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EVERYWHERE SURJECTIONS AND RELATED TOPICS: EXAMPLES AND COUNTEREXAMPLES

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This paper deals with everywhere surjections, i.e. functions defined on a topological space whose restrictions to any non-empty open subset are surjective. We introduce and discuss several constructions in different contexts; some constructions are easy, while others are more involved. Among other things, we prove that there is a vector space of uncountable dimension whose non-zero elements are everywhere surjections from $\mathbb{Q}$ to $\mathbb{Q}$; we give an example of an everywhere surjection whose domain is the set of countably infinite real sequences; we construct an everywhere surjective linear map from the Cantor set into itself. Finally, we prove the existence of functions from $\mathbb{R}$ to $\mathbb{R}$ which are everywhere surjections in stronger senses.

1. Introduction

A function from $\mathbb{R}$ to $\mathbb{R}$ is said to be an everywhere surjection when its restriction to any non-trivial interval is a surjective function. This definition was given in [1]; in the last decade, several other papers have been published about this topic; see for instance [4] and [10].

The history of everywhere surjections is somewhat curious. On the one hand, the behavior of an everywhere surjection is completely different from that

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of a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Indeed, if f is continuous at a point x_0, then the images under f of smaller and smaller neighborhoods of x_0 are contained in smaller and smaller neighborhoods of $f(x_0)$. On the contrary, if g is an everywhere surjection, the images of smaller and smaller neighborhoods of any point are always the whole set $\mathbb{R}$. On the other hand, everywhere surjections may be confused with continuous functions, because both of them satisfy the intermediate value property, in the sense that if $a < b$ then the interval with endpoints $f(a)$ and $f(b)$ is included in $f([a,b])$. In fact, just to point out that continuity is not equivalent to the intermediate value property, in 1904 Henri Lebesgue gave the first example of what we call now an everywhere surjection (Lebesgue defined a function from the interval $[0,1]$ into itself). Note that an everywhere surjection not only is discontinuous at any point, but behaves in a strongly counterintuitive way; for example, its graph is a dense subset of the plane. Let us briefly recall Lebesgue construction.

Example 1.1 (Lebesgue, [9] p. 90)**.** Let $f : [0,1] \to [0,1]$ be the function defined as follows: if $x = 0.a_1a_2a_3\ldots$ is a number in $[0,1]$ written in base 10, we consider the sequence $a_1, a_3, a_5 \ldots$ of digits with odd indices. If this sequence is not ultimately periodic, we put $f(x) = 0$; otherwise, if the first period of the sequence begins with the digit a_{2n-1} for some $n \in \mathbb{N}$, we define $f(x) = 0.a_{2n}a_{2n+2}a_{2n+4}\ldots$. This function is surjective if restricted to any non-empty open subinterval of $[0,1]$.

Example 1.2 (see [12])**.** The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \lim_{n\to\infty} \tan(n!\pi x) & \text{for any } x \text{ for which the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

is an everywhere surjection.

We will recall another construction of an everywhere surjection in the Example 6.1; for other examples, see [2], [10], [11]. For recent results and a more theoretical approach, see [5].

In this paper, we introduce and discuss several examples of everywhere surjections in different contexts that may be of some interest, each one because of its peculiar construction and properties. The definition of an everywhere surjection is generalized to any topological space in the following way:

Definition 1.3 (see [10])**.** Let X be a topological space; a function $f : X \to X$ is said to be an *everywhere surjection on* X if $f(A) = X$ for any non-empty open subset $A \subseteq X$.

What can be said about a topological space X for which there exists an everywhere surjection? The following proposition (see [10], Theorem 5) provides a sufficient condition; the proof requires the axiom of choice.

Proposition 1.4. *Let $(X, \mathcal{T})$ be an infinite topological space of cardinality k. Suppose the following hold:*

a) every open subset of X has cardinality k, and

b) there is a set $\mathcal{B}$ of non-empty open subsets of X so that $|\mathcal{B}| \leq k$, and for any open subset $A \subseteq X$ there exists $U \in \mathcal{B}$ such that $U \subseteq A$.

Then there exists an everywhere surjective function $f : X \to X$.

A straightforward consequence of this theorem is the existence of everywhere surjections from $\mathbb{Q}$ to $\mathbb{Q}$. In the following section we give some explicit examples. Also, we prove that there exists a vector space of uncountable dimension whose non-zero elements are everywhere surjections from $\mathbb{Q}$ to $\mathbb{Q}$ (this is not obvious because in general the sum of two everywhere surjections is not in turn an everywhere surjection). In section 3 we define everywhere surjections between different sets; we deduce the existence of everywhere surjections from $\mathbb{R}$ to $\mathbb{Q}$ and from $\mathbb{R} \setminus \mathbb{Q}$ to $\mathbb{Q}$. In section 4 we prove a theorem which gives two sufficient conditions for a function f defined on a topological space X to be everywhere surjective on its image, i.e. an everywhere surjection from its domain X to $f(X)$. In section 5 we give an example of an everywhere surjection whose domain is $\mathbb{R}^{\mathbb{N}}$, i.e. the set of of countably infinite real sequences. Then we build a sequence of everywhere surjections from $\mathbb{R}$ to $\mathbb{R}$ whose graphs enjoy unexpected properties; this sequence leads to the definition of a family of functions from $\mathbb{N}$ to $\mathbb{R}$ which are, in some weak sense, "surjective". In section 6 we define an everywhere surjection from $\mathbb{R}$ to $\mathbb{Q}$ such that the inverse image of each rational number has a positive Lebesgue measure and we give an example of an everywhere surjective linear function. Finally, in section 7 we prove the existence of a function f from $\mathbb{R}$ to $\mathbb{R}$ which is everywhere surjective in a strong sense: for any non-empty open subset $A \subseteq \mathbb{R}$ there exists a subset $B \subset A$ such that $f|_B$ is a continuous increasing bijection. This result is generalized to prove the existence of an everywhere surjection which "simulates" all continuous functions.

