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On stability and stabilization of periodic discrete-time systems with an application to satellite attitude control ⋆

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Abstract

An alternative stability analysis theorem for nonlinear periodic discrete–time systems is presented. The developed theorem offers a trade–off between conservatism and complexity of the corresponding stability test. In addition, it yields a tractable stabilizing controller synthesis method for linear periodic discrete–time systems subject to polytopic state and input constraints. It is proven that in this setting, the proposed synthesis method is strictly less conservative than available tractable synthesis methods. The application of the derived method to the satellite attitude control problem results in a large region of attraction.

Key words: Periodic systems, periodic control, Lyapunov functions, constrained control, satellite attitude control

1 Introduction

This work deals with stability and stabilization of periodically time-varying systems, or shortly, *periodic systems*. Stability analysis and stabilization of periodic systems is typically handled by means of periodically time–varying standard Lyapunov functions (LFs), see Jiang and Wang (2002) for the nonlinear case and Bittanti and Colaneri (2009) for the linear case. For most of the available controller synthesis methods for periodic systems, existence of a periodically time–varying LF for the closed–loop dynamics can be derived, either directly or by the converse result in Jiang and Wang (2002). Consider methods based on the periodic Riccati equation Bittanti et al. (1991), Varga (2008), output feedback schemes De Souza and Trofino (2000), $\mathcal{H}_2$ synthesis for the case of linear periodic systems with polytopic uncertainties Farges et al. (2007), eigenvalue assignment Brunovský (1970), Kabamba (1986) controllability Longhi and Zulli (1995), model predictive control Böhm (2011), Gondhalekar and Jones (2011), and control with saturation Zhou et al. (2011). In the monograph (Bittanti and Colaneri, 2009, Chapter 13), a thorough exposition of existing results on stabilization techniques, including also frequency domain considerations or lifting techniques, is presented.

In the presence of constraints, however, stability analysis based on periodically time–varying standard LFs can yield a conservative region of attraction, as shown recently in Böhm et al. (2012). Therein, a relaxed stability analysis theorem was derived for autonomous nonlinear periodic systems. The main idea behind this relaxation is that the Lyapunov function is not required to decrease at each time instant, as in Jiang and Wang (2002) or in Bittanti and Colaneri (2009) for the linear case, but at each period. This paper considers stabilization of linear periodic systems with inputs, subject to polytopic state and input constraints, by means of linear periodic state–feedback control laws. The presence of input constraints further motivates the need for a relaxation of the classical stability analysis theorems Jiang and Wang (2002), Bittanti and Colaneri (2009). For the case of periodic systems with inputs, however, the relaxed periodic Lyapunov conditions in Böhm et al. (2012) lead to a nonlinear and non–convex optimization problem which is not tractable.

Motivated by the current status, we propose an alternative stability analysis theorem for nonlinear periodic systems. This new result allows the establishment of a tractable constrained synthesis method for linear periodic systems, by

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choosing quadratic periodic Lyapunov functions. We show how the constrained synthesis problem with linear periodic state–feedback can be solved by decomposing the original non–convex optimization problem in a finite set of semi–definite optimization problems having linear matrix inequalities (LMIs) as constraints. The equivalence between the original non–convex problem and the set of semi–definite optimization problems is formally proven. The method is applied successfully in the challenging magnetic satellite attitude control problem. The developed synthesis method yields a large region of attraction for the resulting closed–loop system while providing non–trivial performance guarantees.

The remainder of this paper is structured as follows. Existing results on Lyapunov stability for periodic systems are briefly discussed in Section 2. The problem formulation as well as solutions from existing approaches are presented in Section 3. The main results are established in Section 4. Application of the established results to the satellite attitude control problem is presented in Section 5, while conclusions are drawn in Section 6.

Notation and basic definitions: Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ we define $\Pi_{\geq c} := \{x \in \Pi \mid x \geq c\}$, and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For $N \in \mathbb{Z}_{\geq 1}$, $\Pi^N := \Pi \times \ldots \times \Pi$. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the i-th element of x and $\|x\|$ denotes its 2-norm, i.e., $\|x\| := \sqrt{\sum_{i=1}^n |[x]_i|^2}$. The transpose of a matrix $X \in \mathbb{R}^{n\times m}$ is denoted by $X^\top$. For a symmetric matrix $Z \in \mathbb{R}^{n\times n}$ let $Z \succ 0 (\succeq 0)$ denote that Z is positive definite (semi-definite). For a positive definite matrix $Z \in \mathbb{R}^{n\times n}$ let $\lambda_{\min(\max)}(Z)$ denote its smallest (largest) eigenvalue. Moreover, for a block symmetric matrix $Z = \left[\begin{smallmatrix} a & b^\top \\ b & c \end{smallmatrix}\right]$, where a, b, c are matrices of appropriate dimensions, the symbol $\star$ is used to denote the symmetric part, i.e., $\left[\begin{smallmatrix} a & \star \\ b & c \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b^\top \\ b & c \end{smallmatrix}\right]$. For the definition of functions of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$, refer to Böhm et al. (2012).

2 Preliminaries

Let $n, m \in \mathbb{Z}_+$ be integers and let $\mathbb{X} : \mathbb{Z}_+ \to \mathbb{R}^n$ and $\mathbb{U} : \mathbb{Z}_+ \to \mathbb{R}^m$ be maps that assign to each $k \in \mathbb{Z}_+$ a subset of $\mathbb{R}^n$ and a subset of $\mathbb{R}^m$ respectively, which contain the origin in their interior. We consider time–varying nonlinear systems of the form

$$x(k+1) = f(k, x(k), u(k)), \quad k \in \mathbb{Z}_+, \tag{1}$$

where $f : \mathbb{Z}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is an arbitrary nonlinear map such that $f(k, 0, 0) = 0$, for all $k \in \mathbb{Z}_+$. The vector $x(k) \in \mathbb{X}(k)$ is the system state at time $k \in \mathbb{Z}_+$ and $u(k) \in \mathbb{U}(k)$ is the system input at time $k \in \mathbb{Z}_+$.

