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Validating the J-value safety assessment tool against pan-national data

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ABSTRACT

The J-value is an objective method for determining when life extending measures are sensible, applicable to both manufacturing and service industries, including public health and healthcare. A model of human decision making based on the J-value is able to explain the shape of the Preston curve that relates life expectancy at birth and gross domestic product (GDP) per head for all the nations in the world. Making a number of reasonable assumptions, a J-value model produces a population-average life expectancy, which may be translated easily into a corresponding life expectancy at birth when life expectancy is not modified by discounting (net discount rate equals zero). The resultant values may be tested against pan-national data, showing a very good match. Thus the shape of the Preston curve has been explained and, at the same time, validation has been provided for the J-value model. A perturbation analysis shows that the J-value explanation for the Preston curve starts to break down as the net discount rate is increased above zero. Thus the Preston curve may be seen to validate the J-value model at a net discount rate of zero, but not at higher net discount rates. The result allows a closed-form expression to be derived for the first time for the pure time discount rate, namely the product of the rate of economic growth and the complement of risk-aversion. A further conclusion from the work is that no discernable limit is apparent before the age of 100 to the process by which people live longer as they get richer; such an intrinsic limit might be overcome by future improved medical technology.

1. Introduction

The J-value (Thomas et al., 2006a,b; 2010a) is an objective method for determining when life extending measures are sensible, applicable to both manufacturing and service industries. Based on the life-quality index (Nathwani and Lind, 1997; Nathwani et al., 2009), the method assumes that a rational trade-off is made between an increase in life expectancy and the cost of the measure that brings about that increase, with the ultimate objective of maintaining or improving life quality.

The J-value has the considerable advantage over conventional cost benefit analysis that no explicit assumptions have to be made about the difficult issue of the monetary value to be attached to saving a human life. Instead of using subjective stated preferences of a small sample of the population exposed to a hazard, it is instead grounded in objective actuarial and economic statistics characterising the lives and behaviours of millions of citizens. It is thus suitable for assessing health and safety measures across all industries, from oil and gas, chemical and nuclear through transport to the National Health Service in the UK. Also, unlike other approaches, the J-value allows immediate fatalities and loss of life in the longer term (e.g. after exposure to a carcinogen) to be measured on the same scale. It is thus particularly suitable for judging nuclear safety, including assessing mitigation strategies following large accidents like Chernobyl and Fukushima Daiichi.
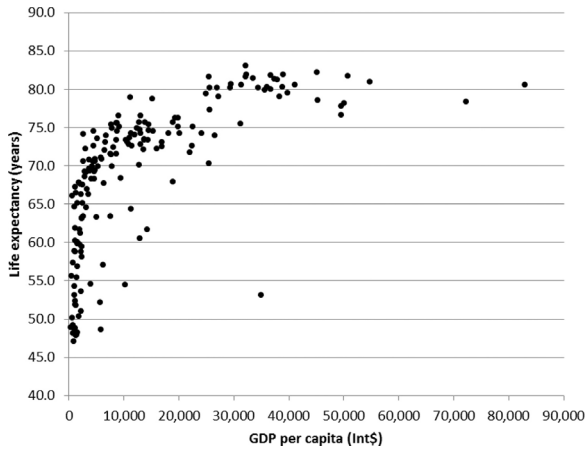
Fig. 1 – The Preston curve—life-expectancy at birth (years) as a function of GDP per capita (international dollars) for the 180 countries for which both datasets are available for 2009 (OECD, 2011; World Bank, 2012a,b).

An ethical principle of J-value analysis is that the next day of life should be valued the same for everyone in the nation, old or young, rich or poor. This principle is reflected in the use of the gross domestic product (GDP) per head as the baseline annual income used in the definition of life quality.

The Preston curve (Preston, 1975) highlights the fact that there is a clear, positive correlation between the GDP per head in different countries in the world and the average length of time people in those countries can expect to live. See Fig. 1, which shows results from 180 out of the 193 nations affiliated to the United Nations. To the authors' knowledge, no successful, theoretical explanation has been put forward for the shape of the Preston curve up to this point.

The paper will suggest how the actions of the peoples in different nations in the world can be characterised by a life-extension assessment procedure based on J-value principles, and the resultant mathematical model will be used to explain the form of the Preston curve. The process of explanation can also be seen as a test of the J-value model, and, indeed a severe test, since the model is required to match data derived from the decisions of almost everyone in the world when grouped together into nations. Hence passing this test will provide a substantial degree of validation (Popper, 1934; Butterfield and Thomas, 1986a,b; Thomas, 1999) for the J-value method. This validation exercise is additional to and complementary to the corroboration reported in Thomas (2017a,b), where the J-value model was tested against UK data on life expectancy at birth.

2. The J-value

The J-value is derived from the life quality index (LQI), Q, (Nathwani and Lind, 1997; Pandey et al., 2006; Nathwani et al., 2009; Thomas et al. 2006a, 2010a):

$$Q = G^{1-\varepsilon} X_d \tag{1}$$

where G is the income per person, taken for ethical reasons to be the GDP per head (£/year) and thus the same for everyone in the same national jurisdiction, while ε is the risk-aversion associated with measures that will extend life expectancy, estimated previously at between 0.82 and 0.85 for the UK (Thomas et al., 2010a,b). See also Blundell et al. (1994) who suggest a figure of 0.83 using a diverse method and Pearce and Ulph (1995), who suggest a range 0.8–0.9. Meanwhile X_d is the discounted life expectancy of the population as a whole (years).

It may be seen from Eq. (45) of Thomas et al. (2010a), that discounting is actually applied to the utility of income in

future years, but the fact that the necessary integrand consists of the product of discount factor, utility and survival probability means that it is convenient to define the integral of {survival probability × discount factor} as the "discounted life expectancy". The choice of title reflects the fact that the discounted life expectancy, X_d, will degenerate to the life expectancy, X, when the net discount rate is zero. The net discount rate, r, is given generally by

$$r = \lambda + g\varepsilon - g = \lambda - g(1 - \varepsilon) \tag{2}$$

where λ is the pure time preference rate and g is the growth rate of the economy, both of which might differ between different nations (Thomas, 2012).

A condition for a life-extending measure to be rationally justified is that the life quality index should not fall as a result of a person spending an annual amount, δG, from his income on the health and safety measure for the rest of his expected life, causing a decrement, $-\delta G$ in his annual income. In line with the Kaldor–Hicks compensation principle (Kaldor, 1939; Hicks, 1939), while the individual would be prepared to spend such an amount, the annual payment might actually be made (and in many cases will be made) by some other person or body. Assuming a constant value of net discount rate, r, the change in LQI due to small changes in income and discounted life expectancy, δX_d, is

$$
\begin{aligned}
\delta Q &= -\frac{\partial Q}{\partial G}\delta G + \frac{\partial Q}{\partial X_d}\delta X_d \\
&= -(1-\varepsilon)G^{-\varepsilon}X_d\delta G + G^{1-\varepsilon}\delta X_d
\end{aligned}
\tag{3}
$$

Dividing by Eq. (1), we find

$$\frac{\delta Q}{Q} = -(1-\varepsilon)\frac{\delta G}{G} + \frac{\delta X_d}{X_d} \tag{4}$$

The maximum rational annual expenditure on life extension will occur when $\delta Q = 0$ and, for non-zero Q, this will occur when

$$\frac{\delta X_d}{X_d} = (1-\varepsilon)\frac{\delta G}{G} \tag{5}$$

which thus defines δG. If the actual annual expenditure is $\delta\hat{G}$, then the J-value is given by:

$$J = \frac{\delta\hat{G}}{\delta G} \tag{6}$$

Thus $J = 1$ defines the locus of a curve in the plane of G versus X_d where life quality, Q, is maintained constant. See Fig. 2.

3. Life expectancy and discounted life expectancy

The life expectancy at age a is given by the integral of the conditional survival probabilities:

$$X(a) = \int_{t=a}^{\infty} S(t|a)\,dt == \int_{t=a}^{\infty} \frac{S(t)}{S(a)}\,dt \tag{7}$$

where $S(t)$ is the cumulative probability of survival to age, t, from age, 0. It may be helpful, by way of example, to observe that the life expectancy at birth, $X(0)$, averaged across the two
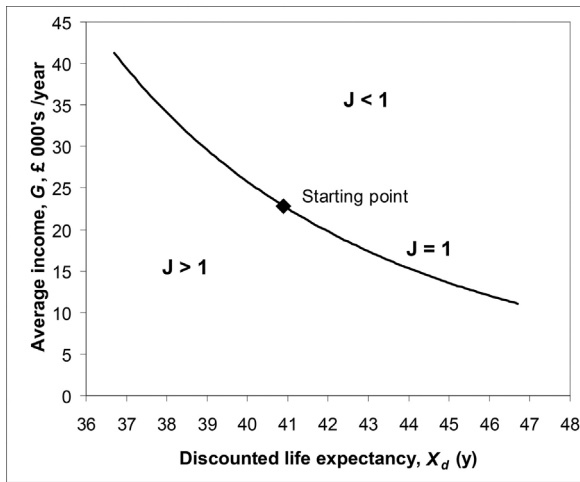
Fig. 2 – J = 1 defines the locus of the line in the plane of G vs. X_d that maintains the life quality index constant.

genders is currently about 81 years in the UK, while the gender-averaged life expectancy of a 60 year old, $X(60)$, is about 24 years.

A further useful expression (e.g. Thomas et al., 2006c) for a steady-state population relates the probability density for age, t, $p(t)$, to the survival probability to age, t, $S(t)$, and the life expectancy at birth, $X(0)$:

$$p(t) = \frac{S(t)}{X(0)} \tag{8}$$

Meanwhile, the discounted life expectancy at age, a, is found by inserting the discounting term, $e^{-r(t-a)}$, into the integral of Eq. (7):

$$
\begin{aligned}
X_d(a) &= \int_{t=a}^{\infty} S(t|a)\, e^{-r(t-a)} dt = \int_{t=a}^{\infty} \frac{S(t)}{S(a)} e^{-r(t-a)} dt \\
&= \int_{t=a}^{\infty} \frac{p(t)}{p(a)} e^{-r(t-a)} dt
\end{aligned}
\tag{9}
$$

The discounted life expectancy for the population as a whole follows from integrating over all possible ages:

$$X_d = \int_{a=0}^{\infty} p(a)\, X_d(a)\, da = \int_{a=0}^{\infty}\int_{t=a}^{\infty} p(t)\, e^{-rt} e^{ra}\, dt\, da \tag{10}$$

Changing the order of integration:

$$
\begin{aligned}
X_d &= \int_{t=0}^{\infty}\int_{a=0}^{t} p(t)\, e^{-rt} e^{ra}\, da\, dt \\
&= \int_{t=0}^{\infty} p(t)\, e^{-rt} \int_{a=0}^{t} e^{ra}\, da\, dt
\end{aligned}
\tag{11}
$$

Now

$$\int_{a=0}^{t} e^{ra}\, da = \begin{cases} \dfrac{1}{r}[e^{ra}]_{a=0}^{a=t} = \dfrac{1}{r}\left(e^{rt}-1\right) & \text{for } r > 0 \\[2mm] t & \text{for } r = 0 \end{cases} \tag{12}$$

Thus,

$$X_d = \begin{cases} \dfrac{1}{r}\displaystyle\int_{t=0}^{\infty} p(t)\, e^{-rt}\left(e^{rt}-1\right) dt = \dfrac{1}{r}\displaystyle\int_{t=0}^{\infty} p(t)\left(1 - e^{-rt}\right) dt & \text{for } r > 0 \\[4mm] \displaystyle\int_{t=0}^{\infty} p(t)\, t\, dt = E(T) = X & \text{for } r = 0 \end{cases} \tag{13}$$

The second line gives the result that if a person is drawn at random from a steady-state population and if the net discount rate is zero, the discounted life expectancy is the expected value $E(T)$ of this individual's random age, T. Moreover, since $X_d|_{r=0} = X$ by definition, this will be equal to the average life expectancy, X, in the population.

