Complexity-driven construction of controlled invariant polytopic sets

Complexity–driven Construction of Controlled Invariant Polytopic Sets

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Abstract— **In this article, the problem of constructing controlled invariant polytopic sets of a specified complexity, for discrete–time linear systems subject to linear state and control constraints, is investigated. First, geometric conditions for enlarging a polytopic set such that the resulting polytopic set has an a priori chosen number of vertices are formulated. Next, results concerning the enlargement of controlled invariant sets such that the resulting set remains controlled invariant are presented. Finally, having established this necessary theoretical background, a method of constructing nondecreasing sequences of admissible controlled invariant sets with complexity specifications is established.**

I. Introduction

A problem of interest in both the analysis and the design of linear systems in the presence of state and control constraints is the computation of the admissible domain of attraction for the autonomous case and of the admissible stabilizability region for the case of systems with inputs. For example, in the model predictive control scheme, the computation of the controlled contractive region determines the domain of state space where the convergence to the target state can be guaranteed. Also, the problem is relevant in many control applications, where the goal is to determine whether a desired set of initial states belongs to the admissible domain of attraction. Since controlled contractive, and as a consequence controlled invariant, subsets of the state space provide an approximation of the admissible domain of attraction, construction of an invariant set is one of the typical approaches to solve the aforementioned problem.

In numerous real–life applications, state constraints are specified by linear inequalities that define bounded polyhedral sets, which, in the linear case, can be equivalently formulated via bounded and closed polyhedral sets that contain the origin in their interior. In this setting, several methods of constructing an admissible invariant polyhedral set for both continuous–time and discrete–time linear systems are available. These methods can be grouped in two categories, according to the approach used. The first category exploits the algebraic necessary and sufficient conditions of existence of invariant sets, stemming mainly from Lyapunov theory [1]–[9]. Among the early works that belong to this category are [1]–[3] for bounded polyhedral sets, [4], [5] for both bounded or unbounded sets and [6] for polytopic sets in vertex representation. These conditions can be used directly to verify invariance of a given set, while exploiting them in order to construct an invariant set requires the analysis of the spectral properties of the system [7], [10], resulting in symmetric polytopic sets. The second category consists in computing convergent sequences of sets, that are mainly related to the inverse reachability map and start from specially chosen sets [11]–[19]. These methods provide polytopic approximations of the maximal controlled invariant set with any desired accuracy, but of arbitrary complexity.

Thus, although a number of works in the control research field deals with the characterization and computation of controlled invariant sets, there is small progress towards characterizing and constructing polytopic controlled invariant sets of bounded complexity and non–trivial size, except the works [20]–[24] that utilize heuristic methods combined with special types of polytopic sets. Motivated by this lack of systematic constructive methods and the need to compute controlled invariant sets of low complexity, the goal of this article is to establish the theoretical foundations for developing design methods of construction of admissible controlled contractive polytopic sets with specified complexity for discrete–time systems. These methods can then be used for solving different kinds of constrained control problems where the complexity of the controlled contractive sets is considered as an additional constraint that must be respected. The main idea behind the approach consists in the addition of vertices to the convex hull of polytopic sets, resulting in conditions that can be easily verified by solving a series of linear programs.

The paper is organized as follows. In Section II, some basic definitions and the problem statement are given. In Section III, the theoretical framework for enlarging polytopic sets with specified complexity while preserving controlled invariance is established. In Section IV, systematic methods for constructing admissible controlled invariant sets by computing sequences of nondecreasing polytopic sets are presented, along with an illustrative example that demonstrates the effectiveness of the approach. Last, conclusions are drawn in Section V.

II. Problem statement

Throughout the paper, capital letters denote real matrices and lower case letters denote column vectors or scalars. $\mathbb{R}^n$ denotes the real n-space, $\mathbb{R}_+$ denotes the set of nonnegative real numbers, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{N}_{[q_1,q_2]}$ denotes the set of integers belonging to the interval $[q_1, q_2]$ and $\mathbb{R}^{n\times m}$ denotes the set of real $n \times m$ matrices. The column and the row vectors of a

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matrix $G \in \mathbb{R}^{s \times n}$ are denoted by g^i and g_i^T respectively, i.e. $G = \left[\begin{array}{cccc} g^1 & g^2 & \cdots & g^n \end{array} \right] = \left[\begin{array}{cccc} g_1 & g_2 & \cdots & g_s \end{array} \right]^T$. Given two real matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $A, B \in \mathbb{R}^{n \times m}$, the inequality $A \leq B$ $(A < B)$ with $A, B \in \mathbb{R}^{n \times m}$ is equivalent to $a_{ij} \leq b_{ij}$ $(a_{ij} < b_{ij})$, for all $i \in \mathbb{N}_{[1,n]}$, $j \in \mathbb{N}_{[1,m]}$. Similar notation holds for vectors. The p-dimensional vector with all its elements equal to one is denoted by e_p and the $n \times m$ real matrix with all its elements equal to zero is denoted by $0_{n \times m}$. Finally, given a subset $S \subset \mathbb{R}^n$ and a real number r, rS denotes the set $rS := \{y \in R^n : (\exists x \in S : y = rx)\}$.

The half-space representation of a convex polyhedral set having the origin as an interior point is defined by a vector inequality $Gx \leq e_p$, $G \in \mathbb{R}^{p \times n}$ and is denoted by $\mathcal{P}(G)$, i.e.

$$\mathcal{P}(G) := \{x \in \mathbb{R}^n : Gx \leq e_p\}.$$

If the set $\mathcal{P}(G)$ is bounded then it is a polytope and can be equivalently defined as the convex hull of a set of vectors $v^1, v^2, .., v^q$, namely

$$\mathcal{Q}(\mathcal{V}) := \mathrm{conv}(v^1, v^2, .., v^q),$$

where $\mathcal{V} = \{v^1, v^2, .., v^q\}$. In such a description, some of vectors $v^1, v^2, .., v^q$ may be redundant. The minimal set of vectors v^i that defines a polytope constitutes the set of its vertices. In this paper, the notation $\mathcal{Q}(\mathcal{V})$ is used for describing polytopes having the origin as an interior point.

