ON KINEMATIC CONSTRAINT IN MICROPLANE THEORY

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ARTICLE INFO

***Article history*:**
Received 23.03.2012.
Received in revised form 25.04.2012.
Accepted 08.05.2012.

Keywords:
Microplane theory
Kinematic constraint
Concrete

Abstract:

In this paper it is shown that the microplane formulation based on the volumetric-deviatoric split (VD split) possesses the property of losing macro information during the transition from macro to micro level, i.e. during the projection of macroscopic strain components on microplanes with various orientations. However, it is also argued that the kinematic constraint principle including microplane lateral strains preserves all the information related to the macroscopic strain tensor.

1 Introduction

The microplane theory is based on the hypothesis that the influence of the macroscopic state of strain can be quantitatively represented at the microplane level. Generally, the macro-micro transition depends on the assumed kinematic constraint principle and is defined via a set of appropriated projections of the macroscopic strain tensor components (e.g. engineering strain tensor **ε**). To bear evidence of some implications provoked by the adopted kinematic constraint principle, it is opportune to introduce the *microplane kinematic operator* **P**. The operator depends on the microplane orientation, defined by an orthonormal basis **n**, **m** and **k**, and relates macro (**ε**) to micro quantity (**e**) through:

$$\mathbf{e} = \mathbf{P}\,(\mathbf{n},\mathbf{m},\mathbf{k})\,\boldsymbol{\varepsilon}. \tag{1}$$

It is worth noting that it is convenient for **P** (1) to perform the *one-to-one mapping*. The statement is to be hereafter appropriately supported and it basically means that the mapping procedure in Eq. (1) is reversible, enabling the inverse operation:

$$\boldsymbol{\varepsilon} = \mathbf{P}^{-1}(\mathbf{n},\mathbf{m},\mathbf{k})\,\mathbf{e}. \tag{2}$$

Namely, the violation of the requirement in Eq. (2) implies that the state of strain at the microplane level is not fully related to the macroscopic level since the system is actually not uniquely defined and undoubtedly possesses an infinite number of solutions. Consequently, the microscopic strain components will not depend on all macroscopic strain tensor components. In other words, if for given microplane deformations and kinematic constrain rules, the macroscopic strain tensor cannot be reconstructed, it can be said that the adopted kinematic constraint principle possesses the property of losing information and during the macro-micro transition performs some filtering of macro data. This paper sets out to test the kinematic constraint principle based on the decomposition of the normal microplane strain vector $\boldsymbol{e}_N$ into its volumetric $\boldsymbol{e}_V$ and deviatoric part $\boldsymbol{e}_D$ (split procedure), which is commonly adopted in the microplane models M2 up to M5 [1, 2, 3], and M2-O [4]. It is, therefore, tested on the preserved macroscopic information during its transition to the

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microplane level. Correspondingly, in order to evidence the usage of a more opportune and robust kinematic constraint principle, the same procedure is adopted to check the requirement in Eq. (2) by testing the "enriched" kinematic constraint principle introduced in Hasegawa-Bažant (HB) microplane model [5,6]. The conclusions drawn from both analyses are thus invariant on the microplane orientation.

2 Kinematic constraint based on VD split

The microplane model basically involves a subdivision (discretization) of a unit microsphere located around the given finite element Gauss point. The discretization process leads to a finite number of integration points [7], the locations of which define the microplanes as tangential planes on the microsphere surface (Fig. 1).

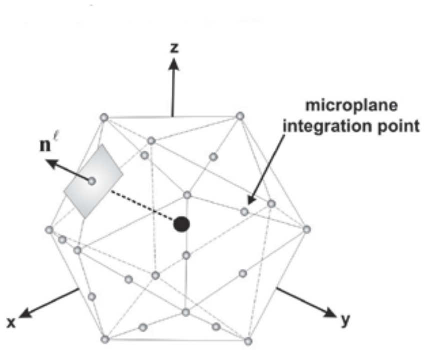

Figure 1. Discretization of a unit sphere

Congruently, each generated microplane is defined by its own local coordinate system with orthonormal basis vectors **n**, **m** and **k** (Fig. 2).