Definition 1 *The system (1) is called periodic if there exists an $N \in \mathbb{Z}_{\geq 1}$ such that for all $k \in \mathbb{Z}_+$ it holds (i) $\mathbb{X}(k) = \mathbb{X}(k+N)$; (ii) $\mathbb{U}(k) = \mathbb{U}(k+N)$; (iii) $f(k, x, u) = f(k+N, x, u)$ for all $x \in \mathbb{X}(k)$, for all $u \in \mathbb{U}(k)$. Furthermore, the smallest such $N \in \mathbb{Z}_{\geq 1}$ is called the period of system (1).*

We consider a periodically time–varying state feedback control law $g : \mathbb{Z}_+ \times \mathbb{R}^n \to \mathbb{R}^m$ such that $g(k, 0) = 0$, for all $k \in \mathbb{Z}_+$, $g(k, x) = g(k+N, x)$, for all $k \in \mathbb{Z}_+$, and $g(k, x(k)) \in \mathbb{U}(k)$, for all $k \in \mathbb{Z}_+$ and for all $x(k) \in \mathbb{X}(k)$. We assume, for simplicity, that the period of the control law is equal to the period of system (1). The corresponding closed–loop system is

$$x(k+1) = f(k, x(k), g(k, x(k))), \quad k \in \mathbb{Z}_+. \tag{2}$$

System (2) is periodic with period N, since $f(k+N, x, g(k+N, x)) = f(k, x, g(k, x))$. In what follows, let $\mathbb{X}_0 := \mathbb{X}(0)$ and define $\overline{\mathbb{X}} := \bigcup_{k=0}^{N-1} \mathbb{X}(k)$. As such, all state trajectories of system (2) with $x(0) \in \mathbb{X}_0$ satisfy $x(k) \in \overline{\mathbb{X}}$, for all $k \in \mathbb{Z}_+$. For clarity of exposition, we will consider constant input and state dimensions for all modes of the periodic system. The classical time-invariant unconstrained state-space and input domain is recovered by setting $\mathbb{X}(k) = \mathbb{R}^n$, $\mathbb{U}(k) = \mathbb{R}^m$, for all $k \in \mathbb{Z}_+$.

We adopt the notions of asymptotic stability in a set $\mathbb{X}_0$ (AS($\mathbb{X}_0$)), exponential stability in a set $\mathbb{X}_0$ (ES($\mathbb{X}_0$)) and region of attraction (ROA) for system (2) from Böhm et al. (2012). Next, the notion of a *periodically positively invariant* (PPI) sequence of sets is recalled. Let $\{\mathbb{D}(\pi)\}_{\pi\in\mathbb{Z}_{[0,N-1]}}$ denote a sequence of sets with $\mathbb{D}(\pi) \subseteq \mathbb{X}(\pi)$ for all $\pi \in \mathbb{Z}_{[0,N-1]}$.

Definition 2 *The sequence $\{\mathbb{D}(\pi)\}_{\pi\in\mathbb{Z}_{[0,N-1]}}$ is called periodically positively invariant for system* (2) *if for each $\pi \in \mathbb{Z}_{[0,N-1]}$, each $k \in \{iN + \pi\}_{i\in\mathbb{Z}_+}$ and $x(k) \in \mathbb{D}(\pi)$, it holds that $x(k+N) \in \mathbb{D}(\pi)$ and $x(k+j) \in \mathbb{X}(k+j)$, for all $j \in \mathbb{Z}_{[1,N-1]}$.*

The following stability Theorems correspond to Jiang and Wang (2002) and Böhm et al. (2012) respectively. These results are adapted for system (2) and modified appropriately in order to provide a framework compatible with the results established in this article.

Theorem 1 Jiang and Wang (2002) *Let $\{\mathbb{X}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ be a PPI sequence of sets w.r.t.* (2)*. Let $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathbb{R}_{[0,1)}$ and let $x(\cdot)$ be a solution to* (2) *with $x(0) := \xi \in \mathbb{X}(0)$. Let $V : \mathbb{Z}_+ \times \overline{\mathbb{X}} \to \mathbb{R}_+$ be a function, such that $V(k, x) = V(k+N, x)$, for all $k \in \mathbb{Z}_+$, and moreover, for all $k \in \mathbb{Z}_+$ it holds that*

$$\alpha_1(\|\xi\|) \leq V(k, \xi) \leq \alpha_2(\|\xi\|), \forall \xi \in \mathbb{X}(k) \tag{3a}$$

$$V(k+1, f(k, x(k), g(k, x(k)))) \leq \rho V(k, x(k)), \quad \forall \xi \in \mathbb{X}(0). \tag{3b}$$

Then, system (2) is AS($\mathbb{X}_0$).

Theorem 2 Böhm et al. (2012) *Let* $\{\mathbb{X}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ *be a PPI sequence of sets w.r.t.* (2). *Let* $\alpha_1, \alpha_2\ \bar{\alpha}_j, j \in \mathbb{Z}_{[1,N-1]}$ *be* $\mathcal{K}_\infty$ *functions,* $\eta \in \mathbb{R}_{[0,1)}$ *and* $x(\cdot)$ *be a solution to* (2) *with* $x(0) := \xi \in \mathbb{X}(0)$. *Let* $V : \mathbb{Z}_+ \times \overline{\mathbb{X}} \to \mathbb{R}_+$ *be a function, such that* $V(k,x) = V(k+N,x)$*, for all* $k \in \mathbb{Z}_+$*, and moreover, for all* $k \in \mathbb{Z}_+$*, for all* $j \in \mathbb{Z}_{[1,N-1]}$*, it holds that*

$$\|x(j)\| \le \bar{\alpha}_j(\|x(j-1)\|), \forall \xi \in \mathbb{X}(0) \quad \text{(4a)}$$
$$\alpha_1(\|\xi\|) \le V(k,\xi) \le \alpha_2(\|\xi\|), \qquad \forall \xi \in \mathbb{X}(k) \quad \text{(4b)}$$
$$V(k+N, x(k+N)) \le \eta V(k, x(k)), \quad \forall \xi \in \mathbb{X}(0). \quad \text{(4c)}$$

Then, system (2) is AS($\mathbb{X}_0$*).*

3 Problem formulation

We consider non-autonomous linear periodic systems

$$x(k+1) = A(k)x(k) + B(k)u(k), \tag{5}$$

where $A(k) \in \mathbb{R}^{n\times n}$, $B(k) \in \mathbb{R}^{n\times m}$, and $A(k) = A(k+N)$, $B(k) = B(k+N)$, for all $k \in \mathbb{Z}_+$. Equivalently to the nonlinear case, by choosing a linear periodic state–feedback control law with period N, i.e.,

$$u(k) = g(k, x(k)) := K(k)x(k), \tag{6}$$

with $K(k) = K(k+N)$, the closed-loop system is

$$x(k+1) = (A(k) + B(k)K(k))x(k). \tag{7}$$

Next, we consider that system (5) is subject to polytopic state periodic constraints

$$\mathbb{X}(k) := \{x \in \mathbb{R}^n : c_i(k)x \le 1, \forall (i,k) \in \mathbb{Z}_{[1,p(k)]} \times \mathbb{Z}_+\}, \tag{8}$$

where $p(k) \in \mathbb{Z}_{\ge 1}$, for all $k \in \mathbb{Z}_+$, is the number of hyperplanes that define set $\mathbb{X}(k)$, and $c_i(k+N) = c_i(k)$, for all $(i,k) \in \mathbb{Z}_{[1,p(k)]} \times \mathbb{Z}_+$. Similarly, we consider polytopic input constraints

$$\mathbb{U}(k) := \{u \in \mathbb{R}^m : d_i(k)u \le 1, \forall (i,k) \in \mathbb{Z}_{[1,q(k)]} \times \mathbb{Z}_+\}, \tag{9}$$

where $q(k) \in \mathbb{Z}_{\ge 1}$, for all $k \in \mathbb{Z}_+$, and $d_i(k+N) = d_i(k) \in \mathbb{R}^{1\times n}$ for all $(i,k) \in \mathbb{Z}_{[1,q(k)]} \times \mathbb{Z}_+$.

