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Descartes' Analytic Method and the Art of Geometric Imagineering in Negotiation and Mediation

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DESCARTES' ANALYTIC METHOD AND THE ART OF GEOMETRIC IMAGINEERING IN NEGOTIATION AND MEDIATION

JOHN W. COOLEY*

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I. INTRODUCTION

It may come as a surprise to some readers that René Descartes, widely regarded as the “father of modern philosophy”¹ and the discoverer of the foundations of analytic or co-ordinate geometry,² was from a family of lawyers and judges. Descartes’ father and his elder brother were magistrates at the High Court of Brittany at Rennes, France, and Descartes himself held a doctoral degree and a license in law, conferred at Poitiers in 1616.³ Although Descartes shunned a career in law and instead chose a life path in mathematics and philosophy,⁴ his intellectual achievements as a *problem solver* in *mathematics* are, paradoxically, more relevant to *law* practitioners now—nearly four hundred years after he received his law degree—than they ever have been in the past. With the advent of alternative dispute resolution and its principal problem-solving processes of collaborative negotiation⁵ and mediation,⁶ Descartes’ lifetime quest to invent a universal method for solving problems assumes an aura

1. Nicholas Jolley, *The Reception of Descartes' Philosophy*, in THE CAMBRIDGE COMPANION TO DESCARTES 393, 416-17 (John Cottingham ed., 1992) [hereinafter CAMBRIDGE COMPANION].

2. JOHN W. COTTINGHAM, THE RATIONALISTS, 4 A HISTORY OF WESTERN PHILOSOPHY 37 (1988) [hereinafter COTTINGHAM, RATIONALISTS].

3. See JACK R. VROOMAN, RENE DESCARTES: A BIOGRAPHY 22-23 (1970); BERNARD WILLIAMS, DESCARTES: THE PROJECT OF PURE ENQUIRY 15 (1978); COTTINGHAM, RATIONALISTS, *supra* note 2, at 12. Descartes’ doctoral dissertation in law has recently been discovered and, as of 1988, was being edited. COTTINGHAM, RATIONALISTS, *supra* note 2, at 189 n.24.

4. VROOMAN, *supra* note 3, at 41.

5. For purposes of this article, “collaborative negotiation” is defined as group decisionmaking in which emotional issues are not present (non-existent or already dissipated) and positional bargaining (right versus wrong; rights and duties) is not present or has been concluded and the parties are focusing on their interests (or needs) and resources to satisfy those interests with the goal of achieving a mutually acceptable solution. This type of negotiation can occur in a dispute or transaction setting.

6. For purposes of this article, mediation is defined as facilitated collaborative negotiation. “Collaborative negotiation” is defined in *supra* note 5.

of special significance for practicing lawyers. This article will explore Descartes' twenty-one *Rules for the Direction of the Mind* and their application in the mediation and collaborative negotiation settings (referred to collectively as "mediational problem solving"). More specifically, Part II of this article will examine Descartes—the person and the problem solver—focusing in the latter topic on Descartes' mental processes as mediator between geometry and algebra that resulted in the unification of those problem-solving methods. Part III will describe Descartes' *Rules* as a paradigm of problem-solving methods, and, through the help of the writings of the late George Polya, one of the most highly regarded teachers of the analytic method of the twentieth century,⁷ demonstrate the application of Descartes' *Rules* in solving real-life problems, both in conflict and transactional settings. Finally, Part IV will propose a problem-solving paradigm, which, while using Descartes' *Rules* as a cognitive foundation, goes beyond the specific contours of his method by suggesting a graphic, representational approach as an aid to resolving conflict and to concluding transactions. This paradigm is a visual-analytical geometry of collaborative negotiation, which I call geometric imagineering.

II. DESCARTES—THE PERSON AND THE PROBLEM SOLVER

A. Descartes—The Person

René Descartes was born on March 31, 1596, in a small town in France between Tours and Poitiers, now called la-Haye-Descartes.⁸ He was a sickly child, and ill health plagued him throughout his life. His mother died shortly after his birth, and he was raised almost exclusively by his maternal grandmother, his faithful nurse, and his sister.⁹ Being the son of a lawyer and magistrate in Brittany and a member of one of the oldest and most respected families in the region, he was brought up amid all the amenities of nobility and upper-class life.¹⁰ At a very early age he exhibited extraordinary curiosity,

7. George Polya, whose life spanned close to 100 years (1887-1985), was a Professor of Mathematics at Stanford University for many years. One commentator referred to Polya as "a brilliant teacher, a teacher's teacher," who believed that "the ability to discover and the ability to invent can be enhanced by skillful teaching which alerts the student to the principles of discovery" and which gives him or her the opportunity to practice these principles. PHILIP J. DAVIS & REUBEN HERSH, *THE MATHEMATICAL EXPERIENCE* 285 (1981).

8. COTTINGHAM, *RATIONALISTS*, *supra* note 2, at 11; WILLIAMS, *supra* note 3, at 15. Descartes was born into and lived during one of the great intellectual periods of civilization. Pierre Fermat and Blaise Pascal were his contemporaries in mathematics; Gilbert, the founder of the science of electromagnetism, died when Descartes was seven; Milton was born when Descartes was twelve; Shakespeare died when Descartes was twenty; Galileo died when Descartes was 46, and Newton and Leibniz were eight and four years old respectively when Descartes died in 1650. E.T. BELL, *MEN OF MATHEMATICS* 36, 117 (1937).

9. COTTINGHAM, *RATIONALISTS*, *supra* note 2, at 11; VROOMAN, *supra* note 3, at 22.

10. T.Z. LAVINE, *FROM SOCRATES TO SARTRE: THE PHILOSOPHIC QUEST* 86 (1984).

which caused his father to refer to him on occasion as “the little philosopher.”¹¹ Between the ages of ten and eighteen, he attended a newly founded Jesuit College of La Flèche, considered at the time to be one of the finest educational institutions in Europe.¹² His father’s selection of the Jesuit institution is considered to be “the most important act in the relationship between father and son.”¹³

At La Flèche, Descartes was introduced to classic literature and traditional classics-based subjects, such as history and rhetoric. He later took courses in mathematics, moral philosophy, and theology, as well as “natural philosophy,” as physical science was then known. Although he held several of his teachers in high esteem, he considered the philosophy and the science that he learned there, “despite being cultivated by many centuries by the best minds, [as] contain[ing] no point that was not disputed and hence doubtful.”¹⁴ After leaving La Flèche in 1614, he embarked on his study of law, which he completed in 1616.¹⁵

Shortly afterwards, at the age of twenty-one, he left France and set out on a series of travels throughout Europe. On his first stop, in Holland, he met the philosopher and mathematician Isaac Beeckman, to whom Descartes dedicated his first essay in 1618, the *Compendium Musicae* or “Summary of Music.”¹⁶ That same year he joined the army of Prince Maurice of Nassau as an unpaid volunteer and travelled to Germany where at age twenty-three he had a remarkable experience in the town of Ulm that set the future course of his life.¹⁷

11. VROOMAN, *supra* note 3, at 23.

12. COTTINGHAM, RATIONALISTS, *supra* note 2, at 11; VROOMAN, *supra* note 3, at 23; LAVINE, *supra* note 10, at 86; John Cottingham, *Introduction* to CAMBRIDGE COMPANION, *supra* note 1, at 3 [hereinafter Cottingham, *Introduction*].

13. VROOMAN, *supra* note 3, at 23.

14. Cottingham, *Introduction*, *supra* note 12, at 3. See also COTTINGHAM, RATIONALISTS, *supra* note 2, at 11; VROOMAN, *supra* note 3, at 41; LAVINE, *supra* note 10, at 86.

15. COTTINGHAM, RATIONALISTS, *supra* note 2, at 12.

16. *Id.*; Geneviève Rodis-Lewis, *Descartes' Life and the Development of His Philosophy*, in CAMBRIDGE COMPANION, *supra* note 1, at 26. The “Summary of Music,” written in Latin, dealt with the mathematical ratios involved in harmony. Descartes always had a “strong inclination for the arts,” and he came to love oratory and poetry—such gifts as come from “inspiration rather than a set of rules.” *Id.*

17. COTTINGHAM, RATIONALISTS, *supra* note 2, at 12; LAVINE, *supra* note 10, at 86-87. At Ulm, on November 10, 1619, he remained for a whole day shut up in a stove-heated room “where he was completely free to converse with himself about his own thoughts” and where he had a vision in a dream of a marvelous science, which evolved into his quest to discover a universal method for solving problems. COTTINGHAM, RATIONALISTS, *supra* note 2, at 12-13; LAVINE, *supra* note 10, at 86-87. At that time he took a vow that he would devote the rest of his life to developing his new science. LAVINE, *supra* note 10, at 86-87. Hector-Pierre Chanut, Descartes’ friend and correspondent, described this period in Descartes’ life in a final epitaph: the young man “‘on his

In the early 1620s, Descartes continued to travel in Europe, but in 1625, he returned to Paris and lived there for three years. During that time, he authored his first major work, the *Rules for the Direction of the Mind*, which, though never completed or published during his lifetime, nonetheless served as a statement of his early views on knowledge and the philosophical method and inaugurated the development of an entire scientific system.¹⁸ During this same period he also completed much of his work on the *Geometry*, eventually published in 1637, which laid the foundations for analytical geometry.¹⁹ In the late 1620s, Descartes began working on two essays, the *Optics* and the *Meteorology*, which applied geometrical techniques to the solution of problems in the real world. In essence, these essays purported to demonstrate how a variety of seemingly diverse phenomena could be explained through reference to a few principles of great simplicity and generality.²⁰

In 1629, Descartes moved to Holland and eventually decided to make it his permanent home.²¹ During the years which followed, he sought a life of tranquility and solitude in the Dutch countryside, content in avoiding public controversy of any kind.²² Although he had few close personal friends, he maintained a wide circle of correspondents with whom he shared detailed commentary and analysis covering virtually every aspect of his philosophical system.²³ He never married, choosing instead to devote his life to advance knowledge in accordance with his vision at Ulm.²⁴

By the year 1632, Descartes was completing his treatise *Le Monde* ("The World" or "The Universe"), an ambitious undertaking the goal of which was to provide a comprehensive account of the whole of physics, applying the same

way to the army/ amid the calm of winter/ combining nature's mysteries with the laws of *mathesis*/, dared to hope/, with one single key, to unlock the secrets of both." Rodis-Lewis, *supra* note 16, at 30-31.

18. COTTINGHAM, RATIONALISTS, *supra* note 2, at 13. See generally 1 THE PHILOSOPHICAL WRITINGS OF DESCARTES (John W. Cottingham et al. trans., 1985) [hereinafter PHILOSOPHICAL WRITINGS].

19. COTTINGHAM, RATIONALISTS, *supra* note 2, at 13.

20. *Id.*

21. Even in Holland, Descartes was never actually able to "settle down" in one place. He changed his place of residence more than a dozen times in as many years. COTTINGHAM, RATIONALISTS, *supra* note 2, at 13.

22. Descartes chose not to be a professor at a university because universities were so censored by the Catholic Church that they were stagnant and, indeed, hostile toward the supporters of the new science that he promoted. LAVINE, *supra* note 10, at 87.

23. 3 THE PHILOSOPHICAL WRITINGS OF DESCARTES: THE CORRESPONDENCE at vii (John W. Cottingham et al. trans., 1991).

24. LAVINE, *supra* note 10, at 87. He did have a liaison, however, with his serving woman, which resulted in the birth of a daughter who died tragically at the age of five. COTTINGHAM, RATIONALISTS, *supra* note 2, at 13-14.

general principles to the explanation of both terrestrial and celestial phenomena.²⁵ Five years later, Descartes prepared for the publication of the *Optics*, the *Meteorology*, and the *Geometry*, to illustrate the richness of his new scientific method, and an extended introduction referred to commonly as the *Discourse on Method*.²⁶ Written in French instead of Latin, and designed to reach an audience beyond the narrow scope of the academic world, these publications allowed Descartes to bring his ideas regarding a universal method for solving problems (which he termed “*mathesis universalis*”) to the general masses.²⁷

It was not until 1641 that Descartes published a more carefully crafted presentation of the metaphysical foundations of his philosophy, entitled the *Meditations on First Philosophy*.²⁸ Containing six separate meditations, and commonly viewed as describing the “Cartesian” method, the *Meditations* consisted of a set of mental exercises that follows a path from preconceived opinion, to doubt, to awareness, and finally to knowledge of the nature and

25. In that same year, Galileo published in Florence, Italy, his *Dialogue on the Two Chief Systems of the Universe*. This work, like Descartes', took a unificatory view of the cosmos, rejecting Aristotle's view that the terrestrial and celestial worlds were entirely different in kind. It also forcefully defended the Copernican opinion (to which Descartes similarly subscribed in *Le Monde*) that the earth rotates daily and revolves annually around the sun. When, in 1633, Galileo's work was formally condemned by the Inquisition, Descartes immediately withdrew his own *Le Monde* from publication, preferring not to antagonize the Catholic Church and desiring to live in peace. COTTINGHAM, RATIONALISTS, *supra* note 2, at 14; LAVINE, *supra* note 10, at 88. See generally Rodis-Lewis, *supra* note 16, at 35-39.

26. Rodis-Lewis, *supra* note 16, at 39. The formal title of the *Discourse* was “Discourse on the Method of rightly conducting one's reason and seeking the truth in the sciences,” and it took two or three months to write. The volume containing the three essays and *Discourse* was published anonymously in Leiden in 1637. *Id.*; COTTINGHAM, RATIONALISTS, *supra* note 2, at 14. The three examples of his method were considered to be quite successful illustrations in that each one provided at least one new fundamental result: in the *Optics*, the sine law of refraction; in the *Meteorology*, the calculation and experimental confirmation of the angles of the bows of the rainbow; and in the *Geometry*, the solution of the early Greek mathematician Pappas' locus problem for four or more lines. See Stephen Gaukroger, *The Nature of Abstract Reasoning: Philosophical Aspects of Descartes' Work in Algebra*, in CAMBRIDGE COMPANION, *supra* note 1, at 91. The solution of Pappas' locus problem is discussed in more detail in part II.B.2. *infra*, in text.

27. COTTINGHAM, RATIONALISTS, *supra* note 2, at 15. The term *mathesis* is derived from the Greek verb *manthanein*, to learn, and corresponds to the Latin *disciplina*, derived from the equivalent Latin verb for “to learn,” *discere*. Descartes contended that his universal discipline or general science would provide the key to a wide range of apparently distinct real-world investigations. *Id.* at 37.

28. COTTINGHAM, RATIONALISTS, *supra* note 2, at 15. Although the six separate parts of the *Meditations* comprised a relatively short work, the published volume was quite large, having been increased by the addition of six sets of “Objections” written by various scholars, philosophers, and theologians, together with Descartes' Replies. *Id.* at 16.

existence of the physical world and its relation to the mind.²⁹ In the Second Meditation Descartes wrote his famous maxim, *Cogito, ergo sum* ("I think, therefore I am"), that he considered the rock of certainty—the self-evident inner "light of reason"—upon which to base his philosophy and to construct a system of knowledge.³⁰

By the mid-1640s, publication of the *Discourse* and the *Meditations* had accomplished just what Descartes had always feared. These works catapulted him into the public eye and sculpted him into an international figure. His arguments favoring doubt that appeared at the beginning of the *Meditations* had made him a target of both the envy and the hostility of many theological scholars.³¹ The storm of protest became so strong that, in 1643, Descartes felt compelled to publish an open letter in self-defense. Despite these difficulties, Descartes endeavored to have his philosophical system accepted and taught in universities. In 1644, he published a comprehensive academic textbook, the *Principles of Philosophy*.³² Eventually, in 1649, Descartes released an extensive study of human life, pertaining to what he referred to as the "substantial union" of body and mind, entitled *The Passions of the Soul*. In the *Passions*, Descartes viewed the emotions and feelings arising from the intermingling of mind and body as constituting one of the principal ingredients of the good life, being responsible for some of the richest and most vivid experiences that humans can enjoy.³³

Descartes had completed the *Passions* shortly before his ill-fated visit to Stockholm at the invitation of Queen Christina of Sweden. She had read his *Principles of Philosophy* and had expressed an interest in his ideas. When Descartes arrived at the Queen's court in September, she was preoccupied with other activities, and she employed Descartes for approximately four months to perform a series of unusual tasks, such as writing verses for a ballet to celebrate

29. COTTINGHAM, RATIONALISTS, *supra* note 2, at 16. See generally Jean-Luc Marion, *Cartesian Metaphysics and the Role of the Simple Natures*, & Louis E. Loeb, *The Cartesian Circle*, in CAMBRIDGE COMPANION, *supra* note 1, at 115 & 200, respectively.

30. Cottingham, *Introduction*, *supra* note 12, at 7-8. See generally Peter Markie, *The Cogito and Its Importance*, in CAMBRIDGE COMPANION, *supra* note 1, at 140.

31. See Jolley, *supra* note 1, at 393.

32. This work contained a detailed introduction to Descartes' metaphysics (part I), a full account of the principles of his physics (part II), his theory of the structure of the universe and the solar system (part III), and an explanation of the origins of the earth and of a wide variety of terrestrial phenomena such as tides, earthquakes, and magnetism (part IV). COTTINGHAM, RATIONALISTS, *supra* note 2, at 16. See generally Daniel Garber, *Descartes' Physics*, in CAMBRIDGE COMPANION, *supra* note 1, at 286.

33. COTTINGHAM, RATIONALISTS, *supra* note 2, at 17; Cottingham, *Introduction*, *supra* note 12, at 16. See generally Gary Hatfield, *Descartes' Physiology and Its Relation to His Psychology*, in CAMBRIDGE COMPANION, *supra* note 1, at 335; Amélie O. Rorty, *Descartes on Thinking with the Body*, in CAMBRIDGE COMPANION, *supra* note 1, at 335.

her birthday. In January, the Queen determined that she was ready to begin her course of instruction in philosophy. She commanded Descartes to conduct the instructional sessions at five o'clock in the morning. The strain of early morning risings, combined with the cold weather and his life-long precarious health condition, caused him to contract pneumonia.³⁴

Ironically, Descartes' surrendering of his treasured solitude caused his demise. Descartes died on February 11, 1650, about a month short of his fifty-fourth birthday. Only a few months earlier he had written that "[w]ithout my solitude, I cannot without great difficulty make any progress in that search wherein consists my chief good in this life, the search for truth."³⁵ His search remained uncompleted.

B. Descartes—The Problem Solver

The genius of Descartes as a problem solver is best demonstrated in the *Geometry*.³⁶ In that work he performs the function of the grand mediator between the two problem solving processes of geometry and algebra, unifying them in such a way as to produce a new mathematical system, analytic geometry—a task closely analogous to that of merging two dispute resolution systems.³⁷

In order to appreciate the magnitude of Descartes' intellectual achievement, one must first have some familiarization with the state of mathematical knowledge (data base) at the time Descartes was performing his unifying work. With this background, the process used by Descartes in reconciling and synthesizing geometry and algebra will then be examined.

1. Descartes' Data Base

From early Greek times a clear distinction had developed between

34. COTTINGHAM, RATIONALISTS, *supra* note 2, at 17.

35. *Id.*

36. See *supra* note 19 and accompanying text.

37. The purpose of this section of this article is *not* to make mathematicians out of readers who have no such interest, nor to test the mathematical prowess of those who are mathematically inclined. Rather, its purpose is to examine the general techniques and thought processes employed by Descartes as a problem solver, as background for our further inquiry, *infra*, into the utility of analytic method for solving problems in collaborative negotiation and mediation contexts. Every effort has been made in this section, and indeed in the remainder of this article, to minimize the pure mathematics aspects of the discussion and to maximize those aspects of analytic method that have direct or indirect application to solving real-life problems presented in transactions or in conflicts. Most everyone having a basic understanding of high school level mathematics should have no trouble understanding the mathematical concepts presented here.

arithmetic and geometry. In early Greek mathematics, geometry was seen to operate with lines, and arithmetic with line *lengths* (or areas or volumes). Even Aristotle noted that a line length was determinate, in that it was potentially divisible into discontinuous parts—that is, a determinate plurality of unit lengths. The line considered simply as a line was indeterminate and thus a continuous magnitude, infinitely divisible. The line therefore became the subject matter of geometry, and line *length*, effectively perceived as a number, became the subject matter of arithmetic. Although he did not explicitly refer to it as such, Aristotle's conception of arithmetic was metrical geometry, an arithmetical discipline, common to the whole of ancient mathematics from the old-Babylonian period to the Alexandrians.³⁸

The principal goal of Greek mathematicians was to discover the inherent properties of various geometric figures or numbers as definite collections of line units,³⁹ and the utility of their mathematics was limited by its very nature. In early Greek mathematics only three spatial dimensions existed. The product of two line lengths was conceived as a plane area, and the product of three line lengths, a solid. Beyond that, the number of available dimensions was exhausted.⁴⁰ As the seventeenth century neared, a need arose to explain curves (particularly the curves of conic sections—the ellipse, parabola, and hyperbola—depicted in Figure 1) beyond the scope of the arithmetic and geometry of the Greeks.⁴¹

The parabola, the path taken by projectiles, was studied in ballistics. Astronomers needed to know more about the elliptical, parabolic, and hyperbolic paths traveled by the planets and comets. Optics concentrated upon knowledge of conic sections required for the construction of lenses and mirrors. In the last decade of the sixteenth century, an important step was made toward overcoming the three-dimensional limitations of the Greek mathematical system through the

38. Stephen Gaukroger, *The Nature of Abstract Reasoning: Philosophical Aspects of Descartes' Work in Algebra*, in *CAMBRIDGE COMPANION*, *supra* note 1, at 101.