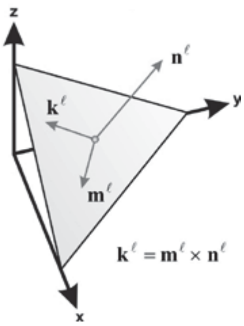

Figure 2. Microplane basis vectors

Once the macroscopic strain tensor $\boldsymbol{\varepsilon}$ has been known (typically at the finite element Gauss point), the adopted kinematic constraint principle dictates the computational procedure needed to obtain the microplane strain vectors. For this purpose, since the related procedures are equal for all microplanes, it is opportune to focus the further considerations on a single microplane. Accordingly, Fig. 3 shows a typical result obtained by the currently considered kinematic constraint principle (VD decomposition).

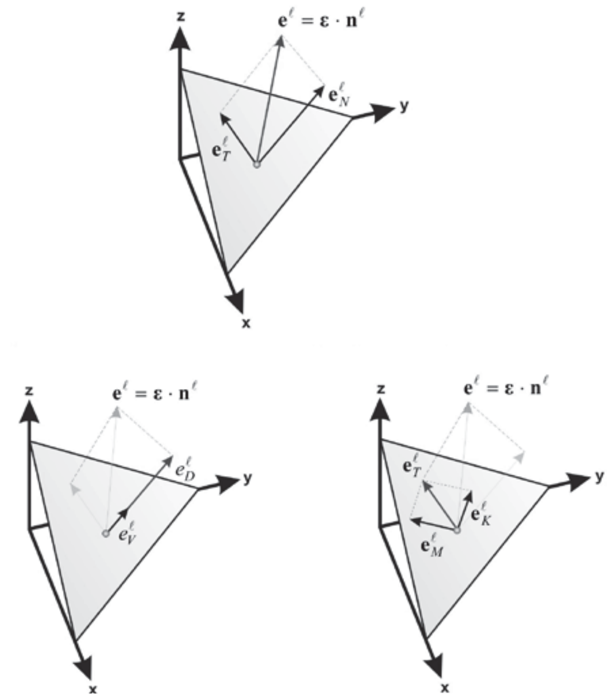

Figure 3. Microplane strain components

As seen, the resulting microplane strain vector **e** has been decomposed into its normal $\mathbf{e}_N$ and tangential part $\mathbf{e}_T$. Furthermore, due to constitutive requirements to model quasi-brittle material such as concrete [4], the normal part has been decomposed into its volumetric $\mathbf{e}_\mathrm{V}$ and deviatoric part $\mathbf{e}_D$, and the tangential component into two perpendicular strain vectors $\mathbf{e}_M$ and $\mathbf{e}_K$ associated with the coordinate axes **m** and **k**, respectively. To consider the preservation aspect of macroscopic strain information (i.e. strain tensor components) and to simplify the microplane system, the microplane with local basis vectors is assumed to be parallel with global axes of the Cartesian coordinated system. (Fig. 4).
The microplane strain components (illustrated in Fig. 3 for an arbitrary oriented microplane) are obtained through the adopted kinematic constraint principle by applying basic vector algebra.

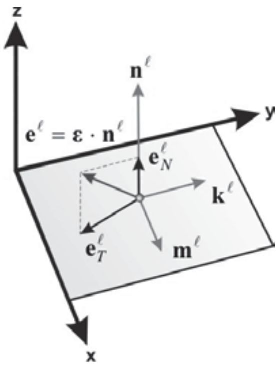

Figure 4. Results of macro-micro transition with the kinematic constraint principle based on VD decomposition

The considered case (Fig. 4) involves three skew-symmetric second order tensors $\mathbf{N}^p$, $\mathbf{M}^p$ and $\mathbf{K}^p$. These projection tensors relate the macroscopic strain components ε_{ij} to a particular microplane strain components, as shown in Eq. (3).

$$e_N = \sum_{i=1}^{3}\sum_{j=1}^{3} N_{ij}^{p}\,\varepsilon_{ij}\ ,\ N_{ij}^{p} = n_i n_j \tag{3a}$$

$$e_M = \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}^{p}\,\varepsilon_{ij}\ ,\ M_{ij}^{p} = m_i n_j \tag{3b}$$

$$e_K = \sum_{i=1}^{3}\sum_{j=1}^{3} K_{ij}^{p}\,\varepsilon_{ij}\ ,\ K_{ij}^{p} = k_i n_j \tag{3c}$$