The *complexity* of a polytopic set can be characterized by the number of its vertices, the number of its faces, the structure of the induced face lattice etc [25]. In this paper, the complexity of a polytopic set is defined as the number of its vertices.

We consider discrete-time linear systems described by difference equations of the form

$$x(t+1) = Ax(t) + Bu(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are the system and input matrices respectively and $t \in \mathbb{N}_+$ is the time variable. Throughout the paper, it will be assumed that the pair (A, B) is stabilizable. Autonomous systems

$$x(t+1) = Ax(t) \tag{2}$$

will also be considered as a special case of (1).

The state vector is constrained to belong to a bounded subset of the state space defined by a vector inequality of the form

$$G_x x \leq e_{p_x}, \tag{3}$$

where $G_x \in \mathbb{R}^{p_x \times n}$. This means that the trajectories $x(t; x_0)$ of the system are confined in the polyhedral set $\mathcal{P}(G_x)$.

Constraints are also imposed on the control input which has to satisfy linear inequalities of the form

$$G_u u \leq e_{p_u}, \tag{4}$$

where $G_u \in \mathbb{R}^{p_u \times m}$.

Definition 1: Given system (1), a set $\mathcal{S} \subset \mathbb{R}^n$ containing the origin as an interior point is said to be a *controlled* $\varepsilon-$*contractive set* with contraction factor ε if and only if $0 \leq \varepsilon < 1$ and there exists a state-feedback control $u = f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^m$ such that for any initial state $x_0 \in \mathcal{S}$ the corresponding trajectory $x(t; x_0)$ of the resulting closed-loop system satisfies the relation $x(t; x_0) \in \varepsilon\mathcal{S}$ for all $x_0 \in \mathcal{S}$, $t_0 \in T$ and $t \geq t_0$.

Definition 2: Given system (1) and constraints (3) and (4), a set $\mathcal{S} \subset \mathbb{R}^n$ containing the origin as an interior point is said to be an *admissible controlled* $\varepsilon-$*contractive set* with contraction factor ε if and only if $0 \leq \varepsilon < 1$ and there exists a state-feedback control $u = f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^m$ such that for any initial state $x_0 \in \mathcal{S}$ the corresponding trajectory $x(t; x_0)$ of the resulting closed-loop system and the control input $f(x(t; x_0))$ respect the constraints (3) and (4) respectively for all $t \geq t_0$ and $x(t; x_0) \in \varepsilon\mathcal{S}$ for all $x_0 \in \mathcal{S}$, $t_0 \in T$ and $t \geq t_0$.

If there exists a ε satisfying the conditions of the above definition then the set $\mathcal{S}$ is said to be an *admissible controlled contractive set*. Finally, if the relation $x(t; x_0) \in \varepsilon\mathcal{S}$ in Definition 2 is satisfied with $\varepsilon = 1$ then the set $\mathcal{S}$ is said to be an *admissible controlled invariant set.*

Definition 3: Given the autonomous system (2) and the state constraints (3), a set $\mathcal{S} \subset \mathbb{R}^n$ containing the origin as an interior point is said to be an *admissible* $\varepsilon-$*contractive set* with contraction factor ε if and only if $0 \leq \varepsilon < 1$ and for any initial state $x_0 \in \mathcal{S}$ the corresponding trajectory $x(t; x_0)$ of the resulting closed-loop system respects the state constraints (3) and satisfies the relation $x(t; x_0) \in \varepsilon\mathcal{S}$ for all $x_0 \in \mathcal{S}$, $t_0 \in T$ and $t \geq t_0$.

Remark 1: Positively invariant and controlled invariant sets are special cases of contractive and controlled contractive sets by setting $\varepsilon = 1$. Thus, the exposition of the results will be made for the general case when $0 \leq \varepsilon \leq 1$.

If the pair (A, B) of an unconstrained system (1) is stabilizable, then it is possible to determine a controlled contractive polytope $\mathcal{Q}(\mathcal{V}^0) = \mathrm{conv}(v^{01}, v^{02}, ..., v^{0q_0})$ [7], [10]. Then, by proper scaling, we can determine a sufficiently "small" admissible controlled contractive polytope $\mathcal{Q}(r\mathcal{V}^0) = \mathrm{conv}(rv^{01}, rv^{02}, ..., rv^{0q_0})$. Moreover, if the pair (A, B) is controllable it is always possible to construct an admissible controlled contractive polytope $\mathcal{Q}(\mathcal{V}^0)$ consisting of $2n$ parallel faces. This, for example, can be done by first determining a stabilizing control $u(t) = Fx(t)$ that assigns the eigenvalues of the resulting closed-loop system matrix $A + BF$ in the unit rhombus of the complex plane and then by computing a contractive polytope $\mathcal{Q}(\mathcal{V})$ of complexity 2^n [4]. Thus, by appropriate scaling, an admissible contractive polytope $\mathcal{Q}(\mathcal{V}^0)$ can be obtained.

However, as it is already underlined, the aforementioned method produces symmetric polytopic sets that might be not practical for constrained control problems, while computing them when the system is not controllable may become involved, especially for high dimensional systems. On the other hand, while the reachability–based methods [11]–[19] converge to the maximal admissible controlled contractive set, the sets produced are usually of high complexity.

Thus, the problem investigated in this article has a different setting: Given the system (1), the state and input constraint sets (3) and (4), an admissible controlled ε–contractive set $\mathcal{Q}(\mathcal{V})$ of complexity q, find a systematic method of computing admissible controlled ε–contractive supersets $\mathcal{Q}(\mathcal{V}^*)$ of a specified complexity q^*.