We are now ready to state the problem of interest.

Problem 1 *Given system* (5)*, state and input constraints* $\mathbb{X}(k)$ (8) *and* $\mathbb{U}(k)$ (9) *respectively, determine a stabilizing linear periodic state–feedback control law* (6) *and a corresponding PPI sequence of sets* $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ *with respect to the closed–loop system* (7)*.*

3.1 Solutions based on existing stability analysis theorems

To solve Problem 1, consider quadratic periodic Lyapunov function candidates

$$V(k,x) = x^\top P(k)x, \tag{10}$$

where $P(k) \in \mathbb{S}^n_{++}$, with $P(k+N) = P(k)$ for all $k \in \mathbb{Z}_+$. The candidate Lyapunov function (10) is upper and lower bounded by $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$,

$$\alpha_1(y) := \min_{i\in\mathbb{Z}_{[0,N-1]}} |\lambda_{\min} P(i)| y^2, \tag{11}$$
$$\alpha_2(y) := \max_{i\in\mathbb{Z}_{[0,N-1]}} |\lambda_{\max} P(i)| y^2, \tag{12}$$

i.e.,

$$\alpha_1(\|\xi\|) \le V(k,\xi) \le \alpha_2(\|\xi\|), \tag{13}$$

for all $\xi \in \mathbb{R}^n$. In this setting, one can apply Theorem 1, which results in the periodic Lyapunov lemma (PLL), as formally stated next.

Lemma 1 Bittanti and Colaneri (2009) *Consider system (7). Let* $\rho(k) \in \mathbb{R}_{[0,1)}$*,* $k \in \mathbb{Z}_{[0,N-1]}$*, and* $P(k) \in \mathbb{S}^n_{++}$*,* $k \in \mathbb{Z}_{[0,N]}$ *be positive definite matrices, with* $P(N) := P(0)$*, which define sets* $\mathbb{E}(k) = \{x \in \mathbb{R}^n : x^\top P(k)x \le 1\}$ *such that* $\mathbb{E}(k) \subseteq \mathbb{X}(k)$*, for all* $k \in \mathbb{Z}_{[0,N-1]}$*. If the matrix inequalities*

$$(A(k) + B(k)K(k))^\top P(k+1)(A(k) + B(k)K(k)) - \rho(k)P(k) \preceq 0, \tag{14}$$

hold for all $k \in \mathbb{Z}_{[0,N-1]}$*, then system (7) is ES(*$\mathbb{E}(0)$*).*

Next, it is shown how Theorem 2 could be applied to solve Problem 1. To this end, for all $k \in \mathbb{Z}_{[0,N-1]}$ define the monodromy matrices Bittanti and Colaneri (2009)

$$\Phi(k) := \prod_{i=0}^{N-1} (A(k+i) + B(k+i)K(k+i)).$$

Lemma 2 *Consider system (7). Let* $\eta \in \mathbb{R}_{[0,1)}$*, and* $P(k) \in \mathbb{S}^n_{++}$*,* $k \in \mathbb{Z}_{[0,N]}$ *be positive definite matrices, with* $P(N) := P(0)$*, which define sets* $\mathbb{E}(k) = \{x \in \mathbb{R}^n : x^\top P(k)x \le 1\}$ *such that* $\mathbb{E}(k) \subseteq \mathbb{X}(k)$*, for all* $k \in \mathbb{Z}_{[0,N-1]}$*. If the matrix inequalities*

$$\Phi(k)^\top P(k)\Phi(k) - \eta P(k) \preceq 0, \tag{15a}$$
$$(A(k) + B(k)K(k))^\top P(k+1)(A(k) + B(k)K(k)) - P(k) \preceq 0, \tag{15b}$$

hold for all $k \in \mathbb{Z}_{[0,N-1]}$*, then system (7) is AS(*$\mathbb{E}(0)$*).*

Remark 1 *The closed–loop system* (7) *is a time–varying linear system with a finite number of time–invariant subsystems. Also, the corresponding periodic Lyapunov function* (10) *is periodically time–varying. Since the upper and lower* $\mathcal{K}_\infty$ *bounds* (11)*,* (12) *of the Lyapunov function* (10) *are time-invariant, exponential stability of the closed-loop system* (7) *can be deduced following the proof of (Böhm et al., 2012, Theorem 9), by exploiting the specific form of* (11) *and* (12)*. Consequently, if conditions* (15a),(15b) *of Lemma 2 hold, then the system* (7) *is ES(*$\mathbb{E}(0)$*). The formal details of this straightforward derivation are omitted for brevity.*

Remark 2 *Lemma 2 is a strict relaxation of the result stated in Lemma 1. Indeed, a feasible set of matrices* $P(k)$*,* $k \in \mathbb{N}_{[0,N-1]}$*, and periodic state feedback gains* $K(k)$*,* $k \in \mathbb{N}_{[0,N-1]}$*, that satisfies* (14)*, satisfies relations* (15) *as well, but the converse is not true. Regarding computational aspects, the conditions* (14) *of Lemma 1 can be reformulated as an equivalent semidefinite program, while finding a solution to the conditions* (15) *of Lemma 2 requires solving a non–convex and nonlinear program. Therefore, it of interest to establish a trade–off between the additional degree of freedom introduced by Lemma 2 and the tractability of the conditions of Lemma 1.*

4 Main results

The first main result of this paper is an alternative stability analysis theorem for periodic nonlinear systems, which provides a trade–off between Theorem 1 and Theorem 2, as formally stated next.

