $\sum\limits_j u_j v_j$ of form $(-b, b)$ are the required isomorphisms. Firstly observe that the maps π, β map summands on each side in a one-to-one relation as A and B satisfy Condition 2.3.1.1. That π is a R_{id_G}-R_{id_G} balanced bimodule map is clear, β is clearly a left R_{id_G}-morphism, to see it on the right:

$$\begin{aligned}
\beta(x_{ab}x_{\mathrm{id}_G}) &= \sum_j x_{ab}x_{\mathrm{id}_G}u_j \otimes v_j = \sum_i\sum_j x_{ab}u_i v_i x_{\mathrm{id}_G}u_j \otimes v_j \\
&= \sum_i\sum_j x_{ab}u_i \otimes v_i x_{\mathrm{id}_G}u_j v_j = \sum_i x_{ab}u_i \otimes v_i x_{\mathrm{id}_G} \\
&= \beta(x_{ab})x_{\mathrm{id}_G}.
\end{aligned}$$

The composition is the identity as $\pi\beta = \mathrm{id}$ trivially and $\beta\pi = \mathrm{id}$ due to

$$x_a \otimes x_b \mapsto x_a x_b \mapsto \sum_j x_a x_b u_j \otimes v_j = x_a \otimes x_b$$

from the fact that $x_b u_j \in R_{\mathrm{id}_G}$. Hence π and β are mutual isomorphisms as required. $\square$

Call the maps $\pi_{a,b}$ the *product maps* and $\beta_{a,b}$ the *splitting maps*.

Throughout this work, when such an isomorphism is required, the sets A, B will be of the form $A' \times \{\mathrm{id}\}, \{\mathrm{id}\} \times B'$ hence satisfying Condition 2.3.1.1 immediately.

Lemma 2.3.2. *The map $\beta_{a,b}$ is independent of the choice of partition of unity.*

Proof. Let $\beta_{a,b}$ and $\beta'_{a,b}$ be two splitting maps using two different partitions of unity $\sum_{j=1}^{q} u_j v_j$ and $\sum_{k=1}^{q'} u'_k v'_k$ of form $(-b, b)$. Then $\beta_{a,b} = \beta'_{a,b}$, as

$$\begin{aligned}\sum_{j=1}^{q} ru_j \otimes v_j &= \sum_{j=1}^{q}\sum_{k=1}^{q'} ru_j \otimes v_j u'_k v'_k \\ &= \sum_{j=1}^{q}\sum_{k=1}^{q'} ru_j v_j u'_k \otimes v'_k = \sum_{k=1}^{q'} ru'_k \otimes v'_k\end{aligned}$$

as $v_j u'_k \in R_0$. This tells us that the map $\beta_{a,b}$ is independent of the choice of partition of unity. □

Definition 2.3.3. For a given $\mathbb{Z}^n$-graded ring R, let $R^{(j)} = \bigoplus_{m\in\mathbb{Z}} R_{me_j}$, where $R^{(j)}_m = R_{me_j}$, be the *restriction of R to the jth axis*, itself a (strongly) $\mathbb{Z}$-graded ring when R is a (strongly) $\mathbb{Z}^n$-graded ring. There are one sided versions, $R^{(j^+)} = \bigoplus_{m\geq 0} R_{me_j}$ and $R^{(j^-)} = \bigoplus_{m\leq 0} R_{me_j}$ that are R_0-R_0-bimodules.

Let $\Theta_j = \{R^{(j)}, R^{(j^+)}, R^{(j^-)}\}$. Consider the collection of R_0-R_0 bimodules

$$H_1 \underset{R_0}{\otimes} H_2 \ldots \underset{R_0}{\otimes} H_n,$$

for $H_j \in \Theta_j$. I claim that there is an one-to-one association between the collection of these tensor products and the collection $R_*[T_F]$ for faces F of $S = [-1,1]^n$, which is indicated by isomorphisms between associated objects.

Lemma 2.3.4. *Every $R_*[T_F]$ is isomorphic to the tensor product of the form*

$$H_1 \underset{R_0}{\otimes} H_2 \ldots \underset{R_0}{\otimes} H_n,$$

where $H_j \in \Theta_j$ is:

1. $H_j = R^{(j)}$ *when* $\pm e_j \in T_F$.

2. $H_j = R^{(j^+)}$ *when* $+e_j \in T_F$.

3. $H_j = R^{(j^-)}$ *when* $-e_j \in T_F$.

In addition the dimension of the face F is precisely the number of times H_j are of the form $R^{(j)}$ in the tensor product.

Proof. Note that for any H_i, H_j, $i \neq j$, the underlying sets of the supports satisfy Condition 2.3.1.1 and have intersection $\{0\}$. Let Q be the R_0-R_0-module

$$Q = \bigoplus_{m_i, m_j \in \mathbb{Z}} R_{m_i e_i + m_j e_j}.$$

From Proposition 2.3.1 there is an isomorphism

$$\beta\colon Q \cong R^{(i)} \underset{R_0}{\otimes} R^{(j)}\ , \quad \beta_{m_i e_i, m_j e_j}\colon r \to \sum_{\ell} r u_\ell \otimes v_\ell$$

for partition of unities of type $(-m_j e_j, m_j e_j)$. The inverse of β is the map $\pi\colon a \otimes b \to ab$.

Given a tensor product $H_1 \underset{R_0}{\otimes} H_2 \ldots \underset{R_0}{\otimes} H_n$ we can repeat the process of applying π for all $1 \leq j \leq n$, so that there is an isomorphism

$$H_1 \underset{R_0}{\otimes} H_2 \ldots \underset{R_0}{\otimes} H_n \cong \overline{Q},$$

where $\overline{Q}$ is an R_0-R_0-module consisting of sums of certain homogenous components of $R = \bigoplus_{\rho \in \mathbb{Z}^n} R_\rho$.

We look at the support of an element $r = \sum_{\rho \in \operatorname{supp}(r)} r_\rho \in \overline{Q}$, $\rho = \sum_{1 \leq j \leq n} m_j e_j \in \mathbb{Z}^n$. For all j there are three possible constraints on each of the m_j:

1. $m_j \in \mathbb{Z}$ when $H_j = R^{(j)}$.

2. $m_j \geq 0$ when $H_j = R^{(j^+)}$.

3. $m_j \leq 0$ when $H_j = R^{(j^-)}$.

It is now a matter of observation that the support of such an element has precisely the same restrictions as the support of an element within $R_*[T_F]$ where F is the unique face such that:

1. $\pm e_j \in T_F$ when $H_j = R^{(j)}$.
2. $+e_j \in T_F$ when $H_j = R^{(j^+)}$.
3. $-e_j \in T_F$ when $H_j = R^{(j^-)}$.

Hence $\overline{Q}$ is equal to $R_*[T_F]$ for a certain face. It remains to note that the dimension of the face is precisely the number of j such that both $\pm e_j \in T_F$, that this number is precisely the number of H_j of the form $R^{(j)}$ provides the result. □

Now let $\Theta_j^k = \{R^{(j)}, \bigoplus_{m \geq -k} R_{me_j}, \bigoplus_{m \leq k} R_{me_j}\}$. We can see from Proposition 2.3.1 in a similar manner to Lemma 2.3.4:

Lemma 2.3.5. *Every $R_*[k \circ T_F]$ is isomorphic as an R_0-R_0 bimodule to the tensor product of the form*

$$H_1 \underset{R_0}{\otimes} H_2 \ldots \underset{R_0}{\otimes} H_n,$$

where $H_j \in \Theta_j^k$. In addition the dimension of the face F is precisely the number of times H_j are of the form $R^{(j)}$ in the tensor product.

Proof. Again, an assessment of the conditions on the support provides the result as in Lemma 2.3.4. Note that for any H_i, H_j, $i \neq j$, the underlying sets of the supports satisfy Condition 2.3.1.1 and have intersection $\{0\}$. Take a face F of dimension $n-1$ such that $e_j \in T_F$, $-e_j \notin T_F$. Then the face is the subspace

$$\{\rho = \sum_{1 \leq i \leq n} m_i e_i \in \mathbb{Z}^n \colon -1 \leq m_i \leq +1, \, m_j = -1\}$$

for some j. Therefore, $k \circ T_F$ is bounded below by the subspace

$$\{\rho = \sum_{1 \leq i \leq n} m_i e_i \in \mathbb{Z}^m : m_j = -k\}$$

and unbounded elsewhere, which is precisely the condition on the support of elements of the tensor product

$$R^{(1)} \underset{R_0}{\otimes} R^{(2)} \dots \underset{R_0}{\otimes} \Big(\bigoplus_{m \geq -k} R_{me_j} \Big) \dots \underset{R_0}{\otimes} R^{(n)}.$$

A similar observation can be made for other choices of face F. Finally, we note that the dimension of F is precisely the number of j such that $\pm e_j \in T_F$, which is also the number of j such that H_j is of the form $R^{(j)}$ as required. □

Let Ω_j^k be the complex concentrated in levels 0 and -1

$$\Big(\bigoplus_{m \leq k} R_{me_j} \oplus \bigoplus_{m \geq -k} R_{me_j} \Big) \overset{-\iota + \iota}{\to} R^{(j)}$$

where ι are inclusions. Since $R^{(j)}$ is a $\mathbb{Z}$-graded ring, we know from Proposition 2.6 of [HS16] that $\bigoplus_{i \in [-ke_j, ke_j]} R_i \simeq \Omega_j^k$ via the inclusion of $\bigoplus_{i \in [-ke_j, ke_j]} R_i$ into both summands of $(\Omega_j^k)_0$, i.e., the diagonal map inclusion, written Δ_j, with a map ρ_j such that $\Delta_j \rho_j \simeq \mathrm{id}$, $\rho_j \Delta_j \simeq \mathrm{id}$.

Lemma 2.3.6. *There is an isomorphism of chain complexes of R_0-R_0 bimodules $\Gamma(D(k)) \cong \Omega_1^k \underset{R_0}{\otimes} \Omega_2^k \dots \Omega_n^k$.*

Proof. For ease of writing, let $H_1 \underset{R_0}{\otimes} H_2 \dots \underset{R_0}{\otimes} H_n = \overline{H}$. For the complex $\Gamma(D(k))$, non-zero only for indexes 0 to $-n$, note that at the $-i$, $0 \leq i \leq n$ chain degree we see precisely the sum of $R_*[T_F]$ for faces of dimension i in particular at the $-n$th level we see R. By quoting Lemma 2.3.5, we observe an isomorphic construction levewise on the right of the isomorphism.

It remains to argue that the chain maps are the same. The isomorphism $\overline{H} \cong R_*[T_F]$, that is the repeated application of maps $\pi \colon a \otimes b \mapsto ab$, will have no effect on the signs of the inclusion maps of the complex. On the right, any given map $\iota\colon \overline{H} \to \overline{H}'$ between summands has sign precisely equal to that of $\Gamma(D(k))$. This is best seen by considering the effect of tensoring onto the left of Ω_n^k by Ω_{n-1}^k, then each Ω_j^k from $n-2$ to 1.

Fix $1 \le j \le n$. Firstly note that, in agreement with $y_{F,F'}$, there is a negative sign in Ω_j^k for $\iota\colon R^{(-j)} \to R^j$. By observation of the convention of Definition 0.3.2, the sign of any map changes only when the individual summand of the tensor product has an entry of degree -1 tensored onto the left, i.e., precisely whenever there is $R^{(j)}$ in the summand where $i < j$, in agreement with $L_{F'\setminus G_j,j}$ for the map $R_*[T_{F'\setminus G_j}] \to R_*[T_{F'}]$ where G_j is an $n-1$ dimensional face such that $\eta_{G_j} \in \{e_j, -e_j\}$. Hence, the sign on a given map between summands on the right $\overline{H}' \to \overline{H}$ is precisely $[F, F']$ where $\overline{H} \cong R_*[T_F], \overline{H}' \cong R_*[T_{F'}]$ and $F = F' \cap G_j$, aligning with the sign on the corresponding map within $\Gamma(D(k))$. □

Lemma 2.3.7. *The complex* $\Omega_1^k \underset{R_0}{\otimes} \Omega_2^k \underset{R_0}{\otimes} \dots \underset{R_0}{\otimes} \Omega_n^k$ *is homotopy equivalent to*

$$\left(\bigoplus_{i\in[-ke_1,ke_1]} R_i\right) \underset{R_0}{\otimes} \left(\bigoplus_{i\in[-ke_2,ke_2]} R_i\right) \underset{R_0}{\otimes} \dots \underset{R_0}{\otimes} \left(\bigoplus_{i\in[-ke_n,ke_n]} R_i\right).$$

Proof. Note that for all j,

$$\Delta_j\rho_j \simeq \mathrm{id},\ \rho_j\Delta_j \simeq \mathrm{id}$$

from Proposition 2.6 of [HS16]. Also note that $\Delta_\ell\rho_\ell\otimes\Delta_j\rho_j = (\Delta_\ell\otimes\Delta_j)(\rho_\ell\otimes\rho_j)$ and $\rho_\ell\Delta_\ell \otimes \rho_j\Delta_j = (\rho_\ell \otimes \rho_j)(\Delta_\ell \otimes \Delta_j)$ for all $\ell \neq j$. Use Corollary 9.2 of [ML95] to see that

$$\Delta_1\rho_1 \otimes \Delta_2\rho_2 \otimes \cdots \otimes \Delta_n\rho_n \simeq \mathrm{id}$$

(simarly for tensor products of $\rho_j\Delta_j$) hence the map consisting of n Δ_j maps tensored together form a homotopy equivalence from

$$\left(\bigoplus_{i\in[-ke_1,ke_1]} R_i\right) \underset{R_0}{\otimes} \left(\bigoplus_{i\in[-ke_2,ke_2]} R_i\right) \underset{R_0}{\otimes} \dots \underset{R_0}{\otimes} \left(\bigoplus_{i\in[-ke_n,ke_n]} R_i\right)$$

to $\Omega_1^k \underset{R_0}{\otimes} \Omega_2^k \underset{R_0}{\otimes} \dots \underset{R_0}{\otimes} \Omega_n^k$ as required. $\square$

Proposition 2.3.8. *The complex of R_0-R_0- bimodules*

$$0 \to \bigoplus_{i \in [-k,k]^n} R_i \to \Gamma(D(k)) \to 0$$

with the non-trivial map being inclusions of $\bigoplus_{i \in [-k,k]^n} R_i$ *into each summand of* $\Gamma(D(k))_0 = \bigoplus_{\dim(F)=0} D(k)(F)$ *is exact for* $k \geq 0$.

Proof. Firstly, note that:

$$\bigoplus_{i \in [-k,k]^n} R_i \cong \Big(\bigoplus_{i \in [-ke_1, ke_1]} R_i \Big) \otimes \Big(\bigoplus_{i \in [-ke_2, ke_2]} R_i \Big) \dots \Big(\bigoplus_{i \in [-ke_n, ke_n]} R_i \Big)$$

via applications of maps of the form of β from Proposition 2.3.1 for the sets $[-ke_j, ke_j]$,

$$\Big(\bigoplus_{i \in [-ke_1, ke_1]} R_i \Big) \otimes \Big(\bigoplus_{i \in [-ke_2, ke_2]} R_i \Big) \dots \Big(\bigoplus_{i \in [-ke_n, ke_n]} R_i \Big) \simeq \Omega_1^k \otimes \Omega_2^k \otimes \dots \otimes \Omega_n^k$$

using the tensor product of maps Δ_j from Lemma 2.3.7 and

$$\Omega_1^k \otimes \Omega_2^k \otimes \dots \otimes \Omega_n^k \cong \Gamma(D(k))$$

via applications of $a \otimes b \to ab$ from Lemma 2.3.6. The composition of these maps can easily be seen to be the claimed inclusions. The result is thus proven. $\square$

2.4 Extending a complex of modules of graded rings to a complex of quasi-coherent diagrams

The next construction is to take a bounded chain complex of finitely generated free R-modules and form a bounded chain complex of quasi-coherent sheaves.

From this point there will be a number of diagrams that have chain complexes at each entry in the poset, that is they are chain complexes of diagrams. For the sake of clarity, given such a diagram Φ, we write $\Phi(p)$

as the chain complex seen at the point indexed by p in the poset while we write Φ_j for the diagram at the jth chain level and $\Phi(p)_j$ for the jth level of the complex at p.

Proposition 2.4.1. *Let C be a bounded chain complex of finitely generated free R-modules. We can form a complex of sheaves where in the centre we have the original complex C and at each level we have a sheaf of the form $D(k_j)^{m_j}$ for some $k_j \geq 0, m_j \in \mathbb{Z}, k_j \geq k_{j+1}$.*

Proof. Given C and $j \in \mathbb{Z}$, note $C_j = R^{m_j}$ for some $m_j \in \mathbb{N}$. Given $k \geq 0$, we have a collection of modules $C(F)_j = R_*[k \circ T_F]^{m_j}$ formed from restricting the support of C to the R_0-R_0 bimodule $R_*[k \circ T_F]$. The obvious inclusions satisfy the adjoint map condition of the structure maps of a quasi-coherent diagram. Hence we can always form a sheaf level-wise for any $k \geq 0$ of the form $D(k)^{m_j}$. We need to show that a chain complex of sheaves can be formed.

Consider that for all j, the boundary map of C, d_j, is an $m_j \times m_{j-1}$ matrix of maps that do not respect the grading of the ring. However, each of these maps will map every homogenous component R_p of the ring R to a finite selection of components in the image based on the image of the identity element of the ring. For each map we can find a number k_{j-1} large enough such that the support of the image of R_p is contained in a n-dimensional cube of sides $2k_{j-1}$ centered on p and a cube of this size will be enough to contain the image of every component R_p, $p \in \mathbb{Z}^n$. It follows that given a restriction $C(F)_j$ for the domain restriction of the boundary to be well defined the image must contain every component in the image of each of the components on the edge of the restriction.

For example, if we were looking at $C(v)_j = R_*[0 \circ T_v]^{m_j}$ for a vertex v, it would be possible to map into the module $R_*[k_{j-1} \circ T_v]^{m_{j-1}}$ with the obvious inclusion. It follows that the sheaf $D(k_{j-1})^{m_{j-1}}$ is a suitable codomain for the extension of the boundary map $d_j \colon C_j \to C_{j-1}$ to the sheaf $D(0)^{m_j}$, hence we have a new map $D(0)^{m_j} \to D(k_{j-1})^{m_{j-1}}$. Boundedness of C

allows a process of iteration to generate a chain complex with $D(k_i)^{m_i}$ at point i with k_i increasing as i decreases. Showing that the maps satisfy the boundary map condition follows from the fact that the central map d_j is a boundary and the other maps are restrictions of this map that commute with inclusion maps. Hence we have the complex of sheaves $\mathcal{Y}_j = D(k_j)^{m_j}$ where for all F we can form a chain complex $\mathcal{Y}(F) = C(F)$ consisting of each of the $C(F)_j$, $j \in \mathbb{Z}$ and restrictions of the boundary maps d_j at the point indexed by F. $\square$

Note that from this point $\mathcal{Y}$ is assumed to have a bounded chain complex of finitely generated free R-modules $\mathcal{Y}(S) = C$ at the point indexed by S.

By applying the same rank and incidence functions as used in Remark 2.2.10, the Čech complex $\Gamma(\mathcal{Y})$ of $\mathcal{Y}$ can be written. This will be a chain bi-complex with commutating differentials, considering the horizontal level to be the Čech complex of the relevant level of $\mathcal{Y}$. Let $\check{\Gamma}(\mathcal{Y})$ be the totalisation of $\Gamma(\mathcal{Y})$.

Corollary 2.4.2. *The complex $\check{\Gamma}(\mathcal{Y})$, considered as the totalisation of the Čech complex of $\mathcal{Y}$ where $\mathcal{Y}_j = D(k_j)^{m_j}$, is homotopy equivalent to the complex D with modules*

$$D_j = \bigoplus_{1 \leq p \leq m_j} \left(\bigoplus_{i \in [-k_j, k_j]^n} R_i \right)$$

and maps consisting of restrictions of the boundary map d of C.

Proof. We can see D is a chain complex for the same reason that $\mathcal{Y}$ is. Consider the map that levelwise is precisely sums of the map in Proposition 2.3.8. These will commute with the boundary maps as they are injective and the boundary of D is a further restriction of the boundary of C beyond that of $\Gamma(\mathcal{Y})$ and will ensure that each row of $D \to \Gamma(\mathcal{Y})$ is exact by Proposition 2.3.8, hence it follows from Lemma 0.3.6 that the totalisation of $\Gamma(\mathcal{Y})$, the chain complex $\check{\Gamma}(\mathcal{Y})$, is homotopy equivalent to D. $\square$

Remark 2.4.3. Observe that D is a bounded complex of finitely generated projective R_0-modules as, levelwise, it is a finite sum of finitely generated projective R_0-modules [HS16, Proposition 1.6]. Boundedness follows as C is bounded. We will use D as our finite domination by showing C is a retract up to homotopy of D and noting that D itself is a retract of a bounded chain complex of finitely generated free R_0-modules.

2.5 The diagram E_F

In this section we introduce the diagram E_F and its Čech complex $\Gamma(E_F)$ which will be important for the proof later on.

Definition 2.5.1 (Star of a face)**.** For $F \subset S$, define the *star of* F, $\mathrm{st}(F)$, as the set of faces of S that contain F.

Definition 2.5.2 (Nerve of $\mathrm{st}(F)$)**.** Let $\mathcal{N}_F$ be the simplicial complex consisting of flags $\mathcal{F} = \{F_0 \subset \cdots \subset F_\ell\}$ where each face F_i is contained within $\mathrm{st}(F)$, that is each face contains F.

Lemma 2.5.3. *For every $\mathcal{F} \in \mathcal{N}_F$, the ring $R_*(\!(\mathcal{F})\!)$ admits a $R_*[T_F]$-$R_*[T_F]$ bimodule structure.*

Proof. Generally speaking, we only need to consider flags where $S \notin \mathcal{F}$. A flag with S in it admits a generalised Novikov ring that can have support across $\mathbb{Z}^n$ and since $R_*[T_F]$ only has elements with finite support there is no possibility of any finiteness conditions being broken.

It remains to show for when $\mathcal{F}$ has a largest face $G \subset S$. Condition (1) becomes a requirement that the support of an element is in T_G. Since $\mathcal{F} \in \mathcal{N}_F$, then $T_F \subseteq T_G$ and since T_G is closed under addition immediately we see that we can always define a well-defined closed $R_*[T_F]$-action on the left or right as required. $\square$

We define a diagram for each $F \subseteq S$:

$$E_F \colon \mathcal{N}_F \to R_0\text{-Mod},\ E_F(\mathcal{F}) = R_*(\!(\mathcal{F})\!)$$

where the structure maps $\alpha_{\mathcal{F},\mathcal{F}'}$, $\mathcal{F} \subseteq \mathcal{F}'$ are inclusions, this follows from Lemma 1.2.21.

This diagram admits a Čech complex whenever $\mathcal{N}_F$ is given a rank map $\mathrm{rk}(\mathcal{F}) = 1 - \dim(\mathcal{F})$ (which has values 0 to $n - \dim(F) = \mathrm{codim}(F)$) and an incidence function as detailed below.

Lemma 2.5.4. *For a flag $\mathcal{F} = \{F_0 \subset \cdots \subset F_j \subset \cdots \subset F_\ell\}$ and a face F_j, let*

$$[-,-]\colon \mathcal{N}_F \times \mathcal{N}_F \to \mathbb{Z}$$

be the map where:

- $[\mathcal{F} \setminus \{F_j\}, \mathcal{F}] = (-1)^j$.
- 0 *otherwise.*

This map satisfies the three conditions (DI1),(DI2),(DI3).

Proof. Condition (DI1) is satisfied trivially. Condition (DI2) follows as for two flags $\mathcal{F}, \mathcal{F}'$ such that $\dim(\mathcal{F}) + 2 = \dim(\mathcal{F}')$, there are two faces $F_g, F_{g'}$ contained within $\mathcal{F}' = \{F_0, \ldots, F_g, \ldots F_{g'}, \ldots F_k\}$ that are not contained in $\mathcal{F}$, and precisely two flags $\mathcal{F} \cup F_g, \mathcal{F} \cup F_{g'}$ contained within $I(\mathcal{F}\colon \mathcal{F}')$. Without loss of generality, let $g' > g$. Now see that $[\mathcal{F} \cup F_g\colon \mathcal{F}'][\mathcal{F}\colon \mathcal{F} \cup F_g] = (-1)^{g'}(-1)^g$ and $[\mathcal{F} \cup F_{g'}\colon \mathcal{F}'][\mathcal{F}\colon \mathcal{F} \cup F_{g'}] = (-1)^g(-1)^{g'-1}$ hence the sum is trivial and (DI2) is satisfied. Finally consider flags $\mathcal{F} = \{F \subset F'\}$ with only two faces. The set $I(< \mathcal{F})$ contains only two flags. Then $[\{F'\}\colon \mathcal{F}]$ and $[\{F\}\colon \mathcal{F}]$ have opposite signs making the sum trivial hence (DI3) is satisfied. ☐

The above collection of signs allow us to form a Čech complex $\Gamma(E_F)$.

Definition 2.5.5. Given E_F, define $\Gamma(E_F)$ as the complex with the sum $\bigoplus_{\dim(\mathcal{F})=k} R_*((\mathcal{F}))$ at level $1-k$ and boundary map d_{1-k} consisting of sums of

$$[\mathcal{F} \setminus \{F_j\}, \mathcal{F}]\alpha_{\mathcal{F}\setminus\{F_j\},\mathcal{F}}\colon E_F(\mathcal{F} \setminus \{F_j\}) \to E_F(\mathcal{F})$$

over the collection of $\mathcal{F}$ with dimension k and all $F_j \in \mathcal{F}$ for each $\mathcal{F}$.

Remark 2.5.6. Lemma 2.5.3 tells us that $\Gamma(E_F)$ is a left-right $R_*[T_F]$-bimodule.

The following Theorem is a major step in the proof of the main result of this chapter.

Theorem 2.5.7. *The diagram E_F indexed by elements $\mathcal{N}_F$, where for $\mathcal{F} \in \mathcal{N}_F$ the entry is $R_*(\!(\mathcal{F})\!)$, has Čech complex quasi isomorphic to $R_*[T_F]$ via the $R_*[T_F]$-$R_*[T_F]$-bimodule map induced by $\sigma_F\colon E_F \to R_*[T_F]$ where $\sigma_F(\{S\})$ is the projection $R_*(\!(\{S\})\!) = R \to R_*[T_F]$ and σ_F is trivial otherwise.*

In solving 2.5.7 we follow a similar proof as seen in [HQ15], that is we split the diagram E_v into 2^n-many diagrams corresponding to the 2^n collection of orthants of $\mathbb{Z}^n$ represented by T_w where w is a vertex of S. The rest of this section is spent managing E_F to bring a bit more clarity to this diagram.

The first goal is to justify limiting our interest only to those E_F for F a vertex of S. Specifically, for a larger dimensional face G we will see that a similar proof of Proposition 2.5.7 for the diagram E_v where v is a vertex of $[-1,1]^{n-\dim(G)}$ will work for E_G.

Lemma 2.5.8. *Given a $\mathbb{Z}^n$-graded ring R and a face $F \in S = [-1,1]^n$ of dimension k there is a $\mathbb{Z}^{n-k}$ graded ring U such that for a vertex w, E_F is the same diagram as E'_w where $(E'_w)(\mathcal{F}) = U_*(\!(\mathcal{F})\!)$.*

Proof. If $k = 0$ there is nothing to prove, so let $k > 0$. Using Remark 1.4.11, we generate a ring U such that for all $\mathcal{F} \in \mathcal{N}_F$, there is a flag $\gamma(\mathcal{F}) \in \mathcal{N}_{\gamma(F)}$ where $U_*(\!(\gamma(\mathcal{F}))\!) = R_*(\!(\mathcal{F})\!)$. □

So, we restrict our treatment of Proposition 2.5.7 only to E_v where v is a vertex.

Next we present how E_F can be split into the 2^n diagrams.

Definition 2.5.9 (Intersection Rings of $R_*(\!(\mathcal{F})\!)$ with M)**.** For a flag $\mathcal{F}$ and set $M \subseteq \mathbb{Z}^n$ closed under addition of $\mathbb{Z}^n$, write $R_*(\!(\mathcal{F} \cap M)\!)$ for the collection of elements of $R_*(\!(\mathcal{F})\!)$ with support entirely contained within M.

If we naively divide each $(E_F)(\mathcal{F}) = R_*(\!(\mathcal{F})\!)$ into R_0-R_0 bimodules with support in each T_w for vertices w, we will not find that $\bigoplus_w R_*(\!(\mathcal{F} \cap T_w)\!)$ is equal to $R_*(\!(\mathcal{F})\!)$, for example R_0 will appear in each summand rather than in only one. So we need to replace T_w with some specifically shifted cones to make this splitting argument work.

Definition 2.5.10. Fix two vertices w and v. Let $F_{v,w}$ be the unique lowest dimensional face such that both v and w are contained within $F_{v,w}$. Let

$$\eta'_{w,v} = \eta_w - \eta_{F_{v,w}}$$

and define the *shifted cone of w in relation to v* as

$$\eta'_{w,v} + T_w.$$

Example 2.5.11. Going back to $S = [-1,1]^2$, if we take $v = v_{bl}$, $w = v_{tl}$, then $F_{v_{tl},v_{bl}} = \epsilon_l$ and $\eta'_{v_{tl},v_{bl}} = \eta_{v_{tl}} - \eta_{\epsilon_l} = (1,-1) - (1,0) = (0,-1)$. More generally, for all n we have $\eta'_{v,v} = \eta_v - \eta_v = 0$ and $\eta'_{-v,v} = \eta_{-v} - \eta_S = \eta_{-v}$. If $F_{w,v}$ is a 1 dimensional face then $\eta'_{w,v}$ is either e_t or $-e_t$ for some t.

What these do is shift T_w depending on how it is aligned with a fixed T_v. Now, what we claim is that

$$\bigoplus_w R_*(\!(\mathcal{F} \cap (\eta'_{w,v} + T_w))\!) = R_*(\!(\mathcal{F})\!)$$

as R_0-R_0 bimodules. We argue this by showing that $\mathbb{Z}^n = \bigcup_w (\eta'_{w,v} + T_w)$ for disjoint sets $\eta'_{w,v} + T_w$.

Proposition 2.5.12. *For a fixed v, the sets $\mathbb{Z}^n$ and $\bigcup_w (\eta'_{w,v} + T_w)$ are equal and each $\eta'_{w,v} + T_w$ are disjoint with one another.*

Proof. Pick $\rho = \rho_t e_t \in \mathbb{Z}^n$ such that each $\rho_t \neq 0$. Then there is a unique vertex w such that $\rho \in T_w$, precisely the choice of w such that for all t, $\mathrm{sgn}(w)_t = |\rho_t|/\rho_t$. Since we set each ρ_t to be non-zero, and the effect of $\eta'_{w,v}$ only removes some elements ρ' of T_w such that at least one ρ'_t is zero, ρ is still contained within $\eta'_{w,v} + T_w$.

Now let there be $k > 0$ many ρ_{u_i}, $1 \leq i \leq k$ such that $\rho_{u_i} \neq 0$ and $n-k$ many ρ_{x_i}, $1 \leq i \leq n-k$ such that $\rho_{x_i} = 0$. It is immediately clear that ρ is contained within T_w whenever $|\rho_{u_i}|/\rho_{u_i} = \mathrm{sgn}(w)_{u_i}$ for all $1 \leq i \leq k$. For every vertex z where $|\rho_{u_i}|/\rho_{u_i} = \mathrm{sgn}(z)_{u_i}$ it follows that $\rho \in T_z$. Whenever z satisfies $\mathrm{sgn}(z)_{x_i} = -\mathrm{sgn}(v)_{x_i}$, it follows that $\rho \notin w'_{z,v} \circ T_z$ as $\eta'_{z,v}$ is a sum of elements $\mathrm{sgn}(z)_t e_t$, one of which will be $-\mathrm{sgn}(v)_{x_i} e_{x_i}$. Hence, ρ is contained in the barrier lattice of the unique vertex w where $\mathrm{sgn}(w)_{u_i} = |\rho_{u_i}|/\rho_{u_i}$ for all $1 \leq i \leq k$ and $\mathrm{sgn}(w)_{x_i} = \mathrm{sgn}(v)_{x_i}$ for all $1 \leq i \leq n-k$. Since we can see that every element of $\mathbb{Z}^n$ is contained within a unique set on the right of the equation and that each $\eta'_{w,v} + T_w$ is a subset of $\mathbb{Z}^n$ we have shown the result. ☐

Corollary 2.5.13. *For a fixed vertex v and a flag $\mathcal{F}$,*

$$\bigoplus_w R_*((\mathcal{F} \cap (\eta'_{w,v} + T_w))) = R_*((\mathcal{F})).$$

Proof. Immediate from Proposition 2.5.12. ☐

For any v, w define $E_v \cap (\eta'_{w,v} + T_w)$ as the diagram indexed by flags of faces of $\mathcal{N}_v$ where for $\mathcal{F} \in \mathcal{N}_v$,

$$(E_v \cap (\eta'_{w,v} + T_w))(\mathcal{F}) = R_*((\mathcal{F} \cap (\eta'_{w,v} + T_w))).$$

Corollary 2.5.14. *For a fixed vertex v,*

$$\bigoplus_w (E_v \cap (\eta'_{w,v} + T_w)) = E_v.$$

Proof. Immediate from Corollary 2.5.13. ☐

What we will find is that whenever $w \neq v$, $\Gamma(E_v \cap (\eta'_{w,v} + T_w))$ will have trivial homology while $\Gamma(E_v \cap (\eta'_{v,v} + T_v)) = \Gamma(E_v \cap T_v)$ is homotopy equivalent to $R_*[T_v]$ as $R_*[T_v]$-$R_*[T_v]$ bimodules.

Definition 2.5.15. We call the R_0-R_0 module $R_*((\mathcal{F} \cap (\eta'_{w,v} + T_w)))$ the *intersection module of $R_*((\mathcal{F}))$ with $\eta'_{w,v} + T_w$*, or just intersection module. From this point onward, we fix v and write $\eta'_{w,v} + T_w = T'_w$ and hence

$$R_*((\mathcal{F} \cap (\eta'_{w,v} + T_w))) = R_*((\mathcal{F} \cap T'_w))$$

and $E_v \cap (\eta'_{w,v} + T_w) = E_v \cap T'_w$. Also $T'_v = T_v$.

Finally, note that a number of the $(E_v \cap T_w)(\mathcal{F}) = R_*((\mathcal{F} \cap T'_w))$ are in fact trivial.

Proposition 2.5.16. *Let $\mathcal{F} = \{F_0 \subset \cdots \subset F_\ell\} \in \mathcal{N}_v$ be such that the skeleton has non empty W (i.e., so that $S \notin \mathcal{F}$). If $\operatorname{sgn}(w)_k = -\operatorname{sgn}(v)_k$ for at least one k where $\operatorname{sgn}(w)_k e_k \in W$, then $R_*((\mathcal{F} \cap T'_w)) = 0$.*

Proof. An element r of $R_*((\mathcal{F}))$ must have support contained within T_{F_ℓ} by condition (1) of the original Novikov definition. In particular, for $\rho = \sum_j \rho_j e_j$ to be in the support of r, we require that $|\rho_k|/\rho_k = \operatorname{sgn}(w)_k$. That is, ρ_k is non-positive or non-negative, without loss of generality let $\operatorname{sgn}(w)_k = 1$ so that $\rho_k \geq 0$. However, for this element to also be within $R_*((\mathcal{F} \cap T'_w))$, then the support must be within T'_w. Since $\operatorname{sgn}(w)_k = -\operatorname{sgn}(v)_k$, the intersection of T_v with T_w is a subspace such that for a $\rho \in T_v \cap T_w$, we need $\rho_k = 0$. The application of $\eta'_{w,v}$ on T_w is precisely to remove this intersection as $\eta'_{w,v}$ is a linear combination containing $\operatorname{sgn}(w)_k e_k$. It follows that $T_v \cap T'_w$ is empty, making $R_*((\mathcal{F} \cap T'_w)) = 0$ as required. $\square$

2.6 Rough skeletons of intersection modules

We now need to describe $R_*((\mathcal{F} \cap T'_w))$ in an enlightening way, particularly since two different flags may have the same intersection with T'_w for a given w. I begin by representing w as a collection of signs

$$\{\operatorname{sgn}(w)_k,\ 1 \leq k \leq n\}$$

such that $\operatorname{sgn}(w)_k e_k \in T_w$. Whenever $\operatorname{sgn}(w)_k$ is $+1$, the support of an element of $R_*((\mathcal{F} \cap T'_w))$ is bounded such that the coefficient of e_k is never negative, the opposite case occurs when $\operatorname{sgn}(w)_k = -1$. Note that whenever $\mathcal{F} \in \mathcal{N}_v$ and $e_k \notin P$, it follows that $\operatorname{sgn}(\mathcal{F})_k = \operatorname{sgn}(v)_k$. Since we are only concerned with flags contained within a given $\mathcal{N}_v$ (it is all we need to solve Proposition 2.5.7) we now will tend to use $\operatorname{sgn}(v)$ in place of $\operatorname{sgn}(\mathcal{F})$.

For a vertex w, consider the collection of maps $\mathfrak{A}_w$

$$f\colon T'_w \to R$$

such that $f(a) \in R_a$. Similar to the Novikov ring definition, we form a precise definition of these intersection modules by restricting the collection $\mathfrak{A}_w$ by applying conditions, specifically restrictions of the original Novikov ring definitions.

For a given face F, we consider a new collection of conditions that combine both (2-F) and the need for the support to be within T'_w.

Definition 2.6.1. For a given face $F \subset S$ and vertex w, let (2-$F \cap T'_w$) refer to the following condition:

A map $f \in \mathfrak{A}_w$ satisfies (2-$F \cap T'_w$) if for all $q \in \mathbb{Z}^n$ there exists $k \gg 0$ such that

$$k\eta_F + \Big(\big(q + (-T_F)\big) \cap \operatorname{supp}(f)\Big) \subset T_F.$$

That is, though we restrict our immediate attention to elements with support contained within T'_w, we still require the ability to shift the intersection with the support into the relevant barrier lattice, even if T'_w and T_F are completely different. I hope it is clear that whenever $T'_w \subseteq T_F$, the condition (2-$F \cap T'_w$) is trivial, since $\big(q + (-T_F)\big) \cap \operatorname{supp}(f) \subseteq T'_w$ as $\operatorname{supp}(f) \subseteq T'_w$ from the definition of $\mathcal{A}_w$ so $k = 0$ is satisfactory.

Note that condition (1), that the support of an element is contained within the barrier lattice of the largest face of a flag, is either an implication of the requirement that the support is within T'_w, or directly opposed to it. In the latter case we get a trivial module $R_*((\mathcal{F} \cap T'_w))$, incidentally this occurs if and only if when there is some element of W such that $\operatorname{sgn}(F)_k e_k \in W$ and $\operatorname{sgn}(F)_k \neq \operatorname{sgn}(w)_k$ (Proposition 2.5.16). We can discuss these ideas in terms of $n-1$-dimensional faces.

Definition 2.6.2. Call a face G of dimension $n-1$ *w-aligned* if $T'_w \subseteq T_G$, making (2-$G \cap T'_w$) trivial. Conversely, if T'_w is not contained within T_G call G *w-unaligned*. Since we understand each vertex to be the intersection of n many faces of dimension $n-1$, there can be no situation where G is not one or the other. We only refer to $n-1$ dimensional faces with this terminology.

While (2-$G \cap T'_w$) is trivial for w-aligned G, we see that the condition is not trivial otherwise. When G is w-unaligned, (2-$G \cap T'_w$) will put a global bound on the support of an element so that for $\rho = \sum_j \rho_j \operatorname{sgn}(w)_j e_j \in \operatorname{supp}(f)$ it follows that $\rho_G \leq k\eta_G$ for some $k > 0$. This bound is in the opposite direction to that provided by T'_w ($\rho_G > 0$).

Now we put this discussion to work. The first thing we do is ascertain the orientation of the flag $\mathcal{F}$ in relation to T_w, in effect working out whether or not the $n-1$-faces that have faces in $\mathrm{CT}(\mathcal{F})$ contained within them are w-aligned or w-unaligned.

Like for the Novikov rings themselves, we seek to represent what the intersection modules $R_*((\mathcal{F} \cap T'_w))$ are with a skeleton structure, firstly we begin by just encoding the orientation of the flag in relation to w.

Definition 2.6.3 (Rough skeleton of the intersection module)**.** Given a flag $\mathcal{F} = \{F_0, \dots F_\ell\} \in \mathcal{N}_v$ and a vertex w, consider the collection of sets

$$\mathrm{RSK}(\mathcal{F} \cap T'_w) := \{A^P_{\min}, A^W_0, A^P_0, \dots A^W_{\ell-1}, A^P_{\ell-1}, A^W_{\max}\}$$

where:

- $A^P_{\min}$ contains elements $-\mathrm{sgn}(w)_k e_k$ of P (we discard $\mathrm{sgn}(w)_k e_k$), for $0 \leq i \leq n$.
- A^W_i contains elements in A_i where $\mathrm{sgn}(v)_k = \mathrm{sgn}(\mathcal{F})_k = \mathrm{sgn}(w)_k$ while conversely.
- A^P_i contains elements in A_i where $\mathrm{sgn}(v)_k = \mathrm{sgn}(\mathcal{F})_k \neq \mathrm{sgn}(w)_k$.
- Finally $A^W_{\max}$ contains elements of W where $\mathrm{sgn}(v)_k = \mathrm{sgn}(\mathcal{F})_k = \mathrm{sgn}(w)_k$.

We call $\mathrm{RSK}(\mathcal{F} \cap T'_w)$ the *rough skeleton of the intersection module $R_*((\mathcal{F} \cap T'_w))$*.

Note that for all i, either A^P_i or A^W_i must be non-empty.

Remark 2.6.4. Once again, the rough skeleton tells us the nature of the faces in a similar way as the skeleton does as described in Remark 1.4.8 However, we have added information. for a flag $\mathcal{F}$, such that $\mathrm{RSK}(\mathcal{F} \cap T'_w)$ has, for some i, non zero A^P_i and A^W_i sets, then

$$F_i = F_{i+1} \cap (\bigcap_{j_a} G_a) \cap (\bigcap_{j_u} G_u)$$

where G_a are w-aligned such that $\eta_{G_a} \in A_i^W$ while G_u are w-unaligned such that $\eta_{G_u} \in A_i^P$. The set $A^W_{\max}$ will tell us the nature of the largest face, precisely:

$$F_\ell = (\bigcap_{j_b} G_b)$$

for w-aligned faces G_b such that $\eta_{G_b} \in A^W_{\max}$ in particular, $W = \emptyset$ implies $F_\ell = S$. The set $A^P_{\min}$ is more interesting. We discard half of the elements of P when forming $A^P_{\min}$, to take into account the two-sided boundedness condition from the Novikov ring itself. We see that no face within $\mathcal{F}$ is contained within G or $-G$ such that either η_G or η_{-G} are in $A^P_{\min}$.

Example 2.6.5. For A_i^P, A_i^W we begin by discussing a case for $\mathbb{Z}^2$. Using the naming conventions from Section 1, consider the flag $\mathcal{F} = \{v_{bl}, S\}$ with $\mathrm{CT}(\mathcal{F}) = \{\epsilon_b, \epsilon_l\}$, the vertex v_{tl} and the intersection modules $R_*((\mathcal{F} \cap T'_{v_{tl}}))$. Then:

$$\mathrm{SK}(\mathcal{F}) = \{A_0 = \{e_x, e_y\}\}$$

while

$$\mathrm{RSK}(\mathcal{F} \cap T'_{v_{tl}}) = \{A_0^W = \{e_x\},\, A_0^P = \{e_y\}\}.$$

The condition of $\mathcal{F}$ in the direction e_x aligns with the condition from $T'_{v_{tl}}$, so the only bound on the support in this direction is that any coefficient on e_x must be non-negative. That is, given an element r of $R_*((\mathcal{F} \cap T'_{v_{tl}}))$ and fixed $q \in \mathbb{Z}^2$, the subspace

$$\{q + m_y e_y \colon m_y \in \mathbb{Z}\}$$

can be shifted infinitely often in the direction of e_x and, in general, the intersection of the subspace and the support of r will be non-trivial.