Meanwhile the discounted life expectancy at birth is found by putting $a = 0$ into Eq. (9) and noting that $S(0) = 1$ for live births:

$$X_d(0) = \int_{t=0}^{\infty} S(t)\, e^{-rt} dt = X(0) \int_{t=0}^{\infty} p(t)\, e^{-rt} dt \tag{14}$$

Hence

$$\int_{t=0}^{\infty} p(t)\, e^{-rt} dt = \frac{X_d(0)}{X(0)} \tag{15}$$

Substituting into Equ. (13) relates the population average discounted life expectancy, X_d, to both the net discount rate, r, and the ratio of discounted, $X_d(0)$, to undiscounted life expectancy, $X(0)$, at birth:

$$
\begin{aligned}
X_d &= \int_{t=0}^{\infty} p(t)\, \frac{1 - e^{-rt}}{r} dt = \frac{1}{r}\left(\int_{t=0}^{\infty} p(t)\, dt - \int_{t=0}^{\infty} p(t)\, e^{-rt} dt\right) \quad \text{for } r > 0 \\
&= \frac{1}{r}\left(1 - \frac{X_d(0)}{X(0)}\right)
\end{aligned}
\tag{16}
$$

A problem with using Eq. (16) to calculate discounted life expectancy, X_d, is that it requires a knowledge of discounted life expectancy at birth, $X_d(0)$. An alternative formulation takes Eq. (13) as its starting point, as is explained in the next section.

4. Relating age-averaged discounted life expectancy, X_d, to life expectancy at birth, $X(0)$

The series expansion for a negative exponential is:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots = 1 - \sum_{i=1}^{n} \frac{(-1)^{n-1}}{n!} x^n \quad \text{as } n \to \infty \text{ for all finite } x \tag{17}$$

This may be applied to Eq. (13) for the case where $r > 0$:

$$X_d = \frac{1}{r} \int\limits_{t=0}^{\infty} p(t) \left(1 - \left(1 - \sum_{i=1}^{n} \frac{(-1)^{n-1}}{n!} r^n t^n \right) \right) dt$$

$$= \frac{1}{r} \left(r \int\limits_{t=0}^{\infty} p(t) \, t \, dt - \frac{r^2}{2!} \int\limits_{t=0}^{\infty} p(t) \, t^2 \, dt + \frac{r^3}{3!} \int\limits_{t=0}^{\infty} p(t) \, t^3 \, dt \ldots \quad \text{as} \, n \to \infty \right.$$

$$\left. + \frac{(-r)^{n-1}}{n!} \int\limits_{t=0}^{\infty} p(t) \, t^n \, dt \right) \tag{18}$$

But

$$\int\limits_{t=0}^{\infty} p(t) \, t \, dt = E(T) \tag{19}$$

where $E(T)$ is the 1st moment of age or mean age of the population. Moreover, from Eq. (13) it is also the population average life expectancy for a steady-state population: $E(T) = X$. Similarly,

$$\int\limits_{t=0}^{\infty} p(t) \, t^2 \, dt = E\left(T^2\right)$$

$$\int\limits_{t=0}^{\infty} p(t) \, t^3 \, dt = E\left(T^3\right) \tag{20}$$

$$\int\limits_{t=0}^{\infty} p(t) \, t^4 \, dt = E\left(T^4\right)$$

$$\vdots$$

where $E(T^n)$ is the nth moment of age about the origin. Therefore:

$$X_d = \frac{1}{r} \left(r E(T) - \frac{r^2}{2!} E\left(T^2\right) + \frac{r^3}{3!} E\left(T^3\right) - \frac{r^4}{4!} E\left(T^4\right) \ldots \right)$$

$$= E(T) - \frac{r}{2!} E\left(T^2\right) + \frac{r^2}{3!} E\left(T^3\right) - \frac{r^3}{4!} E\left(T^4\right) + \ldots \quad \text{as} \, n \to \infty$$

$$+ \frac{(-r)^{n-1}}{n!} E(T^n) \tag{21}$$

We may proceed further by defining the moment ratio, $\rho(n)$, as the ratio to the mean of the nth root of the nth moment of age about the origin:

$$\rho(n) = \frac{(E(T^n))^{\frac{1}{n}}}{E(T)} \tag{22}$$

Starting from $\rho(1) = 1$, $\rho(n)$ is a monotonically increasing function of power, n. See Fig. 3.

The mean age of the population will be a proper fraction, b, of the life expectancy at birth:

$$E(T) = X = bX(0) \tag{23}$$

where b will depend on the GDP per head, $b = b(G)$ (Appendix C derives an approximate relationship). Eq. (22) then becomes,

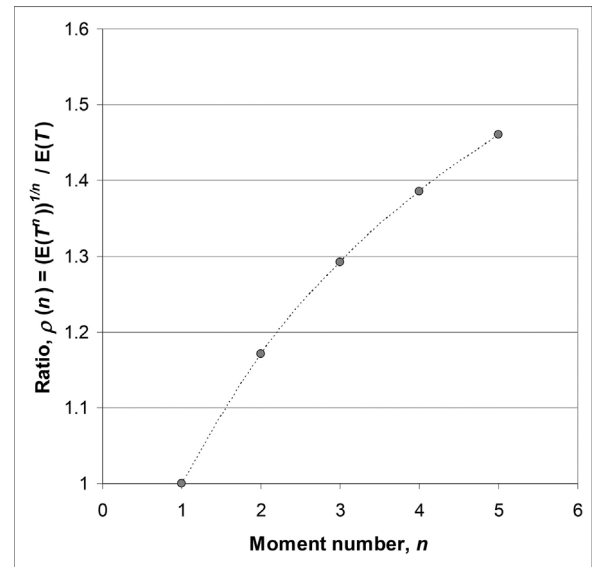$$E(T^n) = \rho^n(n) X^n = (b\rho(n))^n X^n(0) \tag{24}$$

Fig. 3 – Ratio of the nth root of the expected value of the nth moment of age about the origin to the mean age.

Substituting from Eq. (24) into Eq. (21) then gives,

$$X_d = b\rho(1) X(0) - \frac{r}{2!} (b\rho(2))^2 X^2(0) + \frac{r^2}{3!} (b\rho(3))^3 X^3(0)$$

$$- \frac{r^3}{4!} (b\rho(4))^4 X^4(0) + \frac{r^4}{5!} (b\rho(5))^5 X^5(0) + \ldots \quad \text{as} \, n \to \infty \tag{25}$$

$$+ \frac{(-r)^{n-1}}{n!} (b\rho(n))^n X^n(0)$$

Eq. (25), in principle exact as $n \to \infty$, illustrates how population discounted life expectancy, X_d, is related analytically to life expectancy at birth, $X(0)$, with net discount rate, r, as parameter.

5.   The life-expectancy ratio, b, and the moment ratio, $\rho(n)$

An expression for life-expectancy ratio, b, is derived in Appendix C, where, based on World Bank data from 2008 and the piece-wise linear models of Appendix A, the b-value is shown to start at ⅔ for undeveloped countries and decrease towards ½ for highly developed countries. The progression is shown to be represented well by a first-order lag with GDP per head as the base parameter, Eq. (C1), repeated below:

$$b = b_s + (b_0 - b_s) e^{-\frac{G}{G_T}} \tag{C1}$$

where $b_0 = 0.667$, $b_s = 0.52$ and $G_T = 4295$ Int\$ per year. An important feature of this model (matched by the data) is that there is only a small variation in b once GDP per head has reached 12,900 Int\$ per year.

The life-expectancy ratio, b, for the UK for 2009 (ONS, 2016) may be calculated as $79.76/41.07 = 0.515$. Applying Eq. (A28) from Appendix A allows the value of the parameter, k, used in the piecewise-linear survival function to be determined as $k = 0.299$. The estimated nth moment of age about the origin, $E(T^n)^*$, follows from Eq. (A25), repeated in abbreviated form below:

$$E(T^n)^* = \frac{(1+k)^{n+2} - (1-k)^{n+2}}{2k(n+1)(n+2)} X^n(0) \tag{26}$$

Applying Eq. (22) then gives the estimated moment ratio, $\rho^*(n)$. Table 1 compares the moment ratios calculated by Eq.

Table 1 – Moment ratios, $\rho(n)$, $n = 1$–5 for UK 2009.

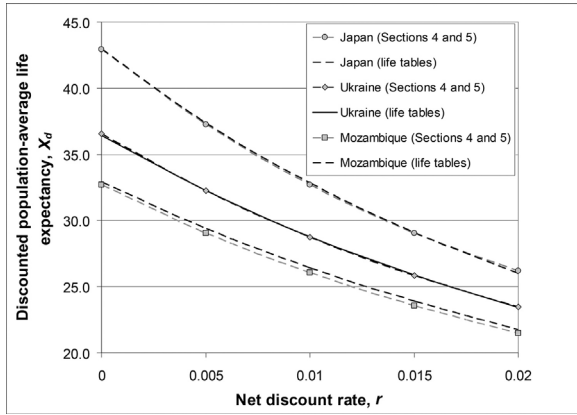| n	$\rho^*(n)$ using Eq. (27)	$\rho(n)$ from life-tables	Correction factor, $\alpha(n)$
1	1	1	1
2	1.170354	1.165885	0.996181
3	1.293044	1.282562	0.991894
4	1.388475	1.371293	0.987625
5	1.466211	1.442172	0.983604

Fig. 4 – Discounted population-average life expectancy: comparison of methods of calculation for Mozambique, Ukraine and Japan.
(It is difficult to distinguish the two curves on this scale for Japan and almost impossible for Ukraine. Meanwhile the curves lie very close together for Mozambique).

(26) with those calculated using the UK 2009 life tables and the CLEARE computer program.

A correction factor, $\alpha(n)$, close to unity, may be applied to bring them exactly into line and it will be assumed henceforth that,

$$\rho(n) \approx \alpha(n)\,\rho^*(n) \tag{27}$$

will be applicable across all nations.

Fig. 4 compares the discounted, population-average life expectancies calculated using the methods of Sections 4 and 5 with a more accurate method that uses the full life-table data. It is clear that the 5th-order polynomial gives a good representation of reality over the full range of nations, from Mozambique, which had a life expectancy at birth of 51 years in 2008 to Japan, with a life-expectancy at birth of 82.6 years.

6. Application of the J-value across different nations

Here it is supposed that a typical nation in the world will attempt to improve the quality of life of each of its citizens by taking a rationally balanced view of measures to increase the population-average life expectancy. Specifically, it is assumed that in all nations:

(i) People will decide to spend the same fraction, a, of their national income per head on life-extending measures (which will include not only medical services but also, for example, the supply of plentiful and clean drinking water, the installation and maintenance of effective sewage disposal systems and the regulation of safe transport systems). The sum spent per person is then,

$$y = aG \tag{28}$$

The value of a will lie in the interval $0 < a \le 1.0$, but it is not necessary to specify the fraction, a, further for the purposes of this paper other than to say that it is likely to lie at the lower end of its range;

(ii) Within the budget defined by Eq. (28), the average person in the nation will spend on health and safety measures resulting in life extension an overall amount such that a J-value of unity will result, implying that this spending will just maintain the life quality index: $\delta Q = 0$;

(iii) The value of risk-aversion, ε, applicable when decisions on life extension are being made, will remain constant as wealth and life expectancy increase in tandem and will be the same for all nations in the world;

(iv) The net discount rate, r, applied to life expectancy will remain constant as wealth and life expectancy increase in tandem.