It is worth pointing out that the components of tensors $\mathbf{N}^p$, $\mathbf{M}^p$ and $\mathbf{K}^p$ are only dependant on the components of microplane basis vectors. Particularly, for the considered microplane orientation (Fig. 4), the orthonormal basis vectors are:

$$\mathbf{n} = \begin{Bmatrix}0\\0\\1\end{Bmatrix}\quad , \quad \mathbf{m} = \begin{Bmatrix}1\\0\\0\end{Bmatrix} \tag{4}$$

$$\mathbf{k} = \begin{Bmatrix}0\\1\\0\end{Bmatrix} = \mathbf{n}\times\mathbf{m}\,.$$

To characterize property of the currently considered kinematic constraint principle, it is opportune to expand Eq. (3) by components (5).

$$\begin{aligned}&N_{11}\,\varepsilon_{11} + N_{22}\,\varepsilon_{22} + N_{33}\,\varepsilon_{33} +\\&\varepsilon_{12}(N_{1,2} + N_{2,1}) +\\&\varepsilon_{13}(N_{13} + N_{31}) +\\&\varepsilon_{23}(N_{23} + N_{32}) = e_N\end{aligned} \tag{5a}$$

$$\begin{aligned}&M_{11}\,\varepsilon_{11} + M_{22}\,\varepsilon_{22} + M_{33}\,\varepsilon_{33} +\\&\varepsilon_{12}(M_{12} + M_{21}) +\\&\varepsilon_{13}(M_{13} + M_{31}) +\\&\varepsilon_{23}(M_{23} + M_{32}) = e_M\end{aligned} \tag{5b}$$

$$\begin{aligned}&K_{11}\,\varepsilon_{11} + K_{22}\,\varepsilon_{22} + K_{33}\,\varepsilon_{33} +\\&\varepsilon_{12}(K_{12} + K_{21}) +\\&\varepsilon_{13}(K_{13} + K_{31}) +\\&\varepsilon_{23}(K_{23} + K_{32}) = e_K\end{aligned} \tag{5c}$$

Rearranging the macroscopic strain components into vector form as (Voigt notation):

$$\boldsymbol{\varepsilon} = \begin{Bmatrix}\varepsilon_{11}\\\varepsilon_{21}\\\varepsilon_{13}\\\varepsilon_{22}\\\varepsilon_{23}\\\varepsilon_{33}\end{Bmatrix}, \tag{6}$$

and the microplane strain components as:

$$\mathbf{e} = \begin{Bmatrix}e_N\\e_M\\e_K\end{Bmatrix}, \tag{7}$$

the microplane kinematic operator $\mathbf{P}$ (1) can be defined as:

$$\mathbf{P} = \begin{bmatrix}N_{11} & N_{12}+N_{21} & N_{13}+N_{31} & N_{22} & N_{23}+N_{32} & N_{33}\\M_{11} & M_{12}+M_{21} & M_{13}+M_{31} & M_{22} & M_{23}+M_{32} & M_{33}\\K_{11} & K_{12}+K_{21} & K_{13}+K_{31} & K_{22} & K_{23}+K_{32} & K_{33}\end{bmatrix}. \tag{8}$$

Resuming, **P** relates macroscopic strain components to microplane strains via:

$$\mathbf{P}\,\boldsymbol{\varepsilon} = \mathbf{e}\,. \tag{9}$$

For the assumed microplane (4), **P** has the following structure (10):

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{10}$$

Now, after performing the matrix multiplication in Eq. (9), it follows that the involved components of **ε** are only those with at least one index equal to 3 (i.e. *z* direction). The other marked components in Eq. (11) are not involved in the kinematic procedure.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{33} \\ \varepsilon_{31} \\ \varepsilon_{23} \end{Bmatrix} = \begin{Bmatrix} e_N \\ e_M \\ e_K \end{Bmatrix}$$

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \boxed{\begin{matrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{matrix}} & \begin{matrix} \varepsilon_{13} \\ \varepsilon_{23} \end{matrix} \\ \begin{matrix} \varepsilon_{31} & \varepsilon_{32} \end{matrix} & \varepsilon_{33} \end{bmatrix} \tag{11}$$