Theorem 3 *Let* $\{\mathbb{X}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ *be a PPI sequence of sets w.r.t.* (2)*. Let* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$*, scalars* $\rho(j) \in \mathbb{R}_{[0,1]}$*,* $j \in \mathbb{N}_{[0,N-1]}$*, and* $x(\cdot)$ *be a solution to* (2) *with* $x(0) := \xi \in \mathbb{X}(0)$*. Let* $V : \mathbb{Z} \times \bar{\mathbb{X}} \to \mathbb{R}_+$ *be a function, such that* $V(k,x) = V(k+N,x)$*, for all* $k \in \mathbb{Z}_+$*, and moreover, for all* $j \in \mathbb{Z}_{[0,N-1]}$*, it holds that*

$$V(j+1, x(j+1)) \leq \rho(j)V(j, x(j)), \quad \forall \xi \in \mathbb{X}(0) \tag{16a}$$
$$\alpha_1(\|\xi\|) \leq V(k,\xi) \leq \alpha_2(\|\xi\|), \quad \forall \xi \in \mathbb{X}(k) \tag{16b}$$
$$0 \leq \prod_{i=0}^{N-1} \rho(i) < 1. \tag{16c}$$

Then, system (2) is AS($\mathbb{X}_0$*).*

Proof From (16a) and (16b), for any $j \in \mathbb{Z}_{[1,N]}$, it holds that

$$\begin{aligned}\alpha_1(\|x(j)\|) \leq V(j,x(j)) &\leq \rho(j-1)V(j-1,x(j-1)) \\ &\leq \rho(j-1)\alpha_2(\|x(j-1)\|),\end{aligned}$$

or

$$\|x(j)\| \leq \alpha_1^{-1}(\rho(j-1)\alpha_2(\|x(j-1)\|)).$$

Thus, relation (4a) of Theorem 2 is satisfied with

$$\bar{\alpha}_j(s) := \alpha_1^{-1}(\rho(j-1)\alpha_2(s)), \quad \forall j \in \mathbb{Z}_{[1,N-1]}.$$

Moreover, from (16a), for any $k \in \mathbb{Z}_+$ and for any $x(k) \in \mathbb{X}(k)$, it holds that

$$V(k+N, x(j+N)) \leq \rho(N-1)V(j+N-1, x(j+N-1)).$$

Applying the previous inequality successively, it holds that

$$\begin{aligned}&V(k+N, x(j+N)) \leq \\ &\quad \rho(N-1)\rho(N-2)V(k+N-2, x(k+N-2)) \leq \\ &\quad\quad \leq \ldots \leq \prod_{i=0}^{N-1} \rho(i)V(k, x(k)).\end{aligned}$$

Taking into account (16c), relation (4c) of Theorem 2 is satisfied with $\eta := \prod_{i=0}^{N-1} \rho(i) \in \mathbb{R}_{[0,1)}$. Thus, by Theorem 2, system (2) is AS($\mathbb{X}_0$). ■

Remark 3 *To compare the available stability analysis theorems observe the following:* (i) *Theorem 1 requires* $V(\cdot)$ *to decrease at every time instant* $k \in \mathbb{Z}_+$*;* (ii) *Theorem 3 requires* $V(\cdot)$ not to increase at every time instant $k \in \mathbb{Z}_+$ *and to decrease at every* N *time instants, i.e., at every period;* (iii) *Theorem 2 requires* $V(\cdot)$ *to decrease at every period. Furthermore, notice that condition* (16c) *does not fix a particular time instant when the decrease should take place, within each period.*

Remark 4 *While existence of a function* $V(\cdot)$ *that satisfies conditions of Theorems 1–3 is necessary and sufficient*[1] *for the system* (2) *to be AS(*$\mathbb{X}_0$*), the feasible solution sets of the underlying conditions are ordered as follows: A function* $V(\cdot)$ *satisfying conditions* (3) *of Theorem 1 satisfies conditions* (16) *of Theorem 3, which in turn satisfy conditions* (15) *of Theorem 2. The opposite is not true.*

Since Theorem 3 still provides a strict relaxation of Theorem 1, it is of further interest to utilize Theorem 3 for solving Problem 1. To this end, consider the following result.

Theorem 4 *Consider system (5) and constraints* $\mathbb{X}(k)$ *(8),* $\mathbb{U}(k)$ *(9). Let* $\rho(k) \in \mathbb{R}_{[0,1]}$*,* $X(k) \in \mathbb{S}^n_{++}$*,* $Y(k) \in \mathbb{R}^{m\times n}$*, for all* $k \in \mathbb{Z}_{[0,N-1]}$*, where* $X(N) := X(0)$*,* $Y(N) := Y(0)$*, be a feasible solution to the following set of matrix inequalities, for all* $k \in \mathbb{Z}_{[0,N-1]}$*, for all* $i \in \mathbb{Z}_{[1,p(k)]}$ *and*

[1] Theorem 1 further requires a continuity assumption on the dynamics of the system (2).

all $j \in \mathbb{Z}_{[1,q(k)]}$:

$$\begin{bmatrix} \rho(k)X(k) & \star \\ A(k)X(k)+B(k)Y(k) & X(k+1) \end{bmatrix} \succeq 0, \tag{17a}$$

$$0 \leq \sum_{l=0}^{N-1} \rho(l) < N, \tag{17b}$$

$$\begin{bmatrix} 1 & \star \\ X(k)c_i(k)^\top & X(k) \end{bmatrix} \succeq 0, \tag{17c}$$

$$\begin{bmatrix} 1 & \star \\ Y(k)^\top d_j(k)^\top & X(k) \end{bmatrix} \succeq 0. \tag{17d}$$

Then, the sequence $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$, where $\mathbb{E}(k) := \{x \in \mathbb{R}^n : x^\top X(k)^{-1}x \leq 1\}$, $k \in \mathbb{Z}_{[0,N-1]}$, is PPI for the closed-loop system (7) with linear periodic state feedback control law $u(k,x) = Y(k)X(k)^{-1}x$, where $u(k+N,x) = u(k,x)$ for all $k \in \mathbb{Z}_+$. Moreover, system (7) is ES($\mathbb{E}(0)$).