Health care spending across the world has been examined in Thomas (2017a), who finds that, while the world-average spend as a fraction of per capital GDP has remained a broadly constant fraction, 10%, of GDP per head for the past 10 years (World Bank, 2012a,b,c), there are significant differences between countries. Thomas (2017a) shows that the health spending per head in less developed countries remains a roughly constant fraction of GDP per head, but that the more developed nations are steadily increasing the fraction of GDP per head devoted to health care. This is consistent with the J-value assumption of overall expenditure per head on health and safety being proportional to GDP per head under the following argument. Having installed many of the public health measures that yield very large improvements in life expectancy, the more developed countries need to turn their attention increasingly to individual health care provision and hence, within the overall envelope of health and safety spending, to spend a growing fraction of GDP per head on individual health care. By contrast it makes sense for less developed countries to devote a higher fraction of their resources to improving public health and infrastructure (for example the supply of clean drinking water and the introduction of effective sewage disposal systems), and, by the same token, a lower fraction of GDP per head on individual health care. Such health care spending can be expected to remain a constant fraction of GDP per head as long the opportunities for extending life expectancy elsewhere are good enough to remove the incentive to expand health care provision.

In view of the many and various demands on resources, health and safety measures will take up a relatively low proportion of the average person's income. Moreover, the average life extension generated is likely to be a small fraction of the average person's current life expectancy. Hence, Eq. (5) may be written in differential form, thus giving the rate of change of life expectancy with GDP per head:

$$\frac{dX_d}{dG} = (1 - \varepsilon)\,\frac{X_d}{G} \tag{29}$$

Differentiating Eq. (28) gives,

$$\frac{dy}{dG} = a = \frac{y}{G} \tag{30}$$

Meanwhile another expression for dX_d/dG may be found from the formal differentiation:

$$\frac{dX_d}{dG} = \frac{dX_d}{dy}\frac{dy}{dG}$$
$$= \frac{dX_d}{dy}\frac{y}{G} \tag{31}$$

where Eq. (30) has been used in the second step.

Substituting from Eq. (31) into Eq. (29) gives:

$$\frac{dX_d}{dy}\frac{y}{G} = (1-\varepsilon)\frac{X_d}{G} \tag{32}$$

and so:

$$\frac{dX_d}{dy} = (1-\varepsilon)\frac{X_d}{y} \tag{33}$$

Now consider nation A, where the average individual has an income (GDP per head), G_A, a corresponding health and safety spend, $y_A = aG_A$, and a discounted life expectancy of X_{dA}. If the average wealth, and so the amount that can be afforded for health and safety, increases, then the basic health and safety measures that initially cost y_A will be augmented so that the total cost gradually increases to a new value, y. Hence we may integrate Eq. (33) according to,

$$\int_{X_{dA}}^{X_d} \frac{1}{X'}dX' = (1-\varepsilon)\int_{y_A}^{y}\frac{1}{y'}dy' \tag{34}$$

This gives

$$\ln\frac{X_d}{X_{dA}} = (1-\varepsilon)\ln\frac{y}{y_A} \tag{35}$$

or

$$X_d = X_{dA}\left(\frac{y}{y_A}\right)^{1-\varepsilon} \tag{36}$$

Furthermore, based on Eq. (28), we can write:

$$\frac{G}{G_A} = \frac{y}{y_A} \tag{37}$$

Hence

$$\frac{X_d}{X_{dA}} = \left(\frac{G}{G_A}\right)^{1-\varepsilon} \tag{38}$$

Thus the population-average life expectancy scales with GDP per head according to Eq. (38), parameterized by the risk-aversion, ε. Eq. (38) constitutes the "J-value model" of the growth in national life expectancy: as GDP per head rises from an initial value, G_A, to a higher figure, G, so population-average discounted life expectancy will rise from X_A to X.

Alternatively, if another country has a greater GDP per head, G_B, that country's discounted life expectancy is predicted to take the higher value, X_{dB}:

$$X_{dB} = X_{dA}\left(\frac{G_B}{G_A}\right)^{1-\varepsilon} \tag{39}$$

Hence we propose that Eq. (38) should apply across all nations, each of which will use the same value of risk-aversion, ε,
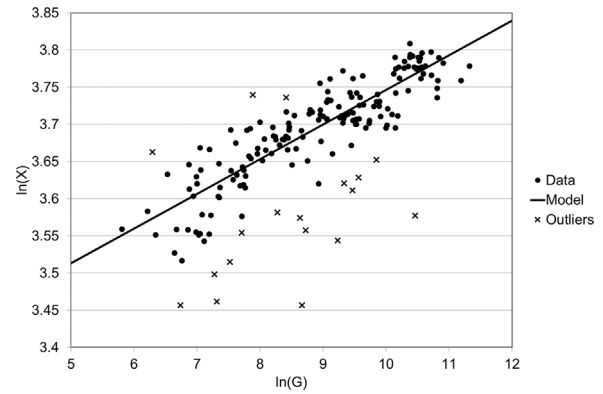

Fig. 5 – Ln X vs ln G for 180 nations in 2009 (the "Bristol curve"). 18 outliers (marked with crosses) were excluded from the final fit (OECD, 2011; World Health Organization, 2012).

and the same value of net discount rate, r (the uniformity of risk-aversion and of net discount rate across nations are mathematical constraints inherent in the derivation of Eq. (39)). This proposal incorporates the egalitarian assumption that people in all nations will act equivalently, given the same level of GDP per head.

By its nature, the model is constrained to be a steady-state description, as large changes in national income might require structural change before feeding through into changes in life expectancy. Hence one would expect its predictions to be best when a country's conditions are settled and to lose accuracy the further those conditions are from a steady state. Poor predictions could thus be expected if the nation's GDP per head has undergone a major change or if it is experiencing war or major unrest. For example, the discovery of significant mineral wealth might boost GDP per head very quickly but not be reflected in improved health and hence life expectancy for many years.

Eq. (38) may be written in the logarithmic form:

$$\ln X = (1-\varepsilon)\ln\left(\frac{G}{G_A}\right) + \ln X_A \tag{40}$$

Given that risk-aversion, ε, is constant in the model, Eq. (40) has the linear form $y = mx + c$. Once a reference nation, A, with GDP per head, G_A, and life expectancy, X_A, has been selected, it is possible to fit a regression line to find the slope and hence the risk-aversion, ε. It is found from the data of 180 countries out of 193 registered with the United Nations that $\varepsilon = 0.95$, with the square of the correlation coefficient, R^2, taking the value of 0.6. When the number of countries is reduced to 162, the risk-aversion stays the same, $\varepsilon = 0.953$ (with a 90% confidence interval of 0.950–0.956), but R^2 rises to 0.8. This is the line shown in Fig. 5, which explains 80% of the variation in the logarithm of population-average life expectancy of the 162 nations in terms of the logarithm of GDP per head. The best-match value of risk-aversion lies within 10% ± 5% of the values derived by diverse methods: 0.82 (Thomas et al., 2010a), 0.83 (Blundell et al., 1994), 0.85 (Thomas et al. 2010b) and 0.8–0.9 (Pearce and Ulph, 1995).

The 18 outliers marked in Fig. 5 are: Afghanistan, Angola, Botswana, Cameroon, Chad, Congo, Cote d'Ivoire, Equatorial Guinea, Eritrea, Gabon, Guinea-Bissau, Lesotho, Malawi, Namibia, Nicaragua, South Africa, Swaziland, Zambia. Arguments can be made for why many of these countries should

be regarded as being in an unsteady state, but further research is desirable on this topic.

7. Validation of the J-value model

The philosophy of model validation is discussed in Thomas (1999). See also Butterfield and Thomas (1986a,b). Suppose we have a mathematical model that makes a set of predictions; how much logical justification can we have for regarding these predictions as correct? The basis for an answer to this question was provided by the philosopher Sir Karl Popper in 1934, who was interested in the characteristics of a scientific theory. (We may note here that a mathematical model, perhaps embodied in a computer programme may be regarded as a scientific theory—and a very well documented theory at that.) Popper's conclusions (Popper, 1934) were that:

1. To be deemed scientific, a theory must be "falsifiable". That is to say, it must be empirically testable, in principle at least, and there must be some test that we can set for the theory in which an unfavourable outcome will prove the theory wrong; and
2. There can never be a rigorous logical justification for any scientific theory. The best we can do is to set empirical tests for the theory – fair tests, but the more severe the better – and continue to use the theory only so long as the theory passes all the tests.

A theory can never be proved by any of its successes, since a new test, perhaps as yet not thought of, may come along that it will fail. Failure in any fair test, on the other hand, indicates a fault in the theory: "falsifies" it. At this stage, effort will need to be spent in improving the theory or devising a completely new theory.

Following Popper's conclusions, we see that we will never be able to prove beyond doubt that any model is correct, a judgement that must be applied to many famous and generally applied theories, from Newton's Laws of Motion through Einstein's General Relativity to Quantum Theory, for example. Model validation does not and cannot prove the model or theory is correct. On the other hand, confidence in the model's use in practical situations is generated when that model passes a fair test, especially a test that is difficult as well as fair.

The fact that the J-value model produces a risk-aversion value close to previous, diverse estimates while generating a high value for the R^2 statistic provides a high degree of corroboration for the J-value model. Meanwhile predicting the socio-economic behaviour of most countries in the world in applying health and safety measures (180 nations at $R^2 = 60\%$ and 162 out of the 193 countries in the United Nations at $R^2 = 80\%$) can hardly be regarded as an easy problem. Hence it is argued that the J-value model has passed an important and difficult validation test. The J-value model may thus be regarded as validated in the sense of having been proved strong (L: *validus*) although, of course, not proven true, the latter being impossible for any scientific theory.

The R^2 value found for the logarithm of population-average life expectancy in terms of the logarithm of GDP per head for the 162 nations is high at 80%, which corresponds to a correlation coefficient of just under 90%. Nevertheless it allows for residual differences in the way in which different nations apply their resources to health and safety spending. As noted in the brief discussion on outliers, some nations may fulfil bet-

ter the requirement to have reached (or be close to) a steady state, perhaps much better than others. But there may also be other differences arising independently as a result of different cultures and different styles of government.

Some might suppose that assumption (ii) in Section 6 implied the conscious use of the J-value method in deciding the level of resources to devote to improving health and safety in any given case. This supposition might then be followed by an ethical objection to taking economic factors into account (as implied by the J-value method) when making decisions on life extension. But in fact no claim has been made that people have been performing J-value calculations in their heads; nevertheless the good match between the J-value model and the observations suggests that the world's nations are allocating societal resources to extending life as if they were using such a method. Regarding such socio-economic models, Bueno de Mesquita (2009) provides useful clarification on the intent of such mathematical descriptions:

> "Real people may not be able to do the cumbersome math that goes into a model, but that doesn't mean they aren't making much more complicated calculations in their heads even if they don't know how to represent their analytic thought processes mathematically."

He illustrates the point further by pointing out that no player could or would solve the plethora of differential equations on angle, velocity, position, wind-speed etc. needed to model the process of playing a normal return shot in tennis. Friedman (1953, 1966) makes a similar point with respect to billiards professionals, and also to whether or not businessmen are aware that their behaviour is well predicted by marginal analysis:

> "The articles on both sides of the controversy largely neglect what seems to me clearly the main issue – the conformity to experience of the implications of the marginal analysis –and concentrate on the largely irrelevant question whether businessmen do or do not in fact reach their decisions by consulting schedules, or curves, or multivariable functions showing marginal cost and marginal revenue."