However, the currently adopted kinematic constraint principle will also involve the macroscopic strain components ε_{11} and ε_{22}. Namely, these components are necessary for the computation of the microplane volumetric strain e_V, defined as [1,2,4]:

$$e_V = \frac{\varepsilon_{kk}}{3}. \tag{12}$$

Nevertheless, it is evident that these macro-micro and vice-versa transitions are not influenced by the microplane orientation. This is the reason why Eq. (12), i.e. a possible additional equation in the system of Eq. (9), is not present in the microplane kinematic operator **P** (8). Since for the considered kinematic constraint principle all the macro-micro relations have been exhausted, we can conclude that the macroscopic strain component ε_{12} (and obviously ε_{21}) will not affect the microplane state of strain.

To single out a possible side effect (in the proceeding of the microplane constitutive description), two microplanes must be considered. Clearly, the microplanes are on different microspheres and have the same spatial orientations. To simplify the consideration, for both microplanes, the local basis vectors are assumed to be coaxial with the global basis vectors. To point out the implications of Eq. (11), two macroscopic strain tensors (one for each microsphere) are presented with equal strain components except for the shear component ε_{12}. Under these circumstances and considering these microplanes the given kinematic procedure reflects different macroscopic states of strain in equal microplane strain components. In other words, both microplanes will be further traded as if they were immersed in the same macroscopic strain environment, which they are apparently not. This is caused because the shear component ε_{12} is not included in the macro-micro transition (11). On the other hand, it is easy to deduce that even if the microplane has some arbitrary direction, i.e. if the coordinate system of the microplanes rotates, the property of losing macro information will still be present and the macroscopic state of strain will not be fully (uniquely) reflected on the microplane level. The statement can be supported by the fact that three microplane strain components (7) are directly related to 6 macroscopic strain components (6). Taking into account the micro-macro transition, the system of Eq. (11) has infinitely many solutions. As a consequence, the macroscopic state of strain cannot be reconstructed from the microplane state of strain, which arises from the fact that the kinematic constraint has not preserved all macro information.

3 Kinematic constraint based on HB split

Apart from the three microplane strain components (*on-plane components*) in Eq. (7), the HB model [5,6] introduces an additional kinematic constraint into macroscopic strain components. The resulting additional microplane strains (*in-plane components*) are the so-called lateral strains (Fig. 4). Indeed, the lateral strains should not be viewed as strains on predefined edges obtained by cutting the microplane (as shown in Fig. 4). On the contrary, in accordance with the on-plane strain components (7), the lateral strains are associated at the integration point, i.e. acting at that point on the unit microsphere.

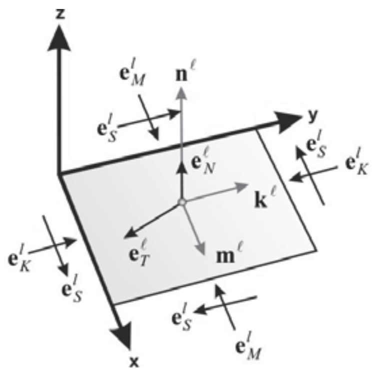

Figure 5. Results of macro-micro transition with the kinematic constraint principle in the HB model

Apart from the on-plane strain components (7), the same as those presented in the previously considered kinematic constraint principle (Fig. 4), the additional lateral strains are analogously obtained through tensor projections on imaginary lateral planes (Fig. 5). Since these surrounding planes are well defined through permutation of the basis vectors **m** and **k**, the lateral strain components $\mathbf{e}_M{}^l$, $\mathbf{e}_K{}^l$ and $\mathbf{e}_S{}^l$ are related to macroscopic strain components ε_{ij} with second order projection tensors $\mathbf{M}^l$, $\mathbf{K}^l$ and $\mathbf{S}^l$, respectively (13).

$$e_M^l = \sum_{i=1}^{3}\sum_{j=1}^{3} M_{ij}^l \, \varepsilon_{ij} \quad , \quad M_{ij}^l = m_i \, m_j \tag{13a}$$

$$e_K^l = \sum_{i=1}^{3}\sum_{j=1}^{3} K_{ij}^l \, \varepsilon_{ij} \quad , \quad K_{ij}^l = k_i \, k_j \tag{13b}$$

$$e_S^l = \sum_{i=1}^{3}\sum_{j=1}^{3} S_{ij}^l \, \varepsilon_{ij} \quad , \quad S_{ij}^l = m_i \, k_j \tag{13c}$$

