



Ichinose, G., & Masuda, N. (2018). Zero-determinant strategies in finitely repeated games. *Journal of Theoretical Biology*, 438, 61-77.
<https://doi.org/10.1016/j.jtbi.2017.11.002>

Publisher's PDF, also known as Version of record

License (if available):
CC BY-NC-ND

Link to published version (if available):
[10.1016/j.jtbi.2017.11.002](https://doi.org/10.1016/j.jtbi.2017.11.002)

[Link to publication record in Explore Bristol Research](#)
PDF-document

University of Bristol - Explore Bristol Research

General rights

This document is made available in accordance with publisher policies. Please cite only the published version using the reference above. Full terms of use are available:
<http://www.bristol.ac.uk/pure/about/ebr-terms>



Zero-determinant strategies in finitely repeated games



Genki Ichinose^a, Naoki Masuda^{b,*}

^a Department of Mathematical and Systems Engineering, Shizuoka University, 3-5-1 Johoku, Naka-ku, Hamamatsu, 432-8561, Japan

^b Department of Engineering Mathematics, University of Bristol, Merchant Venturers Building, Woodland Road, Clifton, Bristol BS8 1UB, United Kingdom

ARTICLE INFO

Article history:

Received 11 July 2017

Revised 5 November 2017

Accepted 6 November 2017

Available online 14 November 2017

Keywords:

Prisoner's dilemma game

Cooperation

Direct reciprocity

Discount factor

ABSTRACT

Direct reciprocity is a mechanism for sustaining mutual cooperation in repeated social dilemma games, where a player would keep cooperation to avoid being retaliated by a co-player in the future. So-called zero-determinant (ZD) strategies enable a player to unilaterally set a linear relationship between the player's own payoff and the co-player's payoff regardless of the strategy of the co-player. In the present study, we analytically study zero-determinant strategies in finitely repeated (two-person) prisoner's dilemma games with a general payoff matrix. Our results are as follows. First, we present the forms of solutions that extend the known results for infinitely repeated games (with a discount factor w of unity) to the case of finitely repeated games ($0 < w < 1$). Second, for the three most prominent ZD strategies, the equalizers, extortioners, and generous strategies, we derive the threshold value of w above which the ZD strategies exist. Third, we show that the only strategies that enforce a linear relationship between the two players' payoffs are either the ZD strategies or unconditional strategies, where the latter independently cooperates with a fixed probability in each round of the game, proving a conjecture previously made for infinitely repeated games.

© 2017 Elsevier Ltd. All rights reserved.

1. Introduction

The prisoner's dilemma game models situations in which two individuals are involved in a social dilemma and each individual selects either cooperation (C) and defection (D) in the simplest setting. Although an individual obtains a larger payoff by selecting D regardless of the choice of the other individual, mutual defection, which is the unique Nash equilibrium of the game, yields a smaller benefit to both players than mutual cooperation does. We now know various mechanisms that enable mutual cooperation in the prisoner's dilemma game and other social dilemma games (Nowak, 2006; Rand and Nowak, 2013; Sigmund, 2010), which inform us how cooperation is probably sustained in society of humans and animals and how to design cooperative organisations and society.

One of the mechanisms enabling mutual cooperation in social dilemma games is direct reciprocity, i.e., repeated interaction, in which the same two individuals play the game multiple times. An individual that defects would be retaliated by the co-player in the succeeding rounds. Therefore, the rational decision for both players in the repeated prisoner's dilemma game is to keep mutual cooperation if the number of iteration is sufficiently large (Axelrod, 1984; Nowak, 2006; Trivers, 1971). Generous tit-for-tat

(Nowak and Sigmund, 1992) and win-stay lose-shift (often called Pavlov) (Kraines and Kraines, 1993; Nowak and Sigmund, 1993) strategies are strong competitors in evolutionary dynamics of the repeated prisoner's dilemma game under noise, and a population composed of them realizes a high level of mutual cooperation.

In 2012, when the study of direct reciprocity seemed to be matured, Press and Dyson proposed a novel class of strategies in the repeated prisoner's dilemma game, called zero-determinant (ZD) strategies (Press and Dyson, 2012). ZD strategies impose a linear relationship between the payoff obtained by a focal individual and its co-player regardless of the strategy that the co-player implements. A special case of the ZD strategies is the equalizer, with which the focal individual unilaterally determines the payoff that the co-player gains regardless of what the co-player does, within a permitted range of the co-player's payoff value (see Boerlijst et al., 1997 and Sigmund, 2010 for the previous accounts for this strategy). As a different special case, the focal individual can set an "extortionate" share of the payoff that the individual gains as compared to the co-player's payoff. The advent of the ZD strategies has spurred new lines of investigations of direct reciprocity. They include the examination and extension of ZD strategies such as their evolution (Adami and Hintze, 2013; Akin, 2017; Chen and Zinger, 2014; Hilbe et al., 2013a; 2013b; 2015b; Liu et al., 2015; Stewart and Plotkin, 2012; Szolnoki and Perc, 2014a; 2014b; Wu and Rong, 2014; Xu et al., 2017), multiplayer games (Hilbe et al., 2014b; 2015b; Milinski et al., 2016; Pan et al., 2015; Stewart

* Corresponding author.

E-mail address: naoki.masuda@bristol.ac.uk (N. Masuda).

et al., 2016), continuous action spaces (McAvoy and Hauert, 2016; 2017; Milinski et al., 2016; Stewart et al., 2016), alternating games (McAvoy and Hauert, 2017), human reactions to computerized ZD strategies (Hilbe et al., 2014a; Wang et al., 2016), and human-human experiments (Hilbe et al., 2016; Milinski et al., 2016).

Most of the aforementioned mathematical and computational studies of the ZD strategies have been conducted under the assumption of infinitely repeated games. While mathematically more elegant and advantageous, finitely repeated games are more realistic than infinitely repeated games and comply with experimental studies. In the present study, we examine the ZD strategies in the finitely repeated prisoner's dilemma game. There are a few studies that have investigated ZD strategies in finitely repeated games. Hilbe and colleagues defined and mathematically characterized ZD strategies in finitely repeated games (Hilbe et al., 2015a) (also see Hilbe et al., 2014a). McAvoy and Hauert analyzed ZD strategies in the finitely repeated donation game (i.e., a special case of the prisoner's dilemma game) in a continuous strategy space (McAvoy and Hauert, 2016; 2017). Given these studies, our main contributions in the present article are summarized as follows. First, we derive expressions for ZD strategies in finitely repeated games that are straightforward extensions of those previously found for the infinitely repeated game. Second, for the three most studied ZD strategies, we derive the threshold discount factor (i.e., how likely the next round of the game occurs in the finitely repeated game) above which the ZD strategy can exist. Third, we prove that imposing a linear relationship between the two individuals' payoffs implies that the focal player takes either the ZD strategy defined for finitely repeated games (Hilbe et al., 2015a) or an unconditional strategy (e.g., unconditional cooperation and unconditional defection), proving the conjecture in Hilbe et al. (2013b) in the case of finitely repeated games.

2. Preliminaries

In this section, we explain the finitely repeated prisoner's dilemma game, the strategies of interest (i.e., memory-one strategies), and the expected payoffs. More thorough discussion of them is found in Nowak et al. (1995), Sigmund (2010) and Hilbe et al. (2015a).

We consider the symmetric two-person prisoner's dilemma game whose payoff matrix is given by

$$\begin{matrix} & C & D \\ C & \begin{pmatrix} R & S \end{pmatrix} \\ D & \begin{pmatrix} T & P \end{pmatrix} \end{matrix} \tag{1}$$

The entries represent the payoffs that the focal player, denoted by X, gains in a single round of a repeated game. Each row and column represents the action of the focal player, X, and the co-player (denoted by Y), respectively. We assume that

$$T > R > P > S, \tag{2}$$

which dictates the prisoner's dilemma game. Both players obtain a larger payoff by selecting D than C because $T > R$ and $P > S$. We also assume that

$$2R > T + S, \tag{3}$$

which guarantees that mutual cooperation is more beneficial than the two players alternating C and D in the opposite phase, i.e., CD, DC, CD, DC, ... where the first and second letter represent the actions selected by X and Y, respectively (Axelrod, 1984; Rapoport and Chammah, 1965). The two players repeat the game whose payoff matrix in each round is given by Eq. (1). A next round given the current round takes place with probability w ($0 < w < 1$), which is called the discount factor.

Consider two players X and Y that adopt memory-one strategies, with which they use only the outcome of the last round to decide the action to be submitted in the current round. A memory-one strategy is specified by a 5-tuple; X's strategy is given by a combination of

$$\mathbf{p} = (p_{CC}, p_{CD}, p_{DC}, p_{DD}) \tag{4}$$

and p_0 , where $0 \leq p_{CC}, p_{CD}, p_{DC}, p_{DD}, p_0 \leq 1$. In Eq. (4), p_{CC} is the conditional probability that X cooperates when both X and Y cooperated in the last round, p_{CD} is the conditional probability that X cooperates when X cooperated and Y defected in the last round, p_{DC} is the conditional probability that X cooperates when X defected and Y cooperated in the last round, and p_{DD} is the conditional probability that X cooperates when both X and Y defected in the last round. Finally, p_0 is the probability that X cooperates in the first round. Similarly, Y's strategy is specified by a combination of

$$\mathbf{q} = (q_{CC}, q_{CD}, q_{DC}, q_{DD}) \tag{5}$$

and the probability to cooperate in the first round, q_0 , where $0 \leq q_{CC}, q_{CD}, q_{DC}, q_{DD}, q_0 \leq 1$.

We refer to the first round of the repeated game as round 0. Because both players have been assumed to use a memory-one strategy, the stochastic state of the two players in round t ($t \geq 0$) is specified by

$$\mathbf{v}(t) = (v_{CC}(t), v_{CD}(t), v_{DC}(t), v_{DD}(t)), \tag{6}$$

where $v_{CC}(t)$ is the probability that both players cooperate in round t , $v_{CD}(t)$ is the probability that X cooperates and Y defects in round t , and so forth. The normalization is given by $v_{CC}(t) + v_{CD}(t) + v_{DC}(t) + v_{DD}(t) = 1$ ($t = 0, 1, \dots$). The initial condition is given by

$$\mathbf{v}(0) = (p_0q_0, p_0(1 - q_0), (1 - p_0)q_0, (1 - p_0)(1 - q_0)). \tag{7}$$

Because the expected payoff to player X in round t is given by $\mathbf{v}(t)\mathbf{S}_X^T$, where

$$\mathbf{S}_X = (R, S, T, P), \tag{8}$$

the expected per-round payoff to player X in the repeated game is given by

$$\pi_X = (1 - w) \sum_{t=0}^{\infty} w^t \mathbf{v}(t)\mathbf{S}_X^T. \tag{9}$$

The transition-probability matrix for $\mathbf{v}(t)$ is given by

$$\mathbf{M} = \begin{pmatrix} p_{CC}q_{CC} & p_{CC}(1 - q_{CC}) & (1 - p_{CC})q_{CC} & (1 - p_{CC})(1 - q_{CC}) \\ p_{CD}q_{DC} & p_{CD}(1 - q_{DC}) & (1 - p_{CD})q_{DC} & (1 - p_{CD})(1 - q_{DC}) \\ p_{DC}q_{CD} & p_{DC}(1 - q_{CD}) & (1 - p_{DC})q_{CD} & (1 - p_{DC})(1 - q_{CD}) \\ p_{DD}q_{DD} & p_{DD}(1 - q_{DD}) & (1 - p_{DD})q_{DD} & (1 - p_{DD})(1 - q_{DD}) \end{pmatrix} \tag{10}$$

By substituting

$$\mathbf{v}(t) = \mathbf{v}(0)\mathbf{M}^t \tag{11}$$

in Eq. (9), one obtains

$$\begin{aligned} \pi_X &= (1 - w)\mathbf{v}(0) \sum_{t=0}^{\infty} (w\mathbf{M})^t \mathbf{S}_X^T \\ &= (1 - w)\mathbf{v}(0)(I - w\mathbf{M})^{-1}\mathbf{S}_X^T, \end{aligned} \tag{12}$$

where I is the 4×4 identity matrix. Similarly, the expected per-round payoff to player Y is given by

$$\pi_Y = (1 - w)\mathbf{v}(0)(I - wM)^{-1}\mathbf{S}_Y^T, \tag{13}$$

where

$$\mathbf{S}_Y = (R, T, S, P). \tag{14}$$

3. Results

We search player X's strategies that impose a linear relationship between the two players' payoffs, i.e.,

$$\alpha\pi_X + \beta\pi_Y + \gamma = 0. \tag{15}$$

When $\alpha \neq 0$, we set $\chi = -\beta/\alpha$ and $\kappa = -\gamma/(\alpha + \beta)$ to transform Eq. (15) to

$$\pi_X - \kappa = \chi(\pi_Y - \kappa). \tag{16}$$

3.1. Equalizer

3.1.1. Expression

By definition, the equalizer unilaterally sets the co-player's payoff, π_Y , to a constant value irrespectively of the co-player's strategy (Boerlijst et al., 1997; Press and Dyson, 2012; Sigmund, 2010). To derive an expression for the equalizer strategies in the finitely repeated game, we proceed along the following idea: If a strategy \mathbf{p} ensures that the payoffs of the two players are on a horizontal line in the $\pi_X - \pi_Y$ space, irrespectively of the co-player's strategy, then the payoffs must be on that horizontal line if the co-player uses unconditional cooperation or unconditional defection. Substituting the co-player's unconditional cooperation and unconditional defection into the payoff formulas gives necessary conditions imposed on X's strategy. A straightforward computation then shows that these necessary conditions are in fact often sufficient; even if the co-player uses strategies that are not unconditional cooperation or defection, the two payoffs lie on the same line. We will use the same idea in Section 3.2 as well.

