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On connectedness and hamiltonicity of direct graph bundles

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Abstract. A necessary and sufficient condition for connectedness of direct graph bundles is given where the fibers are cycles. We also prove that all connected direct graph bundles $X = C_s \times^{\alpha} C_t$ are Hamiltonian.

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1. Introduction

Hamiltonian properties of graphs is one of the fundamental topics in graphs theory [4, 8]. Besides famous related historical problems (Icosian game, chessboard puzzles, etc.), it has important practical applications. For example, the traveling salesman problem [10], which is the most studied problem in combinatorial optimization, asks for a minimal Hamiltonian cycle in an edge weighted graph. It is well known that there is no efficient algorithm for deciding whether a graph is Hamiltonian or not. Therefore it is interesting to ask, given a subclass of graphs, whether the problem may be solved efficiently by designing a polynomial algorithm or by providing a characterization of Hamiltonian graphs within the subclass. Graph products are one of the natural constructions giving more complex graphs from simple ones. Graph bundles, also called twisted products, are a generalization of product graphs, which has been (under various names) frequently used as computer topologies or communication networks, see for example [1]. A famous example is the ILIAC IV supercomputer [2]. While Hamiltonian properties of the cartesian products are well studied, much less is known on Hamiltonian properties of direct products and bundles. The reason may be that the direct product has some, on the first sight not nice properties, for example the direct product of connected graphs is not necessarily connected. In this paper, we consider the direct graph bundles of cycles over cycles and provide a complete characterization of connected and Hamiltonian graphs within this class. We also give a sufficient condition for connectedness of a more general class of graph bundles.

Our less general motivation for this research is the following. It is well-known that the Cartesian product of two Hamiltonian graphs is Hamiltonian, and therefore

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it is interesting to investigate conditions under which the product is Hamiltonian if at least one of the factors is not Hamiltonian. In 1982, Batagelj and Pisanski [3] proved that the Cartesian product of a tree T and a cycle C_n has a Hamiltonian cycle if and only if $n \geq \Delta(T)$, where $\Delta(T)$ denotes the maximum vertex degree of T. They introduced the cyclic Hamiltonicity $cH(G)$ of G as the smallest integer n for which the Cartesian product $C_n \Box G$ is Hamiltonian. More than twenty years later, Dimakopoulos, Palios and Paulakidas [5] proved that $cH(G) \leq \mathcal{D}(G) \leq cH(G) + 1$, as conjectured already in [3]. (Here $\mathcal{D}(G)$ denotes the minimum of $\Delta(T)$ over all spanning trees T of G.) These results can be extended in a certain way to Cartesian graph bundles, see [11] and [9].

It is natural to ask whether a similar theory may be developed for other graph products. In the case of the direct product, the question of Hamiltonicity seems to be much more complicated, because even the direct product of two cycles is not necessarily Hamiltonian ([7] gives a characterization which direct products of Hamiltonian graphs are Hamiltonian). For example, the direct product of two even cycles is not connected so it cannot be Hamiltonian. Furthermore, the product of a tree (on at least 3 vertices) with any graph is not Hamiltonian, However, the direct graph bundle with even cycles as base and as fiber can be connected. When is it Hamiltonian?

In this paper, we study connectedness and Hamiltonicity of direct graph bundles. We give a complete list of necessary and sufficient conditions for connectedness of graph bundles where the fibers are cycles (Theorem 3). In the special case when the base graph is also a cycle, a complete list of connected bundles can be written. More precisely, we prove:

Theorem 1. *The direct graph bundle $C_s \times^\alpha C_t$ with fiber C_t and base C_s, $s, t \geq 3$, is connected:*

1. *when t is odd, for any automorphism $\alpha \in \mathrm{Aut}(C_t)$.*
2. *when t is even and s is odd, if and only if α is identity, even cyclic ℓ-shift or reflection with two fixed points ρ_2.*
3. *when t is even and s is even, if and only if α is odd cyclic ℓ-shift or reflection without fixed points ρ_0.*

Otherwise, $C_s \times^\alpha C_t$ is not connected.

Theorem 1 implies a sufficient condition for connectedness of an arbitrary direct graph bundle with fiber C_t and connected base (see Theorem 3 in Section 4).

Then we study Hamiltonicity of direct graph bundles where both fibers and bases are cycles. We prove that all connected direct bundles of cycles over cycles are Hamiltonian:

Theorem 2. *Let $X = C_s \times^\alpha C_t$ be a direct graph bundle with fibre C_t and base C_s. Then X is Hamiltonian if and only if X is connected.*

The rest of the paper is organized as follows. In the next section we introduce terminology and notation, and recall some basic definitions including the definition of graph bundles. In Section 3, we consider a simple case, bundles over P_2, for a later reference. In Section 4, we study connectedness of direct bundles: first, a complete characterization of bundles of cycles over cycles is given, and then a necessary and sufficient condition for bundles over arbitrary base is proved. Hamiltonicity of direct graph bundles is discussed in Section 5. Finally, constructions of Hamiltonian cycles of direct bundles of cycles over cycles are given, first constructions for shifts (Section 6) and then for reflections (Section 7).

2. Terminology and notation

A finite, simple and undirected graph $G = (V(G), E(G))$ is given by a set of vertices $V(G)$ and a set of edges $E(G)$. As usual, the edges $\{i, j\} \in E(G)$ are shortly denoted by ij. Although here we are interested in undirected graphs, the order of the vertices will sometimes be important, for example when we assign automorphisms to edges of the base graph. Is such case we assign two opposite arcs $\{(i, j), (j, i)\}$ to edge $\{i, j\}$.

Two graphs G and H are called *isomorphic*, in symbols $G \simeq H$, if there exists a bijection φ from $V(G)$ onto $V(H)$ that preserves adjacency and nonadjacency. In other words, a bijection $\varphi : V(G) \to V(H)$ is an *isomorphism* when: $\varphi(i)\varphi(j) \in E(H)$ if and only if $ij \in E(G)$. An isomorphism of a graph G onto itself is called an *automorphism*. The identity automorphism on G will be denoted by id_G or shortly id.

