Regular Two-Graphs and Equiangular Lines

by
Shonda Gosselin

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Abstract

Regular two-graphs are antipodal distance-regular double coverings of the complete graph, and they have many interesting combinatorial properties. We derive a construction for regular two-graphs containing cliques of specified order from their connection to large sets of equiangular lines in Euclidean space. It is shown that the existence of a regular two-graph with least eigenvalue τ containing a clique of order d depends on the existence of an incidence structure on d points with special properties. Quasi-symmetric designs provide examples of these incidence structures.

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Contents

Chapter 1

Introduction

We discuss the relationship between regular two-graphs and large sets of equiangular lines in Euclidean space, and use linear algebra to derive a construction for regular two-graphs with cliques of specified order. Regular two-graphs are antipodal distance-regular graphs of diameter three. These are the first nontrivial class of antipodal distance-regular graphs, and they have many interesting combinatorial properties. They can be viewed as two-fold covers of the complete graph, and doing so will help uncover many of their important characteristics.

1.1 Covering Graphs and Two-Graphs

Let G be a graph, and suppose that there is a partition Π of its vertices into cells such that each cell is an independent set, and between any two cell C_1 and C_2, either there are no edges, or there is an induced matching of size $\max\{C_1, C_2\}$. Let G/Π be the graph with the cells of Π as vertices, in which two cells are adjacent if and only if there is an induced matching between them in G. Then we say that G is a *covering graph* of G/Π. The graph G/Π is called the *quotient graph* of G over Π. The quotient graph G/Π often preserves some of the structure of G. In particular, the characteristic polynomial of G/Π always divides that of G. The map sending each vertex in G to its corresponding cell is called the *covering map*, and the cells are called *fibres*. If G is connected, then each fibre has the same size, called the *index* of the covering. If the index is r, then G is called an *r-fold covering* of G/Π.

Let X be a graph with n vertices. The *switching graph* $\mathrm{Sw}(X)$ of X is the graph with vertex set $V(X) \times \{0, 1\}$, in which (v, i) is adjacent to (w, i)

if and only if $vw \in E(X)$, and (v, i) is adjacent to $(w, 1 - i)$ if and only if $v \neq w$ and $vw \notin E(X)$. A switching graph is also called a *two-graph*. Let Π be the partition of the vertices of $\mathrm{Sw}(X)$ into the n cells of the form $\{(v, 0), (v, 1)\}$, for v in X. Since there is an induced matching in $\mathrm{Sw}(X)$ between any two distinct cells in Π, we see that the two-graph $\mathrm{Sw}(X)$ is a two-fold covering of the complete graph on n vertices, K_n, for any graph X, and if $\mathrm{Sw}(X)$ is connected, it has diameter three. Thus every eigenvalue of K_n is also an eigenvalue of $\mathrm{Sw}(X)$ with the same multiplicity. Hence the two-graph $\mathrm{Sw}(X)$ has the n eigenvalues of K_n, and it has n more eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, called the *nontrivial eigenvalues* of the two-graph. A *regular two-graph* is a two-graph with only two distinct nontrivial eigenvalues.

1.2 Regular Two-Graphs

A connected graph is *distance regular* if for any two vertices u and v, the number of vertices at distance i from u and j from v depends only on i, j, and the distance between u and v. These graphs are necessarily regular, since u may equal v. A connected graph of diameter d is *antipodal* if the vertices at distance d from a given vertex are all at distance d from each other. The antipodal distance-regular graphs are covering graphs (see page 32-33). In Theorem 3.14 we will provide a proof due to Godsil and Hensel [9] that a two-graph is regular if and only if it is an antipodal distance-regular two-fold cover of K_n (page 36). The antipodal distance-regular graphs of diameter one are the complete graphs, and those of diameter two are the complete bipartite graphs $K_{n,n}$, so those of diameter three are the first nontrivial case. These graphs are r-fold coverings of the complete graph, for some integer r. Thus, one reason for studying regular two-graphs is to gain insight into the general structure of antipodal distance-regular graphs. Distance-regular graphs have important relations to other areas of combinatorics, such as finite geometry and coding theory.