Conversely, the condition of $\mathcal{F}$ in the direction e_y does not align with the condition from $T'_{v_{tl}}$. So, as well as any coefficient on e_y needing to be non-positive, there is an additional condition to consider for the intersection that comes from (2-$\epsilon_b \cap T'_{v_{tl}}$) due to ϵ_b being v_{tl}-unaligned. Namely, that given an element r the subspace

$$\{q + m_x e_x \colon m_x \in \mathbb{Z}\}$$

can be shifted finitely often in the direction of $-e_y$ until the intersection of the subspace and the support is trivial. This is akin to a polynomial ring over a power series, e.g., $V[\![x]\!][y^{-1}]$. Note that this is a stronger condition that a similar condition on a single point q, that is the condition that q can only be shifted finitely often in the direction of $-e_y$ until the intersection of the subspace and the support is empty.

The former is a stronger condition on an entire line within $\mathbb{Z}^2$, which provides a global bound on values of y, while the latter is a condition on individual points. The latter, weaker condition is what is seen when taking the intersection of $\mathcal{F}' = \{e_l, S\}$ with $T'_{v_{tl}}$, where

$$\mathrm{SK}(\mathcal{F}) = \{P = \{\pm e_y\},\, A_0 = \{e_x\}\}$$

and

$$\mathrm{RSK}(\mathcal{F} \cap T'_{v_{tl}}) = \{A^P_{\min} = \{e_y\},\, A^W_0 = \{e_x\}\}$$

in this case the intersection will look like the power series of a polynomial, e.g., $V[y^{-1}][\![x]\!]$. The order of the sets in the intersection skeletons, like with the skeletons, are highly important. Note that since the set P contains both the positive and negative elements e_k, we pick the direction that matches $-\mathrm{sgn}(w)_k$, and hence the other A^P_i, to make the direction clear.

While there is a one-to-one correspondence between flags and rough intersection skeleton, there are potentially multiple flags with the same intersection, for example $R_*(\!(\{S\} \cap T'_w)\!) = R_*[T'_w]$ while for any other flag $\mathcal{F}$ such that $\mathrm{RSK}(\mathcal{F} \cap T'_w)$ satisfies $A^W_i = \emptyset$ for all $0 \le i \le \ell - 1$ and satisfies $A^W_{\max} = \emptyset$ it also follows that $R_*(\!(\mathcal{F} \cap T'_w)\!) = R_*[T'_w]$. We will aim to simplify these skeletons so that there is a one-to-one correspondence between simplified flags and intersections. Firstly, we will take a moment to explore the structure by considering the subspace bound conditions (Remark 1.4.10).

What $A^W_{\max}$ tells us. We can observe immediately from Proposition 2.5.16 that if there are elements of W that are not placed within $A^W_{\max}$, then $R_*((\mathcal{F} \cap T'_w))$ is 0. So, from this point we assume that the signs $\mathrm{sgn}(v)_k$ of the elements e_k of W agree with those of w. There are no bounds associated to the support of the elements of the intersection module in these directions. As asides, note that if the flag contains S, the intersection is never trivial, and for $w = v$, the intersections are never trivial.

What $A^P_{\min}$ tells us. For $1 \leq k \leq n$, let $-\mathrm{sgn}(w)_{k_p} e_{k_p} \in A^P_{\min}$. For the original generalised Novikov ring $R_*((\mathcal{F}))$, for $\pm e_{k_p} \in P$, the support of a general element of the ring will have a finiteness condition in the directions of $\pm\mathrm{sgn}(w)_{k_p} e_{k_p}$. Specifically that is for each j where $\pm e_{k_j} \in P$

$$\{q + \sum_{k=k_p} m_k e_k \colon m_k \in \mathbb{Z},\, m_{k_j} = 0\}$$

has the subspace bound condition in both $\pm e_{k_j}$ directions. When taking an element of the intersection module the above subspace will have finiteness conditions in the directions of $\mathrm{sgn}(w)_{k_j} e_{k_j}$ for $\mathrm{sgn}(w)_{k_j} e_{k_j} \in A^P_i$ while in the directions $-\mathrm{sgn}(w)_{k_j} e_{k_j}$ there will be a bound by the subspace

$$\{\sum_{k=1}^{n} \mathrm{sgn}(w)_k m_k e_k \colon m_{k_j} = 0,\, m_k \geq 0\}$$

as the support must be contained within T'_w. This is akin to a polynomial in a single indeterminate as opposed to two sided (i.e., Laurent) polynomial.

What A^P_i, A^W_i tell us, general case. More generally, for A^P_i, A^W_i we focus on $\mathrm{sgn}(v)_{k_a} e_{k_a} \in A_i$, note that the faces G_{k_a} of dimension

$$|P| + (\sum_{1 \leq j \leq i+1} |A_j|) - 1$$

are the relevant faces within $\mathrm{CT}(\mathcal{F})$ such that

$$\mathrm{sgn}(v)_{k_a} e_{k_a} \in T_{G_{k_a}},\ -\mathrm{sgn}(v)_{k_a} e_{k_a} \notin T_{G_{k_a}}.$$

Some of these conditions will place a bound in the same direction as the bound of T_w, others will not. If $\mathrm{sgn}(v)_{k_a} = \mathrm{sgn}(w)_{k_a}$ for some a, then the conditions align, and taking the intersection will replace the local finiteness condition in the $-\mathrm{sgn}(w)_{k_a}$ direction with a global bound by the subspace

$$J = \{\sum_{k=1}^{n} \mathrm{sgn}(w)_k m_k e_k \colon m_{k_a} = 0,\, m_k \geq 0\}$$

above or below depending on whether or not $\mathrm{sgn}(w)_{k_a}$ is $+1$ or -1, this is analogous to a power series condition. If $\mathrm{sgn}(v)_{k_a} \neq \mathrm{sgn}(w)_{k_a}$, then not only is the support bounded by the plane J but there will be a local bound in the opposite direction, akin to a typical polynomial condition.

Therefore the result of intersecting with T_w can be understood in terms of the skeleton. That is, the elements of A_i are split into two groups. The elements $\mathrm{sgn}(v)_{k_a} e_{k_a} \in A_i^W$ are those such that $\mathrm{sgn}(v)_{k_a} = \mathrm{sgn}(w)_{k_a}$. Let G_{k_a} be the $n-1$ dimensional face where $\eta_{G_{k_a}} = \mathrm{sgn}(w)_{k_a}$, where for a given $r \in R_*(\!(\mathcal{F})\!)$, and each $\mathrm{sgn}(v)_{k_a} e_{k_a} \in A_i^W$, the subspace $(F_{i+1} \cap G_{k_a})_q^*$ cannot, in general, be shifted in the direction $\mathrm{sgn}(w)_a e_a$ until the intersection of the support of r and the plane is empty. The elements $-\mathrm{sgn}(w)_{k_a} e_{k_a} \in A_i^P$ are those such that $\mathrm{sgn}(v)_{k_a} \neq \mathrm{sgn}(w)_{k_a}$, and $(F_{i+1} \cap G_{k_a})_q^*$ can be shifted in the direction $\mathrm{sgn}(w)_{k_a} e_{k_a}$ until the intersection of the support of r and the plane is empty.

Explaining the order of sets. Let $|A_i| = 2$ for some $1 \leq i \leq \ell - 1$, with $\mathrm{sgn}(v)_j e_j \in A_i^W$, $\mathrm{sgn}(v)_{j'} e_{j'} \in A_i^P$ and $n-1$ faces G_j, $G_{j'}$ with $\eta_{G_j} = \mathrm{sgn}(v)_j e_j$, $\eta_{G_{j'}} = \mathrm{sgn}(v)_{j'} e_{j'}$ so that G_j is w-aligned and $G_{j'}$ is w-unaligned. Let $\pm e_{k_p} \in A_{\min}^P$, $\mathrm{sgn}(v)_{k_a} e_{k_a} \in \bigcup_{x \leq i-1} A_x$. Noting that $F_i = F_{i+1} \cap G_j \cap G_{j'}$, then $\mathrm{CT}(\mathcal{F})$ has two faces whose conditions are relevant to the elements within A_i, $F_{i+1} \cap G_j$ and $F_{i+1} \cap G_{j'}$.

Consider the condition associated with the face $F_{i+1} \cap G_j$. This tells us that given an element $r \in R_*(\!(\mathcal{F} \cap T_w)\!)$, the subspace $(F_{i+1} \cap G_j)_q^*$ can be shifted only finitely often in the direction $\mathrm{sgn}(w)_j e_j$. However, $(F_{i+1} \cap G_{j'})_q^*$ has no such bound in the direction $\mathrm{sgn}(w)_j e_j$. This means that the condition is analogous to a polynomial of a power series (e.g., something like $V[[x]][y]$), rather than the other way round. If we add a face between F_i and F_{i+1} there are two choices, one of which makes no change to the intersection module and another that does. If we add $F_{i+1} \cap G_{j'}$, then while $F_{i+1} \cap G_{j'}$ is still within $\mathrm{CT}(\mathcal{F} \cup \{F_{i+1} \cap G_{j'}\})$, $F_{i+1} \cap G_j$ is not within $\mathrm{CT}(\mathcal{F} \cup \{F_{i+1} \cap G_{j'}\})$ or $\mathrm{CP}(\mathcal{F} \cup \{F_{i+1} \cap G_{j'}\})$, however we will see later that (2-$F_{i+1} \cap T'_w$) will imply (2-$(F_{i+1} \cap G_j) \cap T'_w$). Let the rough intersection skeleton of $\mathcal{F}$ be the following:

$$\{A_{\min}^P, A_0^W, A_0^P, \ldots A_i^W, A_i^P, \ldots A_{\ell-1}^W, A_{\ell-1}^P, A_{\max}^W\}.$$

The rough intersection skeleton of $\mathcal{F} \cup \{F_{i+1} \cap G_{j'}\}$ can be seen to be the following collection of sets:

$$\{A^P_{\min}, A^W_0, A^P_0, \dots A^W_i, (A')^P_{i+1}, \dots (A')^W_\ell, (A')^P_\ell, A^W_{\max}\}$$

wherefore all $j \geq i$, $A^P_j = (A')^P_{j+1}$ and for $j \geq i+1$, $A^W_j = (A')^W_{j+1}$ i.e., the only change is in the numbering of a few of the sets, which we will see is no change at all in relation to the intersection module.

If we add $F_{i+1} \cap G_j$, then we will see the following rough intersection skeleton for $\mathcal{F} \cup \{F_{i+1} \cap G_j\}$

$$\{A^P_{\min}, A^W_0, A^P_0, \dots (A'')^W_i, A^P_i, (A'')^W_{i+1}, \dots (A')^W_\ell, (A')^P_\ell, A^W_{\max}\}$$

where $(A'')^W_{i+1} = A^W_i$ and $(A'')^W_i = \emptyset$. Now we get a change, the face $F_{i+1} \cap G_{j'}$ is no longer in the caterpillar. Instead, the largest subspace such that the associated subspace bound condition provides a bound in the direction $-\mathrm{sgn}(w)_j e_j$ is associated with the face $F_i \cap G_j$ as opposed to $F_{i+1} \cap G_j$, making the condition strictly weaker. This means that the condition is analogous to a power series of a polynomial (e.g., something like $V[y][[x]]$) in contrast to the case for $\mathcal{F}$.

A similar pattern occurs when $|A_i| > 2$. It follows that when writing out the rough skeleton of the intersection $\mathrm{RSK}(\mathcal{F} \cap T'_w)$, we write the A^W_i to the left of the A^P_i. We will formally justify this idea later on.

2.7 Simplifying the rough skeletons

Now that we have put work into explaining the R_0-R_0 bimodules $R_*((\mathcal{F} \cap T'_w))$ using the rough intersection skeletons $\mathrm{RSK}(\mathcal{F} \cap T'_w)$, we wish to simplify these skeletons to further understand what these modules are. Recall that there can be multiple flags, and hence rough skeletons, that correspond to a particular intersection module. This section will introduce a system of simplifying the rough skeletons, so that there is a one-to-one correspondence between the collection of skeletons and the collection of intersections modules.

From the discussion in the previous section, as long as 'P' elements do not swap order with 'W' elements, we are free to add or take away faces as we wish with no change to the intersection. For example if A_i has two elements both of which are in A^P_i, we can add precisely one face splitting A^P_i into two pieces. Depending on the choice of face we can split A^P_i two ways. The choice of face has an effect on the associated Novikov ring but not the intersection. For A_i with both non-empty A^P_i and A^W_i, we can add faces as long as there is no face within the flag that, for at least one of $\mathrm{sgn}(v)_k e_k \in A_i^W$ and one of $\mathrm{sgn}(v)_{k'} e_{k'} \in A_i^P$, $\pm\mathrm{sgn}(w)_{k'} \in T_F$ and $-\mathrm{sgn}(w)_k \notin T_F$.

Also note, that if there is $\pm e_j \in P$ such that $\mathrm{sgn}(v)_j = \mathrm{sgn}(w)_j$ then adding new faces of lower dimension than what is there can change the intersection - namely if a face is added that places $\mathrm{sgn}(v)_j e_j$ into A_0 of the original skeleton, then taking the intersection with T'_w will put it into A_0^W. Evidentally, swapping a bounded condition with a trivial one changes the intersection non-trivially.

I will now consider what faces can be added or removed from a flag $\mathcal{F} = \{F_0 \subset \cdots \subset F_\ell\}$ whose rough intersection skeleton satisfies a certain condition without changing the intersection with T'_w itself. Broadly speaking, we can remove faces from the flag without changing the intersection whenever there are sets $A_?^P$, $A_?^W$ that are empty within the rough skeleton.

Lemma 2.7.1. *For a given flag $\mathcal{F} = \{F_0 \subset \cdots \subset F_\ell\} \in \mathcal{N}_v$ and vertex w such that $RSK(\mathcal{F} \cap T'_w)$ satisfies $A_{\ell-1}^P = \emptyset$, we have $R_*((\mathcal{F} \cap T'_w)) = R_*((\mathcal{F} \setminus \{F_\ell\} \cap T'_w))$. Similarly, for $\mathcal{J} = \{J_0 \subset \cdots \subset J_m\}$ not containing S, we can add at least one face of larger dimension than J_m without changing the intersection ring, in particular S itself.*

Proof. The rough skeleton tells us that $F_{\ell-1}$ is an intersection of w-aligned $n-1$ faces, and hence so is every face it is contained within. It follows that every condition (2-$H \cap T'_w$) for $F_{\ell-1} \subseteq H$ is trivial and condition (1) for $F_{\ell-1}$ is satisfied as $T'_w \subseteq T_{\ell-1}$. Hence, removing F_ℓ swaps conditions implied by the requirement of the support to be in T'_w with other conditions that are also implied by this requirement, hence the intersection will not change. Similarly, adding a face strictly containing J_m will not remove a condition from the tree that is not already weaker than the requirement that the support is within T'_w. In particular, we can add S. □

The following proposition is needed for the case where some A_i^P is empty.

Proposition 2.7.2. *Let G be a w-aligned $n-1$-dimensional face. Then for a face F not a subface of G or $-G$, the condition (2-$F \cap T'_w$) implies (2-$(F \cap G) \cap T'_w$).*

Proof. Assume (2-$F \cap T'_w$) is true, then for a fixed q there exists some $k > 0$ where

$$k\eta_F + \Big(\big(q + (-T_F)\big) \cap \operatorname{supp}(f)\Big) \subset T_F.$$

Since $T'_w \subset T_G$, condition (2-$G \cap T'_w$) is true for any k. Also note that for all $t \in \mathbb{Z}$, $t\eta_G \in T_F$. It follows that we can combine the two conditions thus:

$$k\eta_F + k\eta_G + \Big(\big(q + \big(-(T_F \cap T_G)\big)\big) \cap \operatorname{supp}(f)\Big) \subset T_F \cap T_G$$

which is precisely (2-$(F \cap G) \cap T'_w$) as $k\eta_F + k\eta_G = k\eta_{F\cap G}$ as required. $\square$

Lemma 2.7.3. *For a given flag $\mathcal{F} = \{F_0, \dots, F_\ell\} \in \mathcal{N}_v$ and vertex w such that* $\mathrm{RSK}(\mathcal{F} \cap T'_w)$ *satisfies $A^P_i = \emptyset$ for some $i < \ell$, we have $R_*(\!(\mathcal{F} \cap T'_w)\!) = R_*(\!(\mathcal{F} \setminus \{F_{i+1}\} \cap T'_w)\!)$.*

Proof. Immediately note that $R_*(\!(\mathcal{F} \setminus \{F_{i+1}\})\!) \subseteq R_*(\!(\mathcal{F})\!)$, so

$$R_*(\!(\mathcal{F} \setminus \{F_{i+1}\} \cap T'_w)\!) \subseteq R_*(\!(\mathcal{F} \cap T'_w)\!).$$

Only the other direction remains. Comparing $\mathrm{CT}(\mathcal{F})$ and $\mathrm{CT}(\mathcal{F} \setminus \{F_{i+1}\})$, we see that by removing the face F_{i+1}, we remove some faces of dimension $\dim(F_{i+1}) - 1 = Q$ from $\mathrm{CT}(\mathcal{F})$ and replace them with faces of dimension $\dim(F_{i+2}) - 1 = Q'$. No other changes will occur, so we focus on the faces of dimension Q and Q'. It is enough to show that every condition associated to faces within $\mathrm{CT}(\mathcal{F} \setminus \{F_{i+1}\})$ is implied by conditions associated with faces of $\mathrm{CT}(\mathcal{F})$. Hence we restrict our consideration to faces of dimension $\dim(F_{i+2}) - 1 = Q'$ as this is the only dimension that can have any new faces in $\mathrm{CT}(\mathcal{F} \setminus \{F_{i+1}\})$ that were not within $\mathrm{CT}(\mathcal{F})$.

If $F \in \mathrm{CT}(\mathcal{F})$ then there is nothing to prove, and $F_{i+1} \subseteq F \subseteq F_{i+2}$. Otherwise, since $A^P_i = \emptyset$, if F contains F_{i+1} it follows that $F_{i+2} \cap G_k = F$ for some w-aligned G_k. Then Remark 2.7.2 tells us that (2-$F_{i+2} \cap T'_w$) implies (2-$(F_{i+2} \cap G_k) \cap T'_w$) which is precisely (2-$F \cap T'_w$). Hence the conditions associated with the faces of $\mathrm{CT}(\mathcal{F} \setminus \{F_{i+1}\})$ are implied by the conditions associated with the faces of $\mathrm{CT}(\mathcal{F})$, so $R_*(\!(\mathcal{F} \cap T'_w)\!) \subseteq R_*(\!(\mathcal{F} \setminus \{F_{i+1}\} \cap T'_w)\!)$ as required. $\square$

Before tackling the $A_?^W = \emptyset$ case, we need a few more results.

Lemma 2.7.4. *Let F be such that $\pm e_t \in T_F$, w a vertex, and G_t an $n-1$ dimensional face that is w-unaligned. Then for $q = \sum_{j=1}^n q_j e_j \in \mathbb{Z}^n$*

$$q + (-T_F) \cap T'_w = (q - q_t e_t) + \big(-(T_F \cap T_{G_t})\big) \cap T'_w.$$

Proof. Consider just T_w rather than T'_w to begin with. Without loss of generality, let $\mathrm{sgn}(w)_t = 1$ so that $\eta_{G_t} = -1$. Firstly note that $T_w \subset q + (-T_{G_t})$ if and only if $q_t \leq 0$. For example $0 \in T_w$ is not contained within $e_t + (-T_{G_t})$ but is contained within $-e_t + (-T_{G_t})$. Hence, for all $q \in \mathbb{Z}^n$, $T_w \subseteq q - q_t e_t + (-T_{G_t})$. Now observe that $q + (-T_F) = q - q_t e_t + (-T_F)$ as an element $\mu = \sum \mu_j e_j \in T_F$ is allowed to have both positive and negative values of μ_t, as $\pm e_t \in T_F$ (so that, in addition $F \notin G_t$). It follows that

$$\begin{aligned} q + (-T_F) \cap T_w &= q - q_t e_t + (-T_F) \cap T_w \\ &= (q - q_t e_t + (-T_F) \cap T_w) \cap (q - q_t e_t + (-T_{G_t}) \cap T_w) \\ &= \Big(q - q_t e_t + \big(-(T_F \cap T_{G_t})\big)\Big) \cap T_w \end{aligned}$$

and simply replacing T_w with T'_w gives the result as required. □

Corollary 2.7.5. *Let $\pm e_t \in T_F$, w a vertex and G_t an $n-1$-dimensional face such that $w_{G_t} = -\mathrm{sgn}(w)_t e_t$, i.e., that is w unaligned. Condition (2-$(F \cap G_t) \cap T'_w$) implies condition (2-$F \cap T'_w$).*

Proof. Fix q. Using (2-$(F \cap G_t) \cap T'_w$), we generate k such that

$$k\eta_{F \cap G_t} + \big(q - q_t e_t + (-T_F \cap T_{G_t})\big) \cap \mathrm{supp}(f) \subseteq T_F \cap T_{G_t}.$$

Note $T_F \cap T_{G_t} \subseteq T_F$ and use Lemma 2.7.4 and the fact that $\mathrm{supp}(f) \in T'_w$ to see that this is equivalent to

$$k\eta_F + k\eta_{G_t} + \Big(q + (-T_F)\Big) \cap \mathrm{supp}(f) \subseteq T_F.$$

Finally, note that if $x \in T_F$, so is $x + k\eta_{G_t}$ for any $k \in \mathbb{Z}$ as $\pm e_t \in T_F$ hence

$$k\eta_F + \Big(q + (-T_F)\Big) \cap \mathrm{supp}(f) \subseteq T_F$$

which is precisely (2-$F \cap T'_w$) as required. □

Lemma 2.7.6. *For a flag $\mathcal{F}$, let $A_0^W = \emptyset$. Then*

$$R_*((\mathcal{F} \cap T'_w)) = R_*((\mathcal{F} \setminus \{F_0\} \cap T'_w)).$$

Proof. The proof follows a similar pattern to that used when $A_?^P$ was empty. Firstly note $R_*((\mathcal{F} \setminus \{F_0\} \cap T'_w)) \subseteq R_*((\mathcal{F} \cap T'_w))$. Consider a face $F \in \mathrm{CT}(\mathcal{F} \setminus \{F_0\})$ of dimension $\dim(F_1) - 1$. We want to prove that the conditions associated with faces in $\mathrm{CP}(\mathcal{F})$ will imply the condition associated with F. Observe that by Corollary 2.7.5, (2-$F \cap T'_w$) is implied by (2-$(F \cap G) \cap T'_w$) where $\eta_G = -\mathrm{sgn}(w)_G e_G$ and $\pm e_G$. Since $F_0 = F \cap (\bigcap_i G_i)$ for $\eta_{G_i} = -\mathrm{sgn}(w)_{G_i} e_{G_i} \in A_0^P$, we see that by iterating this process (2-$F_0 \cap T'_w$) will imply (2-$F \cap T'_w$) as required. □

Lemma 2.7.7. *For a flag $\mathcal{F}$, let $A_i^W = \emptyset$. Then*

$$R_*((\mathcal{F} \cap T'_w)) = R_*((\mathcal{F} \setminus \{F_i\} \cap T'_w)).$$

Proof. Effectively identical to Lemma 2.7.6. □

We now attempt to gleam some information on what faces can be added into a flag.

Corollary 2.7.8. *Given a flag $\mathcal{F}$ such that for $A_i^P = \emptyset$, $A_{i+1}^W = \emptyset$, the intersection modules $R_*((\mathcal{F} \cap T'_w))$ and $R_*((\mathcal{F} \setminus \{F_{i+1}\} \cap T'_w))$ are the same.*

Proof. This follows from 2.7.7 and 2.7.3. □

Proposition 2.7.9. *For a given flag $\mathcal{F}$ with non-zero A_i^W, A_i^P, there is a single possible face of dimension*

$$|A_{\min}^P| + \sum_{0 \le j \le i} |A_j^W| + \sum_{0 \le j \le i-1} |A_j^P|,$$

that can be added without changing the intersection module.

Proof. That a face can be added follows from Corollary 2.7.8, since if we are given a flag where $A_i^P = \emptyset = A_{i+1}^W$ and remove F_{i+1}, we are left with a flag with an intersection skeleton where A_i^W, A_i^P are non-empty. Let $\mathrm{sgn}(w)_{k_t} e_{k_t} \in A_i^W$, $-\mathrm{sgn}(w)_{k_p} e_{k_p} \in A_i^P$. Also let G_{k_t} be the $n-1$ dimensional face such that $\eta_{G_{k_t}} = \mathrm{sgn}(w)_{k_t} e_{k_t}$ and G'_{k_p} be the $n-1$ dimensional face such that $\eta_{G'_{k_p}} = -\mathrm{sgn}(w)_{k_p} e_{k_p}$. I claim that the face H such that $H \cap (\bigcap_t G_{k_t}) = F_i$ is this face, so that its presence or absence from $\mathcal{F}$ makes no difference to the intersection module. The set $\mathrm{CT}(\mathcal{F} \cup H)$ is the same as $\mathrm{CT}(\mathcal{F})$ except faces are removed at dimension $\dim F_{i+1} - 1$ and added at $\dim H - 1 = \dim F_{i+1} - |A_i^P| - 1$. It is enough to show that for any other face $F_i \subset H' \subset F_{i+1}$, $\dim H' = \dim H$, $H' \neq H$, $R_*(\!(\mathcal{F} \cup H')\!)$ cannot possibly be contained within $R_*(\!(\mathcal{F})\!)$.

For any $k_t = k_r$ and $k_p = k_s$, let H' be such that

$$H' \cap (\bigcap_{t \neq r} G_{k_t}) \cap G'_{k_s} = F_i,$$

equivalently

$$H' = F_{i+1} \cap G_{k_r} \cap (\bigcap_{p \neq s} G'_{k_p}),$$

so that H' is not a subface of G'_{k_s}. Then the face $F_{i+1} \cap G'_{k_s}$ is in $\mathrm{CT}(\mathcal{F})$ but not $\mathrm{CT}(\mathcal{F} \cup H')$ or even $\mathrm{CP}(\mathcal{F} \cup H')$. Observe also that $H' \cap G'_{k_s} \in \mathrm{CT}(\mathcal{F} \cup H')$.

The subspace $(F_{i+1} \cap G'_{k_s})^*_q$ is larger than $(H' \cap G'_{k_s})^*_q$ and both provide the strongest subspace bound conditions in the direction $-\mathrm{sgn}(w)_{k_s} e_{k_s}$ for their respective flags ($\mathcal{F}$ and $\mathcal{F} \cup H'$). None of the conditions associated with a face in $\mathrm{CT}(\mathcal{F} \cup H')$ will imply the condition associated with $F_{i+1} \cap G'_{k_s}$ from any argument in our previous Lemmas. To prove the inequality, consider that any map $f_Y \colon T'_w \to R$, $f_Y(a) \in R_a$ with infinite support concentrated in every entry of the following set

$$\{\eta'_{w,v} + x(\mathrm{sgn}(w)_{k_s} e_{k_s} + \mathrm{sgn}(w)_{k_r} e_{k_r}) \colon x \geq 0\}$$

will be in $R_*(\!(\mathcal{F} \cup H')\!)$, as e_{k_r} will not be among the e_k that have unbounded coefficients in the subspace $(H' \cap G'_{k_s})^*_0$ (in paticular it does not contain the line $\{me_{k_r} : m \in \mathbb{Z}\}$) and every smaller face will only put conditions on strictly smaller subspaces. Also there is no condition at all that puts a bound on the support in the direction of $\operatorname{sgn}(w)_{k_r}$ as G_{k_r} is w-aligned and the set is fully bounded in all other directions. However it definitely will not be in $R_*(\!(\mathcal{F})\!)$, as it clashes with the plane bound condition of $(F_{i+1} \cap G'_{k_s})^*_0$ in the direction $\operatorname{sgn}(w)_{k_s} e_{k_s}$ since the larger subspace will have unbounded coefficients of e_{k_r}. That is, for every $y \geq 1$, $(F_{i+1} \cap G'_{k_s})^*_{y(\operatorname{sgn}(w)_{k_r} e_{k_r})}$ will have non empty intersection with $\operatorname{supp}(f_Y)$, making satisfying (2-$F_{i+1} \cap G'_{k_s}$) impossible, while given $q = \sum_k q_k e_k$ the subspace $(H' \cap G'_{k_s})^*_q$ will have trivial intersection with $\operatorname{supp}(f_Y)$ for all $q_{k_s} \neq q_{k_r}$.

It follows that choice of face is unique, precisely the face H that leaves the conditions (2-$F_{i+1} \cap G'_{k_p}$) in $\operatorname{CT}(\mathcal{F} \cup H)$, which is precisely the case where $H = F_{i+1} \cap (\bigcap_p G'_{k_p})$, $H \cap (\bigcap_t G_{k_t}) = F_i$. □

The removed faces would affect the rough skeleton of the intersection by combining adjacent A^W_i, A^W_{i+1} or A^P_i, A^P_{i+1} even if the nature of the intersection module is unchanged. To better represent what intersection is related to a given flag, we simply remove all the faces that can be removed by the above argument, leaving a unique flag with a minimal collection of faces associated with a given intersection module.

Definition 2.7.10. Given vertices v, w, a flag $\mathcal{F} \in \mathcal{N}_v$ and an associated intersection module $R_*(\!(\mathcal{F} \cap T'_w)\!)$ define the *intersection skeleton of* $R_*(\!(\mathcal{F} \cap T'_w)\!)$, written $\operatorname{SK}(\mathcal{F} \cap T'_w)$, by taking the union of adjacent A^W_i or A^P_i within the rough skeleton of $R_*(\!(\mathcal{F} \cap T'_w)\!)$, $\operatorname{RSK}(\mathcal{F} \cap T'_w)$, and renumbering as necessary forming

$$\mathcal{I} = \{I^P_{\min}, I^W_0, I^P_0, \ldots I^W_{m-1}, I^P_{m-1}, I^W_{\max}\},$$

which will resemble the rough skeleton of the unique flag with the minimum possible faces (m) with a given intersection. A similar procedure to Proposition 2.7.9 can be used to show that each intesection skeleton corresponds to an unique intersection module.

Though mentioned previously, the following Corollary is worth spelling out for total avoidance of doubt.

Corollary 2.7.11. *Given any intersection skeleton $\mathcal{I}$, there is a unique minimal face $\mathcal{F}_{\mathcal{I}}$ such that* $\mathrm{RSK}(\mathcal{F}_{\mathcal{I}} \cap T'_w) = \mathrm{SK}(\mathcal{F}_{\mathcal{I}} \cap T'_w) = \mathcal{I}$.

Proof. A result of Proposition 2.7.9. □

2.8 Organising the intersections and their skeletons

To make proper use of the intersection skeletons, we have to fix some vertex v. We now only ever look at flags $\mathcal{F} \in \mathcal{N}_v$, so we use the $\mathrm{sgn}(v)_k$ notation when relevant. The choice of v doesn't matter beyond orientation.

We use the definition of intersection skeleton to illuminate claims on the flags associated to a given intersection. Fixing a vertex w let

$$\mathcal{I} = \{I^P_{\min}, I^W_0, I^P_0, \ldots I^W_{m-1}, I^P_{m-1}, I^W_{\max}\}$$

be the intersection skeleton of any given intersection module $R_*((\mathcal{F} \cap T'_w))$, so that $\mathrm{SK}(\mathcal{F} \cap T'_w) = \mathcal{I}$ with $\mathcal{F} \in \mathcal{N}_v$ and let $\underline{\mathcal{I}}$ be the collection of flags in $\mathcal{N}_v$ that have this intersection module. The collection of skeletons can be ordered by considering both how many elements are contained in $I^W_?$ sets and in what configuration (σ) they are. The largest intersection occurs when every possible element is contained within $I^W_{\max}$, that is the elements $\mathrm{sgn}(v)_k e_k$, where $\mathrm{sgn}(v)_k = \mathrm{sgn}(w)_k$. Given a pair v, w of vertices, let there be y many elements such that $\mathrm{sgn}(v)_k = \mathrm{sgn}(w)_k$.

The way of sorting through these intersections is by taking consideration of the unique minimal flag $\mathcal{F}_{\mathcal{I}}$ associated to an intersection. The flag $\mathcal{F}_{\mathcal{I}}$ will have precisely m many faces within it at dimensions

$$|I^P_{\min}| + \sum_{0 \le q \le m-1} (|I^W_q| + |I^P_q|).$$

If we fix a configuration σ of the n elements and let $K = \{0, 1, \ldots, k\}$, then the possible number of skeletons is precisely the number of elements of $(N-Y)^{y+1}$ such that the sum of the elements of each $N-Y$ is $n-y$ - there are precisely $n-y$ elements to place between or around the y elements. It follows that each intersection can be represented by a unique element of the set

$$Z = \{(a_j)_{1 \le j \le y+1} \in (N-Y)^{y+1}, \sum_{1 \le j \le y+1} a_j = n-y\}.$$

We are interested in the restriction of the lexicographical ordering of $(N-Y)^{y+1}$ to Z.

Definition 2.8.1. Call Z equipped with the sub-ordering of the lexicographical ordering of $(N-Y)^{y+1}$ the *special intersection ordering.*

Among the $y+1$ components of Z (i.e., each $N-Y$), let the 'largest' represent the number of the $n-y$ elements in the set $I^P_{\min}$ 'smaller' than every element in every $I^W_?$ (that is, in a set to the left in the intersection skeleton), the next largest those smaller than all but one, etc. I claim that this ordering corresponds to the ordering of intersection modules under inclusion whenever the maximum number of $\operatorname{sgn}(v)_k e_k$, where $\operatorname{sgn}(v)_k = \operatorname{sgn}(w)_k$ are contained within some $I^W_?$ (i.e., y many of them). Specifically, I claim that an intersection module associated to a higher level of Z cannot be contained to one associated to a lower one.

Lemma 2.8.2. *For two intersections $\mathcal{I}, \mathcal{I}'$ such that under the lexicographical ordering the skeleton of $\mathcal{I}$ is strictly greater than $\mathcal{I}'$ and the configuration (σ^W) and number of the elements within the collections of sets $I^W_?$ and $I'^W_?$ are identical for $\mathcal{I}$ and $\mathcal{I}'$ respectively, $\mathcal{I}$ does not contain $\mathcal{I}'$.*

Proof. For simplicity, let $\mathcal{I}$ be directly above $\mathcal{I}'$ in the special intersection ordering of Z. Write

$$\operatorname{SK}(\mathcal{I}) = \{I^P_{\min}, I^W_0, \dots, I^W_i, I^P_i, I^W_{i+1} \dots, I^P_{m-1}, I^W_{\max}\}.$$

Consider $\mathcal{F}_{\mathcal{I}}$ and $\mathcal{F}_{\mathcal{I}'}$. Begin by taking the case where there exists at least one i and at least one $\operatorname{sgn}(w)_k e_k \in I^W_i$ and $-\operatorname{sgn}(w)_j e_j \in I^P_{i-1}$ such that $\operatorname{sgn}(w)_k e_k \in (I')^W_t, \operatorname{sgn}(w)_j e_j \in (I')^P_s$ where $t \le s$. There is a face H within $\operatorname{CT}(\mathcal{F}_{\mathcal{I}'})$ such that $(H)^*_q$ has a subspace bound condition in the direction $\operatorname{sgn}(w)_j e_j$ and $(H)^*_q$ contains the subspace

$$\{q + m_k e_k \colon m_k \in \mathbb{Z}\}.$$

Similarly to a previous argument, a map $f\colon T'_w \to R$, $f(a) \in R_a$ with infinite support concentrated in every element of the set

$$\{\eta'_{w,v} + x(\operatorname{sgn}(w)_j e_j + \operatorname{sgn}(w)_k e_k) \colon x \ge 0\}$$

cannot possibly satisfy (2-$H \cap T'_w$). However, we can see, owing to the position of the elements in the sets of $\operatorname{RSK}(\mathcal{F}_{\mathcal{I}} \cap T'_w)$, that such a map has no impedement to being included within $R_*((\mathcal{F}_{\mathcal{I}} \cap T'_w))$, since there are no subspace bound conditions that fix a limit on how far a subspace containing $m_k e_k$ can be shifted in the direction $\mathrm{sgn}(w)_j e_j$ before the intersection with the support of an element is trivial.

So it is evident, due to the absence of the stronger subspace bound condition, that $R_*((\mathcal{F}_{\mathcal{I}} \cap T'_w))$ is not contained within $R_*((\mathcal{F}_{\mathcal{I}'} \cap T'_w))$ as required.

If the sets $I^P_{\min}, I^W_{\max}$ are involved, or I and I' are not adjacent in the ordering of Z, the result follows with the same argument.

That the result holds for any pair of flags such that their intersection skeletons are $\mathcal{I}$ and $\mathcal{I}'$ respectively follows from the above and the work in Section 2.7. $\square$

By changing the order of the elements, there may be differing intersections, depending on the configuration of $I^P_?$ and $I^W_?$ sets. If a flag $\mathcal{F}$ within $\mathcal{N}_v$ has the maximum possible number of elements within $I^W_?$ sets (y) (it can have fewer, we cover this later), then to consider the total possibilities there are three things to look at:

- What $I^W_?$ the y elements are in.
- What $I^P_?$ the $n - y$ elements are in.
- The order (configuration) σ^W of the y elements.
- The order (configuration) σ^P of the $n - y$ elements.

We now consider the order of the intersections for differing y and σ^W, σ^P.

Remark 2.8.3. Fix an element z of Z (the special intersection ordering). For two given flags within $\mathcal{N}_v$ with differing configurations σ^W_1, σ^W_2 of the y elements but both at z in Z, either the intersections are equal (i.e., when the ordering has no effect, which will happen, for example, when all of the y elements are in $I^W_{\max}$) or neither are contained in the other (there will be clashing plane bound conditions, i.e., ones present in one but not the other like in Lemma 2.8.2). Similarly, we get the same thing for differing orderings σ^P_1, σ^P_2 of the $n - y$ elements.

Lemma 2.8.4. *Given two intersection modules A, B associated to intersection skeletons $\mathcal{I}_A, \mathcal{I}_B$ such that for all i,*

$$|(I_A)_i^P| = |(I_B)_i^P|,\ |(I_A)_{\min}^P| = |(I_B)_{\min}^P|,$$
$$|(I_A)_i^W| = |(I_B)_i^W|,\ |(I_A)_{\max}^W| = |(I_B)_{\max}^W|$$

but there is some j such that

$$(I_A)_j^P \neq (I_B)_j^P, \qquad (I_A)_{\min}^P \neq (I_B)_{\min}^P,$$
$$(I_A)_j^W \neq (I_B)_j^W, \qquad \text{and/or} \qquad (I_A)_{\max}^W \neq (I_B)_{\max}^W$$

then A cannot contain or be contained within B.

Proof. This is precisely what happens for two intersections at the same level of Z but associated to different configurations. A similar argument as seen in Lemma 2.8.2 tells us that there will be clashing plane bound conditions which tells us enough for this result. □

If there are two flags at different levels of Z with different orderings of the y and $n-y$ elements, it is enough for my uses to note that the larger by Z cannot possibly be contained within the smaller, and in particular they cannot be equal, which follows from the previous two Lemmas.

The final thing to consider is whenever there are fewer than y elements within $I_?^W$ sets. This can only happen if one of the $-\mathrm{sgn}(w)_k e_k$ where $\mathrm{sgn}(w)_k = \mathrm{sgn}(v)_k$ is contained within $I_{\min}^P$ — anything else would put $\mathrm{sgn}(w)_k e_k$ in some $I_?^W$. We can still order the possible intersections by the lexicographical ordering. For example, if there are only $y-1$ elements within $I_?^W$ instead of taking a restriction of $(N-Y)^{y+1}$ we restrict $(N-Y)^y$ to

$$Z' = \{(a_j)_{1\leq j\leq y} \in (N-Y)^y,\ \sum_{1\leq j\leq y} a_j = n-y\}$$

(that is, there is one less place to put the $n-y$ elements, as we fix the position of one of the y). In this case, if we fix an ordering of the y and $n-y$ elements, and pick two flags $\mathcal{F}$ and $\mathcal{F}'$ where for the latter we place any one of the y elements within $I_{\min}^P$, it is clear that $R_*((\mathcal{F}\cap T'_w))$ cannot be contained within $R_*((\mathcal{F}'\cap T'_w))$. This is clearly because there is an extra bound condition in the direction of some element $\mathrm{sgn}(w)_k e_k$ that was not present for $R_*((\mathcal{F} \cap T'_w))$, for example an element within support equal to $\{\eta'_{w,v} + x\mathrm{sgn}(w)_k e_k : x \geq 0\}$ is in $R_*((\mathcal{F} \cap T'_w))$ but not $R_*((\mathcal{F}' \cap T'_w))$. We can use this argument for all $0 \leq i \leq y$.

Comparing two intersection modules with different numbers of elements in $I^W_?$ is covered in the following Lemma.

Lemma 2.8.5. *Let there be y many k such that $\mathrm{sgn}(v)_k = \mathrm{sgn}(w)_k$. Let $\mathcal{I}$ have y and $\mathcal{I}'$ have $y - 1$ elements in $I^W_?$ sets and let the orders of the elements be in any configurations for the two skeletons. Then $R_*((\mathcal{F}_{\mathcal{I}'} \cap T'_w))$ does not contain $R_*((\mathcal{F}_{\mathcal{I}} \cap T'_w))$.*

Proof. Let t be such that $\mathrm{sgn}(w)_t e_t \in I^W_?$ while $-\mathrm{sgn}(w)_t e_t \in (I')^P_{\min}$. Now simply note that due to the presence of a face in the Christmas tree of $\mathcal{F}_{\mathcal{I}'}$ that puts some kind of local bound on the support in the direction of $-\mathrm{sgn}(w)_t e_t$, $R_*((\mathcal{F}_{\mathcal{I}}))$ can't be included into $R_*((\mathcal{F}_{\mathcal{I}'}))$ as required. □

Hence we have shown that for a fixed y and configurations σ^W, σ^P, the special intersection ordering Z given above orders the collection of intersection modules as wished. Similarly, varying σ^W, σ^P does not conflict with the ordering, i.e. being on one level of Z with some configuration of σ^W, σ^P does not make the intersection module smaller to something at the same or a lower level of Z with a different configuration. Also intersection modules associated with elements in Z', with smaller y, are no greater than or equal to those associated with elements in Z. Collectively, this organisation will allow us to construct a filtration on $E_v \cap T'_w$.

2.9 The homology of the Čech complex of E_F

As progress towards proving Proposition 2.5.7, we now wish to find the parity of the number of flags that are associated with a certain intersection.

Proposition 2.9.1. *Given a vertex w, an intersection skeleton $\mathcal{I}$ and a collection of flags $\mathcal{F} \in \underline{\mathcal{I}}$ such that $\mathrm{SK}(\mathcal{F} \cap T'_w) = \mathcal{I}$, the number of flags within $\underline{\mathcal{I}}$ is even for all z, unless $I^W_0 = \emptyset$, and $v = w$.*

Proof. Firstly, let $I^W_{\max} \neq \emptyset$. Given a flag within $\underline{\mathcal{I}}$ that does not contain S, we know that we can add S without affecting the intersection module -

we have assumed that every element of W must satisfy $\operatorname{sgn}(v)_k = \operatorname{sgn}(w)_k$. Conversely, if the flag does contain S, then as long as $I^W_{\max} \neq \emptyset$, it follows that $A^P_{\ell-1} = \emptyset$ hence Lemma 2.7.1 tells us we can remove S.

If $I^W_{\max} = \emptyset$ but $I^W_0 \neq \emptyset$, then from Proposition 2.7.9 we have precisely two choices at dimension $|I^P_{\min}| + |I^W_0|$, either there is an unique face, or no face at all. This is the face F such that only for all $\pm e_k \in I^P_{\min} \cup I^W_0$, $\pm e_k \in T_F$. Anything else would non-trivially 'swap' the order of conditions and change the intersection module. Hence, there are a even number of flags within $\underline{\mathcal{I}}$ that can be paired off in a similar way to before, those with F and those without.

If $I^W_{\max} = \emptyset$ and $I^W_0 = \emptyset$ then every element is within $I^P_{\min}$. If $v \neq w$, there are $y < n$ many k such that $\operatorname{sgn}(v)_k = \operatorname{sgn}(w)_k$. It follows that, at dimension y, there is precisely one possible face F, where for all e_j such that $\operatorname{sgn}(v)_j = \operatorname{sgn}(w)_j$, we have $\pm e_j \in T_F$. This is because the only way these can have a bounded condition in the direction of $\operatorname{sgn}(w)_j$ is if there is a two-sided condition that comes from having both $\pm e_j$ within the smallest face in the flag, so we also now know that this face is the smallest dimension of face possibly present within the flag. However, this face may not be present in the flag (but will always be in the caterpillar), hence we again see that there are even many choices here, by pairing of flags with F and those without.