Because the equalizer is equivalent to $\alpha = 0$ in Eq. (15) and hence not covered by Eq. (16), we start by rewriting Eq. (13) as follows:

$$\begin{aligned} \pi_Y &= (1 - w)\mathbf{v}(0)\mathbf{u}^{\text{eq}} \\ &= (1 - w)(p_0q_0, p_0(1 - q_0), (1 - p_0)q_0, (1 - p_0)(1 - q_0)) \\ &\quad \times \begin{pmatrix} u_1^{\text{eq}} \\ u_2^{\text{eq}} \\ u_3^{\text{eq}} \\ u_4^{\text{eq}} \end{pmatrix}, \tag{17} \\ &= (1 - w)[p_0q_0u_1^{\text{eq}} + p_0(1 - q_0)u_2^{\text{eq}} \\ &\quad + (1 - p_0)q_0u_3^{\text{eq}} + (1 - p_0)(1 - q_0)u_4^{\text{eq}}] \end{aligned}$$

where

$$\mathbf{u}^{\text{eq}} = \begin{pmatrix} u_1^{\text{eq}} \\ u_2^{\text{eq}} \\ u_3^{\text{eq}} \\ u_4^{\text{eq}} \end{pmatrix} \equiv (I - wM)^{-1}\mathbf{S}_Y^T. \tag{18}$$

We denote \mathbf{u}^{eq} when Y's strategy is $\mathbf{q} = (0, 0, 0, 0)$ by $\mathbf{u}^{\text{eq}, 0000}$. Note that $\mathbf{u}^{\text{eq}, 0000}$ is independent of the probability that Y cooperates in the initial round, i.e., q_0 . We denote by $\pi_{Y, 0000}$ the payoff of Y when $\mathbf{q} = (0, 0, 0, 0)$. Similarly, we denote \mathbf{u}^{eq} when Y's strategy is $\mathbf{q} = (1, 1, 1, 1)$ by $\mathbf{u}^{\text{eq}, 1111}$ and by $\pi_{Y, 1111}$ the payoff of Y when $\mathbf{q} = (1, 1, 1, 1)$. The expressions of $\mathbf{u}^{\text{eq}, 0000}$, $\pi_{Y, 0000}$, $\mathbf{u}^{\text{eq}, 1111}$, and $\pi_{Y, 1111}$ are given in Appendix A. If X applies an equalizer strategy, $\pi_{Y, 0000} = \pi_{Y, 1111}$ must hold true regardless of q_0 . Therefore, we obtain

$$\begin{aligned} (1 - w)[p_0q_0u_1^{\text{eq}, 0000} + p_0(1 - q_0)u_2^{\text{eq}, 0000} + (1 - p_0)q_0u_3^{\text{eq}, 0000} \\ + (1 - p_0)(1 - q_0)u_4^{\text{eq}, 0000}] \\ = (1 - w)[p_0q_0u_1^{\text{eq}, 1111} + p_0(1 - q_0)u_2^{\text{eq}, 1111} \\ + (1 - p_0)q_0u_3^{\text{eq}, 1111} + (1 - p_0)(1 - q_0)u_4^{\text{eq}, 1111}], \tag{19} \end{aligned}$$

which leads to

$$\begin{aligned} q_0[p_0(u_1^{\text{eq}, 0000} - u_1^{\text{eq}, 1111}) - p_0(u_2^{\text{eq}, 0000} - u_2^{\text{eq}, 1111}) \\ + (1 - p_0)(u_3^{\text{eq}, 0000} - u_3^{\text{eq}, 1111}) - (1 - p_0)(u_4^{\text{eq}, 0000} - u_4^{\text{eq}, 1111})] \\ + [p_0(u_2^{\text{eq}, 0000} - u_2^{\text{eq}, 1111}) \\ + (1 - p_0)(u_4^{\text{eq}, 0000} - u_4^{\text{eq}, 1111})] = 0. \tag{20} \end{aligned}$$

Eq. (20) must hold true for arbitrary $0 \leq q_0 \leq 1$. Therefore, we obtain

$$p_0(u_1^{\text{eq}, 0000} - u_1^{\text{eq}, 1111}) + (1 - p_0)(u_3^{\text{eq}, 0000} - u_3^{\text{eq}, 1111}) = 0, \tag{21}$$

$$p_0(u_2^{\text{eq}, 0000} - u_2^{\text{eq}, 1111}) + (1 - p_0)(u_4^{\text{eq}, 0000} - u_4^{\text{eq}, 1111}) = 0. \tag{22}$$

Combination of Eqs. (18), (21), and (22) leads to the following necessary conditions:

$$p_{\text{CD}} = \frac{p_{\text{CC}}(T - P) - (\frac{1}{w} + p_{\text{DD}})(T - R)}{R - P}, \tag{23}$$

$$p_{\text{DC}} = \frac{(\frac{1}{w} - p_{\text{CC}})(P - S) + p_{\text{DD}}(R - S)}{R - P}, \tag{24}$$

and p_{CC} , p_{DD} , and p_0 are arbitrary under the constraint $0 \leq p_{\text{CC}}$, p_{CD} , p_{DC} , p_{DD} , $p_0 \leq 1$. Eqs. (23) and (24) extend the results previously obtained for $w = 1$ (Press and Dyson, 2012).

Surprisingly, Eqs. (23) and (24) are also sufficient for \mathbf{p} to be an equalizer strategy. In other words, if a strategy of player X satisfies Eqs. (23) and (24), then every co-player Y's strategy, not restricted to unconditional cooperation or unconditional defection, yields the same payoff of Y. To verify this, we substitute

$$\mathbf{p} = \left(p_{\text{CC}}, \frac{p_{\text{CC}}(T - P) - (\frac{1}{w} + p_{\text{DD}})(T - R)}{R - P}, \frac{(\frac{1}{w} - p_{\text{CC}})(P - S) + p_{\text{DD}}(R - S)}{R - P}, p_{\text{DD}} \right) \tag{25}$$

and $\mathbf{q} = (q_{\text{CC}}, q_{\text{CD}}, q_{\text{DC}}, q_{\text{DD}})$ in Eq. (18) to obtain

$$\begin{aligned} \mathbf{u}^{\text{eq}} &= \frac{1}{(1 - w)(1 - wp_{\text{CC}} + wp_{\text{DD}})} \\ &\quad \times \begin{pmatrix} w(1 - p_{\text{CC}})P + (1 - w + wp_{\text{DD}})R \\ w(1 - p_{\text{CC}})P + (1 - w + wp_{\text{DD}})R \\ (1 - wp_{\text{CC}})P + wp_{\text{DD}}R \\ (1 - wp_{\text{CC}})P + wp_{\text{DD}}R \end{pmatrix}, \tag{26} \end{aligned}$$

which does not contain \mathbf{q} . By substituting Eq. (26) in Eq. (17), we obtain

$$\pi_Y = \frac{(1 - p_0 + wp_0 - wp_{\text{CC}})P + (p_0 - wp_0 + wp_{\text{DD}})R}{1 - wp_{\text{CC}} + wp_{\text{DD}}}, \tag{27}$$

which is independent of \mathbf{q} and q_0 . Therefore, the set of the equalizer strategies is given by Eq. (25), where $0 \leq p_{\text{CC}}$, p_{CD} , p_{DC} , $p_{\text{DD}} \leq 1$, combined with any $0 \leq p_0 \leq 1$.

It should be noted that an equalizer does not require any condition on p_0 . However, Eq. (27) indicates that the payoff that an equalizer enforces on the co-player, π_Y , depends on the value of p_0 . Because Eq. (27) is a weighted average of P and R with non-negative weights, an equalizer can impose any payoff value π_Y such that $P \leq \pi_Y \leq R$. If P is enforced, it holds that $p_0 - wp_0 + wp_{\text{DD}} = 0$, and hence $p_{\text{DD}} = p_0 = 0$. Therefore, the equalizer is a cautious strategy (i.e., never the first to cooperate) (Hilbe et al., 2015a). If R is enforced, it holds that

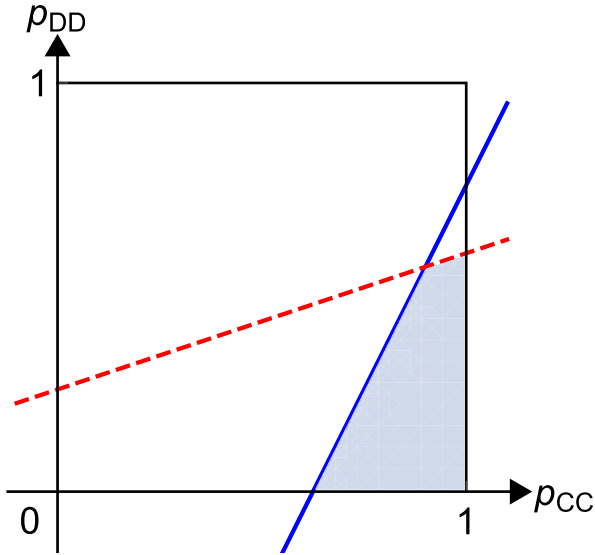


Fig. 1. Region in the p_{CC} - p_{DD} space where the equalizer strategy exists (shaded region). The border line of the half plane specified by Eqs. (30) and (31) are shown by the solid and dashed lines, respectively. We set $R = 3$, $T = 5$, $S = 0$, $P = 1$, and $w = 0.8$.

$1 - p_0 + wp_0 - wp_{CC} = 0$, and hence $p_{CC} = p_0 = 1$. Therefore, the equalizer is a nice strategy (i.e., never the first to detect) (Hilbe et al., 2015a). We remark that the equalizer is a ZD strategy for finitely repeated games as defined in Hilbe et al. (2015a) because it satisfies Eq. (31) of Hilbe et al. (2015a) with $\alpha = 0$.

3.1.2. Minimum discount rate

In this section, we identify the condition for w under which equalizer strategies exist. Eq. (25) indicates that an equalizer strategy exists if and only if

$$0 \leq p_{CC}(T - P) - \left(\frac{1}{w} + p_{DD}\right)(T - R) \leq R - P \quad (28)$$

and

$$0 \leq \left(\frac{1}{w} - p_{CC}\right)(P - S) + p_{DD}(R - S) \leq R - P \quad (29)$$

for some $0 \leq p_{CC}, p_{DD} \leq 1$. Note that we used Eq. (2). Independently of w , any pair of p_{CC} and p_{DD} satisfies the second inequality of Eq. (28) and the first inequality of Eq. (29) because they are satisfied in the most stringent case, i.e., $p_{CC} = 1$ and $p_{DD} = 0$. The first inequality of Eq. (28) and the second inequality of Eq. (29) read

$$p_{DD} \leq \frac{T - P}{T - R} p_{CC} - \frac{1}{w} \quad (30)$$

and

$$p_{DD} \leq \frac{P - S}{R - S} p_{CC} - \frac{1}{w} \frac{P - S}{R - S} + \frac{R - P}{R - S}, \quad (31)$$

respectively. Eqs. (30) and (31) specify a $p_{CC} - p_{DD}$ region in the square $0 \leq p_{CC}, p_{DD} \leq 1$, near the corner $(p_{CC}, p_{DD}) = (1, 0)$ (shaded region in Fig. 1). The feasible set (p_{CC}, p_{DD}) monotonically enlarges as w increases. Therefore, we obtain the condition under which an equalizer exists by substituting $p_{CC} = 1$ and $p_{DD} = 0$ in Eqs. (30) and (31), i.e.,

$$w \geq w_c \equiv \max\left(\frac{T - R}{T - P}, \frac{P - S}{R - S}\right). \quad (32)$$

When $w = w_c$, the unique equalizer strategy is given by $p_{CC} = 1$, $p_{DD} = 0$, and either p_{CD} or p_{DC} is equal to zero, depending on whether $(T - R)/(T - P)$ is larger than $(P - S)/(R - S)$ or vice versa. The condition $w \geq (T - R)/(T - P)$ in Eq. (32) coincides with

that for the GRIM or tit-for-tat strategy to be stable against the unconditional defector (Axelrod, 1984).

Eq. (32) is consistent with the result for the continuous donation game (McAvoy and Hauert, 2016). Their result adapted to the case of two discrete levels of cooperation is $w_c = c/b$, where b and c are the usual benefit and cost parameters in the donation game, respectively. We verify that Eq. (32) with $R = b - c$, $T = b$, $S = -c$, and $P = 0$ yields $w_c = c/b$.

3.2. General cases

All strategies but the equalizer in which a linear relationship is imposed between π_X and π_Y are given in the form of Eq. (16). In this section, we derive expressions of X 's strategy that realizes Eq. (16).