Proof First, we show that the sets $\mathbb{E}(k)$, $k \in \mathbb{Z}_{[0,N-1]}$, are contained in $\mathbb{X}(k)$ and $\mathbb{U}_x(k)$ for all $k \in \mathbb{Z}_{[0,N-1]}$, where

$$\mathbb{U}_x(k) := \{x \in \mathbb{R}^n : d_i(k)Y(k)X(k)^{-1}x \leq 1, \\ \forall (i,k) \in \mathbb{Z}_{[1,q(k)]} \times \mathbb{Z}_+\},$$

for all $k \in \mathbb{Z}_{[0,N-1]}$. Applying the Schur complement in (17c) and exploiting the periodicity of $\mathbb{X}(k)$ we obtain

$$c_i(k)^\top X(k) c_i(k) \leq 1, \quad \forall (i,k) \in \mathbb{Z}_{[1,p(k)]} \times \mathbb{Z}_+. \tag{18}$$

From (Athanasopoulos et al., 2013, Lemma 12), inequality (18) implies $\mathbb{E}(k) \subset \mathbb{X}(k)$, for all $k \in \mathbb{Z}_+$. Equivalently, applying the Schur complement in (17d) we obtain

$$d_j(k)Y(k)X(k)^{-1}Y(k)^\top d_j(k)^\top \leq \\ (d_j(k)Y(k)X(k)^{-1})X(k)(X(k)^{-1}Y(k)^\top d_j(k)^\top) \leq 1, \tag{19}$$

for all $(j,k) \in \mathbb{Z}_{[1,q(k)]} \times \mathbb{Z}_+$. From (Athanasopoulos et al., 2013, Lemma 12), inequality (19) implies $\mathbb{E}(k) \subset \mathbb{U}_x(k)$, for all $k \in \mathbb{Z}_+$.

Next, we show that $V(k,x) = x^\top X(k)^{-1}x$ is a periodic Lyapunov function that satisfies Theorem 3 for the closed–loop system (7). The matrix inequality (17a) is equivalent to

$$(A(k)X(k)+B(k)Y(k))^\top X(k+1)^{-1}(A(k)X(k) \\ +B(k)Y(k)) - \rho(k)X(k) \preceq 0.$$

Pre–multiplying and post–multiplying by $X(k)^{-1}$, the previous inequality becomes

$$(A(k)+B(k)Y(k)X(k)^{-1})^\top X(k+1)^{-1}(A(k) \\ +B(k)Y(k)X(k)^{-1}) - \rho(k)X(k)^{-1} \preceq 0. \tag{20}$$

Thus, condition (16a) of Theorem 3 is satisfied with $V(k,x) = x^\top X(k)^{-1}x$. Also, condition (16b) holds with

$$\alpha_1(y) = \min_{i\in\mathbb{Z}_{[0,N-1]}} |\lambda_{\min}(X(i)^{-1})|y^2 \tag{21}$$

$$\alpha_2(y) = \max_{i\in\mathbb{Z}_{[0,N-1]}} |\lambda_{\max}(X(i)^{-1})|y^2. \tag{22}$$

Lastly, since $\rho(k) \in \mathbb{R}_{[0,1]}$, from (17b) it necessarily holds that

$$0 \leq \prod_{l=0}^{N-1} \rho(l) < 1, \tag{23}$$

thus, condition (16c) of Theorem 3 is also satisfied. Thus, from (20)–(23), Theorem 3 is satisfied, system (7) is AS($\mathbb{E}(0)$) and $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ is a PPI sequence of sets w.r.t. system (7). Moreover, taking into account Remark 1, the system (7) is ES($\mathbb{E}(0)$). ■

Still, conditions (17) of Theorem 4 cannot be used directly to form a tractable synthesis method that solves Problem 1, since (17a) consists of N products between the scalars $\rho(k)$ and matrices $X(k)$, $k \in \mathbb{N}_{[0,N-1]}$. In specific, condition (17a) is a special case of a bilinear matrix inequality that will be denoted in what follows as a *bilinear scalar matrix inequality (BsMI)*. Although $\rho(k)$ is a scalar for each $k \in \mathbb{N}_{[0,N-1]}$, finding a solution to the N joint BsMI conditions corresponding to (17a) is challenging, since the bisection method cannot be used. The second main result of this paper provides an equivalent formulation of the conditions (17) that can be solved by semidefinite programming. To this end, first, consider the following problem.

Problem 2 *Given system (5), constraints $\mathbb{X}(k)$ (8), $\mathbb{U}(k)$ (9), and a fixed $\bar{k} \in \mathbb{Z}_{[0,N-1]}$, solve the feasibility problem*

$$\text{find } X(\bar{k}), Y(\bar{k}), \bar{\rho}, X(k), Y(k),\ k \in \mathbb{Z}_{[0,N-1]} \setminus \{\bar{k}\}. \tag{24}$$

subject to

$$\begin{bmatrix} X(k) & \star \\ A(k)X(k)+B(k)Y(k) & X(k+1) \end{bmatrix} \succeq 0, \tag{25a}$$

$$\begin{bmatrix} \overline{\rho}X(\overline{k}) & \star \\ A(\overline{k})X(\overline{k})+B(\overline{k})Y(\overline{k}) & X(\overline{k}+1) \end{bmatrix} \succeq 0, \tag{25b}$$

$$0 \le \overline{\rho} < 1, \tag{25c}$$

$$\begin{bmatrix} 1 & \star \\ X(l)c_i(l)^\top & X(l) \end{bmatrix} \succeq 0, \tag{25d}$$

$$\begin{bmatrix} 1 & \star \\ Y(l)^\top d_j(l)^\top & X(l) \end{bmatrix} \succeq 0, \tag{25e}$$

with $X(N) := X(0)$*, for all* $k \in \mathbb{Z}_{[0,N-1]} \setminus \{\overline{k}\}$, $l \in \mathbb{Z}_{[0,N-1]}$, $i \in \mathbb{Z}_{[1,p(l)]}$, $j \in \mathbb{Z}_{[1,q(l)]}$.

Lemma 3 *Consider system (5), constraints* $\mathbb{X}(k)$ *(8) and* $\mathbb{U}(k)$ *(9). Then, the matrix inequalities (17) define a nonempty feasible solution set if and only if there exists an index* $\bar{k}^\star \in \mathbb{Z}_{[0,N-1]}$ *such that Problem 2 is feasible with* $\overline{k} = \overline{k}^\star$.