1.3 Equiangular Lines

A set of lines through the origin of $\mathbb{R}^d$ is *equiangular* if the angle between any two distinct lines in the set is the same. Regular two-graphs have an important connection to the problem of finding the maximum number $v(d)$ of equiangular lines in $\mathbb{R}^d$, which is the maximum size of a simplex in real projective space. In the late 1960s, Van Lint and Seidel [22] expressed this geometric problem in graph-theoretic terms. The *Seidel matrix* $S(X)$ of a graph X is defined by

$$S(X) = I + 2A(X) - J,$$

in which $A(X)$ is the adjacency matrix of X, I is the identity matrix, and J is the all ones matrix. Van Lint and Seidel used linear algebra to show that the problem of finding a set of n equiangular lines in $\mathbb{R}^d$ is equivalent to the problem of finding graphs on n vertices whose least Seidel matrix eigenvalue has multiplicity $n - d$ (see page 7 - 9). They derived a bound on the size of a set of equiangular lines in $\mathbb{R}^d$ with mutual angle θ in terms of d and $\cos(\theta)$, called the *relative bound* (see Theorem 2.3, page 11). They showed that sets of equiangular lines which meet this relative bound correspond to graphs X such that $\mathrm{Sw}(X)$ is a regular two-graph (see Theorem 2.8, page 19). Consequently, in the late 1960s and early 1970s, many algebraic graph theorists found constructions for regular two-graphs with the aim of finding large sets of equiangular lines. They also found connections between these structures and finite simple groups.

In Chapter 4, we will take the opposite approach. We will use linear algebraic methods to construct large sets of equiangular lines, and use them to construct regular two-graphs which contain cliques of a specified order. A *positive basis* of a set Ω of unit vectors spanning a set of equiangular lines in $\mathbb{R}^d$ is a subset of Ω of order d in which the inner product of any two vectors is positive. A positive basis of Ω corresponds to a clique in the corresponding two-graph. We will show that a positive basis B is a basis of $\mathbb{R}^d$, and then use linear algebraic methods to characterize the vectors in $\Omega \backslash B$ (see page 44 - 50). By relating these vectors to subsets of $\{1, 2, \ldots, d\}$, we will show that the existence of a regular two-graph with a clique of order d having least eigenvalue τ depends on the existence of an incidence structure with special properties (Theorem 4.4, page 50), of which quasi-symmetric designs are one example.

1.4 Outline of the Thesis

In Chapter 2, we discuss the relationship between large sets of equiangular lines and regular two-graphs. We will first describe the graph-theoretic approach of Van Lint and Seidel to finding a set of n equiangular lines in $\mathbb{R}^d$. We present a linear algebraic proof of Van Lint and Seidel's relative bound on the number of equiangular lines in $\mathbb{R}^d$ which have mutual angle θ, and we show that the switching graphs of graphs corresponding to sets of lines which meet the relative bound are regular two-graphs. We also discuss an *absolute bound* on the number $v(d)$ of equiangular lines in $\mathbb{R}^d$ due to Gerzon, which depends only on the dimension d of the space. Examples will be given for which these bounds are tight.

In Chapter 3 we will define strongly regular graphs and discuss some of their properties, and then provide a proof due to Godsil and Royle [10] that the neighbourhoods of regular two-graphs are all strongly regular with the same parameters. Then we will discuss antipodal distance-regular graphs, and view regular two-graphs as antipodal distance-regular two-fold covers of the complete graph. We will look at results regarding antipodal distance-regular graphs due to Brouwer, Cohen, Neumaier [1] and Gardiner [8], which show that they are covering graphs. We provide a proof due to Godsil and Hensel [9] which implies that regular two-graphs are the distance-regular two-fold covers of K_n. This view of regular two-graphs will help us restrict the parameters of these structures. We will generate and list the feasible parameter sets for regular two-graphs on n vertices, for $n \leq 100$, in Table 3.1 and 3.2.

In Chapter 4, we derive a linear algebraic construction for regular two-graphs with cliques of specified order, and connect these graphs to incidence structures with special properties. We discuss quasi-symmetric designs, which provide examples of these incidence structures. Finally, we construct regular two-graphs on some of the parameter sets generated in Chapter 3.

Chapter 2

Equiangular Lines and Two-Graphs

2.1 Introduction

A *simplex* in a metric space V is a subset of V in which the distance between any two distinct points is the same. One of the founding problems of algebraic graph theory is that of finding the maximum number of points in a simplex in real projective space. The points in real projective space of dimension $d-1$ are the lines through the origin of $\mathbb{R}^d$, and the distance between two such lines is determined by the angle between them. A set of lines is *equiangular* if the angle between any two distinct lines in the set is the same. Hence, the problem of finding the maximum size of a simplex in $(d-1)$-dimensional real projective space is equivalent to that of finding the maximum size of a set of equiangular lines in $\mathbb{R}^d$.