By substituting Eqs. (12) and (13) in Eq. (16), we obtain

$$(1 - w)v(0)(I - wM)^{-1}\mathbf{S}_X^\top - \kappa = \chi[(1 - w)v(0)(I - wM)^{-1}\mathbf{S}_Y^\top - \kappa]. \quad (33)$$

Eq. (33) yields

$$v(0)\{(1 - w)(I - wM)^{-1}[\mathbf{S}_X^\top - \chi\mathbf{S}_Y^\top] + (\chi - 1)\kappa\mathbf{1}\} = 0, \quad (34)$$

where

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}. \quad (35)$$

We set

$$\mathbf{u}^{\text{zd}} = \begin{pmatrix} u_1^{\text{zd}} \\ u_2^{\text{zd}} \\ u_3^{\text{zd}} \\ u_4^{\text{zd}} \end{pmatrix} \equiv (1 - w)(I - wM)^{-1}[\mathbf{S}_X^\top - \chi\mathbf{S}_Y^\top] + (\chi - 1)\kappa\mathbf{1}. \quad (36)$$

Then, Eq. (34) is rewritten as

$$v(0)\mathbf{u}^{\text{zd}} = 0, \quad (37)$$

which is equivalent to

$$q_0[p_0u_1^{\text{zd}} - p_0u_2^{\text{zd}} + (1 - p_0)u_3^{\text{zd}} - (1 - p_0)u_4^{\text{zd}}] + [p_0u_2^{\text{zd}} + (1 - p_0)u_4^{\text{zd}}] = 0. \quad (38)$$

Because Eq. (38) must hold true irrespectively of q_0 , we require

$$p_0u_1^{\text{zd}} + (1 - p_0)u_3^{\text{zd}} = 0, \quad (39)$$

$$p_0u_2^{\text{zd}} + (1 - p_0)u_4^{\text{zd}} = 0. \quad (40)$$

Let us denote by $\mathbf{u}^{\text{zd}, 0000}$ and $\mathbf{u}^{\text{zd}, 1111}$ the vector \mathbf{u} when $\mathbf{q} = (0, 0, 0, 0)$ and $\mathbf{q} = (1, 1, 1, 1)$, respectively. The expressions of $\mathbf{u}^{\text{zd}, 0000}$ and $\mathbf{u}^{\text{zd}, 1111}$ are given in Appendix B. By substituting $\mathbf{u}^{\text{zd}, 0000}$ and $\mathbf{u}^{\text{zd}, 1111}$ in Eqs. (39) and (40), we obtain the four necessary conditions, Eqs. (91)–(94), given in Appendix B.

If we assume $\kappa - S + \chi(T - \kappa) \neq 0$, we can rewrite Eq. (92) as

$$p_{DD} = \frac{(1 - w)p_0[(\chi - 1)P + S - \chi T] + (1 - wp_{CD})(\chi - 1)(\kappa - P)}{w[\kappa - S + \chi(T - \kappa)]}. \quad (41)$$

If we assume $T - \kappa + \chi(\kappa - S) \neq 0$, we can rewrite Eq. (93) as

$$p_{CC} = \frac{-(1-w)p_0[(\chi-1)R+T-\chi S]+T-\chi S+(1+wp_{DC})(\chi-1)\kappa-wp_{DC}(\chi-1)R}{w[T-\kappa+\chi(\kappa-S)]} \tag{42}$$

We will deal with the case $\kappa-S+\chi(T-\kappa)=0$ or $T-\kappa+\chi(\kappa-S)=0$ later in this section.

By substituting Eqs. (41) and (42) in Eq. (91), we obtain an equation containing p_{CD} , p_{DC} , p_0 , κ , and χ as unknowns. This equation can be factorized. By equating each of the two factors with 0, we obtain two types of solutions. The one type of solution is given by

$$\mathbf{p} = \begin{pmatrix} \frac{(1-w)p_0[(\chi-1)R+S-\chi T]-(1-wp_{CD})(\chi-1)R+\chi T-S-wp_{CD}(\chi-1)\kappa}{w[\kappa-S+\chi(T-\kappa)]} \\ p_{CD} \\ \frac{-(1-w)p_0(\chi+1)(T-S)+(1-wp_{CD})[(\chi-1)\kappa+T-\chi S]}{w[\kappa-S+\chi(T-\kappa)]} \\ \frac{(1-w)p_0[(\chi-1)P+S-\chi T]+(1-wp_{CD})(\chi-1)(\kappa-P)}{w[\kappa-S+\chi(T-\kappa)]} \end{pmatrix} \tag{43}$$

Eq. (43) also satisfies Eq. (94). To verify that Eq. (43) is sufficient, we substitute Eq. (43) in Eq. (36) to obtain

$$\mathbf{u}^{zd} = \frac{(1-w)[S+(\chi-1)\kappa-\chi T]}{1-wp_{CD}-(1-w)p_0} \begin{pmatrix} 1-p_0 \\ 1-p_0 \\ -p_0 \\ -p_0 \end{pmatrix} \tag{44}$$

which does not contain \mathbf{q} . Using Eqs. (7) and (44), we verify Eq. (37). Therefore, Eq. (43) is a set of strategies that impose the linear relationship between the payoff of the two players, i.e., Eq. (16).

The strategies given by Eq. (43) are ZD strategies for $w < 1$ as defined in Hilbe et al. (2015a), which is verified as follows. Assume that $\alpha \neq 0$ in Eq. (31) of Hilbe et al. (2015a) because $\alpha = 0$ corresponds to the equalizer. Then, let us substitute $\alpha = \phi$, $\beta = -\phi\chi$, and $\gamma = \phi(\chi-1)\kappa$ in Eq. (31) of Hilbe et al. (2015a) without loss of generality. Note that this transformation is a bijection because (i) $\phi > 0$ and (ii) either $\chi > 1$ or $\chi < 0$ is required (in the notation of Hilbe et al. (2015a), $\phi > 0$ and $\chi < 1$ because their χ is defined as the reciprocal of our χ). Then, we obtain

$$w\mathbf{p} = \begin{pmatrix} 1-\phi(\chi-1)(R-\kappa)-(1-w)p_0 \\ 1+\phi[(\chi-1)\kappa-\chi T+S]-(1-w)p_0 \\ \phi[(\chi-1)\kappa+T-\chi S]-(1-w)p_0 \\ \phi(\chi-1)(\kappa-P)-(1-w)p_0 \end{pmatrix} \tag{45}$$

which is equivalent to Eq. (33) of Hilbe et al. (2015a). Eq. (45) combined with

$$\phi = \frac{1-wp_{CD}-p_0(1-w)}{\kappa-S+\chi(T-\kappa)} \tag{46}$$

is equivalent to Eq. (43). It should also be noted that Eq. (45) extends Eq. (9) of Chen and Zinger (2014), which has been obtained for $w = 1$, to general w , R , and P values.

The other type of solution that we obtain by substituting Eqs. (41) and (42) in Eq. (91) is given by

$$p_0[T-R+\chi(R-S)] = T-\kappa+\chi(\kappa-S) \tag{47}$$

Substitution of Eqs. (41) and (42) in Eq. (94) yields either Eq. (43) or

$$p_0[P-S+\chi(T-P)] = (\chi-1)(\kappa-P) \tag{48}$$

The combination of Eqs. (47) and (48) is equivalent to that of

$$\kappa = p_0^2R + p_0(1-p_0)(T+S) + (1-p_0)^2P \tag{49}$$

and

$$\chi = -\frac{(1-p_0)(T-P)+p_0(R-S)}{(1-p_0)(P-S)+p_0(T-R)} \tag{50}$$

However, Eqs. (41), (42), (49), and (50) do not provide a sufficient condition for Eq. (37) to hold true for arbitrary \mathbf{q} and q_0 . Therefore, we additionally consider the vector \mathbf{u}^{zd} when $\mathbf{q} = (1, 0, 0, 0)$ and $\mathbf{q} = (0, 0, 0, 1)$, which we denote by $\mathbf{u}^{zd, 1000}$ and $\mathbf{u}^{zd, 0001}$, respectively. The calculations shown in Appendix C lead to

$$p_0 = p_{CC} = p_{CD} = p_{DC} = p_{DD} \quad (0 \leq p_0 \leq 1) \tag{51}$$

To verify that the unconditional strategies given by Eq. (51) are a sufficient condition for Eq. (16) to hold true for arbitrary \mathbf{q} and q_0 , we substitute Eqs. (49)–(51) in Eq. (36) to obtain

$$\mathbf{u}^{zd} = \frac{(1-w)(T-S)}{-(1-p_0)P+S+p_0(R-S-T)}$$

$$\times \begin{pmatrix} -(1-p_0)[-(1-p_0)P + (1+p_0)R - p_0(T+S)] \\ -(1-p_0)[-(2-p_0)P + T + S + p_0(R-S-T)] \\ p_0[-(1-p_0)P + (1+p_0)R - p_0(T+S)] \\ p_0[-(2-p_0)P + T + S + p_0(R-S-T)] \end{pmatrix}, \tag{52}$$

which does not contain \mathbf{q} . Using Eqs. (7) and (52), we verify Eq. (37). The unconditional strategy given by Eq. (51) is not a ZD strategy in the sense of Hilbe et al. (2015a) unless $R + P = T + S$ (Appendix D), which is the same condition as that for the infinitely repeated game (Hilbe et al., 2013b).

The obtained solution, i.e., Eq. (51) combined with Eqs. (49) and (50), is equivalent to the previously derived solution for $w = 1$ (Hilbe et al., 2013b). This set of solutions contains the unconditional cooperator and unconditional defector as special cases, and always realizes $\chi < 0$ (Eq. (50)).

When $\kappa - S + \chi(T - \kappa) = 0$ or $T - \kappa + \chi(\kappa - S) = 0$, the calculations shown in Appendix E and Appendix F reveal the following three types of solutions: (i) a subset of the ZD strategies given by Eq. (43) (Appendix F.2), (ii) a subset of the strategies given by Eq. (51) (Appendices E.1, E.2 and F.2), and (iii) the set of strategies given by

$$\mathbf{p} = \left(p_{CC}, 1, \frac{wp_{CC}(\chi + 1)(\kappa - T) - w[R - (\chi + 1)T + \chi\kappa] - (\kappa - R)}{w(\kappa - R)}, \frac{wp_{CC}(\kappa - P) - w(R - P) - (\kappa - R)}{w(\kappa - R)} \right), p_0 = 1, \tag{53}$$

where $0 \leq p_{CC} \leq 1$ and $\kappa \neq R$ (Appendix E.2). Although Eq. (53) is a sufficient condition and the resulting solutions are distinct from those given by Eq. (43), in fact Eq. (53) yields $\chi < 0$ (Appendix E.2).

To summarize, the set of X 's strategies that enforce Eq. (16) is the union of the strategies given by the ZD strategies, Eq. (43), and the non-ZD unconditional strategies, Eq. (51). In the next sections, we examine two special cases, which have been studied in the literature, and derive w_c in each case.

3.3. Extortioner

3.3.1. Expression

The extortioner is defined as a strategy that enforces an extortionate share of payoffs larger than P (Press and Dyson, 2012). We obtain the extortioner by setting $\kappa = P$ in Eq. (16). By setting $\kappa = P$ in Eq. (43), we obtain

$$\mathbf{p} = \begin{pmatrix} \frac{(1-w)p_0[(\chi - 1)R + S - \chi T] - (1-wp_{CD})(\chi - 1)R + \chi T - S - wp_{CD}(\chi - 1)P}{w[P - S + \chi(T - P)]} \\ p_{CD} \\ \frac{-(1-w)p_0(\chi + 1)(T - S) + (1-wp_{CD})[(\chi - 1)P + T - \chi S]}{w[P - S + \chi(T - P)]} \\ \frac{(1-w)p_0}{w} \end{pmatrix}. \tag{54}$$

Because $p_{DD} = -(1-w)p_0/w \geq 0$ and $w < 1$, we obtain $p_0 = 0$ and $p_{DD} = 0$, which is consistent with the previously obtained result (Hilbe et al., 2015a). Therefore, the extortioner is never the first to cooperate and hence a so-called cautious strategy (Hilbe et al., 2015a). By setting $p_0 = 0$ in Eq. (54), we obtain

$$\mathbf{p} = \begin{pmatrix} \frac{-wp_{CD}(\chi - 1)P - (1-wp_{CD})(\chi - 1)R - S + \chi T}{w[P - S + \chi(T - P)]} \\ p_{CD} \\ \frac{(1-wp_{CD})[(\chi - 1)P + T - \chi S]}{w[P - S + \chi(T - P)]} \\ 0 \end{pmatrix}. \tag{55}$$

3.3.2. Minimum discount rate

By setting $\kappa = P$ and $p_0 = 0$ in Eq. (45), we obtain

$$w\mathbf{p} = \begin{pmatrix} 1 - \phi(\chi - 1)(R - P) \\ 1 + \phi[(\chi - 1)P - \chi T + S] \\ \phi[(\chi - 1)P + T - \chi S] \\ 0 \end{pmatrix}. \tag{56}$$

Because $p_{CC} \leq 1$ and $w < 1$, Eq. (56) implies that $\phi(\chi - 1) > 0$ must hold true. We consider the case $\phi > 0$ and $\chi > 1$ in this section. We can exclude the case $\phi < 0$ and $\chi < 1$ because a strategy with $\chi < 0$ is not considered as an extortionate strategy (Chen and Zinger, 2014; Hilbe et al., 2013a; 2013b; 2015a; McAvoyn and Hauert, 2016; 2017; Pan et al., 2015; Press and Dyson, 2012; Stewart and Plotkin, 2013; Xu et al., 2017) and $\chi < 1$ implies $\chi < 0$ (Appendix G.1).

