THE SEMANTICS OF ENTAILMENT

Richard ROUTLEY
Australian National University

Robert K. MEYER
University of Toronto, Indiana University

Once upon a time, modal logics "had no semantics". Bearing a real world G, a set of worlds K, and a relation R of relative possibility between worlds, Saul Kripke beheld this situation and saw that it was formally explicable, and made model structures. It came to pass that soon everyone was making model structures, and some were deontic, and some were temporal, and some were epistemic, according to the conditions on the binary relation R.

None of the model structures that Kripke made, nor that Hintikka made, nor that Thomason made, nor that their co-workers and colleagues made, were, however, relevant. This caused great sadness in the city of Pittsburgh, where dwelt the captains of American Industry. The logic industry was there represented by Anderson, Belnap & Sons, discoverers of entailment and scourge of material impliers, strict impliers, and of all that to which their falsehoods and contradictions led. Yea, every year or so Anderson & Belnap turned out a new logic, and they did call it E, or R, or $E_{\bar{I}}$, or P – W, and they beheld each such logic, and they were called relevant. And these logics were looked upon with favor by many, for they captureth the intuitions, but by many more they were scorned, in that they hadeth no semantics.

Word that Anderson & Belnap had made a logic without semantics leaked out. Some thought it wondrous and rejoiced,[1] that the One True Logic should make its appearance among us in the Form of Pure Syntax, unencumbered by all that set-theoretical garbage. Others said that relevant logics were Mere Syntax. Surveying the situation Routley, and quite independently Urquhart, found an explication of the key concept of relevant implication. Building on Routley [1972], and with a little help from our friends – Dunn and Urquhart

[1] The underlying point is, of course, that there are many ways to explicate informal logical or mathematical notions formally, and that an axiom set counts. So do matrices, rules for natural deduction, correlated algebraic structures, and so forth, which had previously been provided for the relevant logics. The novelty of the present approach, as Belnap has put it, is that like Kripke's semantical reductions of modal logics it provides an extensional – in a significant sense, a *truth-functional* – understanding of relevant logic. Why this kind of understanding turns out particularly illuminating is a matter for *psychology* of logic, in which we profess no competence; in fact, even purely technical problems seem to become much easier – cf., e.g., section VIII below.

in particular, with thanks also due to Anderson, Belnap, V. Routley, and Woodruff – we use these insights to present here a formal semantics for the system R of relevant implication, and to provide it with proofs of consistency and completeness relative to that semantics.

Central to the semantics being developed here is a *ternary* relation R which takes the place for the relevant logics of the Kripke binary relation for standard modal and intuitionistic logics. In subsequent work we shall show how by varying the postulates on R one gets, as is customary in these things, other relevant logics, notably the Anderson-Belnap systems E of entailment and P of ticket entailment and the Ackermann systems of *strenge Implikation*[2]. For the present we stick to R, developing as in Meyer [1968] a theory of entailment by adding an explicit Lewis-style modal operator to R.[3] Since the *modal* part of Ackermann-Anderson-Belnap theories of entailment is essentially S4, while the *relevant* part rests on novel insights, the essential novelties of the semantics developed here will lie in the treatment of the underlying relevant system R, necessity than being analyzed along the lines of Kripke's analysis of S4 in Kripke [1963]. We note that in its implicational part (Church's weak theory of implication R_I), R is the oldest of the relevant logics and perhaps the most naturally motivated. Extensions of R (Dunn's R-mingle) and of its positive fragment R+ (positive logic, intuitionistic logic) fall easily under our account and will be treated in passing.

Consider a natural English rendering of Kripke's binary R. H R H_1 "says" that "world" H_1 is possible relative to world H. An interesting ternary generalization is to read H R H_1,H_2 to say that "worlds" H_1 and H_2 are *compossible* (better, maybe, *compatible*) relative to H. (The reading is suggested by Dunn.) "Worlds" will not detain us, for we apply the terminology of Routley [1972] to speak of "set-ups" rather than "worlds", by which we indicate that what we deal with is not necessarily realized or even realizable in any ordinary sense. [But logic, we think, should have room for *extraordinary* sense, to do justice to physics as well as to philosophy and poetry.] To ply some of the intuitions that make relevant logics philosophically interesting, in addition to the pedestrian set-ups that might count also as worlds, we could include in, perhaps, what is putatively described by a coherent (though not necessarily consistent) set of beliefs, what might be presumed on a certain

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[2] P has been renamed T, in Anderson and Belnap [1972], q.v. for formulations of all logics mentioned.

[3] We develop the semantics of NR in a sequel to this paper, to be published in the *Journal of Philosophical Logic.*

combination of law and observational report (even if the combination engenders difficulties [the scientist's word for inconsistencies] or is incomplete), what would happen if *all* that ought to be done were done. We waive here the question whether the notion of a set-up belongs ultimately to ontology, epistemology, or even perhaps just psychology[4].

Two set-ups are compatible relative to a third, on the intuitions exploited here, provided that whenever sentences A and B hold respectively in the first and the second, a sentence C that asserts directly that A and B are consistent holds in the third. [Dunn recalls the standard model analogue from Kripke – a world is possible relative to a second provided that whenever A holds in the first, a sentence directly asserting that A is possible – namely $\Diamond$ A – holds in the second.]

To make it easier to comprehend postulates and proofs, we switch notation away from that most familiar to readers of Kripke. Henceforth we use *a, b, c,* etc., in place of H, H_1, etc., to indicate set-ups. We also abandon, with a permutation, the old infix notation; R*abc* shall henceforth assert, as we have put it informally, that *a* and *b* are compatible relative to *c*.[5] We also introduce an explicit binary consistency connective –read 'A $\circ$ B', 'A is consistent with B'.[6]

If one examines the syntactical residue of normal modal semantics (as we were doing in brackets a paragraph ago), Kripke's modal structures may be naively viewed as having consistent and complete theories as their elements. Our model structures, on examination of their syntactical residue, will also have theories as their elements. But as in Routley [1972] and in view of previous remarks, regard for relevance requires us to treat theories abnormal from the classical viewpoint. The class of theories to which we shall attend will be, given that constraint, as nearly normal as possible; its members will be what we call here *prime intensional theories.* Besides being closed in a suitable sense under *entailment,* prime intensional theories will respect *conjunction* and *disjunction* as do classically consistent and complete theories.

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4 The same might be said, of course, of a possible world in the semantics of Lewis-style modal logics. But our colleague Nino Cocchiarella finds in ideas like the present ones an epistemological rather than an ontological orientation; we are not so sure, but drop the question as involving matters of philosophical stance too far-reaching for thorough examination here.

5 *c* R *a, b,* in the old notation.

6 An analogue of '$\circ$' goes back to Church's work on R_I in Church [1951]: whether viewed as consistency, compossibility, an intensional kind of conjunction, or simply as an operation in correlated algebraic structures, it has proved very useful; cf., e.g., Meyer et al. [1972].

Negation, on the other hand, requires as in Routley [1972], as in previous work of Dunn, Belnap, and others, the admission of theories that are inconsistent, incomplete, or both. Justification as above can be found in belief contexts, or in evaluative ones, or even, indicatively, when under extreme provocation (e.g., Russell's paradox) we might choose to bite the bullet of explicit contradiction, or must choose (e.g., axiomatized arithmetic) never to savour the sweet fruits of completeness. We save nevertheless something like the familiar recursive treatment of negation by distinguishing a strong and a weak way of affirming a sentence A in a given set-up. The strong way is to assert A; the weak way is to omit the assertion of $\bar{A}$. This yields for each set-up a the complementary set-up a^*, where what is strongly affirmed in a is weakly affirmed in a^* and vice versa. The wanted recursive clause then says that $\bar{A}$ holds in a just when A doesn't hold in a^*; the reader will have noted that under normal circumstances, when we affirm just what we don't deny, a and a^* coincide, whence the account of negation reduces to the usual one.

Relevant implication, though the heart of the matter, derives here its characterization via negation and consistency, given that A relevantly implies B just in case A and $\bar{B}$ are *inconsistent.* As it turns out (on a fixed interpretation of our formal language), A relevantly implies B is a set-up c just in case, for all set-ups a and b, whenever $Rcab$ and A holds in a then B holds in b.[7]

The *real world* plays a distinguished role in our semantical postulates. (Accordingly we call it 0 rather than G; not only does the former *look* better [this is supposed to be, remember, a *mathematical* semantics], but it correctly hints that 0 will play the formal role of an identity.) It's necessary to distinguish 0 for the following reason: Logical truth does *not* turn out to be *truth* in all set-ups; for the strategy which dispatches the paradoxes lies in allowing even logical identities to turn out sometimes false. (What, after all, could be better grounds for denying that q entails $p \to p$ than to admit that sometimes q is true when, essentially on grounds of relevance, $p \to p$ isn't?)[8]

What then is logical truth? Truth in all set-ups, of course, *in which all the logical truths are true!* (That's not, by the way, a tautology; it's possible that a stray non-logical truth might get dragged into all set-ups in which all logical truths are verified.) Frankly, in considering as candidates for the real world only set-ups that verify all logical truths, we are only showing our parochial loyalties as logicians, since as logicians these are the truths we want our formal

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[7] In weaker systems, such as E and T, we shall use this condition to *characterize* the ternary accessibility relation R.

[8] In addition to the remarks immediately following, cf. section 4 below.

semantics to characterize. Our semantics, to be sure, requires the physicist or the economist to *reason* correctly, in the sense that he'd better use valid *arguments,* but the physicist is no more required to prefer *qua physicist* to assert laws of logic than the logician is required to assert *qua logician* the third law of thermodynamics. (We have, by the way, restored a certain parity between the logician and the physicist; that physics deals somehow in only low-grade [on some accounts, high-grade] truths, in that we can imagine – or at least pretend formally – that its laws are false, while laws of logic are always true, turns out to be a view made possible only by a defect of imagination; so much the worse, as Quine would say, for analyticity.)

So much, too, for general motivating remarks. The formal developments to follow are of course independent of them and might be used to ground varying informal intuitions. Chief among these developments are proofs of the semantical consistency and completeness of the sentential logic R of relevant implication. Numerous applications are made of the main result, either newly answering or offering greatly simplified proofs of answers to questions of the sort posed in Anderson [1963]. A characterization of *normal* validity, for example, shows that the set of theorems of R is closed under detachment for *material* implication, the main result of Meyer and Dunn [1969]; borrowing from Meyer [1972a, b], we show moreover that R is *well-axiomatized,* in the sense that in general its fragments got by dropping certain connectives can be got from axioms in which the dropped connectives do not occur (details are in section 10); similarly, extension of our completeness proof to the Dunn-McCall system RM sheds further light on certain results obtained by Meyer and cast in algebraic form by Dunn [1970]. The theory of deMorgan monoids, developed by Dunn in his dissertation (University of Pittsburgh, 1966) to furnish an algebraic counterpart of R and summarized in Meyer et al. [1972], is linked in several places to our semantics, and a useful Stone-type embedding theorem is noted for the former. A theory of propositions is introduced and is used to sketch an extension of the present semantics to the Anderson-Belnap system RP of relevant implication with propositional quantifiers presented in Anderson [1972]; a similar extension is offered for the first order version RQ of R. Alternatively, we offer truth-value semantics (in the sense of Leblanc) in the quantifier cases. We think that proofs of completeness relative to the suggested semantics of the quantificational systems are messy but straightforward, but, our most patient editor's patience having run out, and having just begun to plumb the ramifications of relevance at the quantificational level, we content ourselves here with proofs of semantic consistency for the quantificational systems.

Our introductory remarks conclude with the observation that, as a result of this paper and of Urquhart's [1972] related work, the relevant logics now have a formal semantics; but relating such a semantics to the informal claim that a system of logic has captured one's intuitions is ever a matter of private judgment, and that judgment we leave, as his rightful due, to the reader.

1. Syntactical preliminaries

The *sentential language* SL is a triple $\langle S, O, F\rangle$, where S is a denumerably infinite set of sentential parameters, O is the set whose members are the unary connective $-$ and the binary connectives &, $\vee$, $\rightarrow$, and F is the set of formulas built up as usual from the parameters in S and the connectives in O. (We use 'p', 'q', etc., to refer to sentence parameters in S and 'A', 'B', etc., to refer to arbitrary formulas of F. For ease in reading formulas, the binary connectives, including those immediately to be defined, are to be ranked &, $\circ$, $\vee$, $\rightarrow$, $\leftrightarrow$ in order of increasing scope, otherwise resolving ambiguities by associating to the left.) As definition, axiom, and rule schemata for the system R of relevant implication we enter the following:

A1. $A \rightarrow A$,

A2. $A \rightarrow ((A \rightarrow B) \rightarrow B)$,

A3. $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$,

A4. $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$,

A5. $A \,\&\, B \rightarrow A$,

A6. $A \,\&\, B \rightarrow B$,

A7. $(A \rightarrow B) \,\&\, (A \rightarrow C) \rightarrow (A \rightarrow B \,\&\, C)$,

A8. $A \rightarrow A \vee B$,

A9. $B \rightarrow A \vee B$,

A10. $(A \rightarrow C) \,\&\, (B \rightarrow C) \rightarrow (A \vee B \rightarrow C)$,

A11. $A \,\&\, (B \vee C) \rightarrow A \,\&\, B \vee A \,\&\, C$,

A12. $(A \rightarrow \bar{B}) \rightarrow (B \rightarrow \bar{A})$

A13. $\bar{\bar{A}} \rightarrow A$.

D1. $A \circ B =_{df} \overline{A \rightarrow \bar{B}}$ (Consistency)

D2. $A \leftrightarrow B =_{df} (A \rightarrow B) \circ (B \rightarrow A)$.[9] (Equivalence)

R1. From $A \rightarrow B$ and A, infer B. (*Modus ponens*)

R2. From A and B, infer A & B. (Adjunction)

[9] The *definiens* is equivalent in R to the more usual $(A \rightarrow B) \,\&\, (B \rightarrow A)$.

2. Semantic preliminaries

A *relevant model structure* (henceforth, *r. m. s.*) is a quadruple $\langle 0, K, \mathrm{R}, * \rangle$, where K is a set, $0 \in K$, R is a ternary relation on K, and * is a unary operation on K, satisfying postulates to follow. For ease in stating postulates and for help in interpreting them we define a binary relation $<$ and a 4-ary relation R^2 on K as follows, for all a, b, c, d in K:

d1. $a < b$ =df $\mathrm{R}0ab$,

d2. R^2abcd =df $\exists x(\mathrm{R}abx \;\&\; \mathrm{R}xcd \;\&\; x \in K)$.

d1 and d2 may be explained as follows. We began with a Kripke-style notion of a world being accessible from another. Generalizing to take account of the essentially relational notion of relevance, we passed to the notion of a set-up being accessible from *pairs* of set-ups. Obviously we could go on to trios of set-ups, quadruples of set-ups, and so forth. However, on the intuitions about compossibility formalized in the system R, which rest on the fact that consistency as introduced by D1 is commutative and associative, all the higher order accessibility relations prove definable using just the ternary R; thus R^2, as defined by d2, says that d is accessible from the trio a,b,c, or, if one prefers, that $a,b,$ and c are compossible relative to d. Moreover, just as one can move up levels via the sort of ternary relational product that enters into d2, one can move down thanks to the privileged status accorded to 0; crudely put, to say in the system R that A is compatible with the way things really are is just to say A, and vice versa. Accordingly, and in particular, $\mathrm{R}0ab$ returns us to Kripke's turf by saying simply that b is accessible from a, motivating d1. Since the system R is non-modal, we'd expect $<$ as defined by d1 to be like the binary intuitionistic relation of Kripke [1965]; so it turns out.