The *cycle* C_n on n vertices is defined by $V(C_n) = \{0, 1, \ldots, n-1\}$ and $ij \in E(C_n)$ if $i = j \pm 1 \bmod n$. Denote by P_n the *path* on $n \geq 1$ distinct vertices $0, 1, 2, \ldots, n-1$ with edges $ij \in E(P_n)$ if $j = i + 1, 0 \leq i < n - 1$. (Note that a subgraph isomorphic to the path graph is also called a path.)

An arbitrary connected graph G is said to be *Hamiltonian* if it contains a spanning cycle (called a Hamiltonian cycle).

Let G and H be connected graphs. The *direct product* of graphs G and H is the graph $G \times H$ with a vertex set $V(G \times H) = V(G) \times V(H)$ and whose edges are all pairs $(g_1, h_1)(g_2, h_2)$ with $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$. Other names for the direct product are [6]: Kronecker product, categorical product, tensor product, cardinal product, relational product, conjunction, weak direct product or just product and even Cartesian product. The direct product of graphs is commutative and associative in a natural way. For more facts on the direct product of graphs and other graph products we refer to [6].

Let B and G be graphs and $\mathrm{Aut}(G)$ the set of automorphisms of G. To any ordered pair of adjacent vertices $u, v \in V(B)$ we will assign an automorphism of G. Formally, let $\sigma : V(B) \times V(B) \to \mathrm{Aut}(G)$. For brevity, we will write $\sigma(u, v) = \sigma_{u,v}$ and assume that $\sigma_{v,u} = \sigma_{u,v}^{-1}$ for any $u, v \in V(B)$. Now we construct the graph X as follows. The vertex set of X is the Cartesian product of vertex sets, $V(X) = V(B) \times V(G)$. The edges of X are given by the rule: for any $b_1b_2 \in E(B)$ and any $g_1g_2 \in E(G)$, the vertices (b_1, g_1) and $(b_2, \sigma_{b_1,b_2}(g_2))$ are adjacent in X. We call X a *direct graph bundle* with base B and fibre G and write $X = B \times^{\sigma} G$.

Clearly, if all $\sigma_{u,v}$ are identity automorphisms, the graph bundle is isomorphic to the direct product $X = B \times^{\sigma} G = B \times G$. Furthermore, it is well-known that if

the base graph is a tree, then the graph bundle is always isomorphic to a product, i.e. $X = T \times^\sigma G \simeq T \times G$ for any graph G, any tree T and any assignment of automorphisms σ [12].
A graph bundle over a cycle can always be constructed in a way that all but at most one automorphism are identities. Fixing $V(C_n) = \{0, 1, 2, \ldots, n-1\}$, we denote $\sigma_{n-1,0} = \alpha$, $\sigma_{i-1,i} = id$ for $i = 1, 2, \ldots, n-1$, and $C_n \times^\alpha G \simeq C_n \times^\sigma G$. In this article we will use this fact frequently.

3. Bundles over K_2

Automorphisms of a cycle are of two types. A cyclic shift of the cycle by ℓ elements or briefly a *cyclic ℓ-shift*, $0 \leq \ell < n$, maps u_i to $u_{i+\ell}$ (index modulo n). As a special case we have the identity ($\ell = 0$). Other automorphisms of cycles are *reflections*. If C_n is a cycle on an odd number of vertices, then all the reflections have exactly one fixed point. If the number n is even, then we have reflections without fixed points and reflections with two fixed points.

More formally, we define:

- **cyclic ℓ-shift** σ_ℓ, defined as $\sigma_\ell(i) = (i + \ell) \bmod n$ for $i = 0, 1, \ldots, n-1$.
- **reflection with no fixed points** ρ_0, defined as $\rho_0(i) = n - i - 1$ for $i = 0, 1, \ldots, n-1$. (For n even, there is no fixed point.)
- **reflection with one fixed point** ρ_1, defined as $\rho_1(i) = n - i - 1$ for $i = 0, 1, \ldots, n-1$. (For n odd, there is exactly one fixed point, $\rho_1(\frac{n-1}{2}) = n - \frac{n-1}{2} - 1 = \frac{n-1}{2}$.)
- **reflection with two fixed points** ρ_2, defined as $\rho_2(0) = 0$ and $\rho_2(i) = n - i$ for $i = 1, 2, \ldots, n-1$. (For n even, there is the second fixed point $\rho_2(\frac{n}{2}) = n - \frac{n}{2} = \frac{n}{2}$.)

We first show that the graph bundle $P_2 \times^\alpha C_t$ is either isomorphic to one or to two cycles. (See also Figures 1 and 2.)

Lemma 1. *The direct graph bundle $P_2 \times^\alpha C_t$ for odd t is isomorphic to the cycle C_{2t} for every automorphism α of C_t. If t is even, then for every automorphism α of C_t the graph bundle $P_2 \times^\alpha C_t$ has two connected components that are isomorphic to C_t.*

Proof. First note that each vertex of $P_2 \times^\alpha C_t$ is of degree two, hence the graph is a union of cycles. Now consider vertex $(0, i)$. Observe that the vertices at distance two are $(0, (i+2) \bmod t)$ and $(0, (i-2) \bmod t)$. (Using the fact that $(0, i)$ and $(0, (i+2) \bmod t)$ have a common neighbor $(1, \alpha((i+1) \bmod t))$ and $(0, i)$ and $(0, (i-2) \bmod t)$ have a common neighbor $(1, \alpha((i-1) \bmod t))$.) Hence if t is even, the vertices $(0, i)$ for even i are on one cycle, and consequently it must be of length t. Similarly, vertices $(0, i)$ for odd i are on the other cycle of the same length. If t is odd, then all vertices $(0, i)$ are on the same cycle. $\square$

Let us emphasize that the lemma applies to the product (case $\alpha = id$).