He takes the argument further by stating

> "Truly important and significant hypotheses will be found to have 'assumptions' that are wildly inaccurate descriptive representations of reality, and, in general, the more significant the theory, the more unrealistic the assumptions (in this sense). The reason is simple. A hypothesis is important if it 'explains' much by little, that is, if it abstracts the common and crucial elements from the mass of complex and detailed circumstances surrounding the phenomena to be explained and permits valid predictions on the basis of them alone. To be important, therefore, a hypothesis must be descriptively false in its assumptions; it takes account of, and accounts for, none of the many other attendant circumstances, since its very success shows them to be irrelevant for the phenomena to be explained.

> "To put this point less paradoxically, the relevant question to ask about the 'assumptions' of a theory is not whether they are descriptively 'realistic', for they never are, but whether they are sufficiently good approximations for the purpose in hand. And this question can be answered only by seeing whether the theory works, which means whether it yields sufficiently accurate predictions."

The J-value model, developed to explain pan-national differences in life expectancy with GDP per head as just discussed, has been tested elsewhere in a different role, namely the prediction of future life expectancy at birth within a given country (Thomas, 2017b). Drawing on results specific to the Preston curve that are derived in the next section and including an allowance for the steady reduction over the past 50 years in the gap between male and female life expectancies at birth in industrialised countries, the J-value model incorporating "male catch-up" has been validated against UK data on life expectancy at birth over the 20 year period, 1985–2005. In addition, a close correspondence has also been found between 20 year forecasts for life expectancy at birth in 35 industrialized countries made by the J-value model incorporating male catch-up and those produced in a recent study that applied Bayesian model averaging to 21 demographic projection models (Kontis et al., 2017).

The calculations were made using the figure for risk-aversion, ε_P, derived in the next section for use with the Preston curve, namely $\varepsilon_P = 0.91$, slightly less at than the general figure for the Bristol curve. (See Thomas, 2016, for a general discussion of the risk-aversion parameter, ε.) Obviously the remarks of Bueno de Mesquita and Friedman apply here also: the people of the UK were not making J-value calculations during that 20-year period, but the growth in life expectancy over that time can be modelled well using the J-value assumptions (i)–(iv), listed at the beginning of Section 6.

The J-value model for life expectancy emerges as a scientific theory, which can be tested. It has been validated in two independent tests, first in accounting for the differences in population-average life expectancy amongst nations and second in explaining the growth in life expectancy at birth within the same nation. These successes cannot be taken as implying that the model and its assumptions are true, but they do give a significant degree of confidence in the continuing, future use of the J-value method for assessing whether or not the cost of a safety measure is reasonable and hence whether it should or should not be implemented.

8.   Explaining the Preston curve

As shown above, the J-value model has provided a confirmed prediction for the form of the curve relating population-average life expectancy, X, to GDP per head, G. To facilitate discussion, this curve will be referred to in the rest of the paper using another British place-name, and so the "Bristol curve", reflecting the city in which the two authors of this paper work. But the Preston curve (named after Samuel Preston) comprises something slightly different, namely the relationship of life expectancy at birth, $X(0)$, to GDP per head, G. Do the results carry over from the Bristol to the Preston curve, and if so, how?

8.1.   Deriving the decrement in risk-aversion, ϕ

To answer the question, we shall generalise by considering first what the population-average discounted life expectancy depends on.

Holding the net discount rate, r, constant, the common term, $X(0)$, in the expansion of Eq. (25) means that the population-average discounted life expectancy for a general nation may be written:

$$X_d = X(0) f(X(0)) \tag{41}$$

In the particular case of nation A, Eq. (41) may be written:

$$X_{dA} = X_A(0) f_A(X_A(0)) \tag{42}$$

so that,

$$\frac{X_d}{X_{dA}} = \frac{X(0)}{X_A(0)} \frac{f(X(0))}{f_A(X_A(0))} \tag{43}$$

Using Eq. (38) and rearranging gives:

$$\frac{X(0)}{X_A(0)} = \left(\frac{G}{G_A}\right)^{1-\varepsilon} \frac{f_A(X_A(0))}{f(X(0))} \tag{44}$$

Now let the dimensionless quantity, f_A/f, be represented by the GDP ratio raised to the power, ϕ, where generality is assured by allowing ϕ to take both positive and negative values. In the most general case, $\phi = \phi(G)$ and $\phi = $ constant is a special instance, which produces the test Eq. (45) below:

$$\frac{f_A(X_A(0))}{f(X(0))} = \left(\frac{G}{G_A}\right)^{\phi} \tag{45}$$

Substituting from Eq. (45) into Eq. (44) gives:

$$\frac{X(0)}{X_A(0)} = \left(\frac{G}{G_A}\right)^{1-\varepsilon+\phi} = \left(\frac{G}{G_A}\right)^{1-(\varepsilon-\phi)} \tag{46}$$

showing that ϕ is the decrement in risk-aversion arising from the transposition of life expectancy at birth for population-average life expectancy. The J-value model will produce the Preston curve with the same value of risk-aversion when the risk-aversion decrement is zero, $\phi = 0$, and it will give the Preston curve with a different, constant value of risk-aversion if ϕ is a constant, independent of G. Both these cases will validate the test Eq. (45), and, moreover, the J-value model will have provided an explanation for the Preston curve.

If ϕ varies with G, then the extent of the variation over the range of possible values of GDP per head will give an indication of the degree to which observation and model diverge. A small enough variation of ϕ with G will suggest that the J-value method can explain the Preston curve, in line with the general precepts of the Model Distortion method (Butterfield and Thomas, 1986a,b).

If f and f_A are available (or at least approximations to these functions), then the risk-aversion decrement, ϕ, may be found by considering Eq. (45) and taking logs:

$$\phi = \frac{\ln \frac{f_A(X_A(0))}{f(X(0))}}{\ln \frac{G}{G_A}} \tag{47}$$

Meanwhile, from Eq. (43):

$$\frac{f_A(X(0))}{f(X(0))} = \frac{X_{dA}}{X_d} \frac{X(0)}{X_A(0)} \tag{48}$$

so that, on combining Eq. (48) with Eq. (47), we achieve,

$$\phi = \frac{\ln \frac{X(0)}{X_A(0)} - \ln \frac{X_d}{X_{dA}}}{\ln \frac{G}{G_A}} \tag{49}$$

a form that is useful when the relevant life expectancies are available from life-table calculations.

In the undiscounted case, when $r = 0$, comparing Eqs. (25) and (41) shows that $f(X(0)) = X_d/X(0) = b$, with the
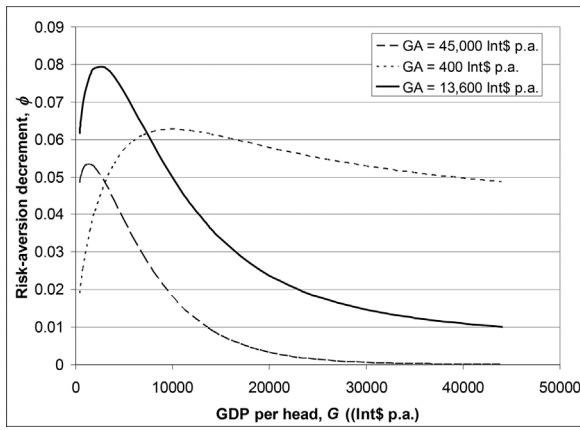
Fig. 6 – Risk-aversion decrement, ϕ, calculated for different GDP per head for nation: 400, 13,600 and 45,000 Int$ p.a.
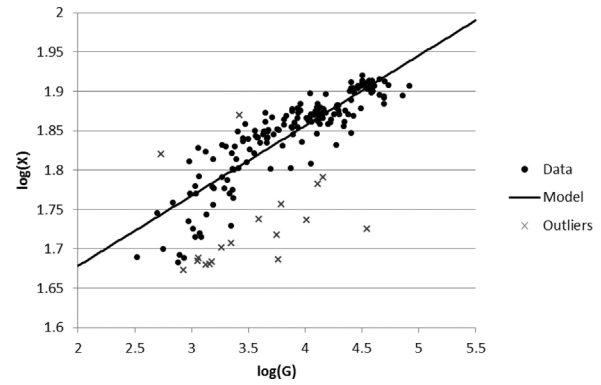

Fig. 7 – The best-fitting power-law model of log life-expectancy at birth ($X(0)$, years) versus log GDP per head (G, international dollars) overlaid on the data from the Preston curve of Fig. 2. Eighteen outliers (marked with crosses) were excluded from the final fit.

corresponding expression for nation A being $f_A(X_A(0)) = X_{dA}/X_A(0) = b_A$, so that, from Eq. (47), and nothing that $\rho(1) = 1.0$

$$\phi = \frac{\ln \frac{b_A}{b}}{\ln \frac{G}{G_A}} \tag{50}$$

Different values of ϕ will result for the same value of G (and hence b, by Eq. (C1)), depending on the characteristics of the reference nation A.

If the chosen reference nation, nation A, has a high GDP per head, e.g. $G_A = 45,000$ Int$ per year and the nation under examination has a lower GDP per head, e.g. $G = 5000$ Int$ per year, then not only will $G/G_A < 1$, but, since the life-expectancy ratio falls with income, then $b_A/b < 1$. Hence both logarithms in Eq. (51) will return a negative value, so that the risk-aversion decrement at G, ϕ, will be positive.

On the other hand, if the chosen reference nation, nation A, has a low GDP per head, e.g. $G_A = 500$ Int$ per year and the nation under examination has a higher GDP per head, e.g. $G = 30,000$ Int$ per year, then $G/G_A > 1$. Now $b_A/b > 1$ as a result of life-expectancy ratio falling with income, so that both logarithms in Eq. (50) will return a positive value. Hence the risk-aversion decrement at G, ϕ, will again be positive. However the larger figures for ϕ will now occur at higher values of GDP per head, rather than at the low values found in the previous paragraphs.

Thus the value of G at which the high and low values of risk-aversion decrement, ϕ, occurs depends on the choice of reference GDP per head. Fig. 6, which will be discussed further in the next subsection, shows that high, positive values of ϕ occur at the low end of the income scale if a high G_A is chosen, whereas if a low G_A is chosen, then ϕ is low when the GDP per head is low, but rises to a peak at about 10,000 Int$ per year before declining rather slowly over the rest of the range.

8.2. Calculating the decrement in risk-aversion, ϕ

The risk-aversion decrement will be zero when $G \neq G_A$ if the life-expectancy ratios are the same: $b = b_A$. In fact, by the model of Appendix C, this situation will occur approximately if G_A and G are both above about 13,000 Int$ (2009) per year.

Using the correlation for life expectancy ratio, b, developed in Appendix C and setting the income per head of reference nation, A, at 45,000, 13,600 and 400 Int$ p.a. produces the graphs of risk-aversion decrement, ϕ, against GDP per head,

G, previously introduced as Fig. 6. The G_A values chosen represent the cases of high and low income, while the mid-range figure is the average GDP per head of the 180 countries surveyed. The corresponding square roots of the means of the integral squared value of ϕ were 0.020, 0.039 and 0.055, with an arithmetic average of 0.038.

The fairly small size of the average risk-aversion decrement, less than 0.04 compared with the risk-aversion of 0.95 found from the J-value model, suggests that a curve of the form,

$$\ln X(0) = (1 - (\varepsilon - \phi)) \ln \frac{G}{G_A} + \ln X_A(0) \tag{51}$$

should provide a reasonable fit to the observed data on life expectancy at birth, $X(0)$ versus, GDP per head, G. With $\varepsilon = 0.95$ and $\phi = 0.04$ it could be predicted that an effective risk-aversion of $\varepsilon_P \approx \varepsilon - \phi = 0.91$ would hold for the Preston curve. Further, the strong variation in risk-aversion decrement below about 13,000 Int$ p.a. and the relative constancy ϕ above that figure for all the G_A curves would suggest that the goodness of fit of Eq. (51) might be slightly worse below this income level and better above it.