The lateral strains are introduced to replicate the influence of confinement on the microplane normal response. In this case, the volumetric compression is individually traded for each microplane, which was not the case in the previously considered kinematic principle (12). Note also that, due to the equilibrium requirement, the microplane value of the lateral component $\mathbf{e}_S{}^l$ is equal on each lateral planes. Indeed, the directions of these strains are not important since these are needed only to compute the principal lateral strains. Namely, to make the microplane response affected by surrounding volumetric strains, the principal lateral strain in the HB model is used as an indicator of confinement and is calculated as [5]:

$$e_L^{max} = \frac{e_K^l + e_M^l}{2} + \sqrt{\left(\frac{e_K^l - e_M^l}{2}\right)^2 + \left(e_S^l\right)^2} \, , \tag{14a}$$

$$e_L^{min} = \frac{e_K^l + e_M^l}{2} - \sqrt{\left(\frac{e_K^l - e_M^l}{2}\right)^2 + \left(e_S^l\right)^2} \, . \tag{14b}$$

To check if the macro-micro kinematic relations presented in the HB microplane model meet the requirement imposed by Eq. (2), it is opportune to expand Eq. (1) in components. So, by appending Eq. (13) to Eq. (3), it follows that:

$$\begin{aligned} & N_{11}\,\varepsilon_{11} + N_{22}\,\varepsilon_{22} + N_{33}\,\varepsilon_{33} + \\ & \varepsilon_{12}(N_{12} + N_{21}) + \\ & \varepsilon_{13}(N_{13} + N_{31}) + \\ & \varepsilon_{23}(N_{23} + N_{32}) = e_N \, , \end{aligned} \tag{15a}$$

$$\begin{aligned} & M_{11}\,\varepsilon_{11} + M_{22}\,\varepsilon_{22} + M_{33}\,\varepsilon_{33} + \\ & \varepsilon_{12}(M_{12} + M_{21}) + \\ & \varepsilon_{13}(M_{13} + M_{31}) + \\ & \varepsilon_{23}(M_{23} + M_{32}) + M_{33}\,\varepsilon_{33} = e_M \, , \end{aligned} \tag{15b}$$

$$\begin{aligned} & K_{11}\,\varepsilon_{11} + K_{22}\,\varepsilon_{22} + K_{33}\,\varepsilon_{33} + \\ & \varepsilon_{12}(K_{12} + K_{21}) + \\ & \varepsilon_{13}(K_{13} + K_{31}) + \\ & \varepsilon_{23}(K_{23} + K_{32}) = e_K \, , \end{aligned} \tag{15c}$$

$$\begin{aligned} & M_{11}^l\,\varepsilon_{11} + M_{22}^l\,\varepsilon_{22} + M_{33}^l\,\varepsilon_{33} + \\ & \varepsilon_{12}(M_{12}^l + M_{21}^l) + \\ & \varepsilon_{13}(M_{13}^l + M_{31}^l) + \\ & \varepsilon_{23}(M_{23}^l + M_{32}^l) = e_M^l \, , \end{aligned} \tag{15d}$$

$$K_{11}^l\,\varepsilon_{11}+K_{22}^l\,\varepsilon_{22}+K_{33}^l\,\varepsilon_{33}+ \\ \varepsilon_{12}(K_{12}^l+K_{21}^l)+ \\ \varepsilon_{13}(K_{13}^l+K_{31}^l)+ \\ \varepsilon_{23}(K_{23}^l+K_{32}^l)=e_K^l\,, \tag{15e}$$

$$S_{11}^l\,\varepsilon_{11}+S_{22}^l\,\varepsilon_{22}+S_{33}^l\,\varepsilon_{33}+ \\ \varepsilon_{12}(S_{12}^l+S_{21}^l)+ \\ \varepsilon_{13}(S_{13}^l+S_{31}^l)+ \\ \varepsilon_{23}(S_{23}^l+S_{32}^l)=e_S^l\,. \tag{15f}$$