39. Michael S. Mahoney, *The Beginnings of Algebraic Thought in the Seventeenth Century*, in *DESCARTES: PHILOSOPHY, MATHEMATICS AND PHYSICS* 141, 142 (Stephen Gaukroger ed., 1980).

40. *Id.* at 142-43. Euclid restricted his study of geometry to two postulates: (1) that a straight line can be drawn between any two points; and (2) that a circle can be drawn with any given point as center to pass through another given point. Later he added a third postulate, namely, that a given cone could be cut by a given plane producing a conic section. Little was known, however, about the curves (ellipse, hyperbola, and parabola) formed by planes passing through a cone. Gaukroger, *supra* note 38, at 92, 102.

41. Figure 1 is reprinted from *BEYOND NUMERACY* by John Allen Paulos (at 200). Copyright © 1991 by John Allen Paulos. Reprinted by permission of Alfred A. Knopf, Inc.

development of algebra.⁴²

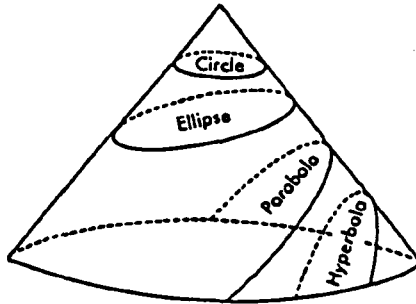


Figure 1

In 1591, mathematician François Viète, in his *Introduction to the Analytic Art* (as algebra was then known), introduced a new algebraic symbolism. He suggested that setting up mathematical equations:

be helped by some art, . . . [so] that the given magnitudes be distinguished from the uncertain ones being sought by a . . . convention, such as by designating the magnitudes being sought by the letter A or some other vowel, E, I, O, U, Y; and the given magnitudes by the letters B, G, D, or other consonants.⁴³

Although not appearing radical by today's standards, this suggestion figuratively shook the foundations of contemporary mathematics. In the decades to follow, the contours of what came to be known as the "algebraic mode of thought" would be further delineated through the efforts of Descartes and others. That mode of thought came to have, even in Descartes' time, three principal characteristics: (1) the use of an operative symbolism that not only consists of abbreviated words, but also of a number of combinatory operations; (2) the combinatory operations pertain to mathematical *relations* rather than to objects;⁴⁴ and (3) an abstract, rather than an intuitive or physical, world basis, depending upon consistent definition within a given axiom system, and

42. Mahoney, *supra* note 39, at 143. Algebra obtained its name from a book entitled *Al-jabr wa'l Muqabalah*, written by Al-Khowarizmi, one of the preeminent mathematicians of an early era of Arabic learning. From his name, the word "algorithm" was derived. Paulos, *supra* note 41, at 7.

43. François Viète, *quoted in* Mahoney, *supra* note 39, at 143.

44. The subject of modern algebra is the *structures* defined by mathematical relations and rests more on a logic of relations than on a logic of predicates. Mahoney, *supra* note 39, at 142.

consisting of “mutually compatible mathematical structures . . . living in peaceful co-existence with mathematics as a whole.”⁴⁵ In this milieu of vibrant challenges to traditional mathematical thought, Descartes published his treatise, the *Geometry*.

2. Descartes' Problem Solving Process

Descartes' treatise, the *Geometry*, had a revolutionary effect upon the development of mathematics.⁴⁶ Considered by some to be mistitled (since it was *in pari materia* a treatise on algebra rather than geometry),⁴⁷ the *Geometry* consisted of three books. The first book dealt with “problems that can be constructed using only circles and straight lines” (the types of problems with which Euclid had concerned himself); the second book discussed the “nature of curves”; and the third book described the construction of “solid and supersolid problems.”⁴⁸

Descartes began the first book of the *Geometry* by making a direct comparison between arithmetic and geometry. Noting that arithmetic consisted of only four or five operations, namely, addition, subtraction, multiplication, division, and extraction of roots (a kind of division), he demonstrated, using actual drawings, how these arithmetic operations could be used together with geometry to find line lengths. Next, he pointed out that actually drawing the lines on paper was unnecessary if the lines were designated, in the abstract, by letters. Descartes wrote:

If, then, we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction,—to those that are unknown as well as to those that are known. Then, making no distinction between known and unknown lines, we must unravel the difficulty in any way that shows most naturally the relations between these lines, until we find it possible to express a single quantity in two ways. This will constitute an equation, since the terms of one of these two expressions are together equal to the terms of the other.

We must find as many such equations as there are supposed unknown lines If there are several equations, we must use each in order, either considering it alone or comparing it with the others, so as to obtain a value for each of the unknown lines; and so we must

45. *Id.*

46. Gaukroger, *supra* note 38, at 92-93.

47. Mahoney, *supra* note 39, at 145.

48. Gaukroger, *supra* note 38, at 93.

combine them until there remains a single unknown line which is equal to some known line, or whose square, cube, fourth power, fifth power, sixth power, etc., is equal to the sum or difference of two or more quantities, one of which is known, while the others consist of mean proportionals between unity and this square, or cube, or fourth power, etc., multiplied by other known lines. I may express this as follows:

$$\begin{aligned} z &= b \\ \text{or } z^2 &= -az + b^2 \\ \text{or } z^3 &= az^2 + b^2z - c^3 \\ \text{or } z^4 &= az^2 - c^3z + d^4, \text{ etc.} \end{aligned}$$

That is, z , which I take for the unknown quantity, is equal to b ; or, the square of z is equal to the square of b diminished by a multiplied by z Thus, all the unknown quantities can be expressed in terms of a single quantity, whenever the problem can be constructed by means of circles and straight lines, or by conic sections, or even by some other curve of degree not greater than the third or fourth.⁴⁹

Prior to Descartes' innovation described here, algebraic equations in two unknowns were traditionally considered to be indeterminate since the two unknowns could not be determined from such an equation. The only available technique was to substitute arbitrarily chosen values for x and then solve the equation for y for each of these values. Descartes' innovation permitted this procedure to be transformed into a general solution. As one commentator has explained:

What he effectively does is to take x as the abscissa of a point and the corresponding y as its ordinate, and then one can vary the unknown x so that to every value of x there corresponds a value of y which can be computed from the equation. We thereby end up with a set of points that form a completely determined curve satisfying the equation.⁵⁰

The power of Descartes' new analytic method was perhaps best demonstrated in the latter part of the first book of the *Geometry*, where he used the method to solve Pappus' locus problem, one that early Greek theorists of

49. René Descartes, *Geometry*, in 31 GREAT BOOKS OF THE WESTERN WORLD 296-97 (Robert M. Hutchins ed., 1952).

50. Gaukroger, *supra* note 38, at 95.

geometry could formulate but could neither generalize nor solve.⁵¹ Pappas proposed the problem in terms of a three- or four-line locus problem. As a four-line problem, four lines and their positions are given, and the task is to determine the locus of points from which four lines can be drawn to the given lines, such that the product of the length of two of the lines bears a constant proportion to the product of the other two. The early Greek geometers knew that the locus was a conic section passing through the intersections of the lines, but they failed to determine a general procedure for solving the problem.

Descartes' procedure for solving this geometry problem used algebra, allowing him to express relations between the lines with the help of only two variables. He showed how the problem could be solved algebraically (and visually) in a way that could be generalizable to n (any number of) lines through algebraic abstraction. In essence, he demonstrated that, through algebra and the reliability of its combinatory operations, all geometrical problems could be reduced to one in which, for resolution, all that needs to be known is the length of certain lines. The lengths of the lines can be designated by placement of points on the coordinate axes (horizontal or abscissae or x axis, and vertical or ordinate or y axis), traditionally referred to as "Cartesian coordinates." He determined that for three or four fixed lines, the solution to Pappus' problem can be expressed as a quadratic equation (for all known values of y , the values of x can be determined and thus the required curve satisfying the locus of points can be determined); for five or six lines, the solution is a cubic equation; for seven or eight lines, a quartic equation, for nine or ten lines, a quintic equation, and so on, increasing one degree with the introduction of every two lines. The procedure Descartes used to solve the Pappus' locus problem was simple and elegant.

In the second book of the *Geometry*, Descartes continued his analysis of the Pappus problem by distinguishing the curves corresponding to the equations of the second degree (namely the ellipse, hyperbola, and parabola). Even though he made no direct contribution to the development of calculus, his method of drawing a tangent to curves was later seen to be the equivalent of finding the slope of a curve at any point, a type of differentiation later cultivated by Leibniz and Newton to bring calculus into full bloom. In the third book of the *Geometry*, Descartes made an important, and then considered even radical, advance beyond the Greek mathematicians by allowing negative roots and imaginary roots in his structural analysis of equations relating to solid and

51. The early Greeks defined a locus to be "the position of a line or a surface producing one and the same property." For Descartes, a locus was a collection of an infinite number of points, all of which satisfied the equation of a curve. DAVID R. LACHTERMAN, *THE ETHICS OF GEOMETRY* 144, 146 (1989).

supersolid problems.⁵²

John Stuart Mill once said that “[a]nalytical geometry . . . immortalized the name of Descartes . . . and constitutes the greatest single step ever made in the progress of the exact sciences.”⁵³ Indeed, Descartes’ combining of algebra and geometry in the *Geometry*, as one commentator has observed, foreshadowed the quite distinct problem-configurations of the later-developed algebraic geometry and the infinitesimal (differential) calculus, and provides “an especially instructive example of the way in which heterogeneous fields are unified in a complex way by strategies for sharing methods, techniques and ways of introducing items and framing problems in the service of problem solving.”⁵⁴ To better understand how Descartes’ analytic strategies for sharing methods and ways of framing problems can be applied in the service of solving problems arising in transactions and disputes, his analytic method as embodied in his *Rules for the Direction of the Mind* must be carefully examined.

52. Gaukroger, *supra* note 38, at 95-98.

53. LACHTERMAN, *supra* note 51, at 141. In fairness, it must be pointed out that Pierre Fermat, a contemporary of Descartes and a person with whom Descartes corresponded, also made significant contributions to the development of analytic geometry. Fermat, known as the “prince of amateurs,” was a King’s councillor (a kind of magistrate) by trade who was an untrained, yet brilliant, mathematician by nature. He was keenly interested in pure mathematics, more as a hobby or avocation, and his greatest work was the foundation of the theory of numbers that guaranteed him immortality in the field of mathematics. BELL, *supra* note 8, at 56-59. Currently, interest in the ingenuity of Fermat has been rekindled in the mathematics community. In about 1637, Fermat posited a theorem (called Fermat’s Last Theorem) that held that although a square can be broken into two smaller squares—e.g., 25 (the square of five) can be broken into 16 (the square of four) plus 9 (the square of three)—a cube cannot be divided into two smaller cubes, nor can any higher power be divided into two smaller numbers of the same power. In his writings, Fermat stated that he had found a truly marvelous demonstration of the theorem that was too expansive to be written in the margin of a page. This proposition, although seemingly simple in nature, is so complex and mathematically profound that the French Academy of Sciences, in 1815—nearly two hundred years after Fermat’s discovery—offered a gold medal and 300 francs for its solution. In 1908, Professor Paul Wolfskehl (German) left 100,000 marks (now worth 7,500 marks) to the person who could provide a complete proof of Fermat’s Last Theorem.

In recent decades countless problem solvers have submitted would-be proofs of the Theorem. None were successful. In June of 1993, Professor Andrew Wiles delivered three lectures on “Modular Forms, Elliptic Curves, and Galois Representations,” at the end of which he contended that his presentation had proved Fermat’s elusive mathematical puzzle. He accompanied his presentation by reasoning contained in 200 pages of written material. Professor Wiles’ written proofs are presently under review by a handful of prestigious mathematicians who are among the very few who are even capable of judging whether his reasoning is valid. *Id.* at 70-72. Charles Krauthammer, *An Age-Old Puzzle Solved Modestly*, CHI. TRIB., July 5, 1993, at A14.

54. Emily Grosholz, *Descartes’ Unification of Algebra and Geometry*, in DESCARTES: PHILOSOPHY, MATHEMATICS AND PHYSICS, *supra* note 39, at 156, 164.

III. THE PARADIGM OF ANALYTIC METHOD IN PROBLEM SOLVING

A. *Descartes' Rules for the Direction of the Mind*

When Descartes set out to write his *Rules for the Direction of the Mind*, his goal was to draft thirty-six rules to be contained in three separate books. The first book (Rules One to Twelve) was to deal with what he called the "simple natures" of problems; the second book (Rules Thirteen to Twenty-four) was to deal with natures *deduced* from the natures that are most simple and self-evident; the third book (Rules Twenty-five to Thirty-six), in turn was to deal with "those natures that presuppose others which experience shows us to be composite in reality."⁵⁵ As actually drafted, Rules One to Eleven provided general guidance for problem solvers, and those that followed described this analytic method in more detail. Descartes provided explanations, some quite lengthy, of Rules One through Eighteen. Rules Nineteen through Twenty-one were written in heading form, but no explanation of them was provided. Historians believe that the absence of explanations indicate that Descartes was distracted by his other work and never completed the project.⁵⁶

The following discussion examines Rules One through Eleven, and provides guidance for negotiators and mediators in problem solving. The following section will explore the analytic method under the rubric of Rules Twelve through Twenty-one as amplified by examples in mathematical problem solving developed by the late Professor George Polya.⁵⁷

B. *Rules 1 to 11: General Guidance for Negotiators and Mediators*⁵⁸

Rule 1: Your goal in problem solving should be to direct your mind with a view to forming true and sound judgments.

In explanation of this rule, Descartes notes that people who see similarity between two things have the habit of ascribing to one what they find true of the other, even when the two in that respect are dissimilar. From this premise he finds a basis to draw what he believes to be clear distinctions between art and

55. René Descartes, as quoted from Rule Eight in *PHILOSOPHICAL WRITINGS*, *supra* note 18, at 32. See also *id.* at 7.

56. Cottingham, *Introduction*, *supra* note 12, at 5-7.

57. I emphasize again that when reading this material, it is unnecessary to concentrate on the computational aspects of mathematical problem solving. Rather, it is much more important to concentrate on those aspects of the method (thinking techniques) that can serve as actual or metaphorical tools for solving "real life" problems in negotiation and mediation.

58. The rules appearing in the windows are simplified, paraphrased statements of Descartes' actual language. The verbatim language of all 21 rules appears in the appendix to this article.

science. He does this to make the point that a properly "scientific" approach to finding truth can be compromised inappropriately by artistic interference.⁵⁹ He sees the separate sciences to be interconnected, and together as constituting human wisdom. For him, "what makes us stray from the correct way of seeking the truth is chiefly our ignoring the general end of universal wisdom and directing our studies towards some particular ends."⁶⁰ Descartes' wrote:

It must be acknowledged that all the sciences are so closely interconnected that it is much easier to learn them all together than to separate one from the other. If, therefore, someone seriously wishes to investigate the truth of things, he ought not to select one science in particular, . . . [h]e should, rather, consider simply how to increase the natural light of his reason, not with a view to solving this or that scholastic problem, but in order that his intellect should show his will what decision it ought to make in each of life's contingencies. He will soon be surprised to find that he has made far greater progress than those who devote themselves to particular studies, and that he has achieved not only everything that the specialists aim at but also goals far beyond any they can hope to reach.⁶¹

Whether or not you subscribe to Descartes' controversial distinction between art and science,⁶² Rule One suggests a very useful problem-solving mindset for mediators and negotiators. In essence, as the key to achieving sound solutions, Descartes advocates a study of the problem-solving *process* as opposed to the particular *substance* of particular problems. The *process* should be applicable to the entire universe of problems confronting the problem-solver, regardless of the topic. The mediator or negotiator should be conditioned to see the whole context of the problem, to spot connections and patterns among problems, and to avoid concentrating unnecessarily on details.⁶³

Rule 2: Attend only to those objects that your mind seems capable of having certain and indubitable cognition.

In the explanation of Rule Two, Descartes defines knowledge as "certain

59. M. GLOUBERMAN, DESCARTES: THE PROBABLE AND THE CERTAIN 50-56 (1986).

60. PHILOSOPHICAL WRITINGS, *supra* note 18, at 9-10.

61. *Id.* at 10.

62. For criticism of this distinction, see GLOUBERMAN, *supra* note 59, at 51-56.

63. In some ways, Descartes' Rule One finds a reflection in Edward de Bono's discussion of "attention areas" in his book *Lateral Thinking*. EDWARD DE BONO, LATERAL THINKING: CREATIVITY STEP BY STEP 181-85 (1970). See also JOHN W. COOLEY, THE APPELLATE ADVOCACY MANUAL 196-98 (Lawyer's Edition 1989) [hereinafter COOLEY, APPELLATE ADVOCACY].

and evident cognition."⁶⁴ At the same time, he recognizes that knowledge, so defined, is difficult to obtain. He notes that whenever two persons make opposite judgments about the same thing, the only thing certain is that at least one of them is mistaken, and it is unlikely that one of them has knowledge. It is likely that neither has knowledge because if the reasoning of one of the two were compelling, the other should be convinced of the certainty of the asserted conclusion. He further contends that it is better never to study at all than to study objects so difficult that it is impossible to distinguish what is true from what is false, because in such case, we are forced to accept the doubtful as certain. These were his words:

That is just what many people do: they ingeniously construct the most subtle conjectures and plausible arguments on difficult questions, but after all their efforts they come to realize too late, that rather than acquiring any knowledge, they have merely increased the number of their doubts.⁶⁵

He further posits that the two ways of arriving at a knowledge of things is through experience and through deduction. According to Descartes, errors in finding truth rarely arise from faulty reasoning; rather, such errors occur because problem solvers take for granted certain poorly understood observations or rely on rash or groundless judgments.⁶⁶ Many people, he points out, feel free to make more confident guesses about matters which are obscure than about matters which are clear. "It is much easier," Descartes explains, "to hazard some conjecture on this or that question than to arrive at the exact truth about one particular question, however straightforward it may be."⁶⁷ From these and other premises, he argues that of all the sciences devised as of that time, only arithmetic and geometry are free of any taint of falsity or uncertainty. In Descartes' view, these two disciplines "are concerned with an object so pure and simple that they make no assumptions that experience might render uncertain."⁶⁸ He concludes his explanation of Rule Two by conceding that arithmetic and geometry are not the only sciences worth studying, but that in seeking the right path to a solution, the problem solver should concern himself or herself "only with objects which admit of as much certainty as the demonstrations of arithmetic and geometry."⁶⁹

64. PHILOSOPHICAL WRITINGS, *supra* note 18, at 10.

65. *Id.* at 12.

66. This viewpoint describes the differences between the Seventeenth Century philosophies of the rationalists (Descartes, Spinoza, and Leibniz) and the empiricists (Locke, Hume, and Berkeley). See COTTINGHAM, RATIONALISTS, *supra* note 2, at 11-30.

67. PHILOSOPHICAL WRITINGS, *supra* note 18, at 12.

68. *Id.*

69. *Id.* at 13.

The message of Rule Two for mediators and negotiators is that when solving a problem, they should suspend judgment about any perceived set of facts and to challenge, at least mentally, any and all assumptions about what is evidence, what is appropriate inference, what is fact, and what is an appropriate conclusion.⁷⁰ The parties' perceptions, derived through their experiences or observation of an event, are not *necessarily* fact regardless of how honest their beliefs are that their perceptions are accurate. It may be an illusion, much in the way a person's perception is "tricked" by an optical illusion.⁷¹ A wise strategy for the mediator or negotiator is "to assume certain information as true" for the purpose of proposing a solution or an avenue for solution rather than to come to any firm conclusion in his or her own mind as to the actual truth or certainty of the information. Mediators and negotiators should keep the problem solving *process* pure, simple, and untainted by the substance of the problem, or by conclusions concerning that substance. They should have faith in the inevitability of the process and let the appropriately applied and gently guided process solve the problem. In many situations, the solution will find itself.

Rule 3: Investigate objects that you can clearly and evidently intuit or deduce with certainty, without regard to the views of others or your own conjecture.

In explaining this rule, Descartes recommends that problem solvers read the writings of the ancients, but cautions against taking what they say at face value. He is quite cynical of their methods and motives. For example, when speaking of the ancients, he observes:

[O]nce writers have . . . heedlessly taken up a position on some controversial question, they are generally inclined to employ the most subtle arguments in an attempt to get us to adopt their point of view. On the other hand, whenever they have the luck to discover something certain and evident, they always present it wrapped up in various obscurities, either because they fear that the simplicity of their argument may depreciate the importance of their finding, or because they begrudge us the plain truth.⁷²

Next he observes that problem solvers are not aided by merely memorizing other people's demonstrations of proof by heart. Instead, the goal should be to

70. See DE BONO, *supra* note 63, at 91-103. See also COOLEY, *APPELLATE ADVOCACY*, *supra* note 63, at 148-51.

71. See John W. Cooley, *Mediation and Joke Design: Resolving the Incongruities*, 1992 J. DISP. RESOL. 287-93 [hereinafter Cooley, *Joke Design*].