These conclusions are broadly confirmed by using linear least-squares curve fitting and finding the slope of the regression line, $1 - \varepsilon$, and hence the risk-aversion, ε, for the data on life expectancy at birth versus GDP per head for the 180 nations discussed previously. See Fig. 7. The best fitting model for 180 nations has $\varepsilon_P = 0.91$ and a correlation statistic of $R^2 = 0.41$, not as high as the figure, $R^2 = 0.60$, found for the Bristol curve, but moderately good nevertheless.

It can be seen from Fig. 7 that there are a number of points that lie significantly away from the trend of the data, and 18 countries (10% of the sample) were excluded to assess any bias introduced to the regression by these points. The quality of the fit is improved to $R^2 = 0.78$ with no change in the best-fitting risk aversion of $\varepsilon = 0.91$ (with a 90% confidence interval of 0.905–0.917):

$$\ln X(0) = (1 - \varepsilon_P) \ln \frac{G}{G_A} + \ln X_A(0) \tag{52}$$

where $G_A = 7600$, $X_A(0) = 70$ and the effective risk-aversion, $\varepsilon_P = 0.91$, is in line with the value predicted using an average value of the risk-aversion decrement, ϕ. Meanwhile the square of the correlation coefficient, $R^2 = 0.78$, is now almost identical

to the figure of $R^2 = 0.80$ found for the Bristol curve. It may thus be seen that the J-value model is able to predict the shape of the Preston curve with some accuracy.

Since $\varepsilon_P < 1.0$, Eq. (52) implies that the life expectancy at birth, $X(0)$, will rise as GDP per head, G, increases. Eq. (52) sets no cut-off point for this process, but a limit may be arrived at when "the inherent life expectancy at birth" is reached, estimated as about 100 years (see Appendix A, paragraphs following Eq. (A130) and Figs. A4 and A5). However it is possible that in future years improvements in medical technology may overcome this apparent limit.

9. Discounting population-average life expectancy

The arguments of the previous section have established that the data making up the Preston curve are well represented an equation of the form:

$$\frac{X(0)}{X_A(0)} = \left(\frac{G}{G_A}\right)^{1-\varepsilon_P} \tag{53}$$

The value of risk-aversion, ε_P, used to explain the Preston curve (which deals with life expectancy at birth) may now be used as a reference point for judging the effect of discounting on the risk-aversion, ε, that should be used in J-value calculations, which deals with the average life expectancy across all ages in the population.

In the reverse procedure of that used in Section 8, Eq. (53) may be substituted into Eq. (43) to give,

$$\frac{X_d}{X_{dA}} = \left(\frac{G}{G_A}\right)^{1-\varepsilon_P} \frac{f(X(0))}{f_A(X_A(0))} \tag{54}$$

Let the dimensionless quantity, f/f_A, be represented by the GDP ratio raised to the power, $-\theta$:

$$\frac{f(X(0))}{f_A(X_A(0))} = \left(\frac{G}{G_A}\right)^{-\theta} \tag{55}$$

where $\theta = \theta(G)$. Substituting from Eq. (55) into Eq. (54) gives:

$$\frac{X_d}{X_{dA}} = \left(\frac{G}{G_A}\right)^{1-\varepsilon_P-\theta} = \left(\frac{G}{G_A}\right)^{1-(\varepsilon_P+\theta)} \tag{56}$$

establishing θ as the increment in risk-aversion arising from the transposition of population-average life expectancy for life expectancy at birth. It may be found by taking logs in Eq. (56),

$$\theta = \frac{-\ln\frac{f(X(0))}{f_A(X_A(0))}}{\ln\frac{G}{G_A}} = \frac{\ln\frac{f_A(X_A(0))}{f(X(0))}}{\ln\frac{G}{G_A}} = \phi \tag{57}$$

where the final step follows from comparing Eqs. (47) and (57). The expected identity between θ and ϕ emerges.

Eq. (57) may be evaluated for a range of GDP per head, G, by using Eq. (53) to give $X(0)$ and applying the methods of Sections 4 and 5 to give the life expectancy ratio, b, and then the discounted life expectancy, X_d. The function $f(X(0)) = X_d/X(0)$ then follows. Fig. 8 gives the results for risk-aversion increment, θ, based on $G_A = 13,600$ Int$ p.a. It can be seen that the effect of a positive net discount rate is generally to cause an increase in the value of risk-aversion, $\varepsilon = \varepsilon_P + \theta$.

Fig. 8 for θ at different values of the net discount rate, r, displays the same distinction between developed and less well-developed countries as was seen with ϕ (=θ) in Fig. 6,
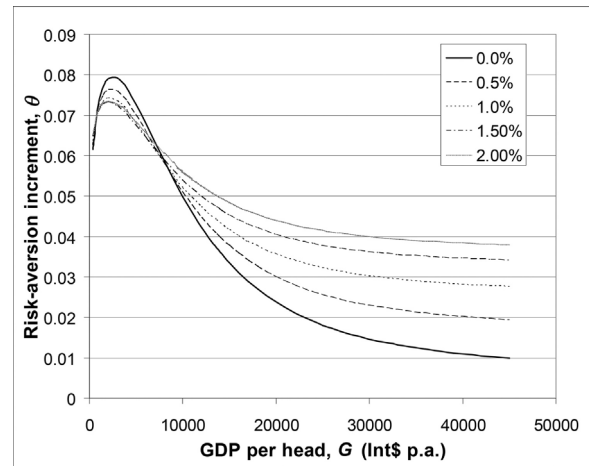

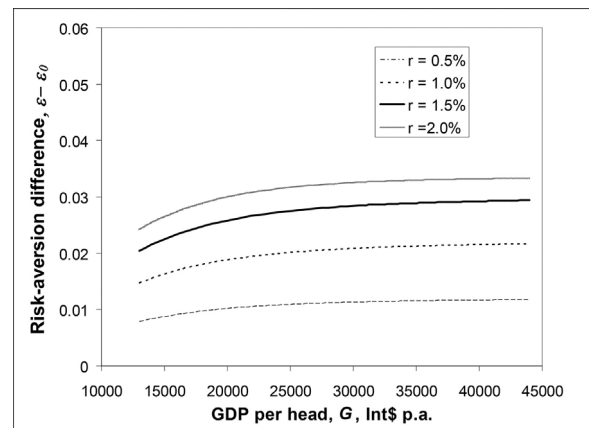Fig. 8 – Risk-aversion increment, θ, at different net discount rates; $G_A = 13,600$ Int$ p.a.

Fig. 9 – Risk-aversion difference versus GDP per head, 13,000 Int$ p.a. to 44,000 Int$ p.a.

where a zero net discount rate is assumed, $r = 0$. The curves of θ and ϕ undergo significantly less variation once GDP is above 13,000 Int$ per head.

The difference between the risk-aversion, ε, at net discount rate, r, and that at a zero net discount rate is:

$$\varepsilon - \varepsilon_0 = \varepsilon_P + \theta - (\varepsilon_P + \theta_0) = \theta - \theta_0 \tag{58}$$

This difference is plotted in Fig. 9 for more developed countries, with GDP per head above 13,000 Int$ per head and the net discount rate, r, set successively at 0.5%, 1.0%, 1.5% and 2.0% p.a. It is clear from the figure that an increase in the net discount rate is accompanied by a corresponding increase in the risk-aversion to be used in J-value analysis, with the one tending to counteract the other. So while a higher net discount rate will cause the amount spent against any particular hazard to be reduced, this reduction will tend to be offset by the higher risk-aversion, which will call for the amount to be increased.

The effect may be quantified numerically. From Fig. 9, $\varepsilon - \varepsilon_0 \approx 0.025$ when the net discount rate is $r = 1.5\%$ p.a. A calculation may be made at $J = 1$ of the amount that should be spent on protecting number of members of the UK public from a radiation dose of 10 mSv per year for a period of 10 years. For example, choosing a risk-aversion, $\varepsilon = 0.91$, for a discount rate, $r = 1.5\%$ p.a., will imply a risk-aversion at a zero discount rate of $\varepsilon|_{r=0} = \varepsilon_0 = 0.91 - 0.025 = 0.885$. Using this value of ε_0 and the 2009 UK life tables values the average future life as £5.18 M at $J = 1$. Increasing the net discount rate to 1.5% and using
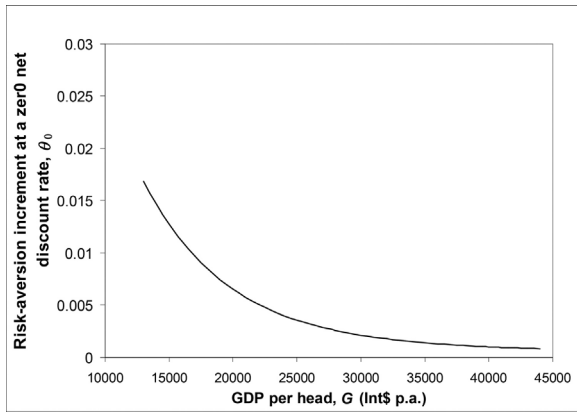
Fig. 10 – Risk-aversion increment, θ_0, at a net discount rate of zero, countries with GDP per head between 13,000 and 44,000 Int\$ p.a.; $G_A = 28, 500$ Int\$ p.a.

the appropriate value of risk-aversion, $\varepsilon = 0.91$, produces an unchanged figure for the average life to come, namely £5.18 M at $J = 1$. The decrease in value due to the higher net discount rate, $r = 1.5\%$ p.a., has been balanced by the increase in value due to the higher risk-aversion.

It may be concluded that there will be little point in adding the complication of a non-zero net discount rate in J-value calculations of the desirable expenditure to protect against loss of life expectancy, since the effect will tend to be balanced out by the rise in the necessary value of risk-aversion, ε.

It is clear that, although the egalitarian model produces an excellent fit to the data, there is nevertheless some difference, between the way that developed and less developed countries respond, as illustrated in Fig. 8. To understand the situation better for more developed countries, the spread of national incomes per head has been limited in Fig. 10 to between 13,000 and 44,000 Int\$ per year, with the reference level of GDP per head set at $G_A = 28, 500$ Int\$ per year. The figure shows the value of risk-aversion increment, θ_0, at $r = 0$, which has a root mean square value of 0.005. Because the life-expectancy ratio is now close to its asymptotic value for countries with a GDP per head above 13,000 Int\$ p.a., the risk-aversion, $\varepsilon = \varepsilon_0$, used in undiscounted J-value calculations is very close to the Preston curve value:

$$\varepsilon_0 = \varepsilon_P + \theta_0 \tag{59}$$

where θ_0 is small. In the limiting case where all countries have the same life expectancy ratio, b, then $\theta_0 = 0$, from Eqs. (57) and (50). As shown in Appendices A and C and Fig. C1, this situation is approximated well as countries' GDP per head rises to values characteristic of the developed world, with $b \to 1/2$ as $G \to \infty$.

Thus for a developed country with GDP per head in excess of 28,500 Int\$ per year, the risk-aversion appropriate for use in J-value calculations is $\varepsilon = \varepsilon_0 = \varepsilon_P = 0.91$. Such a value is roughly 10% higher than the figure of 0.82 derived in Thomas et al. (2010a) and 8% higher than the figure of 0.85 calculated as an average over all risk decisions except those of frequent occurrence for the average UK adult (Thomas et al., 2010b,c). Such increases in risk-aversion do not seem unreasonable *prima facie*, given that the J-value is applied to situations where life expectancy is at stake.