Now consider the remaining case, where $v = w$. The only condition that needs to be satisfied to produce an intersection skeleton with $I^W_0 = \emptyset$ is that all k such that $\operatorname{sgn}(v)_k = \operatorname{sgn}(w)_k$ are contained with $A^P_{\min}$ for the rough skeleton of any flag associated to the intersection module. One other condition that these flags satisfy is that they contain S. As $v = w$, we have $y = n$. Hence, whenever $I^W_{\max} = \emptyset$, $I^W_0 = \emptyset$ it follows from the argument in the previous paragraph that every $\operatorname{sgn}(w)_k e_k$ is contained within $A^P_{\min}$ for the rough skeletons of all flags associated to the intersection module. However, the only satisfactory flag in the case $v = w$ is $\{S\}$ as for all $1 \leq k \leq n$, $\operatorname{sgn}(w)_k = \operatorname{sgn}(v)_k$. $\square$

So at long last we can now prove Theorem 2.5.7.

Theorem 2.5.7. The diagram E_F indexed by elements $\mathcal{N}_F$, where for $\mathcal{F} \in \mathcal{N}_F$ the entry is $R_*((\mathcal{F}))$, has Čech complex quasi isomorphic to $R_*[T_F]$ via the $R_*[T_F]$-$R_*[T_F]$-bimodule map induced by $\sigma_F \colon E_F \to R_*[T_F]$ where $\sigma_F(\{S\})$ is the projection $R_*((\{S\})) = R \to R_*[T_F]$ and σ_F is trivial otherwise.

Proof. Using the argument from Lemma 2.5.8, observe that we only need to consider the cases for $F = v$ where v is a vertex of S. We begin by showing that for $v \neq w$, $E_v \cap T'_w$ has a Čech complex with trivial homology.

To argue that the Čech complex of the diagram has trivial homology, we form a filtration of diagrams

$$X_0 = E_v \cap T'_w \xrightarrow{x_1} X_1 \cdots \to 0$$

which will eventually terminate at the zero diagram. The kernel of each map x_i will have a Čech complex with trivial homology, hence we can show that $\Gamma(E_v \cap T'_w)$ has trivial homology by noting that for all i, the following sequence is short exact:

$$0 \to \Gamma(\ker x_i) \to \Gamma(X_{i-1}) \xrightarrow{\Gamma(x_i)} \Gamma(X_i) \to 0 \qquad (2.9.1.1)$$

making each induced map $\Gamma(x_i)$ a quasi-isomorphism and hence $\Gamma(E_v \cap T'_w)$ quasi-isomorphic to 0 as wished. Begin with $w \neq v$ and $w \neq -v$. Let y be the number of k such that $\operatorname{sgn}(w)_k = \operatorname{sgn}(v)_k$. Broadly speaking, the filtrations follow the following pattern:

1. We take flags such that the entries indexed by the flags in $E_v \cap T_w$ are associated to y, in the sense they have y elements in sets $I^W_?$ (the maximum number of values possible). We also take all the configurations σ^W, σ^P, of the y and $n-y$ elements. Call these entries $(E_v)_y$.

2. Now, we take the $(E_v)_y$ associated to the highest element $z_{\max}$ of the special intersection ordering of Z. Configurations of σ^W and σ^P do not matter for intersection skeletons associated to this level of Z.

3. We now take the entries indexed by maximal flags ($n+1$ faces) and remove them, pairing them off with flags of n faces using Lemma 2.9.1, in this case we use the fact that for all of these flags removing S does not change the intersection to form pairs.

4. Staying at $z_{\max}$, we now remove all the remaining entries indexed by flags with n faces, one of them S and their partner without S, ignoring configurations of the y and $n-y$ elements, associated to this intersection. We continue in this manner until we have cleared all of the entries indexed by the flags with this intersection module, noting that there is an even number of such flags from Lemma 2.9.1.

5. We then repeat for $z_{\max}-1$, until reaching $z_{\min}$.

6. After clearing every z associated with y we repeat for $y-1$, moving down the ordering of $Z' = \{(a_j)_{1\leq j\leq y} \in (N-Y)^y, \sum\limits_{1\leq j\leq y} a_j = n-y\}$ in much the same way.

7. Eventually, we will reach $y=0$, and clear every entry. When we are left with the case of $w=v$, we will end up with $R_*[T_v]$ rather than nothing.

Picking $v \neq w \neq -v$ to begin with, there is at least one k such that $\operatorname{sgn}(v)_k \neq \operatorname{sgn}(w)_k$, let there be y many k in this case. Pick a flag $\mathcal{F}$ such that $\mathrm{SK}(\mathcal{F} \cap T'_w)$ has $I^W_{\max} = \{\operatorname{sgn}(v)_k e_k \colon \operatorname{sgn}(v)_k = \operatorname{sgn}(w)_k\}$ and (as a result) every other $\operatorname{sgn}(v)_k e_k$ is contained in $I^P_{\min}$. There is only one potential intersection skeleton that satisfies this condition, configurations will not matter. Take a flag $\mathcal{F}$ associated to this skeleton with the largest number of faces, in this case there are maximal flags associated with this intersection skeleton (we can add faces at every dimension to the minimal face with only one possible choice of face at dimension $n-y$). Letting $X_0 = E_v \cap T'_w$, define X_1 as the same diagram except the entries at $\mathcal{F}$ and $\mathcal{F}\setminus\{S\}$ are replaced with zero. This forms a valid first step in the filtration, as the entry indexed by $\mathcal{F}$ will not map into any other entry of the diagram, while the entry indexed by $\mathcal{F} \setminus \{S\}$ could only have mapped into the entry indexed by $\mathcal{F}$. Continue the filtration across all maximal flags associated to this intersection.

After removing all entries indexed by maximal flags and their partner without S associated to this intersection skeleton, half of the entries indexed by 'almost' maximal flags with n faces associated with this intersection skeleton will be replaced by zero, specifically those without S. One now repeats the process with the remaining flags of n faces, again pairing with the same flag without S. The entries indexed by these flags have no non-zero entry to map to, as the flags they are contained in with the same intersection skeleton have been removed, and any flags with more entries with a different intersection skeleton can't possibly be the codomain of injective maps from these entries, as we have deliberately picked the entries with the largest intersection skeleton to begin the filtration. We will be left with two flags - the unique minimal one associated with this intersection skeleton and the same flag with S added. This pair is removed also.

Next, continue the argument with a maximal flag with intersection skeleton precisely one step down in the (totally ordered) lexicographical ordering on Z, using Lemma 2.8.2. Specifically, these will be flags associated to an intersection module with intersection skeleton such that all y of the e_k are contained within one or more of the I_i^W sets for some i in any combination. There is no longer a unique intersection module at this level of Z, that is the configurations σ^W, σ^P will matter in terms of the precise nature of the intersection module, but due to Lemma 2.8.2 and Lemma 2.8.4 we ignore them in our filtration beyond fixing one to begin with and working through them at each level of z.

Again, we can form a filtration of diagrams by fixing an intersection and pairing off with a face G that is the only possible choice of face at that dimension in any flag associated with the intersection, other than no face being present at all (as discussed in Proposition 2.9.1). Begin with maximal flags, removing all the possible entries indexed by maximal flags whose intersections are at this level of Z before descending down the dimension of flags as earlier. Like before none of the removed entries can map into entries of different intersection module, as the larger intersection modules have already been removed when we dealt with the largest element of Z, and others at the same level with different ordering of elements cannot be contained within each other by Lemma 2.8.4.

Now continue this process descending down the ordering of Z, with the maximum number of elements (y) within some $I_?^W$.

At this point, we take flags associated to intersection skeletons with only $y-1$ of the potential elements within one or more of the I_i^W sets, These flags cannot be maximal, as the only way for $\mathrm{sgn}(v)_k e_k$ for $\mathrm{sgn}(v)_k = \mathrm{sgn}(w)_k$ to not be in one of the I_i^W sets is for $\mathrm{sgn}(v)_k e_k \in I_{\min}^P$, which happens precisely when $\pm e_k \in T_{F_0}$. In this case, they cannot contain v, so we have actually already removed all entries associated with maximal flags by this point. Note that when $y-k$ elements are within $I_?^W$ sets, each flag cannot have a face of dimension $k-1$ or lower, so we begin the filtration with flags that have a face of each dimension from k to n. So once again we work through a lexicographical ordering at each point descending down the dimension of flags with intersection skeleton at this level of the lexicographical ordering, with one of the y elements taking turns to be within $I_{\min}^P$.

Then, continue the same process across all other collections of intersections until we are left with the unique intersection module with intersection skeleton satisfying $I_?^W = \emptyset$. The final pair of flags associated with this intersection skeleton after filtration of all other entries within $E_v \cap T'_w$ will be the flag with the pair of faces S and the unique face G containing v such that $\pm e_k \in T_G$ if and only if $\mathrm{sgn}(v)_k = \mathrm{sgn}(w)_k$ and the flag containing only S. This final piece of the filtration, X_{final}:

$$0 \to R_*((\{G, S\} \cap T'_w)) \cong R_*((\{S\} \cap T'_w)) \to 0,$$

clearly has Čech complex with trivial homology, immediately telling us by the filtration argument that $\Gamma(E_v \cap T'_w)$ is has trivial homology.

Now consider $w = -v$. Since $y = 0$ in this case, there is only one Z, one level of Z and one configuration - which is associated with the intersection $R_*[T_v]$. We cannot remove S from the flag without the intersection being trivial, but we can use the face v for all of these flags and pair off in the same way as before ending with the pair of flags $\{v, S\}, \{S\}$ to show that $\Gamma(E_v \cap T'_{-v})$ has trivial homology.

Now consider $v = w$, in this case for all $1 \leq k \leq n$, $\operatorname{sgn}(v)_k = \operatorname{sgn}(w)_k$. We now will show that $\Gamma(E_v \cap T'_v) = \Gamma(E_v \cap T_v)$ is quasi-isomorphic to $R_*[T_v]$. Repeat the process for the diagram $E_v \cap T_v$, beginning with flags associated with intersections such that all elements are in $I^W_?$ (i.e., when the only non-zero set is $I^W_{\max}$), then with intersections that have descending numbers of elements in the $I^W_?$ sets at each point. Eventually, we will be left with the flags associated with intersections where $I^W_? = \emptyset$. This happens only when the flag is $\{S\}$ as shown in Proposition 2.9.1. Note that $R_*((\{S\} \cap T_v)) = R_*[T_v]$, as $R((\{S\})) = R$. The composition of the x_i from $E_v \cap T_v$ to $E_v \cap T_v(\{S\}) = R_*((\{S\} \cap T_v)) = R_*[T_v]$ is clearly the map σ_v consisting of the identity map on $E_v \cap T_v(\{S\}) = R_*[T_v]$ and zero elsewhere. Hence, the filtration will tell us that $\Gamma(E_v \cap T_v) \overset{\Gamma(\sigma_v)}{\simeq} R_*[T_v]$.

Now, by noting that $E_v = E_v \cap T_v \oplus \bigoplus_{w \neq v} E_v \cap T'{}_w$, we arrive at the result $\Gamma(E_v) \overset{\Gamma(\sigma_v)}{\simeq} R_*[T_v]$, which is clearly a quasi-isomorphism of $R_*[T_v]$-$R_*[T_v]$ modules.

□

2.10 Contractibility of Novikov homology implies finite domination

We now put together the main proof of this section. From this point let C be a bounded complex of finitely generated free R-modules for a strongly $\mathbb{Z}^n$-graded ring R that has trivial Novikov homology, that is for all flags of the form $(F \subset S)$ where $S = [-1,1]^n$, the following complexes

$$C \underset{R}{\otimes} R_*((\{F \subset S\}))$$

are acyclic. To show that the given complex C is R_0 finitely dominated, we take the following steps:

1. From C, we form a complex of quasi-coherent diagrams, $\mathcal{Y}$ indexed over the faces of $S = [-1,1]^n$, as in Proposition 2.4.1.

2. We use Corollary 2.4.2 to show that the totalisation of the Čech complex of $\mathcal{Y}$, $\check{\Gamma}(\mathcal{Y})$, is quasi-isomorphic to D as defined in Corollary 2.4.2.

3. Proposition 2.5.7 will tell us that $\Gamma(E_F) \overset{\sigma_F}{\simeq} R_*[T_F]$, where E_F is indexed by $\mathcal{N}_F$ for $F \subseteq S$.

4. We observe that we can form a diagram indexed by $F \subseteq S$ with E_F at the point indexed by F and structure maps $\pi_{F,G} \colon E_F \to E_G$ consisting of the projection, a valid diagram map as for $F \subseteq G$, $\mathcal{N}_G \subseteq \mathcal{N}_F$ so that the following diagram

$$\begin{tikzcd} (E_F)(\mathcal{F}) \arrow[r, "\pi_{F,G}"] \arrow[d, "{[\mathcal{F},\mathcal{F}']\iota}"'] & (E_G)(\mathcal{F}) \arrow[d, "{[\mathcal{F},\mathcal{F}']\iota}"'] \\ (E_F)(\mathcal{F}') \arrow[r, "\pi_{F,G}"] & (E_G)(\mathcal{F}') \end{tikzcd}$$

is commutative when we let $(E_F)(\mathcal{J}) = 0$ for $\mathcal{J} \notin \mathcal{N}_F$.

5. We form a new diagram of chain complexes $\mathcal{M}$ also indexed by the faces of S consisting at each point the complex $\mathcal{Y}(F)$ tensored with Čech complex of the diagram E_F indexed by $\mathcal{N}_F$,

$$\mathcal{M}(F) = \mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E_F),$$

and structure maps $\iota \otimes \pi$. This diagram will be quasi-isomorphic to $\mathcal{Y}$ as $R_*[T_F]$-$R_*[T_F]$ bimodules, due to pointwise quasi-isomorphisms

$$\mathrm{id} \otimes \sigma_F : \mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E_F) \to \mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} R_*[T_F].$$

Also note that due to the adjoint map sheaf condition of $\mathcal{Y}$,

$$\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} R_*((\mathcal{F})) = \mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} R \underset{R}{\otimes} R_*((\mathcal{F})) \cong \mathcal{Y}(S) \underset{R}{\otimes} R_*((\mathcal{F})).$$

Since $\mathcal{Y}(S) = C$ and Lemma 1.3.4 tells us that $C \underset{R}{\otimes} R_*((\mathcal{F}))$ have trivial homology for flags containing S and at least one other face if we assume C satisfies the trivial Novikov homology condition, it follows that the entries of E_F indexed by flags containing S and at least one other face are complexes with trivial homology.

6. We note that via the contraction assumptions and implication of which, Lemma 1.3.4, for all $F \subseteq S$, the diagram $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} E_F$ will be quasi-isomorphic to a sub-diagram $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} E'_F$ where each contractible entry, that is those that are indexed by flags containing S and at least one other face, are replaced by 0. By Lemma 2.1.2, the Čech complexes will be quasi-isomorphic, that is

$$\Gamma(\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} E_F) \simeq \Gamma(\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} E'_F)$$

while Lemma 0.3.6 tells us that

$$\check{\Gamma}(\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} E_F) \simeq \check{\Gamma}(\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} E'_F).$$

On adapting the sign convention of Γ for a double complex accordingly, we find that

$$\check{\Gamma}(\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} E_F) = \mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E_F)$$

and similarly for E'_F. Hence

$$\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E_F) \simeq \mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E'_F).$$

7. Form a diagram $\mathcal{M}'$ indexed by $\mathcal{S}$ where the entry at $F \subseteq S$ is $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E'_F)$. From the above it is clear that $\mathcal{M} \simeq \mathcal{M}'$, as are $\Gamma(\mathcal{M}) \simeq \Gamma(\mathcal{M}')$ levelwise hence so are the totalisations $\check{\Gamma}(\mathcal{M}) \simeq \check{\Gamma}(\mathcal{M}')$.

8. The diagram $\mathcal{M}'$ will contain, as a sub-diagram, the entries indexed by the flag $\{S\}$ of E'_F tensored with $\mathcal{Y}(F)$ over $R_*[T_F]$ at each point indexed by $F \subset S$. That is we see a subdiagram with zero entries except at the point indexed by $\{S\}$ where we have the module
$$\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} R$$
which, via the properties of quasi-coherent diagrams, will be isomorphic to $\mathcal{Y}(S) = C$. Hence this subdiagram will be isomorphic to a constant diagram $\mathcal{S}$, where at each point we see the complex C. The complex $\check{\Gamma}(\mathcal{S})$ will be homotopy equivalent to C.

9. At this point, we know that there are quasi-isomorphisms $D \to \check{\Gamma}(\mathcal{Y})$, $\check{\Gamma}(\mathcal{M}) \to \check{\Gamma}(\mathcal{Y})$ and $\check{\Gamma}(\mathcal{M}) \to \check{\Gamma}(\mathcal{M}')$. Using standard results pertaining to the unbounded derived category of R_0, $D(R_0)$, we observe that D is isomorphic to $\check{\Gamma}(\mathcal{M}')$ within $D(R_0)$.

10. Combining the above wth the fact that C and D are both bounded complexes of projective R_0-modules tells us that C is a homotopy retract of D, which will provide an R_0 finite domination of C as D is a bounded complex of finitely generated projective R_0-modules and hence a retract of a bounded complexes of finitely generated free R_0-modules.

In diagram form, firstly just with the diagrams over the faces of S without taking Čech complexes, then with taking totalisations of Čech complexes:

$$\begin{tikzcd}
\mathcal{Y} & \mathcal{M} \arrow[l, "\mathrm{id}\otimes\sigma", "\simeq"'] \arrow[r, "\rho", "\simeq"'] & \mathcal{M}' \\
& & \mathcal{S} \arrow[u, "\supseteq"]
\end{tikzcd} \tag{2.10.0.2}$$

$$\begin{tikzcd}
D \arrow[r, "\simeq"] & \check{\Gamma}(\mathcal{Y}) & \check{\Gamma}(\mathcal{M}) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \check{\Gamma}(\mathcal{M}') \\
& & & \check{\Gamma}(\mathcal{S}) \arrow[u, "\supseteq"] & C. \arrow[l, "\simeq"']
\end{tikzcd} \tag{2.10.0.3}$$

Lemma 2.10.1. *The map* $\mathrm{id} \otimes \sigma \colon \mathcal{M} \to \mathcal{Y}$ *is a quasi-isomorphism.*

Proof. Note that the tensor product $\mathcal{Y}(F)_n \underset{R_*[T_F]}{\otimes} R_*[T_F]$ is well defined ($\mathcal{Y}(F)$ has a right $R_*[T_F]$ action) hence making the tensor product $\mathcal{Y}(F)_n \underset{R_*[T_F]}{\otimes} \Gamma(E_F)$ well defined. It immediately follows that there are levelwise quasi-isomorphisms

$$\mathcal{Y}(F)_n \underset{R_*[T_F]}{\otimes} \Gamma(E_F) \overset{\mathrm{id}\otimes\sigma_F}{\to} \mathcal{Y}(F)_n \underset{R_*[T_F]}{\otimes} R_*[T_F] = \mathcal{Y}(F)_n.$$

That we have a quasi-isomorphism $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E_F) \to \mathcal{Y}(F)$ is clear. Combining the Čech complexes $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E_F)$ into a diagram $\mathcal{M}$ with canonical embeddings as structure maps (note that $\mathcal{M}(S) = C$), there is a quasi-isomorphism $\mathcal{M} \to \mathcal{Y}$. □

As an instant corollary, we now know that $\Gamma(\mathcal{M}) \simeq \Gamma(\mathcal{Y})$ and also $\check{\Gamma}(\mathcal{M}) \simeq \check{\Gamma}(\mathcal{Y})$.

Next we want to construct the diagram $\mathcal{M}'$ with an associated Čech complex containing, as a sub complex, the complex C.

Lemma 2.10.2. *There is a diagram* $\mathcal{M}'$ *such that there is a quasi-isomorphism* $\rho \colon \mathcal{M} \to \mathcal{M}'$ *and a constant subdiagram* $\mathcal{S}$ *with* C *at each point where* $\check{\Gamma}(\mathcal{S}) \simeq C$*.*

Proof. Consider the complex $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E_F)$. Under the contractibility assumption of the main result and the implication Lemma 1.3.4, the following complexes have trivial homology:

$$C \underset{R}{\otimes} R_*(\!(\mathcal{F})\!)$$

where $\mathcal{F}$ is a flag containing S and at least one other face. It follows from an earlier discussion that there is a quasi-isomorphism, consisting of identity maps or zero maps where appropriate, between the complex $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E_F)$ and another where E_F is replaced with another diagram E'_F which is the same as E_F except it is zero at all points indexed by every point where

$S \in \mathcal{F}$ and $|\mathcal{F}| \neq 1$. Hence, we have quasi-isomorphisms $\rho_F \colon \mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E_F) \to \mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E'_F)$, noting that they form a valid chain complex map as the flags containing S and at least one other flag are not contained in any other possible flag in $\mathcal{N}_F$, so mapping into the Čech complex of the diagram with these entries removed respects the $\mathcal{N}_F$ structure of the diagram E_F.

If we set $\mathcal{M}'$ as the diagram with $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E'_F)$ at the point indexed by F we have a quasi-isomorphism $\rho \colon \mathcal{M} \to \mathcal{M}'$, and so $\check{\Gamma}(\mathcal{M}) \simeq \check{\Gamma}(\mathcal{M}')$.

The diagram E'_F splits into two diagrams, one consisting of R at the point indexed by the flag $\{S\}$ and everything indexed by flags that do not contain S. It follows that $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} \Gamma(E'_F)$ contains $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} R$ as a subcomplex. By the quasi-coherent sheaf assumption on $\mathcal{Y}$, we know that $\mathcal{Y}(F) \underset{R_*[T_F]}{\otimes} R \cong \mathcal{Y}(S) = C$ for all F. It follows that $\mathcal{M}'$ contains the constant diagram $\mathcal{S}$ consisting of C at every point in S.

It remains to note that the complex $\check{\Gamma}(\mathcal{S})$ is homotopy equivalent to C. The bounded below and to the left double complex $C \to \Gamma(\mathcal{S})$ follows the same form as the similar complex of [HQ15, Lemma 4.6.4], and as a result the same argument as used in that paper can be used to show that $C \simeq \check{\Gamma}(\mathcal{S})$. □

To find a homotopy retraction from our quasi-isomorphisms, we make use of the derived category. See [Hov99, P41 Theorem 2.3.11] and understand that the homotopy category of chain complexes over R_0 is precisely the derived category. The following result follows from standard model category arguments.

Theorem 2.10.3. *There is a category $D(R_0)$, the* unbounded derived category*, with*

$$\mathrm{Ob}(D(R_0)) = \mathrm{Ob}(\mathrm{Ch}(R_0)),$$

and a functor

$$\gamma \colon \mathrm{Ch}(R_0) \to D(R_0)$$

which maps identically on objects, such that

1. *γ maps quasi-isomorphisms to isomorphisms.*

2. *γ is universal with property (1), that is, given a second functor*

$$F\colon \mathrm{Ch}(R_0) \to \mathcal{C}$$

that maps quasi-isomorphisms to isomorphisms, there is a unique functor $\overline{F}\colon D(R_0) \to \mathcal{D}$ such that $\overline{F}\gamma = F$.

Moreover, if the complexes C, D are bounded below chain complex of projective R_0-modules, then

$$\mathrm{Hom}_{\mathrm{Ch}(R_0)}(C, D) \to \mathrm{Hom}_{\mathrm{Ch}(D(R_0))}(C, D)$$

is onto and is such that

$$f \simeq g \iff \gamma(f) = \gamma(g).$$

The result now breaks down to tactical referencing:

Proposition 2.10.4. *Let C be a bounded complex of finitely generated free R-modules for a strongly $\mathbb{Z}^n$-graded ring R. If C has trivial Novikov homology, that is when the complexes:*

$$C \underset{R_0}{\otimes} R_*(\!(\mathcal{F})\!)$$

for all flags $\{F \subset S\}$ containing S and one other face F have trivial homology, then C is R_0-finitely dominated.

Proof. Lemmas 2.10.1 and 2.10.2 will provide the maps as seen in Diagram (2.10.0.3). Then using Theorem 2.10.3, note that C is a retract of D within the derived category. As both are, in particular, bounded complexes of projective R_0-modules, we have a homotopy retract $\alpha\colon C \to D$, $\beta\colon D \to C$, $\beta\alpha \simeq \mathrm{id}_C$. As D is a bounded complex of finitely generated projective R_0-modules by construction and hence a retract of a bounded complex of finitely generated free R_0-modules, it follows that C is R_0-finitely dominated as required. $\square$

3. THE CATEGORY OF N-CUBES

In this chapter we define the N-cubes that will be used in the final chapter to form a homotopy equivalence between a bounded complex of finitely generated free R-modules and the totalisation of an N-cube that we will show is contractible. These objects, in a slightly different manner, were defined and used in [HQ16]. They can be seen as a variation of a commutative n-dimensional cube of chain complexes of R-modules and chain complex maps, in that each square or cube is not commutative but commutative up to homotopy. We begin by setting a few sign conventions. After defining the N-cubes themselves, we define morphisms between them and hence we can define a category. Afterwards we form a totalisation functor to the category of R-module complexes that will have both left and right adjoints. We also discuss the relationship between these cubes and mapping cones.

3.1 Introduction to sign algebra of N-cubes

This section deals with elements $\langle A : T\rangle$ where A, T are disjoint subsets of a finite totally ordered set N. These are used to calculate the signs given to the relations of the elements of the following N-cubes. A number of useful definitions and results are discussed in this section.

Let $N = \{1, 2, \ldots, n\}$ be a finite ordered set.

Definition 3.1.1. Given a finite ordered set N, a subset A and an element $x \in N$, define the set $A_{>x} = \{y \in A; y > x\}$. Also, given a set $B \subseteq N$, let $P^A_B = \sum_{x\in B} |A_{>x}|$ and set $P^{\emptyset}_B = 0$ and $P^A_{\emptyset} = 0$.

Lemma 3.1.2 (Algebra rule 1)**.** *For sets $A \subseteq B_1, B_2 \subseteq N$ where $B_1 \cap B_2 = \emptyset$, we have that $|P^A_{B_1}| + |P^A_{B_2}| = |P^A_{B_1\cup B_2}|$.*

Lemma 3.1.3 (Algebra rule 2)**.** *For sets $A_1, A_2 \subseteq B \subseteq N$ where $A_1 \cap A_2 = \emptyset$, we have that $|P^{A_1}_B| + |P^{A_2}_B| = |P^{A_1\cup A_2}_B|$.*

Consider a general element in a set-inclusion indexed matrix i.e., a matrix indexed by elements of $P(N)$, with non zero entries only for pairs $A \subseteq B \subseteq N$. If we want to consider a product of two such matrices, say MN, then $(MN)_{B,A} = \sum_{A \subseteq \dot{S} \subseteq B} M_{B,S} N_{S,A}$. We are interested in matrices that have a certain sign applied to each of their entries.

Definition 3.1.4. For sets $A \subseteq B \subseteq N$, label $B \setminus A = \{x_1, ..., x_t\}$. Define the number $\langle A : B \setminus A \rangle$, where

$$\langle A : B \setminus A \rangle = \begin{cases} \sum_{i \leq t} |A_{>x_i}| = |P^A_{B \setminus A}| & \text{if } t \text{ is odd.} \\ |A| + \sum_{i \leq t} |A_{>x_i}| = |A| + |P^A_{B \setminus A}| & \text{if } t \text{ is even.} \end{cases}$$

Examples 3.1.5. Let $N = \{1, 2, 3, 4\}$. Then:

- $\langle \{1\} \colon \{2,3\} \rangle = 1 + |P^{\{1\}}_{\{2,3\}}| = 1 + 0 = 1.$
- $\langle \{2,3\} \colon \{1\} \rangle = 0 + |P^{\{2,3\}}_{\{1\}}| = 2.$
- $\langle \{1,3\} \colon \{2\} \rangle = 0 + |P^{\{1,3\}}_{\{2\}}| = 1.$
- $\langle \{1,3,4\} \colon \{2\} \rangle = 0 + |P^{\{1,3,4\}}_{\{2\}}| = 2.$
- $\langle \{1,3\} \colon \{2,4\} \rangle = 2 + |P^{\{1,3\}}_{\{2,4\}}| = 2 + (1 + 0) = 3.$

The next four lemmas will be often used in later computations.

Lemma 3.1.6. *For all* $B \subseteq N$, $\langle \emptyset : B \rangle \equiv_2 0$.

Proof. Simply note that since $|P^{\emptyset}_{B}|$ is set as zero $\langle \emptyset : B \rangle = 0$ when $|B|$ is odd and $\langle \emptyset : B \rangle = |\emptyset| \equiv_2 0$ when $|B|$ is even. $\square$

Lemma 3.1.7. *For all* $B \subseteq N$, $\langle B : \emptyset \rangle = |B|$.

Proof. Since in this case, $|\emptyset| = 0$ is even then for all B, we have $\langle B : \emptyset \rangle = |B| + |P^{B}_{\emptyset}| = |B|$. $\square$

Lemma 3.1.8. *Given* $A \subseteq B \subset N$, *let* $T \subseteq N$ *such that* $\min(T) > \max(B)$. *Then*

$$\langle A \cup T : B \setminus A \rangle \equiv_2 |T| + \langle A : B \setminus A \rangle.$$

Proof. Note, $|P^T_{B\setminus A}| = |T| \cdot |B \setminus A|$ as T is bigger than everything in B (and hence, everything in $B \setminus A$). Assume $|B \setminus A|$ is odd. Then

$$\begin{aligned}\langle A \cup T : B \setminus A\rangle &= |P^{A\cup T}_{B\setminus A}| \\ &= |P^T_{B\setminus A}| + |P^A_{B\setminus A}| \\ &= |T| \cdot |B \setminus A| + \langle A : B \setminus A\rangle \equiv_2 |T| + \langle A : B \setminus A\rangle\end{aligned}$$

(the last equivalence is as $|B \setminus A|$ is odd) as expected. Similarly, when $|B \setminus A|$ is even

$$\begin{aligned}\langle A \cup T : B \setminus A\rangle &= |A \cup T| + |P^{A\cup T}_{B\setminus A}| \\ &= |A| + |T| + |P^T_{B\setminus A}| + |P^A_{B\setminus A}| \\ &= |T| + |T| \cdot |B \setminus A| + \langle A : B \setminus A\rangle \equiv_2 |T| + \langle A : B \setminus A\rangle\end{aligned}$$

(the last equivalence is as $|B \setminus A|$ is even) once again. □

In particular, for $T = \{n + 1\}$ for an element $n + 1$ bigger than anything else in N, $\langle A \cup T : B \setminus A\rangle \equiv_2 1 + \langle A : B \setminus A\rangle$.

Lemma 3.1.9. *Given $A \subseteq B \subset N$, let $T \subseteq N$ such that $\min T > \max B$. Then*

$$\langle A : B \setminus A \cup T\rangle \equiv_2 |A| + \langle A : B \setminus A\rangle$$

when $|T|$ is odd and

$$\langle A : B \setminus A \cup T\rangle \equiv_2 \langle A : B \setminus A\rangle$$

when $|T|$ is even.

Proof. Firstly, note that $P^A_T = 0$ as everything in T is bigger than anything in A. Assume both $|T|$ and $|B \setminus A|$ be odd (making their sum even). Then

$$\begin{aligned}\langle A : B \setminus A \cup T\rangle &= |A| + |P^A_{B\setminus A\cup T}| \\ &= |A| + |P^A_{B\setminus A}| + |P^A_T| \\ &= |A| + |P^A_{B\setminus A}| = |A| + \langle A : B \setminus A\rangle.\end{aligned}$$

Now, assume both $|T|$ and $|B \setminus A|$ be even (making their sum even). Then

$$\begin{aligned}\langle A : B \setminus A \cup T\rangle &= |A| + |P^A_{B\setminus A\cup T}| \\ &= |A| + |P^A_{B\setminus A}| + |P^A_T| \\ &= |A| + |P^A_{B\setminus A}| = \langle A : B \setminus A\rangle.\end{aligned}$$

Next, assume $|T|$ be odd and $|B \setminus A|$ be even (making their sum odd). Then

$$\begin{aligned}\langle A : B \setminus A \cup T\rangle &= |P^A_{B\setminus A\cup T}| \\ &= |P^A_{B\setminus A}| + |P^A_T| \\ &= |P^A_{B\setminus A}| \equiv_2 |A| + \langle A : B \setminus A\rangle.\end{aligned}$$

Finally, assume $|T|$ be even and $|B \setminus A|$ be odd (making their sum odd). Then

$$\begin{aligned}\langle A : B \setminus A \cup T\rangle &= |P^A_{B\setminus A\cup T}| \\ &= |P^A_{B\setminus A}| + |P^A_T| \\ &= |P^A_{B\setminus A}| = \langle A : B \setminus A\rangle.\end{aligned}$$

The result is clear. □

In particular, if $T = \{n+1\}$ for an element $n+1$ bigger than anything else in N, we have that $\langle A : B \setminus A \cup T\rangle \equiv_2 |A| + \langle A : B \setminus A\rangle$.

Lemma 3.1.10. *Fix $A \subseteq B \subseteq N$. For all S where $A \subseteq S \subseteq B$, the difference modulo 2 between*

$$\langle S : B \setminus S\rangle + \langle A : S \setminus A\rangle$$

and

$$\langle S \setminus A : B \setminus S\rangle + \langle \emptyset : S \setminus A\rangle$$

is precisely $|A| + \langle A\colon B \setminus A\rangle$.

Proof. Note $\langle \emptyset : S \setminus A \rangle = 0$ for all $A \subseteq S$. Firstly, assume $|B \setminus S|$ and $|S \setminus A|$ both be odd, so that $|B \setminus A|$ is even. Then for $A \subseteq S \subseteq B$

$$\langle S \setminus A : B \setminus S \rangle = |P^{S \setminus A}_{B \setminus S}| \tag{3.1.10.1}$$

and

$$\langle S : B \setminus S \rangle + \langle A : S \setminus A \rangle = |P^{S}_{B \setminus S}| + |P^{A}_{S \setminus A}|. \tag{3.1.10.2}$$

Now, using the algebra rules,

$$|P^{A}_{S \setminus A}| = |P^{A}_{B \setminus A}| - |P^{A}_{B \setminus S}|$$

and

$$|P^{S}_{B \setminus S}| = |P^{S \setminus A}_{B \setminus S}| + |P^{A}_{B \setminus S}|$$

hence

$$\begin{aligned} |P^{S}_{B \setminus S}| + |P^{A}_{S \setminus A}| &= |P^{A}_{B \setminus A}| - |P^{A}_{B \setminus S}| + |P^{S \setminus A}_{B \setminus S}| + |P^{A}_{B \setminus S}| \\ &= |P^{A}_{B \setminus A}| + |P^{S \setminus A}_{B \setminus S}| \end{aligned}$$

so the difference between Equations 3.1.10.1 and 3.1.10.2 is $|P^{A}_{B \setminus A}|$ as hoped. Now, assume both $|B \setminus S|$ and $|S \setminus A|$ be even, so that $|B \setminus A|$ is even. Then

$$\langle S \setminus A : B \setminus S \rangle = |P^{S \setminus A}_{B \setminus S}| + |S \setminus A|$$

and

$$\langle S : B \setminus S \rangle + \langle A : S \setminus A \rangle = |P^{S}_{B \setminus S}| + |P^{A}_{S \setminus A}| + |S| + |A|.$$

Again, since $|S \setminus A| \equiv_2 |S| + |A|$, the difference is $|P^{A}_{B \setminus A}|$.

Now, assume $|B \setminus S|$ be even and $|S \setminus A|$ be odd, so that $|B \setminus A|$ is odd. Now

$$\langle S \setminus A : B \setminus S \rangle = |P^{S \setminus A}_{B \setminus S}| + |S \setminus A|$$

and

$$\langle S : B \setminus S \rangle + \langle A : S \setminus A \rangle = |P^{S}_{B \setminus S}| + |P^{A}_{S \setminus A}| + |S|.$$

Here the difference is $|P^{A}_{B \setminus A}| + |A|$.

Finally, assume $|B \setminus S|$ be odd and $|S \setminus A|$ be even, so that $|B \setminus A|$ is odd. Then

$$\langle S \setminus A : B \setminus S\rangle = |P^{S\setminus A}_{B\setminus S}|$$

and

$$\langle S : B \setminus S\rangle + \langle A : S \setminus A\rangle = |P^{S}_{B\setminus S}| + |P^{A}_{S\setminus A}| + |A|,$$

again providing a difference of $|P^{A}_{B\setminus A}| + |A|$.

It follows that for $|B\backslash A|$ even or odd, the difference mod 2 is always $|A| + \langle A\colon B\backslash A\rangle$. □

3.2 Defining N-cubes

Fix a unital ring R. In this section we introduce the N-cubes themselves, building towards forming a category with N-cubes as objects.

Definition 3.2.1 (N-diagram of $\mathbb{Z}$-indexed modules)**.** Let $\mathfrak{D}$ be the following collection of information:

- For all $B \subseteq N$, collections of R-modules indexed by $\mathbb{Z}$,

$$\mathfrak{D}_B = \{(\mathfrak{D}_B)_k,\ k \in \mathbb{Z}\}.$$

- For all $A \subseteq B$ contained within N, collections of R-module maps of degree $|B \setminus A| - 1$ indexed by subsets $A \subseteq B$ of N,

$$\mathfrak{D}_{B,A} = \{(H_{B,A})_k\colon (\mathfrak{D}_A)_k \to (\mathfrak{D}_B)_{k+|B\setminus A|-1},\ k \in \mathbb{Z}\}.$$

It is best to think of these data as an n-dimensional cube where each vertex is indexed by a subset of N. At each vertex we have a collection of modules indexed by $\mathbb{Z}$ which allows us to view the arrows as $\mathbb{Z}$ indexed collections of maps of a certain degree. The edges (the maps $\mathfrak{D}_{B,A}$ for $|B \setminus A| = 1$) are collections of maps of degree 0, the maps that stay at a certain vertex (the maps $\mathfrak{D}_{B,B}$) are degree -1. For the pairs $(\mathfrak{D}_B, \mathfrak{D}_{B,B})$ to be a chain complex and the maps of degree 0 to be chain complex maps they need to satisfy conditions which will follow from Definition 3.2.2. The 'diagonals' (the maps $\mathfrak{D}_{B,A}$ for $|B \setminus A| \leq 2$) are higher degree maps, and the additional conditions of Definition 3.2.2 below will make these maps homotopies or higher homotopy data.

Definition 3.2.2 (N-cube)**.** We call an N-diagram of $\mathbb{Z}$-graded R-modules $\mathfrak{D}$ a *homotopy commutative N-cube*, or just N-cube, if for all pairs $A \subseteq B$ of subsets of N

$$\sum_{A \subseteq \underset{\bullet}{S} \subseteq B} (-1)^{\langle S:B \setminus S \rangle} \mathfrak{D}_{B,S} (-1)^{\langle A:S \setminus A \rangle} \mathfrak{D}_{S,A} = 0 \tag{3.2.2.1}$$

where the signs $(-1)^{\langle X \,:\, Y \rangle}$ are taken from Definition 3.1.4 and the dot underneath S informs that the sum is taken over the collection of S for fixed A, B.

We say that the N-cubes are commutative up to homotopy, by which we mean for $N = \{1, 2\}$, rather than a square of chain complex maps:

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "g_1"'] & D \arrow[d, "g_2"] \\ E \arrow[r, "f_2"] & F \end{tikzcd}$$

being commutative, there is a non-trivial map of degree 1 $H \colon C \to F$ such that $g_2 f_1 - f_2 g_1 = d_F H + H d_C$ where d_C, d_F are the respective boundary maps. We present this as:

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "g_1"'] \arrow[dr, "H"] & D \arrow[d, "g_2"] \\ E \arrow[r, "f_2"] & F \end{tikzcd}$$

in diagram form. For larger dimensions, each face of dimension 2 will be such a homotopy commutative square, but the entire cube will also have a similar homotopy commutative property that is ultimately provided by the conditions of Definition 3.2.2.

We can define a *homotopy totalisation* hTot that associates an N-cube $\mathfrak{D}$ with a chain complex $\mathrm{hTot}(\mathfrak{D})$. This will form part of a functorial mapping later on.

Definition 3.2.3. Given an N-cube $\mathfrak{D}$, let the *homotopy totalisation* $\mathrm{hTot}(\mathfrak{D})$ be the complex $(\mathrm{hTot}(\mathfrak{D})_k, D)$ where

$$\mathrm{hTot}(\mathfrak{D})_k = \bigoplus_{B\subseteq N} (\mathfrak{D}_B)_{k-|N\setminus B|}$$

with boundary $D\colon \mathrm{hTot}(\mathfrak{D})_k \to \mathrm{hTot}(\mathfrak{D})_{k-1}$ such that

$$D_{B,A} = \begin{cases} (-1)^{\langle A:B\setminus A\rangle}\mathfrak{D}_{B,A} & \text{if } A\subseteq B. \\ 0 & \text{otherwise.} \end{cases}$$

The boundary map satisfies $DD = 0$ due to the conditions of Equation 3.2.2, as for all $A \subseteq B$ the entry of DD at A, B is

$$\sum_{A\subseteq \underset{\bullet}{S}\subseteq B} D_{B,S}D_{S,A} = (-1)^{\langle S:B\setminus S\rangle}\mathfrak{D}_{B,S}(-1)^{\langle A:S\setminus A\rangle}\mathfrak{D}_{S,A} = 0.$$

To assist the understanding of these objects it is worth explaining what these objects are when N has $0, 1$ or 2 elements. Firstly, a 0-cube is simply a chain complex and the totalisation is trivial.

A 1-cube $\mathfrak{D}$ is a diagram:

$$\mathfrak{D}_\emptyset \xrightarrow{\mathfrak{D}_{\{1\},\emptyset}} \mathfrak{D}_{\{1\}}$$

and the totalisation is the complex

$$\left((\mathfrak{D}_\emptyset)_{k-1} \oplus (\mathfrak{D}_{\{1\}})_k, \begin{pmatrix} \mathfrak{D}_{\emptyset,\emptyset} & 0 \\ \mathfrak{D}_{\{1\},\emptyset} & -\mathfrak{D}_{\{1\},\{1\}} \end{pmatrix} \right)$$

which can be immediately recognised as the mapping cone of the map $\mathfrak{D}_{\{1\},\emptyset}\colon \mathfrak{D}_\emptyset \to \mathfrak{D}_{\{1\}}$.

A 2-cube $\mathfrak{D}$ is a diagram

$$\begin{tikzcd} \mathfrak{D}_\emptyset \arrow[r, "\mathfrak{D}_{\{1\},\emptyset}"] \arrow[d, "\mathfrak{D}_{\{2\},\emptyset}"'] \arrow[dr, "\mathfrak{D}_{\{1,2\},\emptyset}"] & \mathfrak{D}_{\{1\}} \arrow[d, "\mathfrak{D}_{\{1,2\},\{1\}}"] \\ \mathfrak{D}_{\{2\}} \arrow[r, "\mathfrak{D}_{\{1,2\},\{2\}}"] & \mathfrak{D}_{\{1,2\}} \end{tikzcd}$$

and the totalisation is the complex with modules

$$(\mathfrak{D}_\emptyset)_{k-2} \oplus (\mathfrak{D}_{\{1\}})_{k-1} \oplus (\mathfrak{D}_{\{2\}})_{k-1} \oplus (\mathfrak{D}_{\{1,2\}})_k$$

and boundary map:

$$\begin{pmatrix} \mathfrak{D}_{\emptyset,\emptyset} & 0 & 0 & 0 \\ \mathfrak{D}_{\{1\},\emptyset} & -\mathfrak{D}_{\{1\},\{1\}} & 0 & 0 \\ \mathfrak{D}_{\{2\},\emptyset} & 0 & -\mathfrak{D}_{\{2\},\{2\}} & 0 \\ \mathfrak{D}_{\{1,2\},\emptyset} & \mathfrak{D}_{\{1,2\},\{1\}} & -\mathfrak{D}_{\{1,2\},\{2\}} & \mathfrak{D}_{\{1,2\},\{1,2\}} \end{pmatrix}$$

which can immediately be seen as the mapping cone of the following map:

$$\begin{pmatrix} \mathfrak{D}_{\{2\},\emptyset} & 0 \\ \mathfrak{D}_{\{1,2\},\emptyset} & \mathfrak{D}_{\{1,2\},\{1\}} \end{pmatrix} : \operatorname{cone}(\mathfrak{D}_{\{1\},\emptyset}) \to \operatorname{cone}(\mathfrak{D}_{\{1,2\},\{2\}}).$$

This idea of N-cubes as iterated mapping cones will be discussed further later.