Here are the postulates that an *r. m. s.* $\langle 0, K, \mathrm{R}, * \rangle$ must satisfy, for all a,b,c,d in K:

p1. $\mathrm{R}0aa$,

p2. $\mathrm{R}aaa$,

p3. $\mathrm{R}^2abcd \Rightarrow \mathrm{R}^2acbd$,

p4. $\mathrm{R}^2abcd \Rightarrow \mathrm{R}abc$,

p5. $\mathrm{R}abc \Rightarrow \mathrm{R}ac^*b^*$,

p6. $a^{**} = a$.

We trust that, given our motivating remarks, it is not implausible that each of p1–p6 should be true. (Since, as noted, by varying postulates different relevant logics result, there are of course points of view which exclude one or another of these postulates. Accordingly plausibility is all that is to be wished.)

3. Valuations, interpretations, validity

Let $\langle 0, K, R, *\rangle$ be an *r. m. s.*, $\{T, F\}$ the set of classical truth-values, and $SL = \langle S, O, F\rangle$ the sentential language defined in I. A *valuation* v of SL in $\langle 0, K, R, *\rangle$ will be a function that assigns a truth-value to each parameter in S for each set-up in K, subject to the restriction that v respect the binary accessibility relation $<$ defined by d1. The *interpretation I associated with v* is the unique extension of v to all formulas of F in each set-up K required by the informal explication we have given of the connectives. Formally,

(a) v is a valuation of SL in $\langle 0, K, R, *\rangle$ provided that v is a function from $S \times K$ to $\{T, F\}$ that satisfies the following condition, for all p in S and a,b in K:

(1) $a < b \;\&\; v(p,a) = T \Rightarrow v(p,b) = T.$

(b) I is the interpretation associated with v provided that I is a function from $F \times K$ to $\{T, F\}$ which satisfies the following conditions, for all p in S, A,B in F, and a in K:

i. $I(p,a) = v(p,a)$,

ii. $I(A \;\&\; B,a) = T$ iff $I(A,a) = T$ and $I(B,a) = T$,

iii. $I(A \vee B,a) = T$ iff $I(A,a) = T$ or $I(B,a) = T$,

iv. $I(A \rightarrow B,a) = T$ iff, for all b,c in K, $Rabc$ and $I(A,b) = T \Rightarrow I(B,c) = T$,

v. $I(A \circ B,a) = T$ iff there exist b,c in K such that $Rbca$ and $I(A,b) = T$ and $I(B,c) = T$,[10]

vi. $I(\bar{A},a) = T$ iff $I(A,a^*) = F$.

A formula A is *true* on a valuation v, or on the associated interpretation I, at a point a of K, just in case $I(A,a) = T$; otherwise A is *false* on v, at a.[11]

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[10] Since $\circ$ has been defined by D1, that this definition is correct is determined by applying iv, vi, and the semantical postulates; in section 10 below, when $\circ$ is added as a primitive connective with governing axioms A14–A15, v becomes of course a primitive semantical postulate.

[11] This switches the terminology of Meyer and Dunn [1969]; note accordingly that here if A is false at a on v, it does not follow that $\bar{A}$ is true at a on v; it does follow by vi, in accordance with the introductory remarks, that $\bar{A}$ is true at a^* on v.

Truth at 0 is as noted earlier what counts in verifying logical truths; accordingly we say simply that A is *verified* on v, or on the associated I, just in case $I(A,0) = T$, and otherwise that A is *falsified* on v. A is *valid* in an *r. m. s.* $\langle 0, K, R, * \rangle$ just in case A is verified on all valuations therein. Finally, A is R-*valid* just in case A is valid in all *r. m. s.;* otherwise A is R-*invalid.*

4. Entailment

In the introductory remarks, we allowed, e.g., the physicist to deny some logical *laws*, so long as he *reasoned* correctly. By this we meant that if the physicist asserts A, and A entails B, he is committed also to B. Thus we arrive at the following semantical concepts.

Let $\langle 0, K, R, * \rangle$ be an *r. m. s.* in which v is a valuation and I is the associated interpretation. Then A *entails* B *on* v provided that, for all a in K,

(2) $I(A,a) = T \Rightarrow I(B,a) = T.$

We say simply that A *entails* B in $\langle 0, K, R, * \rangle$ just in case A entails B on all valuations therein. Finally, A R-*entails* B just in case A entails B in all *r. m. s.*

Entailment being the key notion which it is our business here to explicate, a word about the role which we have assigned it in our semantics is in order. First, entailment here is a semantical *relation* between sentences; in the sequel, when we add necessity to R and consider the arrow of E, the means will be at hand to consider an entailment *connective*, but for the moment our system is too poor to *express* the claim that A entails B, in any sense. Indeed, except to keep our terminology uniform we might better have spoken of *implication* on v, or in $\langle 0, K, R, * \rangle$. But as the classical *provability* of $A \supset B$ *indicates* classically that A classically entails B, just so, as we shall see, the provability of $A \rightarrow B$ in R *indicates* that A R-entails B in the sense just defined; i.e., in an absolutely general sense of 'whenever', whenever A is true B is true. Second, entailment on a valuation and, derivatively, entailment in an *r. m. s.* are *not* to be identified with entailment in its logical sense; rather these notions characterize all the truth-preserving arguments in specific contexts; by taking into account all interpretations and all *r. m. s.*, however, we arrive at the desired logical notion.

We now wish to relate entailment and verification. Our remarks about the preferred status of 0 suggest, quite correctly, that $A \rightarrow B$ is to be R-valid iff A R-entails B, and, more specifically, that on a valuation v in an *r. m. s.*, A entails B on v iff $A \rightarrow B$ is verified (i.e., true at 0) on v. These things are to be proved immediately; in that light, we pause to remark that the chief role of 0 and its special postulates lies in the fact that we wish to know in general which sentences are logically true, not just which sentences of the special form A → B are true. Nevertheless, the former problem may be reduced to the latter, making 0 and its accompanying machinery theoretically dispensable.

To show that entailment reduces to verification, we prove first some key lemmas. Until further notice, let A, B, C, be arbitrary formulas, ⟨0, *K*, R, *⟩ be an arbitrary relevant model structure, *a* and *b* be members of *K*, *v* be a valuation of SL in ⟨0, *K*, R, *⟩, and *I* be the interpretation associated with *v*.

Lemma 1. *$a < b$ and I*(A,*a*) = T ⇒ *I*(A,*b*) = T.

Proof. By induction on the length of A. Restriction (1) on p. 206 takes care of the basis case. The argument is trivial on inductive hypothesis where the main connective is & or V. Since ○ is a defined connective, two cases remain. (a) If A is of the form $\overline{\mathrm{B}}$, then if $b > a$ and *I*(A,*a*) = T, by p6 $b^* < a^*$ and so by vi *I*(B, a^*) = F, whence on inductive hypothesis *I*(B,b^*) = F and hence *I*(A, *a*) = T. (b) If A is of the form B → C, then if *I*(A,*a*) = T it is the case by iv that if *I*(B,*d*) = T and R*adc* then *I*(C,*c*) = T, for any *d,c* in *K*. Suppose then that $a < b$ and *I*(B,*d*) = T and R*bdc*. By d1 and d2, $R^2 0adc$, whence by p4 R*adc*, whence on assumption, *I*(C,*c*) = T; so by iv, *I*(A,*b*) = T ending the proof of Lemma 1.

Lemma 2. A *entails* B *on v* ⇒ A → B *is verified on v.*

Proof. Suppose that, for all *c* in *K*, if *I*(A,*c*) = T then *I*(B,*c*) = T. We must show *I*(A → B,0) = T. For arbitrary *a,b* in *K*, suppose *I*(A,*a*) = T and R0*ab*. By d1, $a < b$. By Lemma 1, *I*(A,*b*) = T. By assumption *I*(B,b) = T, which, applying iv, ends the proof.

Lemma 3. A → B *is verified on v* ⇒ A *entails* B *on v.*

Proof. Suppose that for all *a,b* in *K*, if R0*ab* and *I*(A,*a*) = T then *I*(B,*b*) = T. Applying d1 and p1, A entails B on *v*.

The theorem relating entailment and verification is at hand.

Theorem 1. A *entails* B *on v iff* A → B *is verified on v. So* A *entails* B *in* ⟨0, *K*, R, *⟩ *iff* A → B *is valid therein, and* A R-*entails* B *iff* A → B *is* R-*valid.*

Proof by Lemmas 2, 3 and definitions.

5. Semantic consistency of R (adequacy of the postulates)

We show in this section that every theorem of R is R-valid. Notational conventions are as in 4, and again we prove some preliminary lemmas.

Lemma 4. *If* A *is an axiom of* R, $I(\mathrm{A}, 0) = \mathrm{T}$.

Proof. Since each of the axioms of R is of the form $\mathrm{B} \rightarrow \mathrm{C}$, it suffices by Lemma 2 to show for each such axiom that (a) if $I(\mathrm{B},a) = \mathrm{T}$ then $I(\mathrm{C},a) = \mathrm{T}$. Accordingly, for each instance of A1–A13 we assume the antecedent of (a) and prove its conclusion. A1 is trivial. For A2 it suffices to show that if A and $\mathrm{A} \rightarrow \mathrm{B}$ are respectively true on v at points a and b respectively, then if $\mathrm{R}abc$ it follows that C is true at c; inasmuch as $\mathrm{R}abc \Rightarrow \mathrm{R}^2 0abc \Rightarrow \mathrm{R}^2 0bac \Rightarrow \mathrm{R}bac$ by p1, p3, and p4, C is indeed true at c, verifying A2. For A3 it suffices to show that if $\mathrm{A} \rightarrow \mathrm{B}$ is true at a and $\mathrm{R}abc$, then if $\mathrm{B} \rightarrow \mathrm{C}$ is true at b and $\mathrm{R}cde$ and A is true at d on the valuation v then C is true at e; but if $\mathrm{R}abc$ and $\mathrm{R}cde$ then by d2, R^2abde and by p3, R^2adbe, whence if $I(\mathrm{A} \rightarrow \mathrm{B},a) = I(\mathrm{A},d) = I(\mathrm{B} \rightarrow \mathrm{C},b) = \mathrm{T}$ then there is by d2 an x in K such that $I(\mathrm{B},x) = \mathrm{T}$ and $\mathrm{R}xbe$, whence $\mathrm{R}bxe$ [cf. verification of A2], whence C is true at e, verifying A3. For A4 suppose $I(\mathrm{A} \rightarrow (\mathrm{A} \rightarrow \mathrm{B}), a) = \mathrm{T}$ and $I(\mathrm{A},b) = \mathrm{T}$ and $\mathrm{R}abc$; show $I(\mathrm{B},c) = \mathrm{T}$. But $\mathrm{R}abc \Rightarrow \mathrm{R}bac$, as above, whence by p2 and d2, R^2bbac; by p3, R^2babc; applying d2 and commuting, again R^2abbc, whence on assumption, d2, and iv it is clear that $I(\mathrm{B},c) = \mathrm{T}$, verifying A4.

Conjunction and disjunction axioms are trivial, given the characterization (b) above of what counts as an interpretation. Consider, for example, A7; assume both $\mathrm{A} \rightarrow \mathrm{B}$ and $\mathrm{A} \rightarrow \mathrm{C}$ true at a; show $\mathrm{A} \rightarrow \mathrm{B}\ \&\ \mathrm{C}$ true at a. Suppose then $I(\mathrm{A},b) = \mathrm{T}$ and $\mathrm{R}abc$; show B & C true at c. But on assumption $I(\mathrm{A} \rightarrow \mathrm{B},a) = I(\mathrm{A} \rightarrow \mathrm{C},a) = \mathrm{T}$, whence $I(\mathrm{B},c) = I(\mathrm{C},c) = \mathrm{T}$, whence by ii on p. 206, $I(\mathrm{B}\ \&\ \mathrm{C}, c) = \mathrm{T}$, which was to be proved; other verifications among A5-A11 are in like manner.

To verify A12, suppose $\mathrm{A} \rightarrow \overline{\mathrm{B}}$ true at a. Show $\mathrm{B} \rightarrow \overline{\mathrm{A}}$ true at a. Suppose $\mathrm{R}abc$ and B true at b; show $\overline{\mathrm{A}}$ true at c – i.e., show $I(\mathrm{A},c^*) = \mathrm{F}$. Suppose for reductio that $I(\mathrm{A},c^*) = \mathrm{T}$; by p5, $\mathrm{R}ac^*b^*$ and hence $I(\overline{\mathrm{B}},b^*) = \mathrm{T}$, since $\mathrm{A} \rightarrow \overline{\mathrm{B}}$ is true at a. So $I(\mathrm{B},b^{**}) = \mathrm{F}$, by vi, whence by p6, $I(\mathrm{B},b) = \mathrm{F}$, a contradiction. Similarly, verify A13 by assuming $\overline{\overline{\mathrm{A}}}$ true at a, hence $\overline{\mathrm{A}}$ false at a^*, hence A true at a^{**}, hence by p6 A true at a, thus completing the proof of Lemma 4.

In order to consider variations on the underlying logic, it is significant to inquire what in the proof of Lemma 4 depends on what. Among the postulates, only p4 figures essentially in verification of the conjunction-disjunction axioms and of A1, on account of its role in the proof of Lemma 1. The negation postulates p5 and p6, it should be noted, enter essentially only into verification of negation axioms; these postulates may be weakened to produce, if desired, another theory of negation – e.g., an intuitionistically acceptable one, lacking A13. The total reflexivity postulate p2 enters only into the verification of the contraction axiom A4; dropping the postulate amounts to dropping the axiom, and conversely. Finally p3, called by Dunn *Pasch's Law* because of its similarity in form to the famous postulate introduced by Pasch into geometry (reading R*abc*, '*c* is between *a* and *b*'), pays its way in the verification of the transitivity axiom A3; other uses may be replaced, as is clear from the proof of Lemma 4, by weaker postulates.

To complete the proof that our postulates are adequate, we want to show that R1 and R2 preserve verification; we prove a little more, beginning with a definition. Where ⟨0, *K*, R, *⟩ is an *r. m. s.*, $a \in K$, and v is an arbitrary valuation in ⟨0, *K*, R, *⟩ the *theory determined by v at a*, in symbols $T(v, a)$, shall be the set of sentences true at a on v; the *regular theory determined by v* shall be the set of sentences verified on v – i.e., $T(v,0)$, which in context we abbreviate as $T(v)$.

Lemma 5. *The set of sentences $T(v,a)$ true on v at a is closed under adjunction;* i.e., *if* A, B $\in T(v,a)$, A & B $\in T(v,a)$. *Accordingly the set of sentences $T(v)$ verified on v is closed under adjunction.*

Proof. Immediate from definitions.

Lemma 6. *$T(v,a)$ is closed under entailment on v;* i.e., *if* A $\in T(v,a)$ *and* A → B $\in T(v,0)$, *then* B $\in T(v,a)$. *Accordingly $T(v)$ is closed under modus ponens.*

Proof. Immediate from definitions and Theorem 1.

Lemma 7. *$T(v,a)$ is closed under modus ponens;* i.e., *if* A $\in T(v,a)$ *and* A → B $\in T(v,a)$, *then* B $\in T(v,a)$. *Accordingly $T(v)$ is closed under modus ponens.*

Proof. Immediate from definitions and p2.