72. PHILOSOPHICAL WRITINGS, *supra* note 18, at 13.

develop the intellectual aptitude to solve any given problem. He then reviews what he considers the only two "actions of the intellect" by which we are able to arrive at certain knowledge: intuition and deduction. He defines intuition as "the indubitable conception of a clear and attentive mind which proceeds solely from the light of reason."⁷³ As examples of intuition, he notes that everyone can intuit that he or she exists, that he or she is thinking, that a triangle is bounded by just three lines, and a sphere by a single surface. But he quickly adds that apart from apprehending single propositions such as these, intuition assists in apprehending a train of reasoning. As an example he offers the equation 2 plus 2 equals 3 plus 1. Descartes observes that "not only must we intuitively perceive that 2 plus 2 make 4, and that 3 plus 1 make 4, but also that the original proposition follows necessarily from the other two."⁷⁴ He defines deduction as "the inference of something as following necessarily from some other propositions which are known with certainty."⁷⁵ In distinguishing between intuition and deduction, Descartes offers this analogy:

[M]any facts . . . are known with certainty, provided they are inferred from true and known principles through a continuous and uninterrupted movement of thought in which each individual proposition is clearly intuited. This is similar to the way in which we know that the last link in a long chain is connected to the first[.] [E]ven if we cannot take in at one glance all the intermediate links . . . , we can have knowledge of the connection . . . [if] we survey the links one after the other, and keep in mind that each link from first to last is attached to its neighbor. . . . [W]e are aware of . . . a sort of a sequence in . . . [deduction] . . . ; immediate self-evidence is not required for deduction, as it is for intuition.⁷⁶

Finally, Descartes posits that first principles (basic truths or premises) are known only through intuition, and that remote conclusions are known only through deduction.

The advice offered by Descartes in explanation of Rule Three is critical to mediators and negotiators. In essence, he communicates that although the knowledge gained by reading the writings of experts may be of some benefit to problem solvers, it cannot substitute for developing one's own mental ability in problem solving. It is not sufficient, Descartes implies, for problem solvers merely to copy or mimic another problem solver's successful technique in solving a specific problem, or to have a "toolbox" of techniques from which to

73. *Id.* at 14.

74. *Id.* at 15.

75. *Id.*

76. *Id.*

select and to use, mechanically, for trouble-shooting purposes. These things are helpful, but not sufficient. The effective problem solver must first understand how his or her mind works, must recognize the difference between intuition and deduction, and must know when and how to use each. The effective problem solver needs to be comfortable thinking about thinking and applying the appropriate thinking process or processes to the problem presented.

Rule 4: You need a method if you are going to investigate the truth of things.

Descartes urges problem solvers not to leave investigation methods for solutions to chance. He believes that "it is far better never to contemplate investigating the truth about any matter than to do so without method."⁷⁷ He defines method to be "reliable rules which are easy to apply, and such that if one follows them exactly, one will never take what is false to be true or fruitlessly expend one's mental efforts"⁷⁸ Noting that ancient geometers employed an analysis that they applied to the solution of every problem of geometry, he proposes a new analytic method, a universal or general science, that goes beyond the parameters of geometry, arithmetic, algebra, or of mathematics itself. He calls this new method *mathesis universalis*:

I came to see that the exclusive concern of mathematics is with questions of order or measure and that it is irrelevant whether the measure in question involves numbers, shapes, stars, sounds, or any other object whatever. This made me realize that there must be a general science which explains all the points that can be raised concerning order and measure irrespective of the subject-matter, and that this science should be termed *mathesis universalis*

. . . .

This discipline should contain the primary rudiments of human reason and extend to the discovery of truths in any field whatever. Frankly speaking, I am convinced that it is a more powerful instrument of knowledge than any other with which human beings are endowed, as it is the source of all the rest.⁷⁹

77. PHILOSOPHICAL WRITINGS, *supra* note 18, at 16.

78. *Id.*

79. *Id.* at 19, 17. Descartes further describes the details of his analytic method in Rules 12-21, and these are discussed, more thoroughly, in part III.C., *infra*.

Rule Four is instructive for mediators and negotiators because it suggests that mental capacity or thinking ability is not, in itself, sufficient in problem solving, but rather the problem solver must employ a *method* to guide and direct the thinking function.⁸⁰

Rule 5: The method consists in ordering and arranging objects on which you must concentrate your mind's eye in order to discover some truth.

Rule 6: Attend to what is most simple in each series of things in which you have directly deduced some truths and observe how all the remaining things are more or less removed from or equal to the simplest.

Rule 7: Survey every single thing relating to your undertaking in a continuous and wholly uninterrupted sweep of thought and include them in a sufficient and well-ordered enumeration.

Of the eighteen rules for which Descartes provides an explanation, he provides the shortest discussion with respect to Rule Five. This is perhaps because—as he later explains with regard to Rule Seven—Rules Five, Six, and Seven should be read together.⁸¹ Despite the brevity of his related explanation, Descartes states categorically that Rule Five “covers the most essential points in the whole of human endeavor.”⁸² He concedes that the order that is required by Rule Five is often obscure and complicated and that to avoid going astray, the problem solver must carefully observe the message of Rule Six.

In explaining Rule Six, Descartes unabashedly states that it contains the “main secret” of his method and that no more useful Rule exists in his whole treatise.⁸³ In further explanation of Rule Six, he points out that it is quite important not to inspect the isolated natures of things, “but to compare them with each other so that some may be known on the basis of the others.”⁸⁴ Anything useful for comparison in a particular problem-solving project may be termed either “absolute” or “relative.” Descartes defines “absolute” as “whatever is viewed as being independent, a cause, simple, universal, single,

80. A discussion of how Descartes analytic method can be employed in dispute and transactional situations appears in part III.C., *infra*.

81. PHILOSOPHICAL WRITINGS, *supra* note 18, at 27.

82. *Id.* at 20.

83. *Id.* at 21.

84. *Id.*

equal, similar, straight, and other qualities of that sort.”⁸⁵ He calls this “the simplest and the easiest thing when we can make use of it in solving problems.”⁸⁶

To Descartes, “relative” means that which “shares the same nature, or at least something of the same nature, in virtue of which we can relate it to the absolute and deduce it from the absolute in a definite series of steps.”⁸⁷ Within the concept of “relative” occurs other items, which he calls “relations,” consisting of things said to be dependent, an effect, composite, particular, many, unequal, dissimilar, oblique, and so on. In his conception, the farther such relative attributes stray from the absolute, the more mutually dependent relations of the sort itemized that they contain. According to Descartes, the point of Rule Six is the following:

[A]ll these relations should be distinguished, and the interconnections between them, and their natural order, should be noted, so that given the last term we should be able to reach the one that is absolute in the highest degree, by passing through all the intermediate ones.

The secret of this technique consists entirely in our attentively noting in all things that which is absolute in the highest degree. For some things are more absolute than others from one point of view, yet more relative from a different point of view.⁸⁸

With respect to implementing Rule Six, Descartes gives some final cautionary advice not to commence problem solving by investigating difficult matters. He suggests that before tackling a specific problem, the problem solver should first survey a random selection of truths which are at hand, and from them attempt to deduce some other truths, step by step. When this is completed, the problem solver should reflect attentively on the truths so discovered and consider why it was possible to discover some of these truths sooner and more easily than others. This procedure, Descartes suggests, will enable the problem solver to determine, when approaching any specific problem in the future, which points he or she may usefully concentrate on discovering first.

In Rule Seven, Descartes addresses what he perceives to be the limitations of long chains of deductive thinking in problem solving. When the chains of reasoning leading to a conclusion are very long, “it is not easy,” he says “to

85. PHILOSOPHICAL WRITINGS, *supra* note 18, at 21.

86: *Id.*

87. *Id.*

88. *Id.* at 22.

recall the entire route which led us to it."⁸⁹ In such situations, the relations between things considered become obscured. To correct the situation, Descartes suggests mentally running through the items:

several times in a continuous movement of the imagination, simultaneously intuiting one relation and passing on to the next, until [one has] learn[ed] to pass from the first to the last so swiftly that memory is left with practically no role to play, and [one] seem[s] to intuit the whole thing at once.⁹⁰

He calls this technique enumeration. In concluding his explanation of Rule Seven, he points out that these three rules do not always apply simultaneously to every problem-solving task. He gives this example:

[I]f you want to construct a perfect anagram by transposing the letters of a name, there is no need to pass from the very easy to the more difficult, nor to distinguish what is absolute from what is relative, for these operations have no place here. All you need do is to decide on an order for examining permutations of letters so that you never go over the same permutations twice. The number of these permutations should, for example, be arranged into definite classes, so that it becomes immediately obvious which ones present the greater prospect of finding what you are looking for. If this is done, the task will seldom be tedious; it will be mere child's play.⁹¹

A modern interpretation of Rules Five, Six, and Seven, as they relate to problem solving in mediation and negotiation, might be as follows: When approaching a negotiation problem, be prepared. Mediators and negotiators should have in mind some type of overall ordered arrangement or method by which to proceed. Of course, this approach can be modified as one proceeds, but initially one should be prepared to give the problem-solving process structure and direction. In most problem-solving experiences in collaborative negotiation, few absolutes appear—i.e., unchangeable positions, unmodifiable interests, uncompromisable objectives. Most things can be considered relative, not absolute, in negotiation, while some things might *appear* more absolute than others from one point of view, yet more relative from a different point of view. Mediators and negotiators should start with easy, rather than difficult, matters in the problem-solving process. The problem solver should first look for basic characteristics of both the relationship of the parties and the substance of the dispute. From these basic characteristics, other characteristics might be deduced

89. *Id.* at 25.

90. *Id.*

91. *Id.* at 27.

that suggest alternate problem-solving routes.

Problem solvers should not, however, get so carried away with isolated matters, or details related to them, that they cannot see the forest for the trees. Instead, they should step back, mentally, at various intervals throughout the problem-solving process and take a look at the "big picture" all at once. Also, from time to time, they should reflect attentively on previous negotiation and mediation experiences to discover common basic characteristics of relationships or of disputes and consider why it was possible to discover some of these basic characteristics sooner and more easily than others. This procedure may enable problem solvers to determine, when approaching any specific problem in negotiation or mediation, which points or topics they may usefully concentrate on discovering first.

Rule 8: If in your examination of a series of things you encounter something which your intellect is unable to intuit sufficiently well, stop at that point.

In explanation of Rule Eight, Descartes states that the knowledge required in problem solving is divided into two types: knowledge of the faculties of the problem solver available to that problem solver and knowledge of the things it is possible to know to solve the problem. As to the first type of knowledge, Descartes says that while the intellect is capable of knowledge, it can be assisted or impeded by the faculties of imagination, sense-perception, and memory. These three faculties, he urges, should be examined carefully in each situation to see which would be a hindrance and which would be an asset in the problem-solving process. As to the second type of knowledge, Descartes recommends that problem solvers should deal with things it is possible to know in solving a problem only insofar as they are in reach of the intellect. He divides these things into two parts: absolutely simple natures and complex or composite natures. Finally, he observes this in problem solving:

[A]s often as . . . [a problem solver] applies his mind to acquire knowledge of something, either he will be entirely successful, or at least he will realize that success depends upon some observation which is not within his power to make—so he will not blame his intelligence, even though he is forced to come to a halt⁹²

Descartes' advice in Rule Eight holds great significance for mediators and negotiators. Essentially, he is saying that in problem solving, mediators and

92. PHILOSOPHICAL WRITINGS, *supra* note 18, at 32.

negotiators must be consciously aware of their own mental faculties and to understand which of their mental faculties are helpful or unhelpful in the various phases of problem solving. He also puts mediators and negotiators on notice that problems may be incapable of solution because of obstacles presented by the nature of the problem itself or by the human condition.⁹³ Mediators and negotiators who have ably employed their mental faculties should not blame themselves when they are unable to produce a mutually acceptable solution. The discovery that a problem cannot be negotiated or mediated to a solution is knowledge in itself, and may suggest that a solution can be achievable only through the parties' relinquishing their joint decisionmaking (problem solving) function to another decisionmaking person or entity, perhaps an arbitrator, who will make the decision or solve the problem for them.

Rule 9: Concentrate your mind's eye upon the most insignificant and easiest of matters, and dwell on them long enough to acquire the habit of intuiting the truth distinctly and clearly.

After first reminding the reader that intuition and deduction are the actions or operations of intellect on which problem solvers exclusively rely in the acquisition of knowledge, Descartes states that the purpose of Rule Nine is to explain how the problem solver can make the employment of intuition and deduction more skillful and how the problem solver can cultivate two special mental faculties: perspicacity in the distinct intuition of particular things and discernment in the methodical deduction of one thing from another. His explanation of Rule Nine largely relates to perspicacity and his explanation of Rule Ten mostly concerns discernment in the methodical deduction of one thing from another.

In describing perspicacity, Descartes writes this:

We can best learn how mental intuition is to be employed by comparing it with ordinary vision. . . . [C]raftsmen who engage in delicate operations, and who are used to fixing their eyes on a single point, acquire through practice the ability to make perfect distinctions between things, however minute and delicate. The same is true of those who never let their thinking be distracted by many different objects at the same time, but always devote their whole attention to the simplest and easiest of matters: they become perspicacious.⁹⁴

93. *Id.* at 28. This situation would be in the nature of an impossible illusion. See Cooley, *Joke Design*, *supra* note 71, at 293.

94. PHILOSOPHICAL WRITINGS, *supra* note 18, at 33.

As an example of concentrating upon the most insignificant and easiest matters, Descartes further writes this:

[I]f I want to know how one and the same simple cause can give rise simultaneously to opposite effects, I shall not . . . prattle on about the moon's warming things by its light and cooling them by means of some occult quality. Rather, I shall observe a pair of scales, where a single weight raises one scale and lowers the other instantaneously, and similar examples.⁹⁵

For mediators and negotiators, Descartes continues to emphasize the importance of focusing on the easiest and the simplest aspects of the problem first. The ability to focus on and analyze one object (each tree in the forest) at a time—perspicacity—is as important to the problem solver, as being able to step back mentally from time to time to see the “big picture” (the entire forest) as described in Rule Seven. Indirectly, Descartes is also suggesting that using simple analogies (in this case the scale) is a helpful method in initiating the problem-solving process or in finding new avenues to solutions.

Rule 10: Investigate what others have already discovered and methodically survey even the most insignificant products of human skill, especially those which display or predispose order.

In explanation of this Rule, Descartes exhorts problem solvers to recognize the limits of classical dialectic, syllogism, and rhetoric in producing truth. He further urges problem solvers not to take the ancient Greek reasoning tools at face value without questioning their utility and valid application in the particular problem-solving task at hand. Descartes cautions:

Our principal concern here is thus to guard against reason's taking a holiday while we are investigating the truth about some issue; . . . the . . . [classical] art of reasoning contributes nothing whatever to knowledge of the truth [O]n the basis of their method, dialecticians are unable to formulate a syllogism with a true conclusion unless they are already in possession of the substance of the conclusion, i.e. unless they have previous knowledge of the very truth deduced in the syllogism. . . . Its sole advantage is that it sometimes

95. *Id.* at 34.

enables us to explain to others arguments which are already known.⁹⁶

The message here for mediators and negotiators is that, in problem solving, they should always be on their guard to test the validity of all reasoning being used. Not only can the truth of premises be flawed, but also inferences drawn from premises can be defective.⁹⁷

Rule 11: If you deduce something from a number of simple propositions, run through them mentally, reflect on their relations to one another, and form a simultaneous conception of several of them.

In explaining Rule Eleven, Descartes describes the ways in which intuition and enumeration aid and complement one another. Descartes writes:

96. *Id.* at 36-37. For an explanation of the use of classical techniques in mediational problem solving, see John W. Cooley, *A Classical Approach to Mediation — Part I: Classical Rhetoric and the Art of Persuasion in Mediation* 19 U. DAYTON L. REV. (forthcoming Fall 1993) [hereinafter Cooley, *Classical Rhetoric*]; John W. Cooley, *A Classical Approach to Mediation — Part II: The Socratic Method and Conflict Reframing in Mediation* 19 U. DAYTON L. REV. (forthcoming Winter 1994) [hereinafter Cooley, *Socratic Method*].

Others, besides Descartes, have written concerning the limitations of the classical art of reasoning, particularly that embodied in the syllogism. Sextus Empiricus, one of the ancient skeptics, offered an ingenious argument against deductive inference. Consider the following:

A	B
1. If it is day, it is light	It is day
2. It is day	
3. It is light	It is light

Argument *A* is deductive; Argument *B* is nondeductive. Sextus maintained that deductive arguments are always, by their own criteria, flawed. As one commentator explains:

In the present case . . . either (3) follows from (2) or it does not. If it does, the *B* is a perfectly acceptable argument for in *B* we simply infer (3) from (2). But if this is the case then (1) is clearly redundant. On the other hand, if (3) does not follow from (2) then (1) is false, since (1) clearly asserts that it does. So deductive proof is impossible: what *A* tells us over and above *B* is either redundant or false.

Gaukroger, *supra* note 38, at 107. John Stuart Mill also noted that the premises contain the same assertion as the conclusion in deductive arguments, and that in effect is what makes them valid. *Id.*

97. It is of course possible to learn much from deductive proofs. An example is the philosopher Thomas Hobbes' first encounter with Euclid's *Elements*:

Being in a . . . Library, Euclid's *Elements* lay open . . . He read the proposition. By G—, said he . . ., this is impossible! So he read the Demonstration of it, which referred him back to another, which he also read. [And so on] that at last he was demonstratively convinced of that truth. This made him in love with Geometry.

Id.

[Intuition and enumeration] aid and complement . . . each other so thoroughly that they seem to coalesce into a single operation, through a movement of thought, . . . which involves carefully intuiting one thing and passing on at once to the others.

There is . . . a twofold advantage in this fact: it facilitates a more certain knowledge of the conclusion in question, and it makes the mind better able to discover other truths.⁹⁸

It is suggested that mediators and negotiators apply this Rule in connection with Rules Three, Seven, Nine, and Ten.

C. Application of Rules 12 to 21 in Collaborative Negotiation

This section will use Descartes' Rules 12 through 21 as a basis to demonstrate the analytic method as it applies to mathematical problem solving and as that same method can be extrapolated for use in conflict and transactional collaborative negotiation.⁹⁹

98. PHILOSOPHICAL WRITINGS, *supra* note 18, at 38.

99. For a graphical, geometric approach to understanding possible outcomes and settlement ranges in fixed-sum (distributive solution) negotiations, see CHRISTOPHER W. MOORE, THE MEDIATION PROCESS: PRACTICAL STRATEGIES FOR RESOLVING CONFLICT 219-22 (1986).

The vehicle for demonstrating the analytic method in mathematical problem solving will be selected examples from the late Professor George Polya's book *How to Solve It*. GEORGE POLYA, HOW TO SOLVE IT: A NEW ASPECT OF MATHEMATICAL METHOD (Princeton University Press, 2d ed. 1988) [hereinafter POLYA, HOW TO SOLVE IT]. The vehicle for demonstrating how the analytic method can be used in the collaborative negotiation ("real life") setting will be a business dispute fact pattern in which the parties in conflict have an ongoing business relationship. The analytic method will be applied to both the mathematical and real-life problems, in a tandem format, discussed under headings of the four broad stages of what I have defined as the problem-solving process, generally: Problem Design, Process Design, Solution Design, Reflection. See COOLEY, APPELLATE ADVOCACY, *supra* note 63, at 40-49.

These stages correspond, loosely, to George Polya's four stages of mathematical problem solving: Understanding the Problem, Devising a Plan, Carrying Out the Plan, and Looking Back. POLYA, HOW TO SOLVE IT, *supra*, at xvi-xvii. The Problem Design stage relates to Descartes' Rules 13, 14, 15, and 16; the Process Design stage, to Rules 12, 17, and 19; the Solution Design Stage, to Rules 18, 20, and 21; and the Reflection stage, largely to Rules 5, 6, and 7. Definitions of mathematical terms and expressions are provided liberally throughout the remainder of this article to enlighten the reader as to the simple essences of the mathematical method themselves, and in particular, to facilitate the reader's understanding of how these simple essences of method can be translated for use in collaborative negotiation, both mediated and unmediated.

1. Designing the Problem

Rule 13: To perfectly understand a problem, omit every superfluous conception, reduce it to its simplest terms, and divide it into its smallest possible parts.

Rule 14: Re-express the problem in terms of the real extensions of itself and picture those extensions as figures in your imagination.

Rule 15: Draw the imagined figures and display them before your external senses.

Rule 16: Represent those things which do not require immediate attention by concise symbols rather than by complete figures.

a. Basic Definitions

In the preface to the second edition of his book, *How to Solve It*, George Polya quotes from a news article published almost forty years ago. The quotation appearing below has, happily, much less validity today than it did when it was printed in 1956:

[M]athematics has the dubious honor of being the least popular subject in the curriculum Future teachers pass through the elementary schools learning to detest mathematics They return to the elementary school to teach a new generation to detest it.¹⁰⁰

George Polya, and others like him, had a great deal to do with changing this attitude toward mathematics over the years. They saw mathematics, much like Descartes, not only as a systematic deductive science, but also as an experimental, inductive, inventive science. As Polya stated in the preface to the first printing of his book,

A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the discovery.¹⁰¹

100. POLYA, *HOW TO SOLVE IT*, *supra* note 99, at ix.

101. *Id.* at v.