Investigations into the maximum number of equiangular lines in Euclidean space were initiated by Haantjes [13] in 1948 in the terminology of elliptic geometry, and he solved this problem for dimensions at most four using geometric methods. At first glance, this geometric problem seems to have little connection to graphs. However in 1966, Van Lint and Seidel [22] used linear algebraic techniques to show that this problem can be expressed in graph-theoretic terms. They introduced a $(0, \pm 1)$-adjacency matrix for a graph, called the *Seidel matrix*, and associated it to the Gram matrix for a set of unit vectors spanning a set of equiangular lines. They showed that a set of n equiangular lines in $\mathbb{R}^d$ can be viewed as a double cover of the complete graph on n vertices, called a *two-graph*.

Van Lint's and Seidel's graph-theoretic approach to this problem in elliptic geometry led many algebraic graph theorists to find constructions for graphs which give rise to large sets of equiangular lines. Many of these constructions were found in the late 1960's and early 1970's, and are connected to combinatorial designs. Complete descriptions of these constructions can be found in works by Seidel [19], [17], [18], [16], Goethels and Seidel [11], [12], Van Lint and Seidel [22], Bussemaker and Seidel [2], Taylor and Seidel [20], Taylor [21], Delsarte and Goethals [7], Hestenes and Higman [14], and Conway [5]. Many of these constructions also have significance in the theory of finite simple groups, but these connections will not be discussed in this thesis. More recently, in 2000 De Caen [6] found an infinite family of graphs which yield large equiangular line sets.

In this chapter, we discuss the graph-theoretic approach of Van Lint and Seidel to the problem of finding large sets of equiangular lines in $\mathbb{R}^d$. In Section 2.2, it is shown that this problem is equivalent to the problem of finding graphs for which the least Seidel matrix eigenvalue has large multiplicity. In Sections 2.3 and 2.4, we look at two bounds on the maximum number of equiangular lines in $\mathbb{R}^d$. One is an absolute bound which depends only on the dimension d, and the other is a relative bound, which depends on both d and the mutual angle between the lines. Examples of graphs for which these bounds are tight are provided. In Section 2.5, the sharpness of the absolute bound is examined in detail. In Section 2.6, the operation of switching is discussed, and it is demonstrated that a set of equiangular lines is equivalent to a two-graph.

There are several good surveys of equiangular lines and two-graphs in [15], [19] and [20], but the approach taken in this chapter is that of Godsil and Royle in [10].

2.2 Equiangular Lines and Graphs

Given any set of vectors $\Omega = \{u_1, u_2, \ldots, u_n\}$ in $\mathbb{R}^d$, the *Gram matrix* of the vectors in Ω is the $n \times n$ matrix G such that $G_{i,j} = u_i^T u_j$. Observe that if U is a matrix, $U^T U$ is the Gram matrix of the columns of U. This matrix is positive semidefinite and has the same rank as U. Conversely, any symmetric, positive semidefinite matrix of rank d is the Gram matrix for the columns of a $d \times n$ matrix, as Lemma 2.1 shows. This is a standard result in Linear Algebra, and it demonstrates that we can represent any set of vectors by its Gram matrix.

2.1 Lemma. *Let G be an $n \times n$ symmetric matrix. Then G is a positive semidefinite matrix of rank d if and only if there is a $d \times n$ matrix U of rank*

d such that $G = U^T U$.

Proof: Suppose G is an $n \times n$ positive semidefinite matrix of rank d. Since G is symmetric, it is orthogonally diagonalizable, and hence

$$G = L^T \Lambda L,$$

in which Λ is the $n \times n$ diagonal matrix whose i-th diagonal entry is the i-th eigenvalue of G, and L is the $n \times n$ orthogonal matrix whose i-th column is an eigenvector corresponding to the i-th eigenvalue of G. Since G has rank d, the dimension of the kernel of G is $n - d$. Hence zero is an eigenvalue of G with multiplicity $n - d$. We can assume, without loss of generality, that the last $n - d$ diagonal entries of Λ are zero. This implies that

$$G = M^T \Lambda' M,$$

in which Λ' is the $d \times d$ diagonal matrix of non-zero eigenvalues of G, and M is the $d \times n$ matrix whose rows consist of the first d rows of L.