Remark 1. *$P_2 \times C_t \simeq C_{2t}$ if t is odd and $P_2 \times C_t \simeq 2C_t$ if t is even.*

For a later reference define the two cycles of $P_2 \times^{\alpha} C_t$ for even t as follows:

Definition 1. *Let t be even. Let $C_t^{(0)}$ be the component of $P_2 \times^{\alpha} C_t$ containing the vertex $(0,0) \in V(P_2 \times^{\alpha} C_t)$ and $C_t^{(1)}$ the component of $P_2 \times^{\alpha} C_t$ containing the vertex $(0,1) \in V(P_2 \times^{\alpha} C_t)$.*

Let us write explicitly the vertex sets that induce the cycles $C_t^{(0)}$ and $C_t^{(1)}$. Denote the subsets of odd and even vertices of C_t by $W_1 = \{1, 3, \dots, 2\lceil \frac{t-1}{2} \rceil - 1\}$ and $W_0 = \{0, 2, 4, \dots, 2\lfloor \frac{t-1}{2} \rfloor\}$, respectively. Hence $V(C_t) = W_0 \cup W_1$, and recall that $V(P_2) = \{0, 1\}$. From the proof of Lemma 1 the next two remarks directly follow.

Remark 2. *Let t be even and α an identity, an even shift or reflection ρ_2. Then $V(C_t^{(0)}) = Z_0 = (\{0\} \times W_0) \cup (\{1\} \times W_1)$ and $V(C_t^{(1)}) = Z_1 = (\{0\} \times W_1) \cup (\{1\} \times W_0)$.*

Remark 3. *Let t be even and α an odd shift or reflection ρ_0. Then $V(C_t^{(0)}) = \{0, 1\} \times W_0$ and $V(C_t^{(1)}) = \{0, 1\} \times W_1$.*

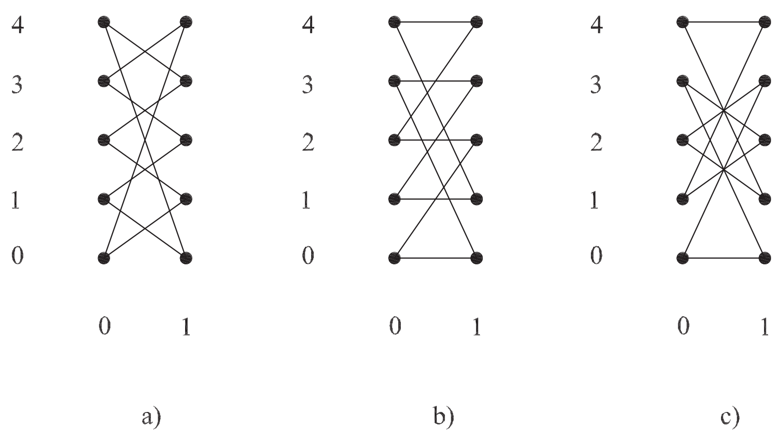

Figure 1: *The direct graph bundles $P_2 \times^{\alpha} C_5$: a) $\alpha = id$, b) $\alpha = \sigma_1$ and c) $\alpha = \rho_1$*

4. Connectedness of direct graph bundles

The fact that the direct product $G \times H$ of connected and bipartite factors G and H has exactly two components was first proved by Weichsel [13]. Hence if G and H are bipartite, then $G \times H$ cannot be Hamiltonian. In particular, the direct product $C_s \times C_t$, where C_s and C_t are even cycles, is not connected and hence not Hamiltonian.

Below we give necessary and sufficient conditions for connectedness of a direct graph bundle $C_s \times^{\alpha} C_t$ and for graph bundles with fibre C_t over arbitrary connected base graph B. The case when t is odd is easier and it is considered first.

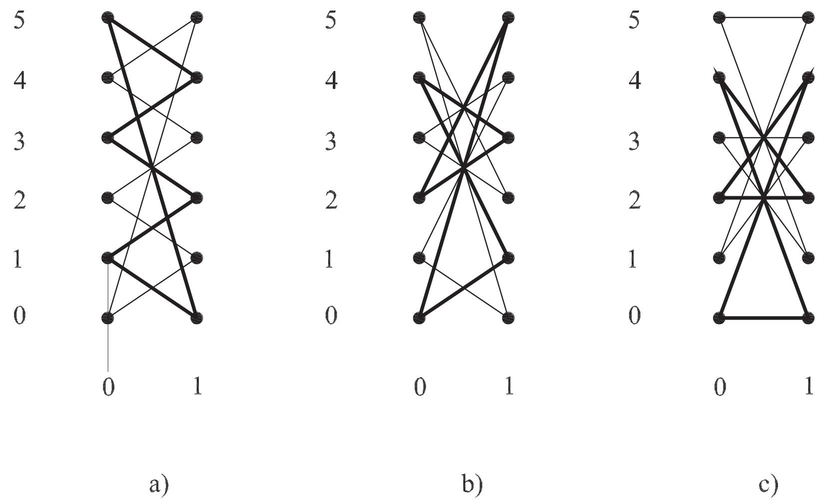

Figure 2: *The direct graph bundles $P_2 \times^{\alpha} C_6$: a) $\alpha = id$, b) $\alpha = \rho_2$ and c) $\alpha = \rho_0$*

Lemma 2. *Let C_t be a cycle on t vertices, where t is odd. Then $B \times^{\alpha} C_t$ is connected for every connected base graph B.*

Proof. Follows directly from Lemma 1. □

As $B = C_s$ is just a special case of interest, we can write

Corollary 1. *Let t be odd. The direct graph bundle $C_s \times^{\alpha} C_t$ is connected for every automorphism $\alpha \in \mathrm{Aut}(C_t)$.*

We now consider the graph bundles with fiber C_t for even t and this time we start with the case when the base graph is a cycle. We first observe a subgraph of $C_s \times^{\alpha} C_t$ in which the edges corresponding to the only (possibly) nontrivial automorphism are missing. As the subgraph $P_s \times C_t$ is not connected, we have to look at the missing edges to see whether $C_s \times^{\alpha} C_t$ is connected.