2. Everywhere surjections from $\mathbb{Q}$ to $\mathbb{Q}$

In the following examples we examine different ways to define an everywhere surjection from $\mathbb{Q}$ to $\mathbb{Q}$. We will need the following lemma (we omit the simple proof).

Lemma 2.1. *Let x and y be two real positive numbers. Suppose that $x/y \notin \mathbb{Q}$; then the set $\{ax - by \mid a, b \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

Example 2.2. Let m/n be an irreducible fraction and let

$$m = \pm p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_{h-1}^{i_{h-1}} \cdot p_h^{i_h} \qquad n = q_1^{j_1} \cdot q_2^{j_2} \cdot \ldots \cdot q_{k-1}^{j_{k-1}} \cdot q_k^{j_k}$$

be the prime decompositions of m and n (prime factors are positive and written in ascending order). We define an everywhere surjection $f : \mathbb{Q} \to \mathbb{Q}$ as follows:

$$f\left(\frac{m}{n}\right) = \begin{cases} 0 & \text{if } m = 0 \text{ or } m = \pm 1 \text{ or } n = 1 \text{ or } \dfrac{m}{n} = \dfrac{2^i}{3^j} \\ +\dfrac{p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_{h-1}^{i_{h-1}}}{q_1^{j_1} \cdot q_2^{j_2} \cdot \ldots \cdot q_{k-1}^{j_{k-1}}} & \text{if } \dfrac{m}{n} \neq \dfrac{2^i}{3^j} \text{ and } p_h > q_k \\ -\dfrac{p_1^{i_1} \cdot p_2^{i_2} \cdot \ldots \cdot p_{h-1}^{i_{h-1}}}{q_1^{j_1} \cdot q_2^{j_2} \cdot \ldots \cdot q_{k-1}^{j_{k-1}}} & \text{if } \dfrac{m}{n} \neq \dfrac{2^i}{3^j} \text{ and } q_k > p_h \end{cases}$$

In other words, except for $m/n = 2^i/3^j$, we get the output of f by deleting the greatest prime factors with its exponent from both the numerator and the denominator (of course, if $h = 1$ or $k = 1$ we assume the empty product to be 1). The function f is everywhere surjective: let r be a non-zero rational number; its inverse image $f^{-1}(\{r\})$ contains all numbers of the form $\pm r \cdot (p^i/q^j)$ such that:

- p and q are prime numbers greater than any prime factor occurring in the decomposition of the numerator and denominator of r;
- $p > q$ if $r > 0$, $p < q$ and $(p,q) \neq (2,3)$ if $r < 0$.

We have to show that, for every r, the set of numbers just described is dense in $\mathbb{Q}$. Let p and q be prime numbers as above; proving that the set of all $|r| \cdot (p^i/q^j)$ is dense in $\mathbb{Q}^+$ will be sufficient. We consider the set of the logarithms of $|r| \cdot (p^i/q^j)$, which is the set M of real numbers $\ln(|r|) + i \ln p - j \ln q$, where $i, j \in \mathbb{N}$. Thanks to Lemma 2.1, M is dense in $\mathbb{R}$: for any non-empty interval $(a,b) \cap \mathbb{Q}^+$ there exists a real number $x \in M$ that belongs to the interval $(\ln a, \ln b)$. It follows that $e^x \in (a,b) \cap \mathbb{Q}$ and $f(e^x) = r$, ending the proof.

If $r = 0$ the argument is the same, choosing $p = 2$ and $q = 3$.

Example 2.3. Let U be the denumerable set of non-empty open intervals with rational endpoints; let us well-order the elements of $U \times \mathbb{Q}$ by $((a_i, b_i), c_i)_{i \in \mathbb{N}}$.

We choose inductively a rational number q_i in each interval (a_i, b_i) in this way:

- $q_0 = \frac{a_0 + b_0}{2}$

- for any $i > 0$ we define $q_i = \frac{a_i+(n-1)b_i}{n}$, where n is the minimum natural number greater than 1 so that $q_i \neq q_j$ for any $j < i$ (there exists such an n because, for any $i \in \mathbb{N}$, only finitely many q_j have been already chosen).

Let us define $f : \mathbb{Q} \to \mathbb{Q}$ as follows:

$$f(q) = \begin{cases} c_i & \text{if } q = q_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

f is well-defined, because $q_i \neq q_j$ if $i \neq j$ and it is everywhere surjective on $\mathbb{Q}$: let y be a rational number and A be a non-empty open subset of $\mathbb{Q}$; there is an interval (a,b) with rational endpoints included in A. By construction, there exists an i so that $((a,b),y) = ((a_i,b_i),c_i)$; it follows that $f(q_i) = c_i = y$, where $q_i \in (a,b)$ as required.

Remark 2.4. The argument of the previous example allows us to build everywhere surjections from X to X where X is any denumerable dense subset of $\mathbb{R}$.

Another way to define everywhere surjections from $\mathbb{Q}$ to $\mathbb{Q}$ consists in finding $f : \mathbb{Q} \to \mathbb{N}$ such that $f(A) = \mathbb{N}$ for every non-empty open subset A of $\mathbb{Q}$; composing f with a bijection between $\mathbb{N}$ and $\mathbb{Q}$ we obtain an everywhere surjection from $\mathbb{Q}$ to $\mathbb{Q}$. We show two examples of "everywhere surjections from $\mathbb{Q}$ to $\mathbb{N}$".

Example 2.5. Let q be a rational number; we call $l(q)$ the number of digits of the minimum repetend of q in its decimal representation. The function $f : \mathbb{Q} \to \mathbb{N}$ defined by $f(q) = l(q) - 1$ is everywhere surjective: given an open interval (a,b) and a natural number n, we can find a rational q in (a,b) whose repetend has $n+1$ digits.