Now since G is positive semidefinite, all of its eigenvalues are non-negative, and hence all of its non-zero eigenvalues are positive. Thus there is a $d \times d$ diagonal matrix D such that $D^2 = \Lambda'$. Now $U = DM$ is a $d \times n$ matrix such that $U^T U = M^T D^2 M = M^T \Lambda' M = G$. Since L^T is orthogonal, its columns are linearly independent, and hence the columns of M^T are linearly independent. Thus $\mathrm{rk}(M) = \mathrm{rk}(M^T) = d$. Since the entries of D are all positive,

$$\mathrm{rk}(U) = \mathrm{rk}(DM) = \mathrm{rk}(M) = d.$$

Conversely, if U is a $d \times n$ matrix such that $G = U^T U$, then for any vector u in $\mathbb{R}^d$, we have

$$u^T G u = u^T U^T U u = (Uu)^T (Uu) \geq 0.$$

Hence G is positive semidefinite. It remains to show that G has the same rank as U.

Suppose that $U^T U x = 0$. Then $x^T U^T U x = (Ux)^T (Ux) = 0$. This implies that $Ux = 0$. Also, if $Ux = 0$, then clearly $U^T U x = 0$. Thus U and $U^T U$ have the same kernel, and since they both have n columns, we must have that $\mathrm{rk}(U) = \mathrm{rk}(U^T U) = \mathrm{rk}(G)$. □

Now let $\Omega = \{x_1, x_2, \ldots, x_n\}$ be a set of unit vectors spanning a set of n equiangular lines in R^d with mutual angle θ. Then for $i \neq j$,

$$\langle x_i, x_j \rangle = x_i^T x_j = \pm \cos(\theta).$$

Let U be the $d \times n$ matrix with the vectors of Ω as its columns. Then $G = U^T U$ is the Gram matrix of the vectors in Ω, and thus G is a symmetric, positive semidefinite matrix of rank d. Let $\alpha = \cos(\theta)$. Then G has the form

$$G = I + \alpha S,$$

in which I is the identity matrix, and S is an $n \times n$ symmetric $(0, \pm 1)$-matrix with all diagonal entries equal to zero, and all off-diagonal entries equal to ± 1. We can view S as a nonstandard adjacency matrix of a graph X on n vertices, in which two distinct vertices i and j of X are adjacent if $S_{i,j} = 1$, and nonadjacent if $S_{i,j} = -1$. This matrix is called the *Seidel matrix* of X, and is denoted by $S(X)$. Clearly, if $\overline{X}$ is the complement of the graph X, then $S(X) = -S(\overline{X})$. The Seidel matrix of X is related to the usual adjacency matrix $A(X)$ of X by

$$S(X) = I + 2A(X) - J,$$

in which J denotes the $n \times n$ matrix with every entry equal to 1. Hence, a set of unit vectors spanning a set of n equiangular lines gives rise to a graph on n vertices.

The *trace* of a square matrix A is the sum of the diagonal entries of A, denoted by $\mathrm{tr}(A)$. Note that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any two $n \times n$ matrices A and B, and if A and B are similar matrices, then $\mathrm{tr}(A) = \mathrm{tr}(B)$.

Now suppose that we are given a graph X on n vertices with Seidel matrix S. Since S is symmetric, it is similar to its diagonal matrix of eigenvalues, D, so $\mathrm{tr}(S) = \mathrm{tr}(D)$. Since $\mathrm{tr}(S) = 0$, the eigenvalues of S sum to zero. Since $S \neq 0$, we must have that the least eigenvalue of S is negative. If this eigenvalue is $-\alpha$, then

$$G = I + \frac{1}{\alpha} S \tag{2.1}$$

is a positive semidefinite matrix. If the rank of G is d, then by Lemma 2.1 it is the Gram matrix for a set Ω of n vectors in $\mathbb{R}^d$, and Equation 2.1 shows that Ω is a set of unit vectors spanning a set of equiangular lines with mutual angle $\arccos(1/\alpha)$. Therefore, any graph X on n vertices gives rise to a set of n unit vectors spanning a set of equiangular lines in $\mathbb{R}^d$, for some positive integer d.