Polya's method of doing and teaching problem solving is based fundamentally on Descartes' Rules,¹⁰² discussed *supra*, and as he himself admits, is equally applicable to solving mathematical problems and the practical problems of everyday life.¹⁰³ He speaks to both students and teachers throughout his book in explaining his method, but as he observes, his book "should interest anybody concerned with the ways and means of invention and discovery."¹⁰⁴ Reading the book from the viewpoint of a student, one can immediately see the relevance of his explanations to negotiators; reading the book from the viewpoint of a teacher, one can similarly see the relevance to the function of a mediator. For example, in the following quoted passage, the word "teacher" could be substituted with the word "mediator" and the word "students" with "parties" with no appreciable loss of meaning:

The teacher who wishes to develop his [or her] students' ability to . . . [solve] problems must instill some interest for problems into their minds and give them plenty of opportunity for imitation and practice. If the teacher wishes to develop in his [or her] students the mental operations which correspond to the *questions and suggestions* [of the method], he [or she] puts these *questions and suggestions* to the students as often as he [or she] can do so naturally. . . . Thanks to such guidance, the student will eventually discover the right use of these questions and suggestions.¹⁰⁵

102. See 1 GEORGE POLYA, *MATHEMATICAL DISCOVERY* 24-45 (1962). See also George Polya, *Induction and Analogy in Mathematics*, in 1 *MATHEMATICS AND PLAUSIBLE REASONING* (1954); George Polya, *Patterns of Plausible Inference*, in 2 *MATHEMATICS AND PLAUSIBLE REASONING* (1954).

103. POLYA, *HOW TO SOLVE IT*, *supra* note 99, at 149.

104. *Id.* at vi. See also EDWARD KASNER & JAMES NEWMAN, *MATHEMATICS AND THE IMAGINATION* (1989); IVARS PETERSON, *THE MATHEMATICAL TOURIST*, (1988); PHILIP DAVIS & REUBEN HERSH, *DESCARTES' DREAM: THE WORLD ACCORDING TO MATHEMATICS* (1986); JACQUES HADAMARD, *THE PSYCHOLOGY OF INVENTION IN THE MATHEMATICAL FIELD* (1954); D. HILBERT & S. COHN-VOSSEN, *GEOMETRY AND THE IMAGINATION* (1952).

105. POLYA, *HOW TO SOLVE IT*, *supra* note 99, at 5 (emphasis added). Polya provides additional advice for teachers in posing questions and suggestions to students, which is equally applicable to mediators:

Begin with a general question or suggestion . . . , and, if necessary, come down gradually to more specific and concrete questions or suggestions till you reach one which elicits a response in the student's mind. . . . The suggestions must be simple and natural because otherwise they cannot be *unobtrusive*. The suggestions must be general, applicable not only to the present problem but to problems of all sorts, if they are to help develop the *ability* of the student [in problem solving] and not just a special technique. The list [of questions] must be short in order that the questions may be often repeated, unartificially, and under varying circumstances; thus, there is a chance that they will be eventually assimilated by the student and will contribute to the development of a *mental habit*.

Id. at 20-21 (emphasis in original).

The questions and suggestions to which Polya refers are the components of his method, called collectively the modern heuristic. Originally, "heuristic" was the name of a certain branch of study, not clearly delineated, belonging to logic, or philosophy, or to psychology, and seldom presented in detail. In antiquity, Euclid and Pappus had dealt with the concept of heuristic in a superficial way, and more recently, Descartes (in the Rules) and Leibniz constructed systems of heuristic.¹⁰⁶ Polya defines his "modern heuristic" as "an endeavor to understand the process of solving problems, especially the *mental operations typically useful* in this process."¹⁰⁷ In further describing this problem solving method, Polya states this:

A serious study of heuristic should take into account both the logical and the psychological background, it should not neglect what such other writers as . . . Descartes . . . [and] Leibniz . . . have to say about the subject, but it should least neglect unbiased experience. Experience in solving problems and experience in watching other people solving problems must be the basis on which heuristic is built.¹⁰⁸

Thus, Polya finds heuristic to be a matter of human individuality. He offers his Descartes-based modern heuristic as a suggested sequence of natural, simple, common sense questions and suggestions which has been useful to him in solving problems and which can be modified and perhaps improved upon when employed by individual problem solver. The heuristic is employed in all four of the problem solving stages (Problem Design, Process Design, Solution Design, and Reflection).

As noted above, the Problem Design stage of mediational problem solving corresponds to Polya's "Understanding the Problem" stage. It is, however, broader than its Polya-defined counterpart. Problem Design is in itself a design problem which encompasses both *finding the problem* and understanding the problem once found. In many types of dispute and transaction situations, *discovering or finding* the problem (i.e., identifying the underlying interests of the parties) is the most difficult task; in others, it is the easiest. Whether difficult or easy, it is a crucial step in the problem-solving process that must be taken prior to application of the analytic method.

106. *Id.* at 112.

107. *Id.* at 129-30 (emphasis in original). "Heuristic" is similar to "algorithm," which is defined as a recursive specification of a procedure by which a given type of problem can be solved in a finite number of mechanical steps. E.J. BOROWSKI & J.M. BORWEIN, *MATHEMATICS* 13 (1991).

108. POLYA, *HOW TO SOLVE IT*, *supra* note 99, at 130.

The table of interests below provides a useful survey of possible interests of parties in any dispute or transaction situation. In negotiation, parties (whether they realize it or not) have certain needs which they seek to have fulfilled. These needs fall generally into the following categories: economic, emotional, psychological, physical, and social.¹⁰⁹ Relating to these basic needs are underlying interests—some compatible, some overlapping, some conflicting. Even where the parties perceive their needs to be purely economic and seek a wholly monetary (distributive) solution, often their underlying interests are compatible and overlapping when identified in terms of “value” instead of dollars. Even in tort cases, traditionally considered to have only purely monetary solutions, the parties may share compatible or overlapping interests, some of which include amount in controversy, cost of recovery, time of payment, exchange rate, method of payment (annuity, etc.), identity of payees, payment in kind, payment in services, payment in real estate, and tax or tariff considerations.

The interests appearing in the table may relate to different needs of the parties and may actually be overlapping or compatible, in achieving an integrative solution (non-monetary, or combined monetary and non-monetary) in both personal and corporate disputes.

Table of Interests in Dispute or Transaction

Time	Words	Secrecy
Place	Apology	Release
Quantity	Control	Reinstatement
Quality	Persons	Assurances
Size	Nature	Procedure
Context	Structure	Opportunity
Distance	Types	Guarantee
Responsibility	Volume	Publicity
Rate	Proportion	Security
Space	Exchange	Share

In determining the parties' interests, the items in this table should be considered, metaphorically, and in the broadest sense possible. For example “volume” in a business dispute could refer to tripling a marketing effort (i.e., turning up the

109. Professor Abraham Maslow of Brandeis University, in his book entitled *Motivation and Personality* identified seven categories of needs as basic factors in human behavior: physiological, safety and security, love and belonging, esteem, self-actualization, to know and understand, aesthetic. ABRAHAM MASLOW, *MOTIVATION AND PERSONALITY* (1954). These seven factors can be condensed to the five needs as presented *supra* in text. See GERARD I. NIERENBERG, *FUNDAMENTALS OF NEGOTIATING* 82-83 (1973).

volume of the corporate message), decreasing the amount of production output, or increasing the amount of storage space in a warehouse. "Rate" could refer to frequency of occurrence, a commission or discount, or evaluation of products, services, or performance.

After the interests of the parties have been at least tentatively identified, then the analytic method can be applied to solve the discovered or found problems (i.e., determining what resource(s) can be used to satisfy the identified interests of the parties). To understand how the analytic method can be employed in such a situation, Polya's topic of "Understanding the Problem," the second aspect of Problem Design in mediational problem solving, is useful.¹¹⁰

b. Understanding the Problem

Polya identifies two types of mathematical problems: problems to find a solution and problems to prove a solution.¹¹¹ He provides the following as an example of a problem to *find* a solution: Construct a triangle with sides a , b , and c . The principal parts of a problem to find a solution are the *unknown*, the *data*, and the *condition*. In the example, the unknown is a triangle; the data is the three lengths a , b , and c ; and the condition is that the triangle has sides of the lengths a , b , and c . An example, on the other hand, of a problem to *prove* a solution would be as follows: If the four sides of a quadrilateral are equal, then the two diagonals are perpendicular to each other. The principal parts of a problem to prove are the *hypothesis* and the *conclusion*. In the example, the first part, starting with "if," is the hypothesis; the second part, starting with "then," is the conclusion. Usually in the Problem Design stage, the problem solver is most often confronted with a problem to *find* a solution; problems to *prove* a solution normally appear in the Solution Design stage. The correlative heuristic for "Understanding the Problem" to be employed by a problem solver with respect to a "real life" mediation as compared to a mathematical problem is shown in the following chart:

110. The analytic method as applied to the mathematics problem is adapted from POLYA, HOW TO SOLVE IT, *supra* note 99, at 6-23. The format which will be used, for the most part, for that topic and the topics that follow in this part will be a side-by-side presentation showing the analogies between the analytic method being applied to solving a mathematics problem (on the left-hand side of the page) and it being applied to a "real life" mediation situation (on the right-hand side of the page).

111. Polya actually calls them "problems to find" and "problems to prove." POLYA, HOW TO SOLVE IT, *supra* note 99, at 154-57. I have used slightly different labels so as to not confuse them with discovered and found problems as described in Cooley, *Socratic Method*, *supra* note 96.

Heuristic for Understanding the Problem

<i>Mathematical Problem</i>	<i>Negotiation/Mediation Problem</i>
What is the unknown?	What is the unknown resource?
What are the data?	What are the data?
What is the condition? (Descartes' Rule 13)	What is the condition governing how the resource(s) must satisfy the interest?
Draw a figure. ¹¹²	Draw a figure.
Introduce suitable notation. ¹¹³ (Descartes' Rules 14,15,16)	Introduce suitable notation.
Is it possible to satisfy the condition?	Is it possible to satisfy the condition?
Is the condition sufficient to determine the unknown? ¹¹⁴	Is the condition sufficiently defined to determine the resource(s)?
Or is it insufficient?	Or is it insufficient?
Or is it redundant?	Or is it redundant?
Or is it contradictory?	Or is it contradictory?
Separate the various parts of the condition.	Separate the various part of the condition.
Can you write them down? (Descartes' Rule 13)	Can you write them down?

This heuristic can be applied to the mathematical and mediation problems as follows:¹¹⁵

112. Polya notes that the initial figure drawn may be assumed and tentative, subject to change. It is important to see a configuration of the unknown and the data, as they are prescribed by the condition of the problem. Specifically, he advises: "In order to understand the problem distinctly, we have to consider each datum and each part of the condition separately; then we reunite all parts and consider the condition as a whole, trying to see simultaneously the various connection required by the problem." POLYA, *HOW TO SOLVE IT*, *supra* note 99, at 104.

113. Polya observes that good notation is unambiguous and easy to remember. It should avoid harmful second meanings and take advantage of useful second meanings. The order and connection of signs should suggest the order and connection of things. Letters at the beginning of the alphabet (*a, b, c*, etc.) are normally used to represent given quantities and constants. Letters at the end of the alphabet (*x, y, z*) are normally used for unknown quantities or variables. See POLYA, *HOW TO SOLVE IT*, *supra* note 99, at 134-41.

114. A condition is called redundant if it contains superfluous parts; it is called contradictory if its parts are mutually opposed and inconsistent so that there is no object satisfying the condition. If a condition is expressed by more linear equations than there are unknowns, it is either redundant or contradictory; if expressed by fewer equations than there are unknowns, it is insufficient to determine the unknowns; if a condition is expressed by just as many equations as there are unknowns, it is usually sufficient to determine the unknowns. POLYA, *HOW TO SOLVE IT*, *supra* note 99, at 72-73.

115. All citations in the following columns are noted by the symbols † and ‡. They are explained in the footnote immediately following the columns.

Statement of Problem

Find the formula for the length of the diagonal of a rectangular parallel-piped† of which the length, the width, and the height are known.

Statement of Problem

Tranex, Inc. has been supplying microcircuit boards for Novatron Inc.'s electronic products for fifteen years. Recently, Novatron has received numerous complaints from retail store managers and individual consumers asserting warranty rights regarding Novatron's new mini-laptop computer. They report that the computer can "crash" without warning, destroying all information in the particular file, even though the file had been previously saved on the hard drive. Tranex supplies part of the circuitry for this product. Novatron's engineers isolated the "crash" problem to a micro-chip supplied by Tranex. The president of Tranex initially denied that its product caused the problem, which infuriated the president of Novatron, who threatened a million dollar lawsuit for breach of contract. After an initial mediation session, Tranex finally admitted that its own tests showed that the micro-transistor in question was defective. Novatron has paid for and received 20,000 of the defective micro-circuits; it has 5000 mini-laptops in production; it has provided 1000 laptops to its distributors for sale. Both parties want to minimize their financial losses with respect to any solution reached. They also wish to continue doing business together, assuming the solution minimizes their financial losses. You are the mediator.

What is the unknown?

The formula for the length of the diagonal of a parallelepiped.

What are the data?

The length, width, and height of the parallelepiped.

What is the condition?

The length of the diagonal must be able to be computed in any instance where the values of the length, width, and height of a parallelepiped are known.

Draw a figure and introduce suitable notation:‡

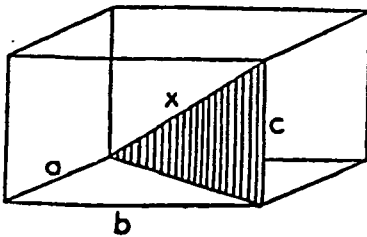


Figure 2

Thus, in Figure 2, x is the diagonal of the parallelepiped of which a , b , and c are the length, width, and height, respectively.

What is the unknown?

The formula of resources for minimizing the parties' financial losses.

What are the data?

Novatron has paid for and received 20,000 of the defective micro-circuits; it has 5000 mini-laptops in production; and it has provided 1000 laptops to its distributors for sale.

What is the condition?

The resources must be able to satisfy both parties' interests where the number of defective laptop units in any stage of production or sale is known.

Draw a figure and introduce suitable notation:

See Part IV of this article.

Is it possible to satisfy the condition?

Yes, it appears to be.

Is the condition sufficient to determine the unknown?

Yes, it is. If we know the values of a , b , and c , we know the parallelepiped. If the parallelepiped is determined, the diagonal can be determined.

Is it possible to satisfy the condition?

Yes, it appears to be.

Is the condition sufficient to determine the unknown?

It appears to be at this point, yes. If we know the number of laptops in the various stages of production and sale, we know the information structure encompassing the parties' economic interests. If that information structure is determined, the resources for satisfying the encompassed interests can be determined.

Once the problem is understood, one can begin to design the process.¹¹⁶

2. Designing the Process

Rule 12: Make use of all aids which intellect, imagination, sense perception, and memory afford in order: (1) to intuit simple propositions distinctly; (2) to combine correctly the matters under investigation with what you already know; and (3) to find out what things should be compared with each other.

Rule 17: Survey the problem to be solved, disregarding the fact that some of its terms are known and others are unknown, and observing their inter-dependence.

Rule 19: Try to find as many magnitudes, expressed in two different ways, as there are unknown terms, which you treat as known in order to make as many comparisons as possible between two equal terms.

116. This footnote explains the information cited in the preceding columns.

† A parallelepiped is defined as a solid with six faces, each of which is a parallelogram. BOROWSKI & BORWEIN, *supra* note 107, at 434.

‡ Figure 2 is reprinted from GEORGE POLYA, *HOW TO SOLVE IT 11* (Princeton University Press, 2d. ed. 1988) with permission of the publisher.

*Heuristic for Devising a Plan*¹¹⁷

<i>Mathematical Problem</i>	<i>Negotiation/Mediation Problem</i>
<p>Do you know a related problem?</p> <p>Look at the unknown. Try to think of a familiar problem having the same or similar unknown.</p> <p>If you identify a related problem previously solved, could you use it?</p> <p>Could you: use its result? use its method? introduce some auxiliary element in order to facilitate its use?[†]</p> <p>(Descartes' Rules 12, 17)</p>	<p>Do you know a related problem?</p> <p>Consider the unknown resource. Try to think of a familiar problem having the same or similar unknown resource.</p> <p>If you identify a related problem previously solved, could you use it?</p> <p>Could you: use its result? use its method? introduce some auxiliary element in order to facilitate its use?</p>
<p>Could you: imagine a more general problem?[‡] imagine a more special problem?^{††} imagine an analogous problem?^{‡‡} introduce an auxiliary problem?^{†††} solve part of the problem? vary the condition? vary the unknown? acquire other data appropriate to determine the unknown?</p> <p>(Descartes' Rule 19)</p>	<p>Could you: imagine a more general problem? imagine a more special problem? imagine an analogous problem? introduce an auxiliary problem? solve part of the problem? vary condition/interest? vary the resource? acquire other data appropriate to determine the unknown resource?</p>
<p>Have you: used all the data? used the whole condition?</p> <p>Could you restate the problem? Go back to definitions.</p> <p>(Descartes' Rule 12)</p>	<p>Have you: used all the data? used the whole condition/interest?</p> <p>Could you restate the problem? Go back to definitions.</p>

117. All citations in the following table are noted by symbols and explained in the footnote immediately following the table.

This heuristic¹¹⁸ can be applied to both the mathematical and mediation problems as follows:¹¹⁹

Do you know a related problem?

Do you know a related problem?

I can't think of one.

I can't think of one immediately.

118. This footnote explains the information cited in the preceding table.

† There are various kinds of auxiliary elements, but in solving geometric problems, the most common kinds are line segments. For example, assume that the present problem does not have a triangle, but a related and previously solved problem involved a triangle. One could add auxiliary line segments forming a triangle to the figure depicting the present problem to see if it will facilitate problem solving. In the problem being solved in the text, the auxiliary element is line segment y , shown in the figure in the text, *supra*. See POLYA, HOW TO SOLVE IT, *supra* note 99, at 46-47.

‡ In some situations, a more general problem may be easier to solve. Take for example the problem being solved in the text, *supra*: Given the three dimensions (length, breadth, and height) of a rectangular parallelepiped, find the diagonal.

A more general problem based on the statement of the original problem would be: find the diagonal of a parallelepiped, being given the three edges issued from an end-point of the diagonal, and the three angles between these three edges. See POLYA, HOW TO SOLVE IT, *supra* note 99, at 67. However, as the problem-solving process continues in the text, *infra*, this more general problem is not immediately helpful in solving the original problem.

†† A more special problem (or special case) related to the problem being solved in the text would be: find the diagonal of a cube with a given edge. The effect of the use of the special problem might be to simplify the original problem for some problem solvers or to bring into focus for others a previously unperceived method of solution. See POLYA, HOW TO SOLVE IT, *supra* note 99, at 67.

‡‡ Problems analogous to that being solved in the text would be: find the diagonal of a regular octahedron with a given edge; find the radius of the circumscribed sphere of a regular tetrahedron with a given edge; given the rectangular coordinates (x_1, y_1, z_1) , (x_2, y_2, z_2) of two points in space, find the distance between these points. These analogous problems, however, are not particularly helpful in solving the original problem. POLYA, HOW TO SOLVE IT, *supra* note 99, at 67.

††† An auxiliary problem is one that is considered not for its own sake, but because it is hoped that its consideration will help solve the present problem. Usually, the auxiliary problem helps to simplify the present problem. Suppose the present problem is to find x , satisfying the equation:

$$x^4 - 13x^2 + 36 = 0.$$

Note that $x^4 = (x^2)^2$ and that there might be some advantage in introducing $y = x^2$. Thus, a new problem arises called the auxiliary problem. The auxiliary problem is to find y satisfying the equation:

$$y^2 - 13y + 36 = 0.$$

The solution of this problem could be used as a means of solving the present problem. The unknown y in the auxiliary problem is called the auxiliary unknown. See POLYA, HOW TO SOLVE IT, *supra* note 99, at 50-51.

119. All citations in the following columns will be noted by symbols and explained immediately after the end of the columns.

Look at the unknown. Try to think of a familiar problem having the same or similar unknown.

I never before solved a problem dealing with the diagonal of a parallelepiped as the unknown, but I have solved a problem dealing with the length of a side of a *right triangle* as the unknown.

If you identify a related problem previously solved, could you use it?

I could try.

Could you introduce some auxiliary element in order to facilitate its use?

Yes. I notice that in Figure 2 above, there is a right triangle. (See shaded area). I also see that the unknown x is the hypotenuse of that right triangle. Height c is given as part of the data. I can introduce line y which is the third side (and hypotenuse) of a right triangle having sides a and b . I can find y by use of the Pythagorean theorem.† Therefore, I have a plan for finding the formula for determining the auxiliary element y . And if I know what y is, I can find the formula for x . Therefore, I have a *plan* for finding a formula for determining x , when a , b , and c are known.‡

Consider the unknown resources. Try to think of a familiar problem requiring the same or similar resources for solution.