When $\phi > 0$, the application of $0 \leq p_{CC}, p_{CD}, p_{DC} \leq 1$ to Eq. (56) yields

$$\frac{(\chi - 1)\frac{R-P}{P-S}}{\frac{1}{w}} \leq \frac{1}{\phi} \leq \frac{(\chi - 1)\frac{R-P}{P-S}}{\frac{1}{w} - 1}, \tag{57}$$

$$\frac{1 + \chi \frac{T-P}{P-S}}{\frac{1}{w}} \leq \frac{1}{\phi} \leq \frac{1 + \chi \frac{T-P}{P-S}}{\frac{1}{w} - 1}, \quad (58)$$

$$\chi + \frac{T-P}{P-S} \leq \frac{1}{\phi}. \quad (59)$$

The condition under which a positive ϕ value that satisfies Eqs. (57)–(59) exists is given by

$$\frac{(\chi - 1) \frac{R-P}{P-S}}{\frac{1}{w}} \leq \frac{1 + \chi \frac{T-P}{P-S}}{\frac{1}{w} - 1}, \quad (60)$$

$$\frac{1 + \chi \frac{T-P}{P-S}}{\frac{1}{w}} \leq \frac{(\chi - 1) \frac{R-P}{P-S}}{\frac{1}{w} - 1}, \quad (61)$$

$$\chi + \frac{T-P}{P-S} \leq \frac{(\chi - 1) \frac{R-P}{P-S}}{\frac{1}{w} - 1}, \quad (62)$$

$$\chi + \frac{T-P}{P-S} \leq \frac{1 + \chi \frac{T-P}{P-S}}{\frac{1}{w} - 1}. \quad (63)$$

Eq. (60) is always satisfied. Eqs. (61)–(63) yield

$$\chi[w(T-P) - (T-R)] \geq R-S - w(P-S), \quad (64)$$

$$\chi[w(T-S) - (P-S)] \geq T-P - w(T-S), \quad (65)$$

$$\chi[w(R-S) - (P-S)] \geq T-P - w(T-R), \quad (66)$$

respectively.

The left-hand side of Eq. (65) is always larger than that of Eq. (66), and the right-hand side of Eq. (65) is always smaller than that of Eq. (66). Therefore, Eq. (65) is satisfied if Eq. (66) is satisfied. The right-hand sides of Eqs. (64) and (66) are positive. Therefore, $w(T-P) - (T-R) > 0$ and $w(T-S) - (P-S) > 0$ are required for χ to be positive. On the other hand, if $w(T-P) - (T-R) > 0$ and $w(T-S) - (P-S) > 0$, Eqs. (64) and (66) guarantee that $\chi > 1$ and that a $\chi (> 1)$ value exists. Therefore, an extortioner with $\chi > 1$ exists if and only if $w > w_c$, where the w_c value coincides with that for the equalizer; it is given by Eq. (32). Under $w > w_c$, Eqs. (64) and (66) imply

$$\chi \geq \chi_c(w) \equiv \max \left(\frac{R-S - w(P-S)}{w(T-P) - (T-R)}, \frac{T-P - w(T-R)}{w(R-S) - (P-S)} \right). \quad (67)$$

Eq. (67) gives the range of χ values for which the extortioner strategy exists. The conditions for the existence of an extortionate strategy are easier to satisfy for large w in the sense that $\chi_c(w)$ monotonically decreases as w increases. In particular, we obtain $\lim_{w \rightarrow w_c+0} \chi_c(w) = \infty$ and $\lim_{w \rightarrow 1} \chi_c(w) = 1$.

For a given χ value, the substitution of $R = b - c$, $T = b$, $S = -c$, and $P = 0$ in Eq. (32) yields

$$w_c = \frac{\chi c + b}{\chi b + c}, \quad (68)$$

which is consistent with Eq. (7) of McAvoy and Hauert (2016).

3.4. Generous strategy

3.4.1. Expression

The generous strategy, also called compliers, is defined as a strategy that yields a larger shortfall from the mutual cooperation payoff R for the player as compared to that for the co-player (Hilbe et al., 2013b; Stewart and Plotkin, 2012; 2013). We obtain the generous strategy by setting $\kappa = R$ in Eq. (16). By setting $\kappa = R$ in Eq. (43), we obtain

$$\mathbf{p} = \begin{pmatrix} \frac{1 - p_0(1-w)}{w} \\ p_{CD} \\ \frac{- (1-w)p_0(\chi+1)(T-S) + (1-wp_{CD})[(\chi-1)R + T - \chi S]}{w[R-S + \chi(T-R)]} \\ \frac{(1-w)p_0[(\chi-1)P + S - \chi T] + (1-wp_{CD})(\chi-1)(R-P)}{w[R-S + \chi(T-R)]} \end{pmatrix}. \quad (69)$$

Because $p_{CC} = [1 - (1-w)p_0]/w \leq 1$, we obtain $p_0 = 1$ and $p_{CC} = 1$, which is consistent with the previously obtained result (Hilbe et al., 2015a). Therefore, the generous strategy is never the first to detect and hence a so-called nice strategy (Axelrod, 1984; Hilbe et al., 2015a).

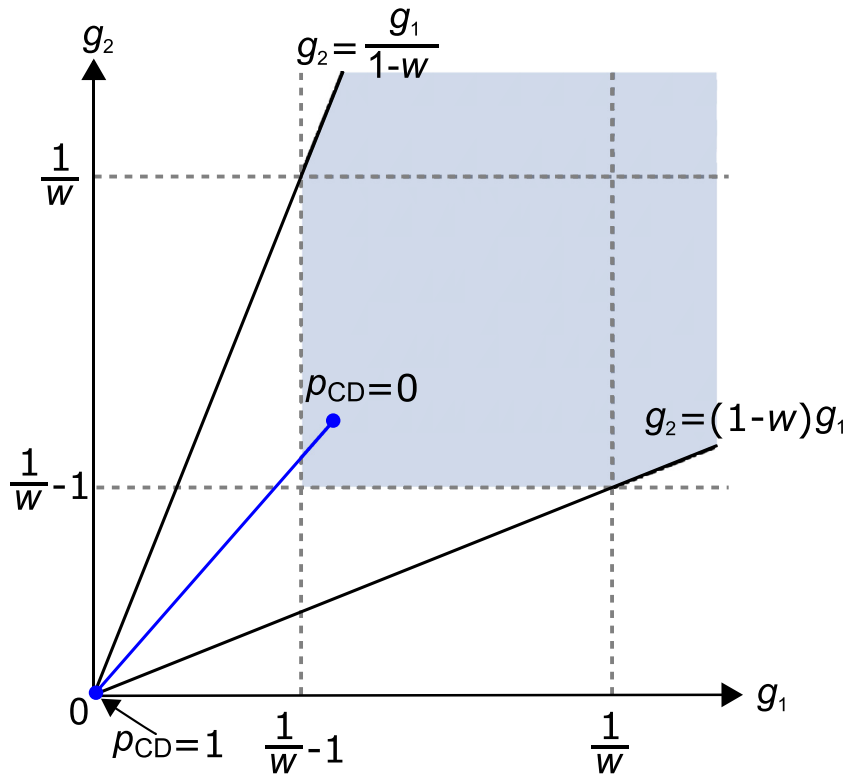


Fig. 2. Region in the g_1 - g_2 space where the generous strategy exists (shaded region). If (g_1, g_2) is located in this region (e.g., filled circle labeled $p_{CD} = 0$), the square given by $1/w - 1 \leq g_1, g_2 \leq 1/w$ intersects the line segment connecting the assumed (g_1, g_2) and the origin. Note that any point on the line segment is realized by the solution by a value of p_{CD} (Eqs. (71) and (72)).

By setting $p_0 = 1$ in Eq. (69), we obtain

$$\begin{aligned}
 \mathbf{p} &= \begin{pmatrix} 1 \\ p_{CD} \\ \frac{-(1-w)(\chi+1)(T-S) + (1-wp_{CD})[(\chi-1)R+T-\chi S]}{w[R-S+\chi(T-R)]} \\ \frac{(1-w)[(\chi-1)P+S-\chi T] + (1-wp_{CD})(\chi-1)(R-P)}{w[R-S+\chi(T-R)]} \end{pmatrix} \\
 &= \begin{pmatrix} 1 \\ p_{CD} \\ 1 - \frac{1}{w} + \frac{(1-p_{CD})[(\chi-1)R+T-\chi S]}{R-S+\chi(T-R)} \\ 1 - \frac{1}{w} + \frac{(1-p_{CD})(\chi-1)(R-P)}{R-S+\chi(T-R)} \end{pmatrix}. \tag{70}
 \end{aligned}$$

3.4.2. Minimum discount rate

By applying $0 \leq p_{DC}, p_{DD} \leq 1$ to Eq. (70), we obtain

$$\frac{1}{w} - 1 \leq (1 - p_{CD})g_1 \leq \frac{1}{w}, \tag{71}$$

$$\frac{1}{w} - 1 \leq (1 - p_{CD})g_2 \leq \frac{1}{w}, \tag{72}$$

where

$$g_1 \equiv \frac{(\chi-1)R+T-\chi S}{R-S+\chi(T-R)}, \tag{73}$$

$$g_2 \equiv \frac{(\chi-1)(R-P)}{R-S+\chi(T-R)}. \tag{74}$$

The necessary and sufficient condition for $0 \leq p_{CD} \leq 1$ that satisfies Eqs. (71) and (72) to exist is given by (Fig. 2)

$$g_1 \geq \frac{1}{w} - 1, \tag{75}$$

$$g_2 \geq \frac{1}{w} - 1, \quad (76)$$

$$1 - w \leq \frac{g_2}{g_1} \leq \frac{1}{1 - w}. \quad (77)$$

In the remainder of this section, we assume $\chi \geq 0$, which a generous strategy requires (Chen and Zinger, 2014; Hilbe et al., 2013b; 2015a; McAvooy and Hauert, 2016; 2017; Stewart and Plotkin, 2013), and examine the conditions given by Eqs. (75)–(77). For mathematical interests, the analysis of the minimum discount rate for $\chi < 0$ is presented in Appendix G.2. First, because $dg_1/d\chi > 0$, which one can derive using Eq. (3), and g_1 is continuous for $\chi \geq 0$, Eq. (75) is equivalent to

$$\chi \geq \frac{R - S - w(T - S)}{-(T - R) + w(T - S)} \quad (78)$$

and

$$w > \frac{T - R}{T - S}. \quad (79)$$

When $w \leq (T - R)/(T - S)$, a positive χ value that satisfies Eq. (75) does not exist. Second, because $dg_2/d\chi > 0$ and g_2 is continuous for $\chi \geq 0$, Eq. (76) is equivalent to

$$\chi \geq \frac{R - S - w(P - S)}{-(T - R) + w(T - P)} \quad (80)$$

and

$$w > \frac{T - R}{T - P}. \quad (81)$$

When $w \leq (T - R)/(T - P)$, a positive χ value that satisfies Eq. (76) does not exist. Third, because $d(g_2/g_1)/d\chi > 0$ and g_2/g_1 is continuous for $\chi \geq 0$, Eq. (77) is equivalent to

$$\chi \geq \frac{T - P - w(T - R)}{-(P - S) + w(R - S)} \quad (82)$$

and

$$w > \frac{P - S}{R - S}. \quad (83)$$

When $w \leq (P - S)/(R - S)$, a positive χ value that satisfies Eq. (77) does not exist.

By combining Eqs. (79), (81), and (83), we find that a generous strategy exists if and only if $w > w_c$, where w_c is given by Eq. (32). Therefore, the threshold w value above which a ZD strategy exists is the same for the equalizer, extortioner, and generous strategy. It should be noted that $w = w_c$ is allowed for the equalizer, but not for the extortioner and the generous strategy. When $w > w_c$, Eq. (80) implies Eq. (78), and hence one obtains

$$\chi \geq \chi_c(w) \equiv \max \left(\frac{R - S - w(P - S)}{-(T - R) + w(T - P)}, \frac{T - P - w(T - R)}{-(P - S) + w(R - S)} \right). \quad (84)$$

Note that $\chi_c(w) > 1$ and $\chi_c(w)$ decreases as $w (> w_c)$ increases. Eq. (84) implies that $\lim_{w \rightarrow w_c + 0} \chi_c(w) = \infty$ and $\lim_{w \rightarrow 1} \chi_c(w) = 1$, which are the same asymptotic as the case of the extortioner.

4. Conclusions

We analyzed ZD strategies in finitely repeated prisoner's dilemma games with general payoff matrices. Apart from the derivation of convenient expressions for ZD strategies, the novel results derived in the present article are two-fold. First, we derived the threshold discount factor value, w_c , above which the ZD strategies exist for three commonly studied classes of ZD strategies, i.e., equalizer, extortioner, and generous strategies. They all share the same threshold value. Similar to the case of the condition for mutual cooperation in direct reciprocity, ZD strategies can exist only when there are sufficiently many rounds. Second, we showed that the memory-one strategies that impose a linear relationship between the payoff of the two players are either ZD strategies (Eqs. (43) and (53)) or an unconditional strategy (Eq. (51)). The latter class includes the unconditional cooperator and unconditional defector as special cases. Therefore, for finitely repeated prisoner's dilemma games (i.e., $w < 1$), we answered affirmatively to the conjecture posed in Hilbe et al. (2013b). With a continuity argument, our results also cover the infinite case, by the consideration of the limit $w \rightarrow 1$. In other words, if the two payoffs are in a linear relationship for any $w = 1 - \epsilon$, where $\epsilon \ll 1$, then the payoffs are also on a line as ϵ goes to 0. For a similar argument, see Eqs. (5) and (6) in Hilbe et al. (2015a). The present results also hold true when the co-player employs a longer-memory strategy, because it is straightforward to apply the proof for the infinite case (Press and Dyson, 2012) to the finite case.