Since there are infinitely many prime numbers, a function between $\mathbb{Q}$ and the set of prime numbers is enough.

Example 2.6. Let $\mathbb{P}$ be the set of prime numbers; we define a function $f : \mathbb{Q} \to \mathbb{P} \cup \{0\}$ as follows:

$$f(q) = \begin{cases} p & \text{if } q = \dfrac{a}{p^n} \text{ for some } a \in \mathbb{Z},\, p \in \mathbb{P},\, n \in \mathbb{N} \text{ and } \gcd(a,p) = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We prove that f is everywhere surjective. Let p be a prime number and let $(q_1,q_2) \cap \mathbb{Q}$ be an open subset of $\mathbb{Q}$, where q_1,q_2 are rationals. We choose a natural number n so that $p^{-n} < q_2 - q_1$; there is a number a such that a/p^{n+1}

and $(a+1)/p^{n+1}$ belong to $(q_1,q_2)\cap\mathbb{Q}$. At least one of a and $a+1$ is coprime with p; we call it a'. The rational number a'/p^{n+1} is in $(q_1,q_2)\cap\mathbb{Q}$ and its image through f is p.

How many everywhere surjections are there from $\mathbb{Q}$ to $\mathbb{Q}$? To answer this question, we recall the following definition:

Definition 2.7 (see [1]). A subset X of a vector space is said to be *lineable* if $X\cup\{0\}$ contains an infinite-dimensional vector space.

Theorem 2.8. *The set $ES(\mathbb{Q})$ of everywhere surjections from $\mathbb{Q}$ to $\mathbb{Q}$ is uncountable and lineable (considering $ES(\mathbb{Q})$ as a subset of the vector space $\mathbb{Q}^{\mathbb{Q}}$ over the field $\mathbb{Q}$).*

Proof. Let $f:\mathbb{Q}\to\mathbb{Q}$ be an everywhere surjection and let $g:\mathbb{Q}\to\mathbb{Q}$ be a function; obviously, if $f(x)\neq g(x)$ only for $x\in\mathbb{N}$, g is everywhere surjective. It follows that every function obtained from a non-zero rational multiple of f modifying only the images of natural numbers is an everywhere surjection. The set consisting of all these functions and 0 is a vector space which is included in $ES(\mathbb{Q})\cup\{0\}$ and whose dimension is uncountable. More precisely, it is the direct sum $\mathbb{Q}\cdot f\oplus\mathbb{Q}^{\mathbb{N}}$, where $\mathbb{Q}^{\mathbb{N}}$ is the set of functions $h\in\mathbb{Q}^{\mathbb{Q}}$ such that $h(r)=0$ if $r\notin\mathbb{N}$. $\square$

3. Everywhere surjections between different sets

As seen in the previous section, in the construction of an everywhere surjection the only meaningful property of the codomain is its cardinality, while choosing a suitable topology for the domain is necessary. This leads to the following definition.

Definition 3.1. Let $(X,\mathcal{T})$ be a topological space and Y be a set. A function $f:X\to Y$ is said to be an *everywhere surjection from X to Y* if $f(A)=Y$ for every $A\in\mathcal{T}$ $(A\neq\varnothing)$. We will denote $ES(X,Y)$ the set of these functions; if $X=Y$, we will write $ES(X)$ instead of $ES(X,X)$.

If $ES(X,Y)$ is non-empty, of course it must be that $|X|\geq|Y|$.

Remark 3.2. Let $(X,\mathcal{T})$ be a topological space and let Y, Y' be two sets such that $|Y|\leq|Y'|$. If $ES(X,Y')$ is non-empty then $ES(X,Y)$ is non-empty.

In particular, we can deduce the existence of everywhere surjections from $\mathbb{R}$ to $\mathbb{Q}$ and from $\mathbb{R}$ to $\mathbb{R}\setminus\mathbb{Q}$. A function in $ES(\mathbb{R},\mathbb{Q})$, seen as a function from $\mathbb{R}$ to $\mathbb{R}$, is *quasi-everywhere surjective* according to the following definition.

Definition 3.3 (see [6])**.** A function $f:\mathbb{R}\to\mathbb{R}$ is said to be *quasi-everywhere surjective* if $f(A)$ is a dense subset of $\mathbb{R}$ for any non-empty open subset $A\subseteq\mathbb{R}$.

Quasi-everywhere surjections can also be obtained adding an everywhere surjection f of $ES(\mathbb{R})$ to a continuous function $g:\mathbb{R}\to\mathbb{R}$. Naïvely $f+g$ could seem to be an everywhere surjection: on small intervals continuous functions behave "almost like" constants, so $f+g$ on a small interval should behave like a translated everywhere surjection. As proved by the following counterexample, this argument is wrong.

Example 3.4. Let $u\in ES(\mathbb{R})$ and let us define a function $f\in\mathbb{R}^{\mathbb{R}}$ as follows

$$f(x)=\begin{cases}u(x) & \text{if } u(x)\neq x\\ x+1 & \text{if } u(x)=x\end{cases}$$

It is easy to prove that f is still an everywhere surjection, but its graph does not intersect the straight line $y=x$; if we add the continuous function $g(x)=-x$ to f we obtain the function $h(x)=f(x)-x$ which has no zeroes: $h=f+g$ is not even a surjection.

Nevertheless, as said before, we can prove the following

Theorem 3.5. *The sum of an everywhere surjection $f:\mathbb{R}\to\mathbb{R}$ and a continuous function $g:\mathbb{R}\to\mathbb{R}$ is a quasi-everywhere surjective function.*

Proof. We have to prove that the restriction of $f+g$ to every non-empty open interval (a,b) has dense image in $\mathbb{R}$. Let us consider an interval $(y-\varepsilon,y+\varepsilon)$, where $\varepsilon>0$, and let us prove that there exists at least a number x^* in (a,b) so that $(f+g)(x^*)\in(y-\varepsilon,y+\varepsilon)$.