Now suppose that X is a graph on n vertices and that its Seidel matrix S has least eigenvalue $-\alpha$ with multiplicity k. Since S is symmetric, the geometric multiplicity of an eigenvalue of S is equal to its algebraic multiplicity. Thus the dimension of the eigenspace associated to the eigenvalue

$-\alpha$ is k. Hence

$$\dim(\ker(S+\alpha I)) = k.$$

Thus

$$n-k = \mathrm{rk}(S+\alpha I) = \mathrm{rk}(I+\frac{1}{\alpha}S) = \mathrm{rk}(G),$$

where G is the Gram matrix for a set of unit vectors spanning a set of equiangular lines in $\mathbb{R}^{n-k}$, with mutual angle $\arccos(1/\alpha)$.

Hence, maximizing the multiplicity of the least eigenvalue of the Seidel matrix of X minimizes the dimension of the Euclidean space in which the associated set of equiangular lines exist. Thus the geometric problem of finding the least integer d such that there are n equiangular lines in $\mathbb{R}^d$ is equivalent to the graph-theoretic problem of finding graphs X on n vertices such that the multiplicity of the least eigenvalue of $S(X)$ is as large as possible.

The set $\Omega = \{x_1, x_2, \ldots, x_n\}$ of spanning unit vectors we choose to represent a set of equiangular lines is not unique, since $-x_i$ spans the same line as x_i, and choosing different spanning sets may lead to different graphs. However, in Section 2.6 it will be shown that these graphs are related.

Example. Let X be the line graph of K_8, denoted by $L(K_8)$. This graph has order 28. The eigenvalues of $S(X)$ are 9 and -3 with multiplicities 7 and 21, respectively. Hence

$$G = I + \frac{1}{3}S(X)$$

is the Gram matrix of a set of 28 unit vectors spanning a set of 28 equiangular lines in $\mathbb{R}^7$, with mutual angle $\arccos(1/3)$.

The 28 equiangular lines in $\mathbb{R}^7$ given by $L(K_8)$ in the above example can also be obtained by taking the $\binom{8}{2}$ unit vectors in $\mathbb{R}^8$ of the form

$$x^T = 1/\sqrt{24}(3,3,-1,-1,-1,-1,-1,-1),$$

with two entries equal to 3 and the remaining six entries equal to -1. One can verify that for two distinct vectors x and y of this form,

$$x^T y = \pm\frac{1}{3},$$

where the positive sign is taken if and only if x and y have an entry of 3 in the same position. Since all of these vectors lie in the space of vectors in $\mathbb{R}^8$ which are orthogonal to the all ones vector $\mathbf{1}$, and since this space has dimension 7, these unit vectors can be mapped to a set of 28 unit vectors spanning a set of equiangular lines in $\mathbb{R}^7$, with mutual angle $\arccos(1/3)$. In the next section, it will be shown that 28 is the maximum number of equiangular lines possible in $\mathbb{R}^7$.

2.3 The Absolute Bound

We present two bounds on the number of equiangular lines in $\mathbb{R}^d$. The first is due to Gerzon in a private communication with Lemmens and Seidel. It is called the *absolute bound* because it does not depend on the common angle between the lines.

2.2 Theorem. (The Absolute Bound) *If there are n equiangular lines in $\mathbb{R}^d$, then*

$$n \leq \binom{d+1}{2}.$$

Proof: Let $\{x_1, x_2, \ldots, x_n\}$ be a set of unit vectors spanning a set of n equiangular lines in $\mathbb{R}^d$, with mutual angle $\arccos(\alpha)$, and let

$$X_i = x_i x_i^T.$$

We show that the matrices X_i $(1 \leq i \leq n)$ form a linearly independent set in the space of symmetric $d \times d$ matrices, which has dimension $\binom{d+1}{2}$.

First, observe that for each i, X_i is a symmetric $d \times d$ matrix and $X_i^2 = X_i$. Thus X_i represents orthogonal projection onto its column space, namely the line spanned by x_i. For $i \neq j$, we have

$$X_i X_j = x_i x_i^T x_j x_j^T = (x_i^T x_j) x_i x_j^T,$$

and thus

$$\mathrm{tr}(X_i X_j) = (x_i^T x_j)^2 = \alpha^2. \tag{2.2}$$

Also

$$\mathrm{tr}(X_i^2) = \mathrm{tr}(X_i) = \mathrm{tr}(x_i x_i^T) = x_i^T x_i = 1. \tag{2.3}$$