Proof Suppose Problem 2 is feasible for a $\overline{k} = \overline{k}^\star \in \mathbb{Z}_{[0,N-1]}$. Then, relations (17) are also feasible setting $\rho(k) = 1$, for all $k \in \mathbb{Z}_{[0,N-1]} \setminus \{\overline{k}^\star\}$, $\rho(\overline{k}^\star) := \overline{\rho}$, and $X(\bar{k}^\star), Y(\bar{k}^\star)$, $X(k), Y(k)$, $k \in \mathbb{Z}_{[0,N-1]}$, the solutions to Problem 2. Conversely, suppose that conditions (17) have a nonempty feasible solution set. Then, there exists at least one $\overline{k}^\star \in \mathbb{Z}_{[0,N-1]}$ such that $\rho(\overline{k}^\star) < 1$. Setting $\bar{k} := \bar{k}^\star$ and $\overline{\rho} := \rho(\overline{k}^\star)$ the corresponding matrix inequalities (25b)–(25e) in Problem 2 are satisfied. Moreover, for any $\hat{k} \in \mathbb{Z}_{[0,N-1]} \setminus \{\bar{k}^\star\}$ such that $\rho(\hat{k}) < 1$, relation (17a) implies

$$\begin{bmatrix} X(\hat{k}) & \star \\ A(\hat{k})X(\hat{k})+B(\hat{k})Y(\hat{k}) & X(\hat{k}+1) \end{bmatrix} \succeq (1-\rho(\hat{k}))X(\hat{k}) \succeq 0.$$

Thus, (25a) is also satisfied, and consequently, Problem 2 has a solution for $\overline{k} = \overline{k}^\star$. ■

Remark 5 *Comparison of conditions (17a) of Theorem 4 with condition (25b) in Problem 2, reveals the significance of the previous result. Lemma 3 shows that existence of a feasible solution to the constraint set (17), which involves N BsMIs, is equivalent to existence of a solution in (at least) one of the* N *feasibility problems (24)–(25), which involve a single bilinear term, i.e., the product of the scalar* $\overline{\rho}$ *and the matrix* $X(\overline{k})$ *in (25b). Furthermore, since the single bilinear term in (25b) consists of a matrix and the constrained nonnegative scalar* $\overline{\rho} \in \mathbb{R}_{[0,1)}$*, solution of Problem 2 is equivalent to solving a series of LMIs via bisection, which is guaranteed to converge to a feasible solution, if a feasible solution exists. Still, it is worth noting that the computational burden induced by the proposed method is higher than the one stemming from the application of the PLL synthesis method, which involves fewer decision variables and strict LMIs, see e.g. Zhou et al. (2011). This is the price to be paid for exploiting the less conservative results of Theorem 4.*

4.1 Additional synthesis objectives

In constrained synthesis, together with computing a stabilizing control law, it is of relevance to aim for a large basin of attraction $\mathbb{E}(0) \subseteq \mathcal{R}(\mathbb{X}_0)$, where $\mathbb{E}(0) = \{x \in \mathbb{R}^n : x^\top X(0)^{-1} x \le 1\}$. To this end, a semi–definite optimization problem that is solved for every $\bar{k} \in \mathbb{Z}_{[0,N-1]}$, maximizes the volume of $\mathbb{E}(0)$ and solves Problem 1, i.e.,

$$\min_{X(\overline{k}),Y(\overline{k}),\overline{\rho},X(k),Y(k),k\in\mathbb{Z}_{[0,N-1]}\setminus\{\overline{k}\}} -\mathrm{trace}(X(0)), \tag{26}$$

subject to (25). Alternative optimization criteria that describe the size of $\mathbb{E}(0)$ can be chosen as well (see e.g. Boyd et al. (1994)).

Moreover, the quantities $\prod_{l=0}^{N-1} \rho(l)$, where $\rho(k)$, $k \in \mathbb{N}_{[0,N-1]}$, obtained from Theorem 4, and $\bar{\rho}$, obtained from Problem 2, represent the exponential decrease of the corresponding periodic Lyapunov functions at each period, and consequently the speed of convergence of the closed–loop system trajectories. Thus, the proposed method offers the possibility to embed performance specifications in the synthesis procedure. In order to achieve a desired decrease $\hat{\rho} \in \mathbb{R}_{[0,1)}$ at each period for the closed–loop system, it is sufficient to replace (25d) with $0 \le \overline{\rho} \le \hat{\rho}$. Similarly, in Theorem 4, relation (17b) can be replaced by $0 \le \sum_{l=0}^{N-1} \rho(l) \le N \sqrt[N]{\hat{\rho}}$.

An additional relevant problem is computation of a PPI sequence of sets $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[0,N-1]}}$ that includes a given set of initial conditions $\mathcal{X}_0 \subseteq \mathbb{X}(0)$. To this end, Problem 2 can be modified such that the first element $\mathbb{E}(0)$ of the PPI sequence includes $\mathcal{X}_0$. First, consider a polytopic set $\mathcal{X}$ which is described as the convex hull of a finite set of vertices $v_i \in \mathbb{R}^n, i \in \mathbb{Z}_{[1,q]}$. i.e., $\mathcal{X} := \mathrm{convhull}(\{v_i\}_{i\in\mathbb{Z}_{[1,p]}})$.

Lemma 4 *Let* $E \in \mathbb{S}^n_{++}$ *and* $v_i \in \mathbb{R}^n$*, for all* $i \in \mathbb{Z}_{[1,q]}$ *with* $q \in \mathbb{Z}_{\ge 1}$*. The polytope* $\mathcal{X} := \mathrm{convhull}(\{v_i\}_{i\in\mathbb{Z}_{[1,p]}})$ *is contained in the ellipsoid* $\mathcal{E} := \{x \in \mathbb{R}^n : x^\top E x \le 1\}$ *if and only if* $v_i^\top E v_i \le 1$, $\forall i \in \mathbb{Z}_{[1,q]}$.

Then, a stabilizing linear periodic state–feedback control law (6) and a PPI sequence of sets $\{\mathbb{E}(k)\}_{k\in\mathbb{Z}_{[1,N-1]}}$ such that $\mathcal{X}_0 \subseteq \mathbb{E}(0)$ can be computed from the solution of Problem 2 having the additional constraint

$$\begin{bmatrix} 1 & \star \\ v_i & X(0) \end{bmatrix} \succeq 0, \quad \forall i \in \mathbb{Z}_{[0,q]}. \tag{27}$$

5 Satellite attitude control

A detailed comparison of the established results with the synthesis method that corresponds to the PLL Bittanti and Colaneri (2009) was provided in Athanasopoulos et al. (2013) for an illustrative, two dimensional academic example. In order to also illustrate the applicability of the results to challenging real–life control problems, we consider the problem (see for example, Lovera and Astolfi (2004), Böhm (2011) and the references therein) of attitude control of a low Earth orbit satellite via magnetic actuators. The linearized continuous–time attitude dynamics of the satellite can be described Böhm (2011) by the following time–varying differential equation