Since g is continuous, if m is the minimum of g in $[a,b]$, there is an open interval $I\subseteq(a,b)$ such that $m\leq g(x)<m+\varepsilon$ for every $x\in I$. Moreover, f is an everywhere surjection, so there exists a number x^* in I such that $f(x^*)=y-m$. It follows that

$$\begin{cases}(f+g)(x^*)=y-m+g(x^*)<y-m+m+\varepsilon=y+\varepsilon\\ (f+g)(x^*)=y-m+g(x^*)\geq y\end{cases}$$

□

Of course $ES(\mathbb{Q},\mathbb{R})$ is empty; however there exist quasi-everywhere surjective functions from $\mathbb{Q}$ to $\mathbb{R}$: any everywhere surjection from $\mathbb{Q}$ to $\mathbb{Q}$, seen as a function from $\mathbb{Q}$ to $\mathbb{R}$, is an example. Also the set $ES(\mathbb{R}\setminus\mathbb{Q},\mathbb{R})$ is not empty; in order to give an example of an everywhere surjection from irrational numbers to real ones the following definition is useful.

Definition 3.6 (see [8]). A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *strongly everywhere surjective* if for any $y_0 \in \mathbb{R}$ the function f takes the value y_0 uncountably many times in any non-empty open subset $A \subseteq \mathbb{R}$.

An explicit example of a strongly everywhere surjective function can be found in [2]. But, as proved in the following theorem, strongly everywhere surjective functions can be easily constructed starting from everywhere surjections.

Theorem 3.7. *Let $f : \mathbb{R} \to \mathbb{R}$ be an everywhere surjection, let $h : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be a bijection and let $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be one of the two projections. The composition $phf : \mathbb{R} \to \mathbb{R}$ is strongly everywhere surjective.*

Proof. We have to prove that for every $y \in \mathbb{R}$ and for every non-empty open set $A \subseteq \mathbb{R}$ there exist uncountably many numbers in A whose image through phf is y. The inverse image of y through p is the set $\{y\} \times \mathbb{R}$, which is uncountable; the inverse image of $\{y\} \times \mathbb{R}$ through h is still uncountable, because h is bijective. Since f is everywhere surjective, for any element in $h^{-1}(\{y\} \times \mathbb{R})$ there exists at least one number in A which belongs to its inverse image. By construction, the subset of A of all these numbers is uncountable. $\square$

Now we give an example of an everywhere surjection from irrational numbers to real ones.

Example 3.8. Let $f : \mathbb{R} \to \mathbb{R}$ be a strongly everywhere surjective function; for any non-empty open subset $A \subseteq \mathbb{R}$ and for any $y_0 \in \mathbb{R}$ there exist $2^{\aleph_0}$ elements in A whose image through f is y_0. One of these elements must be irrational, since $\mathbb{Q}$ is countable; it follows that the restriction of f to irrational numbers is everywhere surjective (and, in fact, strongly everywhere surjective).

4. Functions which are everywhere surjective on their image

According to Definition 3.1, the Dirichlet function is an everywhere surjection from $\mathbb{R}$ to the set $\{0,1\}$. We give the following definition.

Definition 4.1. A function $f : X \to X$ is said to be *everywhere surjective on its image* if it is an everywhere surjection from X to $f(X)$.

The following theorem provides two sufficient conditions for a function to be everywhere surjective on its image. We recall that a function $f : \mathbb{R} \to \mathbb{R}$ is additive if $f(x+y) = f(x) + f(y)$ for every x, y in $\mathbb{R}$; and that, applying the axiom of choice, we obtain the existence of discontinuous additive functions (see [3]).

Theorem 4.2. *i) A periodic function $f : \mathbb{R} \to \mathbb{R}$ with no least positive period is everywhere surjective on its image.*

ii) Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function; if f is not injective then it is everywhere surjective on its image.

Proof. i) The set H of the periods of f is a subgroup of $\mathbb{R}$ (assuming $0 \in H$) with no least positive element, so it is dense in $\mathbb{R}$. It follows that for any element $y_0 = f(x_0)$ belonging to the image of f, the set $x_0 + H$ is dense in $\mathbb{R}$ and its image is y_0.

ii) Since f is not injective, there exist distinct points a and b such that $f(a) = f(b)$; keeping in mind that f is additive, on the one hand $f(a-b) = 0$ and on the other hand every rational multiple of $a-b$ is a period; we apply (i) to conclude.

□

5. Everywhere surjections defined on the set of sequences and a sequence of everywhere surjections

First, we intend to define an everywhere surjective function with the set $\mathbb{R}^{\mathbb{N}}$ of sequences of real numbers as domain; $\mathbb{R}^{\mathbb{N}}$ has the same cardinality as $\mathbb{R}$ but a different topology. Let us consider on $\mathbb{R}$ the discrete topology and then the product topology; if we find an everywhere surjective function from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}^{\mathbb{N}}$ with such a fine topology, we automatically find an everywhere surjection with respect to any other product topology on $\mathbb{R}^{\mathbb{N}}$.

A base for this topology is constituted by all sets of sequences obtained fixing finitely many terms and letting the others free to vary in $\mathbb{R}$; in some sense, this "freedom" makes easier the construction of everywhere surjections.

Example 5.1. Let s be a sequence in $\mathbb{R}^{\mathbb{N}}$ and let us consider the subsequences s_e and s_o

$$s_e(n) = s(2n) \qquad\qquad s_o(n) = s(2n+1)$$

for every $n \in \mathbb{N}$. We define the function $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ in the following way: for any sequence $s \in \mathbb{R}^{\mathbb{N}}$

$$f(s)(n) = \begin{cases} s_o(n + \lfloor t/2 \rfloor) & \text{if } \lim_{n\to\infty} s_e(n) = t \text{ and } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

where $\lfloor t \rfloor$ denotes the floor function of t.