I never before helped to solve a problem dealing with multiple defective electronic products in various stages of production and sale, but I have helped solve a problem concerning the resources appropriate to resolve a conflict arising out of a consumer's purchase of a new car from a car dealership.

If you identify a related problem previously solved, could you use it?

It certainly would be worth a try.

Could you introduce some auxiliary element in order to facilitate its use?

Yes. I notice that both the present problem and the car dealership problem involve a defective component. In the dealership problem, the consumer frequently experienced difficulty in starting the car. The dealer took the position that the consumer was not properly "pumping" the accelerator enough. After much acrimony, the dealership admitted that there was a defect in the car's carburetor.

The consumer had wanted a "brand new" carburetor. The dealership wanted to replace the "butterfly valve" only. The parties compromised by agreeing to have the valve replaced and accompanied by a

written memo of understanding that the carburetor would be replaced if that did not solve the problem. Thus, I may have a resource to be included in the formula of resources for solving the present problem. I will need more data, however.

Can you acquire other data appropriate to determining the unknown?

I believe I have all the data that I need.

Can you acquire other data appropriate to determining the unknown?

Yes. I can talk with the parties to determine whether a replacement part could solve the problem, or part of the problem. That may take care of the 20,000 microcircuits that Novatron has in its possession, and perhaps some of the 5000 laptop units which have not yet been fully assembled. But what about the units that are fully assembled and packed for shipment? It might be costly for Novatron to pay its employees to unpack and disassemble the units and replace the defective part. I would like to find out how Tranex could minimize these costs to Novatron. I would also like to find out how Tranex could minimize Novatron's costs in dealing with the 1000 units shipped to retailers, some of which have been resold to consumers.

I have discussed these matters with the parties in caucuses and in joint sessions and have acquired the information I need regarding the availability of resources to satisfy their respective interests.

This previous section showed how to design the process;¹²⁰ the next section discusses how to design the solution.

3. Designing the Solution

Rule 18: In the solution stage, only four operations are required: addition, subtraction, multiplication, and division.

Rule 20: Once you find the equations using addition and subtraction, carry out the operations of multiplication and division, as appropriate.

Rule 21: Reduce all equations to a single one.

Heuristic for Carrying Out the Plan

<i>Mathematical Problem</i>	<i>Negotiation/Mediation Problem</i>
Perform operations.	Perform operations.
Check each step.	Check each step.
Is each step correct?	Is each step correct?
Can you prove each step is correct?	Can you prove each step is correct?
(Descartes' Rules 18, 20, 21)	

This heuristic can be applied to both the mathematical and mediation problems as shown in the following columns.

120. This footnote explains the information cited in the preceding columns.

† The Pythagorean theorem is "the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of the other two sides." BOROWSKI & BORWEIN, *supra* note 107, at 476.

‡ Another way to get to this point would be to apply the part of the heuristic which asks: Could an analogous problem be imagined? Here, a simpler, analogous problem of plane geometry might be identified, i.e., finding the diagonal of a rectangular parallelogram. The idea is essentially the same, but the *approach* is different. In the text, *supra*, the formerly solved problem was remembered because its *unknown* was the same as the proposed problem; here there is an analogy existing between plain and solid geometry which consists in conceiving the diagonal of the given parallelepiped as the diagonal of a suitable parallelogram which must be introduced into the figure (as the intersection of the parallelepiped with a plane passing through two opposite edges). See POLYA, HOW TO SOLVE IT, *supra* note 99, at 19-20.

Perform Operations:

Using the Pythagorean theorem, these formulae may be stated with respect to Figure 2 as follows:

$$x^2 = y^2 + c^2$$

$$y^2 = a^2 + b^2$$

$$x^2 = a^2 + b^2 + c^2$$

$$x = \sqrt{a^2 + b^2 + c^2}$$

Perform Operations:

Definitions of notations are as follows:

T.R. = total resources for minimizing cost of solution

a = Tranex will provide replacement microchips free of charge (its cost to produce is miniscule).

b = Novatron will replace the microchip (a simple operation) in the 20,000 microcircuits (and in the estimated 2000 unassembled units) prior to assembly of the laptops.

c = Tranex will provide three of its employees to Novatron for three weeks to unpack, disassemble, replace microchips, reassemble, and repack the estimated 3000 assembled and packed units currently in Novatron's warehouse awaiting shipment.

d = Novatron will recall, and bear the related cost of recalling, the estimated 500 laptop units currently held by retailers.

e = Novatron will bear the expense of contacting consumers to offer free replacement of part and related service at service provider of their choice; with a free Novatron-produced personal finances software as an inducement.

Thus, the single formula for total resources is:

$$T.R. = a + b + c + d + e$$

Check each step:

I have checked each step.

Is each step correct?

Yes.

Check each step:

I have checked each step.

Is each step correct?

Each step is correct, but I note that two of them are incomplete. Resource *d* should be augmented with the parties' agreement that Tranex provide the labor to disassemble, replace parts, and reassemble 500 units, with Novatron repacking and reshipping the units to the retailers.

Resource *e* should be augmented by the parties' agreement that Novatron will bear all costs connected with satisfaction of the 500 consumers, even if that ultimately requires Novatron to replace the defective parts, in-house, or to provide, in certain circumstances, replacement of laptop computers.

Can you prove each step is correct?

Yes.

Can you prove each step is correct?

Yes. I believe each step is now correct and complete.

4. Reflection

Heuristic for Looking Back

<i>Mathematical Problem</i>	<i>Negotiation/Mediation Problem</i>
Can you check the result?	Can you check the result?
Can you derive the result differently?	Can you derive the result differently?
Can you use the result, or the method, for some other problem? ¹²¹	Can you use the result, or the method, for some other problem?
(Descartes' Rules 1 and 11)	

This heuristic can be applied to both the mathematical and mediation problems as follows:

Can you check the result?

Yes. Assume that $a = 10.5$; $b = 8$; and $c = 6$. Substituting these values in the derived formula, we obtain:

$$x = \sqrt{(10.5)^2 + (8)^2 + (6)^2}$$

$$x = 14.5$$

Checking the basic equations:

$$y^2 = (10.5)^2 + 8^2$$

Can you check the result?

Yes. Even though I believe each step of the solution is correct and complete, I will insist that the parties carefully review this Total Resource equation and, with my assistance, mentally run through all the possible scenarios (sub-equations) to ensure that the T.R. equation's separate terms are accurate and that it is valid overall.

121. Following are two problems that could now be easily solved knowing the method for solving the original problem in the text: (1) Find the diagonal of a cube with a given edge, and (2) Given the length, the breadth, and the diagonal of a rectangular parallelepiped, find the height. As to (1), the cube is merely a special problem (or special case) related to the original problem. As to (2), it is important to observe that the solution of the original problem consisted essentially in establishing a *relation* among four quantities—the three dimensions of the parallelepiped and its diagonal. If any three of these four quantities are given, the fourth, which is dependent on the relation, can be calculated. Thus, there is a *pattern* available to derive easily solvable new problems from the original problem solved in the text, *supra*. POLYA, HOW TO SOLVE IT, *supra* note 99, at 66-67.

$$y^2=110.25+64=174.25$$

$$x^2=174.25+36=210.25$$

$$x=14.5$$

Can you derive the result differently?

Yes. I could perceive the problem as involving the diagonal of a two-dimensional parallelogram and use knowledge related to the parallelogram to solve the present problem.

Can you use the result or method to solve some other problem?

Yes. For example, if I were given the length, width, and height of a rectangular parallelepiped, I could use the *result* of the problem just solved to find the distance of the center from one of the corners.

Also, I could use the *method* of introducing suitable (two-dimensional

Can you derive the result differently?

Yes. I could perceive the problem as one conducive to fractionation, involving several simple two-dimensional relationships: manufacturer-supplier; manufacturer-retailer; retailer-consumer; manufacturer-consumer; supplier-consumer; supplier-retailer. These relationships could be analogized to the six faces of a rectangular parallelepiped, whose structure encloses potential internal resources for satisfaction of interests. Total Resource Equations could be determined for each of these two-dimensional interest relationships and then those equations could be integrated, simplified, and reduced into a single Combined Total Resource Equation as arrived at here.

Can you use the result or method to solve some other problem?

Yes. For example, if I were given a supplier, a manufacturer, and a retailer, I could use the *result* (or part of the result) of the problem just solved (shared labor, parts provided at no charge, most efficient replacement, bearing of recall costs, etc.) to find the formula of resources satisfying the parties' interests where

right triangles in solving any number of three-dimensional geometrical problems, where the use of the Pythagorean theorem can yield a useful line length or formula for a line length toward solving the overall problem.

I can also imagine many more problems where the *result* and the *method* of the present problem can aid solution. For example, I can solve many problems relating to *pyramids* when I correctly perceive that if I draw the *four* diagonals of the parallelepiped, six pyramids are formed which have the six faces of the parallelepiped as bases, the center of the parallelepiped as the common vertex of the pyramids, and the semidiagonals of the parallelepiped as the edges of the six pyramids.

the manufacturer installed parts incorrectly because of the supplier's erroneous schematics.

Also, I could use the *method* of introducing suitable two-dimensional buyer-seller relationships (i.e., consumer-new car dealer) in solving any number of three-or-more dimensional problems where the use of similar resources for solutions can yield useful resource components for a formula or an equation for solving the overall problem.

I can also imagine many more problems where the *result* and the *method* of the present problem can aid solution. For example, I can now help solve many problems relating to supplying of faulty goods or services, hidden defects, and warranties including those involving construction contracts, real estate contracts, products liability, partnership disputes, landlord-tenant, securities contracts, bank loans, health services, just to name a few.

Now that an example of how Descartes' Rules for the Direction of the Mind and his analytic method can assist in solving problems in mediation and negotiation has been considered, the focus shifts to a part of the method, applicable in mediational problem solving, not fully explained in the example above—drawing a figure. This technique is an aspect of geometrical imagineering.

IV. A PROPOSED PARADIGM FOR MEDIATIONAL PROBLEM SOLVING: A GEOMETRIC IMAGINEERING APPROACH

A. *Geometric Modeling—General Geometrical Perceptions of Real Life Situations*

More than a century ago, Edwin A. Abbott, a headmaster at the City of

London School, published a slim volume entitled *Flatland*.¹²² A pointed satire, the book reflected the widely debated social issues of Victorian Britain, including women's rights, the teaching of higher-dimension (non-Euclidean) geometries and advanced mathematics, and the relationship between scientific proof and religious faith. He accomplished his satirical objectives by creating a geometrical fantasyland in which Flatland's creatures were two-dimensional geometrical figures: Flatland women were Straight Lines; lower-class men were Isosceles Triangles; Squares made up the professional class (of which the narrator was a member); Nobles were polygons with six or more sides; and Priests, the highest-ranking members, were perfect Circles.¹²³ The point of the stories of the escapades of the Flatland creatures was, in part, to emphasize their limited vista. Residents of a three-dimensional world could easily appreciate the two-dimensional Flatlanders' limitations. For example, when viewed from directly above by a three-dimensional creature, a coin sitting on a table clearly looks circular. As the angle of view descends closer to the plane of the table, the coin appears more oval in shape. At the Flatlander's level, along the table's surface, the oval reduces to nothing more than a straight line.

122. EDWIN ABBOTT, *FLATLAND: A ROMANCE OF MANY DIMENSIONS* (Barnes & Noble Books, A Division of Harper & Row, Publishers, 5th ed., 1983).

123. See PETERSON, *supra* note 104, at 82-85. See generally ABBOTT, *supra* note 122.

In the 1940s, Heider and Simmel conducted a classic study using geometric figures. They showed observers a film in which two triangles of different sizes ("T" and "t") and a circle ("c") were seen to move in the vicinity of a rectangular frame (the house) with a moveable flap on one side (the door). The first few frames depicted movements of the geometrical figures as shown next:

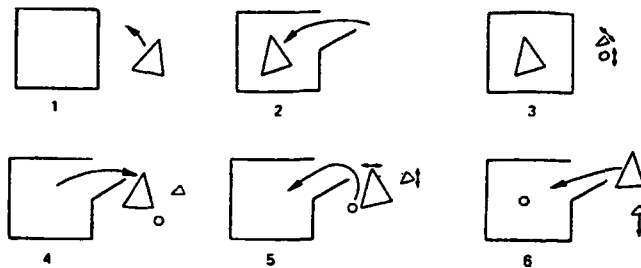


Figure 3

(Figure 3 is reprinted from VICKI BRUCE & PATRICK R. GREEN, *VISUAL PERCEPTION PHYSIOLOGY, PSYCHOLOGY AND ECOLOGY* 303 (1985) and is reprinted with the permission of the publisher, Lawrence Erlbaum Assoc., Ltd., United Kingdom.)

They showed the film to 34 subjects who were asked simply to describe what happened in the pictures. All but one described the film in terms of the movements of animate beings. One typical description was:

A man has planned to meet a girl and the girl comes along with another man. The first man tells the second to go; the second tells the first, and he shakes his head. Then the two men have a fight, and the girl starts to go into the room to get out of the way and hesitates and finally goes in. She apparently does not want to be with the first man.

See *id.* at 302-03.

This, of course, represents the permanent limited perspective of the Flatlanders. Using such examples throughout the book, Abbott not only heightened the reader's understanding of narrow-minded and sometimes ludicrous perspectives of some elements of the Victorian social and political establishment, but he also communicated key ideas regarding needed changes in educational curricula, including the teaching of projective geometry and other important mathematical concepts.

One commentator has observed that "*Flatland* raises the fundamental question of how to deal with something transcendental, especially when recognizing that one will never be able to grasp its full nature and meaning. It's the kind of challenge that pure mathematicians face when they venture into higher dimensions."¹²⁴ In many respects, this is the same challenge that mediators and negotiators face when they venture into the higher dimensions of complex collaborative negotiation. For some mediators and negotiators, the transcendental nature of such problem solving may be more easily overcome through use of the metaphor of algebra and geometry combined with graphic representation. Others will benefit merely by drawing geometric configurations or patterns while concentrating, mentally, on mathematical and perceptual concepts. This I call the geometric imagineering approach to collaborative problem solving.¹²⁵ An example of this approach appeared in solving the Tranex-Novatron problem in Part III. The following is an explanation of some of the various aspects of this still-evolving visualization approach to solving real-life problems.

B. Geometric Imagineering

My development and use of geometric imagineering in mediational problem solving was, in part, inspired by Rudolf Arnheim's book, *Visual Thinking*.¹²⁶ Proceeding from the premise that truly productive thinking takes place in the realm of mental imagery, Arnheim theorized that it is in the perception of *shape*

124. PETERSON, *supra* note 104, at 85. Benedictus Spinoza, a rationalist philosopher (*see supra* note 66) and a contemporary of Descartes, ventured into higher dimensions of mathematics and philosophy in producing his *magnum opus*, the *Ethics*, which he drafted in the 1660s but was not published until after his death in 1677. This remarkable work applied a geometrical presentation to five topics corresponding to the five parts of the treatise: God, the human mind, the affects (psycho-physical states of humanness including emotions), human subjection, and human freedom. Each part opened with a numbered list of "definitions" followed by a numbered list of "axioms." Following those were a long list of numbered "propositions." After each proposition was a demonstration showing how the particular proposition was derived either directly or indirectly from the definitions and axioms. *See* COTTINGHAM, RATIONALISTS, *supra* note 2, at 50.

125. The word "imagineering" here connotes the mental and/or graphical construction and use of visual images in problem solving.

126. RUDOLF ARNHEIM, *VISUAL THINKING* (1969).

that the beginnings of concept formation occur. To Arnheim, perception is fitting together stimulus material by means of templates of relatively simple geometrical shapes.¹²⁷ Arnheim observes this:

The variety of available visual shapes is as great as that of possible speech sounds, but what matters is that they can be organized according to readily definable patterns, of which the geometrical shapes are the most tangible illustration. The principal virtue of the visual medium is that of representing shapes in two-dimensional and three-dimensional space, as compared with the one-dimensional sequence of verbal language. This polydimensional space not only yields good thought models of physical objects or events, it also represents isomorphically the dimensions needed for theoretical reasoning.¹²⁸

Thus, perception can be said to involve the design of information structures or mental constructs. Before a solution can be reached in negotiation and mediation, the initial information structures of the parties have to be transformed into new information structures whose end design is mutually acceptable and pleasing, structurally and aesthetically, to the parties. Negotiators and mediators can be aided in reaching an agreed end design by generating, visualizing, and transforming geometric information structures and by employing certain visualization techniques to identify and to effect transformations appropriate to the situation.

1. Euclidean Visualization Models

The early Greek mathematicians were motivated by a desire to keep geometry, the literal definition of which is "earth measurement,"¹²⁹ simple, harmonious, and aesthetically appealing.¹³⁰ As noted in Part II.B.1., they limited their geometry to a consideration of a straight line and a circle, corresponding to the *physical* counterparts of those two figures, the straight edge and the compass. Even the conic sections (ellipse, circle, hyperbola, and parabola) were obtained by passing a plane through a cone, a figure which itself was generated by moving a straight line.¹³¹ Pythagoras was perhaps the most influential in determining the early nature and content of Greek geometry. His theorem for determining the length of the hypotenuse of a right triangle (the

127. *Id.* at 27.

128. *Id.* at 232.

129. Marjorie Senechal, *Shape*, in ON THE SHOULDERS OF GIANTS: NEW APPROACHES TO NUMERACY 139 (Lynn A. Steen ed., 1990) [hereinafter GIANTS].

130. MORRIS KLINE, MATHEMATICS IN WESTERN CULTURE 51 (1953).

131. *Id.*

square of the hypotenuse is equal to the sum of the squares of the sides—depicted graphically in Figure 4) had a profound impact on the development of mathematics generally in the centuries to follow.¹³²

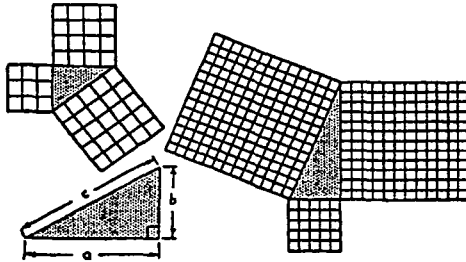


Figure 4

Plato, whose Academy produced some of the most famous philosophers, mathematicians, and astronomers of their age, believed geometry to be a pre-requisite foundation to education. It is said that over the archway to the entrance of Plato's Academy was inscribed the statement, "Let no one ignorant of geometry enter here."¹³³ It was, of course, in recognition of Plato that the five Platonic solids were named. These five geometrical figures, shown below, are three-dimensional solids whose polygonal surfaces are all congruent and whose corners all meet at the same angle.¹³⁴

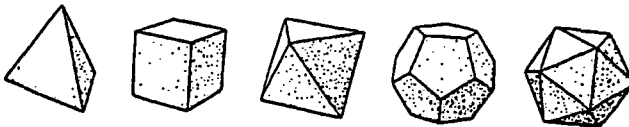


Figure 5

But it was Euclid who unified the work of many mathematicians in one

132. *Id.* at 40-41. Figure 4 is reprinted from *Mathematics and the Imagination* by Edward Kasner and James Newman. Copyright © 1989 by Ruth G. Newman. Reprinted by permission of Microsoft Press. All rights reserved. KASNER & NEWMAN, *supra* note 104, at 121

133. PAULOS, *supra* note 41, at 271.

134. *Id.* at 181. Figure 5 is reprinted from STEWART T. COFFIN, *THE PUZZLING WORLD OF POLYHEDRAL DISSECTIONS 4* (1991) by permission of Oxford University Press, United Kingdom.

masterful treatise called the *Elements*.¹³⁵ From a few wisely chosen axioms, Euclid deduced all the important conclusions (roughly 500 theorems) of the Greek masters of the classical period. These axioms, ten in all, were so obviously true that generations have been willing to agree with them as the basis for further reasoning and secured the construction of the whole system of geometry.¹³⁶ Two of Euclid's early theorems warrant special, but brief, comment here: one, because it concerns the question of *congruency* of component triangles of the isosceles triangle (a triangle with two sides equal); the other, because it concerns *relationship* and *similarity* of shape. Both of these concepts are critical both to the collaborative negotiation process¹³⁷ and to the explanation of my proposed technique of Euclidean visualization described below.

Consider the isosceles triangle, ABC , in Figure 6.¹³⁸

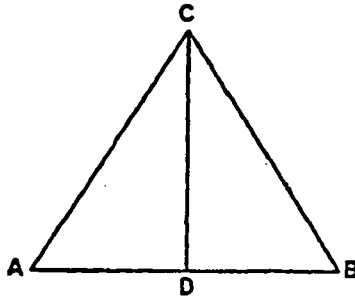


Figure 6

Without detailing the rigor of Euclid's proof, Euclid had previously demon-

135. KLINE, *supra* note 130, at 42. It is reported that Abraham Lincoln studied and nearly mastered the six books of Euclid while a member of Congress. One commentator noted:

He began a course of rigid mental discipline with the intent to improve his . . . powers of logic and language. Hence his fondness for Euclid, which he carried with him on the circuit till he could demonstrate with ease all the propositions in the six books; often studying far into the night, with a candle near his pillow, while his fellow-lawyers, half a dozen in a room, filled the air with interminable snoring.