Let us denote $V(P_s) = V(C_s) = V_0 \cup V_1$, where $V_0 = \{0, 2, 4, \ldots, 2\lfloor \frac{s-1}{2} \rfloor\}$ and $V_1 = \{1, 3, \ldots, 2\lceil \frac{s-1}{2} \rceil - 1\}$ are the sets of even and odd vertices. Similarly, write $V(C_t) = W_0 \cup W_1$, a union of odd and even vertices. Furthermore, write

$$\begin{aligned} Z_0 &= (V_0 \times W_0) \cup (V_1 \times W_1) \\ Z_1 &= (V_0 \times W_1) \cup (V_1 \times W_0). \end{aligned}$$

Lemma 3. *If t is even, then the direct product $P_s \times C_t$ has two connected components, the first induced by the vertices of Z_0 and the second on the vertices from Z_1.*

The proof is straightforward and therefore omitted.

For s odd, the graph $C_s \times^{\alpha} C_t$ will be connected exactly when there is an edge connecting the set $\{s-1\} \times W_0$ with $\{0\} \times W_1$ (or there is an edge connecting the set $\{s-1\} \times W_1$ with $\{0\} \times W_0$). This is true exactly when the automorphism α

is the identity, an even cyclic ℓ-shift or reflection with two fixed points (see Remark 2). On the other hand, when α is an odd cyclic ℓ-shift or reflection without fixed points, there is no such edge by Remark 3.

By analogous reasoning as above, $C_s \times^{\alpha} C_t$ for even s will be connected exactly when there is an edge connecting the set $\{s-1\} \times W_0$ with $\{0\} \times W_0$, or $\{s-1\} \times W_1$ with $\{0\} \times W_1$. This is when the automorphism α is an odd cyclic ℓ-shift or reflection without fixed points (recall Remark 3). On the other hand, if the automorphism α is either the identity, even cyclic ℓ-shift or reflection with two fixed points, there is no such edge (by Remark 2), and therefore $C_s \times^{\alpha} C_t$ is not connected in these cases.

The observations are summarized in Theorem 1 and in Table 4.

t odd	for any automorphism α of C_t	
t even	s odd	**List 1, $\mathcal{L}_1$:** $\alpha = id$ $\alpha = \sigma_\ell$, ℓ is even $\alpha = \rho_2$
	s even	**List 2, $\mathcal{L}_2$:** $\alpha = \sigma_\ell$, ℓ is odd $\alpha = \rho_0$

Table 1: *Connected direct graph bundles $C_s \times^{\alpha} C_t$*

Recall that all graph bundles with connected base B and fibre C_t for odd t are connected. We conclude the section stating a necessary and sufficient condition for connectedness of a graph bundle with connected base B and fibre C_t for even t.

Theorem 3. *Let X be a direct graph bundle with fiber C_t and connected base. If C_t is an odd cycle, then X is connected. If C_t is an even cycle, then X is connected if and only if there is a cycle $C = v_1 v_2 \dots v_k$ in B such that either*

- *$|V(C)| = k$ is odd and $\alpha = \sigma_{v_k,v_1} \circ \sigma_{v_{k-1},v_k} \circ \cdots \circ \sigma_{v_2 v_3} \circ \sigma_{v_1 v_2}$ is an automorphism from $\mathcal{L}_1$, or*

- *$|V(C)| = k$ is even and $\alpha = \sigma_{v_k,v_1} \circ \sigma_{v_{k-1},v_k} \circ \cdots \circ \sigma_{v_2 v_3} \circ \sigma_{v_1 v_2}$ is an automorphism from $\mathcal{L}_2$.*

Proof. (sketch) If the fibre C_t is an odd cycle, X is clearly connected.

Let C_t be an even cycle. (1) First assume that X is connected. Let T be an arbitrary spanning tree of B. Then the subgraph spanned by edges of T, $T \times^{\sigma} C_t$ has two connected components, V_1 and V_2. There must be an edge $e = (b_1, g_1)(b_2, g_2)$ in X that connects two vertices from different components V_1 and V_2. Let $p(e) = b_1 b_2$ be the projection of this edge to B. There is a unique path P that connects b_1 and b_2 in T. The subgraph of X over the cycle $C = P \cup p(e)$, $C \times^{\sigma} C_t$, is connected, hence the automoprhism on the edges of C must be as claimed.
(2) Now assume there is a cycle C in X that fulfills the conditions given in the theorem. Then $C \times^{\sigma} C_t$ is connected, which directly implies that X is connected. $\square$

5. Hamiltonicity of the direct graph bundles

Obviously, a Hamiltonian graph is connected, so from now on we will only be interested in the direct graph bundles that are connected graphs. Among connected graphs, we can easily exclude the direct graph bundles over trees. One can easily prove that the direct product of a tree $T \not\cong P_2$ and an arbitrary graph G is not Hamiltonian. The statement also holds for graph bundles:

Lemma 4. *Let $T \not\cong P_2$ and let G be an arbitrary connected graph. Then the direct graph bundle $T \times^{\sigma} G$ is not Hamiltonian.*

Proof. Let G be a graph on n vertices. Suppose for contradiction that the bundle $T \times^{\sigma} G$ is Hamiltonian and let C be a Hamiltonian cycle. Projection of C to the base graph T spans all vertices of T. Let us walk along C and count how many times each vertex of T is visited and how many times edges will be traversed. Let u be a vertex of T of degree one. As $T \not\cong P_2$, u has a neighbor, say v, with degree more than one. The vertex u has to be visited exactly n times, hence the edge uv is traversed n times in each direction. As v has other neighbors, there is an edge vw that is used at least once, but then the vertex v was visited more than n times, or, equivalently, at least one of the vertices $(v, \star)$ has been visited twice in C. Contradiction. □

Therefore we will start with direct graph bundles of cycles over cycles. In the next two sections several constructions of Hamiltonian cycles will be given, which will prove that all connected graph bundles $X = C_s \times^{\alpha} C_t$ with fibre C_t and base C_s are Hamiltonian. Formally, the constructions that will be given in the last two sections will imply Theorem 2:
Let $X = C_s \times^{\alpha} C_t$ be a direct graph bundle with fibre C_t and base C_s. Then X is Hamiltonian if and only if X is connected.