Theorem 2. *The regular theory $T(v)$ of sentences verified on v, for all relevant model structures and all valuations v therein, contains all theorems of* R. *Accordingly the semantics developed above is adequate for* R, *in the sense that all theorems of* R *are* R*-valid.*

Proof. $T(v)$ being as in the statement of the theorem, it contains by Lemma 4 all axioms of R, and is closed under adjunction and *modus ponens* by Lemmas 5 and 6 (or 7) respectively. So all theorems of R belong to

$T(\nu)$, for arbitrary ν in an arbitrary *r.m.s.* Hence an R-theorem A belongs to all such $T(\nu)$, which amounts to its R-validity.

6. Preliminaries for completeness: The theory of intensional theories

Let the sentential language SL = $\langle S, O, F\rangle$ be as in 1. A subset T of F is an *intensional* R-*theory* provided that T is closed under adjunction and that whenever A ϵ T and A $\rightarrow$ B is a theorem of R, B ϵ T. (The latter condition amounts to closure under R-entailment in the syntactical sense, which coincides, as we must prove, with R-entailment as characterized semantically.)

Let T be an intensional R-theory. T is *prime* provided that whenever A V B ϵ T, A ϵ T or B ϵ T. T is *regular* provided that T contains all theorems of R,[12] and T is *consistent* provided that T does not contain the negation of some theorem of R.[13] Finally, T is *normal*, as in Meyer and Dunn [1969], provided that T is regular, consistent, and prime; normality, we note, implies consistency and completeness in the familiar classical sense.

Contact with the ideas of the last section may be had as follows.

Lemma 8. *Let* ν *be a valuation in an r. m. s.* $\langle 0, K, \text{R}, {}^*\rangle$ *and let* $a \in K$. *Then the theory* $T(\nu,a)$ *determined by* ν *at* a *is an intensional R-theory; moreover,* $T(\nu,a)$ *is prime. If* $0 < a$, $T(\nu,a)$ *is regular; in particular,* $T(\nu)$ *is regular. Finally, a sufficient condition for* $T(\nu)$ *to be normal is that* $0 < 0^*$ *hold in* $\langle 0, K, \text{R}, {}^*\rangle$, $<$ *being defined by* d1 *above.*

Proof. By Lemma 5, $T(\nu,a)$ is closed under adjunction. Suppose then that A ϵ $T(\nu,a)$ and A $\rightarrow$ B is a theorem of R. By Theorem 2, A entails B on ν, when by Lemma 6, B ϵ $T(\nu,a)$. So $T(\nu,a)$ is an intensional R-theory. Moreover, by the recursive conditions on V, $T(\nu,a)$ is prime.

Since by Theorem 2 all theorems of R are verified on ν, if $0 < a$ then by Lemma 1 all theorems of R are true on ν at a; so if $0 < a$, $T(\nu,a)$ is regular. Finally, suppose that $0 < 0^*$. By p2, R0*0*0* always holds, whence by p5-p6 R0*00 and, commuting by p3-p4, R00*0; i.e., $0^* < 0$ holds in all *r. m. s.* Under the hypothesis, then, $T(\nu) = T(\nu,0) = T(\nu,0^*)$ by Lemma 1; i.e., exactly the same formulas are true at 0 and at 0* on ν. Since all theorems of R are true at 0 on ν, they are by hypothesis all true also at 0*; hence no negation of a theorem is true at 0, which suffices for the consis-

[12] What is called here a *regular theory* was called in previous publications – e.g., Meyer and Dunn [1969] and Meyer et al. [1972] – just a *theory*.

[13] Since $\overline{\text{A} \,\&\, \overline{\text{A}}}$ is a theorem of R, at most one of A, $\overline{\text{A}}$ belong to a consistent theory.

tency of $T(\nu)$ and hence, since $T(\nu)$ is regular and prime always, $0 < 0^*$ implies that $T(\nu)$ is normal, ending the proof of the lemma.

We turn now to the development of a *calculus of intensional* R*-theories.* Let $\mathscr{H}$ be the collection of all intensional R-theories. We define an operation $\circ$ on $\mathscr{H}$ by setting, for all S, T in $\mathscr{H}$ $S \circ T$ equal to the set of formulas U such that $C \in U$ iff there are A in S and B in T such that $A \circ B \to C$ is a theorem of R; i.e., $S \circ T = \{C : \exists A \exists B(A \in S \;\&\; B \in T \;\&\vdash_R A \circ B \to C\}$. Then by the calculus of intensional R-theories we mean the structure $\mathscr{H} = \langle \mathscr{H}, \subseteq, \circ, 0\rangle$, where $\mathscr{H}$ and $\circ$ are as just defined and 0 is the set of theorems of R; $\subseteq$ is of course set inclusion.

Lemma 9. *The calculus* $\mathscr{H}$ *just defined of intensional* R*-theories is a partially ordered commutative monoid;* i.e., $\circ$ *is commutative and associative, and* 0 *is an identity with respect to* $\circ$*; furthermore, for all* **a,b,c** *in* $\mathscr{H}$*, if* $a \subseteq b$ *then* $a\circ c \subseteq b \circ c$*. Moreover* $\mathscr{H}$ *is square-decreasing* – i.e., $a \circ a \subseteq a$.

Proof. We verify first that $\circ$ is an operation on $\mathscr{H}$; i.e., that if $a,b \in \mathscr{H}$, $a \circ b \in$. It is trivial that $a\circ b$ is closed under provable R-entailments; to show it closed under adjunction, suppose that both C and D belong to $a \circ b$. Then there are A,A′ in a and B,B′ in b such that $A\circ B \to C$ and $A'\circ B' \to D$ are theorems of R, whence $(A\circ B)\&(A'\circ B') \to C\&D$ is a theorem of R by elementary properties of conjunction. But $(A\&A')\circ(B\&B') \to (A\circ B)\&(A'\circ B')$ is also easily shown an R-theorem; the antecedent is clearly in $a\circ b$ by closure of a, b under adjunction, whence by transitivity and closure of $a\circ b$ under provable R-entailment, $C\&D \in a\circ b$, showing $a \circ b$ an intensional R-theory.

Commutativity and associativity of $\circ$ as an operation on $\mathscr{H}$ fall out easily from the same properties of $\circ$ as a connective of R. That $0\circ a$ contains a is trivial, since if $A \in a$ the R-equivalent $(A \to A) \circ A \in 0\circ a$; conversely, if $B \in 0 \circ a$, there is an R-theorem C and A in a such that $C\circ A \to B$ is provable in R, whence [since $\circ$ exports] $A \to B$ is a theorem of R, whence $B \in a$. The monotonic property of $\circ$ under $\subseteq$ and the square-decreasing property are again trivial, the latter because closure under & implies closure under $\circ$ for an R-theory via the theorem $A \;\&\; B \to A \circ B$ of R, ending the proof of Lemma 9.

Let T be any *regular* intensional R-theory. Let an *intensional T-theory* be any set of formulas of the sentential language SL which is closed under adjunction and *T-entailment* – i.e., such that a is an intensional T-theory provided that a is an intensional R-theory and moreover whenever $A \to B \in T$ and $A \in a$, $B \in a$. The calculus of *intensional T-theories* $\mathscr{H}_T = \langle \mathscr{H}_T, \subseteq, \circ, 0_T\rangle$, for a regular R-theory T, is defined by setting $\mathscr{H}_T$ equal to the set of all intensional T-theories, taking T as 0_T, and defining $\circ$ and $\subseteq$ as before.

Lemma 10. *Where T is a regular* R*-theory, the calculus $\mathscr{H}_T$ just defined is a sub-semigroup of $\mathscr{H}$, and is hence partially ordered, commutative, and square-decreasing in the sense of Lemma 9; moreover $\mathscr{H}_T$ is a monoid with identity* 0_T.

Proof. Since all T-theories are R-theories, clearly $\mathscr{H}_T \subseteq \mathscr{H}$. To show $\mathscr{H}_T$ a sub-semigroup, and hence that commutative, associative, partial ordering, and square-decreasing laws hold, it suffices to show $\mathscr{H}_T$ closed under $\circ$. Suppose then that a and b belong to $\mathscr{H}_T$. We must show $a \circ b$ closed under T-entailment – i.e., that if $C \in a \circ b$ and $C \rightarrow D \in T$, $D \in a \circ b$. But if $C \in a \circ b$ then there are A in a and B in b such that $A \circ B \rightarrow C$ is a theorem of R; then by transitivity $A \circ B \rightarrow D \in T$ (since it is R-entailed by $C \rightarrow D$). By exportation $A \rightarrow (B \rightarrow D)$ belongs to T, whence by closure of a under T-entailment, $B \rightarrow D \in a$. Then $(B \rightarrow D) \circ B$ belongs to $a \circ b$ by definition; but this R-entails D, whence $D \in a \circ b$, which was to be proved.

To show 0_T the identity, note that, since T is regular, $a \subseteq a \circ 0_T$ as before. Conversely, suppose $B \in a \circ 0_T$. Then there are A in a, C in T, such that $A \circ C \rightarrow B$ is a theorem of R. But then, exporting, $A \rightarrow B \in T$, whence by closure of a under T-entailment $B \in a$, ending the proof of Lemma 10.

By an *r+. m. s.*, understand a structure $\langle 0, K, R \rangle$, where K is a set, $0 \in K$, and R is a ternary relation on K satisfying p1–p4. (Clearly all *r. m. s.* are *r+. m. s.*, since the former results from the latter by adding * and its machinery, though not conversely.) Let $\mathbf{M} = \langle M, \leqslant, \circ, 0 \rangle$ be any commutative, partially ordered, square-decreasing monoid in the sense of Lemmas 9 and 10 – i.e., $\circ$ is commutative and associative, 0 is the identity, $\leqslant$ is a partial ordering satisfying $a \leqslant b \Rightarrow a \circ c \leqslant b \circ c$ and $a \circ a \leqslant a$. By the *r+. m. s.* associated with $\mathbf{M}$, we mean the structure $\langle 0, M, R \rangle$ defined by setting 0 equal to the identity of $\mathbf{M}$, identifying M as the underlying set of $\mathbf{M}$, and letting R be a ternary relation on M such that $Rabc$ holds iff $a \circ b \leqslant c$ in M. Then

Lemma 11. *If* $\mathbf{M}$ *is a commutative, square-decreasing, partially ordered monoid, then the r+. m. s.* $\langle 0, M, R \rangle$ *associated with* $\mathbf{M}$ *satisfies* p1–p4, *justifying the terminology. Moreover, for any a, b, c, d in M, $a < b$ in* $\langle 0, M, R \rangle$ *iff $a \leqslant b$ in* $\mathbf{M}$, *and $R^2 abcd$ iff $a \circ b \circ c \leqslant d$ in* $\mathbf{M}$, *thus interpreting* d1 *and* d2. *In particular, the calculus $\mathscr{H}$ of Lemma* 9 *and the $\mathscr{H}_T$ of Lemma* 10 *are associated with r+. m. s.* $\langle 0, \mathscr{H}, R \rangle$ *and* $\langle 0_T, \mathscr{H}_T, R_T \rangle$ *respectively.*

Proof. Let **M** and the ternary relation R be as just characterized. Then $R0ab$ iff $0 \circ a = a \leqslant b$, justifying d1. R^2abcd iff there is an x in M such that $a \circ b \leqslant x$ and $x \circ c \leqslant d$. But then the monotonicity condition on p. o. monoids yields $a \circ b \circ c \leqslant x \circ c \leqslant d$, interpreting d2, in one direction; for the converse, set $a \circ b = x$. Then the postulates hold, as follows: p1, since $a \leqslant a$ in any partial ordering; p2, by the square-decreasing condition $a \circ a \leqslant a$ on **M**; p3, since $\circ$ is commutative and associative; p4, because 0 is the identity of **M**. Thus $\langle 0, \mathscr{H}, R\rangle$ and $\langle 0_T, \mathscr{H}_T, R_T\rangle$ for regular intensional R-theories T are *r+. m. s.* for by Lemmas 9 and 10 the corresponding calculi of theories meet the conditions on **M**, ending proof of Lemma 11.

Development of the calculi of intensional theories takes us most of the way to completeness for our semantics. Two obstacles remain. First, an arbitrary intensional R-theory is not sufficiently discriminating regarding disjunction; second, * is not in a natural way an operation on all of $\mathscr{H}$ or on the various $\mathscr{H}_T$. We cure both problems in the next section by passing to *prime* theories.

7. Semantical completeness of the system R. Prime intensional theories

Let T be a prime, regular intensional R-theory, and let $\langle 0_T, \mathscr{H}_T, R_T\rangle$ be the *r+. m. s.* associated with T by Lemma 11. Let $\mathscr{H}_T'$ be the subset of $\mathscr{H}_T$ consisting of all the *prime* intensional theories in $\mathscr{H}_T$, and let R_T' be the restriction of R_T to $\mathscr{H}_T'$. Then

Lemma 12. *Let T be a prime, regular intensional* R*-theory, and let* $\mathscr{H}_T'$ *and* R_T' *be as just defined. Let* $0_T'$ *be T. Then* $\langle 0_T', \mathscr{H}_T', R_T'\rangle$ *satisfies* p1–p4 – i.e., *it's an* r^+*. m. s.*

Proof. Since $0_T' = T = 0_T$, and since R_T' is the restriction of R_T to the subset $\mathscr{H}_T'$ of $\mathscr{H}_T$, p1, p2, and p4 hold in $\langle 0_T', \mathscr{H}_T', R_T'\rangle$ because they hold in $\langle 0_T, \mathscr{H}_T, R_T\rangle$. Thus only verification of the Pasch Law p3 might pose difficulties, since it makes the existential claim that if $R_T'^2abcd$, then there is an x in $\mathscr{H}_T'$ such that R_Tacx and R_Txbd. The problem is that although such an x certainly belongs to $\mathscr{H}_T$ – by Lemma 11, $a \circ c$ itself will do – there is no guarantee that there is a prime theory x which will serve the mediating function desired for p3; the rest of the proof of Lemma 12 consists of finding a guarantee.

Suppose then that $R_T'^2\ abcd$. By what has been said, there is an x in $\mathscr{H}_T$ such that R_Tacx and R_Txbd; we show that there is an x' in $\mathscr{H}_T'$ such that $R_T'acx'$ and $R_T'x'bd$, whence $R_T'x'bd$, verifying p3. At any rate, by definition of R and hence of R_T on $\mathscr{H}_T$, there is a maximal T-theory x'

satisfying the conditions (i) $x < x'$ and $R_T x'bd$ in $\langle 0_T, \mathscr{H}_T, R_T\rangle$; for x itself satisfies condition (i) while the union of a collection of intensional T-theories satisfying (i) and totally ordered by $\subseteq$ is easily seen to be itself an intensional T-theory satisfying (i), whence by Zorn's Lemma there is an intensional T-theory x' which is a superset of x and such that $x' \circ b \subseteq d$, while no proper supersets of x' are intensional T-theories satisfying (i).