Section 10 offers an alternative but complementary perspective, providing corroboration for a higher figure for risk-aversion, ε, found from the Preston and Bristol curves when human life is at stake.

Fig. 11 – Risk-aversion versus employee's fraction, UK.

10. The risk-aversion for use in J-value analysis in developed countries

A potential weakness associated with the figure for risk-aversion for the UK contained in Thomas et al. (2010a) is that the model used in its derivation assumes that the average person is valuing the extra years of life expectancy solely in terms of the extra years of free time he expects to gain, taking no benefit from his extra years of working, which are no longer his to dispose of as he thinks fit. The assumption is that these have been sold in their entirety to an employer, a stance that is open to the objection that employees might be expected to gain at least some value and satisfaction from the performance of their work duties, with certain employees, perhaps those with a lot of freedom to chose the manner in which they work, gaining a lot of satisfaction.

This potential weakness in the model may be remedied by assuming that the employee values a fraction, the "employee's fraction", ς, of his working time as if it were part of his free time, with the remaining fraction, $1 - \xi$, being regarded as undertaken for the sole benefit of the employer.

The result of this modification is that the effective work fraction for the average employee is then $(1 - \xi) w_0$, which expression replaces the average work fraction, w_0, in the analysis of Thomas et al. (2010a). The value of the exponent, q, then follows from a modified version of Eq. (19) of Thomas et al. (2010a):

$$q = \frac{1}{\beta} \frac{(1 - \varsigma) w_0}{1 - (1 - \varsigma) w_0} \tag{60}$$

where β is the share of wages in the economy. Hence risk-aversion, $\varepsilon = 1 - q$ is given by:

$$\varepsilon = \frac{1 - \frac{\beta+1}{\beta} (1 - \varsigma) w_0}{1 - (1 - \varsigma) w_0} \tag{61}$$

Fig. 11 shows the variation in risk-aversion with employee's fraction, ς, based on UK figures of $w_0 = 0.091$ and $\beta = 0.546$ (Thomas et al., 2010a). An employee's fraction of 1/3 gives a risk-aversion of 0.88, $\varsigma = 1/2$ gives $\varepsilon = 0.91$, while 2/3 gives $\varepsilon = 0.94$.

Thus $\varepsilon = 0.91$ implies an equal, 50:50 deal between the employer and the employee on how much satisfaction the average employee will gain from his work. For such an employee, an hour spent working provides 50% of the enjoyment he would get from an hour spent solely under his own direction. This model, which uses the additional parameter of

Table 2 – GDP per head and life expectancy for countries of widely varying income (2008 data).				
Country	G (Int$/year)	X(0) (year)	X (year)	$b = \frac{X}{X(0)}$
Mozambique	840	51.0	32.9	0.645
Rwanda	1080	57.9	35.4	0.633
Kenya	1570	54.3	34.3	0.612
Ukraine	7300	66.9	36.4	0.545
Mexico	14,800	75.8	40.5	0.537
Saudi Arabia	22,300	72.0	38.7	0.535
Japan	33,800	82.6	43.0	0.521
Germany	37,200	80.0	41.7	0.521

the employee's fraction, ζ, with $\zeta = 1/2$, provides an answer to the reservation raised in the first paragraph of this section, and possesses an attractive degree of verisimilitude. For while there are some people who "live for their work", so that $\varsigma = 1$, and some who regard their work as pure drudgery, for whom $\varsigma = 0$, such extremes will not be representative of the average person, who is more likely to find himself at the "half-way house", with $\zeta = 1/2$. Such would be the expected value of ζ for any symmetrical probability distribution on 0 to 1, such as a uniform distribution or a (truncated) normal distribution.

An explanation is thus available for the process by which a risk-aversion higher than $\varepsilon = 0.82$ will come about for decisions where life expectancy is at stake. The value for a developed nation under the assumption of an equal employer-employee bargain, namely $\varepsilon = 0.91$, corroborates the value suggested from the analysis of the Preston and Bristol curves.

11. The implications of a zero net discount rate

Putting $r = 0$ in Eq. (2) gives the pure time discount rate as:

$$\lambda = g(1 - \varepsilon) \tag{62}$$

Using a long-term average value of UK GDP growth, g:

$$g = 2.41\% \text{ p.a. for 1949 to 2010} \tag{63}$$

and using a risk-aversion of $\varepsilon = 0.91$ gives,

$$\lambda = 0.22\% \text{ p.a.} \tag{64}$$

for the UK.

The dearth of objective evidence for the value of the pure time preference rate led Weitzman (2007) to propose a figure of $\lambda = 2.0$ % p.a. based on his "own rough point-guesstimate of what most economists might think", on the evidence of his informal survey of 2000 economists. Oxera (2002) recommended a value of $\lambda = 1.5$ % p.a. based on a semi-quantitative analysis, and this value was adopted by the UK Treasury (2003, 2011), as valid for payment periods of 30 years or less. However, Stern (2007, 2009) suggests a much lower value of $\lambda = 0.1$ % p.a. Moreover, in his seminal paper, Ramsey (1928) recommends $\lambda = 0.0$ % p.a., saying,

> "One point should perhaps be emphasised more particularly; it is assumed that we do not discount later enjoyments in comparison with earlier ones, a practice which is ethically indefensible and arises merely from the weakness of the imagination; we shall, however, in Section II include such a rate of discount in some of our investigations"

It can be seen that this new, fully quantitative route to determining the pure time preference rate comes up with a value that is significantly closer to Stern and Ramsey than to Oxera and Weitzman.

Meanwhile the social discount rate, r^*, used to discount future payments so as to produce an equivalent up-front lump sum, is given by Ramsey's formula:

$$r^* = \lambda + g\varepsilon \tag{65}$$

Combining Eqs. (62) and (65) gives r^* as:

$$r^* = g \tag{66}$$

Thus we have the pairing for the social discount rate, r^*, and net discount rate, r:

$$r^* = g$$
$$r = 0 \tag{67}$$

for use in J-value assessments. In the case of the UK, $r^* = 2.41\%$ p.a., based on long-term trends.

Applying Eq. (67) together with a risk-aversion of 0.91 to the UK produces an average value for life to come in the UK in 2009 of £6.61 M. This is about 4 times the VPF figure in use in the UK (£1.596 M in 2009, Department for Transport, 2013). Of course, it needs to be borne in mind that the UK VPF has been demonstrated to have no evidential basis (Thomas and Vaughan, 2015a,b,c), despite the figure being in widespread use in the UK since 1999. On the other hand, the figure of £6.61 M is close to the $9.1 M used by the Environmental Protection Agency in the USA in 2010 in deciding how much should be spent on defences against air pollution (Appelbaum, 2011), and later adopted by the US Department of Transportation (Trottenberg, 2013).

12. Conclusions

The J-value method allows a model to be devised whereby a typical nation in the world attempts to improve the quality of life of its citizens by taking a rationally balanced view of measures to improve the the population average life expectancy.

The model has been tested for validity against extensive data on life expectancy and GDP per head. It able to explain the shape of the Bristol curve with an R^2 value of 0.80 for 162 out of the 193 nations recognised by the UN. Thus the J-value model explains 80% of the variation in the logarithm of population-average life expectancy with the logarithm of GDP per head, and so provides a substantial degree of validation for the J-value. An extension of the model explains the Preston curve, which deals with life expectancy at birth rather than the average expectation of life amongst all ages in the population. It is predicted correctly that moving from the Bristol to the Preston curve will cause the risk-aversion to fall by about 5%. The log–log version of the Preston curve is then explained with an almost identical R^2 value, 78%, for the 162 countries. This adds to the validation reported elsewhere of the J-value model when tested against actual UK data on life expectancy at birth.

The J-value approach emerges as essentially a formalisation of an approach being used intuitively all over the world to assess most health and safety spending decisions.

The J-value model is able to explain the Bristol and Preston curves when the net discount rate is zero. Any increase

in the net discount rate is then counteracted by a countervailing increase in the risk-aversion, leaving unchanged the safety spend calculated using the J-value. Hence it is recommended that a zero net discount rate is used when applying the J-value method.

Examination of the detail of the Bristol and Preston curves suggests that there is, in fact, a small difference in risk-aversion between less developed and developed countries. This is supported by a re-examination of the arguments leading to the choice of 0.82 as the value of risk-aversion for use in J-value studies in the UK. Allowing for the average employee gaining satisfaction from his work, equal to half the satisfaction he would have gained had he been free to dispose of his time according to his own wishes, leads to the same value of risk-aversion as produced by a detailed analysis of the Bristol and Preston curves, namely 0.91.

In addition to the validation of the J-valuation just described, the model has enabled a closed-form algebraic expression to be derived for the first time for an important economic parameter, the pure time discount rate estimated up to now only on a subjective basis. The resulting figure emerges as much closer to the estimates of Ramsey and Stern than to the value currently being used by the UK Treasury. The pure time discount rate is particularly relevant to decisions on combating climate change and its effects.

The life expectancy ratio (population-average life expectancy divided by life expectancy at birth) is proposed as an important indicator of development, as detailed in Appendices A to C. It can vary between an upper limit of ⅔ and a lower limit of ½. The upper limit is representative of undeveloped countries, while the lower limit is the asymptote towards which developed countries are moving.

A final conclusion of potential interest to Governments and the insurance industry as regards pension payment, is that no discernable limit is apparent to the process by which people live ever longer as they get ever richer, at least up to the point where the intrinsic life expectancy of ∼100 years, is reached, a limit that improvements in medical technology might in any case bypass.

Acknowledgements

Appendix A. Piecewise linear approximations to the survival probability: the life expectancy ratio, b, as an indicator of development.

Starting at unity at age zero, $t = 0$, the survival probability declines with age, reaching very close to zero at age 100, which constitutes the upper age used by the UK life tables. Survival probability is a decreasing sigmoid function of age that has so far defied fully successful analytical characterisation. However, simple linear expressions may be used to give an approximate description. These have the merit that when limiting values are applied, the resultant functions provide a useful insight into the basic mortality mechanisms that must be at work in the world.

From Eq. (13), the life expectancy of a steady state population, X, is equal to the average age of that population. So using Eq. (8) also, we may write,

$$X = E(T) = \int_{t=0}^{\infty} p(t)\,t\,dt = \frac{1}{X(0)} \int_{t=0}^{\infty} tS(t)\,dt \qquad (A1)$$

where $p(t) = S(t)/X(0)$ is the probability density for age in the steady-state population. Meanwhile it is known that the life expectancy at birth, $X(0)$, is given by the area under the survival probability curve, $S(t)$:

$$X(0) = \int_{t=0}^{\infty} S(t)\,dt \qquad (A2)$$

Eq. (A2) may be expanded formally as:

$$X(0) = \int_{t=0}^{X(0)} S(t)\,dt + \int_{t=X(0)}^{\infty} S(t)\,dt \qquad (A3)$$

Subtracting $\int_{t=0}^{X(0)} S(t)\,dt$ from the left-hand side of Eq. (A3) allows the definition of a difference, Y, between life expectancy at birth and the integral of survival probability up to an age equal to the initial life expectancy:

$$Y = X(0) - \int_{t=0}^{X(0)} S(t)\,dt = \int_{t=0}^{X(0)} dt - \int_{t=0}^{X(0)} S(t)\,dt = \int_{t=0}^{X(0)} (1 - S(t))\,dt \quad (A4)$$

Meanwhile subtracting the same quantity, $\int_{t=0}^{X(0)} S(t)\,dt$, from the right-hand side of Eq. (A3) gives the difference, Y, as:

$$Y = \int_{t=X(0)}^{\infty} S(t)\,dt \qquad (A5)$$

Equating the right hand sides of Eqs. (A4) and (A5) gives,

$$\int_{t=0}^{X(0)} (1 - S(t))\,dt = \int_{t=X(0)}^{\infty} S(t)\,dt \qquad (A6)$$
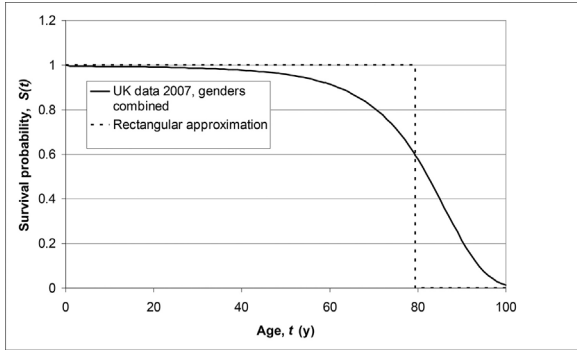
Fig. A1 – Actual and approximate survival probability functions.