We postpone the proof of this theorem to the last two sections.

This theorem, together with Theorem 1, implies

Theorem 4. *Let B and F be Hamiltonian graphs, with $t = |V(F)|$ odd. Then any direct graph bundle X with fiber F and base graph B is Hamiltonian.*

Proof. Consider the subgraph $C_B \times^{\sigma} C_F$ of X that has vertex set $V(C_B) \times V(C_F) = V(B) \times V(F)$ and edges defined by the rule: for any $b_1 b_2 \in E(C_B)$ and any $g_1 g_2 \in E(C_F)$, the vertices (b_1, g_1) and $(b_2, \sigma_{b_1,b_2}(g_2))$ are adjacent. Clearly, $C_B \times^{\sigma} C_F$ is Hamiltonian by Theorem 2 and because it is a spanning subgraph of X, X is Hamiltonian. □

For $t = |V(F)|$ even we are only able to state sufficient conditions for Hamiltonicity.

Theorem 5. *Let B and F be Hamiltonian graphs, with $t = |V(F)|$ even. Then we have:*

- *Let $s = |V(B)|$ be odd. A direct graph bundle X with fiber F and base graph B is Hamiltonian if there is a Hamiltonian cycle $C_B = v_1 v_2 \dots v_s$ in B such that $\alpha = \sigma_{v_s,v_1} \circ \sigma_{v_{s-1},v_s} \circ \cdots \circ \sigma_{v_2 v_3} \circ \sigma_{v_1 v_2}$ is an automorphism from $\mathcal{L}_1$.*

- *Let $s = |V(B)|$ be even. A direct graph bundle X with fiber F and base graph B is Hamiltonian if there is a Hamiltonian cycle $C_B = v_1 v_2 \ldots v_s$ in B such that $\alpha = \sigma_{v_s,v_1} \circ \sigma_{v_{s-1},v_s} \circ \cdots \circ \sigma_{v_2 v_3} \circ \sigma_{v_1 v_2}$ is an automorphism from $\mathcal{L}_2$.*

Proof. (sketch) Consider the spanning subgraph $C_B \times^{\sigma} C_F$ of X as in the proof of Theorem 4. Observe that $C_B \times^{\sigma} C_F \simeq C_B \times^{\alpha} C_F$, where $\alpha = \sigma_{v_s,v_1} \circ \sigma_{v_{s-1},v_s} \circ \cdots \circ \sigma_{v_2 v_3} \circ \sigma_{v_1 v_2}$ and all other automorphisms are identities. If s is even, then by Theorem 1, $C_B \times^{\alpha} C_F$ is Hamiltonian exactly when α is an odd cyclic ℓ-shift or reflection without fixed points, as claimed.

The same Theorem implies the condition for odd s. □

The next two sections provide proofs (constructions) that together imply correctness of Theorem 3. We start with shifts and first give a construction that provides a union of cycles which cover $C_s \times^{\alpha} C_t$ with p cycles. When $p > 1$, another construction will be used to combine the p cycles into one Hamiltonian cycle. Reflections will be considered in the last section: four different constructions will cover all possible cases.

6. Hamiltonicity of the direct graph bundles - cyclic shifts

Construction 1. Let $\bar{X}$ be the subgraph of a connected direct graph bundle $X = C_s \times^{\sigma_\ell} C_t$ in which only edges $(i,j)(i+1,(j+1) \bmod t)$, $i = 0,1,\ldots,s-2$, $j = 0,1,\ldots,t-1$ and $(s-1,j)$ $(0,(j+1+\ell) \bmod t)$, $j = 0,1,\ldots,t-1$ are present. □

Informally, one can also say that in $\bar{X}$, reading from left to right, only edges directed "up" are taken from X.

Lemma 5. *Let $C_s \times^{\sigma_\ell} C_t$ be a connected direct graph bundle. Let $\bar{X}$ be obtained by Construction 1. Then $\bar{X}$ is isomorphic to a union of p cycles of length $\frac{st}{p}$. Moreover, p is an odd number and the i-th cycle meets the first fibre in vertices* $(0,(i+p) \bmod t)$.

Proof. Obviously, vertices of $\bar{X}$ are of degree two, so $\bar{X}$ is a union of cycles. Moreover, $\bar{X}$ is a union of $p = \gcd((s+\ell) \bmod t, t)$ cycles of length $\frac{st}{p}$. If p is even, then both t and $(s+\ell) \bmod t$ must be even. Hence s and ℓ are of the same parity and $C_s \times^{\sigma_\ell} C_t$ is not connected (see Table 4). It follows that p is odd. Due to obvious symmetry, the cycle containing the vertex $(0,i)$ also contains the vertex $(0,(i+p) \bmod t)$. □

If $p = 1$, then $\bar{X}$ gives a Hamiltonian cycle of X, but this is of course not always the case. (Examples with $p = 1$ and $p = 3$ are given in Figure 4.a) and b).) Now we will show how one can always combine the cycles into one by replacing only a few edges.

Construction 2. Let $\bar{X}$ be the subgraph of X that is a union of cycles. Delete edges $(1,i)(2,i+1)$ and $(0,i+1)(1,(i+2) \bmod t)$ and replace them with edges $(0,i+1)(1,i)$ and $(1,(i+2) \bmod t)(2,i+1)$ for $i = 0,1,2,\ldots,p-2$ to obtain $\widetilde{X}$.