It remains to be shown that x' is prime. Suppose it isn't. Then there is a formula A V B such that A V B $\notin x'$, A $\notin x'$, and B $\notin x'$. Let $[x', \text{A}]$ and $[x', \text{B}]$ be respectively the sets of formulas D such that there exists a member C of x' such that respectively C & A → D, C & B → D are members of T. It is easy to see that $[x', \text{A}]$ and $[x', \text{B}]$ are supersets of x' closed under adjunction and T-entailment; accordingly they are intensional T-theories which, by the maximality of x', fail to satisfy the condition (i). So neither $[x', \text{A}]\circ b$ nor $[x', \text{B}]\circ b$ is a subset of d. Accordingly there are E in $[x', \text{A}]$ and F in b such that $\vdash_{\text{R}}$ E∘F → D, but D $\notin d$; moreover by definition of $[x', \text{A}]$ there is a C in x' such that C&A → E ϵ T, whence since by exportation and regularity of T, E → (F → D) ϵ T, by adjunction, transitivity, and importation of ∘, (C&A)∘F → D ϵ T. By parity of reasoning, there are C′ in x', F′ in b, and D′ not in d such that (C′&B)∘F′ → D′ ϵ T. Since T is closed under adjunction, elementary syntactical arguments may then be applied to show that (C&C′&(AVB))∘(F&F′) → DVD′ belongs to T, whence since C&C′&(AVB) belongs to x', F&F′ belongs to b, and $x' \circ b \subseteq d$, D V D′ ϵ d. But d is prime; so D ϵ d or D′ ϵ d, contradicting our selection of D and D′ as non-members of d; the hypothesis that x' is not prime having proved untenable, we conclude that x' is prime, and hence that (along with a, b, c, d) $x' \in \mathscr{H}_T'$.

We have now showed $R_T'x'bd$, $R_T acx$, and $R_T 0_T xx'$. By p3–p4, $R_T acx'$, whence since a, c, x' are all prime, $R_T' acx'$ and so by d2 $R_T'^2 acbd$, completing the verification of p3 and the proof of Lemma 12.

Lemma 12 enables us to throw out of $\mathscr{H}_T$ all but the prime theories, getting $\mathscr{H}_T'$. This enables us to handle negation also, since the *operation may be naturally defined on $\mathscr{H}_T'$. Indeed, where a is a prime intensional theory, let a^* be the set of all formulas A such that $\overline{\text{A}}$ doesn't belong to a – in symbols, $a^* = \{\text{A}: \overline{\text{A}} \notin a\}$. Then

Lemma 13. *Let T be a prime intensional theory, and let the r+. m. s.* $\langle 0_T', \mathscr{H}_T', R_T'\rangle$ *be as in Lemma* 12. *Let * be defined as above, and let *′ be its restriction to* $\mathscr{H}_T'$. *Then* $\langle 0_T', \mathscr{H}_T', R_T', *'\rangle$ *is an r. m. s.*

Proof. We must show (a) that $*'$ applied to members of $\mathscr{K}_T'$ yields values in $\mathscr{K}_T'$ and (b) that p5–p6 are satisfied, since the rest of the lemma follows from Lemma 12. In proving (a), we show first that $*$, applied to prime theories, yields prime theories; second, that if a is closed under T-entailment, so is a^*. Suppose then that a is a prime theory, that $A \in a^*$ and that $A \to B$ is a theorem of R. Suppose, for *reductio*, that $B \notin a^*$. Then $\bar{B} \in a$, whence by contraposition and closure of a under provable R-entailment, $\bar{A} \in a$, contradicting the supposition that $A \in a^*$ and showing a^* closed under provable R-entailment. To show a^* closed under adjunction, suppose A, $B \in a^*$ but that, for *reductio*, $A \& B \notin a^*$. Then $\overline{A \& B} \in a$, whence by the DeMorgan laws and the primeness of a, either $\bar{A}$ or $\bar{B}$ belongs to a, contradicting the assumption that both of A, B are in a^*. Finally, to show a^* prime, if neither A nor B belongs to a^*, then both of $\bar{A}$, $\bar{B}$ are in a, whence by adjunction and DeMorgan $\overline{A \vee B}$ is in a and $A \vee B \notin a^*$. Contraposing, if $A \vee B \in a^*$, $A \in a^*$ or $B \in a^*$, completing the proof that if a is a prime intensional theory so also is a^*. Finally, if a is closed under T-entailment and A belongs to a^* and $A \to B \in T$, if B isn't in a^* then $\bar{B}$ and hence $\bar{A}$ belongs as before to a and A doesn't belong to a^*, which is again absurd. Accordingly (a) is proved – the restriction $*'$ of $*$ to members of $\mathscr{K}_T'$ does not lead out of $\mathscr{K}_T'$.

We verify (b) beginning with p6. $A \in a^{**}$ iff $\bar{A} \notin a^*$, iff $\bar{\bar{A}} \in a$, iff $A \in a$ whenever a is an intensional R-theory, by double negation. To verify p5, suppose that $R_T'abc$. By definition of R_T', $a \circ b \subseteq c$. Suppose that $D \in a \circ c^*$; we show $D \in b^*$. Suppose not; then $\bar{D} \in b$. Since $D \in a \circ c^*$, there are A in a, E in c^* such that $\vdash_R A \circ E \to D$, whence by contraposition $\vdash_R \bar{D} \to (A \to \bar{E})$. Since b is closed under provable R-entailment, $A \to \bar{E} \in b$, whence $A \circ (A \to \bar{E}) \in a \circ b \subseteq c$. But $A \circ (A \to \bar{E})$ provably R-entails $\bar{E}$, whence $\bar{E} \in c$, whence $E \notin c^*$, a contradiction. This shows on our assumptions that $a \circ c^* \subseteq b^*$, and hence by definition that $R_T'ac^*b^*$, completing the verification of p5 and the proof of Lemma 13.

Semantic completeness is now at hand. Where T is any regular prime intensional R-theory, let the *r. m. s.* $\langle 0_T', \mathscr{K}_T', R_T', *' \rangle$ associated with T by Lemma 13 be called the *T-canonical r. m. s.* $\mathscr{K}_T'$. The *T-canonical valuation* v_T will be the function that assigns, for each sentential parameter p, and each prime T-theory a in $\mathscr{K}_T'$, $v_T(p, a) = T$ iff $p \in a$. v_T, in short, makes a parameter true *at* a theory iff that parameter is *in* the theory; we shall show that it does the same for all formulas.

Lemma 14. *Let T, v_T, and $\mathscr{K}_T'$ be as above, and let I be the interpretation in the r. m. s. $\mathscr{K}_T'$ associated with the T-canonical valuation v_T. Then for* *every formula* A *of the sentential language* SL, *and for all a in* $\mathscr{H}_T'$, $I(A, a) = T$ *iff* A ϵ *a. (Succinctly, in the notation of Lemmas 5–7,* $T(\nu_T, a) = a$.*)*

Proof. We suppose on inductive hypothesis that for all a in $\mathscr{H}_T'$ the lemma holds for all formulas A in which fewer than k connectives occur, and we show that it continues to hold for arbitrary A and a if the number of connectives in A is k. The case where $k = 0$ has been decided affirmatively by definition of ν_T. When $k > 0$, there are 4 cases, depending on the main connective of A.

Case 1. A is B & C. Then $I(B\&C, a) = T$ iff $I(B, a) = I(C, a) = T$, iff (applying the inductive hypothesis) B ϵ a and C ϵ a, iff B & C ϵ a.

Case 2. A is B $\vee$ C. Then $I(B \vee C, a) = T$ iff $I(B, a) = T$ or $I(C, a) = T$, iff (applying the inductive hypothesis) B ϵ a or C ϵ a, iff B $\vee$ C ϵ a (since a is prime).

Case 3. A is $\bar{B}$. Then $I(\bar{B}, a) = T$ iff $I(B, a^*) = F$, iff (applying the inductive hypothesis) B $\notin a^*$, iff $\bar{B}$ ϵ a.

Case 4. A is B $\rightarrow$ C. Then $I(B \rightarrow C, a) = T$ iff for all b, c, in $\mathscr{H}_T'$, given $R_T'abc$ and $I(B, b) = T$ then $I(C, c) = T$, iff, applying the inductive hypothesis, (i) for all b, c in $\mathscr{H}_T'$, given $R_T'abc$ and B ϵ b then C ϵ c. We must show that (i) holds iff (ii) B $\rightarrow$ C ϵ a.

We recall that $R_T'abc$ means that $a \circ b \subseteq c$ in the calculus of intensional R-theories. Accordingly that (ii) implies (i) is trivial, since if B $\rightarrow$ C is in a and B is in b, then (B $\rightarrow$ C) $\circ$ B is in $a \circ b$ and hence in c, given $a \circ b \subseteq c$. But (B $\rightarrow$ C)$\circ$B provably R-entails C, whence, since c is an intensional R-theory, C ϵ c.

To show that (i) implies (ii) it suffices to apply the maximizing argument of Lemma 12. Suppose that B $\rightarrow$ C $\notin a$. We show that there exist b, c in $\mathscr{H}_T'$ such that $R_T'abc$, B ϵ b, and C $\notin c$; that (i) implies (ii) then follows by contraposition.

Since B $\rightarrow$ C $\notin a$, by double negation and definition of $\circ$, B$\circ\bar{\bar{C}}$ ϵ a^*. Let [B], [$\bar{C}$] be the sets of formulas which are T-entailed by B and $\bar{C}$ respectively; clearly these are intensional T-theories and [B]$\circ$[$\bar{C}$] $\subseteq a^*$, since we may assume a and hence a^* to be an intensional prime T-theory. But, since a^* is prime, by the argument of Lemma 12 we can find prime intensional T-theories b and d such that [B] $\subseteq b$, [$\bar{C}$] $\subseteq d$, and $b \circ d \subseteq a^*$ applying the argument that verifies p5 and p6, $a \circ b \subseteq d^*$. Taking d^* as the desired c, clearly $R_T'abc$ and B ϵ b; moreover C isn't in $c = d^*$, inasmuch as $\bar{C}$ ϵ $d = c^*$. This is what was promised and hence we've completed the verification of case 4 and with it the proof of Lemma 14 on the inductive argument.

Lemma 14 reduces the problem of proving completeness for R to that of showing that every non-theorem fails to belong to some regular prime R-theory. This was proved in Meyer and Dunn [1969], but will be entered here explicitly for the sake of completeness.

Lemma 15. *Let* A *be a non-theorem of the system* R *of relevant implication. Then there is a prime, regular* R-*theory which does not contain* A.

Proof. On assumption the set of theorems of R is a regular R-theory without A. Ordering the set of all regular R-theories without A by set inclusion, we discover that every chain in this set is bounded by its union and hence, on application of Zorn's Lemma, that there is a maximal regular R-theory without A; call it T. If $B \vee C \in T$, $B \notin T$, $C \notin T$, then by maximality of T the R-theories $[T, B]$ and $[T, C]$ formed as in the proof of Lemma 12 both contain A, whence by elementary properties of disjunction and conjunction so does T, which is impossible and which ends the proof of the lemma.

We record completeness in a theorem.

Theorem 3. *The system* R *of relevant implication is semantically complete, in the sense that all* R-*valid formulas are theorems.*

Proof. We proceed by contraposition. Assume that the formula A of R is not a theorem of R. We prove that A is not R-valid. In fact, since A is not a theorem, there is by Lemma 15 a regular prime R-theory T such that $A \notin T$. Consider then the T-canonical *r. m. s.* yielded by Lemma 13 $\langle 0_T', \mathscr{K}_T', R_T', *' \rangle$. By Lemma 14, $T(v_T, a) = a$ for every member a of $\mathscr{K}_T'$, where ν_T is the T-canonical valuation; in particular, the set $T(\nu_T)$ of formulas verified on ν_T is accordingly 0_T; i.e., T itself. So A in particular is not verified on ν_T and is hence R-invalid, ending the proof of Theorem 3 and accordingly of the semantical completeness of R.

8. Normal relevant semantics – gamma and all that

In Meyer and Dunn [1969] and elsewhere, quite a fuss has been made about showing that whenever A and $\bar{A} \vee B$ are both theorems of R (or of related systems) then B is a theorem of R. The principle, known following Ackermann as γ, required complicated argument in Meyer and Dunn [1969], but it can be disposed of simply here.

Let $\langle 0, K, R, * \rangle$ be an *r. m. s.* We shall call $\langle 0, K, R, * \rangle$ *normal* provided that the following postulate is satisfied.

p0. $0 = 0^*$.

By Lemma 8, we note that the set of formulas verified on a valuation in a normal *r. m. s.* constitute a normal R-theory. Accordingly the set of formulas *valid* in a normal *r. m. s.* is always closed under γ, since if A belongs to each of a collection of normal R-theories $\overline{\mathrm{A}}$ can belong to none of them, since normality presupposes consistency, whence if in addition $\overline{\mathrm{A}} \vee \mathrm{B}$ belong to each member of the collection, B must belong to all of them, since normality also presupposes primeness. Besides, it might be argued, if we want to take really seriously the claim that 0 constitutes the real world, then perhaps we should demand that validity be characterized for the system R as validity in all normal *r. m. s.*, which we shall characterize as *normal* R-*validity*. The work of Meyer and Dunn [1969] shows that validity and normal validity coincide for R; but it is easier to prove that fact directly.

Lemma 16. *Let* $\langle 0, K, \mathrm{R}, * \rangle$ *be an r. m. s. By its normalization understand the structure* $\langle 0', K', \mathrm{R}', *' \rangle$, *where* $0'$ *is a new element,* K' *results from the addition of* $0'$ *to* K, $*$ *is the extension of* $*$ *to* K' *determined by setting* $0'{}^* = 0'$, *and where a, b, c, are elements of* K, R' *holds for triples of elements of* K' *as determined below:*

(i) $\mathrm{R}'0'0'0'$; (ii) $\mathrm{R}'0'0'a$ iff $\mathrm{R}00a$; (iii) $\mathrm{R}'0'a0'$ and $\mathrm{R}'a0'0'$, iff $\mathrm{R}0a0^*$; (iv) $\mathrm{R}'ab0'$ iff $\mathrm{R}ab0^*$; (v) $\mathrm{R}'0'ab$ and $\mathrm{R}'a0'b$, iff $\mathrm{R}0ab$; (vi) $\mathrm{R}'abc$ iff $\mathrm{R}abc$.

Then the normalization $\langle 0', K', \mathrm{R}', *' \rangle$ *is a normal r. m. s.*

Proof. That p0 holds has been settled by fiat, while that p1, p2, and p6 hold follows from their holding for $\langle 0, K, \mathrm{R}, * \rangle$ and the trivial specifications above involving $0'$. p3–p5 require slightly more work, the key to which is the point that R' holds among members of K' iff R holds among members of K on replacing occurrences of $0'$ in the first 2 argument places with 0 and in the final argument place with 0^*, except for the case $\mathrm{R}0'0'0'$. In verifying p3–p4, which require connections, the fact that $0^* < 0$ in all *r. m. s.* is helpful; the rest is busy work, which we leave to the probably uninterested reader.

Theorem 4. R *is consistent and complete with respect to the normal semantics presented in this section – i.e.,* A *is a theorem of* R *iff* A *is normally* R-*valid.*

Proof. If A is a theorem of R it is valid in all *r. m. s.* by Theorem 2, including normal ones, thus establishing normal semantic consistency. Conversely, if A is not a theorem of R it is invalid in some *r. m. s.* $\langle 0, K, \mathrm{R}, * \rangle$, K for short, by Theorem 3; i.e., there is valuation ν, with associated interpretation I, such that $I(\mathrm{A}, 0) = \mathrm{F}$ in K. Let $\langle 0', K', \mathrm{R}', *' \rangle$

be the normalization of K guaranteed by Lemma 16, for short K'. Extend v to a valuation v' in K' by letting v' agree with v where v is defined and setting $v(p,0') = v(p,0)$ for all parameters p. We show (a) that v' continues to satisfy the restrictive condition (1) on p. 206 and hence indeed qualifies as a valuation. Suppose then that for the parameter p and the point a of K' that $v'(p,a) = T$ and that $R'0'ab$, $b \in K'$. We show (b) $v'(p,b) = T$. If a and b are both $0'$, there is nothing to show; if neither are $0'$, by (v), $a < b$ in K, whence since v is a valuation (b) holds; if $a = 0'$ and $b \in K$, by (ii) $0 < b$ in K, whence (b) holds since if p is true at $0'$ by stipulation it's true at 0; finally, if $a \in K$ and $b = 0'$, by (iii) $a < 0^*$ in K, whence since if $v(p,a) = T$ then $v(p,0^*) = v(p,0) = T$, since always $0^* < 0$, whence by stipulation $v'(p,0') = T$, proving (a).