Our analytical approach is different from the previous approaches. Press and Dyson's derivation is based on the linear algebra of matrices (Press and Dyson, 2012). The proof in Hilbe et al. (2013a) considers certain telescoping sums. The approach considered in the present study is more elementary than theirs, i.e., to derive necessary conditions and show that they are sufficient by straightforward calculations.

We mention possible directions of future research. First, we conjecture that the w_c value is the same for all ZD strategies because it takes the same value for the three common ZD strategies. Second, the explicit forms of our solutions (Eqs. (25) and (43)) may be useful for exploring features of ZD strategies in finitely repeated games. For example, investigation of evolutionary dynamics and extensions to multiplayer games, which have been examined for infinitely repeated games (see Section 1 for references), in the case of finitely repeated games may benefit from the present results.

Acknowledgments

We acknowledge Christian Hilbe, Shun Kurokawa, and Kohei Tamura for valuable comments on the manuscript. G.I. acknowledges the support by [HAYAO NAKAYAMA Foundation for Science & Technology and Culture](#). N.M. acknowledges the support provided through JST, [CREST](#), Japan (No. [JPMJCR1304](#)), and JST, [ERATO](#), Kawarabayashi Large Graph Project, Japan ([JPMJER1201](#)).

Appendix A. Expression of $\mathbf{u}^{\text{eq},0000}$, $\pi_{Y,0000}$, $\mathbf{u}^{\text{eq},1111}$, and $\pi_{Y,1111}$

By substituting $\mathbf{q} = (0, 0, 0, 0)$ in [Eq. \(10\)](#) and then substituting the obtained M in [Eq. \(18\)](#), we obtain

$$\mathbf{u}^{\text{eq},0000} = \frac{1}{(1-w)(1-wp_{CD}+wp_{DD})} \times \begin{pmatrix} (1-w)(1-wp_{CD}+wp_{DD})R + w(1-p_{CC}+wp_{CC}-wp_{CD})P + w(p_{CC}-wp_{CC}+wp_{DD})T \\ w(1-p_{CD})P + (1-w+wp_{DD})T \\ (1-w)(1-wp_{CD}+wp_{DD})S + w(1-p_{DC}-wp_{CD}+wp_{DC})P + w(p_{DC}-wp_{DC}+wp_{DD})T \\ (1-wp_{CD})P + wp_{DD}T \end{pmatrix}, \quad (85)$$

which leads to

$$\pi_{Y,0000} = (1-w)\mathbf{v}(0)\mathbf{u}^{\text{eq},0000}. \quad (86)$$

Similarly, by substituting $\mathbf{q} = (1, 1, 1, 1)$ in [Eq. \(10\)](#) and then substituting the obtained M in [Eq. \(18\)](#), we obtain

$$\mathbf{u}^{\text{eq},1111} = \frac{1}{(1-w)(1-wp_{CC}+wp_{DC})} \times \begin{pmatrix} w(1-p_{CC})S + (1-w+wp_{DC})R \\ (1-w)(1-wp_{CC}+wp_{DC})T + w(1-p_{CD}-wp_{CC}+wp_{CD})S + w(p_{CD}-wp_{CD}+wp_{DC})R \\ (1-wp_{CC})S + wp_{DC}R \\ (1-w)(1-wp_{CC}+wp_{DC})P + w(1-p_{DD}-wp_{CC}+wp_{DD})S + w(p_{DD}+wp_{DC}-wp_{DD})R \end{pmatrix}, \quad (87)$$

which leads to

$$\pi_{Y,1111} = (1-w)\mathbf{v}(0)\mathbf{u}^{\text{eq},1111}. \quad (88)$$

Appendix B. Expression of $\mathbf{u}^{\text{zd},0000}$ and $\mathbf{u}^{\text{zd},1111}$, and four necessary conditions in [Section 3.2](#)

By substituting $\mathbf{q} = (0, 0, 0, 0)$ in [Eq. \(10\)](#) and then substituting the obtained M in [Eq. \(36\)](#), we obtain

$$\mathbf{u}^{\text{zd},0000} = \begin{pmatrix} \frac{-w(1-p_{CC}+wp_{CC}-wp_{CD})(\chi-1)P + w(p_{CC}-wp_{CC}+wp_{DD})(S-\chi T)}{1-wp_{CD}+wp_{DD}} + (\chi-1)\kappa - (1-w)(\chi-1)R \\ \frac{-w(1-p_{CD})(\chi-1)P + (1-w+wp_{DD})(S-\chi T)}{1-wp_{CD}+wp_{DD}} + (\chi-1)\kappa \\ \frac{-w(1-p_{DC}-wp_{CD}+wp_{DC})(\chi-1)P + w(p_{DC}-wp_{DC}+wp_{DD})(S-\chi T)}{1-wp_{CD}+wp_{DD}} + (\chi-1)\kappa + (1-w)(T-\chi S) \\ \frac{-(1-wp_{CD})(\chi-1)P + wp_{DD}(S-\chi T)}{1-wp_{CD}+wp_{DD}} + (\chi-1)\kappa \end{pmatrix}. \quad (89)$$

By substituting $\mathbf{q} = (1, 1, 1, 1)$ in [Eq. \(10\)](#) and then substituting the obtained M in [Eq. \(36\)](#), we obtain

$$\mathbf{u}^{\text{zd},1111} = \begin{pmatrix} \frac{w(1-p_{CC})(T-\chi S) - (1-w+wp_{DC})(\chi-1)R}{1-wp_{CC}+wp_{DC}} + (\chi-1)\kappa \\ \frac{w(1-p_{CD}-wp_{CC}+wp_{CD})(T-\chi S) - w(p_{CD}-wp_{CD}+wp_{DC})(\chi-1)R}{1-wp_{CC}+wp_{DC}} + (\chi-1)\kappa + (1-w)(S-\chi T) \\ \frac{(1-wp_{CC})(T-\chi S) - wp_{DC}(\chi-1)R}{1-wp_{CC}+wp_{DC}} + (\chi-1)\kappa \\ \frac{w(1-p_{DD}-wp_{CC}+wp_{DD})(T-\chi S) - w(p_{DD}+wp_{DC}-wp_{DD})(\chi-1)R}{1-wp_{CC}+wp_{DC}} + (\chi-1)\kappa - (1-w)(\chi-1)P \end{pmatrix}. \quad (90)$$

Note that the denominator on the right-hand side of [Eqs. \(89\)](#) and [\(90\)](#) is positive.

By substituting [Eq. \(89\)](#) in [Eq. \(39\)](#), we obtain

$$\begin{aligned} & (1-w)p_0\{(1-wp_{CD}+wp_{DD})[-(\chi-1)R-T+\chi S] + w(p_{CC}-p_{DC})[(\chi-1)P+S-\chi T]\} \\ & + (1-wp_{CD}+wp_{DD})[(\chi-1)\kappa + (1-w)(T-\chi S)] + w[-(1-p_{DC}-wp_{CD}+wp_{DC})(\chi-1)P \\ & + (p_{DC}-wp_{DC}+wp_{DD})(S-\chi T)] = 0. \end{aligned} \quad (91)$$

By substituting [Eq. \(89\)](#) in [Eq. \(40\)](#), we obtain

$$(1 - w)p_0[(\chi - 1)P + S - \chi T] - (1 - wp_{CD})(\chi - 1)P + wp_{DD}(S - \chi T) + (1 - wp_{CD} + wp_{DD})(\chi - 1)\kappa = 0. \tag{92}$$

By substituting Eq. (90) in Eq. (39), we obtain

$$(1 - w)p_0[-(\chi - 1)R - T + \chi S] + (1 - wp_{CC})(T - \chi S) - wp_{DC}(\chi - 1)R + (1 - wp_{CC} + wp_{DC})(\chi - 1)\kappa = 0. \tag{93}$$

By substituting Eq. (90) in Eq. (40), we obtain

$$(1 - w)p_0\{(1 - wp_{CC} + wp_{DC})[(\chi - 1)P + S - \chi T] + w(p_{CD} - p_{DD})[-(\chi - 1)R - T + \chi S]\} + (1 - wp_{CC} + wp_{DC})(\chi - 1)[\kappa - (1 - w)P] + w[(1 - p_{DD} - wp_{CC} + wp_{DD})(T - \chi S) - (p_{DD} + wp_{DC} - wp_{DD})(\chi - 1)R] = 0. \tag{94}$$

Appendix C. Derivation of Eq. (51)

In this section, we derive Eq. (51) from Eqs. (49) to (50).

We obtain

$$u^{zd,1000} = \left(\begin{array}{c} \frac{1}{(1 - wp_{CC})(1 - wp_{CD} + wp_{DD})} \\ \times \{ \{ -(1 - w)(1 - wp_{CD} + wp_{DD})R - w^2(1 - p_{CC})[1 - (1 - w)p_{DC} - wp_{CD}]P \} (\chi - 1) \\ + w(1 - p_{CC})\{ (1 - w)(1 - wp_{CD} + wp_{DD})(T - \chi S) + w[(1 - w)p_{DC} + wp_{DD}](S - \chi T) \} \} + (\chi - 1)\kappa \\ - \frac{w(1 - p_{CD})(\chi - 1)P + (1 - w + wp_{DD})(S - \chi T)}{1 - wp_{CD} + wp_{DD}} + (\chi - 1)\kappa \\ - \frac{w(1 - p_{DC} - wp_{CD} + wp_{DC})(\chi - 1)P + w(p_{DC} - wp_{DC} + wp_{DD})(S - \chi T)}{1 - wp_{CD} + wp_{DD}} + (\chi - 1)\kappa + (1 - w)(T - \chi S) \\ - \frac{(1 - wp_{CD})(\chi - 1)P + wp_{DD}(S - \chi T)}{1 - wp_{CD} + wp_{DD}} + (\chi - 1)\kappa \end{array} \right). \tag{95}$$

Note that the denominator on the right-hand side of Eq. (95) is positive. By substituting Eq. (95) in Eq. (40), we obtain Eq. (92). By substituting Eq. (95) in Eq. (39), we obtain

$$p_0(1 - w)\{ -(1 - wp_{CD} + wp_{DD})R + w[1 - (1 - w)p_{DC} - wp_{CD}]P \} (\chi - 1) - (1 - w)(1 - wp_{CD} + wp_{DD})(T - \chi S) - w[(1 - w)p_{DC} + wp_{DD}](S - \chi T) + (1 - wp_{CC})\{ -w[1 - (1 - w)p_{DC} - wp_{CD}](\chi - 1)P + w[(1 - w)p_{DC} + wp_{DD}](S - \chi T) \} + (1 - wp_{CC})(1 - wp_{CD} + wp_{DD})(\chi - 1)\kappa + (1 - wp_{CC})(1 - wp_{CD} + wp_{DD})(1 - w)(T - \chi S) = 0. \tag{96}$$

Substitution of Eqs. (41) and (42) in Eq. (96) yields either the third entry of Eq. (43) or

$$\frac{(p_0 - p_{DC})(\kappa - R)(1 - w)w(\chi - 1)[(\chi - 1)P + S - \chi T]}{[T - \kappa + \chi(\kappa - S)][\kappa - S + \chi(T - \kappa)]} = 0. \tag{97}$$

The case in which the denominator on the right-hand side of Eq. (97) is equal to 0 is covered in Appendix E and Appendix F. We note that $\chi \neq 1$ because $\chi = 1$ substituted in Eq. (50) yields $T = S$, which contradicts Eq. (2). By combining this observation with $0 < w < 1$, we obtain

$$(p_0 - p_{DC})(\kappa - R)[(\chi - 1)P + S - \chi T] = 0. \tag{98}$$

By substituting Eqs. (49) and (50) in Eq. (98), we obtain the following four possible cases: $p_0 = p_{DC}$, $p_0 = 1$, $p_0 = (R - P)/(T + S - R - P)$, and $p_0 = (T + S - 2P)/(T + S - R - P)$.

First, assume that $p_0 = p_{DC}$. By substituting $p_0 = p_{DC}$ and Eq. (47) in Eq. (93), we obtain $(p_{CC} - p_{DC})[T - \kappa + \chi(\kappa - S)] = 0$. Because we have excluded the case $T - \kappa + \chi(\kappa - S) = 0$, which we deal with in Appendix E, we obtain $p_{CC} = p_{DC}$. Therefore, we obtain

$$p_0 = p_{CC} = p_{DC}. \tag{99}$$

Second, assume that $p_0 = 1$. Substitution of $p_0 = 1$ in Eq. (49) yields $\kappa = R$. Substitution of $p_0 = 1$ and $\kappa = R$ in Eq. (42) yields $p_{CC} = 1$. Substitution of $p_0 = 1$ in Eq. (50) yields $\chi = -(R - S)/(T - R)$. Substitution of $p_0 = 1$, $\chi = -(R - S)/(T - R)$, and $\kappa = R$ in Eq. (92) yields $(1 - p_{CD})(T - S)(R - P) = 0$, which implies $p_{CD} = 1$. Therefore, $p_0 = 1$ combined with Eqs. (49) and (50) results in

$$p_0 = p_{CC} = p_{CD} = 1. \tag{100}$$

Third, we note that

$$p_0 \neq \frac{R - P}{T + S - R - P} \tag{101}$$

because combination of $p_0 = (R - P)/(T + S - R - P)$ and $0 \leq p_0 \leq 1$ leads to $T + S - R - P > 0$ and $2R \leq T + S$, and the latter inequality contradicts Eq. (3).