III. Theoretical foundations

A simple method of enlarging an admissible controlled contractive polytopic set $\mathcal{Q}(\mathcal{V}) = \mathrm{conv}(v^1, ..., v^q)$ is to add a new component $v^* \notin \mathcal{Q}(\mathcal{V})$ in its vertex description to obtain a new admissible controlled contractive polytopic set $\mathcal{Q}(\mathcal{V}^*) = \mathrm{conv}(v^1, ..., v^q, v^*)$. Then, $\mathcal{Q}(\mathcal{V}) \subset \mathcal{Q}(\mathcal{V}^*)$. In this subsection, we establish the necessary and sufficient conditions for such an enlargement approach to produce an admissible controlled contractive polytopic set of specified complexity.

A. Enlargement of polytopes with specified complexity

Addition of a new vertex v^*, situated outside a polytope $\mathcal{Q}(\mathcal{V}) = \mathrm{conv}(v^1, v^2, ..., v^q)$ will generate an enlarged polytope $\mathcal{Q}(\mathcal{V}^*) = \mathrm{conv}(v^1, ..., v^q, v^*)$. The set $\mathcal{Q}(\mathcal{V}^*)$ will not necessarily be of higher complexity because some of the vertices $v^1, ..., v^q$ might be redundant. Consequently, the set $\mathcal{Q}(\mathcal{V}^*)$ may have equal or even lower complexity, depending on where the new vertex v^* is located in the state space. In this subsection, we establish necessary and sufficient conditions for this enlargement procedure to produce polytopic sets with specified complexity.

Let $S \subset \mathbb{R}^n$ be a polytopic set with q vertices and p faces and with vertex and half-plane representations

$$\mathcal{Q}(\mathcal{V}) = \mathrm{conv}(v^1, ..., v^q) \tag{5}$$

and

$$\mathcal{P}(G) = \{x \in \mathbb{R}^n : Gx \leq e_p\} \tag{6}$$

respectively.

With each vertex v^k, $k \in \mathbb{N}_{[1,q]}$ of the polytope $\mathcal{S} = \mathcal{Q}(\mathcal{V})$ we associate the set of indices $N_{\mathcal{S}}(v^k) \subset \mathbb{N}_{[1,q]}$, defined by the relation

$$N_{\mathcal{S}}(v^k) := \{j \in \mathbb{N}_{[1,p]} : g_j^T v^k = 1\}. \tag{7}$$

The set $N_{\mathcal{S}}(v^k)$ represents the set of indices j which correspond to the faces $g_j^T x = 1$ of the polytope $\mathcal{S} = \mathcal{Q}(\mathcal{V})$ that pass through the vertex v^k. Moreover, with each vertex v^k, $k \in \mathbb{N}_{[1,q]}$ of the polytope $S = \mathcal{Q}(\mathcal{V})$ we associate the sets $\mathcal{A}_k$, , $k \in \mathbb{N}_{[1,q]}$, defined by the relation

$$\mathcal{A}_k := \{x \in \mathbb{R}^n : g_j^T x > 1, \quad j \in N_{\mathcal{S}}(v^k)\}. \tag{8}$$

The sets $\mathcal{A}_k$ are polyhedral cones which point outside the set $\mathcal{S}$, formed by the faces of the polytope $\mathcal{S} = \mathcal{Q}(\mathcal{V})$ defined by the equations $g_k^T x = 1$.

In the following theorem, we establish conditions for the proposed enlargement approach not to produce a polytope of higher complexity.

Theorem 1: Given a polytope $\mathcal{S} \subset \mathbb{R}^n$ with vertex and half-space representations (5) and (6) respectively, the polytope

$$\mathcal{Q}(\mathcal{V}^*) = \mathrm{conv}(v^1, ..., v^q, v^*) \tag{9}$$

satisfies the set relation

$$\mathcal{Q}(\mathcal{V}) \subset \mathcal{Q}(\mathcal{V}^*) \tag{10}$$

and is of equal or lower complexity than $\mathcal{Q}(\mathcal{V})$ if and only if

$$v^* \in \mathcal{A}_1 \cup \mathcal{A}_2 ... \cup \mathcal{A}_q. \tag{11}$$

Proof: a) Sufficiency: If $v^* \in \mathcal{A}_1 \cup \mathcal{A}_2 ... \cup \mathcal{A}_q$ then there exists an index $k \in \mathbb{N}_{[1,q]}$ such that $v^* \in \mathcal{A}_k$. By definition (8), this yields

$$g_j^T v^* > 1, \quad j \in N_{\mathcal{S}}(v^k), \tag{12}$$

which implies that $v^* \notin \mathcal{Q}(\mathcal{V})$. Therefore,

$$\mathcal{Q}(\mathcal{V}) \subset \mathcal{Q}(\mathcal{V}^*). \tag{13}$$

Furthermore, (12) yields [26, Theorem 1]

$$\mathrm{conv}(v^1, v^2, ..., v^q) \subset \mathrm{conv}(v^1, v^2, ..., v^{k-1}, v^{k+1}, ...v^q, v^*)$$

which in turn implies that

$$\begin{aligned}&\mathrm{conv}(v^1, v^2, ..., v^{k-1}, v^{k+1}, ...v^q, v^*)\\ &= \mathrm{conv}(v^1, v^2, ..., v^q, v^*).\end{aligned}$$

Therefore, the polytope $\mathcal{Q}(\mathcal{V}^*)$ is of equal or lower complexity than $\mathcal{Q}(\mathcal{V})$.

b) Necessity: If $\mathcal{Q}(\mathcal{V}) \subset \mathcal{Q}(\mathcal{V}^*)$, that is

$$\mathrm{conv}(v^1, v^2, ..., v^q) \subset \mathrm{conv}(v^1, v^2, ..., v^q, v^*), \tag{14}$$

then $v^* \notin \mathcal{Q}(\mathcal{V})$. If, in addition, the polytope $\mathcal{Q}(\mathcal{V}^*)$ is of equal or of lower complexity, then there exists at least one index $k \in \mathbb{N}_{[1,q]}$ such that the corresponding vertex v^k is redundant for the description of set $\mathcal{Q}(\mathcal{V}^*)$. Therefore, relation (14) can be written as

$$\begin{aligned}&\mathrm{conv}(v^1, v^2, ..., v^{k-1}, v^k, v^{k+1}, ...v^q)\\ &\subset \mathrm{conv}(v^1, v^2, ..., v^{k-1}, v^{k+1}, ...v^q, , v^*).\end{aligned}$$

However, this last relation implies [26, Theorem 1] that

$$g_j^T v^* > 1, \quad j \in N_{\mathcal{S}}(v^k),$$

or, equivalently, $v^* \in \mathcal{A}_{k}.$. Consequently, $v^* \in \mathcal{A}_1 \cup \mathcal{A}_2 ... \cup \mathcal{A}_q$. ■

Using this result we can establish conditions for the enlargement of a polytope by adding a new vertex to produce a new polytope with specified lower complexity.