Let I' be the interpretation associated with v'. We show now (c) for all a in K, and for all formulas B, $I'(B,a) = I(B,a)$, and (d) if $I(B,0) = F$, $I'(B,0') = F$.

We show (c) and (d) together by induction on the length of B. The base case, in which B is a parameter, is already settled by (a). 4 cases remain, according as the main connective of B is &, V, –, or →. (c) is trivial on inductive hypothesis in the first three cases, whence (d) follows from Lemma 1 in these cases, since $0' < 0$.

Suppose then that B is of the form C → D, To show (c), suppose first that $a \neq 0'$ and $I(C \to D, a) = F$; then there are b, c in K such that $Rabc$, $I(C,b) = T$, and $I(D,c) = F$. By (vi) $R'abc$, and on inductive hypothesis $I'(C,b) = T$ and $I'(D,c) = F$, whence $I'(C \to D, a) = F$. Suppose conversely for $a \neq 0'$ that $I'(C \to D, a) = F$. Then there are b, c in K' such that $R'abc$, $I'(C,b) = T$, and $I'(D,c) = F$. If $b, c \in K$, $I(C \to D, a) = F$ on inductive hypothesis by reversing the previous argument. Suppose $b = 0'$, $c \in K$. By (d) and inductive hypothesis, $I(C, 0) = T$ and by (v) $R0ac$, which by p3–p4 implies $Ra0c$, which suffices to show $I(C \to D, a) = F$. Suppose $b \in K$, $c = 0'$. Since $0^* < 0'$, by Lemma 1, $I'(D, 0^*) = F$, while by (iv), $Rab0^*$, which suffices to falsify C → D at a on I, given the inductive hypothesis. Finally, suppose $b = c = 0'$. By (iii), commuting again, $Ra00^*$ while on inductive hypothesis and Lemma 1, $I(C,0) = T$ and $I(D,0^*) = F$, falsifying C → D at a on I and establishing that for all a in K and all formulas B, $I'(B,a) = I(B,a)$ on completion of the inductive argument; that's (c), and (d) follows as before by Lemma 1.

We note in conclusion that since for our chosen non-theorem A of R, $I(A,0) = F$, by the principle (d) just proved $I'(A,0') = F$; accordingly A is not normally R-valid, ending the proof of Theorem 4.

Corollary 4.1. *γ holds for* R – i.e., *if both* A *and* $\bar{A} \vee B$ are theorems, so *also is* B.

Proof. The set $T(\nu)$ of formulas verified on a valuation ν in a normal *r. m. s.* is closed under detachment for material implication, whence so is the intersection of all such $T(\nu)$ for *r. m. s.*, which by the theorem is the set of provable formulas of R.

9. R-mingle and beyond. Extensions of the semantics

Dunn and McCall's R-mingle results from R by adding as a new axiom scheme $A \to (A \to A)$. The following will do as a corresponding semantical postulate:

p7. $0 < a$ or $0 < a^*$.

Let accordingly a *Mingle r. m. s.* be any *r. m. s.* satisfying p1–p7, and a *normal Mingle r. m. s.* be a Mingle r. m. s. that also satisfies p0, for all elements of *K*. Let A be RM-valid iff A is valid in all Mingle r. m. s., and normally RM-valid iff A is valid in all normal Mingle r. m. s. We then have as an easy corollary of previous theorems the following.

Theorem 5. *For all sentential formulas* A, *the following conditions are equivalent.* (i) A *is a theorem of* RM; (ii) A *is RM-valid;* (iii) A *is normally* RM-*valid.*

Proof. To show that (i) implies (ii), the proof of Theorem 2 will do provided that we show $A \to (A \to A)$ to be RM-valid. It suffices to note that in RM, though it escapes the most noisome paradoxes, very bad guys nevertheless entail very good guys; i.e., if *both* B and C are RM-valid, $\bar{B}$ RM-entails C. (For proof, assume B and C RM-valid, and let $\bar{B}$ be true at a point a in a Mingle r. m. s. on *I*. Then $a < 0^*$ fails, since otherwise B by Lemma 1 would be false at 0, contradicting its validity; equivalently, by p5 –p6, $0 < a^*$ fails, whence by p7, $0 < a$; but then by Lemma 1 all RM-valid formulas, including C, are true on *I* at a, since by definition of RM-validity they are all true at 0.) So in particular, since $A \to A$ is R-valid and hence RM-valid, its negation RM-entails it. Observe that in the following sequence each member is R-entailed by its predecessor on simple moves: $\overline{A \to A} \to (A \to A)$, $A \to (\overline{A \to A} \to A)$, $A \to (\bar{A} \to (A \to A))$, $A \to (A \to (\bar{A} \to A))$, $A \to (A \to A)$. Since the set of RM-valid formulas is clearly closed under R-entailment, $A \to (A \to A)$ in particular is RM-valid. After this it is trivial that (i) implies (ii), and it *is* trivial that (ii) implies (iii). The method of

Theorem 4 shows that if (ii) implies (i), so does (iii); accordingly, we end the proof of Theorem 5 by showing that (ii) implies (i).

Suppose that C is not a theorem of RM. Lemma 15 delivers a prime, regular RM-theory without C, say T, and Theorem 3 shows that the canonical *r. m. s.* $\langle 0_T', \mathscr{K}_T', R_T', *' \rangle$ invalidates C. All that remains to be proved is that p7 is true for this *r. m. s.* Suppose then that for some $a \in \mathscr{K}_T'$, $0_T' = T \nsubseteq a$; we show $T \subseteq a^*$. Let A belong to T but not to a; by definition of $*$, $\bar{A} \in a^*$. Let B be any member of T. By the standards of T, $\bar{A}$ is a very bad guy and B is a very good guy, whence, since the paradoxical propensities of RM are syntactic as well as semantic, in view of the RM-theorem $A \,\&\, B \rightarrow (\bar{A} \rightarrow B)$, it turns out that $\bar{A} \rightarrow B$ belongs to T. But then since members of $\mathscr{K}_T'$ are closed under T-entailment, $B \in a^*$. B is arbitrary, so $T \subseteq a^*$, which was to be proved. This completes the demonstration that our chosen non-theorem C is RM-invalid, by generalization and contraposition, all RM-valid formulas are theorems of RM, ending the proof of Theorem 5.

p7 is but one of a number of postulates which we might have added to get R-mingle; it implies that $<$ is total, in the sense that if $\langle 0, K, R, * \rangle$ is a *Mingle r.m.s.* either $a < b$ or $b < a$, for all $a, b \in K$. Given the normality postulate p0, total ordering trivially implies p7, though without p0 this does not hold in general[14]. We like p7, however, because it indicates exactly how the imaginative semantic universe for R is cut down to get R-mingle; roughly, the principle is that only additions to, or subtractions from, the "real world" 0 will count as alternatives for RM. If, in particular, we insist that 0 be normal, and hence in principle describable by a consistent and complete theory T, then p7 imposes the conditions that alternatives to 0 be describable in principle either by consistent sub-theories or complete super-theories of T.

RM was proved in Meyer [1968b] to be decidable and to have a simple model in the integers. The present model-theoretic account yields equivalent results – every non-theorem A of RM is refutable in a finite *Sugihara r. m. s.* $\langle 0, K, R, * \rangle$, where K consists of all the integers in some interval $[-m, m]$, 0 is 0, $*$ is $-$, and $Rabc$ holds for integers a, b, c in K iff whichever of a, b is greatest in absolute value is $\leqslant c$; if a, b are alike in absolute value, if the greater of them is no greater than c. This implies the decidability of RM. The only other point worth noting about the semantics of RM given here, in contrast to the account of Meyer [1968b] and Dunn's [1970] extended

[14] We have a concrete counterexample.

account for extensions of RM is that in Meyer [1968b] and in Dunn [1970] to put the integer 0 in a model was to make it *abnormal,* since 0 was its own negation; here the mark of abnormality is to leave 0 out, letting +1 be the real world.

If a logic is an extension of RM, in the sense that it contains all theorems of RM and is closed under adjunction, *Modus ponens,* and substitution, by Dunn's [1970] result it has a finite characteristic matrix. A natural attack on extensions of RM is accordingly to throw in posulates of finitude; by adding to p1–p7 a postulate which says that K shall have no more than 9 members, say, we get a Dunn extension of RM; we leave to Dunn the question whether we get them all that way, noting merely that if we pare down to the single element 0, all distinctions of relevance collapse and we are back to classical logic.

A slightly more interesting way to get classical logic is to strengthen p7 to

p7′. $0 < a$.

Demonstration that p7′ produces classical logic requires a detour through the * postulates and is incomplete without it. For it suffices to show, for all a in K, that $a < 0$ to show that all classical theorems are valid in an *r. m. s.* satisfying p7′. But that's trivial by p5–p6, since $0 < a^*$, whence $a = a^{**} < 0^* < 0$. It then follows, by the monotonicity condition (1) on what counts as a valuation, that for all parameters p and all a, b in K, $v(p,a) = v(p,b)$ for all valuations v whenever p7′ holds, since $a < 0 < b$, and conversely, whence by Lemma 1 for all formulas A, $I(\mathrm{A}, a) = I(\mathrm{A}, b) = I(\mathrm{A}, 0)$, where I is the interpretation associated with v, which suffices to make the real world for all intents and purposes the only world and to validate all tautologies.

10. R+ and within. The semantics of positive logics

We introduced the notion of an *r+. m. s.* on the way to Theorem 3, as a triple ⟨0, K, R⟩ for which p1–p4 hold. The question arises whether *r+. m. s.* furnish a viable semantics for the system R+ determined by the negation-free axioms A1–A11 of R with R1–R2 and in particular for Church's weak theory of implication R_I, determined by A1–A4, R1. Let semantic notions be characterized as before, dropping negation and all that pertains thereto, including the * operation with its postulates; in particular, let a negation-free formula of R be valid in an *r+. m. s.* ⟨0, K, R⟩ iff it is true on every valuation therein, and let it be R+ valid iff valid in all *r+. m. s.* We depend on results from some previous papers in what follows[15].

[15] In particular, Meyer [1972a and b].

Lemma 17. *Let* A *be a negation-free formula of* R. *Then* A *is a theorem of* R *iff* A *is* R+*-valid.*

Proof. Every *r. m. s.* is clearly also an *r+. m. s.*, so if A is R+-valid it is R-valid, whence by Theorem 3, A is a theorem of R. Suppose conversely that A is a theorem of R. In Meyer [1972b] it is proved that A has a proof in a system (call it R++) which results from R+ as defined above by adding the intensional conjunction o as an additional primitive together with axiom schemes

A14. $A \rightarrow (B \rightarrow (A \circ B))$,

A15. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \circ B) \rightarrow C)$.

We take now the recursive condition ν on p. 206 as an additional primitive specification, in accordance with footnote 10. It is easily shown that A14–A15 are R+-valid; the other lemmas leading up to Theorem 2 are not affected, whence the argument of that theorem shows that all negation-free theorems of R, being theorems of R++, are R+-valid, including in particular our chosen theorem A, ending the proof of Lemma 17.

Lemma 18. *If* A *is not a theorem of Church's weak theory of implication* R_I*, and if the only connective occurring in* A *is* $\rightarrow$*, then* A *is not* R+*-valid.*

Proof. In Meyer [1972a] it is proved that under the hypothesis of the lemma, A is not a theorem of R, whence the conclusion follows by Lemma 17.

Lemma 19. *Let* A *be a negation-free formula of* R. *Then* A *is a theorem of* R *iff* A *is a theorem of* R+.

Proof. The 'if' half is trivial. Suppose then that A is a theorem of R. As noted in the proof of Lemma 17, A is then a theorem of the system R++ defined there. What remains to be proved is that R++ is a conservative extension of R+ – i.e., that the addition of o and A14–15 produces no new theorems in the connectives &, V, → of R+.

To do so, we develop now a calculus of intensional theories for R+ as we did above for R. Let an *R+-theory* be any set *T* of negation-free formulas of the sentential language SL which is closed under adjunction and R+-entailment – i.e., such that whenever A → B is derivable from A1–A11, R1–R2, if $A \in T$ then $B \in T$. Let $\mathscr{K}$+ be the set of all R+-theories, and define an operation $\circ$ on $\mathscr{K}$+ as follows, for all a, b in $\mathscr{K}$+:

$a \circ b = \{C$: C is a negation-free formula of R and there exist A in *a*, B in *b* such that $A \rightarrow (B \rightarrow C)$ is a theorem of R+$\}$.

The calculus of intensional R+-theories, in accordance with previous developments, is the structure $\mathscr{K}+ = \langle \mathscr{K}+, \subseteq, \circ, 0+\rangle$, where $\mathscr{K}+$, $\circ$ are as just defined, $\subseteq$ is set inclusion, and 0+ is the set of theorems of R+. We then show, as in Lemma 9, that $\mathscr{K}+$ as just defined is a partially ordered, square-decreasing commutative monoid, with monoid identity 0+; the strategy of the proof of Lemma 9 works; necessary changes in tactics are left to the reader.

We pass now to the sub-collection $\mathscr{K}+'$ of prime R+-theories; as it turns out, unlike before, this includes 0+, as the methods of Meyer [1972c] suffice to establish. Consider accordingly the structure $\langle 0+, \mathscr{K}+', R'\rangle$, where 0+ and $\mathscr{K}+'$ are as indicated and, for all a, b, c in $\mathscr{K}+'$, $R'abc$ iff $a \circ b \subseteq c$. As in Lemma 12, we wish to show $\mathscr{K}+'$ an *r+. m. s.* Verifying p1, p2, and p4 is again immediate; Pasch's law p3 again hinges on a proof that, for all a, b in $\mathscr{K}+$ and c' in $\mathscr{K}+'$, if $a \circ b \subseteq c'$ then, thanks to the primeness of c', there is a prime a' in $\mathscr{K}+'$ such that $a \subseteq a'$ and $a' \circ b \subseteq c'$, which, when proved, assures that Pasch's law will follow from the associativity of $\circ$ as defined on intensional theories. As the reader may verify, the argument of Lemma 12 indeed goes through, showing $\langle 0+, \mathscr{K}+', R'\rangle$ an *r+. m. s.* Accordingly all theorems of R++ are valid in $\langle 0+, \mathscr{K}+', R'\rangle$, and so we may complete the proof of the present lemma by showing all non-theorems of R+ invalid in $\langle 0+, \mathscr{K}+', R'\rangle$.