The microplane kinematic operator **P** (now a 6x6 matrix) can be written as:

$$\mathbf{P}= \begin{bmatrix} N_{11} & N_{12}+N_{21} & N_{13}+N_{31} & N_{22} & N_{23}+N_{32} & N_{33} \\ M_{11} & M_{12}+M_{21} & M_{13}+M_{31} & M_{22} & M_{23}+M_{32} & M_{33} \\ K_{11} & K_{12}+K_{21} & K_{13}+K_{31} & K_{22} & K_{23}+K_{32} & K_{33} \\ M_{11}^l & M_{12}^l+M_{21}^l & M_{13}^l+M_{31}^l & M_{22}^l & M_{23}^l+M_{32}^l & M_{33}^l \\ K_{11}^l & K_{12}^l+K_{21}^l & K_{13}^l+K_{31}^l & K_{22}^l & K_{23}^l+K_{32}^l & K_{33}^l \\ S_{11}^l & S_{12}^l+S_{21}^l & S_{13}^l+S_{31}^l & S_{22}^l & S_{23}^l+S_{32}^l & S_{33}^l \end{bmatrix}, \tag{16}$$

and operates on a supplemented microplane strain vector **e** given by:

$$\mathbf{e}=\begin{Bmatrix} e_N \\ e_M \\ e_K \\ e_M^l \\ e_K^l \\ e_S^l \end{Bmatrix}. \tag{17}$$

To simplify further considerations, the target microplane is assumed again to have a local basis parallel with the axes of the global coordinate system (4). In this case, the components of **P** become:

$$\mathbf{P}=\begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \end{bmatrix}, \tag{18}$$

and accordingly with Eq. (1), the macro-micro transition can be written as:

$$\begin{bmatrix} 0&0&0&0&0&1 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 1&0&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&1&0&0&0&0 \end{bmatrix} \cdot \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{21} \\ \varepsilon_{31} \\ \varepsilon_{22} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{33} \\ \varepsilon_{31} \\ \varepsilon_{23} \\ \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{21} \end{Bmatrix} = \begin{Bmatrix} e_N \\ e_M \\ e_K \\ e_M^l \\ e_K^l \\ e_S^l \end{Bmatrix}. \tag{19}$$

Contrary to the previously considered kinematic constraint principle (11), note that in the current case there are no macroscopic strain components that are to be avoided during the macro-micro transition. Indeed, all the macroscopic strain components are affected by at least one component of **P**. Also, the discussion about eventual rotation of the microplane local coordinate system is redundant because the considered microplane orientation (4) maximizes the number of zeros in **P** while still ensuring that all the components of **ε** are involved in the transition.

Finally, in order to meet the requirement imposed by Eq. (2), i.e. to ensure property preservation of macro information, the determinant of **P** must always be different from zero. Since the matrix rows are multiplications of permutated components of orthogonal basis vectors, they cannot happen to be equal or proportional. This condition is sufficient to ensure that the matrix determinant will always be different from zero. As a consequence, the inverse operation in Eq. (2) is valid, so that we can conclude that the kinematic constraint principle in the HB microplane model preserves all the macroscopic strain data during the skipping form macro to micro.

4 Conclusion

In this paper it is shown that the kinematic constraint principle based on the VD split causes the loss of information of macroscopic strain components while skipping from macro to micro level. At this moment, it is hard to define a true proportion in which the property of losing information affects the macro-mechanical response. However, the possible consequence of losing macro information during the transition to micro scales should be known. It can be concluded that once the macroscopic strain tensor has been uniquely reflected on the microplane level (without losing any information), another, more involving and complex question, arises as how to correctly manipulate the microplane strain components for the purpose of reaching the real macro-mechanical response of quasi-brittle materials such as concrete. Taken as a hole, this paper states that the kinematic constraint principle defined in the HB model should be used for this purpose.

Acknowledgement

I would like to thank the National Foundation for Science, Higher Education and Technological Development of the Republic of Croatia for the financial support during my research activities at the Technological Institute at Northwestern University (USA). Also, I would like to express my gratitude to Prof. Dr. Zdeněk P. Bažant for giving me the opportunity to conduct this research at Northwestern University.

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