BELL, *supra* note 8, at xvi.

136. KLINE, *supra* note 130, at 43-44. Typical of his axioms are: "It shall be possible to draw a straight line joining any two points; It shall be possible to draw a circle with given center and through a given point; The whole is greater than any of its parts." *Id.* at 44.

137. See generally ROGER FISHER & SCOTT BROWN, *GETTING TOGETHER: BUILDING RELATIONSHIPS AS WE NEGOTIATE* (1989).

138. Figure 6 is reprinted from *MATHEMATICS IN WESTERN CULTURE* by Morris Kline on page 44. Copyright © 1953 by Oxford University Press, Inc.; renewed 1981 by Morris Kline. Reprinted by permission of the publisher.

strated that any two triangles which have two sides and the included angle of one equal to two sides and the included angle of the other are congruent (identical in size and shape). Thus, in Figure 6, assuming CD bisects angle C , two component triangles are formed, ADC and DCB . $AC = BC$ by the definition of an isosceles triangle; angle $ACD = \text{angle } DCB$ by the definition of bisector of an angle; and the two component triangles have a common side CD . Therefore, all elements of Euclid's congruency theorem are satisfied. Thus, triangles ADC and DCB are congruent.¹³⁹ Euclid also demonstrated that two triangles are congruent if the sides of one are equal to the sides of the other.

Then Euclid asked, if two triangles are not equal, what significant *relationship* may they bear to each other and what geometric properties can they have in common? He called figures of unequal size but of the same shape, "similar figures." With respect to triangles, similarity meant that the angles of one were equal to the corresponding angles of the other. From this he concluded that the ratio of any two corresponding sides is constant. For example, consider the triangles in Figure 7. If ABC and $A'B'C'$ have equal angles, they are similar. If they are similar, then $AB/A'B'$ equals $BC/B'C'$.¹⁴⁰

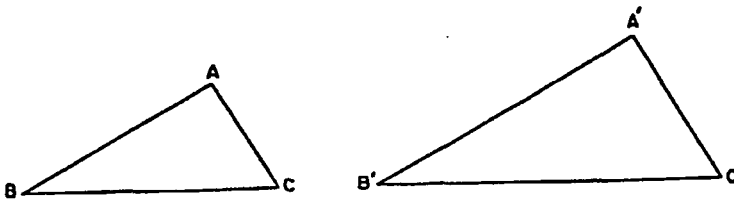


Figure 7

Unlike the triangles depicted in Figure 7, if figures have neither shape nor size in common, they may, of course, have the same area, or in geometrical terms, be "equivalent." Or they may be inscribable in the same circle. The number of possible relationships and questions that can be raised with respect to geometrical figures is indeed infinite.¹⁴¹ Thus, in designing a geometrical model for a particular negotiation or mediation, it is important to define only a few simple geometric shapes and a few related characteristics. Otherwise, the model becomes tedious and impractical. The description of one of my designs—and I emphasize that this is only one of myriad possible designs for a Euclidean

139. *Id.* at 44-45.

140. Figure 7 is reprinted from *MATHEMATICS IN WESTERN CULTURE* by Morris Kline on page 44. Copyright © 1953 by Oxford University Press, Inc.; renewed 1981 by Morris Kline. Reprinted by permission of the publisher.

141. *Id.* at 46.

visualization model of a particular negotiation or mediation—is presented here.

The following Euclidean visualization model could be developed in relation to the Tranex-Novatron scenario discussed in Part III.C. It employs geometric figures as a check to insure that all of the possible relationships of the parties to the dispute and their respective compatible interests are identified for analysis. Other Euclidean models could be designed, for example, to identify: the priority of compatible interests; the equivalence of resources for substitution; the incongruity of resources and interests; commonality of resources by constructing (combining), disassembling (dissecting), decomposing and recombining to form new constructs, or rotation; or similarity, dissimilarity, symmetry, patterns, etc. Some of the visualization techniques related to designing and transforming other types of Euclidean models are discussed in Part IV.¹⁴²

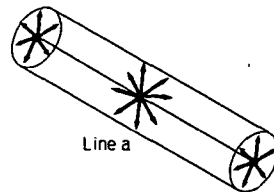
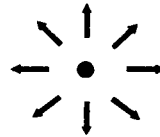
The process of developing a Euclidean visualization model to aid in the solution of any negotiation or mediation problem involves, in the first instance, the *creation* of a set of rules and corresponding graphic representations (sketches of geometrical figures with notations). The rules and related sketches that I developed to analyze the multiple party relationships and the multiple interests in the Tranex-Novatron scenario appear below.

Rules

1. A point = vertex = the endview of a line with emanating radials (vectors) of interests.

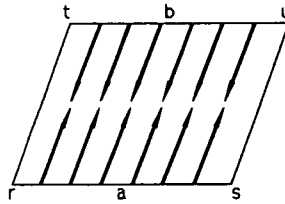
2. A line identifies the total vectors of interests of a party.

Sketches

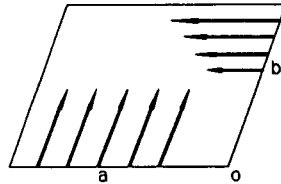


142. For techniques for stretching the imagination to construct four dimensional figures, see DAVIS & HERSH, *supra* note 7, at 400-05.

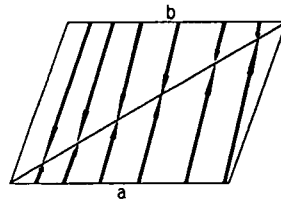
3. No vertex may serve as the origin or endpoint for more than one party's interest—i.e., the interests of only one party may be defined by a single vertex. In the sketch at right, the limits of party *a*'s interest are defined by the vertices "r" and "s." The limits of party *b*'s interests are defined by vertices "t" and "u." The sketch shows the plane of *mutually compatible* interests of varying priorities between parties *a* and *b*.



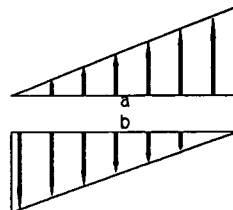
4. Thus, the sketch at right would be impossible in this model because point "o" would be a common vertex between parties *a* and *b*. This sketch represents a plane of *conflicting* interests between parties *a* and *b*.



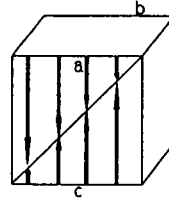
5. A diagonal drawn so as to make a *positive* angle with the line of interests of any party is defined as the *line of satisfactory resources*. This line represents: party *a*'s resources to satisfy its own interests; party *b*'s resources to satisfy its own interests; party *a*'s resources to satisfy *b*'s interests; party *b*'s resources to satisfy party *a*'s interests. A diagonal cuts across interests, stabilizes and strengthens the relational structure and draws parties together.



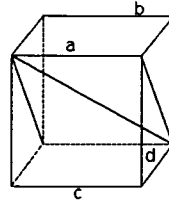
6. The figure at right shows parties *a* and *b* represented as right triangles with included interests and a connector diagonal of satisfactory resources. Each party can connect its radiating triangular plane of interests and resources with any other party *related* to the dispute.



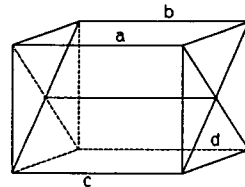
7. The figure at right shows how another triangular plane of interests and resources of party *a* can connect with a triangular plane of interests and resources of party *c*.



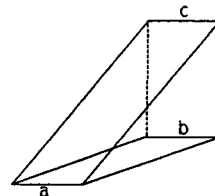
8. The figure at right shows how party *a*'s triangular plane of interests and resources can connect with that of party *d*.



9. The figure at right shows that the respective planes of interests and resources of all four parties intersect in a line that defines the opportunities for optimum solutions.



10. The figure at right (one-half of a cube) shows the interests/resources relationship structure for analyzing the relationships of three of the four parties.



These sketches could be used by the mediator of the Tranex-Novatron to help him or her focus on the relationships and resources of certain parties, to track the progress of the mediation, to identify the potential for cross-relationships, to see the "whole picture" of inter-relationships, and to help him or her not to lose sight of opportunities for optimal solutions.

2. Cartesian Visualization Models

The Cartesian visualization models consist of a broad category of visual modeling devices for use in mediational problem solving, including sketches, graphs and graphical representations except those described above with respect to the Euclidean visualization models. Examples of the Cartesian modeling devices include intuitive co-ordinate geometry, planar and nonplanar graphs,

decision trees,¹⁴³ maps, and matrices. The number and types of Cartesian modeling devices available are limited only by the imagination of the individual problem solver.

a. Intuitive Co-ordinate Geometry

The modeling device of intuitive co-ordinate geometry is similar, conceptually, to Descartes' analytical geometry, but instead of requiring precise plotting of points to form curves and geometric figures, it is intuition-based. It is used primarily as an aid to intuit the nature of parties and their orientation, and to predict the intersections of their interests and resources. Problem solvers with only a minimal understanding of analytical geometry are able to derive perceptual benefit from its use and application. And obviously, those highly intuitive problem solvers who are, additionally, adept at analytical geometry will derive great benefit from its use and applications. The primary components of this device are "signature" equations, their related traces, and the definitions of the geometric figures (curves) identified by the signature equations and their related traces.

To employ the device, one must first realize that to each and every curve there belongs an *equation that uniquely defines the points of that curve* and no other curve. That is, each equation involving the variables x and y can be pictured as a curve by interpreting x and y as co-ordinates of points determined by the insertion of numerical values for x and y . I refer to these equations as "signature" equations. The signature equations of the circle, line, parabola, and ellipse, and their related traces, appear in Figure 8.¹⁴⁴

143. Decision trees as used in negotiation are described in HOWARD RAIFFA, *THE ART AND SCIENCE OF NEGOTIATION* 70-77 (1982).

144. In Figure 8, the figure of the circle is reprinted from *MATHEMATICS IN WESTERN CULTURE* by Morris Kline on page 169. Copyright © 1953 by Oxford University Press, Inc.; renewed 1981 by Morris Kline. Reprinted by permission of the publisher. The figures of the line, parabola, and ellipse are reprinted from *BEYOND NUMERACY* by John Allen Paulos (at 12). Copyright © 1991 by John Allen Paulos. Reprinted by permission of Alfred A. Knopf, Inc.

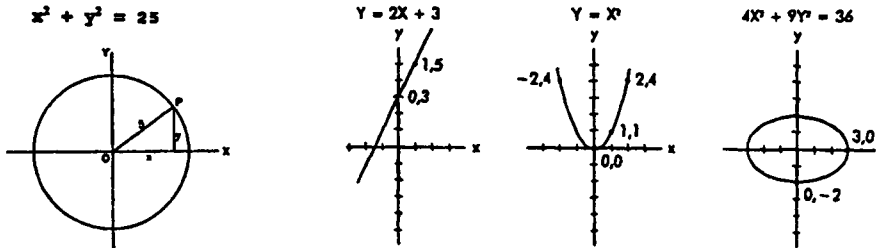


Figure 8

Definitions of the circle, line, parabola, and ellipse are as follows:

- circle: a closed curve, of which the locus of every point is equidistant from a given fixed point, the center¹⁴⁵
- line: a straight, one-dimension locus of points of infinite length and no thickness¹⁴⁶
- parabola: a curve that consists of the locus of points that are at equal distance from a fixed point (a focus) and a fixed straight line (the directrix)¹⁴⁷
- ellipse: a closed curve consisting of a locus of points located in a plane so that the sum of the distances of each point in the locus from two fixed points (the foci) is a constant¹⁴⁸

I have used the intuitive co-ordinate geometry device in mediations as a visual aid to keep me conscious of the types of solutions that may be available in a case. For example, a sketch of a parabola, symmetrical on the vertical axis, with the focus and directrix indicated, and line segments of equal length drawn from the focus to the curve and from the curve to the directrix, serve as a constant reminder to me of the opportunities for an optimal (or even super-optimal) solution. The equation $y = x^2$ and its related trace (depicted in Figure 8) help me to keep in mind that the value of resources to any party is relative. That is, what is of small value to one party may be of great value to another.

145. BOROWSKI & BORWEIN, *supra* note 107, at 81.

146. *Id.* at 343.

147. POLYA, HOW TO SOLVE IT, *supra* note 99, at 86.

148. PAUL K. REES, ANALYTIC GEOMETRY 73 (3d ed. 1970).

The equation for the parabola and the related trace represents the situation where defendant x need only give a small value of a particular resource, yet the value to plaintiff y is great (i.e., the square of the value to x). The locus of points comprising the parabola is, of course, the locus of opportunities for mutually satisfactory solutions. Any solution chosen by the parties will seem fair to them because each will believe it to be a "good deal." Any neutral observer will also believe it to be fair because any solution chosen will be equidistant from a point (focus) on the curve's vertical line of symmetry (which could represent regularity, utility, and durability) and from a base line (directrix) of measurement, for example, dollar value.

By using a sketch-graph of a circle and an inscribed right triangle (as shown in Figure 8), together with a signature equation for a circle (e.g., $x^2 + y^2 = 25$), I have been able to analyze the various combinations of resources which could be proposed by co-defendants (say x and y) in a case, particularly where it has been agreed by plaintiffs and defendants what overall value in settlement the plaintiffs should receive. In such case, the radius represents the constant—the agreed value of resources which will flow to plaintiff. The movement of the radius across the upper right-hand quadrant of the circle creates an array of potential resource contributions by the two defendants. When the radius is at a forty-five degree angle with the horizontal axis, the defendants would be contributing equal value. However, their culpability might be different, so that might not be the appropriate solution. Considering the culpability factor, the solution would be above or below the radius when positioned at a forty-five degree angle, depending on which defendant was more culpable. If the defendants' non-monetary resources to settle the case have differing value to the plaintiffs, then an elliptical sketch-graph could be used to analyze the possibilities for solution.

b. Planar and Nonplanar Graphs

Planar and nonplanar graphs can be used to determine ways to avoid conflict. Try to solve the following problem mentally, without a sketch. If this proves difficult, try sketching it on a piece of paper and then check your solution with the one given below.

You have three employees (A , B , and C) in your department store who argue with one another every time they come in contact. Between 9:00 and 10:00 a.m. every day, each of them has to go to points a , b , and c in the store to pick up merchandise for restocking the shelves. You want to avoid conflict by devising routes for them which will guarantee that they will not cross paths when they are walking to and from points a , b , and c . Is it possible to devise a solution which

meets that condition?¹⁴⁹

The potential for conflict is shown in Figure 9 on the left, and a planar graph solution is shown on the right:¹⁵⁰

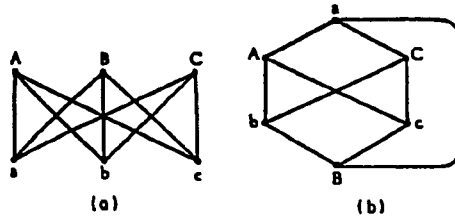


Figure 9

Note that the planar graph solution is only partially satisfactory because it allows one crossover and therefore does not completely avoid conflict. A fully-satisfactory solution can be obtained only by using a nonplanar graph. But the planar graph solution would be highly effective in suggesting ways to *minimize* conflict.

c. Matrices

Matrices can be used as modeling devices in negotiation and mediation to aid in the solution of a variety of problems. At a minimum, they can be used to determine compatible interests, combined resources, and resources to satisfy interests.¹⁵¹ The example which follows demonstrates how *several* types of graphic representations, including the matrix, can be used to aid in problem solving.¹⁵²

Assume that the problem confronting the mediator can be represented by the maze in Figure 10.

149. This problem is an adaptation of a problem which appears in JAY KAPPRAFF, *CONNECTIONS: THE GEOMETRIC BRIDGE BETWEEN ART AND SCIENCE* 127 (1991).

150. Figure 9 is reprinted from JAY KAPPRAFF, *CONNECTIONS: THE GEOMETRIC BRIDGE BETWEEN ART AND SCIENCE* 127 (1991). Copyright © 1991 by McGraw-Hill, Inc. Reproduced with the permission of the publisher, McGraw-Hill, Inc., New York.

151. For an example of the use of matrices in determining resources to satisfy specific and general needs of parties to a negotiation, see COOLEY, *APPELLATE ADVOCACY*, *supra* note 63, at 165-66.

152. This example is adapted from DAVIS & HERSH, *supra* note 7, at 130-33. Figures 10-13 are reprinted from PHILIP J. DAVIS & REUBEN HERSH, *THE MATHEMATICAL EXPERIENCE* 131-33 (1981) with the permission of the publisher, Birkhäuser, Boston.

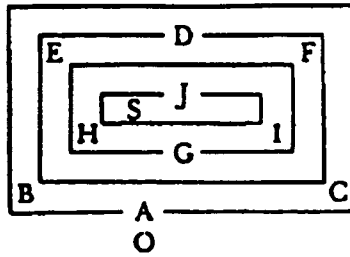


Figure 10

An abstract description of the problem might be as follows:

From the outside *O*, go to the point of entry, *A*. Opening *A* leads to two options, *B* and *C*, both of which lead to opening *D*. Opening *D*, leads to two options, *E* and *F*, both of which lead to opening *G*. Opening *G* leads to two options, *H* and *I*, both of which lead to opening *J*. The opening *J* leads to the solution *S*.

This set of circumstances can also be graphically represented as shown in Figure 11.

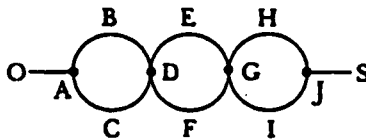


Figure 11

The graphical representation above is equally as useful as the maze-graph, and perhaps provides a simpler depiction with which to work. This graph can be simplified even further, with dots representing the alternatives, and designating the decision points with letters as shown in Figure 12.

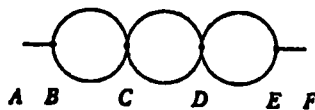


Figure 12

This information can also be represented by a matrix, the so-called "incidence matrix" of Poincaré,¹⁵³ yielding a description in arithmetic fashion as demonstrated in Figure 13.

	A	B	C	D	E	F
A	0	1	0	0	0	0
B	1	0	2	0	0	0
C	0	2	0	2	0	0
D	0	0	2	0	2	0
E	0	0	0	2	0	1
F	0	0	0	0	1	0

Figure 13

Figure 13 shows how intuitive graphing and the precision of Cartesian numeracy can combine to yield an aid to a negotiated or mediated solution of a problem.

3. Visualization Techniques

To employ geometric imagineering effectively, one has to develop an interest in the architecture of problem solving (i.e., in the design of problems and solutions) and in developing skill in using visualization techniques.¹⁵⁴ These visualization techniques lie at the intersection of mathematics and psychology, having their roots in both disciplines. They can be used in developing Euclidean or Cartesian visualization models as described above. They can be used independently to perceive, and to help others to perceive, structures of information in a new way or from a different perspective in order to *transform* mental constructs of inter-personal *situations* toward achieving overall congruence or equivalence of terms in the development of an acceptable Total Resource Equation.¹⁵⁵

153. See generally DAVIS & HERSH, *supra* note 7.

154. For an interesting discussion of design in relation to problem solving in general, see THOMAS HINRICHS, *PROBLEM SOLVING IN OPEN WORLDS* (1992).

155. See Chapter 2, "On the Formation of Structural Descriptions and Meanings for Objects and Events," in DAVID R. OLSON & ELLEN BIALYSTOK, *SPATIAL COGNITION: THE STRUCTURE AND DEVELOPMENT OF MENTAL REPRESENTATIONS OF SPATIAL RELATIONS* 16-31 (1983).

a. Combining

In collaborative negotiation, the parties can sometimes agree on a resource to satisfy their mutual interests, but are unable to determine the existence or availability of such resource. In such a situation, the visualization technique of combining, and its close relative "constructing," can be of benefit to make the resource "materialize." The technique of combining requires a primary skill of perceiving equivalents. A geometric volume construct for combining three resources to achieve a single total resource appears in Figure 14:¹⁵⁶



Figure 14

To perceive the availability of the desired single resource (the cylinder on the left), mediators and negotiators would have to realize that the combined volume of the three cones on the right is equivalent to the cylinder's volume so long as the base and height of each of the three cones is identical to the base and height of the cylinder. In negotiation terms, the desired single resource would be available so long as the three other resources satisfied certain conditions or constraints. Similarly, a geometric volume construct for combining three resources to obtain two desired resources appears in Figure 15:¹⁵⁷

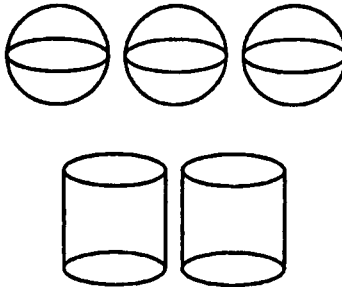


Figure 15

156. Figure 14 is reprinted with permission from *ON THE SHOULDERS OF GIANTS: NEW APPROACHES TO NUMERACY 15* (Lynn A. Steen ed., 1990). Copyright 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, D.C.

For a discussion of dissection of plane geometric models, see H. MARTIN CUNDY & A.P. ROLLET, *MATHEMATICAL MODELS* 19-27 (1961).