Now suppose that

$$Y = \sum_{i=1}^{n} c_i X_i.$$

Then

$$\begin{aligned}
\operatorname{tr}(Y^2) &= \sum_{i,j} c_i c_j \; tr(X_i X_j) \\
&= \sum_{i=1}^{n} c_i^2 + \sum_{i,j:i\neq j} c_i c_j \; \alpha^2 \\
&= \alpha^2 \left(\sum_{i=1}^{n} c_i \right)^2 + (1-\alpha^2) \sum_{i=1}^{n} c_i^2. \qquad (2.4)
\end{aligned}$$

Since $0 < \alpha < 1$, Equation 2.4 implies that $\operatorname{tr}(Y^2) = 0$ if and only if $c_i = 0$ for all i. Since Y is symmetric, $\operatorname{tr}(Y^2) \geq 0$ with equality if and only if $Y = 0$. Therefore $Y = 0$ if and only if $c_i = 0$ for all i, and hence the X_i form a linearly independent set in the space of symmetric $d \times d$ matrices. Hence $n \leq \binom{d+1}{2}$. □

The set of 28 equiangular lines in $\mathbb{R}^7$ associated to the graph $L(K_8)$ of Example 2.2 meets the absolute bound.

2.4 The Relative Bound

We next consider a relative bound on the number of equiangular lines in $\mathbb{R}^d$, which depends on both d and the cosine of the angle between the lines. This result is due to Van Lint and Seidel in [22].

2.3 Theorem. (The Relative Bound) *Suppose that there are n equiangular lines in $\mathbb{R}^d$ with mutual angle* $\arccos(\alpha)$*. If $\frac{1}{\alpha^2} > d$, then*

$$n \leq \frac{d - d\alpha^2}{1 - d\alpha^2}.$$

If $X_1, X_2, \ldots, X_n$ are the projections onto these lines, then equality holds if and only if

$$\sum_{i=1}^{n} X_i = \frac{n}{d} I.$$

Proof: Set

$$Y = I - \frac{d}{n} \sum_{i=1}^{n} X_i.$$

Since Y is symmetric, $tr(Y^2) \geq 0$, with equality if and only if $Y = 0$. Observe that

$$Y^2 = I - \frac{2d}{n}\sum_{i=1}^{n} X_i + \frac{d^2}{n^2}\left(\sum_{i=1}^{n} X_i\right)^2,$$

and thus

$$0 \leq \operatorname{tr}(Y^2) = \operatorname{tr}(I) - \frac{2d}{n}\sum_{i=1}^{n}\operatorname{tr}(X_i) + \frac{d^2}{n^2}\operatorname{tr}\left(\sum_{i=1}^{n} X_i\right)^2.$$

Hence

$$0 \leq d - \frac{2d}{n}(n) + \frac{d^2}{n^2}\left(\sum_{i=1}^{n}\operatorname{tr}(X_i^2) + \sum_{i\neq j}\operatorname{tr}(X_iX_j)\right).$$

Now by Equation 2.2, $\operatorname{tr}(X_iX_j) = \alpha^2$ and by Equation 2.3, $\operatorname{tr}(X_i^2) = 1$. Thus

$$0 \leq -\, d + \frac{d^2}{n^2}(n + n(n-1)\alpha^2).$$

Solving for n in the above inequality, we obtain

$$n \leq \frac{d - d\alpha^2}{1 - d\alpha^2},$$

whenever $1/\alpha^2 > d$. Moreover, equality holds if and only if $Y = 0$, in which case

$$\sum_{i=1}^{n} X_i = \frac{n}{d} I,$$

as required. □

Example. Let X be the Petersen graph, $\overline{L(K_5)}$. The eigenvalues of $S(X)$ are 3 and -3 with equal mutiplicity 5, thus

$$G = I + \frac{1}{3}S(X)$$

is the Gram matrix for a set of unit vectors spanning a set of 10 equiangular lines in $\mathbb{R}^5$, with mutual angle $\arccos(1/3)$. This meets the relative bound, but not the absolute bound.

2.5 Sharpness of the Absolute Bound

If a set of n equiangular lines in $\mathbb{R}^d$ meets the absolute bound, then the proof of Theorem 2.2 shows that the projections $X_1, X_2, \ldots, X_n$ onto the lines form a basis for the space of $d \times d$ symmetric matrices. In particular,

$$I \in \operatorname{span}\{X_1, X_2, \ldots, X_n\}. \tag{2.5}$$

Note that 2.5 also follows if equality holds in the relative bound of Theorem 2.3. The result in Lemma 2.5 is due to Van Lint and Seidel. It shows that having I in the span of the projections onto the lines has significant consequences, whether or not the absolute bound is met. The linear algebraic proof provided here is that of Godsil and Royle in [10], and it uses the following result from [10, page 166], which is stated here without proof.