Theorem 2: Given a polytope $\mathcal{S} \subset \mathbb{R}^n$ of complexity q with vertex and half-space representations (5) and (6) respectively, the polytope

$$\mathcal{Q}(\mathcal{V}^*) = \mathrm{conv}(v^1, ..., v^q, v^*)$$

satisfies the set relation

$$\mathcal{Q}(\mathcal{V}) \subset \mathcal{Q}(\mathcal{V}^*) \tag{15}$$

and is of complexity lower or equal to $q^* = q - l$, $l \in \mathbb{N}_{[1,q-N-1]}$ if and only if there exist at least $l + 1$ indices $k_1, k_2, ..., k_{l+1} \in \mathbb{N}_{[1,q]}$ such that

$$\mathcal{A}_{k_1} \cap \mathcal{A}_{k_2} \cap ... \cap \mathcal{A}_{k_{l+1}} \neq \varnothing \tag{16}$$

and

$$v^* \in \mathcal{A}_{k_1} \cap \mathcal{A}_{k_2} \cap ... \cap \mathcal{A}_{k_{l+1}} \qquad (17)$$

Proof: a) Sufficieny: From (16), (17) it follows that

$$v^* \in \mathcal{A}_{k_i} \text{ for } i = 1, 2, ..., l+1. \qquad (18)$$

Thus, by Theorem 1, it holds that

$$\begin{aligned} &\text{conv}(v^1, v^2, ..., v^q) \subset \\ &\text{conv}(v^1, v^2, ..., v^{k_i - 1}, v^{k_i + 1}, ...v^q, v^*) = \mathcal{Q}(\mathcal{V}^*), \end{aligned}$$

for $i = 1, 2, ..., l+1$. Consequently, the vectors v^{k_i} $i = 1, 2, ..., l+1$ are redundant in the description of $\mathcal{Q}(\mathcal{V}^*)$, thus the polytope $\mathcal{Q}(\mathcal{V}^*)$ is of complexity q^* lower or equal to $q^* = q - l$.

b) Necessity: If $\mathcal{Q}(\mathcal{V}) \subset \mathcal{Q}(\mathcal{V}^*)$, or equivalently

$$\text{conv}(v^1, v^2, ..., v^q) \subset \text{conv}(v^1, v^2, ..., v^q, v^*), \qquad (19)$$

then $v^* \notin \mathcal{Q}(\mathcal{V})$. If, in addition, the polytope is of complexity q^* lower or equal to $q - l$, then there exist at least l indices $k_i \in \mathbb{N}_{[1,q]}$ $i = 1, 2, ..., l+1$, such that the corresponding vertices v^{k_i} are redundant for the description of set $\mathcal{Q}(\mathcal{V}^*)$. Therefore, from relation (19) it follows that

$$\begin{aligned} &\text{conv}(v^1, v^2, ..., v^{k_i - 1}, v^{k_i}, v^{k_i + 1}, ..., v^q) \\ &\subset \text{conv}(v^1, v^2, ..., v^{k_i - 1}, v^{k_i + 1}, ..., v^q, v^*), \end{aligned}$$

for $i = 1, 2, ..., l+1$. This relation implies [26, Theorem 1] that

$$g_j^T v^* > 1, \quad j \in N_{\mathcal{S}}(v^{k_i}), \quad \forall i = 1, 2, ..., l+1,$$

or, equivalently, $v^* \in \mathcal{A}_{k_1} \cap \mathcal{A}_{k_2} \cap ... \cap \mathcal{A}_{k_{l+1}}$. ■

A direct consequence of Theorems 1 and 2, which is of practical importance, follows.

Corollary 1: Given a polytope $\mathcal{S} \subset \mathbb{R}^n$ of complexity q with vertex and half-space representations (5) and (6) respectively, the polytope

$$\mathcal{Q}(\mathcal{V}^*) = \text{conv}(v^1, ..., v^q, v^*)$$

satisfies the set relation

$$\mathcal{Q}(\mathcal{V}) \subset \mathcal{Q}(\mathcal{V}^*) \qquad (20)$$

and is of equal complexity to that of $Q(\mathcal{V})$ if and only if there exists an index $k \in N_{[1,q]}$ such that

$$v^* \in \mathcal{A}_k \setminus (\mathcal{A}_1 \cup \mathcal{A}_2 \cup ... \mathcal{A}_{k-1} \cup \mathcal{A}_{k+1} \cup ... \cup \mathcal{A}_q). \quad (21)$$

Illustrative Example 1: Combining the results established in this subsection enables one to partition the region outside a given polytope $S = \mathcal{Q}(\mathcal{V})$ to subregions where a new vertex v^k must be situated for the enlarged polytope $\mathcal{Q}(\mathcal{V}^*)$ to be of specified complexity. To show this graphically, we consider a polytopic set $S \subset \mathbb{R}^2$ with eight vertices, shown in Figure 1 in white color. We are interested in identifying the regions which correspond to different complexity of the set $\mathcal{Q}(\mathcal{V}^*)$ for the subset $\mathcal{X} \subset \mathbb{R}^2$ of the state space, which is the square of length 2. By calculating first the index sets $N_{\mathcal{S}}(v^k)$, $k \in \mathbb{N}_{[1,8]}$ and next the sets $\mathcal{A}_k \cap \mathcal{X}$, $k \in \mathbb{N}_{[1,8]}$, application of Theorem 1, Theorem 2 and Corollary 1 yields the polytopic regions for which the complexity of the set $\mathcal{Q}(\mathcal{V}^*)$ is identified. Thus, as shown in Figure 1, the complexity of $\mathcal{Q}(\mathcal{V}^*)$ is increased by one for regions that are depicted with $+1$, the complexity remains the same for the regions with 0, and is reduced by one, two and three for the regions shown with $-1, -2$ and -3 respectively.