157. Figure 15 is reprinted with permission from *ON THE SHOULDERS OF GIANTS: NEW APPROACHES TO NUMERACY 16* (Lynn A. Steen ed., 1990). Copyright 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, D.C.

The volume of the two cylinders is equivalent to the volume of the three spheres so long as radius and height of cylinders match those of the spheres.

Geometrical *volume* constructs, as just discussed, are quite useful in determining the components of a *particular* single desired resource where the parties are adamant that only that particular resource can satisfy their interests. However, where parties are willing to see the single desired resource in a different *shape* or *form*, perception of equivalents can be powerfully expanded by using a geometrical *construction* model rather than a geometrical *volume* model.¹⁵⁸ For example, assume that the negotiators have available between them the following six resources represented geometrically by the block figures in Figure 16.¹⁵⁹

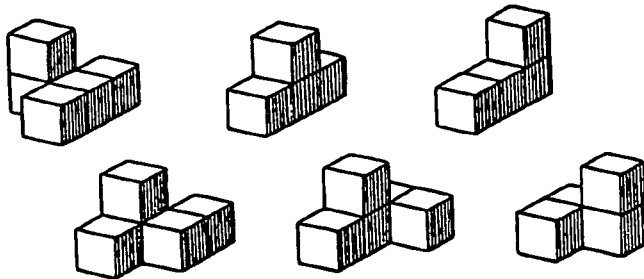


Figure 16

Further assume that they agree tentatively that their interests could be fully satisfied if they could find a way to construct a package solution, represented geometrically by a $3 \times 3 \times 3$ block cube. In attempting to construct the package cube solution (see the first design in Figure 17), the negotiators may discover any number of symmetrical constructions from the six block resources (see the other seven designs in Figure 17) that they can agree to be equivalent and satisfactory.¹⁶⁰

158. I have used children's snap blocks (LEGOS) to enhance the geometric construction model in teaching law students concepts of problem design and solution design and of the differences between facts and arguments. This teaching technique is described more fully in COOLEY, APPELLATE ADVOCACY, *supra* note 63, at § 1:07.

159. Figure 16 is reprinted from STEWART T. COFFIN, THE PUZZLING WORLD OF POLYHEDRAL DISSECTIONS 44 (1991) by permission of Oxford University Press, United Kingdom.

160. Figure 17 is reprinted from STEWART T. COFFIN, THE PUZZLING WORLD OF POLYHEDRAL DISSECTIONS 45 (1991) by permission of Oxford University Press, United Kingdom.

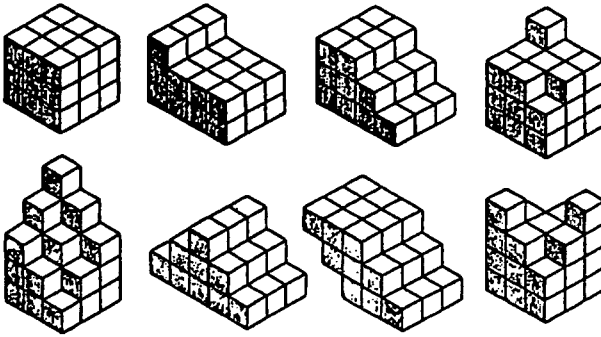


Figure 17

Actually, the six block resources represented geometrically in Figure 16 can be combined into literally hundreds of symmetrical composites, only eight of which are depicted in Figure 17.¹⁶¹ The relevance of symmetry of solution is discussed in more detail below in Part IV.B.3.g.

b. Disassembling

In visualizing the division of a single resource to produce several resources to satisfy the negotiators' separate interests, the technique of disassembling can be used. Graphically representing the single resource geometrically is the first, and perhaps most critical, imagineering challenge. Whether one uses a plane or three-dimensional regular or irregular polygon to depict the single resource will have a significant impact on the size, shape, and form of the portions of the disassembled resource. If a resource is to be divided into three equal parts, it may be easiest to visualize the resource as a cube. This will facilitate the perception of equal or equivalent portions. The mediators or negotiators then should realize that there may be myriad ways to divide a single resource (cube) into three portions. The one which comes to mind immediately is simply to slice the cube into three equal pieces as you would a cubical block of cheese, yielding three regular parallelepipeds. However, *diagonal* subdivision would yield three equal (congruent) component resources of decidedly different form. This pyramidal *form* may be more desirable than the slice *form* to the negotiators for any number of reasons (depicted in the Figure 18).¹⁶²

161. COFFIN, *supra* note 134, at 43-44.

162. Figure 18 is reprinted with permission from ON THE SHOULDERS OF GIANTS: NEW APPROACHES TO NUMERACY 17 (Lynn A. Steen ed., 1990). Copyright 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, D.C.

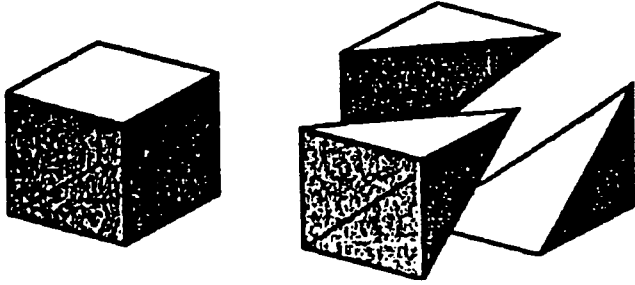


Figure 18

The geometric construct in Figure 18 could be used to depict the result of a partnership dissolution where each of three partners would receive payment in equal *cash* amounts *after sale* of the business. A “three slice” solution could graphically represent a less-preferred division of the total assets of the business into three identical portions. Beginning by visualizing a *rectangular solid* would yield three non-congruent portions of a resource, but all having equal volume as depicted in Figure 19.¹⁶³

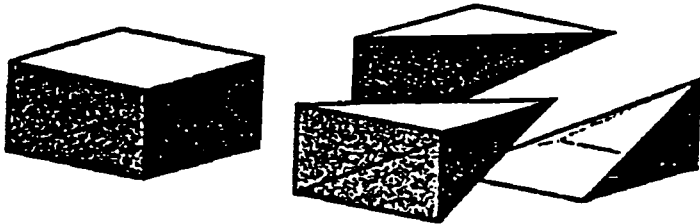


Figure 19

This situation could be replicated in a negotiation setting where all three partners in a partnership dissolution acquire equivalent *values* in the solution, but in different *forms* of partnership assets.

Another way to use the disassembly technique when dealing with an apparent single desired resource is to visualize it as a multi-faceted three-dimensional polyhedron whose faces can be unfolded to form a two-dimensional

163. Figure 19 is reprinted with permission from ON THE SHOULDERS OF GIANTS: NEW APPROACHES TO NUMERACY 18 (Lynn A. Steen ed., 1990). Copyright 1990 by the National Academy of Sciences. Courtesy of the National Academy Press, Washington, D.C.

geometric configuration of regular polygons.¹⁶⁴ An example of the planar configurations resulting from the unfolding of the five Platonic solids (discussed earlier in Part IV.B.1.) is shown in Figure 20.¹⁶⁵

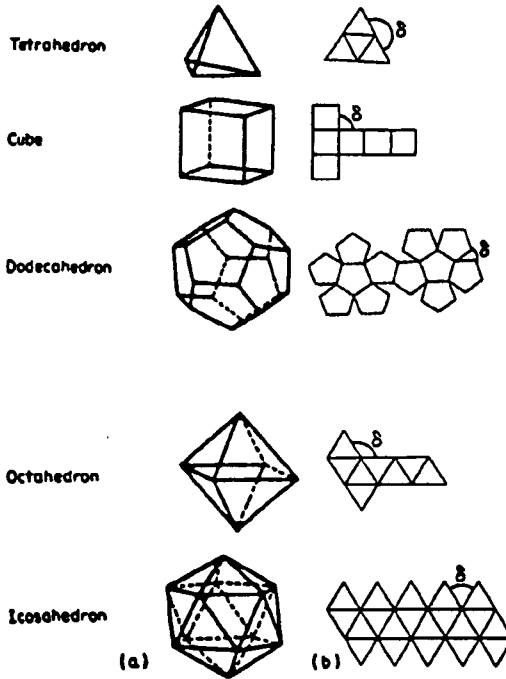


Figure 20

Continuing the analysis of the dissolution of a three-person partnership, Figure 20 shows that in approaching the problem of equal division of the business entity depicted as the tetrahedron, for example, disassembly by unfolding yields a new perspective on the matter. The unfolded tetrahedron (three equilateral triangles surrounding a fourth equilateral triangle) suggests a solution whereby each

164. In working with convex polyhedra, it is helpful to know Euler's Theorem: the sum of the number of faces plus the number of vertices is equal to the number of edges plus two. The simple equation to remember is $F + V = E + 2$. Thus if the value of two of the three terms F, V, E , is known, one can always find the third term. For example, if I count the faces and vertices (points) of a polyhedron and get a total of 10, I know that the number of edges should be 8, i.e., $10 - 2$. GIANTS, *supra* note 129, at 161.

165. Figure 20 is reprinted from JAY KAPRAFF, CONNECTIONS: THE GEOMETRIC BRIDGE BETWEEN ART AND SCIENCE 259 (1991). Copyright © 1991 by McGraw-Hill, Inc. Reproduced with the permission of the publisher, McGraw-Hill, Inc., New York.

partner would immediately take an equal *portion* of the total assets and that they would share over time the proceeds from the remaining portion (the triangle in the center). The center triangle could represent, for example, a building, independently managed, in which they each retained a limited partnership interest and received rental income over an agreed period of years. Other new perspectives and potential solutions could be imagined by examining the geometric configurations generated by unfolding the other four Platonic solids.

c. Disassembling and Recombining to Form a New Construct

Polya points out that in applying the analytic method to solve a problem, it is often necessary to decompose the problem and recombine its elements. He observes:

After having decomposed the problem [identifying its principal parts and going back to definitions if necessary], we may try to recombine its elements in some new manner. Especially, we may try to recombine elements of the problem into some new, more accessible problem which we could possibly use as an auxiliary problem.

There are, of course, unlimited possibilities of recombination. Difficult problems demand hidden, exceptional, original combinations, and the ingenuity of the problem-solver shows itself in the originality of the combination. There are, however, certain usual and relatively simple sorts of combinations, sufficient for simpler problems, which we should know thoroughly and try first, even if we may be obliged eventually to resort to less obvious means.¹⁶⁶

The visualization technique of disassembly and recombining is similar to the technique of fractionation described by Edward de Bono in his book, *Lateral Thinking*.¹⁶⁷ There, he illustrates what Polya would describe as certain usual and relatively simple sorts of combinations. Mediators and negotiators, however, need to be prepared to solve difficult problems demanding hidden, exceptional, and original combinations. This sometimes requires extraordinary ingenuity. Practice in the design of problems and solutions using geometric configurations can greatly enhance one's power of perception and imagination in solving difficult negotiation problems. For example, assume that the parties to a negotiation have identified seven mutual interests, any one of which if

166. POLYA, HOW TO SOLVE IT, *supra* note 99, at 77. The manipulations in which the most usual and useful combinations occur are: (1) keep the unknown and change the data and the condition; (2) keep the data and change the unknown and the condition; (3) change both the unknown and the data. *Id.* at 78.

167. See DE BONO, *supra* note 63, at 131-40.

satisfied, would produce a satisfactory solution. These interests are represented by the geometric figures pictured in Figure 21.¹⁶⁸

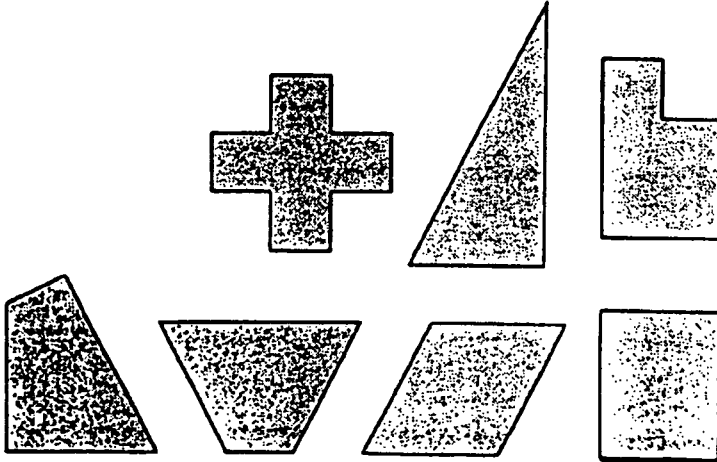


Figure 21

Assume further that the only resource available to satisfy these interests is a set of information and events geometrically represented by a square. As a mediator, how could one show that this resource could be dissected into five parts so that it could be recombined seven separate ways to form a configuration in each recombination that would be *congruent* with each mutual interest identified? This is no simple task. Using a great deal of imagination, one would eventually arrive at the dissection shown in Figure 22.¹⁶⁹

168. Figure 21 is reprinted from STEWART T. COFFIN, *THE PUZZLING WORLD OF POLYHEDRAL DISSECTIONS* 18 (1991) by permission of Oxford University Press, United Kingdom.

169. Figure 22 is reprinted from STEWART T. COFFIN, *THE PUZZLING WORLD OF POLYHEDRAL DISSECTIONS* 18 (1991) by permission of Oxford University Press, United Kingdom.

I have neither the time nor space here to elaborate on the mental processes required to solve this problem, but if the reader is interested in learning more about solving dissection and recombination problems, Stewart Coffin's book, *The Puzzling World of Polyhedral Dissections* is highly recommended. See COFFIN, *supra* note 134.

Every three-dimensional figure composed of points, lines, and planes has a figure dual to it, with planes corresponding to the points of the original figure and vice versa. This is called the principle of duality. For further discussion of this principle as a means of perceiving transformations of geometric figures, see CUNDY & ROLLETT, *supra* note 156, at 116-19; CONSTANCE REID, *A LONG WAY FROM EUCLID* 124-25 (1963); KAPPRAFF, *supra* note 149, at 295-326.

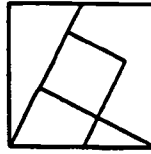


Figure 22

d. Rotating

In some of the practical skills courses that I have taught in law school, I have experimented with the use of a geometric model to emphasize the lawyer's need to examine a problem from all perspectives. Usually, in the first class of the course, I place a simple sculpture of cones, cubes, and a sphere in the center of a tiered classroom. I then ask the students to spend fifteen to twenty minutes drawing what they see. During a short break, I post their finished work so that, when they return, they can see how differently each of them has perceived the same set of information. Because of the large size of the sphere and its position in relation to the students, some students only see one cone, others, two, and still others, all three (or parts of them). For the same reason, some see one cube, others two, and some students draw cubes that are not even there. Because of the tiered nature of the classroom, some students have a top view of the geometric sculpture and others have more of a bottom view. Between those two extremes, students experience a multitude of separate angular perspectives. Although the exercise is simple, it is a powerful learning experience for the students. It teaches each law student that he or she approaches problem solving (views a set of information) from a unique perspective. Each comes to realize that in order to see the "whole picture," one must be willing to *move* from a commitment to one's own unique perspective and view the problem from the unique perspective of others. The more perspectives experienced, the more likely that the student will see the "whole picture." Viewing a problem from the perspectives of others is not always easy for the novice. In order to teach students how to see other perspectives it is sometimes easier to rotate the geometric model. When students have the experience of seeing the sculpture in various rotational frames, it becomes much easier for them to imagine, accurately, what the other side of the sculpture might look like in any stop-frame situation. Practice in mental or actual rotation of geometric models to see structures of information in various perspectives is an important element in developing abilities to see both problems and potential solutions in various perspectives in collaborative negotiation. People have varying degrees of ability

in mentally rotating objects. Consider Figure 23:¹⁷⁰

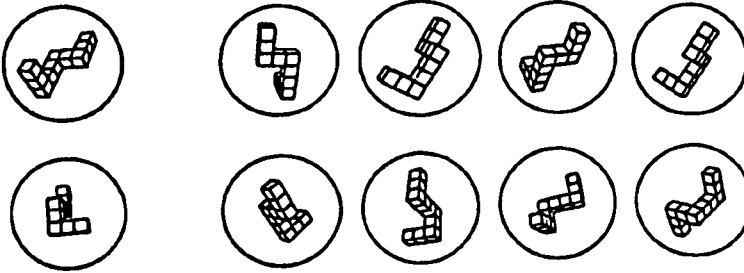


Figure 23

The problem is to determine which of the four alternative views (second through fifth circles) in each row is a rotated view of the original view of the geometric figure appearing in the first circle in each row. Before reading the solution that follows, try to solve the problem. If in the first row, the first and fourth alternative views (the second and fifth circles) were selected, this response is correct. The two distractors in the middle of the correct alternative views are really mirror images of the original view. In the second row, if the second and third alternative views are chosen, this is also correct. The other two distractors are rotated images of other figures. Mental rotation of constructs of information and events is often necessary in negotiation and mediation in order to perceive congruence between interests and available resources.¹⁷¹

e. Identifying Patterns

Patterns occur both in mathematical logic and in visual perception.¹⁷² Discerning patterns in mathematics is a type of *rule finding for relationships*

170. Figure 27 is reproduced with permission of authors and publisher from: Steven G. Vandenberg & Allen R. Kuse, *Mental Rotations, A Group Test of Three Dimensional Spatial Visualization*, PERCEPTUAL AND MOTOR SKILLS, 1978, 47, 599-604 (1978). © PERCEPTUAL AND MOTOR SKILLS 1978. (Figure 27 was cited in accordance with publisher's grant of copyright reprint permission. Legal citation is 47 PERCEPTUAL AND MOTOR SKILLS 599-604 (1978).)

171. Mediators and negotiators who wish to enhance their spatial perception and mental rotation abilities are encouraged to read further about this developing field of psychology. See OLSON & BIALYSTOK, *supra* note 155; Michael C. Corballis, *Mental Rotation: Anatomy of a Paradigm*, in SPATIAL ABILITIES: DEVELOPMENTAL AND PSYCHOLOGICAL FOUNDATIONS 173-97 (1982); Lynn A. Cooper & Roger N. Shepard, *Transformations on Representations of Objects in Space*, in 8 HANDBOOK OF PERCEPTION 105-43 (Edward C. Carterette & Morton Friedman eds., 1978). See generally Roger N. Shepard & Jacqueline Metzler, *Mental Rotation of Three Dimensional Objects*, 171 SCIENCE, Feb. 19, 1971, at 701.

172. Mathematical patterns also define a system of proportion in music. See KAPRAFF, *supra* note 149, at 97-103.

between numbers.¹⁷³ For example, it is simple to find the rule for the following sequence of numbers: 1, 4, 7, 10, 13, etc. The rule is that 3 is added to each term in order to find the next term. But not all patterns are as simple to identify. Sometimes the rules governing numerical relationships are well hidden or, at least, not immediately apparent. Consider the famous Fibonacci sequence of numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, etc.¹⁷⁴ What is the rule which defines the sequential relationship between these numbers? The rule is that each term in the sequence (except the first term) is the sum of its two predecessors.¹⁷⁵

In mathematics, there are also patterns of *proportion*.¹⁷⁶ In explaining his Sixth Rule for the Direction of the Mind, Descartes first set forth the continued proportionals: 3, 6, 12, 24, 48. It is not difficult to determine the rule for finding the proportional relationship between these terms. The rule is that to determine the next term simply multiply the present term by 2; or stated a different way, the ratio between any pair of successive terms in the series is 2:1. But Descartes then asked, what if one was required to find three mean proportionals between 3 and 48 not knowing the sequence just given? The problem would be decidedly more difficult. As Descartes pointed out, to simplify the problem, one could break it down. First, one could look for the single mean proportional between 3 and 48 (i.e., 12). Then, one could look for a further mean proportional between 3 and 12 (i.e., 6), and finally, another between 12 and 48 (i.e., 24). In this way, we would have found a rule for defining a pattern, which rule had to be applied three times to obtain a solution.¹⁷⁷

Identifying patterns in visual perception also involves perceiving *connections* and finding rules for defining relationships and *networks* of relationships. But the relationships with respect to visual perception do not

173. KARL J. SMITH, MATHEMATICS: ITS POWER AND UTILITY 21 (1983).

174. The ratio of a term in the Fibonacci sequence to its predecessor approaches the golden ratio. The Pythagoreans had defined the golden ratio of a line segment to be determined where a line segment AB is divided in between by point C so the ratio of the whole line segment (AB) to the longer part (AC) was equal to the ratio of the longer part (AC) to the shorter part (CB)—i.e., $AB/AC = AC/CB$. In such case, each ratio was said to equal the golden ratio, computed to be 1.61803. Any rectangle whose length-to-width ratio equals the golden ratio is called a golden rectangle. A 3x5 card comes close, with a length-to-width ratio of 1.66666. The Parthenon in Athens can be framed by a golden rectangle, as can many areas inside it. Other Greek art made use of the proportions of the golden rectangle, as have subsequent works of art from da Vinci to Mondrian and Le Corbusier. See PAULOS, *supra* note 41, at 98-99. See also KAPPRAFF, *supra* note 149, at 75-103.