A 3-cube $\mathfrak{D}$ is a cubical diagram consisting of faces consisting of 2-cubes and a higher homotopy map of degree 2, $\mathfrak{D}_{\{1,2,3\},\emptyset}\colon \mathfrak{D}_\emptyset \to \mathfrak{D}_{\{1,2,3\}}$. The totalisation will be the mapping cone of a map between two 2-cubes. Higher N-cubes follow a similar pattern.

3.3 Properties of N-cubes

We now study these objects in some detail.

Remark 3.3.1. For $A = B$

$$(-1)^{|B|}\mathfrak{D}_{B,B}(-1)^{|B|}\mathfrak{D}_{B,B} = 0,$$

hence $(C_B, \mathfrak{D}_{B,B})$ is a chain complex for all $B \subseteq N$. Also for any $i \in N$ and $A \subset N$, $\{i\} \notin A$, since $\langle A : \emptyset \rangle = |A|$ the morphism $\mathfrak{D}_{A\cup\{i\},A}$ satisfies

$$\begin{aligned}(-1)^{|P^A_{\{i\}}|}\mathfrak{D}_{A\cup\{i\},A}(-1)^{|A|}\mathfrak{D}_{A,A} + (-1)^{|A|+1}\mathfrak{D}_{A\cup\{i\},A\cup\{i\}}(-1)^{|P^A_{\{i\}}|}\mathfrak{D}_{A\cup\{i\},A} \\ = (-1)^{|P^A_{\{i\}}|+|A|}(\mathfrak{D}_{A\cup\{i\},A}\mathfrak{D}_{A,A} - \mathfrak{D}_{A\cup\{i\},A\cup\{i\}}\mathfrak{D}_{A\cup\{i\},A}) = 0\end{aligned}$$

so it commutes with these boundary maps, hence as they are of degree 0 they are chain maps $C_A \to C_{A\cup\{i\}}$. Similarly, the maps $\mathfrak{D}_{A\cup\{i,j\},A}\colon C_A \to$

$C_{A\cup\{i,j\}}$ are chain homotopies. Specifically if we assume $j > i$, the map $\mathfrak{D}_{A\cup\{i,j\},A}$ satisfies:

$$\mathfrak{D}_{A\cup\{i,j\},A}\mathfrak{D}_{A,A} + \mathfrak{D}_{A\cup\{i,j\},A\cup\{i,j\}}\mathfrak{D}_{A\cup\{i,j\},A} =$$
$$\mathfrak{D}_{A\cup\{i,j\},A\cup\{i\}}\mathfrak{D}_{A\cup\{i\},A} - \mathfrak{D}_{A\cup\{i,j\},A\cup\{j\}}\mathfrak{D}_{A\cup\{j\},A}.$$

Definition 3.3.2 (The hom complex)**.** Given two chain complexes (C, d_C) and (D, d_D) the *hom-complex* $\underline{\mathrm{Hom}}(C, D)$ is a chain complex with modules

$$\underline{\mathrm{Hom}}(C, D)_k = \prod_{p=-\infty}^{\infty} \mathrm{Hom}(C_p, D_{p+k}),$$

where $\mathrm{Hom}(C_p, D_{p+k})$ is the abelian group of module homomorphisms from C_p to D_{p+k}, (making an element $f \in \underline{\mathrm{Hom}}(C, D)_k$ a collection of maps $f_p \colon C_p \to D_{p+k}$). The boundary map of $\underline{\mathrm{Hom}}(C, D)$ is d such that for $f \in \underline{\mathrm{Hom}}(C, D)_k$, $d_k f$ is a family of maps $(d_k f)_p \colon C_p \to D_{p+k-1}$ where

$$(d_k f)_p = d_D f_p + (-1)^{k+1} f_{p-1} d_C.$$

The following Lemma outlines the nature of the higher homotopy data.

Lemma 3.3.3. *Let $\mathfrak{D}$ be an N-cube and $A \subseteq B \subseteq N$ with $|B \setminus A| \geq 2$. Let $|B \setminus A| - 1 = k$, then the map $\mathfrak{D}_{B,A} \in \underline{\mathrm{Hom}}(\mathfrak{D}_A, \mathfrak{D}_B)_k$ is a preimage of*

$$\zeta_{B,A} = (-1)^{1+|B|+\langle A:B\setminus A\rangle} \sum_{A \subsetneq \underset{\bullet}{S} \subsetneq B} (-1)^{\langle S:B\setminus S\rangle}\mathfrak{D}_{B,S}(-1)^{\langle A:S\setminus A\rangle}\mathfrak{D}_{S,A} \tag{3.3.3.1}$$

of $\underline{\mathrm{Hom}}(\mathfrak{D}_A, \mathfrak{D}_B)_{k-1}$ via the kth level boundary map of the hom-complex $\underline{\mathrm{Hom}}(\mathfrak{D}_A, \mathfrak{D}_B)$, $d_{|B\setminus A|-1} = d_k$. That is $\mathfrak{D}_{B,B}\mathfrak{D}_{B,A} + (-1)^{|B\setminus A|}\mathfrak{D}_{B,A}\mathfrak{D}_{A,A} = \zeta_{B,A}$.

Proof. Observe that for $|B \setminus A|$ odd, (making $\mathfrak{D}_{B,A}$ a map of even degree) then

$$\langle B : \emptyset\rangle + \langle A : B \setminus A\rangle = |B| + |P^A_{B\setminus A}|$$

and

$$\langle A : B \setminus A\rangle + \langle A : \emptyset\rangle = |P^A_{B\setminus A}| + |A|,$$

so the signs differ by $(-1)^{|B\setminus A|} = -1$ hence they disagree.

If instead $|B \setminus A|$ is even (making $\mathfrak{D}_{B,A}$ a map of odd degree) then

$$\langle B : \emptyset \rangle + \langle A : B \setminus A \rangle = |B| + |A| + |P^{A}_{B\setminus A}|$$

and

$$\langle A : B \setminus A \rangle + \langle A : \emptyset \rangle = |A| + |P^{A}_{B\setminus A}| + |A| \equiv_2 |P^{A}_{B\setminus A}|$$

again the signs differ by $(-1)^{|B\setminus A|}$, but now as $|B \setminus A|$ is even, $(-1)^{|B\setminus A|} = 1$ hence the signs agree.

Hence in both cases the signs on the compositions $\mathfrak{D}_{B,B}\mathfrak{D}_{B,A}$ and $\mathfrak{D}_{B,A}\mathfrak{D}_{A,A}$ match the boundary map definition of the hom complex (that is,

$$\begin{aligned}(-1)^{\langle B:\emptyset\rangle}\mathfrak{D}_{B,B}(-1)^{\langle A:B\setminus A\rangle}\mathfrak{D}_{B,A} + (-1)^{\langle A:B\setminus A\rangle}\mathfrak{D}_{B,A}(-1)^{\langle A:\emptyset\rangle}\mathfrak{D}_{A,A}\\ = (-1)^{\langle B:\, \emptyset\rangle + \langle A:\, B\setminus A\rangle} d_{|B\setminus A|}\mathfrak{D}_{B,A}\end{aligned}$$

where $d_{|B\setminus A|}$ is the boundary of the hom complex from $\mathfrak{D}_A$ to $\mathfrak{D}_B$ at the $|B\setminus A|$ level). It remains to rearrange Equation (3.2.2.1) to provide the result — if $\langle B : \emptyset\rangle + \langle A : B \setminus A\rangle$ is odd, we move $(-1)^{\langle B:\emptyset\rangle+\langle A:B\setminus A\rangle} d_{|B\setminus A|}\mathfrak{D}_{B,A}$ to the other side of the equation and leave the rest of the summation, if not we do the opposite, hence providing the result. □

Now, a simplification of the signs.

Lemma 3.3.4. *Let $\mathfrak{D}$ be an N-cube. Given a fixed k the map $\mathfrak{D}_{B,A}$ where $|B \setminus A| = k + 1$ is the preimage via the boundary of the hom complex of the map*

$$\zeta_{B,A} = (-1)^{1+|B\setminus A|} \sum_{A \subsetneq S_{\bullet} \subsetneq B} (-1)^{\langle S\setminus A:B\setminus S\rangle}\mathfrak{D}_{B,S}\mathfrak{D}_{S,A}. \qquad (3.3.4.1)$$

Proof. Using Proposition 3.1.10,

$$\langle S \setminus A : B \setminus S\rangle + P \equiv_2 \langle S : B \setminus S\rangle + \langle A : S \setminus A\rangle$$

for some $P = \langle A\colon B \setminus A\rangle + |A|$ independent of S. Hence, we know from Lemma 3.3.3 that the map $\mathfrak{D}_{B,A}$ is the preimage via the boundary of the hom complex of the map

$$(-1)^{1+|B|+\langle A:B\setminus A\rangle+\langle A:B\setminus A\rangle+|A|} \sum_{A\subsetneq \underset{\bullet}{S}\subsetneq B} (-1)^{\langle S\setminus A:B\setminus S\rangle}\mathfrak{D}_{B,S}\mathfrak{D}_{S,A}$$

equivalently

$$(-1)^{1+|B\setminus A|} \sum_{A\subsetneq \underset{\bullet}{S}\subsetneq B} (-1)^{\langle S\setminus A:B\setminus S\rangle}\mathfrak{D}_{B,S}\mathfrak{D}_{S,A}$$

as required. □

If we set each $\mathfrak{D}_B$ to be the same complex and each chain map to be the same, we see that each map of degree 1 within the cube will satisfy the same condition by observation of Definition 3.2.2. If we fix each map of degree 1 to be the same, again we see that the maps of degree 2 must satisfy the same conditions. Hence, we can define special cubes (Definition 3.3.5), with all self maps and the same information in any direction (i.e., all the morphisms in any one direction are the same, in terms of the indexing sets, the morphism indexed by $\{B, A\}$ is only dependant on the set $B \setminus A$, not A or B, so for example it is equal to the map indexed by the sets $\{B \setminus A, \emptyset\}$).

Definition 3.3.5. Call an N-cube a *special N-cube* if for all B, B', $\mathfrak{D}_B = \mathfrak{D}_{B'}$ (so we only have one chain complex, and every morphism is a self morphism) and in addition all the morphisms are defined only by the union set so that for all $A \subseteq B$ we have that $\mathfrak{D}_{B,A} = \mathfrak{D}_{B\setminus A,\emptyset}$. Note for this case that for T, $T \cap B = \emptyset$,

$$\sum_{\underset{\bullet}{S}\subseteq B} \mathfrak{D}_{B,S}\mathfrak{D}_{S,\emptyset} = \sum_{T\subseteq \underset{\bullet}{S}\subseteq B\cup T} \mathfrak{D}_{B\cup T,S\cup T}\mathfrak{D}_{S\cup T,T}.$$

Therefore for a given degree, there is precisely one map and one condition that follows from the condition of Definition 3.2.2.

The 'trivial' case, where all the homotopies are 0 is called a *trivial N-cube.* Here, as a special case, we do get commutativity on each face. However, note the reverse is not true. If the chain maps commute, we can have non-zero 'null homotopic' information elsewhere.

Definition 3.3.6 (Commutative N-cube.)**.** If an N-cube $\mathfrak{D}$ has non-trivial information only for maps of degree 0,1, then we call the object a *Commutative N-cube*, which is simply a commutative diagram.

3.4 Morphisms of N-cubes

Having defined N-cubes we wish to define maps between these objects.

Definition 3.4.1 (N-diagram morphism.)**.** Given two N-diagrams

$$\mathfrak{D}_1 = \{(\mathfrak{D}_{1,B})_k \colon B \subseteq N,\, k \in \mathbb{Z}\}, \mathfrak{D}_2 = \{(\mathfrak{D}_{2,B})_k \colon B \subseteq N,\, k \in \mathbb{Z}\},$$

let $\mathfrak{F}$ be the collection of maps of $\mathbb{Z}$-indexed modules $\mathfrak{F}_{B,A} \colon \mathfrak{D}_{1,A} \to \mathfrak{D}_{2,B}$ for all $A \subseteq B \subseteq N$, where

$$\mathfrak{F}_{B,A} = \{(\mathfrak{F}_{B,A})_k \colon (\mathfrak{D}_{1,A})_k \to (\mathfrak{D}_{2,B})_{k+|B \setminus A|}\}$$

is a map of $\mathbb{Z}$-indexed modules of degree $|B \setminus A|$.

This can be seen as a map between two N-diagrams (a 'front face', i.e., the domain that has vertices $\mathfrak{D}_{1,B}$ and the 'back face' or codomain that has vertices $\mathfrak{D}_{2,B}$) together forming an $N+1$-diagram — in the sense that each 'arrow' $\mathfrak{F}_{B,B}$ is an edge, forming faces (i.e., squares and cubes), which in turn have diagonal maps across them provided by the other elements of $\mathfrak{F}$, in particular a single diagonal map across the whole $N+1$ dimension cube. We can collate these data and re-index using the set $N+1 = \{1, 2, .., n, n+1\}$ to obtain an $N+1$-diagram as in the definition (here, leave the front face the same, change the index of each $\mathbb{Z}$-indexed module in the back face to $B \cup \{n+1\}$ and rename each arrow accordingly). We can write $\mathfrak{F} \colon \mathfrak{D}_1 \to \mathfrak{D}_2$ to represent $\mathfrak{F}$ as a morphism of N-diagrams.

Remark 3.4.2. We can change Definition 3.4.1 to represent higher (or lower) level maps, by replacing $|B \setminus A|$ with $|B \setminus A| + k$ for whatever k is wished, in this case the N-diagram map has degree k. Above, $\mathfrak{F}$ has degree 0.

Definition 3.4.3 (N-cube morphisms)**.** Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be N-diagrams. We call $\mathfrak{F} \colon \mathfrak{D}_1 \to \mathfrak{D}_2$ a *morphism of homotopy commutative N-cubes*, or just N-cube morphism if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are both N-cubes and for all pairs $A \subseteq B \subseteq N$:

$$\sum_{A \subseteq \underset{\bullet}{S} \subseteq B} \Big((-1)^{|S| + \langle S : B \setminus S \rangle} \mathfrak{F}_{B,S} (-1)^{\langle A : S \setminus A \rangle} \mathfrak{D}_{1,S,A} - (-1)^{\langle S : B \setminus S \rangle} \mathfrak{D}_{2,B,S} (-1)^{|A| + \langle A : S \setminus A \rangle} \mathfrak{F}_{S,A} \Big) = 0. \tag{3.4.3.1}$$

Remark 3.4.4. In particular, for $A = B$, using Lemma 3.1.7,

$$\begin{aligned} (-1)^{|B|+|B|} \mathfrak{F}_{B,B} (-1)^{|B|} \mathfrak{D}_{1,B,B} - (-1)^{|B|} \mathfrak{D}_{2,B,B} (-1)^{|B|+|B|} \mathfrak{F}_{B,B} \\ = (-1)^{|B|} (\mathfrak{F}_{B,B} \mathfrak{D}_{1,B,B} - \mathfrak{D}_{2,B,B} \mathfrak{F}_{B,B}) = 0, \end{aligned}$$

i.e., that the graded module map of degree 0, $\mathfrak{F}_{B,B}$, commutes with the boundary maps $\mathfrak{D}_{1,B,B}$ and $\mathfrak{D}_{2,B,B}$ hence it is a chain map $\mathfrak{D}_{1,B} \to \mathfrak{D}_{2,B}$.

It is worth noting how these signs arise. The general principle is that we want the information of an N-cube morphism to correspond to the information of an $N+1$-cube. If we begin with an $N+1$ cube $\mathfrak{E}$, we can split the information into three collections for $A \subseteq B \subseteq N$:

1. Modules $\mathfrak{E}_B$ and maps $\mathfrak{E}_{B,A}$, the N-cube $\mathfrak{D}_1$ consisting of the front face of the $N+1$ cube.
2. Modules $\mathfrak{E}_{B \cup \{n+1\}}$ and maps $\mathfrak{E}_{B \cup \{n+1\}, A \cup \{n+1\}}$, the N-cube $\mathfrak{D}_2$ consisting of the back face of the $N+1$ cube.
3. Maps $\mathfrak{E}_{B \cup \{n+1\}, A}$, the map $\mathfrak{F}$ from the front face to the back face.

Arriving at Equation 3.4.3.1 is a matter of rewriting Equation 3.2.2.1 for the cube $\mathfrak{E}$, splitting into $\mathfrak{D}_1$, $\mathfrak{D}_2$ and $\mathfrak{F}$ as appropriate and using Lemmas 3.1.8 and 3.1.9 to remove any mention of $n+1$.

In a similar way to the chain complexes and their boundaries, these morphisms can also undergo a 'totalisation' process, and form chain complex maps from the chain complex induced from the domain cube to the chain complex induced from the image cube.

Definition 3.4.5. Given $\mathfrak{F} \colon \mathfrak{D}_1 \to \mathfrak{D}_2$, let $\mathrm{hTot}(\mathfrak{F})$ be the following matrix indexed by pairs of subsets of N:

$$\mathrm{hTot}(\mathfrak{F}) \colon \mathrm{hTot}(\mathfrak{D}_1) \to \mathrm{hTot}(\mathfrak{D}_2), \; \mathrm{hTot}(\mathfrak{F})_{B,A} = (-1)^{|A| + \langle A :\, B \setminus A \rangle} \mathfrak{F}_{B,A}.$$

That $\mathrm{hTot}(\mathfrak{F})$ forms a chain complex follows immediately from the condition of Equation (3.4.3.1).

We now outline the nature of the higher homotopy data of an N-cube morphism.

Proposition 3.4.6. *Let $\mathfrak{D}$ be an N-cube and $A \subseteq B \subseteq N$ with $|B \setminus A| \geq 2$. Let $|B \setminus A| = k$, then the map $\mathfrak{F}_{B,A} \in \underline{\mathrm{Hom}}(\mathfrak{D}_{1,A}, \mathfrak{D}_{2,B})_k$ is a preimage of*

$$\zeta_{B,A} = (-1)^{|B|} \left(\sum_{A \subsetneq \underset{\bullet}{S} \subseteq B} \left((-1)^{|S| + \langle S \setminus A : B \setminus S \rangle} \mathfrak{F}_{B,S} \mathfrak{D}_{1,S,A} \right) \right.$$
$$\left. - \sum_{A \subseteq \underset{\bullet}{S} \subsetneq B} \left((-1)^{|A| + \langle S \setminus A : B \setminus S \rangle} \mathfrak{D}_{2,B,S} \mathfrak{F}_{S,A} \right) \right) \tag{3.4.6.1}$$

of $\underline{\mathrm{Hom}}(\mathfrak{D}_{1,A}, \mathfrak{D}_{2,B})_{k-1}$ via the kth level boundary map of the hom-complex $\underline{\mathrm{Hom}}(\mathfrak{D}_{1,A}, \mathfrak{D}_{2,B})$, $d_{|B\setminus A|} = d_k$. That is $\mathfrak{D}_{B,B}\mathfrak{D}_{B,A} + (-1)^{|B\setminus A|+1}\mathfrak{D}_{B,A}\mathfrak{D}_{A,A} = \zeta_{B,A}$.

Proof. Note that we can rewrite Equation (3.4.3.1) as:

$$(-1)^{\langle A\colon B\setminus A\rangle + |B| + |A| + 1}(\mathfrak{D}_{B,B}\mathfrak{F}_{B,A} - (-1)^{|B\setminus A|}\mathfrak{F}_{B,A}\mathfrak{D}_{A,A}) +$$
$$\left(\sum_{A \subsetneq \underset{\bullet}{S} \subseteq B} (-1)^{|S| + \langle S : B\setminus S\rangle} \mathfrak{F}_{B,S} (-1)^{\langle A : S\setminus A\rangle} \mathfrak{D}_{1,S,A} \right.$$
$$\left. - \sum_{A \subseteq \underset{\bullet}{S} \subset B} (-1)^{\langle S : B\setminus S\rangle} \mathfrak{D}_{2,B,S} (-1)^{|A| + \langle A : S\setminus A\rangle} \mathfrak{F}_{S,A} \right)$$

and note that the same argument as seen in Lemma 3.3.4 shows that $|S| + \langle S\colon B\setminus S\rangle + \langle A\colon S\setminus A\rangle \equiv_2 |S| + \langle S\setminus A\colon B\setminus S\rangle + P$ for $P = |A| + \langle A\colon B\setminus A\rangle$. Finally follow a similar argument to Lemma 3.3.3 to see the result. $\square$

3.5 Compositions of N-cube maps

Some may notice that we have not yet defined compositions of N-cube maps. Interestingly, this is not a trivial matter.

Definition 3.5.1. Let $\mathfrak{F}\colon \mathfrak{D}_1 \to \mathfrak{D}_2$, $\mathfrak{G}\colon \mathfrak{D}_2 \to \mathfrak{D}_3$ be N-cube maps of degree 0. Then define

$$\mathfrak{G} \circ \mathfrak{F}\colon \mathfrak{D}_1 \to \mathfrak{D}_3$$

as the following collections of maps for all $A \subseteq B \subseteq N$:

$$(\mathfrak{G} \circ \mathfrak{F})_{B,A} = \sum_{A \subseteq \underset{\bullet}{S} \subseteq B} (-1)^{|S|+|A|+\langle S \backslash A\colon B \backslash S \rangle} \mathfrak{G}_{B,S} \mathfrak{F}_{S,A}.$$

For notational ease, we will drop the $\circ$.

We want the composition of two N-cube maps to also be an N-cube map, that is it satisfies Equation (3.4.3.1).

Lemma 3.5.2. *Let $\mathfrak{F}\colon \mathfrak{D}_1 \to \mathfrak{D}_2$, $\mathfrak{G}\colon \mathfrak{D}_2 \to \mathfrak{D}_3$ be N-cube maps of degree 0. Then $\mathfrak{G}\mathfrak{F}$ is an N-cube map.*

Proof. We use the fact that $\mathfrak{F}$ and $\mathfrak{G}$ are N-cube maps.

We want to show, by Equation (3.4.3.1) the following:

$$\sum_{A \subseteq \underset{\bullet}{S} \subseteq B} (-1)^{\langle S\colon B \backslash S \rangle + |A| + \langle A\colon S \backslash A \rangle} \mathfrak{D}_{3,B,S} (\mathfrak{G}\mathfrak{F})_{S,A} =$$
$$\sum_{A \subseteq \underset{\bullet}{S} \subseteq B} (-1)^{|S| + \langle S\colon B \backslash S \rangle + \langle A\colon S \backslash A \rangle} (\mathfrak{G}\mathfrak{F})_{B,S} \mathfrak{D}_{1,S,A}.$$

From the definition of composition:

$$(\mathfrak{G}\mathfrak{F})_{S,A} = \sum_{A \subseteq \underset{\bullet}{T} \subseteq S} (-1)^{|T|+|A|+\langle T \backslash A\colon S \backslash T \rangle} \mathfrak{G}_{S,T} \mathfrak{F}_{T,A}.$$

Hence

$$\begin{aligned}
&\sum_{A \subseteq \underset{\bullet}{S} \subseteq B} (-1)^{\langle S\colon B\setminus S\rangle + |A| + \langle A\colon S\setminus A\rangle} \mathfrak{D}_{3,B,S} (\mathfrak{G}\mathfrak{F})_{S,A} \\
= &\sum_{A \subseteq \underset{\bullet}{S} \subseteq B} \sum_{A \subseteq \underset{\bullet}{T} \subseteq S} (-1)^{\langle S\colon B\setminus S\rangle + |A| + \langle A\colon S\setminus A\rangle + |T| + |A| + \langle T\setminus A\colon S\setminus T\rangle} \mathfrak{D}_{3,B,S} \mathfrak{G}_{S,T} \mathfrak{F}_{T,A} \\
&= \sum_{A \subseteq \underset{\bullet}{S} \subseteq B} \sum_{A \subseteq \underset{\bullet}{T} \subseteq S} (-1)^{\langle S\colon B\setminus S\rangle + \langle A\colon S\setminus A\rangle + |T| + \langle T\setminus A\colon S\setminus T\rangle} \mathfrak{D}_{3,B,S} \mathfrak{G}_{S,T} \mathfrak{F}_{T,A} \overset{*}{=}
\end{aligned}$$

From Lemma (3.1.10):

$$\begin{aligned}
&\langle A\colon S \setminus A\rangle + \langle T \setminus A\colon S \setminus T\rangle \\
\equiv_2 &\langle T\colon S \setminus T\rangle + \langle A\colon T \setminus A\rangle + |A|
\end{aligned}$$

hence

$$\overset{*}{=} \sum_{A \subseteq \underset{\bullet}{S} \subseteq B} \sum_{A \subseteq \underset{\bullet}{T} \subseteq S} (-1)^{\langle S\colon B\setminus S\rangle + |T| + \langle T\colon S\setminus T\rangle + \langle A\colon T\setminus A\rangle + |A|} \mathfrak{D}_{3,B,S} \mathfrak{G}_{S,T} \mathfrak{F}_{T,A} \overset{*}{=}$$

Swapping summands around:

$$\overset{*}{=} \sum_{A \subseteq \underset{\bullet}{T} \subseteq B} \sum_{T \subseteq \underset{\bullet}{S} \subseteq B} (-1)^{\langle S\colon B\setminus S\rangle + |T| + \langle T\colon S\setminus T\rangle + \langle A\colon T\setminus A\rangle + |A|} \mathfrak{D}_{3,B,S} \mathfrak{G}_{S,T} \mathfrak{F}_{T,A} \overset{*}{=}$$

As $\mathfrak{G}$ is a chain map, Equation (3.4.3.1) tells us:

$$\begin{aligned}
&\sum_{T \subseteq \underset{\bullet}{S} \subseteq B} (-1)^{|T| + \langle S\colon B\setminus S\rangle + \langle T\colon S\setminus T\rangle} \mathfrak{D}_{3,B,S} \mathfrak{G}_{S,T} \\
= &\sum_{T \subseteq \underset{\bullet}{S} \subseteq B} (-1)^{|S| + \langle S\colon B\setminus S\rangle + \langle T\colon S\setminus T\rangle} \mathfrak{G}_{B,S} \mathfrak{D}_{2,S,T}
\end{aligned}$$

hence

$$\overset{*}{=} \sum_{A \subseteq \underset{\bullet}{T} \subseteq B} \sum_{T \subseteq \underset{\bullet}{S} \subseteq B} (-1)^{\langle S\colon B\setminus S\rangle + |S| + \langle T\colon S\setminus T\rangle + \langle A\colon T\setminus A\rangle + |A|} \mathfrak{G}_{B,S} \mathfrak{D}_{2,S,T} \mathfrak{F}_{T,A} \overset{*}{=}$$

As $\mathfrak{F}$ is a chain map, Equation (3.4.3.1) tells us:

$$\sum_{A \subseteq T \underset{\bullet}{\subseteq} S} (-1)^{|A|+\langle A\colon T\setminus A\rangle+\langle T\colon S\setminus T\rangle} \mathfrak{D}_{2,S,T}\mathfrak{F}_{T,A}$$
$$= \sum_{A \subseteq T \underset{\bullet}{\subseteq} S} (-1)^{|T|+\langle A\colon T\setminus A\rangle+\langle T\colon S\setminus T\rangle} \mathfrak{G}_{S,T}\mathfrak{D}_{1,T,A}$$

hence

$$\stackrel{*}{=} \sum_{A \subseteq T \underset{\bullet}{\subseteq} B} \sum_{T \subseteq S \underset{\bullet}{\subseteq} B} (-1)^{\langle S\colon B\setminus S\rangle+|S|+\langle T\colon S\setminus T\rangle+\langle A\colon T\setminus A\rangle+|T|} \mathfrak{G}_{B,S}\mathfrak{F}_{S,T}\mathfrak{D}_{1,T,A} \stackrel{*}{=}$$

From Lemma 3.1.10:

$$\langle S\colon B\setminus S\rangle + \langle T\colon S\setminus T\rangle$$
$$\equiv_2 \langle S\setminus T\colon B\setminus S\rangle + |T| + \langle T\colon B\setminus T\rangle$$

it follows that

$$\stackrel{*}{=} \sum_{A \subseteq T \underset{\bullet}{\subseteq} B} \sum_{T \subseteq S \underset{\bullet}{\subseteq} B} (-1)^{|S|+\langle A\colon T\setminus A\rangle+|T|+\langle S\setminus T\colon B\setminus S\rangle+|T|+\langle T\colon B\setminus T\rangle} \mathfrak{G}_{B,S}\mathfrak{F}_{S,T}\mathfrak{D}_{T,A}$$
$$= \sum_{A \subseteq T \underset{\bullet}{\subseteq} B} \sum_{T \subseteq S \underset{\bullet}{\subseteq} B} (-1)^{|S|+\langle A\colon T\setminus A\rangle+\langle S\setminus T\colon B\setminus S\rangle+\langle T\colon B\setminus T\rangle} \mathfrak{G}_{B,S}\mathfrak{F}_{S,T}\mathfrak{D}_{1,T,A} \stackrel{*}{=}$$

By the definition of composition:

$$\sum_{T \subseteq S \underset{\bullet}{\subseteq} B} (-1)^{|S|+\langle S\setminus T\colon B\setminus S\rangle} \mathfrak{G}_{B,S}\mathfrak{F}_{S,T}$$
$$= (-1)^{|T|}(\mathfrak{G}\mathfrak{F})_{B,T}$$

hence

$$\stackrel{*}{=} \sum_{A \subseteq T \underset{\bullet}{\subseteq} B} (-1)^{\langle A\colon T\setminus A\rangle+\langle T\colon B\setminus T\rangle+|T|} (\mathfrak{G}\mathfrak{F})_{B,T}\mathfrak{D}_{1,T,A}.$$

Simply renaming T with S and following the trail of $*$ is precisely Equation (3.4.3.1) as required.

$\square$

Lemma 3.5.3. *The composition $\circ$ is associative.*

Proof. Let $\mathfrak{X}\colon \mathfrak{D}_1 \to \mathfrak{D}_2, \mathfrak{Y}\colon \mathfrak{D}_2 \to \mathfrak{D}_3, \mathfrak{Z}\colon \mathfrak{D}_3 \to \mathfrak{D}_4$ be N-cube maps of degree 0. Then

$$\begin{aligned}(\mathfrak{Z}(\mathfrak{Y}\mathfrak{X}))_{B,A} &= \sum_{A\subseteq \underset{\bullet}{S}\subseteq B} (-1)^{|S|+|A|+\langle S\setminus A\colon B\setminus S\rangle} \mathfrak{Z}_{B,S}(\mathfrak{Y}\mathfrak{X})_{S,A} \\ &= \sum_{A\subseteq \underset{\bullet}{T}\subseteq S} \sum_{A\subseteq \underset{\bullet}{S}\subseteq B} (-1)^{|S|+|A|+\langle S\setminus A\colon B\setminus S\rangle} \\ &\qquad {}^{+|T|+|A|+\langle T\setminus A\colon S\setminus T\rangle} \mathfrak{Z}_{B,S}\mathfrak{Y}_{S,T}\mathfrak{X}_{T,A}.\end{aligned}$$

At this point, note that Lemma 3.1.10 tells us that

$$\begin{aligned}\langle S\setminus A\colon B\setminus A\rangle &\equiv_2 \\ |A| + \langle A\colon B\setminus A\rangle + \langle S\colon B\setminus S\rangle + \langle A\colon S\setminus A\rangle.\end{aligned}$$

Hence

$$\begin{aligned}&\langle S\setminus A\colon B\setminus S\rangle + \langle T\setminus A\colon S\setminus T\rangle \\ \overset{3.1.9}{\equiv}_2\ & |A| + \langle A\colon B\setminus A\rangle + \langle S\colon B\setminus S\rangle + \langle A\colon S\setminus A\rangle \\ &+|A| + \langle A\colon S\setminus A\rangle + \langle T\colon S\setminus T\rangle + \langle A\colon T\setminus A\rangle \\ \equiv_2\ & \langle A\colon B\setminus A\rangle + \langle S\colon B\setminus S\rangle + \langle T\colon S\setminus T\rangle + \langle A\colon T\setminus A\rangle.\end{aligned}$$

Using Lemma 3.1.10 again, it follows that

$$\begin{aligned}&\langle A\colon T\setminus A\rangle + \langle A\colon B\setminus A\rangle \\ \equiv_2\ & |A| + \langle T\setminus A\colon B\setminus T\rangle + \langle T\colon B\setminus T\rangle\end{aligned}$$

and similarly

$$\begin{aligned}&\langle S\colon B\setminus S\rangle + \langle T\colon S\setminus T\rangle \\ \equiv_2\ & |T| + \langle S\setminus T\colon B\setminus S\rangle + \langle T\colon B\setminus T\rangle\end{aligned}$$

therefore

$$\begin{aligned}
&\langle A\colon B \setminus A\rangle + \langle S\colon B \setminus S\rangle + \langle T\colon S \setminus T\rangle + \langle A\colon T \setminus A\rangle \\
\overset{3.1.9}{\equiv}_2\ & |A| + \langle T \setminus A\colon B \setminus T\rangle + \langle T\colon B \setminus T\rangle + |T| + \langle S \setminus T\colon B \setminus S\rangle + \langle T\colon B \setminus T\rangle \\
\equiv_2\ & |A| + \langle T \setminus A\colon B \setminus T\rangle + |T| + \langle S \setminus T\colon B \setminus S\rangle.
\end{aligned}$$

So, we can write

$$\begin{aligned}
(\mathfrak{Z}(\mathfrak{Y}\mathfrak{X}))_{B,A} &= \sum_{A\subseteq \underset{\bullet}{T}\subseteq S}\ \sum_{A\subseteq \underset{\bullet}{S}\subseteq B} (-1)^{|S|+|A|+|T|+|A|} \\
&\qquad {}^{+|A|+\langle T\setminus A\colon B\setminus T\rangle+|T|+\langle S\setminus T\colon B\setminus S\rangle}\mathfrak{Z}_{B,S}\mathfrak{Y}_{S,T}\mathfrak{X}_{T,A} \\
&= \sum_{A\subseteq \underset{\bullet}{T}\subseteq S}\ \sum_{A\subseteq \underset{\bullet}{S}\subseteq B} (-1)^{|S|+|A|} \\
&\qquad {}^{+\langle T\setminus A\colon B\setminus T\rangle+\langle S\setminus T\colon B\setminus S\rangle}\mathfrak{Z}_{B,S}\mathfrak{Y}_{S,T}\mathfrak{X}_{T,A}.
\end{aligned}$$

Now consider

$$\begin{aligned}
((\mathfrak{Z}\mathfrak{Y})\mathfrak{X})_{B,A} &= \sum_{A\subseteq \underset{\bullet}{T}\subseteq B} (-1)^{|T|+|A|+\langle T\setminus A\colon B\setminus T\rangle}(\mathfrak{Z}\mathfrak{Y})_{B,T}\mathfrak{X}_{T,A} \\
&= \sum_{A\subseteq \underset{\bullet}{S}\subseteq B}\ \sum_{A\subseteq \underset{\bullet}{T}\subseteq S} (-1)^{|T|+|S|+\langle S\setminus T\colon B\setminus S\rangle} \\
&\qquad {}^{+|T|+|A|+\langle T\setminus A\colon B\setminus T\rangle}\mathfrak{Z}_{B,S}\mathfrak{Y}_{S,T}\mathfrak{X}_{T,A}
\end{aligned}$$

and note that the signs clearly match with those on this summand for $(\mathfrak{Z}(\mathfrak{Y}\mathfrak{X}))$. Hence $\circ$ is associative. $\square$

3.6 Considering N-cubes as iterated mapping cones

We now note that an N cube can be considered as a map between two $N-1$-cubes. That is, if we take the information $\mathfrak{D}_{A,B}$ and $\mathfrak{D}_B$ for $A \subseteq B \subseteq \{1, 2, \ldots, n-1\}$, we see an $N-1$-cube. If we then take the information, maps and modules, $\mathfrak{D}_{A\cup\{n\},B\cup\{n\}}$, $\mathfrak{D}_{B\cup\{n\}}$ where $A \subseteq B \subseteq \{2, \ldots, n\}$, we see another $N-1$-cube and if we take those elements indexed by pairs of sets $A \subset B$ such that $n \notin A$, $n \in B$, which consists only of morphisms, we see an $N-1$-cube map from the first $N-1$-cube to the other.

We can generalise this concept - whenever we take out the information indexed by a subset K of N, we see a K-cube.

Lemma 3.6.1. *For an N-cube $\mathfrak{D}$ and two disjoint sets set $K, L \subseteq N$, the following collection of data:*

$$\mathfrak{D}_{B\cup\{L\}},\ B \subseteq K$$

and

$$\mathfrak{D}_{B\cup\{L\},A\cup\{L\}},\ A \subseteq B \subseteq K$$

forms a K-cube, which we write as $\mathfrak{K}_{[K,L]}$.

Proof. Follows immediately from the fact that these maps satisfy the broader set of conditions of Definition 3.2.2. □

Given an N-cube, we can repeat the argument at the start of this section to note that any K-cube as in Lemma 3.6.1 can be split into two smaller cubes and a map between them.

Remark 3.6.2. Consider the $\{1\}$-cube $\mathfrak{K}_{[\{1\},\emptyset]}$ made from the N-cube $\mathfrak{D}$. We see a 0-cube map (i.e., just a normal chain map) $\mathfrak{F}_0 = \mathfrak{D}_{\{1\},\emptyset} \colon \mathfrak{D}_\emptyset \to \mathfrak{D}_{\{1\}}$ and two 0-cubes (chain complexes) $\mathfrak{D}_\emptyset$ and $\mathfrak{D}_{\{1\}}$. The $\{2\}$ cube $\mathfrak{K}_{[\{1,2\},\emptyset]}$ can be seen as the 1-cube map $\mathfrak{F}_1 \colon \mathfrak{K}_{[\{1\},\emptyset]} \to \mathfrak{K}_{[\{1\},\{2\}]}$. Continuing this process, we can associate the entire N-cube as a selection of triples $(\mathfrak{D}_{\emptyset,\emptyset},\, \mathfrak{F}_0,\, \mathfrak{D}_{\{1\},\{1\}}), (\mathfrak{K}_{[\{1\},\emptyset]},\, \mathfrak{F}_1,\, \mathfrak{K}_{[\{1\},\{2\}]}) \ldots, (\mathfrak{K}_{[N-1,\emptyset]},\, \mathfrak{F}_{n-1},\, \mathfrak{K}_{[N-1,\{n\}]})$.

We see a similar picture for the totalisation $\mathrm{hTot}(\mathfrak{D})$.

Remark 3.6.3. Let $\mathrm{hTot}(\mathfrak{F}_i) = F_i$. Beginning with the $\{1\}$-cube $\mathfrak{K}_{[\{1\},\emptyset]}$, the totalisations of the 0-cubes $\mathfrak{D}_\emptyset$, $\mathfrak{D}_{\{1\}}$ and the map F_0 are easy to see. It is immediate that the boundary map of the mapping cone, $\mathrm{cone}(F_0)$, (taking the convention of 0.3.1) aligns precisely with the boundary map of $\mathrm{hTot}(\mathfrak{K}_{[\{1\},\emptyset]})$, the only change either of them do is to the sign of the back face of $\mathfrak{K}_{[\{1\},\emptyset]}$, that is $\mathfrak{D}_{\{1\},\{1\}}$. We apply the sign -1 to these maps when taking the totalisation of the entire 1-cube $\mathfrak{K}_{[\{1\},\emptyset]}$ and apply the same sign whenever taking the mapping cone of F_0.

More generally, we compare the signs of the three parts

$$\left(\mathfrak{K}_{[K-1,\emptyset]},\, \mathfrak{F}_{k-1},\, \mathfrak{K}_{[K-1,\{k\}]}\right)$$

of the K-cube $\mathfrak{K}_{[\{K\},\emptyset]}$ when totalising the entire cube, and totalising the three parts and taking the mapping cone of $\mathfrak{F}_{k-1}$. The signs on the elements of $\mathfrak{K}_{[K-1,\emptyset]}$ are clearly the same when totalised as part of the K-cube $\mathfrak{K}_{[\{K\},\emptyset]}$ and on its own as a $K-1$ cube, which are

$$(-1)^{\langle A\colon B\backslash A\rangle},\, A \subseteq B \subseteq K-1$$

as in both cases A, B are subsets of $K-1$, taking mapping cones will not change the sign.

The signs on the elements of $\mathfrak{F}_{k-1}$ when totalised individually are:

$$(-1)^{|A|+\langle A\colon B\backslash A\rangle},\, A \subseteq B \subseteq K-1$$

and as part of $\mathfrak{K}_{[\{K\},\emptyset]}$:

$$(-1)^{\langle A\colon B\backslash A\cup\{K\}\rangle},\, A \subseteq B \subseteq K-1.$$

Since mapping cones do not change the sign on the map when the boundary is formed, equality follows from Lemma 3.1.9.

Finally for the elements of $\mathfrak{K}_{[K-1,\{k\}]}$, when totalised on its own

$$(-1)^{\langle A\colon B\backslash A\rangle},\, A \subseteq B \subseteq K-1,$$

taking into account the affect of taking mapping cones means the sign will become:

$$(-1)^{1+\langle A\colon B\backslash A\rangle}.$$

As a part of $\mathfrak{K}_{[\{K\},\emptyset]}$ the sign is:

$$(-1)^{\langle A\cup\{k\}\colon B\backslash A\rangle},\, A \subseteq B \subseteq K-1.$$

By Lemma 3.1.8, the latter can be written as:

$$(-1)^{\langle A\cup\{k\}\colon B\backslash A\rangle} = (-1)^{1+\langle A\colon B\backslash A\rangle}$$

as required.

Hence the boundary of $\mathrm{hTot}(\mathfrak{K}_{[K,\emptyset]})$ is precisely the boundary of the mapping cone of F_{k-1}.

It follows that we can consider the totalisation $\mathrm{hTot}(\mathfrak{K})$ of the entire N-cube $\mathfrak{K}$ as the result of taking iterated mapping cones of the maps

$$F_0, F_1, \ldots F_{n-1},$$

in particular $\mathrm{hTot}(\mathfrak{K}) = \mathrm{cone}(F_{n-1})$.

3.7 The category of N-cubes

Having defined N-cubes and morphisms between them, we have at last acquired the building blocks of the category of N-cubes. We will later look at some interesting properties of this category.

Definition 3.7.1 (The category of N-cubes: ${}_N\mathsf{HC}$)**.** Let the *category of N-cubes*, ${}_N\mathsf{HC}$, have as objects the collection of N-cubes, morphisms consisting of N-cube morphisms of degree 0 and compositions as defined in Definition 3.5.1.

That this actually a category is effectively already presented – we have morphism sets (as opposed to classes) (as they can be embedded into the morphism sets in ${}_R\mathsf{Mod}$), composition is valid from Lemma 3.5.2 and associativity follows from Lemma 3.5.3.

This leaves only the presence of identity morphisms to show — but this is almost trivial, simply consider the data collection $\mathfrak{I}$ where $\mathfrak{I}_{B,B} = \mathrm{id}_{\mathfrak{D}_B}$ and $\mathfrak{I}_{A,B} = 0$ if $A \subset B$. This defines a morphism from $\mathfrak{D}$ to itself which satisfies the properties of an identity morphism in ${}_N\mathsf{HC}$.

Remark 3.7.2. The category ${}_\emptyset\mathsf{HC}$ is simply $\mathsf{Ch}({}_R\mathsf{Mod})$.

Corollary 3.7.3. *We can define a sub category of trivial N-cubes ${}_N\mathsf{THC}$ where the objects are trivial N-cubes and morphisms are 'trivial', that is there is no homotopy data.*

Remark 3.7.4. The category ${}_N\mathsf{HC}$ is additive.

Proof. Note that we can define a null N-cube with all data 0, hence we can define a zero morphism between any two N-cubes. Now, letting $+$ be defined by $(\mathfrak{F}_1 + \mathfrak{F}_2)_{B,A} = \mathfrak{F}_{1B,A} + \mathfrak{F}_{2B,A}$ defines an abelian group operator on the morphism sets which is bi-linear. That $\mathfrak{F}_1 + (-\mathfrak{F}_1) = 0$ for the obvious candidate of $-\mathfrak{F}_1$ is clear. Commutativity is also clear. That the morphism sets are bilinear under addition follows from the fact that morphism composition is bilinear, hence this composition combined with special sign rules must also be bilinear. Finally the category has finite products and coproducts which coincide, represented by $\mathfrak{D}_1 \oplus \mathfrak{D}_2$, where $(\mathfrak{D}_1 \oplus \mathfrak{D}_2)_{B,A} = \mathfrak{D}_{1,B,A} \oplus \mathfrak{D}_{2,B,A}$. $\square$

3.8 The homotopy totalisation functor

Definition 3.8.1 (The homotopy totalisation functor)**.** For an N-cube $\mathfrak{D}$ and N-cube morphism $\mathfrak{F}\colon \mathfrak{D}_1 \to \mathfrak{D}_2$ the associations

$$\mathrm{hTot}_N \colon \mathrm{Ob}({}_N\mathsf{HC}) \to \mathrm{Ob}(\mathsf{Ch}({}_R\mathsf{Mod})),\ \mathfrak{D} \mapsto \mathrm{hTot}_N(\mathfrak{D}),\ \mathfrak{F} \mapsto \mathrm{hTot}_N(\mathfrak{F})$$

from Definition 3.2.3 and Definition 3.4.5 comprise a functor

$$\mathrm{hTot}_N \colon {}_N\mathsf{HC} \to \mathsf{Ch}({}_R\mathsf{Mod})$$

for all finite totally ordered sets N.