Fourth, assume that $p_0 = (T + S - 2P)/(T + S - R - P)$. By substituting $p_0 = (T + S - 2P)/(T + S - R - P)$ in Eqs. (49) and (50), we obtain $\chi = -(P - S)/(T - P)$ and $\kappa = P$, respectively. Then, we obtain $\kappa - S + \chi(T - \kappa) = 0$, which we have decided to deal with later.

To summarize, Eq. (98) leads to either Eq. (99) or (100).

We obtain

$$\mathbf{u}^{\text{zd},0001} = (\chi - 1)\kappa \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{(1+w)(1-wp_{\text{CD}}) + w^2[p_{\text{DC}}(1-p_{\text{DD}}) + p_{\text{CC}}p_{\text{DD}}]} \times \begin{pmatrix} \{[-1 + wp_{\text{CD}} + w^2(1-p_{\text{DC}})(1-p_{\text{DD}}) - w^3(p_{\text{CD}}-p_{\text{DC}})(1-p_{\text{DD}})]R \\ + w[-1 + (1-w)p_{\text{CC}} + wp_{\text{CD}}]P\}(\chi - 1) + w\{w(1-p_{\text{DD}})[(1-wp_{\text{CD}}) - (1-w)p_{\text{CC}}](T - \chi S) \\ + [p_{\text{CC}} - w^2(1-p_{\text{DD}})(p_{\text{CC}} - p_{\text{DC}})](S - \chi T)\} \\ w(1-p_{\text{CD}})[- (P + wp_{\text{DD}}R)(\chi - 1) + w(1-p_{\text{DD}})(T - \chi S)] \\ + [1 - w^2(1-p_{\text{DC}} - p_{\text{CC}}p_{\text{DD}} + p_{\text{DC}}p_{\text{DD}})](S - \chi T) \\ w[-1 + wp_{\text{CD}} + (1-w)p_{\text{DC}}](P + wp_{\text{DD}}R)(\chi - 1) \\ + \{1 - w^2p_{\text{DD}}[1 - (1-w)p_{\text{CC}}] - wp_{\text{CD}}(1 - w^2p_{\text{DD}})\}(T - \chi S) + w[p_{\text{DC}} + w^2(p_{\text{CC}} - p_{\text{DC}})p_{\text{DD}}](S - \chi T) \\ - [(1-wp_{\text{CD}})P + wp_{\text{DD}}(1-wp_{\text{CD}})R](\chi - 1) + w\{(1-p_{\text{DD}})(1-wp_{\text{CD}})(T - \chi S) \\ + w(p_{\text{DC}} + p_{\text{CC}}p_{\text{DD}} - p_{\text{DC}}p_{\text{DD}})(S - \chi T)\} \end{pmatrix}. \tag{102}$$

Note that the denominator on the right-hand side of Eq. (102) is positive. By substituting Eq. (102) in Eq. (40), we obtain

$$(1-w)p_0\{(P + wp_{\text{DD}}R)(\chi - 1) - w(1-p_{\text{DD}})(T - \chi S) + (1+w)(S - \chi T)\} - [(1-wp_{\text{CD}})P \\ + wp_{\text{DD}}(1-wp_{\text{CD}})R](\chi - 1) + w\{(1-p_{\text{DD}})(1-wp_{\text{CD}})(T - \chi S) + w[p_{\text{DC}} + p_{\text{DD}}(p_{\text{CC}} - p_{\text{DC}})](S - \chi T)\} \\ + \{1 + w(1-p_{\text{CD}}) - w^2[p_{\text{CD}} - p_{\text{DC}} - p_{\text{DD}}(p_{\text{CC}} - p_{\text{DC}})]\}(\chi - 1)\kappa = 0. \tag{103}$$

Substitution of Eqs. (41) and (42) in Eq. (103) yields either the third entry of Eq. (43) or

$$\frac{1}{[T - \kappa + \chi(\kappa - S)][\kappa - S + \chi(T - \kappa)]} \times \{w(\chi - 1)^2\kappa^2 - [1 - (1-w)p_0 - wp_{\text{CD}}](\chi - 1)[T - R + \chi(R - S)]P \\ + \{w(T - \chi S) + (1-w)p_0[T - R + \chi(R - S)]\}(S - \chi T) - \{-(1-wp_{\text{CD}})(\chi - 1)R \\ - [1 + w(1-p_{\text{CD}} - \chi)]T + [\chi - w(1 - \chi + p_{\text{CD}}\chi)]S\}(\chi - 1)\kappa\} = 0. \tag{104}$$

We examine the case in which the denominator on the right-hand side of Eq. (104) is zero in Appendix E and Appendix F. Therefore, we ignore the denominator and substitute Eqs. (49) and (50) in Eq. (104) to obtain $p_0 = p_{\text{CD}}$, $p_0 = 0$, $p_0 = (R - P)/(T + S - R - P)$, or $p_0 = (T + S - 2P)/(T + S - R - P)$. Among these four possible options, we have excluded $p_0 = (R - P)/(T + S - R - P)$ and $p_0 = (T + S - 2P)/(T + S - R - P)$ in the course of the analysis of $\mathbf{u}^{\text{zd},1000}$.

First, assume that $p_0 = p_{\text{CD}}$. By substituting $p_0 = p_{\text{CD}}$ and Eq. (48) in Eq. (92), we obtain $(p_{\text{CD}} - p_{\text{DD}})[\kappa - S + \chi(T - \kappa)] = 0$. Because we have excluded the case $\kappa - S + \chi(T - \kappa) = 0$, which we deal with in Appendix E, we obtain $p_{\text{DD}} = p_{\text{CD}}$. Therefore, we obtain

$$p_0 = p_{\text{CD}} = p_{\text{DD}}. \tag{105}$$

Second, assume that $p_0 = 0$. Substitution of $p_0 = 0$ in Eq. (49) yields $\kappa = P$. Substitution of $p_0 = 0$ and $\kappa = P$ in Eq. (41) yields $p_{\text{DD}} = 0$. Substitution of $p_0 = 0$ in Eq. (50) yields $\chi = -(T - P)/(P - S)$. Substitution of $p_0 = 0$, $\chi = -(T - P)/(P - S)$, and $\kappa = P$ in Eq. (93) yields $wp_{\text{DC}}(R - P) = 0$, which implies $p_{\text{DC}} = 0$. Therefore, $p_0 = 0$ combined with Eqs. (49) and (50) results in

$$p_0 = p_{\text{DC}} = p_{\text{DD}} = 0. \tag{106}$$

A solution must simultaneously satisfy either Eq. (99) or (100), and either Eq. (105) or (106). The combination of Eqs. (99) and (105) provides the set of unconditional strategies, i.e., Eq. (51). The combination of Eqs. (99) and (106) provides a subset of the strategies given by Eq. (51). The combination of Eqs. (100) and (105) also provides a subset of the strategies given by Eq. (51). Eqs. (100) and (106) are inconsistent with each other. Therefore, the set of solutions is given by Eq. (51).

Appendix D. An unconditional strategy is not a ZD strategy unless $R + P = T + S$

In this section, we show that the unconditional strategy given by Eq. (51) is not a ZD strategy in the sense of Hilbe et al. (2015a) if $R + P \neq T + S$.

By substituting $p_{\text{DC}} = p_{\text{DD}}$ in Eq. (45), we obtain

$$\frac{\phi[(\chi - 1)\kappa + T - \chi S] - (1-w)p_0}{w} = \frac{\phi(\chi - 1)(\kappa - P) - (1-w)p_0}{w}, \tag{107}$$

which leads to

$$\phi[(\chi - 1)\kappa + T - \chi S] = \phi(\chi - 1)(\kappa - P).$$

If $\phi = 0$, we substitute $\phi = 0$ in the expression of p_{DD} in Eq. (45) to obtain $p_{\text{DD}} = -(1-w)p_0/w$. This equation holds true if and only if $p_0 = p_{\text{DD}} = 0$. Next, we substitute $\phi = 0$ in the expression of p_{CC} in Eq. (45) to obtain $p_{\text{CC}} = [1 - (1-w)p_0]/w$. This equation holds true if and only if $p_0 = p_{\text{CC}} = 1$, which contradicts $p_0 = 0$. Therefore, we obtain $\phi \neq 0$. Given $\phi \neq 0$, Eq. (107) implies

$$\chi = -\frac{T - P}{P - S}. \tag{108}$$

By setting $p_{CC} = p_{CD}$ in Eq. (45) and using $\phi \neq 0$, we obtain

$$\chi = -\frac{R-S}{T-R}. \quad (109)$$

By combining Eqs. (108) and (109), we obtain

$$R + P = T + S. \quad (110)$$

Eq. (110) is a sufficient condition for the unconditional strategy to be a ZD because substitution of Eqs. (49), (50), (110) and

$$\phi = -\frac{T-R}{(T-S)(R-P)} \quad (111)$$

in Eq. (45) yields Eq. (51).

Appendix E. Case $\kappa - S + \chi(T - \kappa) = 0$

In this section, we assume

$$\kappa - S + \chi(T - \kappa) = 0 \quad (112)$$

and derive the set of strategies that satisfy Eq. (16).

By substituting Eq. (112) in Eq. (92), we obtain

$$(\chi - 1)(\kappa - P)[(1 - w)p_0 - 1 + wp_{CD}] = 0. \quad (113)$$

Eq. (112) does not allow $\chi = 1$ because substitution of $\chi = 1$ in Eq. (112) yields $T = S$, which contradicts Eq. (2). Substitution of $\kappa = P$ in Eq. (112) yields $\chi = -(P - S)/(T - P)$. Alternatively, if we set $(1 - w)p_0 - 1 + wp_{CD} = 0$, we obtain $p_0 = p_{CD} = 1$. Therefore, we consider the following two subcases, i.e., subcase (A) specified by

$$\kappa = P \quad (114)$$

and

$$\chi = -\frac{P-S}{T-P}, \quad (115)$$

and subcase (B) specified by

$$\kappa - S + \chi(T - \kappa) = 0 \quad (116)$$

and

$$p_0 = p_{CD} = 1. \quad (117)$$

E1. Subcase (A): $\kappa = P$ and $\chi = -(P - S)/(T - P)$

By substituting Eqs. (114) and (115) in Eq. (91), we obtain

$$\frac{(1-w)[1-w(p_{CD}-p_{DD})][-p_0(T+S-R-P)+T+S-2P](T-S)}{T-P} = 0. \quad (118)$$

Because $T > P > S$, $0 < w < 1$, and there exists no pair of p_{CD} and p_{DD} ($0 \leq p_{CD}, p_{DD} \leq 1$) that satisfies $p_{CD} - p_{DD} = 1/w$, we obtain

$$p_0(T+S-R-P) = T+S-2P. \quad (119)$$

If we set $T+S-R-P = 0$, we obtain $T+S-2P = R-P > 0$, which contradicts Eq. (119). Therefore, Eq. (119) leads to $T+S-R-P \neq 0$, and hence

$$p_0 = \frac{T+S-2P}{T+S-R-P}. \quad (120)$$

If $T+S-R-P > 0$, the condition $p_0 \leq 1$ applied to Eq. (120) yields $R \leq P$, which contradicts Eq. (2). Therefore, we obtain $T+S-R-P < 0$ and hence $T+S-2P \leq 0$.

By substituting Eqs. (114) and (115) in Eq. (96), we obtain

$$\frac{(1-w)[1-w(p_{CD}-p_{DD})]\{p_0[R-(1-w)(T+S)]+(1-wp_{CC})(T+S)-[2-(1-2w)p_0-2wp_{CC}]P\}(T-S)}{T-P} = 0. \quad (121)$$

Because $1-w(p_{CD}-p_{DD}) > 0$, Eq. (121) implies

$$p_0[R-(1-w)(T+S)]+(1-wp_{CC})(T+S)-[2-(1-2w)p_0-2wp_{CC}]P = 0. \quad (122)$$

Substitution of Eq. (120) in Eq. (122) yields

$$\frac{w[-p_{CC}(T+S-R-P)+T+S-2P](T+S-2P)}{T+S-R-P} = 0. \quad (123)$$

We will deal with the case $T+S-2P = 0$ later in this section. Therefore, by assuming $T+S-2P < 0$, we obtain

$$p_{CC} = \frac{T+S-2P}{T+S-R-P}. \quad (124)$$

By substituting Eqs. (114) and (115) in Eq. (93), we obtain

$$\frac{\{(1-w)p_0(R-S-T)+wp_{DC}R-wp_{CC}(T+S)-[2-(1-w)p_0-2wp_{CC}+wp_{DC}]P+T+S\}(T-S)}{T-P} = 0. \quad (125)$$

By substituting $p_0 = p_{CC} = (T + S - 2P)/(T + S - R - P)$ in Eq. (125), we obtain

$$\frac{w[-p_{DC}(T + S - R - P) + T + S - 2P](P - R)(T - S)}{(T - P)(T + S - R - P)} = 0, \tag{126}$$

which leads to

$$p_{DC} = \frac{T + S - 2P}{T + S - R - P}. \tag{127}$$

By substituting Eqs. (114) and (115) in Eq. (103), we obtain

$$w[1 - wp_{CD} - (1 - w)p_0][-p_{DD}(T + S - R - P) + T + S - 2P](T - S) = 0. \tag{128}$$