175. PAULOS, *supra* note 41, at 99.

176. Patterns of proportion also exist in architecture. See KAPPRAFF, *supra* note 149, at 1-34.

177. PHILOSOPHICAL WRITINGS, *supra* note 18, at 23-24.

concern numbers, but rather shape. Study Figure 24 for a moment:¹⁷⁸

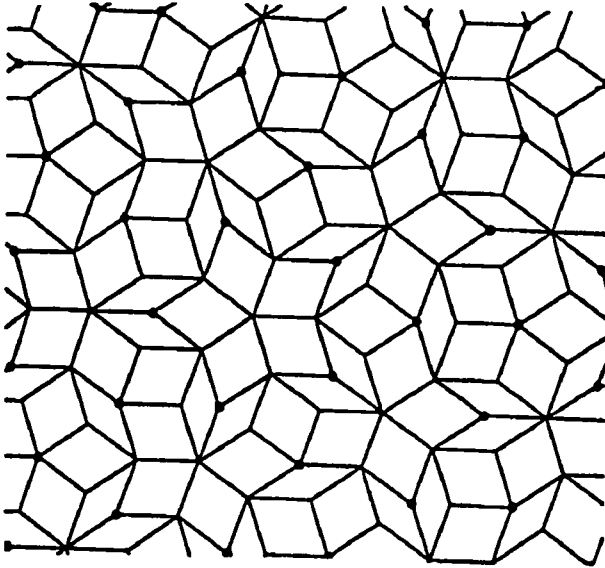


Figure 24

What geometrical patterns does one perceive? Some will see a pattern of 10-sided planar polygons called “decagons.” Some will see innumerable cubes in various orientations. Some will perceive three or more cubes crammed inside a decagon. Others will see several star-shaped figures formed by several diamond-shaped figures. And still others will perceive this whole tiling to be made up of only diamond-shaped figures of various sizes and orientations. And a few will see still other designs. Actually the number of patterns that are perceived is limited only by the number of rules found defining geometric figures and their relationships with the same or other geometric figures.

Pattern is also an important aspect of *stability* in structural design of all types. Some structural patterns are inherently stable, others are not. For example, in experiments involving surface tension of soap film in water, connections were made between the same four points on the surface in four

178. Figure 24 is reprinted from JAY KAPPAFF, *CONNECTIONS: THE GEOMETRIC BRIDGE BETWEEN ART AND SCIENCE* 245 (1991). Copyright © 1991 by McGraw-Hill, Inc. Reproduced with the permission of the publisher, McGraw-Hill, Inc., New York.

different ways to form networks as shown in Figure 25.¹⁷⁹ It was determined that networks (a) and (b) were stable, but that networks (c) and (d) were unstable.

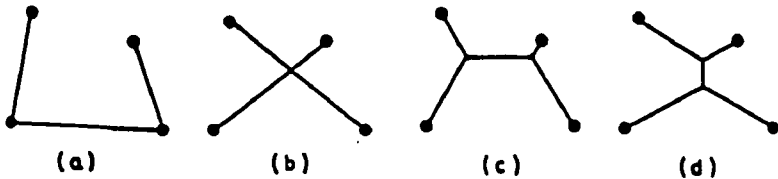


Figure 25

Thus, the relative stability of patterns or networks can depend on the nature of the connections and the orientation of their elements.

These same considerations hold true for identifying and analyzing patterns in the negotiation and mediation settings. But the universe of connections to be perceived and patterns to be identified is much larger and more complex than those in the mathematical or visual-geometrical settings. To facilitate problem solving, mediators and negotiators need to be able to identify connections and multiple patterns existing or potentially existing in each of the following: human relationships,¹⁸⁰ interest relationships, and resource relationships. They must also be able to *integrate* and help others to *integrate* selected patterns of these kinds of relationships into a stable solution mosaic that is acceptable to all concerned. Thus, developing skills in how to perceive connections and how to find rules governing relationships and networks of relationships should be a goal of every negotiator and mediator. Much of this, of course, comes with experience.¹⁸¹

f. Perceiving Similarity

Perhaps the most elementary transformation of a geometric figure is a *similarity* in which the shape of a figure is preserved, but its size is

179. Figure 25 is reprinted from JAY KAPPAFF, *CONNECTIONS: THE GEOMETRIC BRIDGE BETWEEN ART AND SCIENCE* 213 (1991). Copyright © 1991 by McGraw-Hill, Inc. Reproduced with the permission of the publisher, McGraw-Hill, Inc., New York.

180. Patterns of existing and potential human relationships are often quite complex because they include patterns of information structures of past and future events and circumstances.

181. For information on finding rules for identifying human relationships, see FISHER & BROWN, *supra* note 137.

changed.¹⁸² In mathematics, the rules for determining similarity are straightforward. In Euclidean geometry, as pointed out above in Part IV.B.1., two plane figures are similar if their corresponding angles are equal. If they are similar, their corresponding *pairs of sides* are in *proportion*.¹⁸³ In visual perception, however, the rules for perceiving similarity are not as clear-cut or as well defined. Very frequently one perceives similar displays as dissimilar and vice versa. Also, it is not unusual for one to perceive similar displays as *identical*, when they are not identical. Consider Figure 26:¹⁸⁴

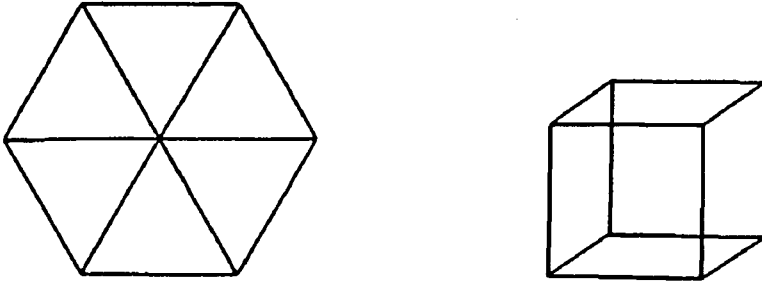


Figure 26

To most viewers, the form on the left appears as a hexagon, while the form on the right resembles a cube. The form on the left, of course, is also a legitimate view of a cube. Even though one person might initially view these forms as totally dissimilar, another person (seeing the hexagon as a cube) might initially view them as quite similar. What about the displays appearing in Figure 27? Are they identical, or merely similar?¹⁸⁵

182. KAPPRAFF, *supra* note 149, at 36.

183. For an excellent discussion of mathematical similarity, see KAPPRAFF, *supra* note 149, at 35-73.

184. The figures in Figure 26 are reprinted from VICKI BRUCE & PATRICK R. GREEN, *VISUAL PERCEPTION PHYSIOLOGY, PSYCHOLOGY AND ECOLOGY* 101 (1985) with the permission of the publisher, Lawrence Erlbaum Assoc., Ltd., United Kingdom.

185. Figures 27 and 28 are reprinted from DAVID R. OLSON & ELLEN BIALYSTOK, *SPATIAL COGNITION: THE STRUCTURE AND DEVELOPMENT OF MENTAL REPRESENTATIONS OF SPATIAL RELATIONS* 4 (1983) with the permission of the publisher, Lawrence Erlbaum Associates, Inc., New York.

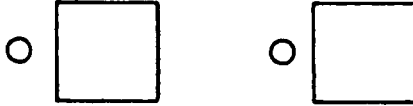


Figure 27

If the displays are superimposed, as shown in Figure 28, they do not fit. They might be similar, but they are not identical.

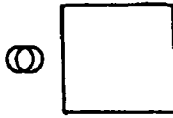


Figure 28

Perceiving similarity in negotiation and mediation is much like that of visual perception. The rules for finding similarity are not well-defined and both similarity (of events, conduct, relationships, interests, resources, etc.) and dissimilarity can be misperceived. Effective mediators and negotiators must be open to see the similar as dissimilar or the dissimilar as similar, depending on which perception is conducive to achieving transformations during the course of designing mutually acceptable solutions.

g. Achieving Symmetry

It has been said that “geometry mediates between the harmony and unity of forms in the natural world and our human capabilities to grasp them with our senses.”¹⁸⁶ Symmetry, or self-congruence, is the aesthetic aspect of geometry which is a meaningful organizing principle in the analysis of structure.¹⁸⁷ One commentator has observed:

Symmetry is a concept that . . . is the common root of artistic and scientific endeavor. To an artist or architect symmetry conjures up feelings of order, balance, harmony, and an organic relation between the whole and its parts. On the other hand, making these notions useful to a mathematician or scientist requires a precise

186. KAPPAFF, *supra* note 149, at 446.

187. GIANTS, *supra* note 129, at 150. Symmetry is pervasive in nature. Nature builds modular structures that organize themselves according to certain rules (e.g., crystals). Repetition of the rules lead to arrangement of modules that are symmetrical. *Id.* See also IAN STEWART & MARTIN GOLUBITSKY, FEARFUL SYMMETRY: IS GOD A GEOMETER? (1992).

definition. Although such a definition may make the idea of symmetry seem less flexible than the artist's intuitive feeling of it, that precision can actually help designers unravel the complexities of a design and see greater possibilities for symmetry in their own work. It can also lead to practical techniques for generating patterns.¹⁸⁸

Symmetry is an effect, not a cause.¹⁸⁹ In geometry, certain transformations exist for producing the effect of symmetry in figures, objects, or mental constructs. These transformations are procedures for "moving things around."¹⁹⁰ Any shape is said to be bilaterally symmetric if there exists some reflection that leaves it invariant, that is, unchanged in appearance.¹⁹¹ But one should be aware that a figure, object, or geometrical construct ("displays") may have multiple symmetries. For example, a five-pointed star fish has ten symmetries, while a human being has two symmetries, and an equilateral triangle has six symmetries.¹⁹² Following are the transformation procedures for determining symmetries in a plane:

translation:	sliding the displays in two distances, one in each of two mutually perpendicular directions;
rotation:	turning the display through an angle around a fixed point;
reflection:	flipping the display over, creating reflections in some line, also referred to as mirror isometries;
glide reflection:	flipping the display over with translation parallel to the

188. KAPPRAFF, *supra* note 149, at 405.

189. GIANTS, *supra* note 129, at 153.

190. STEWART & GOLUBITSKY, *supra* note 187, at 32.

191. This characteristic of invariance is not only an essential aspect of symmetry, but it is also one of the three essences of all types of geometries. A geometry is defined as the study of those properties of figures which remain *invariant* under a given *group* of *transformations*. Transformations can yield different geometries if at least one property of a particular geometric figure remains invariant during the transformations. Thus, a right triangle can be subjected to a series of transformations, and depending on which property or properties of the right triangle remain invariant during the transformations, characteristics of one of the following geometries can be produced during each transformation: Euclidean, projective, elliptic, non-Euclidean, and topology ("rubber sheet geometry"). See REID, *supra* note 169, at 188-91.

192. See STEWART & GOLUBITSKY, *supra* note 187, at 30; MICHAEL LEYTON, SYMMETRY, CAUSALITY, MIND 207 (1992).

mirror.¹⁹³

It should also be noted that designs can be *created* by *destroying* symmetry. Figure 29 shows how the design of a tetrahedron can emerge from successive truncation which destroys the symmetry of the cube.¹⁹⁴

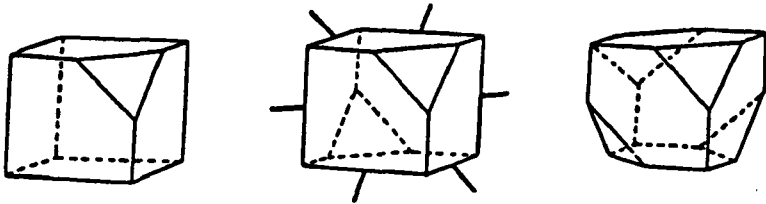


Figure 29

In visual perception, transformations can be combined to produce point similarity symmetry. Figure 30 contains similar polygons and is said to be symmetrical because it is invariant under sixfold rotations about the center.¹⁹⁵

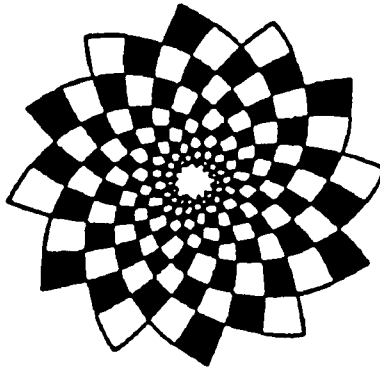


Figure 30

193. STEWART & GOLUBITSKY, *supra* note 187, at 34-35. Technically speaking, isometries, mentioned in the description of reflection, are defined to be transformations that preserve distances between points. KAPPRAFF, *supra* note 149, at 383. For a thorough discussion of isometries and solid and plane symmetry, see *id.* at 295-345 and 383-454.

194. Figure 29 is reprinted from JAY KAPPRAFF, CONNECTIONS: THE GEOMETRIC BRIDGE BETWEEN ART AND SCIENCE 306 (1991). Copyright © 1991 by McGraw-Hill, Inc. Reproduced with the permission of the publisher, McGraw-Hill, Inc., New York.

195. Figure 30 is reprinted from JAY KAPPRAFF, CONNECTIONS: THE GEOMETRIC BRIDGE BETWEEN ART AND SCIENCE 450 (1991). Copyright © 1991 by McGraw-Hill, Inc. Reproduced with the permission of the publisher, McGraw-Hill, Inc., New York.

Perceived self-congruity or symmetry is an important aspect of solution acceptance both in visual perception and in negotiation and mediation.¹⁹⁶ Solutions which in the end preserve and satisfy at least one invariant of the interest-type information structures of each of the parties through successive transformations will normally be symmetrical, i.e., most pleasing to all of the parties and the most stable and durable. The key to obtaining such satisfactory solutions is determining which interest (or interests) of each party is deemed by each of them to be invariant. Normally, the invariant will be what each party initially believes its *primary* interest or interests to be. But occasionally, parties will not consciously realize what their invariant interests are until at some point during the course of the negotiation or mediation process. Mediators and negotiators should be alert to recognize invariant interests, whether always apparent or discovered, during the course of the process. To achieve symmetry of solution, mediators and negotiators should ensure that these invariant interests of the parties are ultimately satisfied in the final solution design.

V. CONCLUSION

Figuratively, this article draws the line of intersection between the planes of mathematical and interpersonal problem solving. Its purpose has been primarily to demonstrate the use of analytical-imaginative thought processes of mathematics to enhance the effectiveness of interpersonal problem solving—to illustrate the very real interface between mathematics and psychology. And, in actuality, this article is as much a proposed paradigm for revolutionizing the teaching of mathematics¹⁹⁷ as it is for introducing a model for solving problems more productively in mediation and negotiation. But this is just the beginning. It is predictable that in the not too distant future, mediators and negotiators will be using computer-generated complex geometric models to assist them in arriving at optimal and super-optimal solutions of disputes and transactions. Already such computer-aided visualization and solid model computer graphics techniques are being used to help solve all sorts of design problems in the fields of science and engineering—and even in the behavioral sciences.¹⁹⁸ It will require a very short step on the pathway of technological

196. As to visual perception, see I. ROCK, *THE LOGIC OF PERCEPTION* 117-32 (1987).

197. In the last decade, the branch of mathematics known as discrete mathematics has rapidly grown in prominence, and great strides have been made toward introducing it in all levels of education, kindergarten through college. Discrete mathematics emphasizes the application of mathematical principles to real-world situations by emphasizing problem-solving skills and problem-solving strategies and by fostering critical and creative thinking and mathematical reasoning. See MARGARET J. KENNEY & CHRISTIAN R. HIRSCH, *DISCRETE MATHEMATICS ACROSS THE CURRICULUM, K-12, 1991 YEARBOOK* (1991). The instant article hopefully will inspire a new and socially beneficial dimension to the teaching of discrete mathematics.

198. See generally *SCIENTIFIC VISUALIZATION AND GRAPHICS SIMULATION*, (Daniel Thalman ed., 1990); *SCIENTIFIC VISUALIZATION OF PHYSICAL PHENOMENA* (N.M. Patrikalakis ed., 1991).

and sociological progress to computerize the concept of geometric imagineering. But until that step is taken, mediators and negotiators can enjoy the delights of putting to work the full power of our intellects and imaginations in mediational problem solving, while recalling Descartes' sage advice tucked away in his explanation to Rule Eight of his Rules for the Direction of the Mind:

Within ourselves we are aware that, while it is the intellect alone that is capable of knowledge, it can be helped or hindered by three other faculties, viz. imagination, sense perception, and memory. We must . . . look at these faculties in turn, to see in what respect each of them could be a hindrance, . . . and in what respect an asset [A]nyone who has mastered the whole method, however mediocre his intelligence, may see that there are no paths closed to him that are open to others [T]his discovery amounts to knowledge, no less than any other.¹⁹⁹

QED.²⁰⁰

199. PHILOSOPHICAL WRITINGS, *supra* note 18, at 32-33.

200. QED is an abbreviation for *Quod erat demonstrandum*, used in mathematical proofs to signify "That which was to be demonstrated" and to mark the end of the communication on the particular topic. PAULOS, *supra* note 41, at 196. Use of the abbreviation here is intentionally ironic, because this author hopes that this article will mark the *beginning*, not the end, of the communication on the topic of the application of analytic method and geometric imagineering in negotiation and mediation.

APPENDIX: DESCARTES' RULES FOR THE DIRECTION OF THE MIND²⁰¹*Rule One*

The aim of our studies should be to direct the mind with a view to forming true and sound judgments about whatever comes before it.

Rule Two

We should attend only to those objects of which our minds seem capable of having certain and indubitable cognition.

Rule Three

Concerning objects proposed for study, we ought to investigate what we can clearly and evidently intuit or deduce with certainty, and not what other people have thought or what we ourselves conjecture. For knowledge can be attained in no other way.

Rule Four

We need a method if we are to investigate the truth of things.

Rule Five

The whole method consists entirely in the ordering and arranging of the objects on which we must concentrate our mind's eye if we are to discover some truth. We shall be following this method exactly if we first reduce complicated and obscure propositions step by step to simpler ones, and then, starting with the intuition of the simplest ones of all, try to ascend through the same steps to a knowledge of all the rest.

Rule Six

In order to distinguish the simplest things from those that are complicated and to set them out in an orderly manner, we should attend to what is most simple in each series of things in which we have directly deduced some truths from others, and should observe how all the rest are more, or less, or equally removed from the simplest.

201. PHILOSOPHICAL WRITINGS, *supra* note 18, at 9-76.

Rule Seven

In order to make our knowledge complete, every single thing relating to our undertaking must be surveyed in a continuous and wholly uninterrupted sweep of thought, and be included in a sufficient and well-ordered enumeration.

Rule Eight

If in a series of things to be examined we come across something which our intellect is unable to intuit sufficiently well, we must stop at that point, and refrain from the superfluous task of examining the remaining items.

Rule Nine

We must concentrate our mind's eye totally upon the most insignificant and easiest of matters, and dwell on them long enough to acquire the habit of intuiting the truth distinctly and clearly.

Rule Ten

In order to acquire discernment we should exercise our intelligence by investigating what others have already discovered, and methodically survey even the most insignificant products of human skill, especially those which display or presuppose order.

Rule Eleven

If, after intuiting a number of simple propositions, we deduce something else from them, it is useful to run through them in a continuous and completely uninterrupted train of thought, to reflect on their relations to one another, and to form a distinct and, as far as possible, simultaneous conception of several of them. For in this way our knowledge becomes much more certain, and our mental capacity is enormously increased.

Rule Twelve

Finally we must make use of all the aids which intellect, imagination, sense-perception, and memory afford in order, firstly, to intuit simple propositions distinctly; secondly, to combine correctly the matters under investigation with what we already know, so that they too may be known; and thirdly, to find out what things should be compared with each other so that we make the most thorough use of all our human powers.

Rule Thirteen

If we perfectly understand a problem we must abstract it from every superfluous conception, reduce it to its simplest terms and, by means of an enumeration, divide it up into the smallest possible parts.

Rule Fourteen

The problem should be re-expressed in terms of the real extension of bodies and should be pictured in our imagination entirely by means of bare figures. Thus it will be perceived much more distinctly by our intellect.

Rule Fifteen

It is generally helpful if we draw these figures and display them before our external senses. In this way it will be easier for us to keep our mind alert.

Rule Sixteen

As for things which do not require the immediate attention of the mind, however necessary they may be for the conclusion, it is better to represent them by very concise symbols rather than by complete figures. It will thus be impossible for our memory to go wrong, and our mind will not be distracted by having to retain these while it is taken up with deducing other matters.

Rule Seventeen

We should make a direct survey of the problem to be solved, disregarding the fact that some of its terms are known and others unknown, and intuiting, through a train of sound reasoning, the dependence of one term on another.

Rule Eighteen

For this purpose only four operations are required: addition, subtraction, multiplication and division. The latter two operations should seldom be employed here, for they may lead to needless complication, and they can be carried out more easily later.

Rule Nineteen

Using this method of reasoning, we must try to find as many magnitudes, expressed in two different ways, as there are unknown terms, which we treat as known in order to work out the problem in the direct way. That will give us as many comparisons between two equal terms.

Rule Twenty

Once we have found the equations, we must carry out the operations which we have left aside, never using multiplication when division is in order.

Rule Twenty-one

If there are many equations of this sort, they should all be reduced to a single one, viz to the equation whose terms occupy fewer places in the series of magnitudes which are in continued proportion, i. e., the series in which the order of the terms is to be arranged.