That hTot_N is a functor follows from previous results, the object and morphism mappings are clear and well defined, that the identity morphism $\mathfrak{I}$ is mapped to the identity morphism of $\mathsf{Ch}({}_R\mathsf{Mod})$ follows from the fact that for all $B \in N$, $|B| + \langle B : \emptyset \rangle = |B| + |B| = 0$ (Lemma 3.1.7) so we do not change the signs of the non-empty entries of $\mathfrak{I}$. The composition rule

$$\mathrm{hTot}(\mathfrak{G}\mathfrak{F}) = \mathrm{hTot}(\mathfrak{G})\mathrm{hTot}(\mathfrak{F})$$

is satisfied as for any $A \subseteq B$ and a specific set $A \subseteq S \subseteq B$ between them, the signs

$$(-1)^{|A| + \langle A:B\backslash A\rangle + |S| + |A| + \langle S\backslash A\colon B\backslash S\rangle}$$

(found from totalising $\mathfrak{G}\mathfrak{F}$ and looking at the entry indexed by B, A) and the signs

$$(-1)^{|S| + \langle S:B\backslash S\rangle}(-1)^{|A| + \langle A:S\backslash A\rangle}$$

(found from taking the product of $\mathrm{hTot}(\mathfrak{G})\mathrm{hTot}(\mathfrak{F})$ and taking the entry indexed by B, A are equal from Lemma 3.1.10.

In addition, observe that hTot_N is a faithful, covariant functor for all N.

3.9 Adjoints of the functor hTot

In this section, I will discuss adjoints of the functor

$$\mathrm{hTot}\colon {}_N\mathsf{HC} \to \mathrm{Ch}({}_R\mathsf{Mod}).$$

Firstly, consider that for $n = 0$ (i.e., for $N = \emptyset$), the functor is in fact an isomorphism, making both left and right adjointness entirely trivial.

For $n \geq 1$, things are more interesting.

Definition 3.9.1. Define the functor $\imath\colon \mathrm{Ch}({}_R\mathsf{Mod}) \to {}_N\mathsf{HC}$ as having the object mapping

$$C \mapsto \imath(C);\ \imath((C)_\emptyset)_k = C_{k+n},\ \imath(C)_{\emptyset,\emptyset} = d_C,$$

that is the shift of the chain complex C at the point indexed by $\emptyset$, with zero for all other $B \subseteq N$, and the morphism mapping

$$f\colon C \to D \mapsto \imath(f)\colon \imath(C) \to \imath(D);\ \imath(f)_{\emptyset,\emptyset} = f.$$

That $\imath$ satisfies the conditions of a functor, that identities are mapped to identities in the image and that the functor is compatible of morphisms, follows by easy observation.

Proposition 3.9.2. *The functor $\imath$ is left adjoint to the functor* hTot.

Proof. Taking objects $\mathfrak{D} \in {}_N\mathsf{HC}$, $C \in \mathrm{Ch}({}_R\mathsf{Mod})$, note that a morphism f in the set $\mathrm{Mor}_{\mathrm{Ch}({}_R\mathsf{Mod})}(C, \mathrm{hTot}(\mathfrak{D}))$ is a 2^n-tuple

$$(f_B)_{B\subseteq N}$$

where $f_B\colon C \to \mathrm{hTot}(\mathfrak{D})_B$ is a map of degree $|B| - |N|$.

Let $\alpha(f)$ be the morphism in $\mathrm{Mor}_{{}_N\mathsf{HC}}(\imath(C), \mathfrak{D})$ such that $\alpha(f)_{\emptyset,B} = f_B$ for $B \subseteq N$ and $\alpha(f)_{A,B} = 0$ for $\emptyset \subset A \subseteq B \subseteq N$.

From the definition of totalisation, the map f must satisfy the following conditions for all $B \subseteq N$:

$$\sum_{\emptyset \subseteq S \subseteq B} (-1)^{\langle S \colon B \setminus S \rangle} \mathfrak{D}_{B,S} f_S = f_B d$$

while the map $\alpha(f)$ must satisfy the following from Equation (3.4.3.1) for all $B \subseteq N$:

$$\sum_{\emptyset \subseteq S \subseteq B} (-1)^{|\emptyset| + \langle \emptyset \colon S \rangle + \langle S \colon B \setminus S \rangle} \mathfrak{D}_{B,S} \alpha(f)_S = (-1)^{|\emptyset| + \langle \emptyset \colon \emptyset \rangle + \langle \emptyset \colon B \rangle} \alpha(f)_B \imath(C)_{\emptyset,\emptyset}.$$

Since $|\emptyset| = \langle \emptyset \colon S \rangle = \langle \emptyset \colon B \rangle = 0$ these are the same conditions hence $\alpha(f)$ is a valid N-cube map.

Therefore it is clear that $\alpha \colon \mathrm{Mor}_{\mathrm{Ch}({}_R\mathsf{Mod})}(C, \mathrm{hTot}(\mathfrak{D})) \to \mathrm{Mor}_{{}_N\mathsf{HC}}(\imath(C), \mathfrak{D})$ is an isomorphism as required. Hence the functor $\imath$ is left adjoint to the functor hTot. $\square$

Note that for $N = \emptyset$, we have that $\mathrm{hTot}^{-1} = \imath$, as hoped.

A similar functor forms a right adjoint to hTot.

Definition 3.9.3. Define the functor $\gamma \colon \mathrm{Ch}({}_R\mathsf{Mod}) \to {}_N\mathsf{HC}$ as having the object mapping

$$C \mapsto \gamma(C);\ \gamma(C)_N = C,\ \gamma(C)_{N,N} = (-1)^{|N|} d_C,$$

that is the unshifted complex C at the point indexed by N, with zero for all other $B \subseteq N$, and the morphism mapping

$$f \colon C \to D \mapsto \gamma(f) \colon \gamma(C) \to \gamma(D);\ \gamma(f)_{N,N} = f.$$

That γ satisfies the conditions of a functor, that identities are mapped to identities in the image and that the functor is associative on morphisms, follows by easy observation.

Proposition 3.9.4. *The functor γ is right adjoint to the functor* hTot.

Proof. Taking objects $\mathfrak{D} \in {}_N\mathsf{HC}$, $C \in \mathrm{Ch}({}_R\mathsf{Mod})$, note that a morphism g in the set $\mathrm{Mor}_{\mathrm{Ch}({}_R\mathsf{Mod})}(\mathrm{hTot}(\mathfrak{D}), C)$ is a 2^n-tuple

$$(g_B)_{B \subseteq N}$$

where $g_B \colon \mathrm{hTot}(\mathfrak{D})_B \to C$ is a map of degree $|N| - |B|$. Let $\beta(g)$ be the morphism in $\mathrm{Mor}_{{}_N\mathsf{HC}}(\mathfrak{D}, \gamma(C))$ such that $\beta(g)_{N,B} = (-1)^{|B|+\langle B:N\backslash B\rangle} g_B$ for $B \subseteq N$ and $\beta(g)_{B,A} = 0$ for $\emptyset \subseteq A \subseteq B \subset N$.

The map g must satisfy the following:

$$\sum_{A \subseteq S \subseteq N} (-1)^{\langle A\colon S\backslash A\rangle} g_S \mathfrak{D}_{S,A} = dg_B$$

while $\beta(g)$ must satisfy:

$$\sum_{A \subseteq S \subseteq N} (-1)^{\langle A\colon S\backslash A\rangle + |S| + \langle S\colon N\backslash S\rangle} \beta(g)_S \mathfrak{D}_{S,A} = (-1)^{|B|+\langle B\colon N\backslash B\rangle + |N|} \gamma(C)_{N,N} \beta(g)_B$$

which by the definition of $\beta(g)_B = (-1)^{|B|+\langle B:N\backslash B\rangle} g_B$ and $\gamma(C)_{N,N} = (-1)^{|N|} d$ are the same conditions.

It is clear that $\beta \colon \mathrm{Mor}_{\mathrm{Ch}({}_R\mathsf{Mod})}(\mathrm{hTot}(\mathfrak{D}), C) \to \mathrm{Mor}_{{}_N\mathsf{HC}}(\mathfrak{D}, \gamma(C))$ is an isomorphism and hence γ is right adjoint to htot.

□

Again, note that for $N = \emptyset$, we have that $\mathrm{hTot}^{-1} = \gamma = \imath$, as hoped.

3.10 Comparing this paper's sign convention to that used in [HQ16]

The homotopy commutative N-cubes first appeared within a paper of Hüttemann and Quinn [HQ16, Definition I.3.1]. Finally for this section, we will show that the convention used here for hTot and Tot in [HQ16] are equivalent.

Definition 3.10.1. Let N be a finite well ordered set. Let $A \subseteq B \subseteq N$ be subsets with suborderings of the ordering of N. Write $\overset{B}{\sum} A$ to represent the sum of the position of the elements of A in relation to the subordering of N held by B. That is, for a set N, subset $B = \{y_1 < y_2 < ... < y_b\} \subseteq N$ and a subset $A = \{y_{i_1} < y_{i_2} < \ldots < y_{i_a}\} \subseteq B$ where $1 \leq i_j \leq b$ we have that $\overset{B}{\sum} A = \sum_{1\leq j\leq a} i_j$.

Remark 3.10.2. In particular, $\overset{B}{\sum} B$ is the $|B|$th triangular number and if $A \subseteq B$ then $\overset{B}{\sum}(B \setminus A) = \overset{B}{\sum} B - \overset{B}{\sum} A$.

We can therefore associate $[B\colon A]$, $A \subseteq B$ from [HQ16, Definition I.1.1] with

$$(-1)^{|B|-|A|+\overset{B}{\sum}(B\setminus A)} = (-1)^{|B|-|A|+\overset{B}{\sum}B-\overset{B}{\sum}A}.$$

Proposition 3.10.3. *For $A \subseteq B$, label $B \setminus A = \{x_1 < .. < x_i < .. < x_t\}$ for some $t \in \mathbb{N}$. There is an equality*

$$\sum^{B} A = \sum^{A} A + |P^{A}_{B\setminus A}|.$$

Proof. Firstly, write $A = A_0 \cup A_1 \cup A_2 \cup \ldots \cup A_t$, where A_k is the collection of all elements of A which are greater than precisely k many elements of B/A. Observe immediately that

$$\sum^{B} A = \sum_{k=0}^{t} (\sum^{B} A_k).$$

It also follows that

$$\sum^{B} A_k = \sum^{A} A_k + k|A_k|,$$

as the difference between the position of any element of A_k in relation to the ordering of B and their position of that element in relation to the ordering of A is precisely k by the definition of A_k. Hence

$$\sum^{B} A = \sum_{k=0}^{t} \left(\sum^{A} A_k + k|A_k| \right) = \sum_{k=0}^{t} \sum^{A} A_k + \sum_{k=0}^{t} k|A_k|.$$

Now, clearly $\sum_{k=0}^{t} \overset{A}{\sum} A_k = \overset{A}{\sum} A$. It remains to observe that for all i, $\sum_{k=i}^{t} |A_k| = |P^{A}_{x_i}|$. On the left, we have the number of elements of A that are bigger than at least i elements of $B \setminus A$, or equivalently no smaller than

x_i, and on the right is the set of elements of A that are bigger that x_i, hence equality is clear. Finally, observe that

$$\sum_{k=0}^{t} k|A_k| = \sum_{i=1}^{t}\sum_{k=i}^{t} |A_k| = \sum_{i=1}^{t} |P^A_{x_i}| = |P^A_{B\setminus A}|$$

and therefore

$$\sum^{B} A = \sum_{k=0}^{t}\sum^{A} A_k + \sum_{k=0}^{t} k|A_k| = \sum^{A} A + |P^A_{B\setminus A}|$$

as required. ◻

Now we can write

$$[B\colon A] = (-1)^{|B|-|A|+\sum^{B} B - \sum^{A} A + |P^A_{B\setminus A}|}.$$

Proposition 3.10.4 (Sign Convention Comparison)**.** *There is an equality modulo 2:*

$$(-1)^{|A||B|}[B\colon A] \equiv_2 (-1)^{|B|+|A|+\sum^{B} B+\sum^{A} A+\langle A\colon B\setminus A\rangle}$$

where $[B\colon A] = (-1)^{|B|-|A|+\sum^{B} B-\sum^{A} A+|P^A_{B\setminus A}|}$ *is taken from [HQ16, Definition I.1.1].*

Proof. Observe that:

$$(-1)^{|A||B|}[B : A] = (-1)^{|A||B|+|B|-|A|+\sum^{B} B-\sum^{A} A-|P^A_{B\setminus A}|}.$$

If we assume $|B| - |A|$ is odd, then

$$(-1)^{|A||B|+|B|-|A|+\sum^{B} B-\sum^{A} A-|P^A_{B\setminus A}|} = (-1)^{|A||B|-1+\sum^{B} B-\sum^{A} A-\langle A:B\setminus A\rangle}$$

now note that in this case $|A| \equiv_2 |B| + 1$ so

$$\begin{aligned} |A||B| - 1 &\equiv_2 (|B| + 1)|B| + |B| - |A| \\ &\equiv_2 |B|^2 + |B| + |B| - |A| \\ &\equiv_2 |B|^2 - |A| \equiv_2 |B| - |A| \end{aligned}$$

hence

$$(-1)^{|A||B|}[B:A] \equiv_2 (-1)^{|B|-|A|+\sum^{B} B-\sum^{A} A-\langle A:B\setminus A\rangle}$$
$$\equiv_2 (-1)^{|B|+|A|+\sum^{B} B+\sum^{A} A+\langle A:B\setminus A\rangle}$$

as hoped. If we assume $|B|-|A|$ is even, then

$$(-1)^{|A||B|+|B|-|A|+\sum^{B} B-\sum^{A} A-|P^{A}_{B\setminus A}|} = (-1)^{|A||B|+|B|+\sum^{B} B-\sum^{A} A-|A|-|P^{A}_{B\setminus A}|}$$
$$= (-1)^{|A||B|+|B|+\sum^{B} B-\sum^{A} A-\langle A\colon B\setminus A\rangle}$$

and as $|A| \equiv_2 |B|$

$$|A||B|+|B| \equiv_2 |B|^2+|B|$$
$$\equiv_2 0 \equiv_2 |B|-|A|$$

so

$$(-1)^{|A||B|}[B:A] \equiv_2 (-1)^{|B|-|A|+\sum^{B} B-\sum^{A} A-\langle A\colon B\setminus A\rangle}$$
$$\equiv_2 (-1)^{|B|+|A|+\sum^{B} B+\sum^{A} A+\langle A\colon B\setminus A\rangle}$$

in this case also. Hence the result is shown. □

Definition 3.10.5 (Totalisation from Definition I.2.2 [HQ16])**.** Let $\mathfrak{D}$ be an N-diagram, then the *totalisation* of $\mathfrak{D}$ consists of the graded R-module $\mathrm{Tot}(\mathfrak{D})$ where

$$\mathrm{Tot}(\mathfrak{D})_\ell = \bigoplus_{A\subseteq N} (\mathfrak{D}_A)_{\ell+|A|}$$

and module homomorphisms consisting of matrices

$$D' = (D'_{B,A})_{A\subseteq B\subseteq N}\colon \mathrm{Tot}(\mathfrak{D})_\ell \to \mathrm{Tot}(\mathfrak{D})_{\ell-1}$$

indexed by pairs of subsets $A \subseteq B \subseteq N$ where for $A \subseteq B$,

$$D'_{B,A} = (-1)^{|A||B|}[B:A]\cdot \mathfrak{D}_{B,A} \qquad (3.10.5.1)$$

and 0 in all other entries.

In Definition 3.10.5, a graded R-module refers to a collection of R-module modules indexed by $\mathbb{Z}$. Also note that [HQ16] deals with cochain complexes not chain complexes, so suitable changes have been made here to fit into the chain complex setting.

Proposition 3.10.6. *Given an N-cube $\mathfrak{D}$ the diagonal matrix map $\mathfrak{P}$ where $\mathfrak{P}_{B,B} = \mathfrak{P}_B = (-1)^{|B|+\sum^{B} B}$ is an isomorphism of graded modules from the chain complex* $(\mathrm{hTot}(\mathfrak{D}),\, D)$ *to* $(\mathrm{Tot}(\mathfrak{D}),\, D')$*, such that $\mathfrak{P}D = D'\mathfrak{P}$.*

Proof. Note that $\mathfrak{P}$ is its own inverse, that it is an isomorphism of graded modules is clear. We need to show that

$$\mathfrak{P}D\mathfrak{P} = D'.$$

Since the underlying maps of both boundaries are the same, only the comparison of signs takes effort. The sign of the entry indexed by B, A of $\mathfrak{P}D\mathfrak{P}$ is

$$|B| + \sum^{B} B + \langle A : B \setminus A\rangle + |A| + \sum^{A} A.$$

However, Proposition 3.10.4 immediately tells us that this is equivalent modulo 2 to $(-1)^{|A||B|}[B\colon A]$ as required. $\square$

This means that the images of hTot and Tot are the same up to isomorphism, in particular one is a chain complex if and only if the other is as well. Also, the definitions of homotopy commutative cubes in Definition 3.2.2 and in Definition I.3.1 of [HQ16] coincide.

4. FINITE DOMINATION IMPLIES CONTRACTIBLITY OF NOVIKOV HOMOLOGY

Now we work towards the opposite implication of the main result, that an R_0-finitely dominated chain complex of R-modules has trivial Novikov homology. Throughout this section let C be a bounded complex of finitely generated free R-modules homotopy equivalent to a bounded complex of finitely generated projective R_0-modules, an equivalent condition to R_0-finite domination via Theorem 0.1.4.

4.1 Resolution of R

We begin by forming a canonical resolution of a strongly $\mathbb{Z}^n$-graded ring R.

Let R be a strongly graded $\mathbb{Z}^n$-graded ring for $n \in \mathbb{Z}^n$. Let $\{e_k; 1 \leq k \leq n\}$ be a basis for $\mathbb{Z}^n$. Recall the strongly $\mathbb{Z}$-graded rings from Definition 2.3.3, $R^{(k)} = \bigoplus\limits_{m \in \mathbb{Z}} R_{me_k}$, where $R^{(k)}_m = R_{me_k}$.

More generally, we can define other graded rings with support in planes of dimension $1 \leq \ell \leq n$.

Definition 4.1.1. For $1 \leq \ell \leq n$, $1 \leq k_j \leq n$, $1 \leq j \leq \ell$, let

$$R^{(k_1,k_2,\ldots,k_\ell)} = \bigoplus_{(m_j)_{1 \leq j \leq \ell} \in \mathbb{Z}^\ell} R_{\sum_{j=1}^{l} m_j e_{k_j}},$$

where $R^{(k_1,k_2,\ldots,k_\ell)}_{m_1,\ldots,m_\ell} = R_{\sum_{j=1}^{\ell} m_j e_{k_j}}$, be the *restriction of* R *to the* $\{1 \leq j \leq \ell\}$ *axes*, itself a strongly $\mathbb{Z}^\ell$-graded ring.

Having defined partition of unities and some maps that make use of them, namely splitting maps β and product maps π, we need to add a few more similar maps into our arsenal.

Definition 4.1.2 (Torus map of degree ρ)**.** Given a $\mathbb{Z}^n$ graded unital ring R, R-module M, $\rho \in \mathbb{Z}^n$ and a partition of unity $1 = \sum_{j=1}^{q} u_j v_j$ of type $(-\rho, \rho)$, define the *torus map of degree* ρ, $\mu_\rho \colon M \underset{R_0}{\otimes} R \to M \underset{R_0}{\otimes} R$, as the map

$$m \otimes r \mapsto \sum_{j=1}^{q} m u_j \otimes v_j r.$$

This map is R_0-balanced as for $s \in R_0$,

$$\begin{aligned}\mu_\rho(m \otimes sr) = \sum_{j=1}^{q} m u_j \otimes v_j sr &= \sum_{j=1}^{q}\sum_{k=1}^{q} m u_j \otimes v_j s u_k v_k r \\ &= \sum_{j=1}^{q}\sum_{k=1}^{q} m u_j v_j s u_k \otimes v_k r \\ &= \sum_{k=1}^{q} m s u_k \otimes v_k r = \mu_\rho(ms \otimes r)\end{aligned}$$

as $v_j s u_k \in R_0$.

Let μ_ρ and μ'_ρ be two torus maps of degree ρ using two partitions of unity $\sum_{j=1}^{q} u_j v_j$ and $\sum_{j=1}^{q'} u'_j v'_j$. Then $\mu_\rho = \mu'_\rho$, as

$$\begin{aligned}\sum_{j=1}^{q} m u_j \otimes v_j r &= \sum_{j=1}^{q}\sum_{j=1}^{q'} m u_j \otimes v_j u'_j v'_j r \\ &= \sum_{j=1}^{q}\sum_{j=1}^{q'} m u_j v_j u'_j \otimes v'_j r = \sum_{j=1}^{q'} m u'_j \otimes v'_j r\end{aligned}$$

as $v_j u'_j \in R_0$. This tells us that the map μ_p is independent of the choice of partition of unity. It is easy to see that μ_p is an R_0-R_0-bimodule map.

Remark 4.1.3. For a chain complex C of right R-modules, μ_ρ is a chain map:

$$d\mu_\rho(c \otimes r) = \sum_j d(c u_j) \otimes v_j r = \sum_j d(c) u_j \otimes v_j r = \mu_\rho d(c \otimes r).$$

Corollary 4.1.4. *Let μ_ρ, μ_σ be torus maps of degree ρ and σ respectively. Then $\mu_\rho \mu_\sigma = \mu_{\rho+\sigma}$ and thus $\mu_\rho \mu_\sigma = \mu_\sigma \mu_\rho$.*

Remark 4.1.5. For all $p, q \in \mathbb{Z}$, $1 \leq j, k \leq n$, the following diagram is commutative:

$$\begin{tikzcd} R \otimes R \arrow[r, "\mathrm{id} - \mu_{pe_k}"] \arrow[d, "\mathrm{id} - \mu_{qe_j}"'] & R \otimes R \arrow[d, "\mathrm{id} - \mu_{qe_j}"'] \\ R \otimes R \arrow[r, "\mathrm{id} - \mu_{pe_k}"] & R \otimes R \end{tikzcd}$$

Proof. Observe that $(\mathrm{id} - \mu_{pe_k}) \circ (\mathrm{id} - \mu_{qe_j}) = \mathrm{id} - \mu_{pe_k} - \mu_{qe_j} + \mu_{pe_k + qe_j} = (\mathrm{id} - \mu_{qe_j}) \circ (\mathrm{id} - \mu_{pe_k})$. □

Corollary 4.1.6. *Let $1 \leq k_\alpha, j_\beta \leq n$, $k_\alpha \neq j_\beta$. Then* $R^{(k_1,\ldots,k_l)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R^{(j_1,\ldots,j_m)} \cong R^{(k_1,\ldots,k_l,j_1,\ldots,j_m)}$ *for all $l + m \leq n$.*

Proof. We can quote Proposition 2.3.1 immediately, as the sets of the supports satisfy the necessary conditions. □

For all $1 \leq k \leq n$, let

$$1 = \sum_{\alpha=1}^{q} x_\alpha^{(k)} y_\alpha^{(k)}$$

be a partition of unity of form $(-e_k, e_k)$. There are maps

$$\mu_{(k)} \colon R^{(k)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R^{(k)} \to R^{(k)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R^{(k)},\ r \otimes s \mapsto \sum_{\alpha=1}^{q} r s x_\alpha^{(k)} \otimes y_\alpha^{(k)}.$$

Observe by [HS16] Proposition 3.2, there are exact sequences for all $1 \leq k \leq n$:

$$0 \to R^{(k)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R^{(k)} \overset{\mathrm{id}-\mu_{(k)}}{\to} R^{(k)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R^{(k)} \overset{\pi}{\to} R^{(k)} \to 0.$$

Let

$$\Lambda^{(k)} = \left(R^{(k)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R^{(k)} \overset{\mathrm{id}-\mu_{(k)}}{\to} R^{(k)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R^{(k)} \right)$$

considered as a chain complex concentrated in degrees 0 and 1. The complexes $\Lambda^{(k)}$ have trivial homology at every point except H_0. Since $H_0(\Lambda^{(k)})$ are the left projective and right projective R_0-modules $R^{(k)}$ and the maps

$\mathrm{id} - \mu_{(k)}$ have trivial kernels the Künneth theorem [ML95, P166 Theorem 10] tells us that $\Lambda^{(j)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} \Lambda^{(k)}$ has trivial homology except at 0, where

$$H_0(\Lambda^{(j)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} \Lambda^{(k)}) = H_0(\Lambda^{(j)}) \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} H_0(\Lambda^{(k)}) = R^{(j)} \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R^{(k)} \cong R^{(j,k)}.$$

Definition 4.1.7 (Ordered tensor product)**.** Given complexes X_k, $1 \leq k \leq n$ of R-R bimodules, call

$$\overline{\bigotimes_{\substack{R_{0\mathbb{Z}^n}\\ 1\leq k\leq n}}} X_k = \overline{\bigotimes_{1\leq k\leq n}} X_k = X_1 \underset{R_{0\mathbb{Z}^n}}{\otimes} X_2 \underset{R_{0\mathbb{Z}^n}}{\otimes} \ldots \underset{R_{0\mathbb{Z}^n}}{\otimes} X_n$$

the *ordered tensor product* of X_k, $1 \leq k \leq n$.

Proposition 4.1.8. *The complex*

$$\overline{\bigotimes_{1\leq k\leq n}} \Lambda^{(k)}$$

is a resolution of R as R-R-bimodules.

Proof. Note firstly that the modules in each $\Lambda^{(k)}$ are finitely generated complexes of $R_{0_{\mathbb{Z}^n}}$ modules. Next, argue with induction. It is true for $n = 2$ from usage of the Künneth theorem as argued before. Let it be true for all $k < n - 1$. Then, noting that $R^{(1,2,\ldots,n-1)}$ is both a left projective and right projective R_0-module, we see that

$$\left(\overline{\bigotimes_{1\leq k\leq n-1}} \Lambda^{(k)}\right) \underset{R_{0\mathbb{Z}^n}}{\otimes} \Lambda^{(n)}$$

is a resolution of $R^{(1,2,\ldots,n-1)} \underset{R_{0\mathbb{Z}^n}}{\otimes} R^{(n)}$ via the Künneth theorem which is isomorphic to R by a product map via Corollary 4.1.6 as required. $\square$

4.2 Representing the resolution of R as an N-cube

This section will introduce and make use of a number of N-cubes. Firstly, we express the resolution of R as the totalisation of a commutative N-cube.

Then we argue that there is a commutative N-cube, with $C \underset{R_0}{\otimes} R$ at each vertex, with totalisation homotopy equivalent to C. We begin by introducing a new map that makes use of the partition of unities, U, which will allow us to show that $\overline{\bigotimes\limits_{1\leq k\leq n}} \Lambda^{(k)}$ is isomorphic to the totalisation of an N-cube with $R \underset{R_0}{\otimes} R$ at every vertex.

Proposition 4.2.1. *There is an isomorphism*

$$U\colon \overline{\bigotimes_{1\leq k\leq n}} (R^{(k)} \underset{R_{0\mathbb{Z}^n}}{\otimes} R^{(k)}) \cong R \underset{R_{0\mathbb{Z}^n}}{\otimes} R$$

of R_0-R_0-bimodules. This isomorphism U will take the form

$$U\colon \overline{\bigotimes_{1\leq k\leq n}} (R^{(k)}_{m_k} \otimes R^{(k)}_{m'_k}) \to R_{\sum_{k=1}^n m_k e_k} \otimes R_{\sum_{k=1}^n m'_k e_k},$$

$$\overline{\bigotimes_{1\leq k\leq n}} (r^{(k)}_{m_k} \otimes s^{(k)}_{m'_k}) \mapsto \sum_{j=1}^{p} r^{(1)}_{m_1} s^{(1)}_{m'_1} r^{(2)}_{m_2} s^{(2)}_{m'_2} ... r^{(n)}_{m_n} s^{(n)}_{m'_n} \delta_j \otimes \gamma_j$$

where $1 = \sum_{j=1}^p \delta_j \gamma_j$ is a partition of unity of form $(-\sum_{k=1}^n m'_k e_k, \sum_{k=1}^n m'_k e_k)$.

Proof. Argue by induction. For a strongly $\mathbb{Z}^2$-graded ring, consider the composition of isomorphisms:

$$\begin{aligned}
&R^{(x)}_n \otimes R^{(x)}_{n'} \otimes R^{(y)}_m \otimes R^{(y)}_{m'} \xrightarrow{\mathrm{id}\otimes\pi_{n'e_1,me_2}\otimes\mathrm{id}} \\
&R^{(x)}_n \otimes R_{n',m} \otimes R^{(y)}_{m'} \xrightarrow{\mathrm{id}\otimes\beta_{me_2,n'e_1}\otimes\mathrm{id}} \\
&R^{(x)}_n \otimes R^{(y)}_m \otimes R^{(x)}_{n'} \otimes R^{(y)}_{m'} \xrightarrow{\pi_{ne_1,me_2}\otimes\pi_{n'e_1,m'e_2}} \\
&R_{n,m} \otimes R_{n',m'}
\end{aligned}$$

where

$$\begin{aligned}
r_x \otimes r_{x'} \otimes r_y \otimes r_{y'} &\overset{\mathrm{id}\otimes\pi_{n'e_1,me_2}\otimes\mathrm{id}}{\mapsto} r_x \otimes r_{x'} r_y \otimes r_{y'} \\
&\overset{\mathrm{id}\otimes\beta_{me_2,n'e_1}\otimes\mathrm{id}}{\mapsto} \sum_{j=1}^{q} r_x \otimes r_{x'} r_y u_j \otimes v_j \otimes r_{y'} \\
&\overset{\pi_{ne_1,n'e_1}\otimes\mathrm{id}\otimes\pi_{me_2,m'e_2}}{\mapsto} \sum_{j=1}^{q} r_x r_{x'} r_y u_j \otimes v_j r_{y'} \\
&= \sum_{k=1}^{p}\sum_{j=1}^{q} r_x r_{x'} r_y u_j \otimes v_j r_{y'} \delta_k \gamma_k \\
&= \sum_{k=1}^{p}\sum_{j=1}^{q} r_x r_{x'} r_y u_j v_j r_{y'} \delta_k \otimes \gamma_k \\
&= \sum_{k=1}^{p} r_x r_{x'} r_y r_{y'} \delta_k \otimes \gamma_k
\end{aligned}$$

where $\sum\limits_j u_j v_j$ is a partition of unity of type $\big((-n',0),(n',0)\big)$, $\sum_{k=1}^{p}\delta_k\gamma_k$ is a partition of unity of type $\big((-n',-m')(n',m')\big)$ (as required) which makes $v_j r_{y'} \delta_k \in R_0$.

It is clear that via repeated applications of twist maps we can show that

$$\overline{\bigotimes_{1\le k\le n-1}} (R^{(k)} \underset{R_0}{\otimes} R^{(k)}) \underset{R_0}{\otimes} (R^{(n)} \underset{R_0}{\otimes} R^{(n)}) \cong \overline{\bigotimes_{1\le k\le n}} R^{(k)} \underset{R_0}{\otimes} \overline{\bigotimes_{1\le k\le n}} R^{(k)}.$$

Let it be true for $n-1$. Note that the induction hypothesis tells us that

$$U' \otimes \mathrm{id} \otimes \mathrm{id} \colon \overline{\bigotimes_{1\le k\le n}} (R^{(k)} \otimes R^{(k)}) \cong R^{(1,..n-1)} \otimes R^{(1,..n-1)} \otimes (R^{(n)} \otimes R^{(n)})$$

via a map $U' \otimes \mathrm{id} \otimes \mathrm{id}$ where $U' \colon \overline{\bigotimes\limits_{1\le k\le n}} (R^{(k)} \otimes R^{(k)}) \to R^{(1,..n-1)} \otimes R^{(1,..n-1)}$. For a primitive tensor $r \in \overline{\bigotimes\limits_{1\le k\le n}} (R^{(k)} \otimes R^{(k)})$, the image of r by U' is $U'(r) = \sum_{j=1}^{p'} r\delta'_j \otimes \gamma'_j$ where $\sum_{j=1}^{p'} \delta'_j \gamma'_j$ is a partition of unity of type $(-\rho',\rho')$ for some $\rho' \in \mathbb{Z}^{n-1}$. For $r \in \overline{\bigotimes\limits_{1\le k\le n}} (R^{(k)} \otimes R^{(k)})$ and $s \otimes s' \in R^{(n)}_{\omega} \otimes R^{(n)}_{\omega'}$, there is an isomorphism

$$\begin{aligned}\sum_{j=1}^{p'} r\delta'_j \otimes \gamma'_j \otimes s \otimes s' &\overset{\mathrm{id}\otimes\pi_{\rho',\omega e_n}\otimes\mathrm{id}}{\mapsto} \sum_{j=1}^{p'} r\delta'_j \otimes \gamma'_j s \otimes s' \\ &\overset{\mathrm{id}\otimes\beta_{\omega e_n,\rho'}\otimes\mathrm{id}}{\mapsto} \sum_{k=1}^{q}\sum_{j=1}^{p'} r\delta'_j \otimes \gamma'_j s u_k \otimes v_k \otimes s' \\ &\overset{\pi_{\omega e_n,\rho}\otimes\mathrm{id}\otimes\pi_{\rho',\omega' e_n}}{\mapsto} \sum_{k=1}^{q} r s u_k \otimes v_k s' \\ &= \sum_{j=1}^{p}\sum_{k=1}^{q} r s u_k \otimes v_k s' \delta_j \gamma_j = \sum_{j=1}^{p} r s s' \delta_j \otimes \gamma_j\end{aligned}$$

where we know that $1 = \sum_{j=1}^{p} \delta_j \gamma_j$ is a partition of unity of the required type (i.e., $(-\rho' - \omega', \rho' + \omega')$ so that $v_k s' \delta_j \in R_0$) and hence

$$U\colon r \otimes s \otimes s' \mapsto \sum_{j=1}^{p} r s s' \delta_j \otimes \gamma_j$$

satisfies the definition of U we want. $\square$

Remark 4.2.2. We can see with a similar argument used to show splitting and torus maps are independent of the choice of partition of unity that the maps U also have this property.

Definition 4.2.3. Given a strongly $\mathbb{Z}^n$-graded ring R, $N = \{1, 2.., n\}, \emptyset \subseteq A \subseteq N$ define $\mathfrak{R}$ as the following N-cube:

$$\mathfrak{R}_A = R \underset{R_0}{\otimes} R,\ \mathfrak{R}_{A\cup\{i\},A} = \mu_{e_i} \text{ if } i \notin A$$

and 0 elsewhere. Since every two-dimensional face of the cube is a diagram of the form of that in Corollary 4.1.4, we know that the entire cube is commutative and hence the N-cube conditions are met.

Definition 4.2.4. Given $\mathfrak{R}$ as above, we can define another N-cube, a *twisted N-cube* $\mathrm{Tw}(\mathfrak{R})$, with the same modules as $\mathfrak{R}$ and only non-zero maps

$$\mathrm{Tw}(\mathfrak{R})_{A\cup\{i\},A} = \mathrm{id} - \mu_{e_i} \text{ if } i \notin A.$$

This is a valid N-cube from Remark 4.1.5.

Write $\mathfrak{T}(\mathfrak{R}) = \mathrm{hTot}\,(\mathrm{Tw}(\mathfrak{R}))$ (using the notation hTot defined In Definition 3.2.3). Note that $\mathfrak{T}(\mathfrak{R})$ is a chain complex concentrated in degrees 0 to n, with the module

$$\bigoplus_{\binom{n}{j}} R \underset{R_0}{\otimes} R$$

at level j.

We wish to show that the totalisation of the N-cube $\mathfrak{R}$ is homotopy equivalent to R. This is done by showing that $\mathfrak{T}(\mathfrak{R})$ is isomorphic to $\overline{\bigotimes_{1\le k\le n}} \Lambda^{(k)}$, which we know is a resolution of R via Proposition 4.1.8.

Proposition 4.2.5. *The map*

$$\overline{U}\colon \overline{\bigotimes_{1\le k\le n}} \Lambda^{(k)} \to \mathfrak{T}(\mathfrak{R})$$

where

$$\overline{U}_j = \bigoplus_{\binom{n}{j}} U\colon \Big(\overline{\bigotimes_{1\le k\le n}} \Lambda^{(k)}\Big)_j \to (\mathfrak{T}(\mathfrak{R}))_j$$

is an isomorphic chain map.

Proof. To show that $\overline{U}$ is a chain map it is enough to show that the following diagram

$$\begin{array}{ccc}
\overline{\bigotimes_{1\le k\le n}} \left(R^{(k)} \underset{R_0}{\otimes} R^{(k)}\right) & \xrightarrow{U} & R \underset{R_0}{\otimes} R \\
\Big\downarrow{\scriptstyle \overline{\mu}_{(l)}} & & \Big\downarrow{\scriptstyle \mu_{e_l}} \\
\overline{\bigotimes_{1\le k\le n}} \left(R^{(k)} \underset{R_0}{\otimes} R^{(k)}\right) & \xrightarrow{U} & R \underset{R_0}{\otimes} R
\end{array}$$

where $\overline{\mu}_{(l)} = \mathrm{id}_{R^{(1)}\otimes R^{(1)}} \otimes \cdots \otimes \mu_{(l)} \otimes \cdots \otimes \mathrm{id}_{R^{(n)}\otimes R^{(n)}}$ for a torus map $\mu_{(\ell)}\colon R^{(\ell)} \otimes R^{(\ell)} \to R^{(\ell)} \otimes R^{(\ell)}$ of degree $(-1, 1)$, is commutative for all $1 \le \ell \le n$.

Given $r = \overline{\bigotimes_{1\leq k\leq n}} r^{(k)}_{m_k} \otimes r^{(k)}_{m'_k} \in \overline{\bigotimes_{1\leq k\leq n}} R^{(k)}_{m_k} \otimes R^{(k)}_{m'_k}$, we know that

$$\begin{aligned}\mu_{e_l} U(r) &= \mu_{e_l}\Big(\sum_{j=1}^{p} r^{(1)}_{m_1} r^{(1)}_{m'_1}, \ldots r^{(n)}_{m_n} r^{(n)}_{m'_n} \delta_j \otimes \gamma_j\Big) \\ &= \sum_{i=1}^{q}\sum_{j=1}^{p} r^{(1)}_{m_1} r^{(1)}_{m'_1}, \ldots r^{(n)}_{m_n} r^{(n)}_{m'_n} \delta_j \alpha_i \otimes \beta_i \gamma_j\end{aligned}$$

where $1 = \sum_{i=1}^{q}\sum_{j=1}^{p} \delta_j \alpha_i \beta_i \gamma_j$ is a partition of unity of type

$$\Big(-\Big(e_l + \sum_{k=1}^{n} m'_k e_k\Big), e_l + \sum_{k=1}^{n} m'_k e_k\Big)$$

and

$$\begin{aligned}U\mu_{(l)}(r) = U\Big(\sum_{i=1}^{q} r^{(1)}_{m_1} \otimes r^{(1)}_{m'_l} \otimes \ldots \otimes r^{(l)}_{m_l}\alpha_i \otimes \beta_i r^{(l)}_{m'_l} \otimes \ldots \otimes r^{(n)}_{m_n} \otimes r^{(n)}_{m'_n}\Big) \\ = \sum_{j=1}^{p'} r^{(1)}_{m_1} r^{(1)}_{m'_1}, \ldots r^{(n)}_{m_n} r^{(n)}_{m'_n} \delta'_j \otimes \gamma'_j\end{aligned}$$

where $1 = \sum_{j=1}^{p} \delta'_j \gamma'_j$ is also a partition of unity of type

$$\Big(-\Big(e_l + \sum_{k=1}^{n} m'_k e_k\Big), e_l + \sum_{k=1}^{n} m'_k e_k\Big)$$

hence by Remark 4.2.2 we know that $\mu_{e_l} U = U\overline{\mu}_{(l)}$ as required.

Now, it follows that the following diagram must also be commutative

$$\begin{tikzcd} \overline{\bigotimes_{1\leq k\leq n}} \Big(R^{(k)} \otimes R^{(k)}\Big) \arrow[r, "U"] \arrow[d, "\mathrm{id} - \overline{\mu}_{(l)}"'] & R \otimes R \arrow[d, "\mathrm{id} - \mu_{e_l}"'] \\ \overline{\bigotimes_{1\leq k\leq n}} \Big(R^{(k)} \otimes R^{(k)}\Big) \arrow[r, "U"] & R \otimes R \end{tikzcd}$$

meaning all that is left to observe is that the diagram

$$\begin{array}{ccc}
\overline{\bigotimes_{1\leq k\leq n}} \Lambda^{(k)} & \xrightarrow{\overline{U}} & \mathfrak{T}(\mathfrak{R}) \\
{\scriptstyle d\left(\underset{R_{0_{\mathbb{Z}^n}}}{\otimes} \Lambda^{(k)}\right)} \downarrow & & \downarrow {\scriptstyle d_{\mathfrak{T}(\mathfrak{R})}} \\
\overline{\bigotimes_{1\leq k\leq n}} \Lambda^{(k)} & \xrightarrow{\overline{U}} & \mathfrak{T}(\mathfrak{R})
\end{array}$$

will consist of, at each chain level that it is non zero, direct sums of diagrams of the form above, showing that the map $\overline{U}$ is a valid chain map, which is an isomorphism from Proposition 4.2.1. $\square$

We now know that $\mathfrak{T}(\mathfrak{R})$ is a resolution of R as R-R-bimodules from Proposition 4.1.8 and Proposition 4.2.5.

Recall that we can define torus maps on $M \underset{R_0}{\otimes} R$ for right R-modules M. Considering the complex $\mathfrak{T}(\mathfrak{R})$ and a right R_0-module M, we can define a commutative N-cube $\mathfrak{M}$, where for $N = \{1, 2.., n\}, \emptyset \subseteq A \subseteq N$,

$$\mathfrak{M}_A = M \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R,\ \mathfrak{M}_{A\cup\{i\},A} = \mu_{e_i} \text{ if } i \notin A$$

and 0 elsewhere.

If M is an R-module, we can see that

$$\mathrm{hTot}\,(\mathfrak{M}) = M \underset{R}{\otimes} \mathrm{hTot}\,(\mathfrak{R})\,.$$

Similarly, we can define $\mathrm{Tw}(\mathfrak{M})$, with the same modules and $\mathfrak{M}_{A,A\cup\{i\}} = \mathrm{id} - \mu_{e_i}$ if $i \notin A$, and we can also see that

$$\mathrm{hTot}\,(\mathrm{Tw}(\mathfrak{M})) = M \underset{R}{\otimes} \mathrm{hTot}\,(\mathrm{Tw}(\mathfrak{R}))\,,$$

equivalently we can write

$$\mathfrak{T}(\mathfrak{M}) = M \underset{R}{\otimes} \mathfrak{T}(\mathfrak{R}).$$

This also holds for a chain complex C.