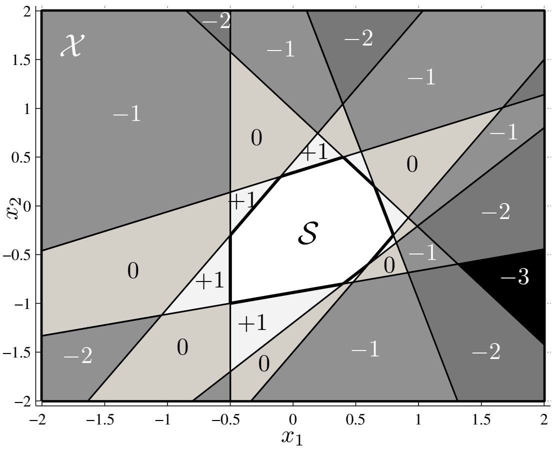

Fig. 1. The polytopic sets $S = \mathcal{Q}(\mathcal{V})$ and X, and the partition of the set $X \setminus \mathcal{Q}(\mathcal{V})$ in regions according to the resulting complexity of the resulting set when a vertex is added to the convex hull of S.

B. Enlargement of contractive polytopes

The results stated in the previous subsection can be utilized to identify, given a polytope, the regions of the state space where a new vertex can be added to its convex hull, such that the resulting enlarged polytope has a desired complexity. In this subsection, we establish the additional conditions that must be satisfied for the enlarged polytope $\mathcal{Q}(\mathcal{V}^*) = \text{conv}(v^1, ..., v^q, v^*)$ to be controlled ε–contractive when the initial set $\mathcal{Q}(\mathcal{V}) = \text{conv}(v^1, ..., v^q)$ is controlled ε–contractive [27]–[29].

Theorem 3: Given a controlled $\varepsilon-$contractive set $\mathcal{Q}(\mathcal{V}) = \text{conv}(v^1, ..., v^q)$ with respect to system (1) and a vector $v^* \in \mathbb{R}^n$, the polytope $\mathcal{Q}(\mathcal{V}^*) = \text{conv}(v^1, ..., v^q, v^*)$ is a controlled $\varepsilon-$contractive set if and only if there exist vectors $u^* \in \mathbb{R}^m$, $p^* \in \mathbb{R}^q$ and a scalar p^*_{q+1}, such that

$$Av^* + Bu^* = Vp^* + p^*_{q+1} v^* \qquad (22)$$

$$e_q^T p^* + p^*_{q+1} \leq \varepsilon \qquad (23)$$

$$p^* \geq 0 \qquad (24)$$

$$p^*_{q+1} \geq 0, \qquad (25)$$

where $V = \begin{bmatrix} v^1 & v^2 & \cdots & v^q \end{bmatrix}$.

Proof: If $\mathcal{Q}(\mathcal{V})$ is a controlled invariant set then there exist a nonnegative matrix $P \in \mathbb{R}_+^{q \times q}$ and a matrix $U \in \mathbb{R}^{m \times q}$ [30, Theorem 4.41] such that

$$AV + BU = VP \qquad (26)$$

$$e_q^T P \leq \varepsilon e_q^T, \qquad (27)$$

where $V = [v^1 \; ... \; v^q]$. Let $P^\star \in \mathbb{R}_+^{(q+1)\times(q+1)}$, where

$$P^\star := \begin{bmatrix} P & p^* \\ 0_{1 \times q} & p^*_{q+1} \end{bmatrix}$$

and matrices $V^\star := [V \; v^*]$, $V^\star := [U \quad u^*]$. Taking into account relations (22)–(25), it follows that conditions (26), (27) are also satisfied for $\mathcal{Q}(\mathcal{V}^*)$, with $P = P^\star$, $V = V^\star$. Thus, $\mathcal{Q}(\mathcal{V})$ is also a controlled ε–contractive set. Conversely, if $\mathcal{Q}(\mathcal{V})$ is a controlled ε–contractive set, there exists a nonnegative matrix $\hat{P} \in \mathbb{R}_+^{(q+1)\times(q+1)}$ satisfying conditions (26), (27) with $V = V^\star$ and $U = U^\star$. Then, relations (22)–(25) are satisfied with $p_i^* = \hat{P}_{(q+1)i}$, $i = 1, ..., q$ and $p_{q+1}^* = \hat{P}_{(q+1)(q+1)}$. ■

A direct consequence of Theorem 3 is the following result which concerns the ε–contractiveness with respect to autonomous discrete–time systems.

Corollary 2: Given a $\varepsilon-$contractive set $\mathcal{Q}(\mathcal{V}) = \mathrm{conv}(v^1, ..., v^q)$ with respect to system (2) and a vector $v^* \in \mathbb{R}^n$, the polytope $\mathcal{Q}(\mathcal{V}^*) = \mathrm{conv}(v^1, ..., v^q, v^*)$ is a $\varepsilon-$contractive set if and only if there exist a vectors $p^* \in \mathbb{R}^q$ and a scalar p_{q+1}^*, such that

$$Av^* = Vp^* + p_{q+1}^* v^* \tag{28}$$
$$e_q^T p^* + p_{q+1}^* \leq \varepsilon \tag{29}$$
$$p^* \geq 0 \tag{30}$$
$$p_{q+1}^* \geq 0, \tag{31}$$

where $V = \begin{bmatrix} v^1 & v^2 & \cdots & v^q \end{bmatrix}$.