2.4 Lemma. *Let A and B be matrices such that AB and BA are both defined. Then AB and BA have the same nonzero eigenvalues with the same multiplicities.* □

2.5 Lemma. *Suppose that $X_1, X_2, \ldots, X_n$ are the projections onto a set of equiangular lines in $\mathbb{R}^d$ with mutual angle* $\arccos(\alpha)$. *If*

$$I = \sum_{i=1}^{n} c_i X_i,$$

then $c_i = d/n$ for all i, and

$$n = \frac{d - d\alpha^2}{1 - d\alpha^2}.$$

The Seidel matrix determined by any set of n unit vectors spanning these lines has eigenvalues

$$-\frac{1}{\alpha}, \quad \frac{n-d}{d\alpha}$$

with multiplicities $n - d$ and d, respectively. If $n \neq 2d$, then $1/\alpha$ is an integer.

Proof: For any j in $\{1, 2, \ldots, n\}$, we have

$$X_j = X_j I = X_j \sum_{i=1}^{n} c_i X_i = \sum_{i=1}^{n} c_i X_i X_j.$$

Hence

$$1 = \operatorname{tr}(X_j) = \sum_{i=1}^{n} c_i \operatorname{tr}(X_i X_j) = (1-\alpha^2)c_j + \alpha^2 \sum_{i=1}^{n} c_i.$$

Thus

$$1 = (1-\alpha^2)c_j + \alpha^2 \sum_{i=1}^{n} c_i, \tag{2.6}$$

so

$$c_j = \frac{1 - \alpha^2 \left(\sum_{i=1}^{n} c_i\right)}{1-\alpha^2},$$

for all j. Hence, the c_i are all equal $(1 \le i \le n)$. Since

$$d = \operatorname{tr}(I) = \sum_{i=1}^{n} c_i \operatorname{tr}(X_i) = \sum_{i=1}^{n} c_i = nc_i,$$

we must have that $c_i = d/n$ for all i. Now Equation 2.6 implies that

$$n = \frac{d - d\alpha^2}{1 - d\alpha^2}.$$

Now let $\Omega = \{x_i, x_2, \ldots, x_n\}$ be a set of unit vectors spanning the n equiangular lines. Then $X_i = x_i x_i^T$. Let U be the $d \times n$ matrix with the vectors in Ω as its columns. Then

$$UU^T = \sum_{i=1}^{n} x_i x_i^T = \sum_{i=1}^{n} X_i = \frac{n}{d} I,$$

and

$$U^T U = I + \alpha S,$$

where S is the Seidel matrix determined by Ω. By Lemma 2.4, the matrices UU^T and $U^T U$ have the same nonzero eigenvalues with the same multiplicities. Now UU^T is $d \times d$, so it follows that $U^T U = I + \alpha S$ has eigenvalues 0 and n/d with multiplicities $n-d$ and d, respectively. We obtain the required eigenvalues of S.

Since S has integer entries, the eigenvalues of S are algebraic integers, so they are either integers or algebraically conjugate, and so have the same multiplicity. If $n \neq 2d$, the multiplicities are different, so in this case $1/\alpha$ is an integer. $\square$

d	n	$1/\alpha$	Graph
2	3	2	K_3
3	6	$\sqrt{5}$	$C_5 \cup K_1$
7	28	3	$\overline{L(K_8)}$
14	105	4	- - -
23	276	5	Section 4.4

Table 2.1: Sharpness of the Absolute Bound

If the absolute bound of Theorem 2.2 is met, then there are $n = \binom{d+1}{2}$ equiangular lines in $\mathbb{R}^d$. In this case, Lemma 2.5 implies that $d+2 = 1/\alpha^2$. If $d \neq 3$, then $n \neq 2d$, so $1/\alpha$ is an integer and hence $d+2$ must be a perfect square. Table 2.1 lists some integers $d \geq 2$ for which there could be a set of equiangular lines in $\mathbb{R}^d$ meeting the absolute bound.