Suppose we now define a function, $S^*(t)$, by,

$$S^*(t) = \begin{cases} 1 & \text{for } t \leq X(0) \\ 0 & \text{for } t > X(0) \end{cases} \tag{A7}$$

We may substitute from Eq. (A7) into Eq. (A6) to give:

$$\int_{t=0}^{X(0)} (S^*(t) - S(t))\, dt = \int_{t=X(0)}^{\infty} S(t)\, dt \tag{A8}$$

Add $\int_{t=0}^{X(0)} S(t)\, dt$ to both sides of Eq. (A8),

$$\int_{t=0}^{X(0)} S^*(t)\, dt = \int_{t=0}^{\infty} S(t)\, dt \tag{A9}$$

But, from the definition of Eq. pair (A7), $\int_{t=0}^{X(0)} S^*(t)\, dt = \int_{t=0}^{\infty} S^*(t)\, dt$.

Thus,

$$\int_{t=0}^{\infty} S^*(t)\, dt = \int_{t=0}^{\infty} S(t)\, dt \tag{A10}$$

Thus the function, $S^*(t)$, has the same area under its curve as the true survival probability, $S(t)$, and hence may be regarded as a rectangular approximation to the survival probability. Moreover, the life expectancy at birth is the integral of the survival probability (Eq. (A2)) and so

$$\int_{t=0}^{\infty} S^*(t)\, dt = X(0) \tag{A11}$$

Fig. A1 shows the actual and the approximate survival probability for functions for UK combined gender data for 2007 (ONS, 2016). The increase in the area under the actual curve above $t = X(0)$ compensates exactly for the deficit in area between the actual and the approximated curves up to age, $t = X(0)$.
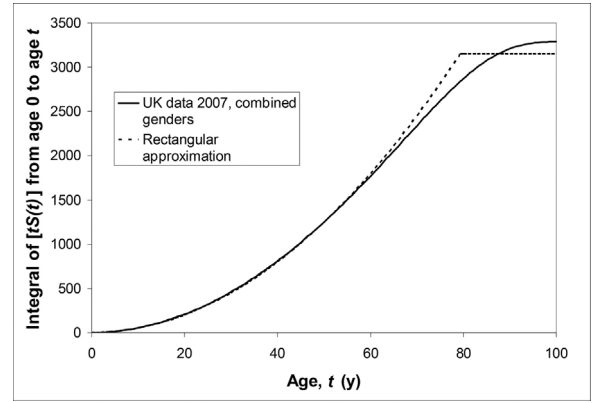

Fig. A2 – $\int_0^t tS(t)\, dt$ and $\int_0^t tS^*(t)\, dt$.

If the approximate, rectangular survival probability is used in Eq. (A1) then an estimate, X^*, of the life expectancy averaged over all ages will be:

$$X^* = \frac{1}{X(0)} \int_{t=0}^{\infty} tS^*(t)\, dt \tag{A12}$$

Fig. A2 compares the integral, $\int_{t=0}^{t} tS^*(t)\, dt$, with the actual integral, $\int_{t=0}^{t} tS(t)\, dt$, for combined genders for 2007. It is clear that the integral based on the rectangular survival probability provides a good match to the actual not only for low values of t, but also as $t \to \infty$. Note, however, that,

$$\int_{t=0}^{\infty} tS^*(t)\, dt < \int_{t=0}^{\infty} tS(t)\, dt \tag{A13}$$

where $S(t)$ is the true survival probability, so that $X^* < X$ by Eqs. (A1) and (A9). Inequality Eq. (A13) is proved in Appendix B.

Integrating the left-hand side of Eq. (A13) gives:

$$\int_{t=0}^{\infty} tS^*(t)\, dt = \int_{t=0}^{X(0)} t\, dt = \left[\frac{t^2}{2}\right]_0^{X(0)} = \frac{X(0)^2}{2} \tag{A14}$$

Hence, from Eq. (A12), the population average life expectancy, X^*, under the rectangular survival probability is,

$$X^* = \frac{X(0)}{2} \tag{A15}$$

Thus the ratio, b^*, of the life expectancy at birth to the population-average life expectancy under this rectangular survival probability is

$$b^* = \frac{X^*}{X(0)} = 0.5 \tag{A16}$$

In fact the rectangular survival probability described by Eq. pair (A7) represents a limiting case for a generalised survival probability that remains at unity from age 0 to age $(1 - k)X(0)$,
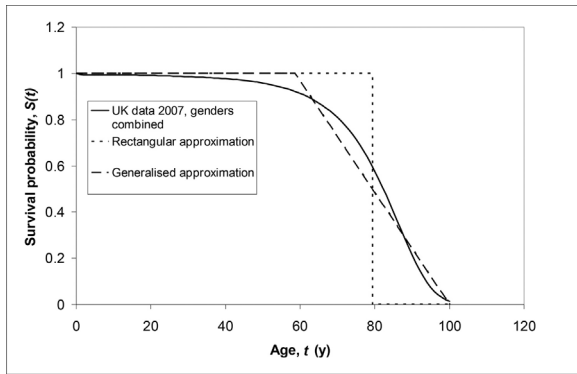
Fig. A3 – Generalised approximations to survival probability.

then begins to fall in a linear fashion with slope, $1/(2kX(0))$, so that it passes through the point $(X(0), 0.5)$ before reaching zero at $t = (1 + k)X(0)$, where $0 \leq k \leq 1$. See Fig. A3. Eq. pair (A7) is then generalised to,

$$S^*(t) = \begin{cases} 1 & \text{for } t \leq (1-k)X(0) \\ \dfrac{1+k}{2k} - \dfrac{t}{2kX(0)} & \text{for } (1-k)X(0) < t \leq (1+k)X(0) \\ 0 & \text{for } t > (1+k)X(0) \end{cases} \quad \text{(A17)}$$

The parameter value, $k = 0$, gives rise to the rectangular survival probability function already described, while the other limit, $k = 1$, leads to a survival probability that starts at unity and then declines linearly with a constant slope, $-1/(2X(0))$, until it reaches zero when age has reached twice the life expectancy at birth.

It is clear from the geometry of Fig. A3 that the area under the generalised survival probability of Eq. set (A17) will be the same as the area under the rectangular survival probability of Eq. pair (A7). Hence Eq. (A11) will hold, implying that the true life expectancy at birth will be retained under Eq. set (A17) for all $k : 0 \leq k \leq 1.0$. Moreover, the extra degree of freedom associated with the parameter, k, allows the adjustment of $S^*(t)$ so that,

$$\int_{t=0}^{\infty} tS^*(t)\, dt = \int_{t=0}^{\infty} tS(t)\, dt \quad \text{(A18)}$$

Since for a steady-state population,

$$S(t) = X(0)p(t) \quad \text{(A19)}$$

from Eq. (8), it follows that $\displaystyle\int_{t=0}^{\infty} tS(t)\, dt$ represents a scaled version of the first moment or expected value of age, T, randomly selected from the population:

$$E(T) = \int_{t=0}^{\infty} tp(t)\, dt = \frac{1}{X(0)} \int_{t=0}^{\infty} tS(t)\, dt \quad \text{(A20)}$$

Moreover, since from Eq. (13), $E(T) = X$, it follows from Eqs. (A18) and (A20) that,

$$X = \frac{1}{X(0)} \int_{t=0}^{\infty} tS^*(t)\, dt \quad \text{(A21)}$$

Thus it possible, under the generalised survival probability model of Eq. set (A17), to ensure that both the life expectancy at birth and the population-average life expectancy can be made equal to the true values, $X(0)$ and X.

The full set of estimated moments, $E(T^n)^* = \displaystyle\int_{t=0}^{\infty} t^n p^*(t)\, dt$, about the origin may be found from:

$$\int_{t=0}^{\infty} t^n p^*(t)\, dt = \frac{1}{X(0)} \int_{t=0}^{\infty} t^n S^*(t)\, dt \quad n = 0, 1, 2, 3, \ldots \quad \text{(A22)}$$

Meanwhile the general integral, $\displaystyle\int_{t=0}^{\infty} t^n S^*(t)\, dt$, may be found using Eq. (A17) as:

$$\begin{aligned} \int_{t=0}^{\infty} t^n S^*(t)\, dt &= \int_{t=0}^{(1-k)X(0)} t^n dt + \frac{1+k}{2k} \int_{t=(1-k)X(0)}^{(1+k)X(0)} t^n dt - \frac{1}{2kX(0)} \int_{t=(1-k)X(0)}^{(1+k)X(0)} t^{n+1} dt \\ &= \frac{1}{2k(n+1)} X^{n+1}(0) \left((1-k)^{n+1}(k-1) + (1+k)^{n+2}\right) \\ &\quad - \frac{1}{2k(n+2)} X^{n+1}(0) \left((1+k)^{n+2} - (1-k)^{n+2}\right) \end{aligned} \quad \text{(A23)}$$

Hence

$$\int_{t=0}^{\infty} t^n S^*(t)\, dt = \frac{(1+k)^{n+2} - (1-k)^{n+2}}{2k(n+1)(n+2)} X^{n+1}(0) \quad \text{(A24)}$$

Thus,

$$E(T^n)^* = \int_{t=0}^{\infty} t^n p^*(t)\, dt = \frac{1}{X(0)} \int_{t=0}^{\infty} t^n S^*(t)\, dt = \frac{(1+k)^{n+2} - (1-k)^{n+2}}{2k(n+1)(n+2)} X^n(0) \quad \text{(A25)}$$

Putting $n = 0$ gives $\displaystyle\int_{t=0}^{\infty} p^*(t)\, dt = 1$, as required, while setting $n = 1$ gives:

$$\begin{aligned} \int_{t=0}^{\infty} tS^*(t)\, dt &= \frac{(1+k)^3 - (1-k)^3}{12k} X^2(0) \\ &= \frac{1 + 3k + 3k^2 + k^3 - \left(1 - 3k + 3k^2 - k^3\right)}{12k} X^2(0) \\ &= \frac{6k + 2k^3}{12k} X^2(0) \\ &= \frac{X^2(0)}{2} \left(1 + \frac{k^2}{3}\right) \end{aligned} \quad \text{(A26)}$$
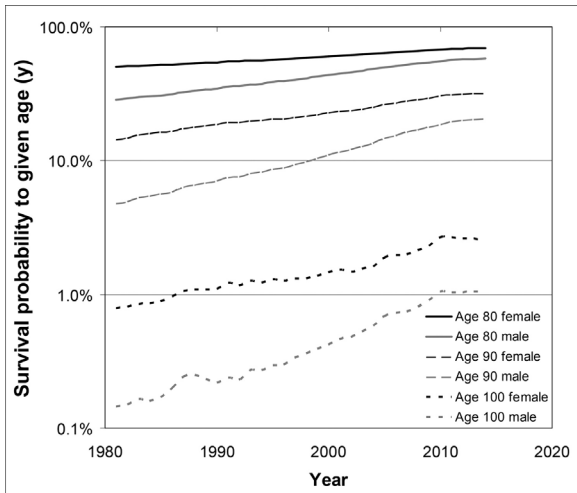
Fig. A4 – Survival probabilities to ages 80, 90 and 100 in the UK.