Let the *canonical valuation* $v+$ assign, for each parameter p and prime R+-theory a in $\mathscr{K}+'$, $v+(p, a) = T$ iff $p \in a$. As in Lemma 14, we wish to show that, where $I+$ is the interpretation associated with $v+$, $I+(A, a) = T$ iff $A \in a$, for all formulas A of R+ and members a of $\mathscr{K}+'$. Proof is again by induction, with all cases being obvious unless A is of the form $B \to C$; again the inductive hypothesis reduces this case to the problem of showing that $B \to C \in a$ iff, for all b, c in $\mathscr{K}+'$, $B \in b$ and $R'abc$ implies $C \in c$. But $R'abc$ means that $a \circ b \subseteq c$, which by definition means that for all formulas C, A, and B, if $A \to (B \to C)$ is a theorem of R+ and $A \in a$ and $B \in b$ then $C \in c$. Accordingly the implication to be shown is trivial from left to right, taking $B \to C$ itself as the desired instance of A. Suppose conversely that $B \to C \notin a$; we wish to show that there are b, c in $\mathscr{K}+'$ such that $R'abc$, $B \in b$, but $C \notin c$. At any rate, there are in $\mathscr{K}+$ such animals; take [B] to be the set of formulas provably R+-entailed by B, and take $c+$ to be $a \circ [B]$; then if C were in $c+$ there would be elements A in a, B′ provably entailed by B, such that $A \to (B' \to C)$ is a theorem of R+, from which it quickly follows by closure of a under provable R+-entailment that $B \to C$ belongs to a, contra hypothesis. But then extending $c+$ to a maximal and, by the usual argument, prime theory c that does not contain C, and then extending [B] to a prime theory b such that

$a \circ b \subseteq c$, courtesy of the result cited in the middle of the last paragraph, all our goals are reached – b and c are both prime, $R'abc$ holds, $B \in b$ and $C \notin c$. This completes the proof that $v+$ does what we asked it to do – it makes an arbitrary formula A true at a just in case $A \in a$. In particular, since 0+ is the set of theorems of R+, $v+$ doesn't make any non-theorems of R+ at all true at 0+, completing the proof that non-theorems of R+ remain non-theorems of R++ and hence non-theorems of R. Given the converse above, Lemma 19 is proved.

Theorem 6. *Let* A *be a negation-free formula of* R. *Then* A *is a theorem of* R+ (i.e., *derivable from negation-free axioms, excluding those for* $\circ$) *iff* A *is* R+*-valid. Moreover, if* A *is a pure implicational formula,* A *is a theorem of Church's weak theory of implication* R_I *iff* A *is* R+*-valid.*

Proof. By Lemmas 17 and 19, if A is negation-free, A is a theorem of R+ iff A is a theorem of R, iff A is R+-valid, proving the first part of the theorem. The second part follows from the theorem and Lemma 18.

The theorem, together with the related results of Meyer [1972a and b], enables us to answer affirmatively all significant questions of conservative extension for the system R.

Theorem 7. R *is well-axiomatized; for each of the following combinations of connectives, all theorems in those connectives are derivable from axiom schemes and rules among* A1–A15 *which contain only those connectives explicitly.* [$\circ$ *axioms are* A14–A15.]

$$\rightarrow;\ \rightarrow, -;\ \rightarrow, \circ;\ \rightarrow, -, \circ;\ \rightarrow, \&;\ \rightarrow, \vee, \&;\ \rightarrow, \circ, \&;\ \rightarrow, \circ, \&, \vee;\ \rightarrow, -, \&, \vee.$$

Proof. In each case either the proof has just been given, or it can be found in Meyer [1972a] or Meyer [1972b], or it is an easy consequence of the results given in one of those places.

Theorem 7 answers for R questions asked about E by Anderson [1963].

11. Intuitionism. Extending the positive semantics

We noted above that the result of enriching the postulates for an *r. m. s.* with $0 < a$ led to classical logic. But that argument led through the * postulates – i.e., through negation. A less drastic collapse occurs if we add p7′, $0 < a$, to the postulates for an *r+. m. s.*; this validates, not all classically valid negation-free formulas, but only those that are intuitionistically valid. We dignify this result as a theorem.

Theorem 8. *Let an i. m. s. be a structure* ⟨0, *K*, R⟩, *as above, satisfying* p1–p4, p7′. *Then a formula* A *is valid in all i. m. s. iff* A *is a negation-free theorem of the intuitionistic sentential calculus J. By extending the notion of an interpretation so that* $I(\neg A, a) = T$ *iff for all b such that* $a < b$, $I(A, a) = F$, *for all* $a \in K$, *we can characterize intuitionistic negation also in Kripke's style, whence all and only the theorems of J are valid in all i. m. s.*

Proof. Given Kripke [1965], one is hardly necessary; we note simply that R*abc* implies $b < c$ and $a < c$, given p7′, p3–p4, and d1–d2. Conversely, if $b < c$ and $a < c$, since R*ccc* by p2 it follows that R*abc* by p3–p4, p7′ Accordingly R can be defined in terms of <, and a simple verification shows that all the postulates require is that < shall be reflexive, transitive, and that 0 bear it to all *a* in *K*. Though Kripke doesn't require the final property of his binary accessibility relation for intuitionistic semantics, his *tree m. s.* have it, whence the Kripke relation can be identified with our < and the proof completed that just the intuitionistically valid formulas are valid in all *i. m. s.*

Theorem 8 suggests exactly the respect in which the intuitions on which the relevant logics rest go beyond those that ground the sentential part of intuitionism. In terms of our initial motivation, it turns out that what intuitionism lacks, and *a fortiori* classical logic lacks, is a plausible way of saying that two sentences A and B are *consistent.* One way of doing so is to assert A & B; another way is to assert $\neg(A \supset \neg B)$, equivalent to $\neg\neg(A \& B)$, which seems only a little better. At any rate, as the proof of Theorem 8 shows, we weaken the positive relevant intuitions to intuitionistic ones if we hold that two theories are compatible relative to a third if whenever A holds in the first and B holds in the second, A & B holds in the third. No wonder, we might add, that intuitionistic logic does not block fallacies of relevance.

Still, intuitionism provides, particularly in its negation-free part, a convenient halfway house between classical and relevant logics. (A halfway house in another sense is R-mingle; cf. Meyer [1968].) Indicative of this fact is the following simple corollary of Theorem 8 (proved in Meyer [1972b], but worth notice here).

Corollary 8.1. *All negation-free theorems of* R *are intuitionistically valid.*

Proof by Theorems 6 and 8, since all *i. m. s.* are *r+. m. s.*

Other systems intermediate between R+ and classical sentential logic – e.g., Dummett's LC, the positive part RM+ of Dunn's R-mingle – can be got by adding varied postulates.

12. Prelude to sentential quantification. A theory of propositions

Much interest has been shown – by Anderson, Belnap, Grover, and others – in equipping relevant logics with a theory of sentential quantification[16]. On account of this interest, and for the sake of philosophical completeness, we accordingly sketch in succeeding sections a couple of ways of fitting sentential quantification into our semantics.

Before entering into the formalism of sentential quantification, however, the question naturally arises, "Over what are sentential quantifiers to range?" We have two answers: (a) propositions; (b) sentences. People tend to wince when given the first answer, or at least they used to; nowdays they recover their composure when it is explained that propositions can be construed simply as *sets*, since hardly anybody objects any more to useful *extensional* fictions.

Ultimate ontological commitments aside, it is in the modern style to view a proposition as something that corresponds to a certain distribution of truth-values over worlds, and to identify propositions whose truth-values are the same in all worlds. That being the fashion, one might as well *construe* a proposition, familiarly, as a set – namely, the set of all the worlds in which the proposition is true.

Accordingly, let $\mathbf{K} = \langle 0, K, R, * \rangle$ be an *r. m. s.* Respecting the condition (1) and in accordance with motivating remarks, a *proposition in* **K** shall be any subset $J \subseteq K$ which is *closed upward* – i.e., which is such that whenever $a \in J$, $b \in K$, and, invoking d1, $a < b$, $b \in J$. Let the *algebra of propositions* $\Pi(\mathbf{K})$ *determined by* **K** *be the quintuple* $\langle \Pi, \circ, \cup, -, 1 \rangle$, characterized as follows:

(i) Π is the set of all propositions in K.

(ii) $\cup$ is the generalized operation of set-theoretical union on subsets Γ of Π.

(iii) $1 = \{a\colon a \in K \text{ and } 0 < a\}$.

(iv) For $F, G \in \Pi$,
 (a) $F \circ G = \{c\colon \exists a \exists b (a \in F \;\&\; b \in G \;\&\; Rabc)\}$
 (b) $\bar{F} = \{c\colon c \in K \;\&\; c^* \notin F\}$.

The first serious attempt to equip the relevant logics R and E – or rather their first degree parts – with a theory of propositions was made by Belnap

[16] Cf. Anderson and Belnap [1961], Meyer [1972d], Anderson [1972], and Grover's doctoral dissertation (U. of Pittsburgh, 1969).

[1967]. Belnap didn't make it very clear what he conceived propositions to *be* – he associated propositions with the logical content of sentences[17], which doesn't help a lot – but he was crashingly clear about the kind of algebraic structure which on his view characterized them; namely, propositions were structured as intensionally complemented distributive lattices with truth-filter[18].

Belnap's algebraic ideas were developed and refined with particular reference to the system R to produce the notion of a *DeMorgan monoid*, used as mentioned earlier by Dunn to provide R with proofs of algebraic completeness[19]. Moreover, on the theory of propositions developed here these ideas turn out to be essentially correct; given any *r. m. s.*, the algebra of propositions just associated therewith is in fact a DeMorgan monoid, as after recalling relevant definitions will be shown below.

Roughly, a structure $\mathbf{D} = \langle D, \circ, \vee, -, 1 \rangle$ is a DeMorgan monoid if D is a set, $1 \in D$, and $\circ$, $\vee$ are binary and $-$ is a unary operation on D such that, when $a \wedge b$ is defined by the DeMorgan law as $-(-a \vee -b)$, D is a distributive lattice under $\wedge$ and $\vee$, a commutative monoid under $\circ$ (with 1 as monoid identity), and the following definitions and postulates hold in addition:

q1. $a < b$ iff $a \vee b = b$

q2. $a \rightarrow b =_{\mathrm{df}} -(a \circ -b)$

q3. $a^2 =_{\mathrm{df}} a \circ a$

q4. $a < a^2$ (square-*increasing* postulate)

q5. $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$

q6. $a \circ (a \rightarrow b) < b$

q7. $--a = a$

For more on DeMorgan monoids see Meyer et al. [1972], and the further work of Dunn cited there.

A DeMorgan monoid is *complete* provided that it is complete as a lattice– i.e., provided that for every subset S of D, greatest lower and least upper

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17 In Belnap [1967], p. 8.

18 For the terminology and its application, cf. again Belnap [1967]. Meyer has elsewhere called these structures, "Belnap lattices". Belnap suggests, "normal DeMorgan lattice", as another alternative.

19 Most accessibly in Meyer et al. [1972]; a fuller account taken from Dunn's dissertation will appear in Anderson and Belnap [1972].

bounds $\bigwedge S$ and $\bigvee S$ exist and belong to D, and provided that the infinite distributive laws hold in the form, for all $a \in D$ and $S \subseteq D$,

q8. $a \wedge \bigvee S = \bigvee \{a \wedge s : s \in s\}$
q9. $a \circ \bigvee S = \bigvee \{a \circ s : s \in s\}$

Theorem 9. The algebra of propositions determined by an *r. m. s.* $\langle 0, K, R, {}^*\rangle$ is a complete DeMorgan monoid.
Proof. Let $\Pi(K) = \langle \Pi, \circ, \cup, -, 1\rangle$ be the algebra of propositions defined above. We note that $\circ$ and $-$, applied to members of Π, and $\cup$, applied to subsets of Π, yield closed upward subsets of K— —i.e., members again of Π. Defining, for members F and G and subsets Γ of Π,

(c) $F \vee G =_{df} \cup \{F, G\}$
(d) $\cap \Gamma =_{df} \overline{\cup} \{F : \overline{F} \in \Gamma\}$
(e) $F \wedge G =_{df} \cap \{F, G\}$

we note that all the postulates and definitions for a complete DeMorgan monoid are satisfied by $\Pi(K)$, ending the proof.

Corollary 9.1.[20] *Every prime DeMorgan Monoid is embeddable in a prime, complete DeMorgan monoid whose elements are sets, with generalized monoid meet $\cap$, join $\cup$, and order $\subseteq$ to be identified with corresponding set-theoretic intersection, union, and inclusion.*

Proof. Let $\mathbf{D} = \langle D, \circ, \vee, -, 1\rangle$ be a prime DeMorgan monoid. $F \subseteq D$ is as usual a *filter* on D provided that, for all $x, y \in D$, $x \wedge y \in F$ iff both x and y are in F; F is moreover a *prime filter* iff, for all $x, y \in D$, $x \vee y \in F$ iff at least one of x, y belongs to F. Let K be the set of all prime filters on D; 0 the principal filter determined by 1 (i.e., $0 = \{x : 1 < x \text{ in } D\}$; R, the ternary relation which holds among members a, b, c of K iff, for all x, y in D, whenever $x \in a$ and $y \in b$, $x \circ y \in c$; *, the operation on K such that, for all $x \in D$ and $a \in K$, $x \in a^*$ iff $\bar{x} \notin a$. All of this simply puts in algebraic form what was developed syntactically in sections 6 and 7, and it is accordingly easily proved by the methods of those sections that $\langle 0, K, R, {}^*\rangle$ is an *r. m. s.*

Consider now the algebra of propositions $\Pi(K)$ defined by (i)–(iv) of this section. $\Pi(K)$ is a complete DeMorgan monoid by Theorem 9, and $\cap$, $\cup$,

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[20] This corollary and the next use Stone's methods (cf. Rasiowa and Sikorski [1963]) to furnish an embedding of DeMorgan monoids in complete DeMorgan monoids. As in Meyer et al. [1972], a DeMorgan monoid is prime if whenever $1 < a \vee b$ then $1 < a$ or $1 < b$.

and $\subseteq$ are by definition intersection, union, and inclusion. Suppose that for some subset J of $\Pi(K)$, the identity $1_{\Pi(K)}$ of $\Pi(\mathbf{K})$ is included in $\cup J$. Since $1_{\Pi(K)}$ is by definition the set of supersets of 0 which belong to $\Pi(K)$, 0 in particular belongs to $1_{\Pi(K)}$, whence on assumption 0 belongs to some J_i in J. But J_i by definition is closed upward, and so not only $\{0\}$ but all of $1_{\Pi(K)} \subseteq J_i$. This proves not only that $\Pi(K)$ is prime but that it is completely so – whenever $1_{\Pi(K)}$ is a subset of a generalized join of elements of $\Pi(K)$, it is a subset of one of these elements.

Let h now be the function from D to $\Pi(K)$ which takes each element x of D into the set $h(x)$ of prime filters on D to which x belongs. By Stone's representation theorem for distributive lattices, h is an injection which preserves $\wedge$ and $\vee$; the methods of Białynicki-Birula and Rasiowa's [1957] have been adapted for our handling of –, whence an easy verification shows $-h(x) = h(-x)$; finally, to show that h preserves $\circ$, observe that $a \in h(x \circ y)$ iff $x \circ y \in a$, iff (algebraizing the argument of Lemma 12) there are prime filters b, c on D such that $x \in b$, $y \in c$, and $Rbca$, iff $a \in h(x) \circ h(y)$. This shows that h is an isomorphism from $\mathbf{D}$ into the prime, complete DeMorgan monoid $\Pi(\mathbf{K})$, ending the proof of Corollary 9.1.