We claim that f is everywhere surjective. Let us choose an open set U of the base by fixing $a_0, a_1, \ldots a_m$ as the first $m+1$ terms of sequences of U;

without loss of generality we can assume that m is even (if necessary, we can further restrict the open set). Let r be a sequence in $\mathbb{R}^{\mathbb{N}}$; we have to define a sequence s such that $s(0) = a_0, \dots, s(m) = a_m$ and $f(s) = r$. For any $n \in \mathbb{N}$, we set $s(m+2n) = m$ and $s(m+2n+1) = r(n)$: in this way $\lim_{n\to\infty} s_e(n) = m$ and so, for any n, $f(s)(n) = s_o(n+m/2) = s(2n+m+1) = r(n)$.

Now, let us examine a different situation: instead of everywhere surjections in the set of sequences, we construct a sequence of everywhere surjections from $\mathbb{R}$ to $\mathbb{R}$.

Let E be the set $\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid n < m\}$. For any $m \in \mathbb{N}$, the straight line of equation $x = m$ intersects E in finitely many points, while, for any $n \in \mathbb{N}$, the intersection between E and the straight line of equation $y = n$ is a denumerable set. A similar asymmetry can be found in the context of everywhere surjective functions: every straight line parallel to the x-axis intersects the graph G_f of an everywhere surjection f in a dense subset, while every straight line parallel to the y-axis intersects G_f in a single point.

We can define a subset A of the plane which has an analogous property in the following way:

$$A = \{(x,y) \in \mathbb{R}\times\mathbb{R} \mid y \prec x\}$$

where $\prec$ is a well-ordering on $\mathbb{R}$, with 0 being the least element (here the axiom of choice is needed).

Accepting the continuum hypothesis, we can assume that every real number has at most countably many predecessors, so that each straight line parallel to the y-axis intersects A in a countable set (hence in a set of Lebesgue measure 0); on the contrary, each straight line parallel to the x-axis meets A in an uncountable set. Even more can be said: the intersection between A and a straight line of equation $y = k$ is “almost the whole line”, because its complementary is countable (and has measure 0).

This remark leads us to try to describe A as the denumerable union of function graphs: can we find a sequence of everywhere surjections so that the union of their graphs coincides with A? The answer is yes.

We fix a well-ordering of type ω_1 (the smallest uncountable ordinal number) in the product $\mathbb{N}\times U\times\mathbb{R}$, where U is the set of non-empty open intervals with rational endpoints; each element of $\mathbb{N}\times U\times\mathbb{R}$ can be represented in the form $(n_\alpha, (a_\alpha, b_\alpha), y_\alpha)$, where α is a finite or denumerable ordinal, $n_\alpha \in \mathbb{N}$, $a_\alpha \in \mathbb{Q}$, $b_\alpha \in \mathbb{Q}$, $a_\alpha < b_\alpha$ and $y_\alpha \in \mathbb{R}$.

By induction, for any $\alpha < \omega_1$, let us pick the least (w.r.t. $\prec$) element x_α in (a_α, b_α) for which the following hold:

1. $y_\alpha \prec x_\alpha$;

2. $x_\alpha \neq x_\beta$ for any $\beta < \alpha$.

It is readily seen that such an x_α exists in (a_α, b_α); indeed, by the previous conditions 1 and 2, only countable many elements of (a_α, b_α) are excluded.

Now we can define the sequence $(f_n)_{n\in\mathbb{N}}$:

- If $x = x_\alpha$ for some $\alpha < \omega_1$, we put $f_{n_\alpha}(x_\alpha) = y_\alpha$ and for any n less than n_α we put $f_n(x) = 0$. Besides, let us consider the set $C_\alpha = \{y \in \mathbb{R} \mid y \neq y_\alpha \text{ and } y \prec x_\alpha\}$ which is countable and so can be ordered in a single sequence (since this has to be done for all α, we need again the axiom of choice). Therefore, for any natural k, we put $f_{n_\alpha+k}(x_\alpha)$ equal to the k-th element in C_α; if $Card(C_\alpha) = m < \aleph_0$, for every $n > n_\alpha + m$ we put $f_n(x) = 0$.
- If $x \neq x_\alpha$ for any $\alpha < \omega_1$, let us consider the set $D = \{y \in \mathbb{R} \mid y \prec x\}$, which is countable. If D is denumerable, we order it in a single sequence and define $f_n(x)$ to be the n-th element in D; if $Card(D) = m < \aleph_0$ the definition of $f_n(x)$ is similar, but we put $f_n(x) = 0$ for any $n > m$.

Clearly, each function f_n is well defined; now we prove that it is everywhere surjective. Let y be a real number and let (a,b) be a non-empty interval; there exists an ordinal α such that $n = n_\alpha$, $(a_\alpha, b_\alpha) \subseteq (a,b)$ and $y_\alpha = y$, so x_α is in (a_α, b_α) and $f_{n_\alpha}(x_\alpha) = y$. Moreover, the union of the graphs of all functions in the sequence coincides with A by construction.

Remark 5.2. We can read the previous result in a different way. For any real number x we define the function $\psi_x : \mathbb{N} \to \mathbb{R}$ with $\psi_x(n) = f_n(x)$. By construction, for any $y \in \mathbb{R}$ and for any $x \in \mathbb{R}$ such that $y \prec x$ (all real numbers but a countable amount) there is an n such that $f_n(x) = y$; therefore, for any y and for every real number x but a countable amount, there exists a natural n such that $\psi_x(n) = y$. The functions ψ_x, even if defined from $\mathbb{N}$ to $\mathbb{R}$, are “surjective” in this weak sense.