If $1 - wp_{CD} - (1 - w)p_0 = 0$, we obtain $p_0 = p_{CD} = 1$, which contradicts Eq. (120). Therefore, Eq. (128) implies

$$p_{DD} = \frac{T + S - 2P}{T + S - R - P}. \tag{129}$$

To derive another condition, we use the vector \mathbf{u} when player Y adopts the tit-for-tat strategy, i.e., $\mathbf{q} = (1, 0, 1, 0)$. This vector, denoted by $\mathbf{u}^{zd,1010}$, is given by

$$\mathbf{u}^{zd,1010} = (\chi - 1)\kappa \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{(1 - wp_{CC})(1 - w^2 p_{DC}) + w^2 p_{CD} p_{DC}(1 - w) + w(1 + w)p_{DD}(1 - wp_{CC}) + w^3 p_{CD} p_{DD}} \times \begin{pmatrix} \{-[1 - w(1 - p_{DD}) - w^2 p_{DC}(1 - p_{CD}) + w^3(1 - p_{CD})(p_{DC} - p_{DD})]R \\ -w^2(1 - p_{CC})(1 - p_{DC})P\}(\chi - 1) + w(1 - p_{CC})[1 - w(1 - p_{DD})](T - \chi S) \\ +w^2(1 - p_{CC})[p_{DC} - w(p_{DC} - p_{DD})](S - \chi T) \\ \{-w^2(1 - p_{DC})[1 - (1 - w)p_{CD} - wp_{CC}]P - wp_{CD}[1 - w(1 - p_{DD})]R\}(\chi - 1) \\ +w[1 - (1 - w)p_{CD} - wp_{CC}][1 - w(1 - p_{DD})](T - \chi S) + (1 - wp_{CC})[1 - w(1 - p_{DD})](S - \chi T) \\ w\{-(1 - p_{DC})(1 - wp_{CC})P - wp_{CD}[(1 - w)p_{DC} + wp_{DD}]R\}(\chi - 1) \\ + (1 - wp_{CC})[1 - w(1 - p_{DD})](T - \chi S) + w[(1 - w)p_{DC} + wp_{DD}](1 - wp_{CC})(S - \chi T) \\ \{-[1 - w^2 p_{DC}[1 - p_{CD}(1 - w)] - wp_{CC}(1 - w^2 p_{DC})\}P - w^2 p_{CD} p_{DD} R\}(\chi - 1) \\ +w^2 p_{DD}[1 - p_{CD}(1 - w) - wp_{CC}](T - \chi S) + wp_{DD}(1 - wp_{CC})(S - \chi T) \end{pmatrix}. \tag{130}$$

Note that the denominator on the right-hand side of Eq. (130) is positive. By substituting Eq. (130) in Eq. (40), we obtain

$$(1 - w)p_0 \left\{ -wp_{CD}R + [1 + w(1 - p_{CC}) - w^2(p_{CC} - p_{CD})]P \right\}(\chi - 1) + w[1 - (1 - w)p_{CD} - wp_{CC}](T - \chi S) \\ + (1 - wp_{CC})(S - \chi T) + \left\{ -1 + w^2 p_{DC}[1 - (1 - w)p_{CD}] + wp_{CC}(1 - w^2 p_{DC}) \right\}P - w^2 p_{CD} p_{DD} R \right\}(\chi - 1) \\ + w^2 p_{DD}[1 - (1 - w)p_{CD} - wp_{CC}](T - \chi S) + wp_{DD}(1 - wp_{CC})(S - \chi T) \\ + \left\{ 1 - w^2 p_{DC} + (1 - w)w^2 p_{CD} p_{DC} + wp_{DD}(1 + w + w^2 p_{CD}) - wp_{CC}[1 - w^2 p_{DC} + (1 + w)wp_{DD}] \right\}(\chi - 1)\kappa = 0. \tag{131}$$

By substituting $\kappa = P$ and $\chi = -(P - S)/(T - P)$ in Eq. (131), we obtain

$$w[(1 - w)p_0 + wp_{DD}]\{p_{CD}[R - (1 - w)(T + S)] + (1 - wp_{CC})(T + S) - [2 - p_{CD} - 2w(p_{CC} - p_{CD})]P\}(T - S) = 0. \tag{132}$$

By substituting $p_0 = p_{CC} = p_{DD} = (T + S - 2P)/(T + S - R - P)$ in Eq. (132), we obtain

$$\frac{[-p_{CD}(T + S - R - P) + T + S - 2P][-w(T + S - 2P) + T + S - R - P](T + S - 2P)}{(T + S - R - P)^2} = 0. \tag{133}$$

If $-w(T + S - 2P) + T + S - R - P = 0$, Eq. (120) implies that $w = 1/p_0$, i.e., $w = p_0 = 1$, which contradicts $0 < w < 1$. Because we decided to treat the case $T + S - 2P = 0$ later, Eq. (133), implies

$$p_{CD} = \frac{T + S - 2P}{T + S - R - P}. \tag{134}$$

In sum, we obtain $p_0 = p_{CC} = p_{CD} = p_{DC} = p_{DD} = (T + S - 2P)/(T + S - R - P)$ if $T + S - 2P < 0$. Substitution of p_0 in Eqs. (49) and (50) yields $\chi = -(P - S)/(T - P)$ and $\kappa = P$, respectively, coinciding with the condition for subcase (A). Therefore, the strategy $p_0 = p_{CC} = p_{CD} = p_{DC} = p_{DD} = (T + S - 2P)/(T + S - R - P)$, where $T + S - 2P < 0$, is a special case of Eq. (51).

Finally, let us consider the case $T + S - 2P = 0$. By combining this condition with Eq. (120), we obtain $p_0 = 0$. By substituting $T + S - 2P = 0$ and $p_0 = 0$ in Eq. (125), we obtain $w(R - P)p_{DC} = 0$, which implies that $p_{DC} = 0$. By substituting $T + S - 2P = 0$ and $p_0 = 0$ in Eq. (128), we obtain $(1 - wp_{CD})(R - P)p_{DD} = 0$, which implies that $p_{DD} = 0$. Because $p_0 = p_{DC} = p_{DD} = 0$, the focal player X never uses p_{CC} and p_{CD} . Therefore, $p_0 = p_{DC} = p_{DD} = 0$ specifies a strategy. By substituting $p_0 = 0$ in Eqs. (49) and (50) and using $T + S - 2P = 0$, we obtain $\chi = -(T - P)/(P - S) = -(P - S)/(T - P) = -1$ and $\kappa = P$, respectively, coinciding with the condition for subcase (A). Therefore, the strategy $p_0 = p_{DC} = p_{DD} = 0$ is a special case of Eq. (51).

E2. Subcase (B): $\kappa - S + \chi(T - \kappa) = 0$ and $p_0 = p_{CD} = 1$

By substituting Eqs. (116) and (117) in Eq. (91), we obtain

$$(1 - w)(\chi - 1)[wp_{DD}(\kappa - R) - wp_{CC}(\kappa - P) + w(R - P) + \kappa - R] = 0. \tag{135}$$

Note that $0 < w < 1$. Because $\chi = 1$ is inconsistent with $\kappa - S + \chi(T - \kappa) = 0$, Eq. (135) yields

$$p_{DD} = \frac{wp_{CC}(\kappa - P) - w(R - P) - (\kappa - R)}{w(\kappa - R)} \tag{136}$$

provided that $\kappa \neq R$. We will deal with the case $\kappa = R$ later in this section. By substituting Eqs. (116) and (117) in Eq. (93), we obtain

$$(\chi - 1)\{wp_{DC}(\kappa - R) - wp_{CC}(\chi + 1)(\kappa - T) + w[R - (\chi + 1)T + \chi\kappa] + \kappa - R\} = 0, \tag{137}$$

which yields

$$p_{DC} = \frac{wp_{CC}(\chi + 1)(\kappa - T) - w[R - (\chi + 1)T + \chi\kappa] - (\kappa - R)}{w(\kappa - R)} \tag{138}$$

provided that $\kappa \neq R$. Therefore, we obtain

$$\mathbf{p} = \left(p_{CC}, 1, \frac{wp_{CC}(\chi + 1)(\kappa - T) - w[R - (\chi + 1)T + \chi\kappa] - (\kappa - R)}{w(\kappa - R)}, \frac{wp_{CC}(\kappa - P) - w(R - P) - (\kappa - R)}{w(\kappa - R)} \right), p_0 = 1, \tag{139}$$

i.e., Eq. (53), as a necessary condition for the linear relationship between the payoff of the two players, i.e., Eq. (16).

To verify that Eq. (53) is sufficient, we substitute Eq. (53) (i.e., Eq. (139)) in Eq. (36) to obtain

$$\mathbf{u}^{zd} = \left(0, 0, -\frac{(1 - w)(\chi - 1)(\kappa - R)}{w(1 - p_{CC})}, -\frac{(1 - w)(\chi - 1)(\kappa - R)}{w(1 - p_{CC})} \right), \tag{140}$$

which is independent of \mathbf{q} . By combining Eqs. (7), (140), and $p_0 = 1$, we obtain $\mathbf{v}(0)\mathbf{u}^{zd} = 0$, i.e., Eq. (37). Therefore, Eq. (53) is a solution that satisfies Eq. (16).

The strategy given by Eq. (53) is expressed in the form of Eq. (45) if we set $\phi = -w(1 - p_{CC})/[(\kappa - R)(\chi - 1)]$ (and use $\kappa - S + \chi(T - \kappa) = 0$ and $p_0 = 1$). As an example, we consider the repeated PD game defined by $R = 3$, $T = 5$, $S = -2$, $P = 1$, and $w = 0.8$. We set $\kappa = 2$. Because this solution requires $\kappa - S + \chi(T - \kappa) = 0$ (Eq. (116)), we obtain $\chi = -4/3$. If we set $p_{CC} = 0$, we obtain $\mathbf{p} = (0, 1, 3/4, 3/4)$ and $p_0 = 1$. This solution cannot be represented in the form of Eq. (43) because Eq. (43) requires $\kappa - S + \chi(T - \kappa) \neq 0$. Consistent with this example, Eq. (45) combined with $\phi = -w(1 - p_{CC})/[(\kappa - R)(\chi - 1)]$, $\kappa - S + \chi(T - \kappa) = 0$, and $p_0 = 1$ yields $\chi < 0$. This can be shown as follows. By substituting $\kappa - S + \chi(T - \kappa) = 0$ and $p_0 = 1$ in Eq. (45), we obtain

$$\mathbf{p} = \begin{pmatrix} 1 - \frac{\phi(\chi - 1)(R - \kappa)}{w} \\ 1 \\ 1 - \frac{1}{w} + \frac{\phi[(\chi - 1)\kappa + T - \chi S]}{w} \\ 1 - \frac{1}{w} + \frac{\phi(\chi - 1)(\kappa - P)}{w} \end{pmatrix}. \tag{141}$$

Because $p_{CC} \leq 1$ must hold true in Eq. (141), we obtain

$$\phi(\chi - 1)(R - \kappa) \geq 0. \tag{142}$$

Because $p_{DD} \geq 0$ must hold true in Eq. (141), we obtain

$$\phi(\chi - 1)(\kappa - P) \geq 0. \tag{143}$$

Given $\phi(\chi - 1) \neq 0$ (Section 3.2) and $R > P$, we find that $P \leq \kappa \leq R$ must hold true for Eqs. (142) and (143) to be simultaneously satisfied. Therefore, using $\kappa - S + \chi(T - \kappa) = 0$ we obtain $\chi = -(\kappa - S)/(T - \kappa) < 0$.

Finally, let us consider the case $\kappa = R$. By substituting $\kappa = R$ in Eq. (135), we obtain $wp_{CC}(R - P) = w(R - P)$, which implies that $p_{CC} = 1$. By combining this result with Eq. (117), we obtain $p_0 = p_{CC} = p_{CD} = 1$, which implies that player X never uses p_{DC} and p_{DD} . Therefore, $p_0 = p_{CC} = p_{CD} = 1$ specifies a strategy. By substituting $p_0 = 1$ in Eqs. (49) and (50), we obtain $\chi = -(R - S)/(T - R)$ and $\kappa = R$, respectively, and the former equality coincides with Eq. (117) when $\kappa = R$. Therefore, the strategy $p_0 = p_{CC} = p_{CD} = 1$ is a special case of Eq. (51).

Appendix F. Case $T - \kappa + \chi(\kappa - S) = 0$

In this section, we assume

$$T - \kappa + \chi(\kappa - S) = 0 \tag{144}$$

and derive the set of strategies that satisfy Eq. (16).

By substituting Eq. (144) in Eq. (93), we obtain

$$(\chi - 1)(\kappa - R)[(1 - w)p_0 + wp_{DC}] = 0. \tag{145}$$

Eq. (144) does not allow $\chi = 1$ because substitution of $\chi = 1$ in Eq. (144) yields $T = S$, which contradicts Eq. (2). Substitution of $\kappa = R$ in Eq. (144) yields $\chi = -(T - R)/(R - S)$. Alternatively, if we set $(1 - w)p_0 + wp_{DC} = 0$, we obtain $p_0 = p_{DC} = 0$. Therefore, we consider the following two subcases, i.e., subcase (C) specified by

$$\kappa = R \tag{146}$$

and

$$\chi = -\frac{T - R}{R - S}, \tag{147}$$

and subcase (D) specified by

$$T - \kappa + \chi(\kappa - S) = 0 \tag{148}$$

and

$$p_0 = p_{DC} = 0. \tag{149}$$

F1. Subcase (C): $\kappa = R$ and $\chi = -(T - R)/(R - S)$

By substituting Eqs. (146) and (147) in Eq. (94), we obtain

$$\frac{(1 - w)[1 - w(p_{CC} - p_{DC})][-p_0(T + S - R - P) + R - P](T - S)}{R - S} = 0. \tag{150}$$

Eq. (150) does not hold true because $0 < w < 1$, $(R - P) - p_0(T + S - R - P) \neq 0$ due to Eq. (101), and $1 - w(p_{CC} - p_{DC}) > 0$. Therefore there is no solution in this case.