IV. Admissible controlled contractive sets with specified complexity

Using the results developed in the subsections III-A and III-B we can establish necessary and sufficient conditions for the enlargement of a polytopic admissible contractive set $\mathcal{Q}(\mathcal{V}) = \mathrm{conv}(v^1, v^2, ...v^q)$ by adding one new vector v^* to its vertex description $\mathcal{Q}(\mathcal{V}) = \mathrm{conv}(v^1, v^2, ...v^q)$ to produce an admissible contractive set $\mathcal{Q}(\mathcal{V}^*) = \mathrm{conv}(v^1, v^2, ...v^q, v^*)$ of specified complexity.

Theorem 4: Let $\mathcal{Q}(\mathcal{V}) = conv(v^1, ..., v^q)$ be an admissible controlled ε–contractive set of the discrete–time system (1) with respect to state and input constraints (3), (4), of complexity q and with half-space representation $\mathcal{P}(G)$, $G \in \mathbb{R}^{p\times n}$. Then, given an index of complexity $q_{\max}$ and a vector $v^* \in \mathbb{R}^n$, the set $\mathcal{Q}(\mathcal{V}^*) = \mathrm{conv}(v^1, ..., v^q, v^*)$ is an admissible controlled ε-contractive invariant set of complexity $q_{\max}$ if and only if there exist vectors $u^* \in \mathbb{R}^m$, $p^* \in \mathbb{R}^q$, a scalar $p_{q+1}^* \in \mathbb{R}$ and indices $i = 1, 2, ..., q - q_{\max} + 1$ satisfying the algebraic relations

$$Av^* + Bu^* = Vp^* + v^* p_{q+1}^* \tag{32}$$
$$p^* \geq 0 \tag{33}$$
$$p_{q+1}^* \geq 0 \tag{34}$$
$$e_q^T p^* + p_{q+1}^* \leq \varepsilon \tag{35}$$
$$G_x v^* \leq e_{p_x} \tag{36}$$
$$G_u u^* \leq e_{p_u} \tag{37}$$
$$g_j^T v^* > 1, \quad \forall j \in N_{\mathcal{S}}(v^{k_i}), \tag{38}$$

where $V = \begin{bmatrix} v^1 & v^2 & \cdots & v^q \end{bmatrix}$.

Proof: From Theorem 3, relations (32)-(35) are necessary and sufficient conditions for the set $\mathcal{Q}(\mathcal{V}^*)$ to be controlled ε–contractive. Moreover, since $\mathcal{Q}(\mathcal{V})$ is admissible controlled ε–contractive set, it holds that, for all vertices v^i, $i \in \mathbb{N}_{[1,q]}$ there exist control inputs $u^i \in \mathbb{R}^m$, $i \in \mathbb{N}_{[1,q]}$, such that $G_x v^i \leq e_{p_x}$, $G_u u^i \leq e_{p_u}$. Since (37) holds, it follows that the set $\mathcal{Q}(\mathcal{V}^*)$ is also an admissible controlled ε–contractive set [30, Theorem 4.41]. Last, from Theorem 2 and (38), it holds that there exists indices k_i, $i = 1, ..., q - q_{\max} + 1$ such that $v^\star \in \mathcal{A}_{k_1} \cap \mathcal{A}_{k_2} \cap ... \cap \mathcal{A}_{k_{q-q_{\max}+1}}$, and consequently, the complexity of the set $\mathcal{Q}(\mathcal{V}^\star)$ is $q^* = q - (q - q_{\max}) = q_{\max}$. ■

Corollary 3: Let $\mathcal{Q}(\mathcal{V}) = conv(v^1, ..., v^q)$ be an admissible ε–contractive set with respect to system (2) and state constraints (3), of complexity q and with half-space representation $\mathcal{P}(G)$, $G \in \mathbb{R}^{p\times n}$. Then, given an index of complexity $q_{\max}$ and a vector $v^* \in \mathbb{R}^n$, the set $\mathcal{Q}(\mathcal{V}^*) = \mathrm{conv}(v^1, ..., v^q, v^*)$ is an admissible controlled ε–contractive invariant set of complexity $q_{\max}$ if and only if there exist a vector $p^* \in \mathbb{R}^q$, a scalar $p_{q+1}^* \in \mathbb{R}$ and indices $i = 1, 2, ..., q - q_{\max} + 1$ satisfying the algebraic relations

$$Av^* = Vp^* + v^* p_{q+1}^* \tag{39}$$
$$p^* \geq 0 \tag{40}$$
$$p_{q+1}^* \geq 0 \tag{41}$$
$$e_q^T p^* + p_{q+1}^* \leq \varepsilon \tag{42}$$
$$G_x v^* \leq e_{p_x} \tag{43}$$
$$g_j^T v^* > 1, \quad \forall j \in N_{\mathcal{S}}(v^{k_i}), \tag{44}$$

where $V = \begin{bmatrix} v^1 & v^2 & \cdots & v^q \end{bmatrix}$.

Remark 2: The importance of Theorem 4 as well as Corollary 3 for the autonomous case lies in the fact that from a given or determined admissible ε–contractive polytopic set $\mathcal{Q}(\mathcal{V}) = \mathrm{conv}(v^1, ..., v^q)$ we can construct an enlarged admissible ε–contractive polytopic set $\mathcal{Q}(\mathcal{V}^*) = \mathrm{conv}(v^1, ..., v^q, v^*)$ with specified complexity by solving the set of algebraic relations (32)–(38) and determine the unknown variables v^*, u^*, p_q^*, and p_{q+1}^* for different sets of integers$\{k_1, k_2, ..., k_{q-q^*+1}\}$. Relations (33)–(38) are linear, while relation (32) involves a bilinear scalar–vector product, where the scalar p_{q+1}^* is bounded between zero and one. Thus, for each index selection k_i, $i \in \mathbb{N}_{[1,q-q_{\max}+1]}$ such that relation (38) holds, and considering p_{q+1}^* as a scalar parameter, the algebraic relations become linear with respect to the unknown variables v^*, u^*, p_q^*.