$$\dot{x}(t) = A_c x(t) + B_c(t) u(t), \tag{28}$$

where [2] $A_c \in \mathbb{R}^{6\times 6}$, $[A_c]_{13} = (1 - s_1)\,\omega_0$, $[A_c]_{14} = -4 s_1 \omega_0^2$, $[A_c]_{25} = -3\omega_0 s_2$, $[A_c]_{13} = -(1 + s_3)\,\omega_0$, $[A_c]_{16} = s_3 \omega_0^2$, $[A_c]_{41} = [A_c]_{52} = [A_c]_{63} = 1$ and $[A_c]_{ij} = 0$ for all the other index pairs (i, j). Furthermore, $B_c(t) \in \mathbb{R}^{3\times 6}$, $[B_c(t)]_{12} = -\frac{b_3(t)}{J_2}$, $[B_c(t)]_{13} = \frac{b_2(t)}{J_3}$, $[B_c(t)]_{21} = \frac{b_3(t)}{J_1}$, $[B_c(t)]_{23} = -\frac{b_1(t)}{J_3}$, $[B_c(t)]_{31} = -\frac{b_2(t)}{J_1}$, $[B_c(t)]_{32} = \frac{b_1(t)}{J_2}$ and $[B_c(t)]_{ij} = 0$ for all the other index pairs (i, j). In detail, $s_1 := \frac{J_2 - J_3}{J_1}$, $\quad s_2 := \frac{J_3 - J_1}{J_2}$, $\quad s_3 := \frac{J_1 - J_3}{J_3}$, where the constants J_i, $i \in \mathbb{Z}_{[1,3]}$ are the moments of inertia, with values $J_1 = 1250\text{kgm}^2$, $J_2 = 2800\text{kgm}^2$, $J_3 = 2600\text{kgm}^2$. The input matrix $B_c(t)$ depends on the components $b_i(t)$, $i \in \mathbb{Z}_{[1,3]}$, of the Earth magnetic field, which are approximated by the trigonometric functions

$$b_i(t) = \alpha_i \cos(\omega_0 t) + \beta_i \sin(\omega_0 t) + \gamma_i, \quad i \in \mathbb{Z}_{[1,3]}.$$

The satellite follows a circular orbit with altitude of 600km and inclination angle of 77^o. A full rotation around the Earth requires $T_0 = 96.7\text{min}$, resulting in the frequency of $\omega_0 = 0.0649\frac{\text{rad}}{\text{min}}$. The relevant coefficients that describe sufficiently the components of the Earth magnetic field are $\alpha_1 = 2.2365 \cdot 10^{-5}$, $\alpha_2 = -8.2537 \cdot 10^{-8}$, $\alpha_3 = 7.7377 \cdot 10^{-6}$, $\beta_1 = -3.9411 \cdot 10^{-6}$, $\beta_2 = -3.8422 \cdot 10^{-7}$, $\beta_3 = 4.4820 \cdot 10^{-5}$, $\gamma_1 = -2.8863 \cdot 10^{-8}$, $\gamma_2 = -4.5491 \cdot 10^{-6}$, $\gamma_3 = -1.4166 \cdot 10^{-7}$. Thus, the input matrix $B_c(t)$ is periodic with period T_0 such that $B_c(t + T_0) = B_c(t)$, for all $t \in \mathbb{R}_+$. The state vector $x \in \mathbb{R}^6$ consists of the three angular rates ω_i, $i \in \mathbb{Z}_{[1,3]}$ and the three angles of the pointing error ϕ_i, $i \in \mathbb{Z}_{[1,3]}$ with respect to each principal axis (referred to as roll, pitch, and yaw angle), i.e.,

$$x = [\omega_1 \quad \omega_2 \quad \omega_3 \quad \phi_1 \quad \phi_2 \quad \phi_3]^\top.$$

The input vector $u \in \mathbb{R}^3$ consists of the magnetic dipole moments which are induced by three coils placed along the axes of the satellite. Each input is subject to hard constraints $-400\text{Am}^2 \leq u_i(t) \leq 400\text{Am}^2$, $i \in \mathbb{Z}_{[1,3]}$, which can be written in the form (9) with $d_1 = [1 \quad 0 \quad 0]$, $d_2 = [0 \quad 1 \quad 0]$, $d_3 = [0 \quad 0 \quad 1]$, $d_4 = -d_1$, $d_5 = -d_2$, $d_6 = -d_3$. The continuous–time system (28) is discretized with $N = 10$ discretization steps at each period, i.e., with a sampling rate $\delta := \frac{T_0}{N} = 9.67\text{min}$. Consequently, the discretized non–autonomous system is of the form (5) with system matrices $A(k) = e^{A_c \delta} \quad k \in \mathbb{Z}_+$, $B(k) = \int_{k\delta}^{(k+1)\delta} \mathrm{e}^{B((k+1)\delta - \tau)} B_c(\tau)\,\mathrm{d}\tau, \quad k \in \mathbb{Z}_{[0,9]}$. The

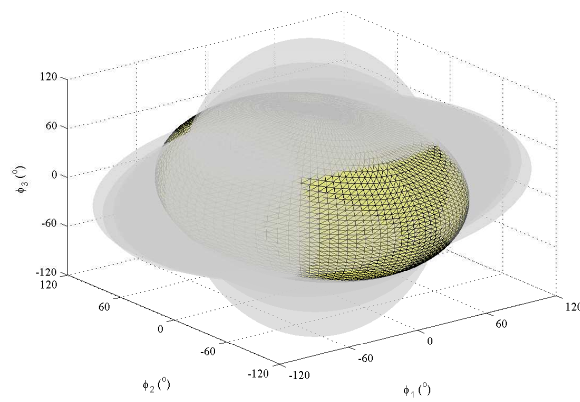

Fig. 1. The elements $\mathbb{E}_s(s)$, $s \in \mathbb{Z}_{[0,9]}$ of the resulting PPI sequences, for $\omega_i = 0$, $i \in \mathbb{Z}_{[1,3]}$. The set $\mathbb{E}_0(0)$ is depicted in yellow color, while sets $\mathbb{E}_s(s)$, $s \in \mathbb{Z}_{[0,9]} \setminus \{0\}$ are depicted in grey.

control problem consists of computing a stabilizing state feedback control law and an estimation of the region of attraction of the closed–loop system. The desired decrease rate at each period of the closed–loop system is set to $\hat{\rho} := 0.4^{10}$. For the problem under study, each mode $s \in \mathbb{Z}_{[0,9]}$ of the periodic system, i.e., $x(k+1) = Ax(k) + B(s)u(k)$, describes the dynamics of the satellite in an area of its orbit. Thus, it is relevant to assume that the initial condition can be applied to any mode of the periodic system (5), which corresponds to all instances of the orbit of the satellite. Furthermore, we consider a preassigned set $\mathcal{X}_0 \subset \mathbb{R}^6$ of initial conditions of interest, where $\mathcal{X}_0 := \{x \in \mathbb{R}^6 : x_i = 0, -60^o \leq x_j \leq 60^o, i \in \mathbb{Z}_{[1,3]}, j \in \mathbb{Z}_{[4:6]}\}$. Setting $\omega_i = 0$, $i \in \mathbb{Z}_{[1,3]}$ in the initial condition set $\mathcal{X}_0$ is a reasonable choice since the rotational energy of the satellite can be minimized using a rate damping controller Böhm (2011); Silani and Lovera (1998). The set $\mathcal{X}_0$ is a three dimensional cube in the subspace of the state variables x_j, $j \in \mathbb{Z}_{[4,6]}$ and can be equivalently written in the form $\mathcal{X}_0 = \text{convhull}(\{v_i\}_{i \in \mathbb{Z}_{[1,8]}})$, where $v_i \in \mathbb{R}^6$, $i \in \mathbb{Z}_{[1,8]}$. In order to meet the preassigned initial