Assuming that the edges of $\bar{X}$ between fibres 0,1, and 2 are as given by Construction 1, (i.e. all edges go "up") we have the situation in Figure 3a) and 3b). The result of Construction 2 on the graph from Figure 4.b) is given in Figure 4.c).

By Lemma 5, the edges $(1,i)(2,i+1)$ and $(0,i+1)(1,(i+2) \bmod t)$ are on the $i-1$-th and $i+1$-th cycle. The replacement thus combines the two cycles into a larger one. Note that the edges involved in Construction 2 for different i are disjoint. Therefore

Lemma 6. *Let $\bar{X}$ be obtained by Construction 1 and assume it has $p > 1$ cycles. Then $\widetilde{X}$, the result of Construction 2 (replacing $p-1$ pairs of edges) gives a Hamiltonian cycle.*

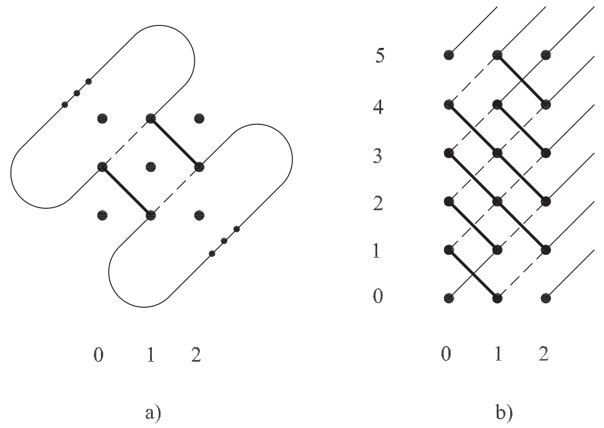

Figure 3: *a) A switch that joins two parallel cycles into one cycle, b) $p-1$ switches that connect p parallel cycles into one (Hamiltonian) cycle*

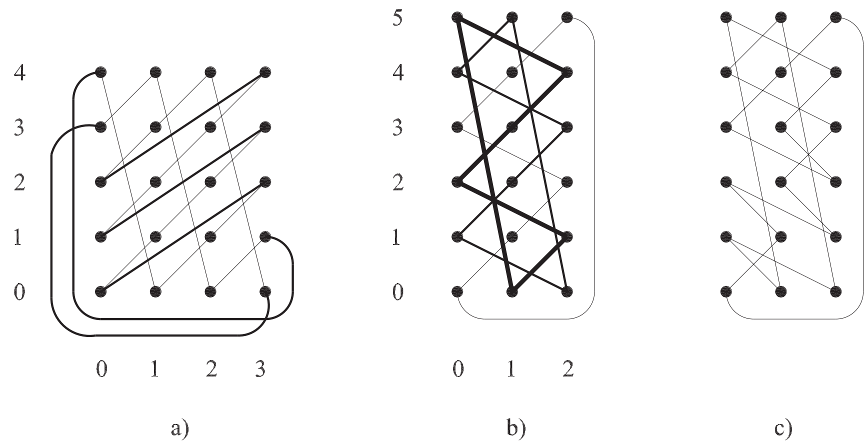

Figure 4: *a) $\bar{X}$ is a Hamiltonian cycle in $C_4 \times^{\sigma_2} C_5$, b) $\bar{X}$ in $C_3 \times C_6$ has 3 cycles, c)a Hamiltonian cycle in $C_3 \times C_6$*

7. Hamiltonicity of the direct graph bundles - reflections

In this section we give constructions of Hamiltonian cycles for connected graph bundles of cycles over cycles where the nontrivial automorphism is a reflection. The four propositions treat cases according to parity of the lengths of cycles, s and t.

Proposition 1. *Let C_s, C_t be two cycles, where $s, t \geq 3$ and s is odd and t even. Let $\alpha = \rho_2$ be a reflection with two fixed points. Then $C_s \times^{\alpha} C_t$ is Hamiltonian.*

Proof. The Hamiltonian cycle is constructed as follows. Form t disjoint paths of length $s-1$ from $(0, j)$ to $(s-1, j)$, $j = 0, 1, \ldots, t-1$, by taking (for example) edges $(i, j)(i+1, (j+1) \bmod t))$ for even i and edges $(i, j)(i+1, (j-1) \bmod t)$ for odd i (and $j = 0, 1, \ldots, t-1$). The edges between fibres $s-1$ and 0 are chosen from $C_t^{(0)}$: $(0, i)(1, \rho_2(i+1)), i \in W_0$, and from $C_t^{(1)}$: $(0, i)(1, \rho_2(i-1)), i \in W_1$, or, equivalently, from $C_t^{(0)}$: $(0, i)(1, \rho_2(i) - 1), i \in W_0$, and from $C_t^{(1)}$: $(0, i)(1, \rho_2(i) + 1), i \in W_1$

(recall the partition of edges of $P_2 \times^{\rho_2} C_t$ from Remark 2), see Figure 5.)

The claim that these edges form a Hamiltonian cycle is easy to check, for example by observing that the edges $(0, i)(1, \rho_2(i)-1), i \in W_0$, and $(0, i)(1, \rho_2(i)+1), i \in W_1$, give rise to a permutation of the set $\{0, 1, \ldots, t-1\}$ with one cycle. We omit the details. $\square$

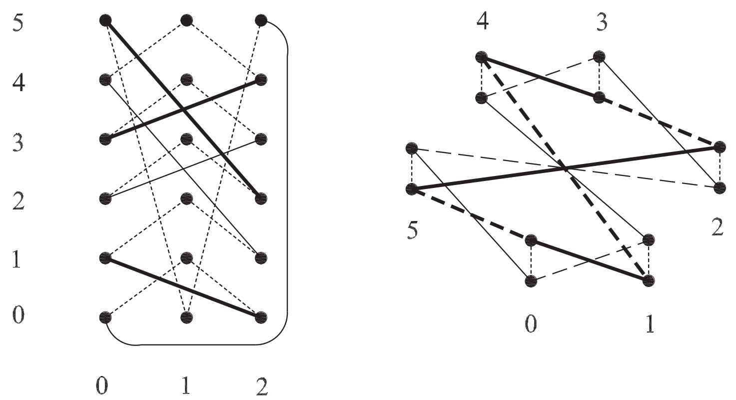

Figure 5: *Hamiltonian cycle in the direct graph bundle $C_3 \times^{\rho_2} C_6$ (left), and the cycles $C_6^{(0)}$, $C_6^{(1)}$ (right) (note that edges of the bundle which are not on the Hamiltonian cycle are not drawn)*