F2. Subcase (D): $T - \kappa + \chi(\kappa - S) = 0$ and $p_0 = p_{DC} = 0$

By substituting Eqs. (148) and (149) in Eq. (91), we obtain

$$w(\chi - 1)[wp_{DD}(\kappa - S)(\chi + 1) + (1 - wp_{CD})(\kappa - P)] = 0. \tag{151}$$

We obtain $\chi \neq 1$ because $\chi = 1$ substituted in Eq. (148) yields $T = S$, which contradicts Eq. (2). Therefore, Eq. (151) implies

$$p_{CD} = \frac{wp_{DD}(\chi + 1)(\kappa - S) + \kappa - P}{w(\kappa - P)} \tag{152}$$

provided that $\kappa \neq P$. We will deal with the case $\kappa = P$ later in this section. By substituting Eqs. (148) and (149) in Eq. (94), we obtain

$$(\chi - 1)(1 - w)[wp_{DD}(\kappa - R) + (1 - w)p_{CC}(\kappa - P)] = 0. \quad (153)$$

Because $0 < w < 1$ and $\chi \neq 1$, we obtain

$$p_{CC} = \frac{wp_{DD}(\kappa - R) + \kappa - P}{w(\kappa - P)} \quad (154)$$

provided that $\kappa \neq P$. Therefore, we obtain

$$\mathbf{p} = \left(\frac{wp_{DD}(\kappa - R) + \kappa - P}{w(\kappa - P)}, \frac{wp_{DD}(\chi + 1)(\kappa - S) + \kappa - P}{w(\kappa - P)}, 0, p_{DD} \right), \quad p_0 = 0, \quad (155)$$

where $0 \leq p_{DD} \leq 1$ is a necessary condition for the linear relationship between the payoff of the two players, i.e., Eq. (16). In fact, we substitute p_{CD} given by Eq. (155) in p_{CD} given by Eq. (43) and use Eqs. (148) and (149) to find that p_{CC}, p_{DC}, p_{DD} given by Eq. (43) coincide with those given by Eq. (155). Therefore, Eq. (155) is a special case of ZD strategies given by Eq. (43).

Finally, let us consider the case $\kappa = P$. By substituting $\kappa = P$ in Eq. (153), we obtain $wp_{DD}(R - P) = 0$, which implies that $p_{DD} = 0$. By combining this result with Eq. (149), we obtain $p_0 = p_{DC} = p_{DD} = 0$, which implies that player X never uses p_{CC} and p_{CD} . Therefore, $p_0 = p_{DC} = p_{DD} = 0$ specifies a strategy. By substituting $p_0 = 0$ in Eqs. (49) and (50), we obtain $\chi = -(T - P)/(P - S)$ and $\kappa = P$, respectively, and the former equality coincides with Eq. (151) when $\kappa = P$. Therefore, the strategy $p_0 = p_{DC} = p_{DD} = 0$ is a special case of Eq. (51).

Appendix G. Minimum discount rate for $\chi < 0$

G1. ZD strategies with $\kappa = P$

Let us consider Eq. (56) under $\phi < 0$ and $\chi < 1$. In this case, we obtain Eqs. (57)–(59), but with all the inequalities flipped (i.e., \geq instead of \leq). Then, we obtain

$$\frac{(\chi - 1)\frac{R-P}{P-S}}{\frac{1}{w}} \geq \frac{1 + \chi\frac{T-P}{P-S}}{\frac{1}{w} - 1}, \quad (156)$$

$$1 + \chi\frac{T-P}{P-S} \geq \frac{(\chi - 1)\frac{R-P}{P-S}}{\frac{1}{w} - 1}, \quad (157)$$

$$\chi + \frac{T - P}{P - S} \geq \frac{(\chi - 1)\frac{R-P}{P-S}}{\frac{1}{w} - 1}, \quad (158)$$

$$\chi + \frac{T - P}{P - S} \geq \frac{1 + \chi\frac{T-P}{P-S}}{\frac{1}{w} - 1}. \quad (159)$$

Eqs. (156)–(159) yield

$$\chi \leq -\frac{P - S + (1 - w)(R - P)}{T - R + w(R - P)} < 0, \quad (160)$$

$$[(R - P) - (1 - w)(T - P)]\chi \leq R - P + (1 - w)(P - S), \quad (161)$$

$$[w(R - P) - (1 - w)(P - S)]\chi \leq w(R - P) + (1 - w)(T - P), \quad (162)$$

$$[-(P - S) + w(T - S)]\chi \leq -w(P - S) + (1 - w)(T - P), \quad (163)$$

respectively. When w is sufficiently large, the coefficients of χ on the left-hand sides of Eqs. (161)–(163) are positive. In this situation, Eqs. (160)–(163) are satisfied by a sufficiently negative large $\chi (< 0)$. This result is consistent with the previously obtained result (Hilbe et al., 2015a).

G2. ZD strategies with $\kappa = R$

In this section, we examine Eq. (75)–(77) under the assumption that $\chi < 0$. First, because $dg_2/d\chi > 0$, g_2 is discontinuous at $\chi = -(R - S)/(T - R)$, and $g_2 < 0$ for $-(R - S)/(T - R) < \chi < 0$, Eq. (76) is equivalent to

$$\chi < -\frac{R - S}{T - R} \quad (164)$$

if $w \geq (T - R)/(T - P)$ and

$$\frac{R - S - w(P - S)}{-(T - R) + w(T - P)} < \chi < -\frac{R - S}{T - R} \quad (165)$$

if $w < (T - R)/(T - P)$. Second, using Eq. (164), $dg_1/d\chi > 0$, and that g_1 is discontinuous at $\chi = -(R - S)/(T - R)$, we find that Eq. (164) implies Eq. (75) if $w \geq (T - R)/(T - S)$ and that Eq. (75) is equivalent to

$$\frac{R - S - w(T - S)}{-(T - R) + w(T - S)} < \chi < -\frac{R - S}{T - R} \quad (166)$$

if $w < (T - R)/(T - S)$. Third, because $d(g_2/g_1)/d\chi > 0$, g_2/g_1 is discontinuous at $\chi = -(T - R)/(R - S)$, and $g_2/g_1 < 0$ for $-(T - R)/(R - S) < \chi < 0$, Eq. (77) is equivalent to

$$\chi < -\frac{T - R}{R - S} \quad (167)$$

if $w \geq (P - S)/(R - S)$ and

$$\frac{T - P - w(T - R)}{-(P - S) + w(R - S)} \leq \chi < -\frac{T - R}{R - S} \quad (168)$$

if $w < (P - S)/(R - S)$.

To summarize these results, if $w \geq w_c$, generous strategies with

$$\chi < \min\left(-\frac{R - S}{T - R}, -\frac{T - R}{R - S}\right) < -1 \quad (169)$$

exist because Eq. (169) yields Eqs. (75)–(77). This result is consistent with the previously obtained results (Hilbe et al., 2015a). Note that we have used Eq. (3) to derive the last inequality in Eq. (169). Even if $w < w_c$, negative χ values that satisfy all the conditions, i.e., the set of equations out of Eqs. (164)–(168), corresponding to the given value of w , may exist.

References

Adami, C., Hintze, A., 2013. Evolutionary instability of zero-determinant strategies demonstrates that winning is not everything. *Nat. Comm.* 4, 2193.
 Akin, E., 2017. The iterated prisoner's dilemma: good strategies and their dynamics, arXiv:1211.0969v3.
 Axelrod, R., 1984. *Evolution of Cooperation*. Basic Books, NY.
 Boerlijst, M.C., Nowak, M.A., Sigmund, K., 1997. Equal pay for all prisoners. *Am. Math. Month.* 104, 303–305.
 Chen, J., Zinger, A., 2014. The robustness of zero-determinant strategies in Iterated Prisoner's Dilemma games. *J. Theor. Biol.* 357, 46–54.
 Hilbe, C., Hagel, K., Milinski, M., 2016. Asymmetric power boosts extortion in an economic experiment. *PLOS One* 11, e0163867.
 Hilbe, C., Nowak, M.A., Sigmund, K., 2013. Evolution of extortion in Iterated Prisoner's Dilemma games. *Proc. Natl. Acad. Sci. USA* 110, 6913–6918.
 Hilbe, C., Nowak, M.A., Traulsen, A., 2013. Adaptive dynamics of extortion and compliance. *PLOS One* 8, e77886.
 Hilbe, C., Röhl, T., Milinski, M., 2014. Extortion subdues human players but is finally punished in the prisoner's dilemma. *Nat. Comm.* 5, 3976.
 Hilbe, C., Traulsen, A., Sigmund, K., 2015. Partners or rivals? Strategies for the iterated prisoner's dilemma. *Games Econ. Behav.* 92, 41–52.
 Hilbe, C., Wu, B., Traulsen, A., Nowak, M.A., 2014. Cooperation and control in multiplayer social dilemmas. *Proc. Natl. Acad. Sci. USA* 111, 16425–16430.
 Hilbe, C., Wu, B., Traulsen, A., Nowak, M.A., 2015. Evolutionary performance of zero-determinant strategies in multiplayer games. *J. Theor. Biol.* 374, 115–124.
 Kraines, D., Kraines, V., 1993. Learning to cooperate with Pavlov an adaptive strategy for the iterated Prisoner's Dilemma with noise. *Theory Decis.* 35, 107–150.
 Liu, J., Li, Y., Xu, C., Hui, P.M., 2015. Evolutionary behavior of generalized zero-determinant strategies in iterated prisoner's dilemma. *Physica A* 430, 81–92.
 McAvoy, A., Hauert, C., 2016. Autocratic strategies for iterated games with arbitrary action spaces. *Proc. Natl. Acad. Sci. USA* 113, 3573–3578.

- McAvoy, A., Hauert, C., 2017. Autocratic strategies for alternating games. *Theor. Popul. Biol.* 113, 13–22.
- Milinski, M., Hilbe, C., Semmann, D., Sommerfeld, R., Marotzke, J., 2016. Humans choose representatives who enforce cooperation in social dilemmas through extortion. *Nat. Comm.* 7, 10915.
- Nowak, M., Sigmund, K., 1993. A strategy of win-stay, lose-shift that outperforms tit-for-tat in the Prisoner's Dilemma game. *Nature* 364, 56–58.
- Nowak, M.A., 2006. *Evolutionary Dynamics*. Belknap Press of Harvard University Press, Cambridge, MA.
- Nowak, M.A., Sigmund, K., 1992. Tit for tat in heterogeneous populations. *Nature* 355, 250–253.
- Nowak, M.A., Sigmund, K., El-Sedy, E., 1995. Automata, repeated games and noise. *J. Math. Biol.* 33, 703–722.
- Pan, L., Hao, D., Rong, Z., Zhou, T., 2015. Zero-determinant strategies in iterated public goods game. *Sci. Rep.* 5, 13096.
- Press, W.H., Dyson, F.J., 2012. Iterated Prisoner's Dilemma contains strategies that dominate any evolutionary opponent. *Proc. Natl. Acad. Sci. USA* 109, 10409–10413.
- Rand, D.G., Nowak, M.A., 2013. Human cooperation. *Trends Cogn. Sci.* 17, 413–425.
- Rapoport, A., Chammah, A.M., 1965. *Prisoner's Dilemma: A Study in Conflict and Cooperation*. Michigan University Press, Ann Arbor, MI.
- Sigmund, K., 2010. *The Calculus of Selfishness*. Princeton University Press, Princeton, NJ.
- Stewart, A.J., Parsons, T.L., Plotkin, J.B., 2016. Evolutionary consequences of behavioral diversity. *Proc. Natl. Acad. Sci. USA* 113, E7003–E7009.
- Stewart, A.J., Plotkin, J.B., 2012. Extortion and cooperation in the Prisoner's Dilemma. *Proc. Natl. Acad. Sci. USA* 109, 10134–10135.
- Stewart, A.J., Plotkin, J.B., 2013. From extortion to generosity, evolution in the iterated Prisoner's Dilemma. *Proc. Natl. Acad. Sci. USA* 110, 15348–15353.
- Szolnoki, A., Perc, M., 2014. Defection and extortion as unexpected catalysts of unconditional cooperation in structured populations. *Sci. Rep.* 4, 5496.
- Szolnoki, A., Perc, M., 2014. Evolution of extortion in structured populations. *Phys. Rev. E* 89, 022804.
- Trivers, R.L., 1971. The evolution of reciprocal altruism. *Q. Rev. Biol.* 46, 35–57.
- Wang, Z., Zhou, Y., Lien, J.W., Zheng, J., Xu, B., 2016. Extortion can outperform generosity in the iterated prisoner's dilemma. *Nat. Comm.* 7, 11125.
- Wu, Z.X., Rong, Z., 2014. Boosting cooperation by involving extortion in spatial prisoner's dilemma games. *Phys. Rev. E* 90, 062102.
- Xu, X., Rong, Z., Wu, Z.X., Zhou, T., Tse, C.K., 2017. Extortion provides alternative routes to the evolution of cooperation in structured populations. *Phys. Rev. E* 95, 052302.