[2] For a matrix $A \in \mathbb{R}^{n\times m}$, its element in the i-th row and j-th column is denoted by $[A]_{ij}$.

condition set specifications, different linear periodic control laws were computed to cover all the cases where the initial condition is applied. Moreover, in order to satisfy the performance requirements, Problem 2 was modified in order a decrease $\overline{\rho}$ of the Lyapunov function to be enforced in five out of the ten modes of the system, with index $\mathcal{I}_s$, resulting in $\overline{\rho} = 0.4^2$. Thus, the decrease rate in one period is $\hat{\rho} = (0.4^2)^5$. This yields the following optimization problems, which were solved for each $s \in \mathbb{Z}_{[0,9]}$:

$$\min_{X(\overline{k}),Y(\overline{k}),X(k),Y(k),k\in\mathbb{Z}_{[0,9]}\setminus\mathcal{I}_s} -\mathrm{trace}(X(s)) \tag{29}$$

subject to

$$\forall k \in \mathbb{Z}_{[0,9]} \setminus \mathcal{I}_s :$$

$$\begin{bmatrix} X(k) & \star \\ AX(k) + B(k)Y(k) & X(k+1) \end{bmatrix} \succeq 0, \tag{30a}$$

$$\forall \overline{k} \in \mathcal{I}_s :$$

$$\begin{bmatrix} \overline{\rho}X(\overline{k}) & \star \\ AX(\overline{k}) + B(\overline{k})Y(\overline{k}) & X(\overline{k}+1) \end{bmatrix} \succeq 0, \tag{30b}$$

$$\forall (l,j) \in \mathbb{Z}_{[0,9]} \times \mathbb{Z}_{[1,6]} :$$

$$\begin{bmatrix} 1 & \star \\ X(l)c_i(l)^\top & X(l) \end{bmatrix} \succeq 0, \tag{30c}$$

$$\begin{bmatrix} 1 & \star \\ Y(l)^\top d_j(l)^\top & X(l) \end{bmatrix} \succeq 0, \tag{30d}$$

$$\forall i \in \mathbb{Z}_{[1,8]} :$$

$$\begin{bmatrix} 1 & \star \\ v_i & X(s) \end{bmatrix} \succeq 0, \tag{30e}$$

with $X(N) := X(0)$. The optimization constraints (30b) enforce a decrease $\overline{\rho}$ of the Lyapunov function in all modes $\overline{k} \in \mathcal{I}_s$, while the optimization constraints (30e) guarantee that the initial condition set will be included in the region of attraction, i.e., $\mathcal{X}_0 \subset \mathbb{E}_s(s)$. All optimization problems were feasible for a choice of each index set $\mathcal{I}_s$, $s \in \mathbb{Z}_{[0,9]}$. The solution of problem (29),(30) resulted in ten stabilizing linear periodic state feedback control laws $u(k) = K_s(k)x(k)$, $s \in \mathbb{Z}_{[0,9]}$ and corresponding PPI sequences $\{\mathbb{E}_s(k)\}_{k\in\mathbb{Z}_{[0,9]}}$, $s \in \mathbb{Z}_{[0,9]}$.

In Figure 1, the elements $\mathbb{E}_s(s)$, $s \in \mathbb{Z}_{[0,9]}$ of the resulting PPI sequences are depicted for $\omega_i = 0$, $i \in \mathbb{Z}_{[1,3]}$. The solution of each optimization problem (29),(30) was computed in Matlab R2011b, using the YALMIP interface and the semidefinite quadratic programming solver SDPT3–4.0. The control law is implemented in the following fashion. First, the mode $s \in \mathbb{Z}_{[0,9]}$ of the periodic system where the initial condition lies is identified. Second, the control strategy $u(k) = K_s x(k)$ is applied, for all $k \in \mathbb{Z}_+$. The continuous–time closed–loop system was simulated in Matlab R20011b, for the initial condition $x_0 = [0 \;\; 0 \;\; 0 \;\; 60^o \;\; -60^o \;\; -60^o]^\top$ and mode $s = 0$. In Figure 2, the state response of the state variables ϕ_i, $i \in \mathbb{Z}_{[1,3]}$ is shown.

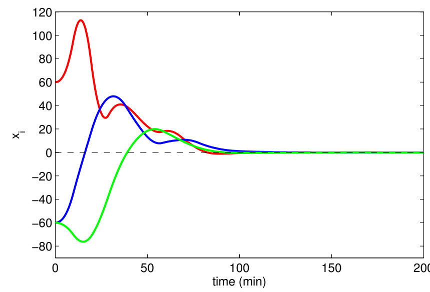

Fig. 2. State trajectories for ϕ_i, $i \in \mathbb{Z}_{[1,3]}$ (ϕ_1 -red, ϕ_2 -blue, ϕ_3 -green) and initial condition $x_0 = [0 \;\; 0 \;\; 0 \;\; 60^o \;\; -60^o \;\; -60^o]^\top$ applied in mode $s = 0$.

Remark 6 *For the considered problem setting, a comparison with relevant methods was made. To this end, application of the PLL synthesis method (Lemma 1) did not result in a feasible solution, due to the preassigned initial condition set specification. On the other hand, modification of the stability analysis method in Böhm et al. (2012) to synthesis (Lemma 2) did not return a solution. In contrast, the computed region of attraction by the developed method spans a range of [-60, 60] in all three angles of the pointing error, which is a significant range, while using only* 10 *feedback gain matrices. Last, comparing with predictive control approaches, the explicit model predictive control solution of (Böhm, 2011, Section 5.4), which employs* 30 *feedback gain matrices, reports a feasible solution for the initial condition* $x_0 = [0 \;\; 0 \;\; 0 \;\; 30^o \;\; 30^o \;\; 30^o]^\top$.

6 Conclusions

An alternative stability analysis theorem for nonlinear periodic discrete–time systems was presented. In addition, the derived theorem was used to devise a tractable stabilizing controller synthesis method for linear periodic discrete–time systems subject to polytopic state and input constraints. The application of the derived method to satellite attitude control resulted in a large region of attraction.

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