Proposition 2. *Let C_s, C_t be two cycles, where $s, t \geq 3$ and both s and t are even. Let $\alpha = \rho_0$ be a reflection without fixed points. Then $C_s \times^{\alpha} C_t$ is Hamiltonian.*

Proof. The subgraph induced on two consecutive fibres i and $i+1$ (for $i = 0, 1, \ldots, s-2$) has two connected components (the first on the vertices from Z_0 and the second on the vertices from Z_1) that are isomorphic to C_t. One of this cycles contains the edge $(i, \frac{t}{2})((i+1) \bmod s, \frac{t}{2} - 1)$, the other the edge $(i, \frac{t}{2} - 1)((i+1) \bmod s, \frac{t}{2})$.

Deleting edges $(i, \frac{t}{2})((i+1) \bmod s, \frac{t}{2} - 1)$ and $(i, \frac{t}{2} - 1)((i+1) \bmod s, \frac{t}{2})$ thus gives two disjoint paths, that span all vertices (and all edges except the two deleted) of fibres i and $i+1$.

Furthermore, the subgraph induced on fibres $s-1$ and 0 has two connected components that are isomorphic C_t, by Lemma 1. The first is induced by the vertices of $\{s-1, 0\} \times W_0$, the second by the vertices of $\{s-1, 0\} \times W_1$, by Remark

3. Two disjoint paths that span all vertices (and all edges but two) of fibres $s-1$ and 0 can be constructed by deleting the edges $(s-1, \frac{t}{2})(0, \frac{t}{2})$ and $(s-1, \frac{t}{2}-1)(0, \frac{t}{2}-1)$ (because $\rho_0(\frac{t}{2}-1) = \frac{t}{2}$ and $\rho_0(\frac{t}{2}) = \frac{t}{2}-1$).

A Hamiltonian cycle on $C_s \times^\alpha C_t$ is constructed as follows. On each of the pairs of fibres: 1 and 2, 3 and 4,..., $s-3$ and $s-2$, we take the two spanning paths. Add the edges $((i+1) \bmod s, \frac{t}{2}-1)((i+2) \bmod s, \frac{t}{2})$ and the edges $((i+1) \bmod s, \frac{t}{2})((i+2) \bmod s, \frac{t}{2}-1)$.

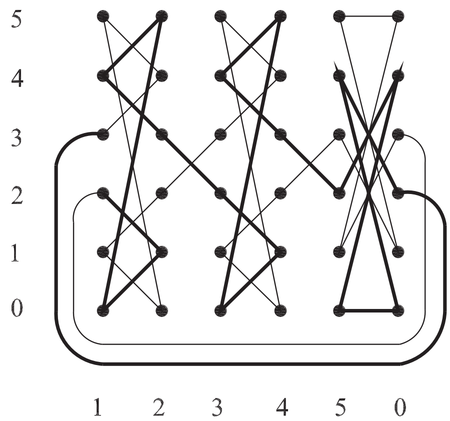

Figure 6: *Hamiltonian cycle in the direct graph bundle $C_6 \times^{\rho_0} C_6$. Note that edges of the bundle which are not on the Hamiltonian cycle are not drawn*

Observation that the edges connect vertices from Z_0 with vertices from Z_0 (and vertices from Z_1 with vertices from Z_1) for $i = 1, 3, \ldots, s-3$ and that the edges between fibres $s-1$ and 0 connect Z_0 to Z_1 and Z_1 to Z_0 implies that a Hamiltonian cycle is constructed (see Figure 6). □

Proposition 3. *Let C_s, C_t be two cycles, where $s, t \geq 3$ and s is even and t odd. Let $\alpha = \rho_1$ be a reflection with one fixed point. Then $C_s \times^\alpha C_t$ is Hamiltonian.*

Proof. Note that the edges between two consecutive fibres i and $i+1$ (for $i = 0, 1, \ldots, s-2$) form a cycle of length $2t$, because the subgraph induced on two consecutive fibres is isomorphic to $P_2 \times C_t$. Also the subgraph induced on fibres $s-1$ and 0 is isomorphic to $P_2 \times^{\rho_1} C_t \simeq C_{2t}$, by Lemma 1.

Each of these subgraphs contains the two edges $(i, \lfloor \frac{t}{2} \rfloor)((i+1) \bmod s, \lfloor \frac{t}{2} \rfloor + 1)$ and $(i, \lfloor \frac{t}{2} \rfloor + 1)((i+1) \bmod s, \lfloor \frac{t}{2} \rfloor)$.

Deleting edge $(i, \lfloor \frac{t}{2} \rfloor)((i+1) \bmod s, \lfloor \frac{t}{2} \rfloor + 1)$ thus gives a path that spans all vertices (and all edges except the deleted) of fibres i and $i+1$.