Lemma 4.2.6. *Let M be a finitely presented R-module and $\mathfrak{M}$ the relevant commutative diagram. The complex $\mathfrak{T}(\mathfrak{M})$ has trivial homology except at 0 where it is M.*

Proof. The functor $M \underset{R}{\otimes} -$ is left exact as M is flat, hence the functor is exact. It follows immediately that

$$H_0(M \underset{R}{\otimes} \mathfrak{T}(\mathfrak{R})) = H_0(M) \underset{R}{\otimes} H_0(\mathfrak{T}(\mathfrak{R})) = M \underset{R}{\otimes} R \simeq M,$$

so, in particular, $\mathfrak{T}(\mathfrak{M})$ is a resolution of M and $0 \to \mathfrak{T}(\mathfrak{M}) \to M \to 0$ is an exact sequence. □

Lemma 4.2.7. *Let C be a bounded complex of finitely generated free R-modules. The totalisation of the cube $\mathfrak{C}$, $\mathfrak{T}(\mathfrak{C}) = C \underset{R}{\otimes} \mathfrak{T}(\mathfrak{R})$, is homotopy equivalent to C.*

Proof. Recall that $\mathfrak{T}(\mathfrak{R})$ is a resolution of R as R-R-bimodules, then

$$0 \to \mathfrak{T}(\mathfrak{C}_n) \to C_n \underset{R}{\otimes} R \to 0$$

is an exact sequence from Lemma 4.2.6 as C_n is a free R-module. Hence the double complex with $\mathfrak{T}(\mathfrak{C}_n) \to C_n$ at row n is made up of exact sequences levelwise. If follows from Lemma 0.3.6 that $\mathfrak{T}(\mathfrak{C})$ is quasi-isomorphic to C ($\mathfrak{T}(\mathfrak{C})$ is precisely the totalisation up to sign of the double complex with $\mathfrak{T}(\mathfrak{C}_n)$ at level n). Homotopy equivalence follows as the complexes are bounded complexes of finitely generated free modules. □

4.3 The strongly $\mathbb{Z}^n$-graded version of the Mather trick

Let us assume that C is a bounded complex of finitely generated free R-modules that is R_0-finitely dominated. Using Theorem 0.1.4 ([Ran85, Proposition 3.2. (ii)]), we let D be homotopy equivalent to a bounded complex of finitely generated projective R_0-modules D, so that $H\colon \mathrm{id}_C \simeq \beta\alpha$, $dH + Hd = \beta\alpha - \mathrm{id}$ and $J\colon \mathrm{id}_D \simeq \alpha\beta$, $dJ + Jd = \alpha\beta - \mathrm{id}$ for chain maps $\alpha\colon C \to D$, $\beta\colon D \to C$. I wish now to use the finite domination assumption to show that $\mathfrak{T}(\mathfrak{C})$ is homotopy equivalent to the totalisation of an N-cube with the module $D \otimes R$ at each vertex. This is the $\mathbb{Z}^n$-graded version of similar results [HQ16, Lemma II.2.6] and [HS16, Lemma 3.7]. For a map f let $f^* = f \otimes \mathrm{id}_R$.

Definition 4.3.1. Given complexes C, D where $H\colon \mathrm{id}_C \simeq \beta\alpha$, $dH + Hd = \beta\alpha - \mathrm{id}_C$, define the N-diagram $\alpha\mathfrak{C}\beta$ as the following, where $\emptyset \subseteq A \subset B \subseteq N$, σ is an ordering of $K \subseteq N$, $\sigma_1, \sigma_2 ... \sigma_K$ are the elements of K and T_k is the kth triangular number:

$$
\begin{aligned}
\alpha\mathfrak{C}\beta_A &= D \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R. \\
\alpha\mathfrak{C}\beta_{A,A} &= d^*. \\
\alpha\mathfrak{C}\beta_{A\cup\{i\},A} &= \alpha^*\mu_{e_i}\beta^*,\ i \notin A. \\
\alpha\mathfrak{C}\beta_{A\cup\{i,j\},A} &= \alpha^*\mu_{e_i}H^*\mu_{e_j}\beta^* - \alpha^*\mu_{e_j}H^*\mu_{e_i}\beta^*,\ i, j \notin A,\ i < j. \\
\alpha\mathfrak{C}\beta_{B,A} &= \sum_\sigma (-1)^{|B\setminus A| + T_{|B\setminus A|} + \sum_{2\le i\le |B\setminus A|} |P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|} \\
&\qquad \alpha^*\mu_{e_{\sigma_{|B\setminus A|}}}H^*...H^*\mu_{e_{\sigma_1}}\beta^*,\ A \subset B.
\end{aligned}
$$

In particular, for $A = \emptyset, B = N$,

$$
\alpha\mathfrak{C}\beta_{N,\emptyset} = \sum_\sigma (-1)^{n + T_{|N|} + \sum_{2\le i\le n} |P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|} \alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_1}}\beta^*.
$$

Proposition 4.3.2. *The N-diagram $\alpha\mathfrak{C}\beta$ is an N-cube.*

Proof. This can be shown to be an N-cube by a proof by induction. For $n = 1$, the case is simple, there are only three maps, $\alpha\mathfrak{C}\beta_{\{1\},\{1\}}, \alpha\mathfrak{C}\beta_{\emptyset,\emptyset}$ and $\alpha\mathfrak{C}\beta_{\{1\},\emptyset}$, while there are only three conditions:

$$
\begin{aligned}
\alpha\mathfrak{C}\beta_{\{1\},\{1\}}\alpha\mathfrak{C}\beta_{\{1\},\{1\}} = 0 = \alpha\mathfrak{C}\beta_{\emptyset,\emptyset}, \alpha\mathfrak{C}\beta_{\emptyset,\emptyset}, \\
\alpha\mathfrak{C}\beta_{\{1\},\emptyset}\alpha\mathfrak{C}\beta_{\emptyset,\emptyset} - \alpha\mathfrak{C}\beta_{\{1\},\{1\}}\alpha\mathfrak{C}\beta_{\{1\},\emptyset} = 0,
\end{aligned}
$$

which are clearly correct as $\alpha\mathfrak{C}\beta_{\{1\},\{1\}}, \alpha\mathfrak{C}\beta_{\emptyset,\emptyset}$ are boundary maps and $\alpha\mathfrak{C}\beta_{\{1\},\emptyset}$ is a chain complex map.

If true for $n - 1$, then observe by Lemma 3.6.1, that we can construct valid $N - 1$ cubes for sets $N \setminus \{i\}$ for all $i \in N$, we need only check the following is satisfied

$$\begin{aligned}
&d^*\alpha\mathfrak{C}\beta_{N,\emptyset} + (-1)^n\alpha\mathfrak{C}\beta_{N,\emptyset}d^* \\
&= \sum_{\emptyset\subsetneq S\subsetneq N}\sum_{\sigma}(-1)^{(1+n+\langle S\colon N\setminus S\rangle)} \\
&\quad {}^{+((n-s)+T_{n-s}+\sum_{s+2\le i\le n}|P^{\{\sigma_{s+1},\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|)} \\
&\quad {}^{+(s+T_s+\sum_{2\le i\le s}|P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|)} \\
&(\alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_{s+1}}}\beta^*)(\alpha^*\mu_{e_{\sigma_s}}H^*...H^*\mu_{e_{\sigma_1}}\beta^*)
\end{aligned} \tag{4.3.2.1}$$

which is Equation (3.3.4.1) for $B = N, A = \emptyset$ (the sign $(1 + n + \langle S\colon N \setminus S\rangle)$ comes from (3.3.4.1) and the other signs come from the definition of $\alpha\mathfrak{C}\beta$).

I show this by fixing a summand of $\alpha\mathfrak{C}\beta_{N,\emptyset}$. Firstly fix a permutation σ of N. Let

$$H_\sigma = \alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_1}}\beta^*$$

and note that we can write $\alpha\mathfrak{C}\beta_{N,\emptyset} = \sum_\sigma(-1)^{n+T_n+\sum_{2\le i\le n}|P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|}H_\sigma$. Using $dH = \beta\alpha - \mathrm{id} - Hd$ note that by iterated substitutions

$$\begin{aligned}
d^*H_\sigma &= d^*\alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_1}}\beta^* = \alpha^*\mu_{e_{\sigma_n}}d^*H^*...H^*\mu_{e_{\sigma_1}}\beta^* \\
&= \alpha^*\mu_{e_{\sigma_n}}\beta^*\alpha^*\mu_{e_{\sigma_{n-1}}}H^*...H^*\mu_{e_{\sigma_1}}\beta^* - \alpha^*\mu_{e_{\sigma_n}}\mu_{e_{\sigma_{n-1}}}H^*...H^*\mu_{e_{\sigma_1}}\beta^* \\
&\quad -\alpha^*\mu_{e_{\sigma_n}}H^*d^*\mu_{e_{\sigma_{n-1}}}H^*...H^*\mu_{e_{\sigma_1}}\beta^* \\
&= \Big(\sum_{s=1}^{n-1}(-1)^{n-s}(\alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_{s+1}}}\mu_{e_{\sigma_s}}H^*...H^*\mu_{e_{\sigma_1}}\beta^* \\
&\quad -\alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_{s+1}}}\beta^*\alpha^*\mu_{e_{\sigma_s}}H^*...H^*\mu_{e_{\sigma_1}}\beta^*)\Big) \\
&\quad +(-1)^{n-1}\alpha^*\mu_{e_{\sigma_n}}H^*\mu_{e_{\sigma_{n-1}}}H^*...H^*d^*\mu_{e_{\sigma_1}}\beta^*.
\end{aligned}$$

As

$$(-1)^{n-1}\alpha^*\mu_{e_{\sigma_n}}H^*\mu_{e_{\sigma_{n-1}}}H^*...H^*d^*\mu_{e_{\sigma_1}}\beta^* = (-1)^{n-1}H_\sigma d^*,$$

it follows that the left side of Equation (4.3.2.1) becomes

$$d^*H_\sigma + (-1)^n H_\sigma d^* = \sum_{s=1}^{n-1} (-1)^{(n-s)+(n+T_n+\sum_{2\le i\le n}|P^{\{\sigma_1,\ldots,\sigma_{i-1}\}}_{\{\sigma_i\}}|)} (\alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_{s+1}}}\mu_{e_{\sigma_s}}H^*...H^*\mu_{e_{\sigma_1}}\beta^* - \alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_{s+1}}}\beta^*\alpha^*\mu_{e_{\sigma_s}}H^*...H^*\mu_{e_{\sigma_1}}\beta^*). \tag{4.3.2.2}$$

Observe that $n - s + n \underset{2}{\equiv} s$.

Firstly, I will show that, given a fixed s, and set $S \in N$, $|S| = s$, the term $\alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_{s+1}}}\beta^*\alpha^*\mu_{e_{\sigma_s}}H^*...H^*\mu_{e_{\sigma_1}}\beta^*$, that appears on both sides of Equation (4.3.2.1) disappears.

I do this by comparing signs. I want the signs on the term on the right of (4.3.2.1) and the right of (4.3.2.2) to be equivalent modulo 2 (as they are on opposite sides of the equation (4.3.2.1)). I begin by noting that, from (4.3.2.2), the sign on the relevant term of $d^*\alpha\mathfrak{C}\beta_{N,\emptyset}$ is:

$$1 + s + T_n + \sum_{i=2}^{n} |P^{\{\sigma_1,\ldots,\sigma_{i-1}\}}_{\{\sigma_i\}}|.$$

Now, note that $T_n = T_{n-s} + T_s + s(n-s)$.

Next observe that

$$\sum_{i=2}^{n} |P^{\{\sigma_1,\ldots,\sigma_{i-1}\}}_{\{\sigma_i\}}| = \sum_{i=2}^{s} |P^{\{\sigma_1,\ldots,\sigma_{i-1}\}}_{\{\sigma_i\}}| + \sum_{i=s+1}^{n} |P^{S\cup\{\sigma_{s+1},\ldots,\sigma_{i-1}\}}_{\{\sigma_i\}}|$$

and

$$\sum_{i=s+1}^{n} |P^{S\cup\{\sigma_{s+1},\ldots,\sigma_{i-1}\}}_{\{\sigma_i\}}| = \sum_{i=s+1}^{n} |P^{S}_{\{\sigma_i\}}| + |P^{\emptyset}_{\{\sigma_{s+1}\}}| + \sum_{i=s+2}^{n} |P^{\{\sigma_{s+1},\ldots,\sigma_{i-1}\}}_{\{\sigma_i\}}|$$

and finally

$$\sum_{i=s+1}^{n} |P^{S}_{\{\sigma_i\}}| = |P^{S}_{N\setminus S}|.$$

Substituting into the above, these tell us that

$$\begin{aligned}1 + s + T_n + &\sum_{i=2}^{n} |P_{\{\sigma_i\}}^{\{\sigma_1,\ldots,\sigma_{i-1}\}}| \\ &= 1 + s + T_{n-s} + T_s + s(n-s) \\ &\qquad + \sum_{i=2}^{s} |P_{\{\sigma_i\}}^{\{\sigma_1,\ldots,\sigma_{i-1}\}}| + \sum_{i=s+2}^{n} |P_{\{\sigma_i\}}^{\{\sigma_{s+1},\ldots,\sigma_{i-1}\}}| + |P_{N\setminus S}^{S}|.\end{aligned}$$

Removing even elements and identical elements from the signs, showing the result is equivalent to showing that

$$s + s(n-s) + |P_{N\setminus S}^{S}| \underset{2}{\equiv} \langle S\colon N \setminus S\rangle.$$

There are two cases, firstly when $|N \setminus S|$ is even. Then $s + s(n-s) \equiv_2 s$ and $\langle S\colon N \setminus S\rangle = |P_{N\setminus S}^{S}| + |S|$ as required. When $|N \setminus S|$ is odd, then $s + s(n-s) \equiv_2 s + s \equiv_2 0$ and $\langle S\colon N \setminus S\rangle = |P_{N\setminus S}^{S}|$ as required.

It remains to show that on the left of Equation 4.3.2.1 for a fixed s the terms $\alpha^*\mu_{e_{\sigma_n}}H^*...H^*\mu_{e_{\sigma_{s+1}}}\mu_{e_{\sigma_s}}H^*...H^*\mu_{e_{\sigma_1}}\beta^*$ cancel. We do this by noting that $\mu_{e_{\sigma_i}}\mu_{e_{\sigma_{i+1}}} = \mu_{e_{\sigma_{i+1}}}\mu_{e_{\sigma_i}}$, then observing that as a result the relevant term formed by taking the id term in the sth substitution of $dH = \beta\alpha - \mathrm{id} - Hd$ for a given permutation σ is the same as the term formed from the sth relevant substitution of a permutation σ' such that $\sigma_s = \sigma'_{s+1}$, $\sigma_{s+1} = \sigma'_s$. So, I want to find that these signs differ modulo 2.

For σ the sign is

$$(n-s) + (n + T_n + \sum_{i=2}^{n} |P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_{i-1}\}}|).$$

while for σ' the sign is

$$\begin{aligned}(n-s) + (n + T_n + &\sum_{i=2}^{s-1} |P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_{i-1}\}}| + |P_{\{\sigma_{s+1}\}}^{S\setminus\{\sigma_s\}}| \\ &+ |P_{\{\sigma_s\}}^{S\setminus\{\sigma_s\}\cup\{\sigma_{s+1}\}}| + \sum_{i=s+2}^{n} |P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_{i-1}\}}|).\end{aligned}$$

Clearly, the difference is $|P^{\{\sigma_s\}}_{\{\sigma_{s+1}\}}|-|P^{\{\sigma_{s+1}\}}_{\{\sigma_s\}}|$ which is 1 or -1 as required. Hence Equation 4.3.2.1 is satsified, hence $\mathrm{htot}(\alpha\mathfrak{C}\beta)$ is an N-cube. $\square$

Definition 4.3.3. Given complexes C, D where $dH+Hd=\beta\alpha-\mathrm{id}_C$, and an N-cube $\alpha\mathfrak{C}\beta$ define the *twisted N-diagram* $\mathrm{Tw}(\alpha\mathfrak{C}\beta)$ as the following, where $\emptyset \subseteq A \subset B \subseteq N$, σ is an ordering of $K \subseteq N$ with elements $\sigma_1, \sigma_2, ...\sigma_K$.

$$\begin{aligned}
\mathrm{Tw}(\alpha\mathfrak{C}\beta)_A =& D \underset{R_{0_{\mathbb{Z}^n}}}{\otimes} R.\\
\mathrm{Tw}(\alpha\mathfrak{C}\beta)_{A,A} =& d^*.\\
\mathrm{Tw}(\alpha\mathfrak{C}\beta)_{A\cup\{i\},A} =& \mathrm{id} - \beta^*\mu_{e_i}\alpha^*,\ i \notin A.\\
\mathrm{Tw}(\alpha\mathfrak{C}\beta)_{A\cup\{i,j\},A} =& \beta^*\mu_{e_i}H^*\mu_{e_j}\alpha^* - \beta^*\mu_{e_j}H^*\mu_{e_i}\alpha^*,\ i,j \notin A,\ i<j.\\
\mathrm{Tw}(\alpha\mathfrak{C}\beta)_{B,A} =& \sum_{\sigma}(-1)^{(|B\setminus A|)+|B\setminus A|+T_{|B\setminus A|}+\sum_{2\le i\le |B\setminus A|}|P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|}\\
& \beta^*\mu_{e_{\sigma_{|B\setminus A|}}}H^*...H^*\mu_{e_{\sigma_1}}\alpha^*,\ A \subset B,\ |B\setminus A| \ge 2.
\end{aligned}$$

Proposition 4.3.4. *The twisted N-diagram,* $\mathrm{Tw}(\alpha\mathfrak{C}\beta)$*, is in fact an N-cube.*

Proof. Ignoring the id portion of the definition to begin with the changes between $\alpha\mathfrak{C}\beta$ and $\mathrm{Tw}(\alpha\mathfrak{C}\beta)$ cancel out. When $|B\setminus A|$ is odd, the sign of each summand in Equation (3.3.4.1) changes as each is changed by $|B\setminus S| = |S\setminus A| = |B\setminus A|$. Similarly when $|B\setminus A|$ is even, the sign of each summand remains the same for the same reason hence the relevant conditions are satisfied.

The compositions of id with other maps also cancel. These terms will only appear on the right side of (4.3.2.1). Fix $j \in N$ and note that $\mathrm{Tw}(\alpha\mathfrak{C}\beta_{B,B\setminus\{j\}})$ and $\mathrm{Tw}(\alpha\mathfrak{C}\beta_{B\setminus\{j\},\emptyset})$ are the same map up to sign. Fix a permutation σ of $B\setminus\{j\}$ and an element $H_\sigma = \beta^*\mu_{e_{\sigma_{|B|\setminus\{j\}}}}H^*...H^*\mu_{e_{\sigma_1}}\alpha^*$. For all $B \subseteq N$, the only terms of (3.3.4.1) we are interested in are those where $S = \{j\}, B \setminus \{j\}$, and only the id parts at that:

$$(-1)^{(1+|B|+\langle \{j\}\colon B\setminus\{j\}\rangle)+(|B|-1+T_{|B|-1}+\sum_{2\leq i\leq b-1}|P_{\sigma_i}^{\sigma_1,\dots,\sigma_{i-1}}|)}H_\sigma \mathrm{id}$$
$$+(-1)^{(1+|B|+\langle B\setminus\{j\}\colon \{j\}\rangle)+(|B|-1+T_{|B|-1}+\sum_{2\leq i\leq b-1}|P_{\sigma_i}^{\sigma_1,\dots,\sigma_{i-1}}|)}\mathrm{id}H_\sigma$$

noting that the sign on id is positive. We want this sum to be zero, so the result breaks down to showing that:

$$\langle \{j\}\colon B \setminus \{j\}\rangle \equiv_2 1 + \langle B \setminus \{j\}\colon \{j\}\rangle.$$

Firstly, note that
$$\langle B \setminus \{j\}\colon \{j\}\rangle = |P_{\{j\}}^{B\setminus\{j\}}|.$$
Assume $|B|$ is even. Then

$$\langle \{j\}\colon B \setminus \{j\}\rangle = |P_{B\setminus\{j\}}^{\{j\}}|.$$

Going back to Definition 3.1.4, we observe that $|P_{\{j\}}^{B\setminus\{j\}}| = B \setminus \{j\}_{>\{j\}}$ must differ in sign to $|P_{B\setminus\{j\}}^{\{j\}}| = \sum_{a\in B\setminus\{j\}}\{j\}_{>a}$ as there are odd many elements in $B \setminus \{j\}$, and an element can only be either larger or smaller than j, so $|P_{\{j\}}^{B\setminus\{j\}}| + |P_{B\setminus\{j\}}^{\{j\}}| \equiv_2 1$ as required. Now assume $|B|$ is odd. Then

$$\langle \{j\}\colon B \setminus \{j\}\rangle = 1 + |P_{\{j\}}^{B\setminus\{j\}}|$$

so we need $|P_{\{j\}}^{B\setminus\{j\}}| \equiv_2 |P_{\{j\}}^{B\setminus\{j\}}|$. This follows for similar reasons as before, the even many elements of $B \setminus \{j\}$ can either be higher or lower than j, hence the difference of $|P_{\{j\}}^{B\setminus\{j\}}|$ and $|P_{\{j\}}^{B\setminus\{j\}}|$ is even as required. Hence,

$$(-1)^{(1+|B|+\langle \{j\}\colon B\setminus\{j\}\rangle)+(|B|-1+T_{|B|-1}+\sum_{2\leq i\leq b-1}|P_{\sigma_i}^{\sigma_1,\dots,\sigma_{i-1}}|)}H_\sigma \mathrm{id}$$
$$+(-1)^{(1+|B|+\langle B\setminus\{j\}\colon \{j\}\rangle)+(|B|-1+T_{|B|-1}+\sum_{2\leq i\leq b-1}|P_{\sigma_i}^{\sigma_1,\dots,\sigma_{i-1}}|)}\mathrm{id}H_\sigma = 0$$

and so over all sets $B \subseteq N$ and $j \in B$ and permuations σ of $B\setminus\{j\}$, the sum disappears. This is the case for $A = \emptyset$, general A follows similarly. Hence the twisted N-diagram $\mathrm{Tw}(\alpha\mathfrak{C}\beta)$ is an N-cube, which we call a *twisted N-cube.* $\square$

For simplicity write $\mathrm{hTot}(\mathrm{Tw}(\alpha\mathfrak{C}\beta)) = \mathfrak{T}(\alpha\mathfrak{C}\beta)$.

The following collection of results will show that given satisfactory $C \simeq D$ as set at the start of this section it is the case that $C \simeq \mathfrak{T}(\mathfrak{C}) \simeq \mathfrak{T}(\alpha\mathfrak{C}\beta)$.

Proposition 4.3.5. *Let C, D be chain complexes and $\mathfrak{C}$, $\mathfrak{D}$ their associated N-diagrams. Given $\alpha\colon C \to D$ and $\beta\colon D \to C$ such that $H\colon \beta\alpha \simeq \mathrm{id}_C$ there is a map $\mathfrak{C} \to \alpha\mathfrak{C}\beta$, represented by the map $\mathfrak{F}$ where for $N = \{1, 2.., n\}$, $\emptyset \subseteq A \subset N$, $i \notin A$.*

$$\begin{aligned}
\mathfrak{F}_{A,A} &= \alpha^*. \\
\mathfrak{F}_{A\cup\{i\},A} &= \alpha^*\mu_{e_i}H^*,\ i \notin A. \\
\mathfrak{F}_{A\cup\{i,j\},A} &= \alpha^*\mu_{e_i}H^*\mu_{e_j}H^* - \alpha^*\mu_{e_j}H^*\mu_{e_i}H^*,\ i, j \notin A,\ i < j. \\
\mathfrak{F}_{B,A} &= \sum_{\sigma}(-1)^{|B\setminus A| + T_{|B\setminus A|} + \sum_{2\le i\le |B\setminus A|}|P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|} \\
&\qquad \alpha^*\mu_{e_{\sigma_{|B\setminus A|}}}H^*...H^*\mu_{e_{\sigma_1}}H^*,\ A \subseteq B.
\end{aligned}$$

Proof. Firstly, consider Proposition 3.4.6. If $\mathfrak{F}$ is an N-cube map then it must satisfy

$$\begin{aligned}
&\alpha\mathfrak{C}\beta_{B,B}\mathfrak{F}_{B,A} + (-1)^{1+|B\setminus A|}\mathfrak{F}_{B,A}\mathfrak{C}_{A,A} = \\
&\qquad (-1)^{|B|}\left(\sum_{A\subsetneq \underset{\bullet}{S}\subseteq B}\left((-1)^{|S| + \langle S\setminus A : B\setminus S\rangle}\mathfrak{F}_{B,S}\mathfrak{C}_{S,A}\right)\right. \\
&\qquad\qquad \left. - \sum_{A\subseteq \underset{\bullet}{S}\subsetneq B}\left((-1)^{|A| + \langle S\setminus A : B\setminus S\rangle}\alpha\mathfrak{C}\beta_{B,S}\mathfrak{F}_{S,A}\right)\right)
\end{aligned}$$

so, as we did when we proved the N-cube $\alpha\mathfrak{C}\beta$ satisfies the definition, we substitute and compare signs.

Note that $\mathfrak{C}_{S,A}$ are zero maps except for when $|S\setminus A| = 0, 1$, specifically $\mathfrak{C}_{A,A} = d_C \otimes \mathrm{id} = d^*_C$ and $\mathfrak{C}_{S,S\setminus\{i\}} = \mu_{e_i}$. Also let $d^*_D = d_D \otimes \mathrm{id}$.

For $n = 1$, we note that there are only three conditions to consider. For $B = A = \{1\}$ and $B = A = \emptyset$, these conditions amount to an observation that α^* is a chain map. For $B = \{1\}, A = \emptyset$, we get something more complicated:

$$\begin{aligned} & d_D^*(\alpha^* \mu_{e_i} H^*) + (\alpha^* \mu_{e_i} H^*) d_C^* \\ = (-1)^1 & \left((-1)^{0+0} (\alpha^*) \mu_{e_i} + (-1)^{0+\langle \{1\} \colon \emptyset \rangle} (\alpha^* \mu_{e_i} \beta^*) \alpha^* \right) \end{aligned}$$

which is shown due to the statements $\langle \{1\} \colon \emptyset \rangle = 1$ and $dH + Hd = \beta\alpha - \mathrm{id}$. Hence the result is true for $n = 1$.

If we note that this is shown for $n = 1$ and noting that the map $\mathfrak{F}$ splits into K-cube maps for all $K \subseteq N$, it remains to show the result for $A = \emptyset, B = N$.

The equation for this case is:

$$\begin{aligned} & \alpha \mathfrak{C} \beta_{N,N} \mathfrak{F}_{N,\emptyset} + (-1)^{1+n} \mathfrak{F}_{N,\emptyset} \mathfrak{C}_{\emptyset,\emptyset} = \\ & \qquad (-1)^n \left(\sum_{\emptyset \subsetneq \underset{\bullet}{S} \subseteq N} \left((-1)^{(|S| + \langle S : N \setminus S \rangle)} \mathfrak{F}_{N,S} \mathfrak{C}_{S,\emptyset} \right) \right. \\ & \qquad\qquad \left. - \sum_{\emptyset \subseteq \underset{\bullet}{S} \subsetneq N} \left((-1)^{|\emptyset| + \langle S : N \setminus S \rangle} \alpha \mathfrak{C} \beta_{N,S} \mathfrak{F}_{S,\emptyset} \right) \right) \end{aligned} \tag{4.3.5.1}$$

and taking account the definition of $\alpha\mathfrak{C}\beta$ we get:

$$
\begin{aligned}
&\alpha\mathfrak{C}\beta_{N,N}\mathfrak{F}_{N,\emptyset}+(-1)^{1+n}\mathfrak{F}_{N,\emptyset}\mathfrak{C}_{\emptyset,\emptyset}=\\
&(-1)^n\Big(\sum_{\emptyset\subsetneq \underset{\bullet}{S}\subseteq N}\sum_{\sigma}(-1)^{(|S|+\langle S:N\setminus S\rangle)+(|N\setminus S|+T_{|N\setminus S|}+\sum_{s+2\leq i\leq|N\setminus S|}|P_{\{\sigma_i\}}^{\{\sigma_s+1,\dots,\sigma_{i-1}\}}|)}\\
&\quad{}^{+(|S|+T_{|S|}+\sum_{2\leq i\leq|S|}|P_{\{\sigma_i\}}^{\{\sigma_1,\dots,\sigma_{i-1}\}}|)}\\
&\quad\alpha^*\mu_{e_{\sigma_n}}H^*\dots H^*\mu_{e_{\sigma_{n-s}}}H^*\mathfrak{C}_{S,\emptyset}\\
&\quad-\sum_{\emptyset\subseteq \underset{\bullet}{S}\subsetneq N}\sum_{\sigma}(-1)^{(|\emptyset|+\langle S:N\setminus S\rangle)+(|N\setminus S|+T_{|N\setminus S|}+\sum_{s+2\leq i\leq|N\setminus S|}|P_{\{\sigma_i\}}^{\{\sigma_s+1,\dots,\sigma_{i-1}\}}|)}\\
&\quad{}^{+(|S|+T_{|S|}+\sum_{2\leq i\leq|S|}|P_{\{\sigma_i\}}^{\{\sigma_1,\dots,\sigma_{i-1}\}}|)}\\
&\quad\alpha^*\mu_{e_{\sigma_n}}H^*\dots H^*\mu_{e_{\sigma_{s+1}}}\beta^*\alpha^*\mu_{e_{\sigma_s}}H^*\dots H^*\mu_{e_{\sigma_1}}H^*\Big)
\end{aligned}
\tag{4.3.5.2}
$$

so, as we did when we showed the N-cube $\alpha\mathfrak{C}\beta$ satisfies the definition, we substitute and compare signs.

Again, note that $\mathfrak{C}_{S,\emptyset}$ are zero maps except for when $|S|=0,1$.

As in the proof that the N-cube $\mathrm{Tw}(\alpha\mathfrak{C}\beta)$ satisfies the definition of an N-cube, we fix a permutation, apply the substitution $dH=\beta\alpha-\mathrm{id}-Hd$ to the summand of $\alpha\mathfrak{C}\beta_{N,N}\mathfrak{F}_{N,\emptyset}$ associated to σ and compare the signs between matching terms in Equation (4.3.5.2).

First of all, note that there are n many H^* terms in each summand of $\alpha\mathfrak{C}\beta_{N,N}\mathfrak{F}_{N,\emptyset}$ hence $(-1)^{1+n}\mathfrak{F}_{N,\emptyset}\mathfrak{C}_{\emptyset,\emptyset}$ is cancelled by taking the Hd substitution at the nth time of $\alpha\mathfrak{C}\beta_{N,N}\mathfrak{F}_{N,\emptyset}$, which gives it a sign of $(-1)^n$.

Now fix a permutation σ of N and a set $S\subseteq N$, but $S\notin\{\emptyset,N\}$. Let $F_{B,\sigma}=\mu_{e_{\sigma_{|B|}}}H...H\mu_{e_{\sigma_1}}$.

Firstly, I want to show that the signs on the term

$$\alpha^*F_{N\setminus S,\sigma}\beta^*\alpha^*F_{S,\sigma}H^*=\alpha^*\mu_{\sigma_n}H^*...H^*\mu_{e_{\sigma_{s+1}}}\beta^*\alpha^*\mu_{e_{\sigma_s}}H^*...H^*\mu_{e_{\sigma_1}}H^*$$

match on each side of the Equation (4.3.5.2). The term on the left appears after $n-s$ substitutions of $dH=\beta\alpha-\mathrm{id}-Hd$ and taking the $\beta\alpha$ summand on the $n-s$th substitution. The relevant term of $\alpha\mathfrak{C}\beta_{N,N}\mathfrak{F}_{N,\emptyset}$ after the substitutions has sign

$$\begin{aligned}(n-s)+1+n+T_n+\sum_{i=2}^{n}|P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_{i-1}\}}|\\ &\equiv_2 1+s+T_{n-s}+T_s+s(n-s)\\ &\quad+\sum_{i=s+1}^{n}|P_{\{\sigma_i\}}^{S\cup\{\sigma_{s+1},\dots\sigma_{i-1}\}}|+\sum_{i=2}^{s}|P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_{i-1}\}}|.\end{aligned}$$

Observe that

$$\begin{aligned}\sum_{i=s+1}^{n}|P_{\{\sigma_i\}}^{S\cup\{\sigma_{s+1},\dots\sigma_{i-1}\}}| &= \sum_{i=s+1}^{n}|P_{\{\sigma_i\}}^{S}|+\sum_{i=s+1}^{n}|P_{\{\sigma_i\}}^{\{\sigma_{s+1},\dots\sigma_{i-1}\}}|\\ &= |P_{N\setminus S}^{S}|+\sum_{i=s+1}^{n}|P_{\{\sigma_i\}}^{\{\sigma_{s+1},\dots\sigma_{i-1}\}}|.\end{aligned}$$

For the relevant term on the other side of the equation, we see that it has the sign:

$$\begin{aligned}&n{+}1+|\emptyset|+\langle S:N\setminus S\rangle\\ &+n-s+T_{n-s}+\sum_{i=s+1}^{n}|P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_s,\dots\sigma_{i-1}\}}|\\ &+s+T_s+\sum_{i=2}^{s}|P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_{i-1}\}}|\\ &\equiv_2 1+\langle S:N\setminus S\rangle+T_{n-s}+\sum_{i=s+1}^{n}|P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_s,\dots\sigma_{i-1}\}}|\\ &+T_s+\sum_{i=2}^{s}|P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_{i-1}\}}|.\end{aligned}$$

Ignoring the terms that appear in both, it is enough to check whether or not the values

$$|P_{N\setminus S}^{S}|+s+s(n-s)$$

and

$$\langle S\colon N\setminus S\rangle$$

agree modulo 2, but this is exactly the result seen in the analogous case where we investigated the N-cube, hence this case is known too.

Next, I see where the terms produced when substituting $dH = \beta\alpha - \mathrm{id} - Hd$ and taking the id summand the $n-s$th time go. Here, given two orderings σ and σ', that only differ by $\sigma_s = \sigma'_{s+1}$, $\sigma_{s+1} = \sigma'_s$, I will show that the signs on $\alpha^* F_{N\backslash S,\sigma} F_{S,\sigma} H^*$ and $\alpha^* F_{N\backslash S,\sigma'} F_{S,\sigma'} H^*$, both on the left of the equation (4.3.5.2), differ hence cancelling both terms.

The sign on the term $(-1)^{n-s}\alpha^* F_{N\backslash S,\sigma} F_{S,\sigma}$ is:

$$|N| + T_n + \sum_{i=2}^{n} |P_{\{\sigma_i\}}^{\{\sigma_1,\ldots,\sigma_{i-1}\}}|$$

while for $(-1)^{n-s}\alpha^* F_{N\backslash S,\sigma'} F_{S,\sigma'}$ the sign is:

$$\begin{aligned} &|N| + T_n + \sum_{i=2}^{s-1} |P_{\{\sigma_i\}}^{\{\sigma_1,\ldots,\sigma_{i-1}\}}| + |P_{\{\sigma_{s+1}\}}^{S\backslash\{\sigma_s\}}| \\ &+ |P_{\{\sigma_s\}}^{S\backslash\{\sigma_s\}\cup\{\sigma_{s+1}\}}| + \sum_{i=s+1}^{n} |P_{\{\sigma_i\}}^{S\cup\{\sigma_{s+1},\ldots,\sigma_{i-1}\}}|. \end{aligned}$$

It is clear through applying the algebra of these sets that the only terms that differ are

$$|P_{\{\sigma_{s+1}\}}^{\{\sigma_s\}}|$$

and

$$|P_{\{\sigma_s\}}^{\{\sigma_{s+1}\}}|.$$

Clearly,

$$|P_{\{\sigma_{s+1}\}}^{\{\sigma_s\}}| =_2 1 + |P_{\{\sigma_s\}}^{\{\sigma_{s+1}\}}|$$

as required.

It remains to investigate what happens to the terms from the final use of the substitution $dH = \beta\alpha - \mathrm{id} - Hd$, that is the terms $\alpha^* F_{N,\sigma}$ and $\alpha^* F_{N,\sigma}\beta^*\alpha^*$.

For a given σ, $\alpha^* F_{N,\sigma}$ is cancelled by the term $\mathfrak{F}_{N,N\setminus\{\sigma_1\}}\mathfrak{C}_{\{\sigma_1\},\emptyset}$, which are on opposite sides of Equation (4.3.5.2). I therefore want the signs to agree. The sign on $\alpha^* F_{N,\sigma}$ after the n substitutions is

$$(n) + n + T_N + \sum_{i=2}^{n} |P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|$$

and the sign on $\mathfrak{F}_{N,N\setminus\{\sigma_1\}}\mathfrak{C}_{\{\sigma_1\},\emptyset}$ is

$$(n) + (1) + \langle\{\sigma_1\} : N \setminus \{\sigma_1\}\rangle + (n - 1 + T_{n-1} + \sum_{i=3}^{n} |P^{\{\sigma_2,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|).$$

It is clear that the difference modulo 2 between the two signs is

$$n + \sum_{i=2}^{n} |P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}| + \langle\{\sigma_1\} : N \setminus \{\sigma_1\}\rangle + \sum_{i=3}^{n} |P^{\{\sigma_2,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|.$$

Since

$$\sum_{i=2}^{n} |P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}| = \sum_{i=2}^{n} |P^{\{\sigma_1\}}_{\{\sigma_i\}}| + \sum_{i=3}^{n} |P^{\{\sigma_2,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|$$

and $\sum_{i=2}^{n} |P^{\{\sigma_1\}}_{\{\sigma_i\}}| = |P^{\{\sigma_1\}}_{N\setminus\{\sigma_1\}}|$ the difference is equivalent modulo 2 to

$$n + |P^{\{\sigma_1\}}_{N\setminus\{\sigma_1\}}| + \langle\{\sigma_1\} : N \setminus \{\sigma_1\}\rangle.$$

When $|N|$ is even, $|N| - 1$ is odd hence $\langle\{\sigma_1\} : N \setminus \{\sigma_1\}\rangle = |P^{\{\sigma_1\}}_{N\setminus\{\sigma_1\}}|$ and when $|N|$ is odd, $\langle\{\sigma_1\} : N \setminus \{\sigma_1\}\rangle = 1 + |P^{\{\sigma_1\}}_{N\setminus\{\sigma_1\}}|$, in both cases the difference modulo 2 is 0 as required.

Finally, its time to consider the terms $\alpha^* F_{N,\sigma}\beta^*\alpha^*$ formed by taking the $\beta\alpha$ term in the nth substitution. These terms are cancelled by the term $(-1)^{1+\langle\emptyset:N\rangle}\alpha\mathfrak{C}\beta_{N,\emptyset}\mathfrak{F}_{\emptyset,\emptyset}$ on the opposite side of the equation. Therefore, we want the signs to agree. The sign on $\alpha^* F_{N,\sigma}\beta^*\alpha^*$ after n substitutions is

$$(n + 1) + n + T_n + \sum_{i=2}^{n} |P^{\{\sigma_1,\dots,\sigma_{i-1}\}}_{\{\sigma_i\}}|$$

and the sign on the term $\alpha\mathfrak{C}\beta_{N,\emptyset}\mathfrak{F}_{\emptyset,\emptyset}$ is

$$1 + \langle\emptyset : N\rangle + n + T_n + \sum_{i=2}^{n} |P_{\{\sigma_i\}}^{\{\sigma_1,..,\sigma_{i-1}\}}|$$

and since $\langle\emptyset : N\rangle = n$ the signs match as required.

The result follows from induction.

□

Proposition 4.3.6. *Given the map $\mathfrak{F}\colon \mathfrak{C} \to \alpha\mathfrak{C}\beta$ from the previous proposition, there is a map $\mathrm{Tw}(\mathfrak{F})\colon \mathrm{Tw}(\mathfrak{C}) \to \mathrm{Tw}(\alpha\mathfrak{C}\beta)$ defined as:*

$$\begin{aligned}
\mathrm{Tw}(\mathfrak{F})_{A,A} &= -\alpha^*. \\
\mathrm{Tw}(\mathfrak{F})_{A\cup\{i\},A} &= \alpha^*\mu_{e_i}H^*,\ i \notin A. \\
\mathrm{Tw}(\mathfrak{F})_{A\cup\{i,j\},A} &= -(\alpha^*\mu_{e_i}H^*\mu_{e_j}H^* - \alpha^*\mu_{e_j}H^*\mu_{e_i}H^*),\ i,j \notin A,\ i<j. \\
\mathrm{Tw}(\mathfrak{F})_{B,A} &= \sum_{\sigma}(-1)^{(1+|B\setminus A|)+|B\setminus A|+T_{|B\setminus A|}+\sum_{2\leq i\leq |B\setminus A|}|P_{\{\sigma_i\}}^{\{\sigma_1,\dots,\sigma_{i-1}\}}|} \\
&\quad \alpha^*\mu_{e_{\sigma_{|B\setminus A|}}}H^*...H^*\mu_{e_{\sigma_1}}H^*,\ A \subseteq B.
\end{aligned}$$

Proof. Ignoring the id terms of the two N-cubes $\mathfrak{T}(\mathfrak{C}), \mathfrak{T}(\alpha\mathfrak{C}\beta)$, the other terms cancel for the same reason they do for $\mathfrak{T}(\alpha\mathfrak{C}\beta)$ - for a given B, A, the effect of $1 + |B \setminus A|$ is to change each term in Equation (3.4.6) by the same sign. When $|B \setminus A|$ is odd , then $(-1)^{|B\setminus A|+1} = 1$ and the right side compositions of Equation (4.3.5.2) do not change signs, either being a component of $\mathrm{Tw}(\mathfrak{F})$ with odd $|B\setminus A|$ combined with a component of $\mathrm{Tw}(\mathfrak{D}_?)$ with even $|B \setminus A|$, so two unchanged signs, or a component of $\mathrm{Tw}(\mathfrak{F})$ with even $|B \setminus A|$ combined with a component of $\mathrm{Tw}(\mathfrak{D}_?)$ with odd $|B \setminus A|$, again changing the sign twice hence leaving it unchanged. The case for when $|B \setminus A|$ is even follows similarly, hence we can focus solely on the id terms that come from $\mathrm{Tw}(\mathfrak{D}_?)$.

For a given B and $j \in B$, only when $S = \{j\}, B \setminus \{j\}$ will id terms appear, and only on the right of Equation (3.4.6). The relevant terms for a fixed σ are:

$$
\begin{aligned}
&(-1)^{(|B|)+(|\{j\}|+\langle\{j\}\colon B\setminus\{j\}\rangle)+(1+T_{|B|-1}+\sum_{2\le i\le|B\setminus\{j\}|}|P_{\{\sigma_i\}}^{\{\sigma_1,\dots,\sigma_{i-1}\}}|)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha^* F_{B\setminus\{j\},\sigma} H^* \mathrm{id} \\
&+(-1)^{1+(|B|)+(|\emptyset|+\langle B\setminus\{j\}\colon \{j\}\rangle)+(1+T_{|B|-1}+\sum_{2\le i\le|B\setminus\{j\}|}|P_{\{\sigma_i\}}^{\{\sigma_1,\dots,\sigma_{i-1}\}}|)} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{id}\alpha^* F_{B\setminus\{j\},\sigma} H^*
\end{aligned}
$$

(the first summand has $S = \{j\}$, the second has $S = B \setminus \{j\}$) and a similar proof to that used for $\mathrm{Tw}(\alpha\mathfrak{C}\beta)$ will suffice, namely showing that

$$\langle\{j\}\colon B \setminus \{j\}\rangle \equiv_2 \langle B \setminus \{j\}\colon \{j\}\rangle + 1.$$

Again, we note that for $A \neq \emptyset$, the case is symmetrical. Hence, we have shown that $\mathrm{Tw}(\mathfrak{F})$ is a valid N-cube map. □

The map $\mathrm{Tw}(\mathfrak{F})$ can be totalised to form a map $\mathfrak{T}(\mathfrak{F})\colon \mathfrak{T}(\mathfrak{C}) \to \mathfrak{T}(\alpha\mathfrak{C}\beta)$, similarly $\mathfrak{F}$ forms $\mathrm{hTot}(\mathfrak{F})$.

Theorem 4.3.7. *The map* $\mathrm{hTot}(\mathfrak{F})\colon \mathrm{hTot}(\mathfrak{C}) \to \mathrm{hTot}(\alpha\mathfrak{C}\beta)$ *is a homotopy equivalence.*

Proof. Considering $\mathrm{hTot}(\mathfrak{F})$ as a matrix, we see that the maps on the diagonal, $\alpha \otimes \mathrm{id}$, are quasi-isomorphisms (as R_0-module maps). It follows from [HQ15, Lemma 2.13] that the map $\mathrm{hTot}(\mathfrak{F})$ is a quasi-isomorphism also. Next, observe that both $\mathrm{hTot}(\mathfrak{C})$ and $\mathrm{hTot}(\alpha\mathfrak{C}\beta)$ are bounded complexes of projective R_0-modules, as both C and D are bounded complexes of projective R_0-module complexes. It follows that $\mathrm{hTot}(\mathfrak{F})$ is a homotopy equivalence of R_0-modules. □

Theorem 4.3.8 (Mather Trick)**.** *The map* $\mathfrak{T}(\mathfrak{F})\colon \mathfrak{T}(\mathfrak{C}) \to \mathfrak{T}(\alpha\mathfrak{C}\beta)$ *is a homotopy equivalence.*

Proof. Note that $-\alpha \otimes \mathrm{id}$ is a quasi-isomorphism and repeat the argument of Theorem 4.3.7. □

4.4 Finite domination implies contractiblity of Novikov homology

At this point, when C is a bounded complex of finitely generated free R-modules homotopy equivalent to a bounded complex of finitely generated projective R_0-modules D, we know that $C \simeq \mathfrak{T}(\alpha\mathfrak{C}\beta)$.