6. Everywhere surjections and Cantor sets

A typical example of an everywhere surjection from $\mathbb{R}$ to $\mathbb{R}$ is based on Cantor sets; in fact, Cantor set properties are well-suited to construct interesting examples in our context.

Example 6.1 (see [4])**.** Let $(I_n)_{n\in\mathbb{N}}$ be a sequence of all non-empty open intervals with rational endpoints; we consider a sequence $(C_n)_{n\in\mathbb{N}}$ of pairwise disjoint Cantor sets such that $C_n \subset I_n$ for any $n \in \mathbb{N}$. Choosing a bijection $\phi_n : C_n \to \mathbb{R}$ for any $n \in \mathbb{N}$, the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} \phi_n(x) & \text{if } x \in C_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

is an everywhere surjection.

In a sense, this example is "standard": each value $y_0 \in \mathbb{R} \setminus \{0\}$ is taken countably many times, exactly one for each interval I_n. It follows that, for any $y_0 \in \mathbb{R}$ except zero, the inverse image of y_0 is dense in $\mathbb{R}$ but has Lebesgue measure 0. Of course it is impossible to find an everywhere surjection from $\mathbb{R}$ to $\mathbb{R}$ such that the inverse image of each value has a positive measure; however we will show it is possible provided that we refer to a quasi-everywhere surjective function.

We recall the standard construction of a *fat Cantor set* (see [7]), which is a subset of $\mathbb{R}$ homeomorphic to the Cantor set but with positive Lebesgue measure. First of all we remove an open interval of length $1/4$ in the middle of the interval $[0,1]$; we obtain two closed intervals of length $3/8$. The n-th step consists in removing intervals of length 2^{-2n} from each of the 2^{n-1} closed intervals obtained at the previous step. We call $K \subset [0,1]$ the set of points which have not been removed in any step. The Lebesgue measure of K is

$$\mu(K) = 1 - \sum_{n=1}^{\infty} 2^{n-1} \cdot \frac{1}{2^{2n}} = 1 - \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2}.$$

K is closed and uncountable like the Cantor set.

Now we give an example of an everywhere surjection from $\mathbb{R}$ to $\mathbb{Q}$ based on K. We fix a sequence of all non-empty open intervals with rational endpoints $(a_i, b_i)_{i \in \mathbb{N}}$; by similarity, we build a fat Cantor set K_0 starting from a non-trivial closed interval $[a_0', b_0']$ included in (a_0, b_0). Going on by induction, we define a sequence of pairwise disjoint fat Cantor sets $(K_i)_{i \in \mathbb{N}}$ such that $K_i \subset (a_i, b_i)$ for any $i \in \mathbb{N}$. This is possible because fat Cantor sets are closed and do not include an open interval. Indeed, since the finite union of nowhere dense sets is nowhere dense, once first $n-1$ terms of the sequence K_i are defined, the set $(a_n, b_n) \setminus \bigcup_{i=0}^{n-1} K_i$ must contain a non-empty open interval, which in turn includes a non-trivial closed interval $[a_n', b_n']$. Therefore we can define K_n starting from $[a_n', b_n']$.

Once the whole sequence (K_i) is defined, we can prove the following theorem.

Theorem 6.2. *There exists an everywhere surjection $f : \mathbb{R} \to \mathbb{Q}$ such that, for any $y_0 \in \mathbb{Q}$, $f^{-1}(\{y_0\})$ has a positive Lebesgue measure.*

Proof. Referring to the sequence (K_i) described above, we define a sequence (ϕ_i) of functions $\phi_i : K_i \to \mathbb{Q}$. Let us order rational numbers in a sequence $(q_i)_{i \in \mathbb{N}}$. For simplicity's sake, we assume that we can choose $[a_0', b_0'] = [0,1]$. We put $\phi_0(x) = q_n$ for any $x \in K_0^{(n)}$, where $K_0^{(n)}$ is the intersection between K_0 and the second-last segment (from left to right) obtained at the $(n+1)$-th step

Figure 1: Construction of ϕ_0

of the algorithm in the construction of K_0; for example, $\phi_0(x) = q_0$ for any $x \in K_0^{(0)} = K_0 \cap [0, \frac{3}{8}]$, $\phi_0(x) = q_1$ for any $x \in K_0^{(1)} = K_0 \cap [\frac{5}{8}, \frac{25}{32}]$ and so on (see figure 1).

For any n, the inverse image of q_n through ϕ_0 is the set $K_0^{(n)}$ which has positive measure by construction.

Lastly, since there is no natural n such that $1 \in K_0^{(n)}$, we set $\phi_0(1) = 0$.

All the other functions $\phi_i : K_i \to \mathbb{Q}$ are defined through similarity to ϕ_0. We are ready to define the function $f : \mathbb{R} \to \mathbb{Q}$ whose existence we want to prove:

$$f(x) = \begin{cases} \phi_i(x) & \text{if } x \in K_i \text{ for some } i \in \mathbb{N} \\ 0 & \text{if } x \notin \bigcup_{i=0}^{\infty} K_i \end{cases}$$

f is everywhere surjective: for any non-empty open interval (a,b) there is an n such that $(a_n, b_n) \subseteq (a,b)$; so

$$f((a,b)) \supseteq f((a_n,b_n)) \supseteq f(K_n) = \mathbb{Q}.$$

Moreover, the inverse image of any rational number has a positive Lebesgue measure. $\square$

The Cantor set is homeomorphic to $\mathbb{Z}_2^{\mathbb{N}}$; therefore it can be regarded as a vector space over the field $\mathbb{Z}_2$. In the next theorem we prove the existence of an *everywhere surjective linear map* from $\mathbb{Z}_2^{\mathbb{N}}$ to itself; the construction is different from the ones we have seen in the previous sections. Note that a function $f : \mathbb{Z}_2^{\mathbb{N}} \to \mathbb{Z}_2^{\mathbb{N}}$ is a linear map if and only if it is additive, in the sense that $f(x+y) = f(x) + f(y)$ for all x and y where the symbol $+$ denotes addition in $\mathbb{Z}_2^{\mathbb{N}}$ and not in $\mathbb{R}$.