Fig. A5 – Life expectancies at ages 80, 90 and 100 in the UK.

As a result, the population-average life expectancy emerges as the following function of k:

$$X^* = \frac{1}{X(0)} \int_{t=0}^{\infty} t S^*(t)\, dt = \frac{X(0)}{2}\left(1 + \frac{k^2}{3}\right) \tag{A27}$$

while the ratio of population-average life expectancy to life expectancy at birth is,

$$b^* = \frac{X^*}{X(0)} = \frac{1}{2}\left(1 + \frac{k^2}{3}\right) \tag{A28}$$

From Eq. set (A17) the lower limit, $k = 0$ produces the rectangular survival probability, which may be regarded as the ideal, since it defines a population where all the people attain what might be described as their 'inherent life expectancy at birth', with no early deaths due to accidents or disease. Since the hazard rate, $h(t)$, may be defined the ratio of the rate of decline of the survival probability to the existing survival probability:

$$h(t) = -\frac{1}{S(t)}\frac{dS^*(t)}{dt} \tag{A29}$$

the hazard rate, $h(t)$, may be characterised for the rectangular case by:

$$h(t) = \begin{cases} 0 & \text{for } t < X(0) \\ \infty & \text{for } t = X(0) \end{cases} \tag{A30}$$

It is only when the inherent life expectancy at birth, $X(0)$, has been reached, that these people experience "wear out" due to aging, perhaps in much the same way as described by Oliver Wendell Holmes in his poem, "The Wonderful One-Hoss Shay" (Encyclopaedia Britannica, 2015), with all bodily systems, despite all interventions and repairs, failing together. There is evidence for the value of the inherent life expectancy at birth being currently about 100 years for both genders, based on UK data. While the number of centenarians is steadily increasing, their life expectancy once they have reached this age is not. Fig. A4 shows the growth over time of the probability of survival to the three ages: 80 years, 90 years and 100 years, graphed on a logarithmic scale to allow the changes to the survival probability to age 100 to be distinguished clearly (ONS, 2016). The probability of survival has increased markedly at
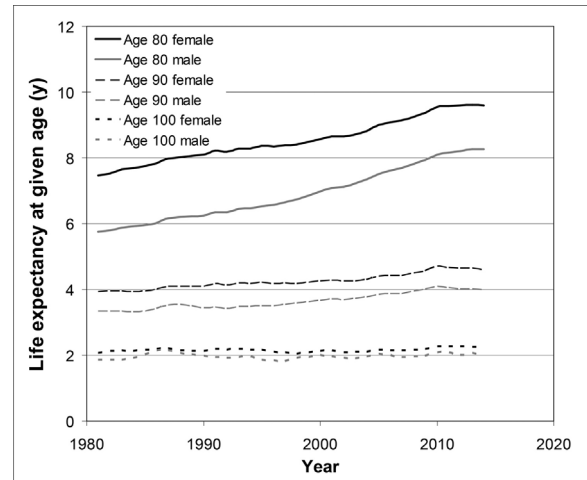
each of these ages over the past 30 years or so, with male catch-up being a further pronounced phenomenon. (For example, the chances of a baby boy reaching the age of 80 rose from 28% in 1981 to 58% in 2014.) But as shown in Fig. A5, while the life expectancy at ages 80 and 90 has risen steadily over the past 30 years, the life expectancy at age 100 has remained approximately static for both men and women.

[While the "inherent life expectancy at birth" may currently be about 100, this may be regarded as (roughly) the ultimate life expectancy at birth only against the background of the current technology and culture. Future developments in medical technology might cause the ultimate life expectancy at birth to rise higher in developed countries, as a result, for example, of greater access to artificial organs. In this case the inherent and ultimate life expectancies at birth would diverge.]

The rectangular survival probability leads to $b^* = 0.5$, as shown in Eq. (A16), and it can be further shown that the most highly developed countries have a life expectancy at birth of about 80 years or more and a value of b that approaches 0.5, typically 0.52.

On the other hand, the upper limit, $k = 1$, defines the triangular survival probability, where, from Eq. set (A17):

$$S^*(t) = 1 - \frac{t}{2X(0)} = \frac{2X(0) - t}{2X(0)} \tag{A31}$$

and so,

$$\frac{dS^*(t)}{dt} = -\frac{1}{2X(0)} \tag{A32}$$

As a result, using equation (A.29), the hazard rate, $h(t)$, in this case is

$$h(t) = \frac{1}{2X(0)} \times \frac{2X(0)}{2X(0) - t} = \frac{1}{2X(0) - t} \tag{A33}$$

For these people, the hazard rate approaches infinity only when their age has reached twice their life expectancy at birth: $h(t) \to \infty$ as $t \to 2X(0)$. The age, $2X(0)$, under the triangular survival probability model is equivalent to the age, $X(0)$, under the rectangular survival probability, in the sense that it is the ultimate age beyond which no-one in the population can survive. No-one is going to live to be older than $2X(0)$, but on average they will die before they get half-way to this age, well before they approach their intrinsic wear-out age. The triangular survival probability leads to $b^* = \frac{2}{3}$, a value

that fits approximately the values of b characterising under-developed countries, where the life expectancy at birth, $X(0)$, is around 50 years. The triangular survival probability model would thus suggest the inherent, limiting age a person from such a country would be about 100 years, which tallies well with the observed maximum age in developed countries.

Thus the life-expectancy ratio, b, emerges as an indicator of national development. It will start near 2/3 for an undeveloped country then decrease as the country gets richer, with the ratio for a highly developed country coming close to the limiting value of ½. The process by which the life expectancy ratio progresses from 2/3 towards the asymptotic value of ½ is modelled empirically in Appendix C.

Appendix B. Proof that $\displaystyle\int_{t=0}^{\infty} tS^*(t)\,dt < \int_{t=0}^{\infty} tS(t)\,dt$.

It is obvious that $t < X(0)$ for all $t : 0 \leq t < X(0)$, and this implies that

$$\int_{t=0}^{X(0)} t\left(S^*(t) - S(t)\right) dt < \int_{t=0}^{X(0)} X(0)\left(S^*(t) - S(t)\right) dt \tag{B1}$$

But

$$\int_{t=0}^{X(0)} X(0)\left(S^*(t) - S(t)\right) dt = X(0)\int_{t=0}^{X(0)} \left(S^*(t) - S(t)\right) dt = X(0)\,Y \tag{B2}$$

where Y is the difference given in Eq. (A4) and the last step follows from applying Eqs. (A5) and (A8). Hence, using Eq. (B2) in inequality Eq. (B1), it follows that,

$$\int_{t=0}^{X(0)} t\left(S^*(t) - S(t)\right) dt < X(0)\,Y \tag{B3}$$

Meanwhile it is clear that, $t > X(0)$ for all $t : X(0) < t < \infty$, and so,

$$\int_{t=X(0)}^{\infty} tS(t)\,dt > \int_{t=X(0)}^{\infty} X(0)\,S(t)\,dt \tag{B4}$$

But,

$$\int_{t=X(0)}^{\infty} X(0)\,S(t)\,dt = X(0)\int_{t=X(0)}^{\infty} S(t)\,dt = X(0)\,Y \tag{B5}$$

where the last step follows from Eq. (A5). Substituting Eq. (B5) into inequality Eq. (B4) gives,

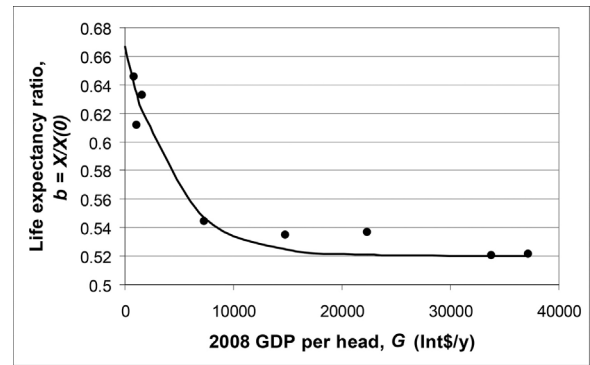$$\int_{t=X(0)}^{\infty} tS(t)\,dt > X(0)\,Y \tag{B6}$$

Fig. C1 – The behaviour of life expectancy ratio, b.

Using condition Eq. (B6) in condition Eq. (B3) gives:

$$\int_{t=0}^{X(0)} t\left(S^*(t) - S(t)\right) dt < \int_{t=X(0)}^{\infty} tS(t)\,dt \tag{B7}$$

Adding $\displaystyle\int_{t=0}^{X(0)} tS(t)\,dt$ to both sides of inequality Eq. (B7) gives,

$$\int_{t=0}^{X(0)} tS^*(t)\,dt < \int_{t=0}^{X(0)} tS(t)\,dt + \int_{t=X(0)}^{\infty} tS(t)\,dt \tag{B8}$$

But since $S^*(t) = 0$ for $t > X(0)$, it follows that $\displaystyle\int_{t=0}^{X(0)} tS^*(t)\,dt = \int_{t=0}^{\infty} tS^*(t)\,dt$. Moreover, $\displaystyle\int_{t=0}^{X(0)} tS(t)\,dt + \int_{t=X(0)}^{\infty} tS(t)\,dt = \int_{t=0}^{\infty} tS(t)\,dt$. Substituting these two forms into equation (B.8) gives the desired result:

$$\int_{t=0}^{\infty} tS^*(t)\,dt < \int_{t=0}^{\infty} tS(t)\,dt \tag{B9}$$

Appendix C. Dependence of the life-expectancy ratio, b, on GDP per head.

Table 2 presents data from 2008 on life expectancy and GDP per capita, G, for countries of widely varying income, including the life-expectancy ratio, b. It may be seen that the value of b corresponds well to the figure predicted by the model of Appendix A, being close to 0.667 for undeveloped countries and approaching 0.5 for highly developed countries. The parameter, b, is plotted against G in Fig. C1, as is the model fitted to it, which has the form.

$$b = b_s + (b_0 - b_s)\, e^{-\frac{G}{G_T}} \tag{C1}$$

where $b_0 = 0.667$ and $b_s = 0.52$, while $G_T = 4295$ Int\$/year is chosen to minimise the sum of the squared errors.

Eq. (C1) is that for a targeted growth model, specifically a proportional feedback control system around an integrator, which is fed a constant target or set-point of b_s, starting from an initial b-value of b_0. The aspiration inherent in this math-

ematical description of a system trying to reach its set point would seem to correspond well with what can plausibly be assumed of populations seeking to survive and prosper.

The data suggest that it is currently very difficult to achieve a b-value much below 0.52, a fact reflected in the choice of b_s in the model. But Eq. (C1) suggests that 95% of the discrepancy between b_0 and b_s will have been eliminated by the time the country's income per head has risen to 13,000 Int\$/year. Hence b should be roughly constant for incomes above the latter level.

Eq. (C1) may be expressed in the alternative form:

$$\frac{b - b_s}{b_0 - b_s} = e^{-\frac{G}{G_T}} \tag{C2}$$

so that,

$$\ln \frac{b - b_s}{b_0 - b_s} = -\frac{1}{G_T} G \tag{C3}$$

Given observed values of life expectancy ratio, $b = X/X(0)$, and GDP per head, G, a best-fit value of G_T could be found based on all the world's nations.

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