Note that the module $M \underset{R_0}{\otimes} R$ can be viewed as a $\mathbb{Z}^n$-graded right R module with $(M \underset{R_0}{\otimes} R)_\rho = M \underset{R_0}{\otimes} R_\rho$. We can define polynomial, power series and Novikov structures on this module analogous to those for rings. We can therefore define for any flag $\mathcal{F}$ Novikov tori $\mathfrak{T}(\alpha\mathfrak{C}\beta)(\!(\mathcal{F})\!)$ for a given torus $\mathfrak{T}(\alpha\mathfrak{C}\beta)$, with matrices:

$$\bigoplus_{B \subseteq N} ((D \underset{R_0}{\otimes} R)_*(\!(\mathcal{F})\!))_{k-|N|+|B|}$$

and a boundary similar to that of the torus $\mathfrak{T}(\alpha\mathfrak{C}\beta)$.

We firstly need to show that there is a homotopy equivalence between $C \underset{R}{\otimes} R(\!(\mathcal{F})\!)$ and $\mathfrak{T}(\alpha\mathfrak{C}\beta)(\!(\mathcal{F})\!)$. Owing to the above, it is enough to show that there is an isomorphism $\mathfrak{T}(\alpha\mathfrak{C}\beta) \underset{R}{\otimes} R(\!(\mathcal{F})\!) \cong \mathfrak{T}(\alpha\mathfrak{C}\beta)(\!(\mathcal{F})\!)$.

Proposition 4.4.1. *Let M be a finitely presented right R_0-module. The map*

$$\Psi_M \colon M \underset{R_0}{\otimes} R_*(\!(\mathcal{F})\!) \to (M \underset{R_0}{\otimes} R)_*(\!(\mathcal{F})\!),\ m \otimes \sum_{\rho \in \mathbb{Z}^n} r_\rho \mapsto \sum_{\rho \in \mathbb{Z}^n} (m \otimes r_\rho)$$

is an isomorphism.

Proof. See [HQ15][Lemma 3.1.1], and note that the above is a similar case. □

Proposition 4.4.2. *There is an isomorphism*

$$\mathfrak{T}(\alpha\mathfrak{C}\beta) \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) \cong \mathfrak{T}(\alpha\mathfrak{C}\beta)(\!(\mathcal{F})\!).$$

Proof. It follows from the isomorphism seen in Proposition 4.4.1, firstly note that $\mathfrak{T}(\alpha\mathfrak{C}\beta) \underset{R}{\otimes} R_*(\!(\mathcal{F})\!)$ has modules $D \underset{R_0}{\otimes} R \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) = D \underset{R_0}{\otimes} R_*(\!(\mathcal{F})\!)$ so $\Psi_D(D \underset{R_0}{\otimes} R \underset{R}{\otimes} R_*(\!(\mathcal{F})\!)) = (D \underset{R_0}{\otimes} R)_*(\!(\mathcal{F})\!)$ is a module isomorphism. Let

$$\overline{\Psi}_D \colon \mathfrak{T}(\alpha\mathfrak{C}\beta) \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) \cong \mathfrak{T}(\alpha\mathfrak{C}\beta)(\!(\mathcal{F})\!)$$

be such that

$$(\overline{\Psi}_D)_B = \Psi_D \colon \mathfrak{T}(\alpha\mathfrak{C}\beta)_B \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) \cong \mathfrak{T}(\alpha\mathfrak{C}\beta)(\!(\mathcal{F})\!)_B.$$

Consider the map μ_{e_i} and a similar map $\mu'_{e_i} \colon (D \underset{R_0}{\otimes} R)_*(\!(\mathcal{F})\!) \to (D \underset{R_0}{\otimes} R)_*(\!(\mathcal{F})\!)$. It can be seen that the following diagram is commuative:

$$\begin{array}{ccc} D \underset{R_0}{\otimes} R \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) & \xrightarrow{\Psi_D} & (D \underset{R_0}{\otimes} R)_*(\!(\mathcal{F})\!) \\ {\scriptstyle \mu_{e_i} \otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu'_{e_i}} \\ D \underset{R_0}{\otimes} R \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) & \xrightarrow{\Psi_D} & (D \underset{R_0}{\otimes} R)_*(\!(\mathcal{F})\!) \end{array}$$

and since the components of the map $\mathfrak{T}(\alpha\mathcal{C}\beta)_{B,A}$ are sums of compositions of maps $\mu_{e_i}, \alpha \otimes \mathrm{id}, \beta \otimes \mathrm{id}$ and $H \otimes \mathrm{id}$, it is clear that $\overline{\Psi}_D$ will commute with the boundary maps hence it is a chain complex isomorphism as required. □

Corollary 4.4.3. *From Lemma 4.2.7, Proposition 4.3.8 (the Mather trick) and the preceding Proposition 4.4.2, there is a homotopy equivalence*

$$C \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) \simeq \mathfrak{T}(\alpha\mathfrak{C}\beta)(\!(\mathcal{F})\!)$$

consisting of maps like so:

$$C \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) \underset{4.2.7}{\simeq} \mathfrak{T}(\mathfrak{C}) \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) \underset{4.3.8}{\simeq} \mathfrak{T}(\alpha\mathfrak{C}\beta) \underset{R}{\otimes} R_*(\!(\mathcal{F})\!) \underset{4.4.2}{\cong} \mathfrak{T}(\alpha\mathfrak{C}\beta)(\!(\mathcal{F})\!).$$

Now we show that the complexes $\mathfrak{T}(\alpha\mathfrak{C}\beta)(\!(\mathcal{F})\!)$ have trivial homology for all $\mathcal{F}$ where $\mathcal{F}$ contains only two faces, one of which is $S = [-1,1]^n$, and hence show that $C \underset{R}{\otimes} R_*(\!(\mathcal{F})\!)$ is contractible also.

Let $E = \{\pm e_j \colon 1 \le j \le n\}$ be a standard basis of $\mathbb{Z}^n$. Let G_i be the $n-1$-dimensional face of S such that $\eta_{G_i} = e_i$, which means that $\eta_{-G_i} = -e_i$.

Proposition 4.4.4. *Let $\mathcal{F} = \{G_i \subset S\}$ for $e_i \in E$. The infinite sum*

$$P_i = \mathrm{id} - \sum_{k \ge 0} (\alpha\mu_{e_i}\beta)^k,$$

is a well-defined chain map on the complex $(D \underset{R_0}{\otimes} R)_*(\!(\mathcal{F})\!)$.

Proof. Take an element $d = \sum_{\rho \in \mathbb{Z}^n} d_q \otimes r_\rho \in (D_q \underset{R_0}{\otimes} R)_*((\mathcal{F}))$ and let $\rho = \sum_{\ell=1}^n \rho_\ell e_\ell, \rho_\ell \in \mathbb{Z}$. By the definition of $(D \underset{R_0}{\otimes} R)_*((\mathcal{F}))$, the support of d has a global lower bound $k \in \mathbb{Z}$ on ρ_i such that for ρ where $\rho_i < k$, $r_\rho = 0$. I wish to show that the image of the above element d is contained in $(D_q \underset{R_0}{\otimes} R)_*((\mathcal{F}))$ and that the at each ρ, the image is a finite sum. Evidentally, the support of $P_i(d)$ also satisfies the bound condition as required as the effect of applying P_i does not lower the possible coefficients of e_i in the support of the element.

Now take the $\rho = \sum_{l=1}^n \rho_l e_l, \rho_l \in \mathbb{Z}$ component of $P_i(d)$ for $\rho_i \geq k$ (if $\rho_i < k$, then $P_i(d_q \otimes r_\rho) = 0$ and there is nothing to show). There are precisely $\rho_i - k + 1$ non-zero elements in the sum that constitutes the element $P_i(d)_\rho$, namely the images of the summands of d at $\sum_{\ell=1}^n \rho_\ell e_\ell - qe_i$ by $(\alpha\mu_{e_i}\beta)^q$ for $1 \leq q \leq \rho_i - k$ and the summand of d at ρ mapped by the identity map.

That P_i commutes with the boundary maps follows from the fact that α, μ_{e_i} and β are chain maps. □

Proposition 4.4.5. *Let $\mathcal{F} = \{F \subset S\}$ for a face $F = \cap_t G_{i_t}$, $1 \leq t \leq n$. The infinite sum $P_{i_q} = \mathrm{id} - \sum\limits_{k \geq 0} (\alpha\mu_{e_{i_q}}\beta)^k$, is a well-defined chain map on the complex $(D \underset{R_0}{\otimes} R)_*((\mathcal{F}))$ for all $1 \leq q \leq t$.*

Proof. Repeat the above argument for each individual P_{i_q}, considering conditions (2-G_{i_q}) of the collection of $n-1$-dimensional faces G_{i_q} contained within $\mathrm{CT}(\mathcal{F})$. □

Proposition 4.4.6. *Let $\mathcal{F} = \{F \subset S\}$ where $F \subseteq G_1$. The complexes $C \underset{R}{\otimes} R_*((\mathcal{F}))$ and $\mathfrak{T}(\alpha\mathfrak{C}\beta)((\mathcal{F}))$ are contractible.*

Proof. Let $P_1 = \mathrm{id} - \sum\limits_{k \geq 0} (\alpha\mu_{e_1}\beta)^k$, which is a self map on the complex $(D \underset{R_0}{\otimes} R)_*((\mathcal{F}))$ that is well defined due to the Novikov structure on the module. Let Q be a $2n$ by $2n$ matrix indexed by subsets of $N = \{1, ..., n\}$, with zero entries except for the entries indexed by $(A \cup \{1\}, A)$ for $1 \notin A \subset N$ which each have $(-1)^{(\langle A \colon \{1\} \rangle)} P_1$. This is a self map on $\mathfrak{T}(\alpha\mathfrak{C}\beta)((\mathcal{F}))$. The composition $d_{\mathfrak{T}(\alpha\mathfrak{C}\beta)((\mathcal{F}))}Q + Qd_{\mathfrak{T}(\alpha\mathfrak{C}\beta)((\mathcal{F}))} = X$ is lower triangular with the identity on the diagonal. This can be seen as the entries on the diagonal of X are compositions of P_1 with the entries indexed by $A, A\cup\{1\}$ of $d_{\mathfrak{T}(\alpha\mathfrak{C}\beta)}$ which are

$$(-1)^{(\langle A\colon \{1\}\rangle)}P_1(-1)^{(\langle A\colon \{1\}\rangle)}(\mathrm{id} - \alpha\mu_{e_1}\beta) = \mathrm{id}$$

at A, A and

$$(-1)^{(\langle A\colon \{1\}\rangle)}(\mathrm{id} - \alpha\mu_{e_1}\beta)(-1)^{(\langle A\colon \{1\}\rangle)}P_1 = \mathrm{id}$$

at $A\cup\{1\}, A\cup\{1\}$ of X and the only non trivial entries that appear above the diagonal appear on the entries indexed by $A, A\cup\{1\}$ of X which are

$$(-1)^A d_{(D\otimes R)_*((\mathcal{F}))}(-1)^{(\langle A\colon \{1\}\rangle)}P_1 + (-1)^{(\langle A\colon \{1\}\rangle)}P_1(-1)^{A+1}d_{(D\otimes R)_*((\mathcal{F}))} = 0$$

(as P_1 is a chain map). It follows that X is an invertible matrix that commutes with $d_{\mathfrak{T}(\alpha\mathfrak{C}\beta)((\mathcal{F}))}$, therefore $X^{-1}Q$ satisfies $d(X^{-1}Q) + (X^{-1}Q)d = \mathrm{id}$ hence it is a contraction making $\mathfrak{T}(\alpha C\beta)((\mathcal{F}))$ contractible, making $C \underset{R}{\otimes} R_*((\mathcal{F}))$ contractible. □

Proposition 4.4.7. *Let $\mathcal{F} = \{F \subset S\}$ where $F \subseteq -G_1$. The complex $C \underset{R}{\otimes} R_*((\mathcal{F}))$ is contractible.*

Proof. Let $\ddot{R}$ be a ring such that $\ddot{R}_{ke_1} = R_{-ke_1}$. This is a strongly $\mathbb{Z}^n$-graded ring as R is. Let $\mathcal{F}'$ be the same flag as $\mathcal{F}$ except swap -1 and 1 (for example, the two faces of $\mathcal{F}'$ will be S and $F'\cap G_1$ where $F'\cap -G_1 = F$). Repeat the entire proof with this new ring, using Proposition 4.4.6 to show that $C\underset{R}{\otimes}\ddot{R}_*((\mathcal{F}'))$ is contractible and observe that $C\underset{R}{\otimes}\ddot{R}_*((\mathcal{F}')) = C\underset{R}{\otimes}R_*((\mathcal{F}))$. □

Proposition 4.4.8. *Let $\mathcal{F} = \{F \subset S\}$ where $F \subseteq G_i$ for some $n-1$ dimensional face G_i with positive η_{G_i}. The complex $C \underset{R}{\otimes} R_*((\mathcal{F}))$ is contractible.*

Proof. Define a new ring $\tilde{R}$ that is the same as R except with basis elements of $\mathbb{Z}^n$ re-ordered, so that $\tilde{e}_1 = e_{i_k}$. Repeat Proposition 4.4.6 with the flag $\mathcal{F}'$ consisting of the two faces $\{S\}$ and $F'\cap G_1$ where $F'\cap G_i = F$ to see that $C \underset{R}{\otimes} \tilde{R}_*((\mathcal{F}'))$ and observe that $C \underset{R}{\otimes} \tilde{R}_*((\mathcal{F}')) = C \underset{R}{\otimes} R_*((\mathcal{F}))$ making $C \underset{R}{\otimes} R_*((\mathcal{F}))$ contractible as required. $\square$

Proposition 4.4.9. *Let $\mathcal{F} = \{F \subset S\}$ where $F = \cap_q(-G_{i_q})$ for $n-1$ dimensional faces $-G_{i_q}$. The complex $C \underset{R}{\otimes} R_*((\mathcal{F}))$ is contractible.*

Proof. Pick $k \in \{i_1, ..., i_t\}$ and form a new $\mathbb{Z}^n$-graded ring $\ddot{\tilde{R}}$ where $\ddot{\tilde{R}}_{j\tilde{e}_1} = R_{-je_k}$ for $j \in \mathbb{Z}$ and a reordered basis. Take a suitable new flag $\mathcal{F}'$ then argue as in Proposition 4.4.7 and Proposition 4.4.8 for this new ring, observing that $C \underset{R}{\otimes} \ddot{\tilde{R}}_*((\mathcal{F}')) = C \underset{R}{\otimes} R_*((\mathcal{F}))$. $\square$

We can now prove the main result of this thesis.

Theorem 4.4.10. *Main result. Let R be a strongly $\mathbb{Z}^n$-graded ring and write $R_{0_{\mathbb{Z}^n}} = R_0$. Let $S = [-1,1]^n$ and C be a bounded complex of finitely generated free R-modules. The complex C is R_0-finitely dominated if and only if for every flag $\mathcal{F}$ of the form $\mathcal{F} = \{F \subset S\}$, the complexes*

$$C \underset{R}{\otimes} R_*((\mathcal{F}))$$

are acyclic.

Proof. The forward direction follows from combining Propositions 4.4.6, 4.4.7, 4.4.8 and 4.4.9. The reverse is precisely Proposition 2.10.4. $\square$

APPENDIX

A. STRONGLY $\mathbb{Z}$-GRADED CASE

The following paper was worked on by Thomas and I during the time of my Phd. It will appear in the Israel Journal of Mathematics.

FINITE DOMINATION AND NOVIKOV HOMOLOGY OVER STRONGLY $\mathbb{Z}$-GRADED RINGS

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ABSTRACT. Let L be a unital $\mathbb{Z}$-graded ring, and let C be a bounded chain complex of finitely generated L-modules. We give a homological characterisation of when C is homotopy equivalent to a bounded complex of finitely generated projective L_0-modules, generalising known results for twisted LAURENT polynomial rings. The crucial hypothesis is that L is a *strongly* graded ring.

1. FINITE DOMINATION OVER STRONGLY $\mathbb{Z}$-GRADED RINGS

Finite domination and Novikov homology. Let L be a unital ring, and let K be a subring of L. A bounded chain complex C of (right) L-modules is *K-finitely dominated* if C, considered as a complex of K-modules, is a retract up to homotopy of a bounded complex of finitely generated free K-modules; this happens if and only if C is homotopy equivalent, as a K-module complex, to a bounded complex of finitely generated projective K-modules [Ran85, Proposition 3.2. (ii)]. The following result of RANICKI gives a complete homological characterisation of finite domination in an important special case:

Theorem 1.1 (RANICKI [Ran95, Theorem 2]). *Let R be a unital ring, and let $R[t,t^{-1}]$ denote the* LAURENT *polynomial ring in the indeterminate t. Let C be a bounded chain complex of finitely generated free $R[t,t^{-1}]$-modules. The complex C is R-finitely dominated if and only if both*

$$C \underset{R[t,t^{-1}]}{\otimes} R((t^{-1})) \qquad \textit{and} \qquad C \underset{R[t,t^{-1}]}{\otimes} R((t))$$

have vanishing homology in all degrees. Here $R((t)) = R[[t]][t^{-1}]$ is the ring of formal LAURENT *series in t, and similarly $R((t^{-1})) = R[[t^{-1}]][t]$ stands for the ring of formal* LAURENT *series in t^{-1}.*

The cited paper [Ran95] also contains a discussion of the relevance of finite domination in topology. — The rings $R((t))$ and $R((t^{-1}))$ are known as NOVIKOV rings. The theorem can be formulated more succinctly: *The chain complex C is R-finitely dominated if and only if it has trivial* NOVIKOV *homology.*

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In the present paper we formulate and prove a surprising strengthening: *Theorem 1.1 remains valid if $R[t,t^{-1}]$ is replaced by an arbitrary strongly $\mathbb{Z}$-graded ring*, provided the definition of NOVIKOV rings is adapted suitably. We start by recalling the requisite definitions.

Definition 1.2. A *$\mathbb{Z}$-graded ring* is a (unital) ring L equipped with a direct sum decomposition into additive subgroups $L = \bigoplus_{k\in\mathbb{Z}} L_k$ such that $L_k L_\ell \subseteq L_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$, where $L_k L_\ell$ consists of the finite sums of ring products xy with $x \in L_k$ and $y \in L_\ell$. The summands L_k are called the *(homogeneous) components* of L; elements of L_k are called *homogeneous of degree k*. — Following DADE [Dad80] we call L a *strongly $\mathbb{Z}$-graded ring*, or simply a *strongly graded ring*, if $L_k L_\ell = L_{k+\ell}$ for all $k, \ell \in \mathbb{Z}$.

A specific example of a strongly $\mathbb{Z}$-graded ring is $L = R[t,t^{-1}]$, the ring of LAURENT polynomials; the nth component is $\{rt^n \mid r \in R\}$. The reader may wish to keep this motivating example in mind.

We will use the symbol $R_*[t,t^{-1}] = \bigoplus_{k\in\mathbb{Z}} R_k$ to denote an arbitrary $\mathbb{Z}$-graded ring in this paper. One may think of the elements of $R_*[t,t^{-1}]$ as formal LAURENT polynomials $\sum_{j=m}^{n} a_j t^j$ with $a_j \in R_j$, but note that this is a purely notational device; in general the ring $R_*[t,t^{-1}]$ does not contain an element called t. The point of using this notation is that we have a rather suggestive way of denoting various rings and modules constructed from $R_*[t,t^{-1}]$. For example, we can introduce the NOVIKOV rings

$$R_*((t^{-1})) = \prod_{n\le 0} R_n \oplus \bigoplus_{n>0} R_n \quad \text{and} \quad R_*((t)) = \bigoplus_{n<0} R_n \oplus \prod_{n\ge 0} R_n$$

and think of their elements as formal power series $\sum_{j\in\mathbb{Z}} a_j t^j$ with $a_j \in R_j$ such that $a_j = 0$ whenever $j \gg 0$ or $j \ll 0$, respectively.

In any $\mathbb{Z}$-graded ring the unit element is necessarily homogeneous of degree 0 [Dad80, Proposition 1.4] so that R_0 is a subring of $R_*[t,t^{-1}]$. With these preliminaries in place we can formulate our main result:

Theorem 1.3. *Let $R_*[t,t^{-1}] = \bigoplus_{k\in\mathbb{Z}} R_k$ be a strongly $\mathbb{Z}$-graded ring, and let C be a bounded chain complex of finitely generated free $R_*[t,t^{-1}]$-modules. The complex C is R_0-finitely dominated if and only if both*

$$C \underset{R_*[t,t^{-1}]}{\otimes} R_*((t^{-1})) \qquad \text{and} \qquad C \underset{R_*[t,t^{-1}]}{\otimes} R_*((t))$$

have vanishing homology in all degrees.

As a special case this says that Theorem 1.1 holds for twisted Laurent polynomial rings [HK07, Theorem 6], or even for the more general case of crossed products (which are characterised by having homogeneous invertible elements of arbitrary degrees). However, this is not the complete extent of the generalisation as there are strongly graded rings which are not crossed products. Possibly the easiest example to write down is the following: Let $K[A,B,C,D]$ be a polynomial ring over the field K, considered as a $\mathbb{Z}$-graded ring by giving A and C degree 1, and giving B and D degree -1. Let $R_*[t,t^{-1}]$ be the quotient $K[A,B,C,D]/(AB+CD-1)$; as the relation is homogeneous, this results in a $\mathbb{Z}$-graded ring which is actually strongly graded since $AB+CD = BA+DC = 1$ by construction. It can be shown, using ideas from Gröbner basis theory, that the only units are $K^\times \subset R_*[t,t^{-1}]$ so that our ring is not a crossed product. Now consider the following 2-step chain complex:

$$R_*[t,t^{-1}] \xrightarrow{\binom{1-A}{1-B}} R_*[t,t^{-1}] \oplus R_*[t,t^{-1}] \xrightarrow{(1-B,-(1-A))} R_*[t,t^{-1}]$$

The map $1-A$ becomes invertible in $R_*(\!(t)\!)$, with inverse $(1-A)^{-1} = \sum_{j\geq 0} A^j$; similarly, the map $1-B$ becomes invertible in $R_*(\!(t^{-1})\!)$. Hence the complex becomes acyclic after tensoring with $R_*(\!(t^{\pm 1})\!)$, and Theorem 1.3 asserts that it is in fact R_0-finitely dominated.

Structure of the paper. For the proof of Theorem 1.3 we combine techniques from strongly graded algebra with homotopy-theoretic methods and homological algebra of bicomplexes. We start by introducing various rings associated to a $\mathbb{Z}$-graded ring, and discuss partitions of unity which are the main technical tool from graded algebra to be used throughout the paper. This will occupy the remainder of §1.

In §2 we prove the "if" implication of Theorem 1.3, based on the homotopy-theoretic methods used in [Ran95] for the case of a Laurent polynomial ring. The organisation follows the pattern laid out by the first author in [Hüt15], where a description of the algebro-geometric background of the procedure is given. It is of interest to note that the $\mathbb{Z}$-graded structure of our ring allows us in Proposition 2.9 to construct complexes of sheaves from the given complex of modules C, while the *strong* grading ensures that certain chain complexes consist of finitely generated projective R_0-modules, cf. Corollary 2.7.

In §3 we attack the reverse implication of Theorem 1.3, using double complex techniques as documented in [Hüt11]. The graded structure is used at various places. Most notably, the definition and the properties of the "algebraic torus", a substitute for the more usual algebraic mapping torus of a self-map of a chain complex, depend crucially on extra data which can be chosen only in view of the strong grading. In addition, passage to NOVIKOV rings involves a certain "twisting" operation on powers of modules that is defined in terms of the grading.

The results of this paper were obtained as part of the second author's PhD thesis.

Rings associated with $\mathbb{Z}$-graded rings. We make the following conventions for the rest of the paper: All rings, graded or otherwise, are assumed unital and all modules right unless stated otherwise. We let $R_*[t,t^{-1}]$ stand for an arbitrary unital $\mathbb{Z}$-graded ring, with nth homogeneous component denoted by R_n. That is, we have a graded ring $R_*[t,t^{-1}] = \bigoplus_{n\in\mathbb{Z}} R_n$. In many cases we will assume this ring to be strongly graded, but will take care to indicate where this hypothesis is really needed.

Given a $\mathbb{Z}$-graded ring $R_*[t,t^{-1}]$, it is known that the unit element 1 must be homogeneous of degree 0 [Dad80, Proposition 1.4]. It is then immediate from the definition that R_0 is a subring of $R_*[t,t^{-1}]$, and that all the homogeneous components R_k are R_0-bimodules.

Given the $\mathbb{Z}$-graded ring $R_*[t,t^{-1}]$ we define two $\mathbb{Z}$-graded subrings by setting $R_*[t^{-1}] = \bigoplus_{k\leq 0} R_k$ and $R_*[t] = \bigoplus_{k\geq 0} R_k$. These graded rings have trivial components in all positive and negative degrees, respectively. Elements can be thought of as formal polynomials in t^{-1} and t, respectively, with the coefficient of t^j an element of R_j.

We can also define the analogues of power series rings, $R_*[[t^{-1}]] = \prod_{n\leq 0} R_n$ and $R_*[[t]] = \prod_{n\geq 0} R_n$. Elements are of course formal power series in t^{-1} and t, respectively, with coefficient of t^j an element of R_j. We have previously defined the NOVIKOV rings $R_*((t^{-1}))$ and $R_*((t))$. Note that power series and NOVIKOV rings are *not* considered as graded rings; in fact, they do not admit a natural $\mathbb{Z}$-grading.

The collection of rings fits into the commutative diagram of ring inclusions displayed in Fig. 1.

Partitions of unity and strongly graded rings.

Definition 1.4. Given $n \in \mathbb{Z}$, an expression of the form $1 = \sum_j u_j v_j$ with $u_j \in R_n$, $v_j \in R_{-n}$ is called a *partition of unity of type* $(n,-n)$. This is understood to be a finite sum; we do not specify the summation range unless we need it explicitly.

A partition of unity of type $(n,-n)$ exists if and only if $1 \in R_n R_{-n}$. Partitions of unity are our main technical tool; their existence is intimately related to properties of the graded structure of the ring:

$$\begin{tikzcd}
 & R_*((t^{-1})) & & R_*((t)) & \\
R_*[[t^{-1}]] \arrow[ur, hook] & & R_*[t,t^{-1}] \arrow[ul, hook] \arrow[ur, hook] & & R_*[[t]] \arrow[ul, hook] \\
 & R_*[t^{-1}] \arrow[ul, hook] \arrow[ur, hook] & & R_*[t] \arrow[ul, hook] \arrow[ur, hook] & \\
 & & R_0 \arrow[ul, hook] \arrow[ur, hook] & &
\end{tikzcd}$$

FIGURE 1. The collection of rings and their inclusion relation

Proposition 1.5 (Characterisation of strongly graded rings)**.** *The following statements are equivalent:*

(1) *The ring $R_*[t, t^{-1}]$ is strongly graded.*
(2) *For every $n \in \mathbb{Z}$ there is at least one partition of unity of type $(n, -n)$.*
(3) *There is at least one partition of unity of type $(1, -1)$, and at least one of type $(-1, 1)$.*

Proof. For the equivalence of statements (1) and (2) see Proposition 1.6 of [Dad80]. That (2) implies (3) is trivial. For the converse, suppose that $1 = \sum_{j=1}^{q} u_j v_j$ is a partition of unity of type $(\pm 1, \mp 1)$; then for $m \geq 2$ the q^m pairs of elements

$$u_{\mathbf{j}} = u_{j_1} u_{j_2} \cdots u_{j_m} \quad \text{and} \quad v_{\mathbf{j}} = v_{j_m} v_{j_{m-1}} \cdots v_{j_1} \ ,$$

where $\mathbf{j} = (j_1, j_2, \cdots, j_m) \in \{1, 2, \cdots, q\}^m$, form a partition of unity of type $(\pm m, \mp m)$. $\square$

The following Proposition is well known; we include a proof because of its fundamental importance for this paper.

Proposition 1.6. *If $R_*[t, t^{-1}]$ is strongly graded, then each of the R_k is finitely generated projective both as a left R_0-module and as a right R_0-module.*

Proof. We treat the case of right R_0-modules only. Let $1 = \sum_j u_j v_j$ be a partition of unity of type $(k, -k)$; existence is guaranteed by Proposition 1.5. The maps $g_j \colon R_k \to R_0$ with $g_j(y) = v_j y$ are maps of right R_0-modules and satisfy $\sum_j u_j g_j(r) = \sum_j u_j v_j r = r$ for any $r \in R_k$. Thus R_k is a finitely generated projective right R_0-module by the dual basis lemma, cf. Proposition 12 of [Bou98, §II.2.7]. $\square$

Corollary 1.7. *Suppose $R_*[t,t^{-1}]$ is strongly graded. Then any projective left or right $R_*[t,t^{-1}]$-module is also projective when considered as a left or right R_0-module.* □

Given numbers $q, p \in \mathbb{Z}$ we define the symbols

$$t^q \cdot R_*[t^{-1}] = \bigoplus_{j \leq q} R_j \quad \text{and} \quad t^{-p} \cdot R_*[t] = \bigoplus_{j \geq -p} R_j \ ;$$

the former is an $R_*[t^{-1}]$-bimodule, the latter an $R_*[t]$-bimodule. The induced $R_*[t,t^{-1}]$-modules behave as expected in the strongly graded case:

Lemma 1.8. *Let $q, p \in \mathbb{Z}$. The $R_*[t,t^{-1}]$-linear maps*

$$\gamma \colon t^q \cdot R_*[t^{-1}] \underset{R_*[t^{-1}]}{\otimes} R_*[t,t^{-1}] \longrightarrow R_*[t,t^{-1}] \ , \quad r \otimes s \mapsto rs$$

and

$$\alpha \colon t^{-p} \cdot R_*[t] \underset{R_*[t]}{\otimes} R_*[t,t^{-1}] \longrightarrow R_*[t,t^{-1}] \ , \quad r \otimes s \mapsto rs$$

are isomorphisms provided the ring $R_*[t,t^{-1}]$ is strongly graded.

Proof. Suppose $R_*[t,t^{-1}]$ is strongly graded. Then we may choose a partition of unity of type $(-p,p)$, say $1 = \sum_j u_j v_j$ with $u_j \in R_{-p}$ and $v_j \in R_p$. The $R_*[t,t^{-1}]$-linear map

$$\beta \colon R_*[t,t^{-1}] \longrightarrow t^{-p} \cdot R_*[t] \underset{R_*[t]}{\otimes} R_*[t,t^{-1}] \ , \quad r \mapsto \sum_j u_j \otimes v_j r$$

satisfies $\alpha\beta(r) = \sum_j u_j v_j r = r$ so that $\alpha\beta = \mathrm{id}$. Also

$$\beta\alpha(r \otimes s) = \beta(rs) = \sum_j u_j \otimes v_j rs \underset{(*)}{=} \sum_j u_j v_j r \otimes s = r \otimes s$$

(where the equality labelled $(*)$ is true since $v_j r \in R_*[t]$ for any $r \in t^{-p} \cdot R_*[t]$), whence $\beta\alpha = \mathrm{id}$. — The case of γ is similar. □

The proof of Proposition 1.6 applies with minor modifications to give the following result:

Lemma 1.9. *Suppose that $R_*[t,t^{-1}]$ is strongly graded. Then $t^q \cdot R_*[t^{-1}]$ is a finitely generated projective left and right $R_*[t^{-1}]$-module. Similarly, $t^{-p} \cdot R_*[t]$ is a finitely generated projective left and right $R_*[t]$-module.* □

2. Trivial Novikov homology implies finite domination

Sheaves and their cohomology. We will have occasion to study diagrams of the form

$$\mathfrak{M} = \Big(M^- \xrightarrow{\mu^-} M \xleftarrow{\mu^+} M^+ \Big) \; ; \tag{2.1}$$

the entries will be modules, or chain complexes of modules. The maps μ^- and μ^+ are called the *structure maps of* $\mathfrak{M}$. A *map of diagrams* consists of a triple of maps (f^-, f, f^+) which is compatible with the structure maps of source and target.

Definition 2.2. Let as before $R_*[t, t^{-1}]$ be a $\mathbb{Z}$-graded ring. A *pre-sheaf* is a diagram $\mathfrak{M}$ of the form (2.1) where M^- is an $R_*[t^{-1}]$-module, M is an $R_*[t, t^{-1}]$-module, M^+ is an $R_*[t]$-module, f^- is $R_*[t^{-1}]$-linear and f^+ is $R_*[t]$-linear. The pre-sheaf $\mathfrak{M}$ is called a *sheaf* if the adjoints of the structure maps f^- and f^+ are isomorphisms of $R_*[t, t^{-1}]$-modules:

$$M^- \underset{R_*[t^{-1}]}{\otimes} R_*[t, t^{-1}] \xrightarrow{\cong} M \xleftarrow{\cong} M^+ \underset{R_*[t]}{\otimes} R_*[t, t^{-1}]$$

Of particular importance will be the pre-sheaves

$$\mathcal{O}(q, p) = \Big(t^q \cdot R_*[t^{-1}] \xrightarrow[\iota_q]{\subset} R_*[t, t^{-1}] \xleftarrow[p^\iota]{\supset} t^{-p} \cdot R_*[t] \Big) \tag{2.3}$$

which depend on the numbers $q, p \in \mathbb{Z}$. In case $R_*[t, t^{-1}]$ is strongly graded these pre-sheaves are actually sheaves by Lemma 1.8, and are then called *twisting sheaves*.

Back to a general diagram $\mathfrak{M}$ of modules of the form (2.1), we define:

Definition 2.4. The R_0-module chain complex

$$H(\mathfrak{M}) = \Big(M \xleftarrow{-f^- + f^+} M^- \oplus M^+ \Big)$$

(concentrated in chain degrees -1 and 0) is called the *cohomology chain complex* of $\mathfrak{M}$. We write $H^q(\mathfrak{M})$ for the $(-q)$th homology of $H(\mathfrak{M})$.

In fact, $H^q(\mathfrak{M}) = \lim^q(\mathfrak{M})$. — The definitions of pre-sheaf and sheaf apply to chain complexes instead of modules *mutatis mutandis*; in effect, a (pre-)sheaf of chain complexes is the same as a chain complex of (pre-)sheaves. Given any diagram of chain complexes $\mathfrak{N} = \big(N^- \xrightarrow{g^-} N \xleftarrow{g^+} N^+ \big)$ we obtain a double complex $H(\mathfrak{N})$ by applying the cohomology chain complex construction levelwise. (The double complex is concentrated in columns -1 and 0, and has commuting differentials.)

Definition 2.5. Given a diagram of chain complexes $\mathfrak{N}$ we define its *hypercohomology complex* $\mathbb{H}(\mathfrak{N})$ by setting $\mathbb{H}(\mathfrak{N}) = \mathrm{Tot} H(\mathfrak{N})$, the totalisation of $H(\mathfrak{N})$.

The totalisation is the usual one: $\mathbb{H}(\mathfrak{N})_n = N_n^- \oplus N_n^+ \oplus N_{n+1}$, with differential induced by $-g^-$, g^+, the differentials of N^- and N^+, and the negative of the differential of N. Up to shift and sign conventions $\mathbb{H}(\mathfrak{N})$ is the mapping cone of the map $-g^- + g^+$.

Proposition 2.6. *Let $R_*[t,t^{-1}]$ be a $\mathbb{Z}$-graded ring, and let $q,p \in \mathbb{Z}$.*

1. *For $p+q \geq 0$, the complex $H\big(\mathcal{O}(q,p)\big)$ is homotopy equivalent to the chain complex having $\bigoplus_{k=-p}^{q} R_k$ in chain level 0 as its only non-trivial chain module.*
2. *For $p+q = -1$, the complex $H\big(\mathcal{O}(q,p)\big)$ is contractible.*
3. *For $p+q \leq -2$, the complex $H\big(\mathcal{O}(q,p)\big)$ is homotopy equivalent to the chain complex having $\bigoplus_{k=q+1}^{-p-1} R_k$ in chain level -1 as its only non-trivial chain module.*

Proof. We consider the case $p+q \geq 0$ only, the others being similar (and quite irrelevant for our purposes). It is enough to show that the R_0-module sequence

$$0 \longrightarrow \bigoplus_{k=-p}^{q} R_k \underset{\rho}{\overset{\Delta}{\rightleftarrows}} t^q \cdot R_*[t^{-1}] \oplus t^{-p} \cdot R_*[t] \underset{\sigma}{\overset{-\iota_q + {}_p\iota}{\rightleftarrows}} R_*[t,t^{-1}] \longrightarrow 0$$

is split exact, where ι_q and ${}_p\iota$ denote the inclusions, and where Δ is the "diagonal" map $r \mapsto (r,r)$; the splitting maps ρ and σ will be defined presently. — The sequence can be re-written in more explicit terms:

$$0 \longrightarrow \bigoplus_{k=-p}^{q} R_k \xrightarrow{\Delta} \bigoplus_{k \leq q} R_k \oplus \bigoplus_{k \geq -p} R_k \xrightarrow{-\iota_q + {}_p\iota} \bigoplus_{k \in \mathbb{Z}} R_k \longrightarrow 0$$

The composition $(-\iota_q + {}_p\iota) \circ \Delta$ is trivial. We define σ by the formula

$$\sigma \colon \bigoplus_{k \in \mathbb{Z}} R_k \longrightarrow \bigoplus_{k \leq q} R_k \oplus \bigoplus_{k \geq -p} R_k \ , \quad \sum_{k \in \mathbb{Z}} r_k \mapsto \Big(-\sum_{k \leq q} r_k, \sum_{k \geq q+1} r_k \Big)$$

(note that $p+q \geq 0$ implies $q+1 > -p$) and ρ by

$$\rho \colon \bigoplus_{k \leq q} R_k \oplus \bigoplus_{k \geq -p} R_k \longrightarrow \bigoplus_{k=-p}^{q} R_k \ , \quad \Big(\sum_{k \leq q} r_k, \sum_{\ell \geq -p} s_\ell \Big) \mapsto \sum_{\ell=-p}^{q} s_\ell \ .$$

They satisfy the identities

$$\rho \circ \Delta = \mathrm{id} \ ,$$
$$\sigma \circ (-\iota_q + {}_p\iota) + \Delta \circ \rho = \mathrm{id} \ ,$$
$$(-\iota_q + {}_p\iota) \circ \sigma = \mathrm{id} \ ,$$

as can be verified by direct calculation; thus the sequence is split exact as required. □

Corollary 2.7. *If $R_*[t,t^{-1}]$ is strongly graded then the cohomology chain complex $H\big(\mathcal{O}(q,p)\big)$ is R_0-finitely dominated.*

Proof. By Proposition 2.6, $H\big(\mathcal{O}(q,p)\big)$ is homotopy equivalent to a chain complex with at most one non-zero entry consisting of a finite sum of homogeneous components R_k of $R_*[t,t^{-1}]$. Since the R_k are all finitely generated projective right R_0-modules by Proposition 1.6, $H\big(\mathcal{O}(q,p)\big)$ is R_0-finitely dominated. □

Building chain complexes of pre-sheaves from chain complexes of modules. Thanks to the graded structure of our ring $R_*[t,t^{-1}]$ one can extend a given chain complex of $R_*[t,t^{-1}]$-modules to a complex of pre-sheaves. We start with the case of a single module homomorphism.

Lemma 2.8. *Let $q,p \in \mathbb{Z}$, and let $f\colon R_*[t,t^{-1}]^n \longrightarrow R_*[t,t^{-1}]^m$ be an $R_*[t,t^{-1}]$-linear map. For all sufficiently large numbers $p',q' \in \mathbb{Z}$ there exists a map of pre-sheaves*

$$(f^-, f, f^+)\colon \bigoplus_{k=1}^{n} \mathcal{O}(q,p) \longrightarrow \bigoplus_{k=1}^{m} \mathcal{O}(q',p') \ ,$$

depending on q' and p', which has the given f as its middle component. In other words, the module homomorphism f can be extended to a map of pre-sheaves.

Proof. Consider the following diagram, where q' and p' are, for the moment, unspecified integers:

$$\begin{tikzcd}
\bigoplus_{k=1}^{n} t^q \cdot R_*[t^{-1}] \arrow[r, "\iota_q"] & \bigoplus_{k=1}^{n} R_*[t,t^{-1}] \arrow[d, "f"] & \bigoplus_{k=1}^{n} t^{-p} \cdot R_*[t] \arrow[l, "{}_p\iota"'] \\
\bigoplus_{k=1}^{m} t^{q'} \cdot R_*[t^{-1}] \arrow[r, "\iota_{q'}"] & \bigoplus_{k=1}^{m} R_*[t,t^{-1}] & \bigoplus_{k=1}^{m} t^{-p'} \cdot R_*[t] \arrow[l, "{}_{p'}\iota"']
\end{tikzcd}$$

The map f yields $R_*[t,t^{-1}]$-linear maps ${}_kf_j\colon R_*[t,t^{-1}] \longrightarrow R_*[t,t^{-1}]$ by restriction to the kth summand of the source and the jth summand of the target, and the (finite) collection of these maps determines f. — For now fix indices j and k. The element ${}_kf_j(1) \in R_*[t,t^{-1}]$ is a finite sum of non-zero homogeneous elements. Let $-a$ be the minimal occurring degree if ${}_kf_j(1) \neq 0$, and an arbitrary integer otherwise. As ${}_kf_j(r) = {}_kf_j(1) \cdot r$, the image of $t^{-p} \cdot R_*[t]$ under ${}_kf_j$ is contained in $t^{-(p+a)} \cdot R_*[t] \subseteq R_*[t,t^{-1}]$, hence is contained in $t^{-p'} \cdot R_*[t]$ provided p' is sufficiently large in the sense that $p' \geq a + p$. — Allowing arbitrary indices j and k now, we may choose p' sufficiently large for all j and k. Then the map $f \circ {}_p\iota$ factors as

$$\bigoplus_{k=1}^{n} t^{-p} \cdot R_*[t] \xrightarrow{f^+} \bigoplus_{k=1}^{m} t^{-p'} \cdot R_*[t] \xrightarrow{{}_{p'}\iota} \bigoplus_{k=1}^{m} R_*[t,t^{-1}]$$

where f^+ is actually the map f, suitably restricted in source and target. — The component f^- is dealt with in a similar manner. $\square$

Proposition 2.9 (Extending chain complexes of modules to chain complexes of pre-sheaves)**.** *Let C be a bounded above chain complex of finitely generated free $R_*[t,t^{-1}]$-modules together with specified isomorphisms $C_n \cong R_*[t,t^{-1}]^{k_n}$. Then C is the "middle" component of a chain complex of pre-sheaves. More precisely, there exists a chain complex of pre-sheaves $\mathfrak{D} = \big(D^- \longrightarrow D \longleftarrow D^+\big)$ such that*

$$\mathfrak{D}_n = \bigoplus_{k_n} \mathcal{O}(q_n, p_n)$$

for certain $q_n, p_n \in \mathbb{Z}$ with $q_n + p_n \geq 0$, with $D \cong C$ via the specified isomorphisms.

In case $R_*[t,t^{-1}]$ is strongly graded, $\mathfrak{D}$ is a chain complex of sheaves in the sense of Definition 2.2, with $D^\pm$ consisting of finitely generated projective $R_*[t^\pm]$-modules.

Proof. We identify the chain modules C_n with direct sums $R_*[t,t^{-1}]^{k_n}$ via the given isomorphisms. The boundary maps then take the form of homomorphisms $d_n \colon R_*[t,t^{-1}]^{k_n} \longrightarrow R_*[t,t^{-1}]^{k_{n-1}}$.

Let m be the maximal index of a non-zero entry of C. Choose $q_m = p_m = 0$.

Now for $\ell = m, m-1, m-2, \cdots$ we use Lemma 2.8 to extend the boundary map d_ℓ to a map of pre-sheaves

$$\mathfrak{D}_\ell = \bigoplus_{k_\ell} \mathcal{O}(q_\ell, p_\ell) \xrightarrow{(d_\ell^-, d_\ell, d_\ell^+)} \bigoplus_{k_{\ell-1}} \mathcal{O}(q_{\ell-1}, p_{\ell-1}) = \mathfrak{D}_{\ell-1}$$

with $q_{\ell-1} + p_{\ell-1} \geq 0$.