Corollary 9.2. *Every DeMorgan monoid is embeddable in a complete DeMorgan monoid.*

Proof. Let D be a DeMorgan monoid, and let $\{F_i\}_{i \in I}$ be the indexed set of all the prime filters that contain the identity 1 of **D**. Define structures $\langle 0_i, K_i, R_i, *_i\rangle$ for each i in I as follows: 0_i is F_i. K_i is the set of all prime filters on D that are closed modulo F_i – i.e., such that the prime filter G belongs to K_i iff whenever $x \in G$ and $x \rightarrow y \in F_i$ then $y \in G$; note that $0_i \in K_i$; R_i is a ternary relation on K_i such that R_iabc holds iff whenever $x \in a$ and $y \in b$ then $x \circ y \in c$; $*_i$ is an operation on K_i such that $x \in a^*$ iff $-x \notin a$; note that R_i and $*_i$ are just the restrictions to K_i of R and * as defined above, and proof that for each i in I, $\langle 0_i, K_i, R_i, *_i\rangle$ is an *r. m. s.* may be had as before.

We define now a *product structure* $\times_{i \in I} \langle 0_i, K_i, R_i, *_i\rangle = \langle 0, K, R, *\rangle$ by letting K be the Cartesian product $\times_{i \in I} I_i$, letting 0 be the element of K which is 0_i on every coordinate i, and letting $Rabc$ and $a^* = b$ hold respectively iff for each i in I, $R_ia_ib_ic_i$ and $a_i^* = b_i$. It is readily verified that the product structure is an *r. m. s.*, given that each component structure is. Consider again the algebra of propositions $\Pi(K)$, which by the theorem is a complete DeMorgan monoid. Define a function h from D into $\Pi(K)$ by letting the ith component of $h(a)$ be the set of members of K_i to which a belongs.

Noting that if $a \nleq b$ in **D**, $1 \nleq a \rightarrow b$, there is by Stone's theorem a prime F_i such that $1 \in F_i$ but $a \rightarrow b \notin F_i$; considering the set of all filters on **D** that are closed modulo F_i and which contain a but not b, application of Zorn's Lemma produces a maximal one, which turns out prime and so belongs to K_i; so if $a \neq b$, on some component $h(a)$ and $h(b)$ differ – i.e., h is 1–1. It remains only to be proved that h preserves the operations of **D**, which, since they are defined pointwise, may be proved as in the previous corollary, ending the proof of Corollary 9. 2.

Since DeMorgan monoids algebraize R, our corollaries may be viewed as conveying syntactical information as well – e.g., that the system RP of R with sentential quantifiers is a conservative extension of the quantifier-free system. But we pursue these matters no further here.

Our theory of propositions turns out pretty algebraically (it would, on what was known already, have been surprising if it hadn't), but there is more to logic than developing the right form of the Stone representation theorems. How, for example, does our theory of propositions stack up against the claim that every proposition is true or false, a notable feature of the Belnap theory sketched above? In fact, what is it for a proposition to be true or false?

As might be expected, we allow for various answers. Regular prime theories, we saw when we were proving completeness, give rise to relevant model structures, and these may or may not be normal. If the underlying *r. m. s.* is normal, i.e., if $0 = 0^*$, then the algebra of propositions $\Pi(\mathbf{K})$ will be normal in the sense that, for each proposition F in $\Pi(\mathbf{K})$, we have, exclusively either $1 \subseteq F$ or $1 \subseteq \overline{F}$, where the algebraic identity 1 functions as what Cocchiarella calls the *world-proposition* – i.e., that proposition which is true at the "real" world 0 and all that contains it, and is false otherwise. So taking $1 \subseteq F$ as our standard for the *truth* of the proposition F, it turns out that normal theories, normal *r. m. s.*, and normal propositional structures all march together. (That normal theories determine normal *r. m. s.* was seen above; suppose that $\langle 0, K, R, {*}\rangle$ is normal; for proof that $\Pi(\mathbf{K})$ is normal, suppose both that $1 \subseteq F$ and $1 \subseteq \overline{F}$, for *reductio.* Then in particular, since $0 \in 1$, $0 \in F$ and $0 \in \overline{F}$, i.e., $0^* \notin F$; but $0 = 0^*$ on the assumption that $\langle 0, K, R, {*}\rangle$ is normal, a contradiction. On the other hand, suppose in general that $1 \nsubseteq F$. Then some member of 1 does not belong to F, and, since 1 is closed upward and has a least member 0, in particular $0 \notin F$. So $0^* \in \overline{F}$, by definition. But $0^* < 0$, whence, since $\overline{F}$ is closed upward, $0 \in \overline{F}$ and hence $1 \subseteq \overline{F}$.)

The last part of the proof just presented seems unfairly asymmetrical.

Whatever the *r. m. s.*, normal or not, our criterion of truth for propositions seems to allow *both* a proposition and its negation to be true but to disallow the possibility that *neither* should be true. There's something to be said for this, too, we think – inconsistency seems to *force* itself on us in a way that incompleteness doesn't (e.g., in the liar paradox) – but there's no need to insist upon it; we characterized the truth of propositions relative to the "real" world 0, and our conditions on the real world permitted inconsistency but not incompleteness; one gets all the incompleteness one might wish – even no truths at all – simply by taking our characterization of propositions as true or false relative to some other set-up. We can do this even if we insist that the real world be normal – i.e., that $0 = 0^*$. Alternatively, we can make the real world even more abnormal – allowing incompleteness – by weakening the postulate to prevent the derivation of $0^* < 0$; the total reflexivity postulate p2 – licensing *reductio* and excluded middle – seems the place to start.

At any rate, having propositions we now have something to quantify over. Let us get to it.

13. A propositional semantics for RP

Having developed a theory of propositions in the last section, in this section we consider the result of adding to R the machinery of sentential quantification. The *propositional language* PL will be a quintuple $\langle S, V, O, Q, F\rangle$, where S is as before a set of parameters, V is a denumerably infinite set of sentential *variables*, O is as before the set of connectives $\{\rightarrow, \&, \vee, -\}$, Q is the set of universal quantifiers (P), one for every variable P in V, and F is the smallest set such that $S \cup V \subseteq F$ and such that F contains A & B, A $\vee$ B, A $\rightarrow$ B, $\bar{\text{A}}$, and (P)A whenever it contains A and B, for all P in V. We continue to use 'p', 'q', etc., for sentence parameters and we shall use 'P', 'Q', etc., for sentential variables. Members of F are called *formulas* of PL; formulas in which no variables occur free are called *sentences.* (Syntactically we shall be interested only in sentences, the syntactical role which would otherwise have been assigned to free variables going instead to parameters, which may occur in sentences.) Let A be any formula of PL. A *closure* of A, as usual, will be any sentence B which results from A by prefacing zero or more universal sentential quantifiers.

Let B be any formula. A[B/P] shall be the result of substituting the formula B for each free occurrence of P in A, rewriting bound variables if necessary to avoid confusion.

We now characterize the relevant *propositional* calculus RP. (The first explicit formulation of RP occurs in Anderson [1972].) As axioms we take *all closures* of formulas of the kinds A1–A13, together with the following new axiom schemes.

A16. $(P)A \rightarrow A\,[B/P]$,
A17. $(P)(A \rightarrow B) \rightarrow ((P)A \rightarrow (P)B)$,
A18. $(P)(A \vee B) \rightarrow ((P)A \vee B)$, if P is not free in B,
A19. $B \rightarrow (P)B$, if P is not free in B,
A20. $(P)A \mathbin{\&} (P)B \rightarrow (P)(A \mathbin{\&} B)$.

The rules are, as before, *modus ponens* and adjunction.

The connectives o and $\leftrightarrow$ are introduced as before. Also defined are $\exists$ and some useful constants[21]:

D3. $\exists PA =\text{df}\ \overline{(P)\overline{A}}$,
D4. $t\ =\text{df}\ (P)(P \rightarrow P)$,
D5. $f\ =\text{df}\ \bar{t}$,
D6. $F\ =\text{df}\ (P)P$,
D7. $T\ =\text{df}\ \bar{F}$.

Let now $\langle 0, K, R, * \rangle$ be an *r. m. s.* and let $\Pi(K) = \langle \Pi, o, \cup, -, 1 \rangle$ be the corresponding algebra of propositions defined in the preceding section. An *assignment of propositions in* **K** is a function α defined on $S \cup V$ with values in Π. We adapt Leblanc's technique by characterizing an assignment α as a P-variant of α' provided that α and α' agree on $S \cup V - \{P\}$; i.e., two assignments are P-variants of each other if they differ *at most* in assigning different propositions to P.

An assignment of propositions α in the *r. m. s.* **K** determines an associated *valuation* ν and *interpretation* I in the sense of section III; i.e., for each sentential variable or parameter A in PL we shall have on the valuation ν associated with α

i. $\nu(A, \alpha) = \text{T}$ iff $a \in \alpha(A)$.

Recursive clauses ii-vi of p. 206 may be adopted as is. To handle sentential quantifiers we add, where Π is as above,

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[21] Cf. Meyer [1972d]; there are inessential differences in the formulation of RP there and here, as also with Anderson [1972].

vii. $I((P)A, a) = T$ iff for each interpretation I' determined by a P-variand α' of α, $\alpha'(P) \in \Pi$, $I'A, a) = T$.

viii. $I(\exists PA, a) = T$ iff for some interpretation I' determined by a *P-variant* α' of α, $\alpha'(P) \in \Pi$, $I'(A, a) = T$. (Since we have not take $\exists$ as primitive, note that vii with vi implies viii.)

Notions of *truth on an assignment or on the associated valuation, verification, entailment,* and ***validity*** may be adapted from section 3. In particular, a formula A is RP-*valid* iff it is true on all assignments of propositions to its sentential variables in all *r. m. s.* $\langle 0, K, R, * \rangle$.

It is to be noted that, since propositions are by definition closed upward, the restriction (1) of 3 is automatically satisfied.

14. Consistency of propositional semantics. Secondary r. m. s.

To prove RP consistent relative to its intended semantical interpretation is just a matter of repeating the arguments of sections 4 and 5, bringing all the lemmas up to date to keep pace with the enlarged vocabulary. We accordingly state immediately.

Theorem 10. *All theorems of* RP *are* RP-*valid.*

Proof. It suffices, extending Lemma 1 and Theorem 1 to the enlarged context, to show again that the axioms are valid and that the rules preserve validity. Only A16 is slightly interesting. Assume for arbitrary a in an arbitrary *r. m. s.* that $I((P)A, a) = T$; it suffices by our updated Theorem 1 to show $I(A[B/P], a) = T$ for arbitrary B. In fact, let $\beta(B)$ be the set of all b in K such that $I(B, b) = T$. By Lemma 1, $\beta(B)$ is closed upward, so $\beta(B) \in \Pi$. But since $(P)A$ is true at a on I, A is true on the interpretation I' that is like I on sentential variables except for setting $I'(P, b) = T$ iff $b \in \beta(B)$– i.e., iff $I(B, b) = T$, for all b in K. An obvious inductive argument on the length of A yields in conclusion $I(A[B/P]) = T$, which ends our proof of the theorem.

Unfortunately, though we have at the present time no proof, there is good reason to believe that the converse of Theorem 10 is false. The reason may be variously located, but the easiest thing to say is that sentential quantification is an essentially second-order matter, sentence letters being parsed as 0-ary *predicates;* so, it would seem, the way to prove completeness for RP is to adapt the techniques by which Henkin [1950] proved completeness for higher-order logics, not with reference to the intended *primary* interpretation alone but including in a class of *secondary* interpretations, in which the quantifiers range not over *all* propositions but over the propositions in some *subset* of the set of all propositions. Not every subset, as is well-known, will do.

This *cut-down* principle was actually applied by Bull [1969] to prove completeness for propositional versions of S4 and S5.[22] We apply it here by characterizing a *secondary r. m. s.* as a pair $\langle \mathbf{K}, \Gamma \rangle$, where $\mathbf{K} = \langle 0, K, R, {}^* \rangle$ is an *r. m. s.* and $\Gamma \subseteq \Pi$, where Π as above is the set of all propositions in **K**. An assignment is characterized as before, on the restriction that its values under α shall always be members of Γ; similarly, α' does not count as a *P*-variant of α unless $\alpha'(P) \in \Gamma$. Valuations, interpretations, etc., may then be characterized as before, except that the (previously vacuous) clause $\alpha'(P) \in \Pi$ in vii and viii gets changed to $\alpha'(P) \in \Gamma$. A formula is then valid in a secondary model $\langle \mathbf{K}, \Gamma \rangle$ iff it is verified on all assignments in that model.

We have at any rate

Theorem 11. *Every theorem of* RP *whose proof does not require* A16 *is valid in all secondary r. m. s.* $\langle \mathbf{K}, \Gamma \rangle$.

Proof like Theorem 10.

Our sidestepping A16 is the usual move in these cases; in fact, though A16 holds when another variable or parameter is put for the universally quantified *P*, we can find a secondary model $\langle K, \Gamma \rangle$ that will refute such a simple consequence thereof as $\exists R(R \leftrightarrow p \,\&\, q)$. (Cf. Henkin [1953] for related relevant discussion.) Since specification is also the point that forces one to put strange conditions on what counts as a model for 2nd order logic and type theory in general – essentially, we select *favored secondary r. m. s.* (i.e., the ones in which all the axioms, in particular A16, are valid). This is not *quite* as arbitrary as it looks, since what it amounts to is as usual the requirement that in favoured *r. m. s.* the propositions be closed in certain reasonable ways. I.e., the choice of $X \subseteq \Pi$, cutting down the range of the sentential quantifier, is not arbitrary, for we must make allowance not only for the propositions assigned to sentential variables but also for the complex propositions that may be built from them; in the case in point, e.g., failure came from failure to close the set of propositions under intersection, to allow for the logical operation of conjunction.

So much for a sketch of our propositional semantics; we turn now to consider the interpretation of sentential quantifiers as ranging over sentences.

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[22] The phrase, 'cut-down', is so far as we know Cocchiarella's.

15. Substitution semantics. First-order semantics

We get a much simpler technique by adopting the substitution interpretation of the sentential quantifier. Specifically, we keep as they were the semantic definitions of section 3 (*not* extending to sentential variables the valuations ν defined there), adding to allow for sentential quantification simply the clause, where B is a sentence and no variables except possible P are free in A,

ix. $I((P)\mathrm{A}, a) = \mathrm{T}$ iff for all sentences B of PL, $I(\mathrm{A}[\mathrm{B}/P], a) = \mathrm{T}$;
x. $I((\exists P\mathrm{A}, a) = \mathrm{T}$ iff got some sentence B of PL, $I(\mathrm{A}[\mathrm{B}/P], a) = \mathrm{T}$.

Again, x follows by definitions from ix and vi. Note that our substitution semantics, in contrast to the propositional semantics, defines semantic notions only on sentences, rather than on arbitrary formulas. This is, in both cases, a matter convenience decides, partly anticipated here by the ruling that made only sentences theorems. Validity, etc., are defined as before. There is, on this approach, very little to do on the side of consistency. We accordingly conclude forthwith.

Theorem 12. *All theorems of* RP *are valid on the substitution semantics.*
Proof omitted.

As was argued in Leblanc and Meyer [1969], there is certainly a great deal to be said on higher-order levels for substitution, or truth-value, semantics; indeed, the very style of the completeness proof of Henkin [1950] suggests, as Henkin himself has pointed out, a preference for linguistic rather than ontological interpretation; though it is interesting to muse, as we did above, about propositions, a typical completeness proof for calculi like RP will deal only with sentences – from the onotological viewpoint, with the *named* propositions only; for the more skeptical, with all that is there to quantify over.