Theorem 6.3. *There exists an everywhere surjection $f : \mathbb{Z}_2^{\mathbb{N}} \to \mathbb{Z}_2^{\mathbb{N}}$ which is a linear map when $\mathbb{Z}_2^{\mathbb{N}}$ is regarded as a vector space over $\mathbb{Z}_2$.*

Proof. Call A the set of elements of $\mathbb{Z}_2^{\mathbb{N}}$ which have exactly one component equal to 1. Then consider a basis $\mathcal{B}$ of $\mathbb{Z}_2^{\mathbb{N}}$ such that $A \subseteq \mathcal{B}$; of course, $\mathcal{B}$ has the cardinality of the continuum. Now define the linear map f in such a way that:

- $f(v) = 0$ for every $v \in A$
- $f|_{\mathcal{B}\setminus A}$ is a bijection between $\mathcal{B}\setminus A$ and the whole $\mathbb{Z}_2^{\mathbb{N}}$.

We claim that f is an everywhere surjection. Indeed, let a number $y \in \mathbb{Z}_2^{\mathbb{N}}$ and an interval (a,b) be given. There is a finite sequence $s = (x_0,\dots,x_k)$ such that, if s represents the first components of a number, then this number belongs to the interval (a,b). By the definition of f, there is an $x^* \in \mathcal{B}\setminus A$ such that $f(x^*) = y$. Now adding to x^* suitable elements $v_1, \dots, v_i$ of A (more precisely, adding those elements of A that have the digit 1 where x^* differs from s), we obtain an element x whose first $k+1$ components coincide with s, while the other components are the same as in x^*. We conclude that $x \in (a,b)$ and $f(x) = f(x^*)+f(v_1)+\dots+f(v_n) = y$.

□

7. Everywhere surjections in a strong sense

We have seen that $ES(C) \neq \varnothing$, where C is the usual Cantor set; here is also an example of a function in $ES(C,\mathbb{R})$ that will be useful for further constructions.

Example 7.1. First we define a function $\psi : [0,1] \to [0,1]$. Write any real number $x \in [0,1]$ in ternary notation and then we change the digits of x according to the following rules:

1. if x has the digit 1 in its ternary expansion, we preserve the first occurrence of the digit 1 and convert all the following digits into 0;
2. we convert any digit 2 into 1;

reading the outcome in binary notation we get $\psi(x)$. The function $\psi : [0,1] \to [0,1]$ just defined is nothing but the well-known Cantor function; note that the definition works also for numbers with two expansions in base-3 (e.g. 0.1 and $0.0\overline{2}$).

Let $g : [0,1] \to \mathbb{R}$ be the restriction to $[0,1]$ of an everywhere surjection on $\mathbb{R}$. The function $f = g \circ \psi|_C : C \to \mathbb{R}$ is everywhere surjective. Indeed, let y_0 be a real number and $A = (a,b)\cap C$ be a non-empty open subset of C; since $\psi|_C$ is surjective and non-decreasing by construction, the image $\psi|_C(A)$ is a non-trivial interval $I \subseteq [0,1]$; since g is everywhere surjective, there exists $z \in I$ such that $g(z) = y_0$. It follows that we can choose $x \in \psi|_C^{-1}(z)$ so that x belongs to A and $f(x) = y_0$.

A similar construction can be used to define a function $\mathbb{R}$ to $\mathbb{R}$ that is everywhere surjective in a stronger sense.

Theorem 7.2. *There exists an everywhere surjection $f : \mathbb{R} \to \mathbb{R}$ such that, for any non-empty open subset $A \subseteq \mathbb{R}$, there is a subset $B \subseteq A$ such that $f|_B : B \to \mathbb{R}$ is a continuous increasing bijection.*

Proof. We aim to restrict the Cantor function ψ to a subset $H \subset C$ so that $\psi|_H : H \to [0,1]$ is an increasing bijection; we define H by removing each element of C that admits an infinite sequence of consecutive digits 2 in its ternary expansion. Geometrically, this means to remove the denumerable set of left endpoints of the open intervals removed during the construction of the Cantor set. $\psi|_H$ is non-decreasing and bijective by construction; moreover, it is easy to prove that $\psi|_H$ is continuous.

Now, let us order in a sequence $(I_n)_{n\in\mathbb{N}}$ all non-empty intervals with rational endpoints and let us define subsets $H_n \subset I_n$ analogous to the set H constructed above in $[0,1]$, so that $H_n \cap H_m = \varnothing$ if $n \neq m$ (which is possible because H is nowhere dense). For any n, let $\psi_n : H_n \to \mathbb{R}$ be a function defined as above starting from the Cantor function; the function we are looking for is $f : \mathbb{R} \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} \psi_n(x) & \text{if } x \in H_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

□

In order to generalize the previous result, we give the following definition. We consider functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition 7.3. A function f *everywhere simulates* a function g if, for every non-empty interval (a,b), there exists a strictly increasing function $h_g : \mathbb{R} \to (a,b)$ such that $g = f \circ h_g$.

Since h_g is an order-preserving function, in some sense, in every interval (a,b) there exists a set A included in (a,b) such that the graph of $f|_A$ has a "shape" similar to the graph of g.

$$\begin{tikzcd} \mathbb{R} \arrow[rr, "g"] \arrow[dr, "h_g"'] & & \mathbb{R} \\ & \mathbb{R} \arrow[ur, "f"'] & \end{tikzcd}$$

Theorem 7.4. *i) There is no function that everywhere simulates all functions.*

ii) There is a function which everywhere simulates all continuous functions.

iii) A function that everywhere simulates all continuous functions is strongly everywhere surjective.