We have defined a (possibly infinite) sequence of maps of pre-sheaves $(d_\ell^-, d_\ell, d_\ell^+)$. These maps are actually boundary maps of a chain complex of pre-sheaves. Indeed, $d_{\ell-1} \circ d_\ell = 0$ easily implies $d_{\ell-1}^+ \circ d_\ell^+ = 0$ and $d_{\ell-1}^- \circ d_\ell^- = 0$ as the structure maps of the diagrams $\mathcal{O}(q_\ell, p_\ell)$ are injective.

The last sentence of the Proposition holds as the pre-sheaves $\mathcal{O}(q,p)$ are actually sheaves by Lemma 1.8, consisting of projective modules by Lemma 1.9, if $R_*[t,t^{-1}]$ is strongly graded. □

From trivial Novikov homology to finite domination. With the machinery of sheaves set up we can implement the programme of [Hüt15] to prove that trivial Novikov homology implies finite domination. The strong grading proves to be crucial in two places. It is the very fact that twisting sheaves are sheaves (and not just pre-sheaves), combined with finiteness of their cohomology, that makes the proof work.

Notation 2.10. Given a chain complex $\mathfrak{D} = (D^- \longrightarrow D \longleftarrow D^+)$ of pre-sheaves let $\mathfrak{D}^+$ denote the diagram of chain complexes

$$D^+ \underset{R_*[t]}{\otimes} R_*[t,t^{-1}] \longrightarrow D^+ \underset{R_*[t]}{\otimes} R_*((t)) \longleftarrow D^+ \underset{R_*[t]}{\otimes} R_*[[t]] \ ;$$

similarly, let $\mathfrak{D}^-$ denote the diagram

$$D^- \underset{R_*[t^{-1}]}{\otimes} R_*[t,t^{-1}] \longrightarrow D^- \underset{R_*[t^{-1}]}{\otimes} R_*((t^{-1})) \longleftarrow D^- \underset{R_*[t^{-1}]}{\otimes} R_*[[t^{-1}]] \ .$$

In addition, we introduce the variants

$$\mathfrak{D}'^+ = \Big(D^+ \underset{R_*[t]}{\otimes} R_*[t,t^{-1}] \longrightarrow 0 \longleftarrow D^+ \underset{R_*[t]}{\otimes} R_*[[t]]\Big)$$

and

$$\mathfrak{D}'^- = \Big(D^- \underset{R_*[t^{-1}]}{\otimes} R_*[t,t^{-1}] \longrightarrow 0 \longleftarrow D^- \underset{R_*[t^{-1}]}{\otimes} R_*[[t^{-1}]]\Big) \ ,$$

and write $\zeta^\pm \colon \mathfrak{D}^\pm \longrightarrow \mathfrak{D}'^\pm$ for the obvious maps of diagrams:

$$\begin{tikzcd}
D^\pm \underset{R_*[t^{\pm 1}]}{\otimes} R_*[t,t^{-1}] \arrow[r] \arrow[d, "\mathrm{id}"'] & D^\pm \underset{R_*[t^{\pm 1}]}{\otimes} R_*((t^{\pm 1})) \arrow[d, "0"'] & D^\pm \underset{R_*[t^{\pm 1}]}{\otimes} R_*[[t^{\pm}1]] \arrow[l] \arrow[d, "\mathrm{id}"'] \\
D^\pm \underset{R_*[t^{\pm 1}]}{\otimes} R_*[t,t^{-1}] \arrow[r] & 0 & D^\pm \underset{R_*[t^{\pm 1}]}{\otimes} R_*[[t^{\pm}1]] \arrow[l]
\end{tikzcd} \tag{2.11}$$

We wish to analyse the hypercohomology complexes of $\mathfrak{D}^{\pm}$. To begin with, the sequence

$$0 \longrightarrow R_*[t] \xrightarrow{\Delta} R_*[t,t^{-1}] \oplus R_*[[t]] \xrightarrow{\rho} R_*((t)) \longrightarrow 0 \ , \qquad (2.12)$$

where $\Delta(r) = (r,r)$ and $\rho(r,s) = s - r$, is split exact as a sequence of right R_0-modules, with splitting maps

$$R_*[t] \xleftarrow{\kappa} R_*[t,t^{-1}] \oplus R_*[[t]] \xleftarrow{\lambda} R_*((t))$$

specified by the formulæ

$$\kappa\colon \Big(\sum_{k\in\mathbb{Z}} r_k, \sum_{k\geq 0} s_k\Big) \mapsto \sum_{k\geq 0} r_k \ ,$$

$$\lambda\colon \sum_{k\geq n} r_k \mapsto \Big(-\sum_{k<0} r_k, \sum_{k\geq 0} r_k\Big) \ .$$

Therefore the sequence (2.12) is exact (but not split) as a sequence of $R_*[t]$-bimodules. If the complex D^+ consists of projective $R_*[t]$-modules, tensoring (2.12) results in an exact sequence of right $R_*[t]$-module chain complexes

$$0 \longrightarrow D^+ \underset{R_*[t]}{\otimes} R_*[t] \longrightarrow D^+ \underset{R_*[t]}{\otimes} R_*[t,t^{-1}] \oplus D^+ \underset{R_*[t]}{\otimes} R_*[[t]] \longrightarrow D^+ \underset{R_*[t]}{\otimes} R_*((t)) \longrightarrow 0 \ .$$

This means that $H^0(\mathfrak{D}^+) = D^+ \otimes_{R_*[t]} R_*[t] \cong D^+$ and $H^1(\mathfrak{D}^+) = 0$ (levelwise application of H^0 and H^1). The latter implies that the natural map $\Delta^+\colon H^0(\mathfrak{D}^+) \longrightarrow \mathbb{H}(\mathfrak{D}^+)$ is a quasi-isomorphism [Hüt15, Lemma 4.2]. — It can be shown by analogous arguments that the natural map $\Delta^-\colon H^0(\mathfrak{D}^-) \longrightarrow \mathbb{H}(\mathfrak{D}^-)$ is a quasi-isomorphism, with source $D^- \otimes_{R_*[t^{-1}]} R_*[t^{-1}] \cong D^-$, provided D^- consists of projective modules. We have shown:

Lemma 2.13. *If D^+ consists of projective $R_*[t]$-modules the map*

$$\Delta^+\colon H^0(\mathfrak{D}^+) \longrightarrow \mathbb{H}(\mathfrak{D}^+)$$

is a quasi-isomorphism. Similarly, if D^- consists of projective $R_*[t^{-1}]$-modules the map $\Delta^-\colon H^0(\mathfrak{D}^-) \longrightarrow \mathbb{H}(\mathfrak{D}^-)$ is a quasi-isomorphism. □

Now let us start with a bounded chain complex C of finitely generated free $R_*[t,t^{-1}]$-modules. For each chain module $C_n \neq \{0\}$ we choose an isomorphism with $R_*[t,t^{-1}]^{k_n}$. Let $\mathfrak{D} = \big(D^- \longrightarrow D \longleftarrow D^+\big)$ denote the resulting complex of pre-sheaves according to Proposition 2.9, and let $\mathfrak{D}^{\pm}$ and $\mathfrak{D}'^{\pm}$ be the diagrams defined at the beginning of this section.— The structure map $D \longleftarrow D^+$ has an $R_*[t,t^{-1}]$-linear adjoint, $D \longleftarrow D^+ \otimes_{R_*[t]} R_*[t,t^{-1}]$, which induces a map of diagrams

$$\mathfrak{D}^+ \quad \longrightarrow \quad \Big(0 \longrightarrow 0 \longleftarrow D\Big) \ ;$$

upon application of $\mathbb{H}$ this yields a map $\pi^+ \colon \mathbb{H}(\mathfrak{D}^+) \longrightarrow D$. We have similarly a map $\pi^- \colon \mathbb{H}(\mathfrak{D}^-) \longrightarrow D$, and analogous maps using $\mathfrak{D}'^{\pm}$ denoted $\pi'^{\pm}$. All these fit into the commutative diagram displayed in Fig. 2.

$$\begin{tikzcd}
D^- \arrow[r] \arrow[d, "\Delta^-"'] & D \arrow[d, "\mathrm{id}"'] & D^+ \arrow[l] \arrow[d, "\Delta^+"'] \\
\mathbb{H}(\mathfrak{D}^-) \arrow[r, "\pi^-"] \arrow[d, "\mathbb{H}(\zeta^-)"'] & D \arrow[d, "\mathrm{id}"'] & \mathbb{H}(\mathfrak{D}^+) \arrow[l, "\pi^+"'] \arrow[d, "\mathbb{H}(\zeta^+)"'] \\
\mathbb{H}(\mathfrak{D}'^-) \arrow[r, "\pi'^-"] & D & \mathbb{H}(\mathfrak{D}'^+) \arrow[l, "\pi'^+"']
\end{tikzcd} \tag{2.14}$$

FIGURE 2. Commutative diagram

Lemma 2.15. *If $R_*[t,t^{-1}]$ is strongly graded, and if the two complexes $C \otimes_{R_*[t,t^{-1}]} R_*(\!(t)\!)$ and $C \otimes_{R_*[t,t^{-1}]} R_*(\!(t^{-1})\!)$ have trivial homology, then the maps $\mathbb{H}(\zeta^-)$ and $\mathbb{H}(\zeta^+)$ are quasi-isomorphisms.*

Proof. There is a chain of isomorphisms

$$\begin{aligned} D^{\pm} \underset{R_*[t^{\pm 1}]}{\otimes} R_*(\!(t^{\pm 1})\!) &\cong D^{\pm} \underset{R_*[t^{\pm 1}]}{\otimes} R_*[t,t^{-1}] \underset{R_*[t,t^{-1}]}{\otimes} R_*(\!(t^{\pm 1})\!) \\ &\cong D \underset{R_*[t,t^{-1}]}{\otimes} R_*(\!(t^{\pm 1})\!) \cong C \underset{R_*[t,t^{-1}]}{\otimes} R_*(\!(t^{\pm 1})\!) \ , \end{aligned}$$

the second one due to the fact that D is a sheaf in the strongly graded setting. By hypothesis the last complex is acyclic. This means that all vertical maps in the diagram (2.11) are quasi-isomorphisms, that is, $\zeta^{\pm}$ consists of quasi-isomorphisms. Hence application of $\mathbb{H}$ results in a quasi-isomorphism $\mathbb{H}(\zeta^{\pm})$ by [Hüt15, Lemma 4.2]. $\square$

Recall that, by construction, $\mathfrak{D}_n$ is a finite direct sum of diagrams of the form $\mathcal{O}(q,p)$, with $q+p \geq 0$. It follows from the calculation in Proposition 2.6 that $H^1(\mathfrak{D}) = 0$ (levelwise) so that the inclusion

$H^0(\mathfrak{D}) \longrightarrow \mathbb{H}(\mathfrak{D})$ is a quasi-isomorphism [Hüt15, Lemma 4.2]. With Proposition 1.6 this yields the following result:

Lemma 2.16. *The bounded chain complex $H^0(\mathfrak{D})$ is quasi-isomorphic to the complex $\mathbb{H}(\mathfrak{D})$. If $R_*[t,t^{-1}]$ is strongly graded, $H^0(\mathfrak{D})$ consists of finitely generated projective R_0-modules.* □

Proof of Theorem 1.3, "if" part. As before we start with a bounded chain complex C of finitely generated free $R_*[t,t^{-1}]$-modules, and construct a complex of sheaves $\mathfrak{D} = \big(D^- \longrightarrow D \longleftarrow D^+\big)$ according to Proposition 2.9, with $D \cong C$. We will also use the diagrams $\mathfrak{D}^\pm$ and $\mathfrak{D}'^\pm$ as defined at the beginning of this section.

Our hypothesis now is that $R_*[t,t^{-1}]$ is strongly graded. In this situation all the vertical maps in diagram (2.14) are quasi-isomorphisms, by Lemmas 2.13 and 2.15. So by applying $\mathbb{H}$ to the rows of the diagram we obtain a chain of maps

$$H^0(\mathfrak{D}) \xrightarrow{\simeq} \mathbb{H}(\mathfrak{D}) \xrightarrow{\simeq} \mathbb{H}\Big(\mathbb{H}(\mathfrak{D}'^-) \xrightarrow{\pi'^-} D \xleftarrow{\pi'^+} \mathbb{H}(\mathfrak{D}'^+)\Big) \ ; \tag{2.17}$$

the first one is a quasi-isomorphism by Lemma 2.16, the second because the functor $\mathbb{H}$ preserves quasi-isomorphisms [Hüt15, Lemma 4.2].

By explicitly spelling out the definitions, we see that the chain complex $\mathbb{H}\big(\mathbb{H}(\mathfrak{D}'^-) \xrightarrow{\pi'^-} D \xleftarrow{\pi'^+} \mathbb{H}(\mathfrak{D}'^+)\big)$ contains the complex

$$\mathbb{H}\Big(D^- \underset{R_*[t^{-1}]}{\otimes} R_*[t,t^{-1}] \longrightarrow D \longleftarrow D^+ \underset{R_*[t]}{\otimes} R_*[t,t^{-1}]\Big)$$

as a retract. But the diagram $\mathfrak{D}$ is a sheaf, making use of the strong grading again, so the maps $D^\pm \otimes_{R_*[t^{\pm 1}]} R_*[t,t^{-1}] \longrightarrow D$ are isomorphisms. It follows that the previous chain complex is isomorphic to $\mathbb{H}\big(D \xrightarrow{=} D \xleftarrow{=} D\big)$, and thus quasi-isomorphic to $D \cong C$.

Combined with (2.17), we thus see that in the derived category of R_0 the complex C is a retract of $H^0(\mathfrak{D})$. Both are bounded complexes of R_0-projective modules, the former by Corollary 1.7, the latter by Lemma 2.16. It follows from general theory of derived categories that there are chain maps $\alpha\colon C \longrightarrow H^0(\mathfrak{D})$ and $\beta\colon H^0(\mathfrak{D}) \longrightarrow C$ with $\beta\alpha \simeq \mathrm{id}$. As $H^0(\mathfrak{D})$ consists of finitely generated projective R_0-modules (Lemma 2.16 again), this proves that C is R_0-finitely dominated as desired. □

3. Finite domination implies trivial Novikov homology

From now on, and for the remainder of the paper, we suppose that the $\mathbb{Z}$-graded ring $R_*[t,t^{-1}]$ admits a partition of unity $1 = \sum_j x_j^{(-1)} y_j^{(1)}$ of type $(-1,1)$, which we choose once and for all.

Canonical resolution and algebraic tori. For a given $R_*[t,t^{-1}]$-module C, or a given chain complex C of such modules, we use the chosen partition of unity to define an $R_*[t,t^{-1}]$-linear map

$$\mu\colon C \underset{R_0}{\otimes} R_*[t,t^{-1}] \longrightarrow C \underset{R_0}{\otimes} R_*[t,t^{-1}] \ ,$$
$$c\otimes r \mapsto c\otimes r - \sum_j c x_j^{(-1)} \otimes y_j^{(1)} r \ . \tag{3.1}$$

Note that for any $s\in R_0$ and any partition of unity $1=\sum_\ell u_\ell v_\ell$ of type $(-1,1)$ there are equalities

$$\sum_j c x_j^{(-1)} \otimes y_j^{(1)} sr = \sum_{j,\ell} c x_j^{(-1)} \otimes y_j^{(1)} s u_\ell v_\ell r$$
$$= \sum_{\ell,j} c x_j^{(-1)} y_j^{(1)} s u_\ell \otimes v_\ell r = \sum_\ell c s u_\ell \otimes v_\ell r \ .$$

Specialising to $u_\ell = x_\ell^{(-1)}$ and $v_\ell = y_\ell^{(1)}$ yields that the map μ is R_0-balanced, and hence well-defined. On the other hand, specialising to $s=1$ shows that, contrary to appearance, the map μ does actually not depend on the choice of partition of unity. — It might be worth pointing out that the map μ cannot be defined in the absence of additional data; the strongly graded structure of the ring enters the picture in a rather subtle form here.

Proposition 3.2 (Canonical resolution)**.** *For any $R_*[t,t^{-1}]$-module M there is a sequence of $R_*[t,t^{-1}]$-modules*

$$0 \longrightarrow M \underset{R_0}{\otimes} R_*[t,t^{-1}] \xrightarrow{\mu} M \underset{R_0}{\otimes} R_*[t,t^{-1}] \xrightarrow{\pi} M \longrightarrow 0 \ , \tag{3.3}$$

where $\pi(m\otimes r) = mr$ and μ is as in (3.1)*. The sequence is natural in M. If $R_*[t,t^{-1}]$ is strongly graded then the sequence is split exact as a sequence of right R_0-modules, and hence is exact (but possibly non-split) as a sequence of right $R_*[t,t^{-1}]$-modules.*

Proof. We first note that $\pi\mu=0$ as

$$\pi\mu(m\otimes r) = \pi\big(m\otimes r - \sum_j m x_j^{(-1)} \otimes y_j^{(1)} r\big)$$
$$= mr - \sum_j m x_j^{(-1)} y_j^{(1)} r = mr - m1r = 0 \ .$$

Let us now suppose that $R_*[t,t^{-1}]$ is strongly graded. In addition to our fixed partition of unity $1=\sum_{\ell_{-1}} x_{\ell_{-1}}^{(-1)} y_{\ell_{-1}}^{(1)}$ we choose for all $k\in\mathbb{Z}$, $k\neq 1$, a partition of unity $1=\sum_{\ell_{-k}} x_{\ell_{-k}}^{(-k)} y_{\ell_{-k}}^{(k)}$ of type $(-k,k)$; as before this is understood to be a finite sum with $x_{\ell_{-k}}^{(-k)} \in R_{-k}$ and $y_{\ell_{-k}}^{(k)} \in R_k$. Such partitions of unity exist by Proposition 1.5.

We denote by ι the right R_0-linear map

$$\iota\colon M \longrightarrow M \underset{R_0}{\otimes} R_*[t,t^{-1}] \ , \quad m \mapsto m \otimes 1 \ ;$$

clearly $\pi\iota = \mathrm{id}_M$. Next, we define an R_0-linear map

$$\tau\colon M \underset{R_0}{\otimes} R_*[t,t^{-1}] \longrightarrow M \underset{R_0}{\otimes} R_*[t,t^{-1}] \ ;$$

as $M \otimes_{R_0} R_*[t,t^{-1}] \cong \bigoplus_{n\in\mathbb{Z}} M \otimes_{R_0} R_n$ as a right R_0-module it will be enough to specify the restrictions $\tau_n = \tau|_{M \otimes R_n}$. For $m \in M$ and $r_n \in R_n$ these are given by

$$\tau_n(m \otimes r_n) = \begin{cases} -\sum_{k=1}^{n} \ \ \sum_{\ell_k} \left(mx_{\ell_k}^{(k)} \otimes y_{\ell_k}^{(-k)} r_n\right) & \text{if } n > 0 \ , \\ \ 0 & \text{if } n = 0 \ , \\ \ \sum_{k=0}^{-n-1} \sum_{j_{-k}} \left(mx_{j_{-k}}^{(-k)} \otimes y_{j_{-k}}^{(k)} r_n\right) & \text{if } n < 0 \ . \end{cases}$$

The map τ satisfies $\tau\mu = \mathrm{id}$; we will verify $\tau\mu(m \otimes r_n) = m \otimes r_n$ for $n \geq 1$, the case $n \leq 0$ being similar. So let $m \in M$ and $r_n \in R_n$, for some $n \geq 1$. Then

$$\begin{aligned} \tau\mu(m \otimes r_n) &= \tau\Big(m \otimes r_n - \sum_j mx_j^{(-1)} \otimes y_j^{(1)} r_n\Big) \\ &= \tau(m \otimes r_n) - \tau\Big(\sum_j mx_j^{(-1)} \otimes y_j^{(1)} r_n\Big) \\ &= \tau_n(m \otimes r_n) - \tau_{n+1}\Big(\sum_j mx_j^{(-1)} \otimes y_j^{(1)} r_n\Big) \ . \end{aligned}$$

Now by definition

$$\tau_n(m \otimes r_n) = -\sum_{k=1}^{n} \sum_{\ell_k} \left(mx_{\ell_k}^{(k)} \otimes y_{\ell_k}^{(-k)} r_n\right)$$

while

$$\tau_{n+1}\Big(\sum_j mx_j^{(-1)} \otimes y_j^{(1)} r_n\Big) = -\sum_{k=1}^{n+1} \Big(\sum_{\ell_k} \sum_j mx_j^{(-1)} x_{\ell_k}^{(k)} \otimes y_{\ell_k}^{(-k)} y_j^{(1)} r_n\Big) .$$

The last term in parentheses, for any fixed k, can be simplified:

$$\begin{aligned}
&\sum_{\ell_k}\sum_j m x_j^{(-1)} x_{\ell_k}^{(k)} \otimes y_{\ell_k}^{(-k)} y_j^{(1)} r_n \\
&\qquad \underset{(\ddagger)}{=} \sum_{\ell_k}\sum_j m x_j^{(-1)} x_{\ell_k}^{(k)} \otimes y_{\ell_k}^{(-k)} y_j^{(1)} \cdot \Big(\sum_{\ell_{k-1}} x_{\ell_{k-1}}^{(k-1)} y_{\ell_{k-1}}^{(-k+1)}\Big) \cdot r_n \\
&\qquad = \sum_{\ell_{k-1}}\sum_{\ell_k}\sum_j m x_j^{(-1)} x_{\ell_k}^{(k)} \otimes y_{\ell_k}^{(-k)} y_j^{(1)} x_{\ell_{k-1}}^{(k-1)} y_{\ell_{k-1}}^{(-k+1)} r_n \\
&\qquad \underset{(\dagger)}{=} \sum_{\ell_{k-1}}\sum_{\ell_k}\sum_j m x_j^{(-1)} x_{\ell_k}^{(k)} y_{\ell_k}^{(-k)} y_j^{(1)} x_{\ell_{k-1}}^{(k-1)} \otimes y_{\ell_{k-1}}^{(-k+1)} r_n \\
&\qquad \underset{(\ddagger)}{=} \sum_{\ell_{k-1}} m x_{\ell_{k-1}}^{(k-1)} \otimes y_{\ell_{k-1}}^{(-k+1)} r_n
\end{aligned}$$

where at (†) we have used that $y_{\ell_k}^{(-k)} y_j^{(1)} x_{\ell_{k-1}}^{(k-1)} \in R_0$, and at (‡) we have used that $\sum_{\ell_k} x_{\ell_k}^{(k)} y_{\ell_k}^{(-k)} = \sum_{\ell_{k-1}} x_{\ell_{k-1}}^{(k-1)} y_{\ell_{k-1}}^{(-k+1)} = \sum_j x_j^{(-1)} y_j^{(1)} = 1$. It follows together with the previous expressions that $\tau\mu(m \otimes r_n)$ equals

$$\begin{aligned}
-\sum_{k=1}^{n}\sum_{\ell_k}\big(m x_{\ell_k}^{(k)} \otimes y_{\ell_k}^{(-k)} r_n\big) + \sum_{k=1}^{n+1}\sum_{\ell_{k-1}}\big(m x_{\ell_{k-1}}^{(k-1)} \otimes y_{\ell_{k-1}}^{(-k+1)} r_n\big) \\
= \sum_{\ell_0} m x_{\ell_0}^{(0)} \otimes y_{\ell_0}^{(0)} r_n = m \otimes r_n \ .
\end{aligned}$$

To show that our sequence (3.3) is split exact when considered as a sequence of R_0-modules it remains only to prove that $\mu\tau + \iota\pi = \mathrm{id}_{M \otimes R_*[t,t^{-1}]}$. The calculation is similar to the one just finished, making use of existence of partitions of unity in exactly the same manner. We omit the details. □

Corollary 3.4. *For any chain complex C of $R_*[t,t^{-1}]$-modules there is a quasi-isomorphism* $\mathrm{cone}(\mu) \xrightarrow{\sim} C$.

Proof. By the previous Proposition there is a short exact sequence of chain complexes

$$0 \longrightarrow C \underset{R_0}{\otimes} R_*[t,t^{-1}] \xrightarrow{\mu} C \underset{R_0}{\otimes} R_*[t,t^{-1}] \xrightarrow{\pi} C \longrightarrow 0 \ .$$

Thus the canonical map $\mathrm{cone}(\mu) \longrightarrow C$ is a quasi-isomorphism. □

Definition 3.5. The mapping cone of μ in the previous Corollary is called the *algebraic torus of* C and denoted $\mathfrak{T}(C)$.

The Mather trick for the algebraic torus. Let C be an $R_*[t,t^{-1}]$-module chain complex, and let D be an R_0-module chain complex. Let $\alpha\colon C \longrightarrow D$ and $\beta\colon D \longrightarrow C$ be R_0-linear chain maps and H a chain homotopy such that $H\colon \beta\alpha \simeq \mathrm{id}_C$; that is, $dH + Hd = \mathrm{id}_C - \beta\alpha$ where d is the differential of C. Define

$$\nu\colon D \underset{R_0}{\otimes} R_*[t,t^{-1}] \longrightarrow D \underset{R_0}{\otimes} R_*[t,t^{-1}] \tag{3.6}$$

by the formula $\nu = (\alpha \otimes \mathrm{id}) \circ \mu \circ (\beta \otimes \mathrm{id})$. Then the diagram

$$\begin{tikzcd}
C \underset{R_0}{\otimes} R_*[t,t^{-1}] \arrow[r, "\mu"] \arrow[d, "\alpha\otimes\mathrm{id}"'] & C \underset{R_0}{\otimes} R_*[t,t^{-1}] \arrow[d, "\alpha\otimes\mathrm{id}"'] \\
D \underset{R_0}{\otimes} R_*[t,t^{-1}] \arrow[r, "\nu"] & D \underset{R_0}{\otimes} R_*[t,t^{-1}]
\end{tikzcd}$$

is homotopy commutative with homotopy

$$J = (\alpha \otimes \mathrm{id}) \circ \mu \circ (H \otimes \mathrm{id})\colon \nu \circ (\alpha \otimes \mathrm{id}) \simeq (\alpha \otimes \mathrm{id}) \circ \mu \ .$$

This homotopy induces a preferred map of $R_*[t,t^{-1}]$-module chain complexes

$$\alpha_* = \begin{pmatrix} \alpha \otimes \mathrm{id} & 0 \\ J & \alpha \otimes \mathrm{id} \end{pmatrix} : \mathfrak{T}(C) = \mathrm{cone}(\mu) \longrightarrow \mathrm{cone}(\nu) \ .$$

If α is a quasi-isomorphism and $R_*[t,t^{-1}]$ is strongly graded then $\alpha \otimes \mathrm{id}$ is a quasi-isomorphism as well; indeed, the functor $\cdot \otimes_{R_0} R_*[t,t^{-1}]$ is exact in the strongly graded case by Proposition 1.6. We obtain the following result analogous to the Mather trick in the topological context [Ran95, "Whitehead Lemma", §2]:

Lemma 3.7 (Mather trick)**.** *Let C be an $R_*[t,t^{-1}]$-module chain complex, and let D an R_0-module chain complex. Let $\alpha\colon C \longrightarrow D$ and $\beta\colon D \longrightarrow C$ be R_0-linear chain maps such that $\beta\alpha \simeq \mathrm{id}_C$ via a specified homotopy. Then there is a preferred map $\alpha_*\colon \mathfrak{T}(C) \longrightarrow \mathrm{cone}(\nu)$. If in addition α is a quasi-isomorphism and $R_*[t,t^{-1}]$ is strongly graded, $\alpha_*\colon \mathfrak{T}(C) \longrightarrow \mathrm{cone}(\nu)$ is a quasi-isomorphism.* □

Corollary 3.8. *Suppose $R_*[t,t^{-1}]$ is strongly graded. Given a bounded below chain complex C of projective $R_*[t,t^{-1}]$-modules, a bounded below chain complex D of projective R_0-modules, and an R_0-homotopy equivalence $\alpha\colon C \xrightarrow{\simeq} D$, there is a homotopy equivalence*

$$C \underset{R_*[t,t^{-1}]}{\otimes} R_*(\!(t^{\pm 1})\!) \simeq \mathrm{cone}(\nu) \underset{R_*[t,t^{-1}]}{\otimes} R_*(\!(t^{\pm 1})\!) \ .$$

Proof. From the previous Lemma and Corollary 3.4 we know that there are quasi-isomorphisms $C \longleftarrow \mathfrak{T}(C) \xrightarrow{\alpha_*} \mathrm{cone}(\nu)$. As both C and $\mathrm{cone}(\nu)$ are bounded below and consist of projective $R_*[t,t^{-1}]$-modules, these two complexes are actually homotopy equivalent. As taking tensor products preserves homotopy equivalences we have proven the claim. $\square$

Bicomplexes and truncated powers. We extend our portfolio of homological techniques further by re-writing the complex $\mathrm{cone}(\nu)$ as the totalisation of a bicomplex, and by introducing twisted truncated powers.

Let C be an $R_*[t,t^{-1}]$-module chain complex, and let D an R_0-module chain complex. Let $\alpha\colon C \longrightarrow D$ and $\beta\colon D \longrightarrow C$ be R_0-linear chain maps. Define $\nu = (\alpha\otimes\mathrm{id})\circ\mu\circ(\beta\otimes\mathrm{id})$ as in (3.6). Let $\zeta_{n,m}$ denote the R_0-linear map

$$D_m \underset{R_0}{\otimes} R_n \longrightarrow D_m \underset{R_0}{\otimes} R_{n+1}\ , \quad z\otimes r \mapsto \sum_j \alpha\big(\beta(z)x_j^{(-1)}\big)\otimes y_j^{(1)} r\ ,$$

and let $E_{\bullet,\bullet}$ denote the bicomplex of right R_0-modules given by

$$E_{n,m} = \big(D_{n+m-1} \underset{R_0}{\otimes} R_{-n}\big) \,\oplus\, \big(D_{n+m} \underset{R_0}{\otimes} R_{-n}\big) \tag{3.9}$$

with differentials

$$d_H = \begin{pmatrix} 0 & 0 \\ \zeta_{-n,n+m} & 0 \end{pmatrix} \colon E_{n,m} \longrightarrow E_{n-1,m}$$

and

$$d_V = \begin{pmatrix} -d\otimes\mathrm{id} & 0 \\ \alpha\beta\otimes\mathrm{id} & d\otimes\mathrm{id} \end{pmatrix} \colon E_{n,m} \longrightarrow E_{n,m-1} \tag{3.10}$$

where d is the differential of the chain complex D.

The *totalisation* $\mathrm{Tot}(E_{\bullet,\bullet})$ is the chain complex with $\mathrm{Tot}(E_{\bullet,\bullet})_\ell = \bigoplus_{n+m=\ell} E_{n,m}$ and differential $d_H + d_V$. More explicitly, we have an identification

$$\begin{aligned}\mathrm{Tot}(E_{\bullet,\bullet})_\ell = \bigoplus_{n\in\mathbb{Z}} E_{-n,\ell+n} &= \bigoplus_{n\in\mathbb{Z}} \Big(\big(D_{\ell-1} \underset{R_0}{\otimes} R_n\big)\Big) \,\oplus\, \big(D_\ell \underset{R_0}{\otimes} R_n\big) \\ &= \big(D_{\ell-1} \underset{R_0}{\otimes} R_*[t,t^{-1}]\big) \,\oplus\, \big(D_\ell \underset{R_0}{\otimes} R_*[t,t^{-1}]\big)\ ,\end{aligned}$$

under which the differential $d = d_H + d_V$ coincides with the differential of $\mathrm{cone}(\nu)$. A straightforward calculation then shows that d_H and d_V are anti-commuting differentials. We summarise the construction:

Lemma 3.11. *The data listed above yields a bicomplex in the sense that* $d_H \circ d_H = 0$, $d_H \circ d_V = -d_V \circ d_H$ *and* $d_V \circ d_V = 0$. *Its totalisation* $\mathrm{Tot}(E_{\bullet,\bullet})$ *is isomorphic to* $\mathrm{cone}(\nu)$. □

We wish to analyse the tensor product $\mathrm{cone}(\nu) \otimes_{R_*[t,t^{-1}]} R_*((t))$ using the bicomplex above. For this, we need to digress a little and talk about truncated powers, or rather a "twisted" version thereof that takes the graded structure of the ring into account.

Definition 3.12. Given a right R_0-module M, we define the *twisted left truncated power of* M, denoted ${}^{\mathrm{lt}}\prod\limits_{\sim} M$, by

$$ {}^{\mathrm{lt}}\prod_{\sim} M = \bigoplus_{n<0} \big(M \underset{R_0}{\otimes} R_n\big) \oplus \prod_{n\geq 0} \big(M \underset{R_0}{\otimes} R_n\big) \ , $$

and the *twisted right truncated power of* M, denoted $\prod\limits_{\sim}^{\mathrm{rt}} M$, by

$$ \prod_{\sim}^{\mathrm{rt}} M = \prod_{n\leq 0} \big(M \underset{R_0}{\otimes} R_n\big) \oplus \bigoplus_{n>0} \big(M \underset{R_0}{\otimes} R_n\big) \ . $$

We note that ${}^{\mathrm{lt}}\prod\limits_{\sim} M$ has a right $R_*((t))$-module structure; if we write elements of ${}^{\mathrm{lt}}\prod\limits_{\sim} M$ as formal LAURENT series $\sum_{n\geq m} xt^n$ with $x_n \in M \otimes_{R_0} R_n$ and elements of $R_*((t))$ as formal LAURENT series $\sum_{n\geq p} r_n t^n$ with $r_n \in R_n$, it is given by the obvious multiplication of series formula using $x_k r_n \in M \otimes_{R_0} R_{k+n}$ via the assignment $(m \otimes s_k)\cdot r_n = m \otimes (s_k r_n)$. — Similarly, $\prod\limits_{\sim}^{\mathrm{rt}} M$ carries a natural right $R_*((t^{-1}))$-module structure.

Proposition 3.13. *For a finitely presented* R_0*-module* M*, there is an isomorphism of right* $R_*((t))$*-modules*

$$ \Phi_M \colon M \underset{R_0}{\otimes} R_*((t)) \longrightarrow {}^{\mathrm{lt}}\prod_{\sim} M \ , \quad m \otimes \sum_k r_k t^k \mapsto \sum_k (m \otimes r_k) t^k $$

and an isomorphism of right $R_*((t^{-1}))$*-modules*

$$ \Psi_M \colon M \underset{R_0}{\otimes} R_*((t^{-1})) \longrightarrow \prod_{\sim}^{\mathrm{rt}} M \ , \quad m \otimes \sum_k r_k t^k \mapsto \sum_k (m \otimes r_k) t^k \ . $$

Both isomorphisms are natural in M.

Proof. We show that Φ_M is bijective, the case of Ψ_M being similar. — Suppose first that $M = F$ is free on the basis $e_1, e_2, \cdots, e_q$. Then $F \otimes_{R_0} R_*((t))$ is a free $R_*((t))$-module with $e_1 \otimes 1, e_2 \otimes 1, \cdots, e_q \otimes 1$ as basis. Thus any $x \in F \otimes_{R_0} R_*((t))$ can uniquely be written in the form

$$x = \sum_{j=1}^{q} \Big(e_j \otimes \sum_k r_{jk} t^k \Big)$$

with $r_{jk} \in R_k$, and $r_{jk} = 0$ if k is sufficiently small. Suppose that $x \in \ker \Phi_F$ so that

$$0 = \Phi_F(x) = \sum_{j=1}^{q} \sum_k (e_j \otimes r_{jk}) t^k = \sum_k \sum_{j=1}^{q} (e_j \otimes r_{jk}) t^k$$

in the twisted left truncated power of F. This implies the equality $\sum_{j=1}^{q} e_j \otimes r_{jk} = 0 \in F \otimes_{R_0} R_k \subseteq F \otimes_{R_0} R_*[t, t^{-1}]$ for all k; as the last module is free on basis elements $e_j \otimes 1$ we conclude that $r_{jk} = 0$ for all k and j. Consequently $x = 0$ which proves that Φ_F is injective.

Now let $z = \sum_{k \geq n} z_k t^k \in \widetilde{\prod}^{\mathrm{lt}} F$ with $z_k \in F \otimes_{R_0} R_k$; using that F is free on basis elements e_j as before we see that we can write z_k in the form $z_k = \sum_j e_j \otimes z_{jk}$ with $z_{jk} \in R_k$. Then

$$x = \sum_j \Big(e_j \otimes \sum_k z_{jk} t^k \Big)$$

is an element of $F \otimes_{R_0} R_*((t))$ satisfying $\Phi_F(x) = z$. Thus Φ_F is seen to be surjective.

For the general case consider a presentation $G \longrightarrow F \longrightarrow M \longrightarrow 0$ of M by finitely generated free modules F and G; standard homological algebra, using that the functors $X \mapsto X \otimes_{R_0} R_*((t))$ and $X \mapsto \widetilde{\prod}^{\mathrm{lt}} X$ are right exact, shows that Φ_M is bijective, cf. [Hüt11, Lemma 2.1]. $\square$

The *right truncated totalisation* of $E_{\bullet,\bullet}$, denoted $\mathrm{Tot}^{\mathrm{rt}}(E_{\bullet,\bullet})$, is the chain complex with

$$\mathrm{Tot}^{\mathrm{rt}}(E_{\bullet,\bullet})_\ell = \prod_{n \leq 0} E_{n,\ell-n} \oplus \bigoplus_{n \geq 0} E_{n,\ell-n}$$

and differential $d_H + d_V$. Plugging in the definition of $E_{n,m}$ this can be re-written as

$$\begin{aligned}\mathrm{Tot}^{\mathrm{rt}}\,(E_{\bullet,\bullet})_\ell = \quad & \prod_{n\leq 0}\Big(\big(D_{\ell-1}\underset{R_0}{\otimes} R_{-n}\big)\,\oplus\,\big(D_\ell\underset{R_0}{\otimes} R_{-n}\big)\Big)\\ & \oplus\bigoplus_{n>0}\Big(\big(D_{\ell-1}\underset{R_0}{\otimes} R_{-n}\big)\,\oplus\,\big(D_\ell\underset{R_0}{\otimes} R_{-n}\big)\Big)\\ = {} & {}^{\mathrm{lt}}\prod_{\sim} D_{\ell-1}\,\oplus\,{}^{\mathrm{lt}}\prod_{\sim} D_\ell\ ;\end{aligned}$$

if the complex D consists of finitely presented R_0-modules we can thus use Proposition 3.13 to identify $\mathrm{Tot}^{\mathrm{rt}}\,(E_{\bullet,\bullet})_\ell$ with the module $\big(D_{\ell-1}\otimes_{R_0} R_*(\!(t)\!)\big)\oplus\big(D_\ell\otimes_{R_0} R_*(\!(t)\!)\big)$. When combined with the isomorphisms $\mathrm{cone}(\nu)\otimes_{R_*[t,t^{-1}]} R_*(\!(t)\!)\cong\mathrm{cone}(\nu\otimes\mathrm{id}_{R_*(\!(t)\!)})$ and

$$D_\ell\underset{R_0}{\otimes} R_*[t,t^{-1}]\underset{R_*[t,t^{-1}]}{\otimes} R_*(\!(t)\!)\cong D_\ell\underset{R_0}{\otimes} R_*(\!(t)\!)\ ,$$

a straightforward calculation with the differentials d_H and d_V yields:

Proposition 3.14. *If D consists of finitely presented R_0-modules, there is an isomorphism of $R_*(\!(t)\!)$-module chain complexes* $\mathrm{Tot}^{\mathrm{rt}}\,(E_{\bullet,\bullet})\cong\mathrm{cone}(\nu)\otimes_{R_*[t,t^{-1}]} R_*(\!(t)\!)$. □

From finite domination to trivial Novikov homology. We are finally in a position to finish the proof of our main result.

Proof of Theorem 1.3, "only if" part. Suppose that the ring $R_*[t,t^{-1}]$ is strongly graded. Let C be a bounded complex of finitely generated free $R_*[t,t^{-1}]$-modules; suppose that C is R_0-finitely dominated. Then there is a bounded complex D of finitely generated projective R_0-modules together with a homotopy equivalence $\alpha\colon C\longrightarrow D$ of R_0-module complexes. Let β be a homotopy inverse of α. According to Corollary 3.8 this data can be used to manufacture a homotopy equivalence $C\otimes_{R_*[t,t^{-1}]} R_*(\!(t)\!)\simeq\mathrm{cone}(\nu)\otimes_{R_*[t,t^{-1}]} R_*(\!(t)\!)$, where ν is a chain complex self-map of $D\otimes_{R_0} R_*[t,t^{-1}]$ as in (3.6). We can use Proposition 3.14 to identify $\mathrm{cone}(\nu)\otimes_{R_*[t,t^{-1}]} R_*(\!(t)\!)$ with $\mathrm{Tot}^{\mathrm{rt}}\,(E_{\bullet,\bullet})$, the right truncated totalisation of the double complex defined in (3.9), as D consists of finitely presented R_0-modules. The vertical differential of this complex, defined in (3.10), is the mapping cone of $\alpha\beta\otimes\mathrm{id}$. As $\alpha\beta\simeq\mathrm{id}$ this means that the columns of $E_{\bullet,\bullet}$ are acyclic, hence $\mathrm{Tot}^{\mathrm{rt}}\,(E_{\bullet,\bullet})$ is acyclic [Hüt11, Proposition 1.2]. This shows that $C\otimes_{R_*[t,t^{-1}]} R_*(\!(t)\!)$, being homotopy equivalent to an acyclic complex, has trivial homology.

To prove that $C \otimes_{R_*[t,t^{-1}]} R_*((t^{-1}))$ has trivial homology too we cannot simply swap the roles of "left" and "right" as we did not analyse whether the *rows* of $E_{\bullet,\bullet}$ are acyclic. Instead, we can quote what we proved so far, applied to the strongly $\mathbb{Z}$-graded ring $\bar{R}_*[t,t^{-1}]$ with nth homogeneous component R_{-n} (which as a ring, neglecting the grading, coincides with $R_*[t,t^{-1}]$). We then conclude that $C \otimes_{R_*[t,t^{-1}]} R_*((t^{-1})) = C \otimes_{\bar{R}_*[t,t^{-1}]} \bar{R}_*((t))$ has trivial homology as required. $\square$

References

[Bou98] Nicolas Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation.

[Dad80] Everett C. Dade. Group-graded rings and modules. *Math. Z.*, 174(3):241–262, 1980.

[HK07] J. A. Hillman and D. H. Kochloukova. Finiteness conditions and PD_r-group covers of PD_n-complexes. *Math. Z.*, 256(1):45–56, 2007.

[Hüt11] Thomas Hüttemann. Double complexes and vanishing of Novikov cohomology. *Serdica Math. J.*, 37(4):295–304 (2012), 2011.

[Hüt15] Thomas Hüttemann. Vector bundles on the projective line and finite domination of chain complexes. *Math. Proc. R. Ir. Acad.*, 115A(1):Art. 2, 12, 2015.

[Ran85] Andrew Ranicki. The algebraic theory of finiteness obstruction. *Math. Scand.*, 57(1):105–126, 1985.

[Ran95] Andrew Ranicki. Finite domination and Novikov rings. *Topology*, 34(3):619–632, 1995.

INDEX

BIBLIOGRAPHY

[Dad80] Everett C. Dade. Group-graded rings and modules. *Math. Z.*, 174(3):241–262, 1980.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[HQ15] Thomas Hüttemann and David Quinn. Finite domination and Novikov rings: Laurent polynomial rings in two variables. *J. Algebra Appl.*, 14(4):1550055, 44, 2015.

[HQ16] Thomas Hüttemann and David Quinn. Finite domination and Novikov rings. Laurent polynomial rings in several variables. *J. Pure Appl. Algebra*, 220(7):2648–2682, 2016.

[HS16] Thomas Hüttemann and Luke Steers. Finite domination and Novikov homology over strongly $\mathbb{Z}$-graded rings. To appear in Israel Journal of Mathematics, 2016.

[ML95] Saunders Mac Lane. *Homology*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1975 edition.

[Ran85] Andrew Ranicki. The algebraic theory of finiteness obstruction. *Math. Scand.*, 57(1):105–126, 1985.

[Ran95] Andrew Ranicki. Finite domination and Novikov rings. *Topology*, 34(3):619–632, 1995.

[Sch06] Dirk Schütz. Finite domination, Novikov homology and nonsingular closed 1-forms. *Math. Z.*, 252(3):623–654, 2006.