The first-order version RQ of R may be handled like RP, except that in this case the peculiar second-order difficulties we've been mulling don't come up. We suppose RQ formulated as in Meyer et al. [1972]; for readers without this article at their elbow, this means with an ordinary first-order language without identity, with predicate letters G, etc., parameters b, etc., and individual variables x, etc., formulas and sentences being built up as usual from connectives and individual quantifiers. One gets an adequate axiomatization by putting 'x' everywhere for 'P' in A16–A20 and taking 'B' in A16 instead as an arbitrary term 't' to be properly substituted for 'x'. A plausible semantics is the following.

A *relevant quantificational model structure (r. q. m. s.)* is a pair ⟨**K**, D⟩, where **K** is an *r. m. s.* and D is a non-empty set. A *valuation* v in an *r. q. m. s.* is a function which assigns all variables and parameters members of D and all n-ary predicate letters G at each point a of K, n-ary relations[23] on D; v then assigns T or F to an atomic sentence $Gt_1 \ldots t_n$ at a iff $\langle v(t_1, \ldots, v(t_n)\rangle \in v(G, a)$. An interpretation I is associated with v as before, satisfying ii-vi. (The sentential constant f having been made primitive for RQ in Meyer et al. [1972], we note that $I(\mathrm{f}, a) = \mathrm{T}$ iff $0 \not< a^*$ suffices.) Defining v and v' as x-variants, again following Leblanc, iff they agree on all variables and parameters except possibly at x, we get for the quantifiers, letting I, I' be x-variants if v, v' are,

xi. $I((x)\mathrm{A}, a) = \mathrm{T}$ iff $I'(\mathrm{A}, a) = \mathrm{T}$, for all x-variants I' of I;

xii. $I(\exists x\mathrm{A}, a) = \mathrm{T}$ iff $I'(\mathrm{A}, a) = \mathrm{T}$, for some x-variant I' of I.

Defining truth, verification, etc., as before, we get

Theorem 12. *All theorems of* RQ *are valid in all r. q. m. s.*

Proof. As in the last section, extend Lemmas 1–3 and Theorem 1 and verify the axioms on an arbitrary valuation in an arbitrary *r. q. m. s.* The rules are no problem (generalization not being among them, on account of our taking only theorems as sentences), whence all theorems are true on an arbitrary v and are hence valid in all *r. q. m. s.*

Our quantificational semantics is pretty heavy-handed, and it's just possible that it validates some conspicuous non-theorem of RQ. We doubt it, however. More interesting, of course, is to build relevance in to our quantificational semantics by associating different domains with different set-ups, perhaps modifying xi and xii and the axioms of RQ in the process. For all the remarks we have made comparing R with the intuitionist calculus J, on the quantificational level RQ certainly goes beyond JQ with respect to the principle $(x(p \vee Fx) \rightarrow p \vee (x)Fx$, discussed at some length in Kripke [1965]. But in this case it's because the principle has been taken as an axiom of RQ, near enough; changing the semantics in an intuitionistically acceptable direction appears to offer no difficulties of principle; whether such a change is relevantly desirable is another question, discussed in Meyer et al. [1972].

A first-order truth-value semantics, corresponding to that of the previous section for RP, may be had along the same lines. Again, semantic consistency is no problem; we presume that completeness isn't either.

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23 (1) must continue to be respected, in the sense $a < b \Rightarrow v(G, a) \subseteq v(G, b)$.

16. Final remarks. Open problems. Trivia

Our main promise has been kept. The system R does have a Kripke semantics. So do many of its close neighbors. We have motivated that semantics from many points of view, but there are others that we have not touched. We have shown a few things that our semantics is good for; the ease with which we've got some of our results leads us to hope that other long intractable technical problems, in particular the decision question, will prove more amenable to analysis and solution. A few extra tidbits are, with some problems, perhaps in order here.

The algebraic essence of our completeness proof lay in the fact that it gives us a recipe to turn prime DeMorgan monoids into relevant model structures; one simply surveys the set K of prime filters of these monoids and uses the monoid operation to define R in a natural way; * is usually no problem. Defining an *r. m. s.*, on the other hand, can if done directly be inordinately time-consuming; the reader interested in making up a few is advised to beware in particular of the Pasch postulate p3. So the recipe of our completeness proof, boiled down most clearly in Corollary 9.1, is welcome; in particular, by turning DeMorgan monoids into *r. m. s.* we can see how problems once settled by matrix methods may be handled in our semantics.

Consider the following *r. m. s.* $\mathbf{K} = \langle 0, K, \mathrm{R}, * \rangle$.

$K = \{0, 1, 2\}$

$0^* = 0$; $1^* = 2$; $2^* = 1$.

R holds of the following triples, and fails otherwise: 000, 011, 101, 022, 202, 111, 222, 121, 122, 120, 211, 212, 210.

K was independently discovered by Urquhart in applying these semantical methods to the DeMorgan monoid[24] $\mathbf{M}_0$, which in Belnap [1960], Anderson and Belnap [1962], and elsewhere has been used to establish central semantical facts about the relevant logics; we note also that $\mathbf{K}_0$, as perhaps we should call it, is in some appropriate sense minimal, in that it is the smallest *r. m. s.* for which all of the following hold: (a) normality, in the sense $0^* = 0$; (b) $<$ is not a linear ordering (so **K** isn't a Mingle *r. m. s.*); (c) * is not the identity. Given these conditions, we note that the postulates force the ternary relation to hold ***at least*** of the listed triples.

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[24] Strictly speaking, $\mathbf{M}_0$ is in Belnap [1967] and in Dunn and Belnap [1968] a DeMorgan lattice; i.e., $\circ$ is not defined on it; but Belnap [1960] marks its original appearance, as a matrix; defining $\circ$ via $-(a \rightarrow -b)$ makes it a monoid, there.

The first use to which $\mathbf{M}_0$ was put, in Belnap [1960], was establishing the *relevance principle* – if A R-entails B, then (at the sentential level) A and B share a parameter. Show this in **K** by assuming that A and B do not share; if p occurs in A, set $\nu(p, 1) = T$; if in B, set $\nu(p, 2) = \mathrm{T}$; except as determined by these conditions, set $\nu(p, a) = \mathrm{F}$ for all a in K. Show by induction that, for every subformula A′ of A and B′ of B, $I(\mathrm{A}', 1) = \mathrm{T}$ and $I(\mathrm{B}', 1) = \mathrm{F}$, and the other way round at 2. We need consider only the inductive case; & and V are inductively trivial; if A′ is F at 2, $\overline{\mathrm{A}'}$ is T at $2^* = 1$, displaying the strategy of the ⌐ case. Finally, consider $\mathrm{A}' \rightarrow \mathrm{A}''$; first, it's surely F at 2, since R212 and on inductive hypothesis $I(\mathrm{A}', 1) = \mathrm{T}$ and $I(\mathrm{A}'', 2) = \mathrm{F}$. On the other hand, it's true at 1; for in the 5 cases in which 1 is the first argument of R – 101, 111, 121, 122, 120 – in the first 3 cases no counterexample arises because A″ is T at 1, and in the last 3 because A′ is false at 2, invoking the inductive hypothesis, so $\mathrm{A}' \rightarrow \mathrm{A}''$ is T at 1; the B case is handled symmetrically, ending the inductive argument and showing that $I(\mathrm{A}, 1) = \mathrm{T}$ and $I(\mathrm{B}, 1) = \mathrm{F}$, which suffices to show that A does not entail B on I, and hence that it does not R-entail it. Contraposing, we get the relevance principle.

The second use of $\mathbf{M}_0$ occurred in Anderson and Belnap [1962]; at the essential level, it was used to show that $A_1 \,\&\, \ldots \,\&\, A_m \rightarrow B_1 \vee \ldots \vee B_n$ is a theorem iff, where all the A_i and B_j are sentential variables or negates thereof, some A_i is identically the same as some B_j. The 'if' is trivial, so we assume as before that there is no sharing, in the present case asserting $A_i \neq B_j$ for all $i, j,$ to show the 'only if' by contraposition. If for some p, $A_i = p$, set $\nu(A_i, 1) = \mathrm{T}$; if $A_i = \bar{p}$, set $\nu(A_i, 2) = \mathrm{F}$; if $B_j = p$, set $\nu(B_j, 1) = \mathrm{F}$; if $B_j = \bar{p}$, set $\nu(B_j, 2) = \mathrm{T}$. Since there is no sharing these specifications don't conflict, and making ν arbitrary except as specified then again we have made the antecedent true and the consequent false at 1 on ν, falsifying the entailment and establishing contrapositively what was to be shown. (Reasoning of this sort is justified at greater length in Routley [1972]; our purpose here is just to show off $\mathbf{K}_0$.)

Well, so much for some of the little things that can be done; it's a lot easier, of course, once they've been done already, and taking seriously some of the propositional motivation given by Anderson and Belnap in their papers and recapitulated in Anderson and Belnap [1972] these insights don't seem essentially improved on our playing with $\mathbf{K}_0$ – certainly not at the first degree level. The superb Anderson-Belnap literary style may have something to do with it – it's easy to laugh at the jokes and to miss the profound insights. A few other things have been done which hadn't been done before;

for example, by taking Cartesian products of *r. m. s.* as in Corollary 9. 2 it's possible to prove R reasonable in the sense of Halldén – i.e., if A ⊃ B is a theorem, where ⊃ is defined truth-functionally, either one of $\bar{A}$, B is a theorem or else, at the sentential level, there is a parameter in common between A and B; this implies, by a result of Kripke, that R has a normal characteristic matrix, other conditions being fulfilled. A final tidbit is used in Meyer [1972d], so we make it explicit; as hinted, RP is a conservative extension of R. For proof, take a non-theorem A of R and find an *r. m. s.* that rejects it; assigning each parameter in A the set of worlds in which that parameter is true, we get a propositional *assignment* in the sense of 13 in the same *r. m. s.* which still falsifies A, whence by Theorem 10, A remains a non-theorem of RP, ending the proof.

Some problems which we would like to see solved, and with respect to which we hope that the present semantics will help, are the following.

1. The decision question for the relevant sentential logics.
2. Extension of these methods to higher-order logics, with non-trivial mathematical applications, and an appropriate proof of completeness. (We'd prefer the applications, if there's a choice.)
3. Deeper analysis of the relation between intutitionistic and relevant logics. To what extent, in particular, do relevance safeguards render *harmless* excursions through non-constructive arguments? (Some results on this topic appear in Meyer [1972e].)
4. Is RP decidable? (Meyer thinks he has an argument that says "No", but it's not tight.) And in general, what's RP like? Can our theory of propositions be characterized in a tighter way? What about the results of Meyer [1972d], using sentential quantification to relate familiar logics like J and Curry's D to R? Do they have easy proofs in the present semantics?
5. Necessity will be added to R in a sequel. But what about other modalities, propositional attitudes, etc.? Are these to be added in straightforward analogy to other treatments, or should they take deeper account of relevant implication?
6. Semantic tableaux. Gentzen formulations. How does the present semantics help to mechanize and to draw pictures of relevant deduction? As opposed to *natural deduction* formulations of R, these have

been hard to come by; Dunn's Gentzenization of R+ is the best result. Can it be extended to all of R?[25]

References

Anderson, A. R., 1963, Some Open Problems Concerning the System E of Entailment, *Acta Philosophica Fennica*, vol. 16, pp. 7-18.
Anderson, A. R., 1972, *An Intensional Interpretation of Truth-Values*, forthcoming.
Anderson, A. R. and Belnap Jr., N. D., 1961, Enthymemes, *The Journal of Philosophy*, vol. 58, pp. 713-23.
Anderson, A. R. and Belnap Jr., N. D., 1962, Tautological Entailments, *Philosophical Studies*, vol. 13, pp. 9-24.
Anderson, A. R. and Belnap Jr., N. D., 1972, *Entailment* (Princeton) in preparation.
Belnap Jr., N. D., 1960, Entailment and Relevance, *The Journal of Symbolic Logic*, vol. 25, pp. 144-46.
Belnap Jr., N. D., 1967, Intensional Models for First Degree Formulas, *The Journal of Symbolic Logic*, vol. 32, pp. 1-22.
Bialnicki-Birula, A. and Rasiowa, H., 1957, On the Representation of Quasi-Boolean Algebras, *Bull. Acad. Polon. Sci.*, vol. 5, pp. 259-61.
Bull, R. A., 1969, On Modal Logic with Propositional Quantifiers, *The Journal of Symbolic Logic*, vol. 34, pp. 257-263.
Church, A., 1951, The Weak Theory of Implication, *Kontrolliertes Denken*, eds. A. Menne, A. Wilhelmy, and H. Angsil, in honor of Prof. W. Britzelmayr, (Munich), pp. 22-37.
Dunn, J. M., 1970, Algebraic Completeness Results for R-Mingle and its Extensions, *The Journal of Symbolic Logic*, vol. 35, pp. 1-13.
Dunn, J. M. and Belnap Jr., N. D., 1968, Homomorphisms of Intensionally Complemented Distributive Lattices, *Mathematische Annalen*, vol. 176, pp. 23-38.
Henkin, L., 1950, Completeness in the Theory of Types, *The Journal of Symbolic Logic*, vol. 15, pp. 81-91.
Henkin, L., 1953, Banishing the Rule of Substitution for Functional Variables, *The Journal of Symbolic Logic*, vol. 18, pp. 201-208.
Kripke, S., 1963, Semantical Analysis of Modal Logic I. *Zeitschrift für mathematische Logik und Grundlagen der Mathematick*, vol. 9, pp. 67-96.
Kripke, S., 1965, Semantical Analysis of Intuitionistic Logic I, *Formal Systems and Recursive Functions*, eds. J. N. Crossley and M. A. E. Dummett (Amsterdam, North-Holland), pp. 92-129.
Leblanc, H. and Meyer, R. K., 1969, Truth-Value Semantics for the Theory of Types, *Philosophical Problems in Logic, Some Recent Developments*, ed. K. Lambert (Dordrecht), pp. 77-101.

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Meyer, R. K., 1968, Entailment and Relevant Implication, *Logique et Analyse no. 44*, pp. 472-9.
Meyer, R. K., 1968b, *A Characteristic Matrix for RM*, Mimeographed.
Meyer, R. K., 1972a, Conservative Extension in Relevant Implication, *Studia Logica*, forthcoming.
Meyer, R. K., 1972b, On Conserving Positive Logics, *Notre Dame Journal of Formal Logic*, forthcoming.
Meyer, R. K., 1972c, *Metacompleteness*, forthcoming.
Meyer, R. K., 1972d, Entailment, Intuitionism, Negation, *this volume*, pp. 168-198.
Meyer, R. K., 1972e, Negation Disarmed, abstract, *The Journal of Symbolic Logic*, forthcoming.
Meyer, R. K. and Dunn, J. M., 1969, E. R. and γ, *The Journal of Symbolic Logic*, vol. 34, pp. 460-474.
Meyer, R. K., Dunn, J. M. and Leblanc, H., 1972, Completeness of Relevant Quantification Theories, *Notre Dame Journal of Formal Logic*, forthcoming.
Rasiowa, H. and Sikorski, R., 1963, *The Mathematics of Metamathematics* (Warsaw).
Routley, R., 1972, The Semantics of First-degree Entailment, forthcoming.
Urquhart, A., 1972, Semantics for Relevant Logics, *The Journal of Symbolic Logic*, forthcoming.