A possible approach to the determination of the unknown variables $v^*, u^*, p_q^*, p_{q+1}^*$, is to pose an optimization problem having (32)-(38) as constraints. If a linear optimization criterion is chosen, this optimization problem can be solved by a series of linear programming problems as indicated in Remark 2. For example, choosing as optimization criterion the minimization of the parameter ε, we can compute a new vertex v^* making the enlarged polytope $\mathcal{Q}(\mathcal{V}) = conv(v^1, ..., v^q, v^*)$ an admissible controlled ε–contractive set of complexity q^* by solving the optimization problem

$$\max_{v^*, u^*, p_q^*, p_{q+1}^*} \{e_q^T p^* + p_{q+1}^*\}$$

under constraints (32)-(38).

Remark 3: Additional linear constraints can be considered in order to satisfy design requirements. For example, if an enlargement of the polytope $\mathcal{Q}(\mathcal{V}) = \text{conv}(v^1, ..., v^q)$ to a specific direction $g \in \mathbb{R}^n$ of the state space is desired, this specification can be achieved by considering an additional linear constraint describing this direction. Thus, if the new vertex v^* is desired or required to be located in a half space defined by the inequality

$$g^T v^* \geq 1,$$

where $g \in \mathbb{R}^n$, this inequality must be considered as an additional linear constraint of the optimization problem. For this particular case, solving the constrained optimization problem (32)–(38) having

$$\max_{v^*, u^*, p_q^*, p_{q+1}^*} \{g^T v^*\} \tag{45}$$

as optimization criterion, an enlarged admissible controlled ε–contractive set $\mathcal{Q}(\mathcal{V}^*)$ of complexity $q_{\max}$ with the new vertex v^* belonging to the half space defined by the inequality $g^T v^* \geq 1$ and located as far as possible from the boundary $g^T v^* = 1$.

Illustrative Example 2: We demonstrate how the results of Section IV can be used to enlarge an initial admissible controlled ε –contractive set $\mathcal{Q}(\mathcal{V}^*)$ by respecting a priori given complexity requirements. To this end, we consider the benchmark example of the discretized double integrator. The double integrator is of the form (1) with system matrices

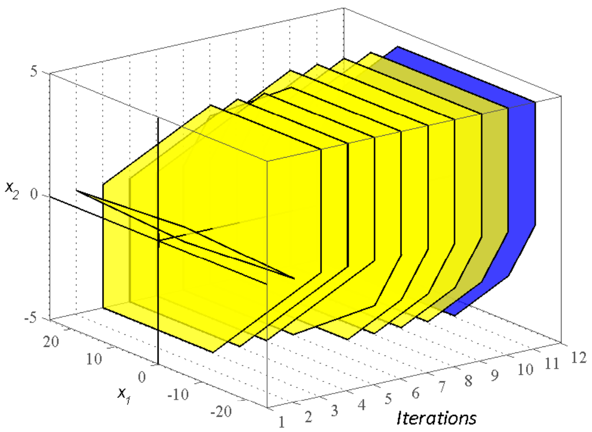

Fig. 2. Sequence of the admissible controlled–invariant sets $\mathcal{X}_i$, $i = 1, ..., 11$, applying the method proposed in this article.

$$A = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} \frac{T_s^2}{2} \\ T_s \end{bmatrix},$$

and sampling time $T_s = 0.1$sec. The system is subject to hard state and input constraints (3) and (4) respectively, with

$$G_x = \begin{bmatrix} 25^{-1} & 0 \\ 0 & 5^{-1} \\ -25^{-1} & 0 \\ 0 & -5^{-1} \end{bmatrix}, G_u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

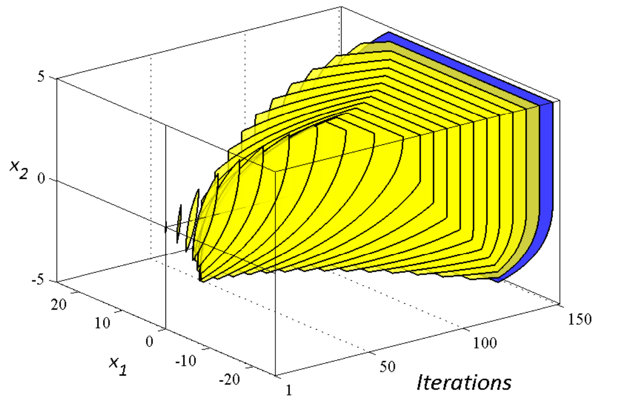

Fig. 3. Sequence of admissible controlled–invariant sets $\mathcal{Y}_i$, $i = 1, ..., 148$, produced by applying iteratively the one–step backward reachability method starting from the singleton $\{0_2\}$, [12].

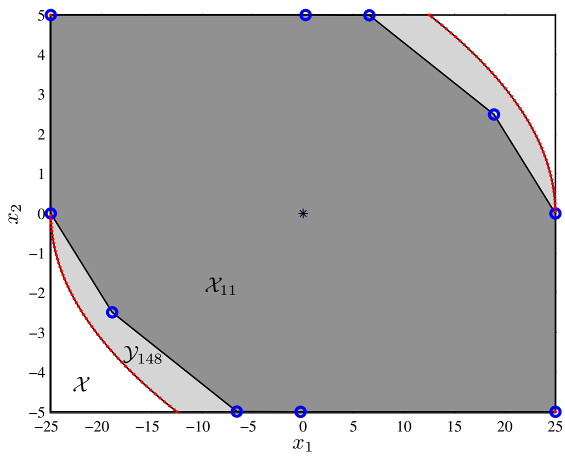

Fig. 4. The state constraint set $\mathcal{X}$, the maximal controlled invariant set $\mathcal{Y}_{148}$ [12] (106 vertices, red dots) and the low-complexity set $\mathcal{X}_{11}$ produced by iteratively applying the method presented in this article (10 vertices, blue circles).