Now we can construct a Hamiltonian cycle on $C_s \times^\alpha C_t$ by taking the spanning paths on pairs of fibres 1 and 2, 3 and 4, ..., $s-2$ and $s-1$ and 0, and connecting them with edges $(i, \lfloor \frac{t}{2} \rfloor + 1)(i+1, \lfloor \frac{t}{2} \rfloor)$, $i = 0, 2, 4, \ldots, s-2$ (see Figure 7.) □

Proposition 4. *Let C_s, C_t be two cycles, where $s, t \geq 3$ and both s and t are odd. Let $\alpha = \rho_1$ be reflection with one fixed point. Then $C_s \times^\alpha C_t$ is Hamiltonian.*

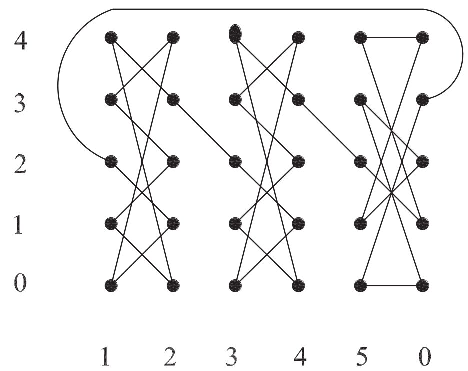

Figure 7: *Hamiltonian cycle in the direct graph bundle $C_6 \times^{\rho_1} C_5$. Note that edges of the bundle which are not on the Hamiltonian cycle are not drawn*

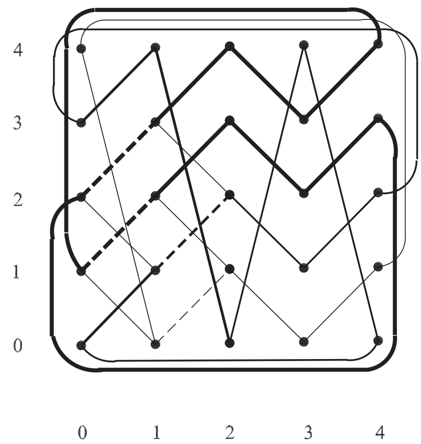

Figure 8: *Hamiltonian cycle in the direct graph bundle $C_5 \times^{\rho_1} C_5$. Note that edges of the bundle which are not on the Hamiltonian cycle are not drawn*

Proof. Consider the following subset of edges (all additions in the second coordinate are modulo t):

(a) $(i,j)(i+1,j+1)$ for $i = 0,1,3,5,\ldots,s-2$ and $j = 0,1,\ldots,t-1$,

(b) $(i,j)(i+1,j-1)$ for $i = 2,4,6,\ldots,s-3$ and $j = 0,1,\ldots,t-1$, and

(c) $(s-1,j)(0,\rho_1(j-1))$ for $j = 0,1,\ldots,t-1$.

Observe that edges from (a) and (b) form t parallel paths that join $(0,j)$ with $(s-1,(j+2) \bmod t)$. As $\rho_1(j-1) = t-(j-1)-1 = t-j$, the edges defined in (c) can be written simpler as $(s-1,j)(0,t-j)$.

Clearly, the edges meet each vertex exactly twice, so they form a union of cycles. More precisely, we have one (short) cycle

$$(s-1,1) \to (0,t-1) = (0,-1) \to \cdots \to (s-1,1)$$

and $\lfloor \frac{t}{2} \rfloor$ longer cycles, namely for $j = 2, 3, \dots, \lfloor \frac{t}{2} \rfloor, \lceil \frac{t}{2} \rceil$

$$(s-1, j) \to (0, t-j) \to \cdots \to (s-1, t-j+2) \to (0, t-(t-j+2) = (0, j-2) \to$$
$$\to \cdots \to (s-1, j).$$

Note that by construction each of the $\lceil \frac{t}{2} \rceil$ cycles, for $j = 1, 2, \dots, \lceil \frac{t}{2} \rceil$, includes a path $(0, (j-2) \bmod t) \to (1, j-1) \to (2, j)$. Hence we can use the same idea as before (Construction 2 in Section 6) to obtain a Hamiltonian cycle from the $\lceil \frac{t}{2} \rceil$ "parallel" cycles. □

An example is given in Figure 8.

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References

[1] I. Banič, R. Erveš, J. Žerovnik, *The edge fault-diameter of Cartesian graph bundles*, European J. Combin. **30**(2009), 1054–1061.

[2] G. H. Barnes, R. M. Brown, M. Kato, D. J. Kuck, D. L. Slotnick, R. A. Stokes, *The ILLIAC IV Computer*, IEEE Trans. Comput. **C-17**(1968), 746–757.

[3] V. Batagelj, T. Pisanski, *Hamiltonian cycles in the cartesian product of a tree and a cycle*, Discrete Math. **38**(1982), 311–312.

[4] J. C. Bermond, *Hamiltonian graphs*, in: *Selected Topics in Graph Theory*, (Beineke and Wilson, Eds.), Academic Press, London, 1978, 127–167.

[5] V. V. Dimakopoulos, L. Palios, A. S. Poulakidas, *On the hamiltonicity of the cartesian product*, Inform. Process. Lett. **96**(2005), 49–53.

[6] R. H. Hammack, W. Imrich, S. Klavžar, *Handbook of Product Graphs*, CRC Press, Boca Raton, FL, 2011.

[7] S. Gravier, *Hamiltonicity of the cross product of two Hamiltonian graphs*, Discrete Math. **170**(1997), 253–257.

[8] R. J. Gould, *Updating the hamiltonian problem – A survey*, J. Graph Theory **15**(1991), 121–157.

[9] I. Hrastnik Ladinek, J. Žerovnik, *Cyclic Bundle Hamiltonicity*, Int. J. Comput. Math. **89**(2012), 129–136.

[10] E. L. Lawler, J. K. Lenstra, A. H. G. Rinnooy Kan, D. B. Shmoys, *The Traveling Salesman Problem: A Guided Tour of Combinatorial Optimization*, (Wiley Series in Discrete Mathematics & Optimization) John Wiley and Sons, Chichester, 1995.

[11] T. Pisanski, J. Žerovnik, *Hamilton cycles in graph bundles over a cycle with tree as a fibre*, Discrete Math. **309**(2009), 5432–5436.

[12] T. Pisanski, J. Vrabec, *Graph bundles*, Prep. Ser. Dep. Math. **20**(1982), 213–298.

[13] P. Weichsel, *The Kronecker product of graphs*, Proc. Amer. Math. Soc. **13**(1962), 47–52.