Proof. i) We can prove this statement by cardinality reasons, starting from the following known lemma. As usual, the letter $\mathfrak{c}$ denotes the cardinality of the continuum.

Lemma 7.5. *The set of increasing functions has cardinality $\mathfrak{c}$. Therefore, for any function f, the cardinality of the set of functions simulated by f is at most $\mathfrak{c}$.*

Proof. There are several ways to prove the lemma. A quick way refers to Borel sets: an increasing function is a measurable function and, as a consequence, its graph is a Borel set. But the set of Borel sets has cardinality $\mathfrak{c}$. However, the last claim is not completely trivial, so we prefer a different proof. Call G_h the graph of an increasing function h. The difference $\overline{G_h} \setminus G_h$ is finite or denumerable, since, for every x_0, at most two points of the kind (x_0, y) can belong to that difference. If at least one point (x_0, y) belongs to $\overline{G_h} \setminus G_h$, the interval $(\sup_{x<x_0} h(x), \inf_{x>x_0} h(x))$ is non-empty; but this fact can happen only denumerable many times since the considered intervals are pairwise disjoint for different values of x_0. Now the claim follows because the set of closed sets in the plane has cardinality $\mathfrak{c}$, and so the same holds for the set of graphs of increasing functions. □

ii) More generally we will prove that, for any set F of functions such that $Card(F) = \mathfrak{c}$, there is a function f which everywhere simulates all functions in F. First note that, if C is the Cantor set, we have

$$C \approx 2^{\mathbb{N}} \approx 2^{\mathbb{N}+\mathbb{N}} \approx C \times C$$

In other words, we obtain a bijection ϕ from C to $C \times C$ as follows: if $x = 0.x_0x_1x_2\dots$ is an element of C (written in base 3, so that every x_i is either 0 or 2), we define $\phi(x) = (0.x_0x_2\dots, 0.x_1x_3\dots)$. Note that ϕ is a continuous function. In this way we have decomposed C in a continuum of sets, which are isomorphic to C and pairwise disjoint: indeed $C \times C = \bigcup_{w \in C} \{w\} \times C$. Now, since the set F has cardinality $\mathfrak{c}$, we can use the elements of C for indexing the set of functions: $F = \{g_w \mid w \in C\}$. As usual, for any interval (a_i, b_i) with rational endpoints, we can construct a Cantor set $C_i \subseteq (a_i, b_i)$, so that these Cantor sets C_i are pairwise disjoint. Moreover, keeping in mind the previous remark, we can write every C_i as $\bigcup_{w \in C} C_{i,w}$ where all $C_{i,w}$ are pairwise disjoint Cantor sets. So, for every i and w there is an increasing function from $C_{i,w}$ to $\mathbb{R}$, which is obtained by identifying ”consecutive” endpoints of $C_{i,w}$. We call $k_{i,w}$ the restriction of this function to the set obtained removing all right endpoints from $C_{i,w}$. In this way $k_{i,w}$ is an injective increasing function from $C_{i,w}$ to $\mathbb{R}$. We are in a position to define the function f. Given a function g_w and a non-empty interval (a,b), let C_i be such that $C_i \subseteq (a,b)$. Then define h_{g_w} to be $k_{i,w}^{-1}$; of course, h_{g_w} is an increasing function from $\mathbb{R}$ to $C_{i,w} \subseteq (a,b)$. Finally, we set

$$f(x) = \begin{cases} g_w \circ h_{g_w}^{-1}(x) & \text{if } x \in C_{i,w} \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $f \circ h_{g_w}(x) = g_w(x)$.

iii) This statement is nearly obvious: a function that everywhere simulates all the constant functions is strongly everywhere surjective.

□

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REFERENCES

[1] R. M. Aron, V. I. Gurariy, J. B. Seoane-Sepúlveda, *Lineability and spaceability of functions on* $\mathbb{R}$, Proceedings of the American Mathematical Society 133 (2005), 795-803.

[2] C. Bernardi, *About the region below the graph of a function*, Bollettino dell’Unione Matematica Italiana 8 (2015), 1-8.

[3] C. Bernardi, *Discontinous additive functions: regular behavior vs. pathological features*, Expositiones Mathematicae 33 (2015), 295-307.

[4] L. Bernal-González, D. Pellegrino, J. B. Seoane-Seplveda, *Linear subsets of non-linear sets in topological vector spaces*, Bulletin of the American Mathematical Society 51 (2014), 71-130.

[5] K. C. Ciesielski, J. L. Gámez-Merino, L. Mazza, J. B. Seoane-Sepúlveda, *Cardinal coefficients related to surjectivity, Darboux, and Sierpiński-Zygmund maps*, Proceedings of the American Mathematical Society 145 (2017), 1041-1052.

[6] A. Farokhinia, *Lineability of space of quasi-everywhere surjective functions*, Journal of Mathematical Extension 6 (2012), 45-51.

[7] B. R. Gelbaum, J. M. H. Olmsted, *Theorems and counterexamples in mathematics*, Springer, New York (1990).

[8] J. L. Gámez-Merino, G. A. Muñoz-Fernández, V. M. Sánchez, J. B. Seoane-Sepúlveda, *Sierpiński-Zygmund functions and other problems on lineability*, Proceedings of the American Mathematical Society 138 (2010), 3863-3876.

[9] H. Lebesgue, *Leçons sur l'intgration et la recherche des fonctions primitives*, Gauthier-Villars (1904).

[10] G. Oman, *The converse of the intermediate value theorem from Conway to Cantor to cosets and beyond*, Missouri Journal of Mathematical Sciences 26 (2014), 134-150.

[11] V. Pambuccian, *Another example of an exotic function*, American Mathematical Monthly 96 (1989), 913-914.

[12] D. G. Radcliffe, *A function that is surjective on every interval*, American Mathematical Monthly 123 (2016), 88-89.

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