The objective in this case is to compute an admissible controlled invariant polytopic set of a non–trivial size, whose complexity does not exceed $q_{\max} = 10$ vertices. For comparison purposes with other methods in the literature, we also apply the method described in [12] for the computation of the largest controlled invariant set. The method in [12] is based on computing a sequence $\{\mathcal{Y}_i\}$ of monotonically increasing controlled invariant sets $\mathcal{Y}_i$, using the one–step inverse reachability mapping starting from the singleton set $\mathcal{Y}_0 := \{0_2\}$.

The results from Theorem 4 were utilized in order to produce a monotonically increasing sequence of sets $\{\mathcal{X}_i\}$ of bounded complexity $q_{\mathcal{X}_i} \leq q_{\max}$, where $\mathcal{X}_i = \mathcal{P}(G_i) = \mathcal{Q}(\mathcal{V}_i)$ and $\mathcal{P}(G_i) = \{x \in \mathbb{R}^2 : G_i x \leq e_{p_i}\}$, $\mathcal{Q}(\mathcal{V}_i) = \text{conv}(v_i^1, v_i^2, ..., v_i^{q_i})$. For this particular case, the set $\mathcal{X}_0$ was set equal to $\mathcal{Y}_2$, which is guaranteed to be an admissible controlled invariant full-dimensional polytopic set that in-

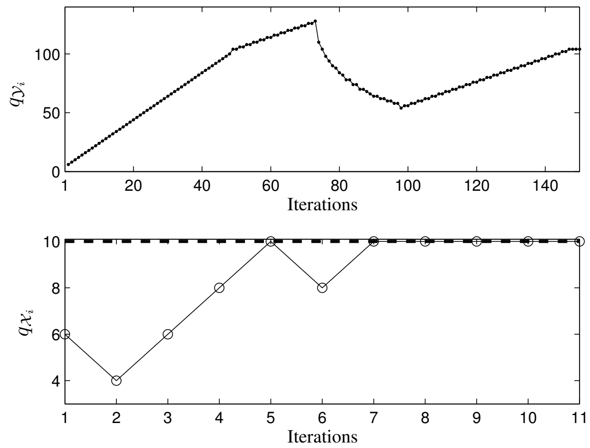

Fig. 5. Number of vertices for the sets produced applying the one-step backward reachability map [12] (upper plot), and the sets produced by applying the method proposed in this article (lower plot). The black dotted line in the lower plot indicates the complexity constraints.

cludes the origin in its interior [13]. At each iteration i, the optimization problem with cost (45) and constraints (32)–(38) was solved p_i times, choosing the vector g in the optimization cost (45) to be equal to each row of matrix G_i. For this case, the sequence of sets $\{\mathcal{X}_i\}$ converges at the 11th iteration and the resulting set $\mathcal{X}_{11}$ is of complexity $q_{\mathcal{X}_{11}} = 10$. In Figure 2 the sets $\mathcal{X}_i$, $i = 1, .., 10$ are shown in yellow color while set $\mathcal{X}_{11}$ is depicted in blue. Applying the method in [12], the maximal controlled invariant set was reached in 148 iterations, resulting in the maximal controlled invariant set $\mathcal{Y}_{148}$ of complexity $q_{\mathcal{Y}_{148}} = 106$. In Figure 3 sets $\mathcal{Y}_{7i}$, $i = 1, .., 20$ are shown in yellow color while the set $\mathcal{Y}_{148}$ is depicted in blue. In Figure 4 the sets $\mathcal{X}_{11}$ and $\mathcal{Y}_{148}$ are shown, along with the state constraint set $\mathcal{X}$. It is worth noticing that the two sets are of comparable volume, while their complexity differs significantly. Lastly, in Figure 5, the complexities of the sequences $\{\mathcal{Y}_i\}$, $\{\mathcal{X}_i\}$ are shown as functions of the set iterations. In the lower plot, the dotted line corresponds to the complexity constraints set by the problem specifications.

A. A note on computational complexity

At each iteration of the enlargement procedure described in Example 2, the optimization problem (45), (32)–(38) is solved separately for (at most) all feasible index selections k_i, $i \in \mathbb{N}_{[1, q-q_{\max}+1]}$, i.e., the index selections for which (38) defines a non–empty constraint set. Each of these separate optimization problems involves a single bilinear product between a bounded scalar variable and a vector. This problem can be reduced to a sequence of linear programming problems using the bisection method. Thus, the complexity of this optimization problem is similar to the one related to the enlargement procedure described in [29]. However, since at each iteration the optimization problem (45), (32)–(38) has to be solved for each feasible index selection k_i, $i \in \mathbb{N}_{[1, q-q_{\max}+1]}$, the number of optimization problems can be larger. Moreover, since this number depends on the number of half–spaces that describe the polytopic set produced at each iteration, it is not directly controlled by the method (although there exists an upper bound of the number of half–spaces as a function of the number of vertices and the polytope dimension, see e.g. the discussion in [24, Section 4.4.4]). In fact, a small number of half–spaces describing a polytopic set does not imply the same for the vertices and vice versa. It is the object of future research to define a complexity measure for polytopes which offers a balance between the complexities of the two representations.

Moreover, at each iteration, a transformation from the vertex representation to the half–space representation, which is known to be computationally expensive, is required. On the other hand, at each iteration, the redundant vertices can be directly computed exploiting Theorems 1 and 2, while the added vertices are the solutions of the optimization problem (45), (32)–(38). Thus, it is not needed to perform neither the transformation from the half–space representation to the vertex representation nor the computation of the minimal vertex representation.

V. Conclusions

The problem of constructing controlled invariant polytopic sets of a specified complexity, for discrete–time linear systems subject to linear state and control constraints was investigated. Geometric conditions for enlarging a polytopic set such that the resulting polytopic set has an a priori chosen number of vertices were formulated. Also, conditions concerning the enlargement of controlled invariant sets such that the resulting set preserves the controlled invariance property were presented. The efficacy of the established results was illustrated in the benchmark example of the double integrator where an iterative approach was taken in order to construct a controlled invariant set of non–trivial size and pre–specified complexity. It is worth noting that the results presented here can be extended, under suitable modifications, to the continuous–time case.

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