In [15], a result due to Peter Neumann shows that if $n > 2d$, then $1/\alpha$ is an odd integer, so there do not exist 105 equiangular lines in $\mathbb{R}^{14}$. The last column of Table 2.1 lists graphs which give rise to equiangular line sets meeting the absolute bound in each of the other four cases. In [12], Goethals and Seidel constructed a strongly regular graph which gives rise to a set of 276 equiangular lines in $\mathbb{R}^{23}$, and they showed that the two-graph associated to this graph is the unique regular two-graph on 276 vertices. This graph is also constructed in [10] by Godsil and Royle using the Witt design on 23 points. In Chapter 4, we will take a different approach to constructing this graph, and show that it contains a clique of order 23.

It is not know whether there are any other examples of equiangular line sets for which the absolute bound is tight. In fact, not much is known about the maximum number $v(d)$ of equiangular lines in $\mathbb{R}^d$ in general. The best asymptotic result to date is de Caen's construction in [6] for an infinite family of graphs which give rise to $2/9(d+1)^2$ equiangular lines in $\mathbb{R}^d$, for integers d of the form $d = 3(2^{2t-1}) - 1$. This is the first known constructive lower bound on $v(d)$ of order d^2.

2.6 Switching and Two-Graphs

Let U be a set of n equiangular lines in $\mathbb{R}^d$, and let $\Omega = \{x_1, x_1, \ldots, x_d\}$ be a set of unit vectors spanning U. Of course Ω is not unique, since the unit vector x spans the same line as $-x$, so there are 2^n possible choices for Ω. Different choices for Ω may have different Gram matrices, and hence may yield different graphs. However, we shall see that these graphs are related.

Suppose that $\sigma \subseteq \{1, 2, \ldots, n\}$ and that we form Ω' from Ω be replacing each vector x_i in Ω by $-x_i$ whenever $i \in \sigma$. Let G be the Gram matrix for the vectors in Ω and let G' be the Gram matrix for the vectors in Ω'. Then G' is obtained from G by multiplying the rows and columns of G indexed by σ by -1. The graph X' corresponding to Ω' arises from the graph X corresponding to Ω by changing all of the edges between σ and $V(X) \setminus \sigma$ to non-edges, and all of the non-edges between σ and $V(X) \setminus \sigma$ to edges. We call this operation *switching* on the subset σ.

For a graph X and a subset σ of $V(X)$, let X^σ denote the graph obtained from X by switching on σ. The set of graphs which can be obtained from X by switching on any subset of $V(X)$ is called the *switching class* of X. If Y is isomorphic to X^σ for some subset σ of $V(X)$, we say that X and Y are *switching equivalent*. The set of all graphs on n vertices can be partitioned into a finite number of switching classes, and a set of n equiangular lines determines one of these switching classes.

In Section 2.2 we saw that the set of equiangular lines determined by any graph on n vertices depends only on the eigenvalues of the Seidel matrix of the graph. Lemma 2.6 shows that the Seidel matrices of all graphs in the same switching class have the same eigenvalues with the same multiplicities. Hence these eigenvalues can be considered constant parameters of the switching class.

2.6 Lemma. *Let X be a graph and let σ be a subset of $V(X)$. Then $S(X)$ and $S(X^\sigma)$ have the same eigenvalues.*

Proof: We show that $S(X)$ and $S(X^\sigma)$ are similar matrices. Let D be the diagonal matrix whose rows and columns are indexed by the vertices of X, in which $D_{uu} = -1$ if $u \in \sigma$, and $D_{uu} = 1$ otherwise. Then $D^2 = I$, and

$$S(X^\sigma) = DS(X)D.$$

Thus $S(X^\sigma)$ and $S(X)$ are similar matrices, and therefore have the same eigenvalues with the same multiplicities. □

Let X be a graph with n vertices and let $N(v)$ denote the neighbourhood of the vertex v in X. Then $X^{N(v)}$ is a graph with the vertex v isolated. Let X_v denote the graph on $n-1$ vertices obtained from $X^{N(v)}$ by deleting the isolated vertex v. For any vertex v, there is a unique graph in the switching class of X with v isolated, so the collection of graphs

$$\{X_v : v \in V(X)\}$$

is independent of the choice of X, and depends only on the switching class of X. We call this collection of graphs the *neighbourhoods* of the switching class.

We now define a graph on $2n$ vertices associated to any switching class of graphs on n vertices. Given a graph X, the *switching graph* $\mathrm{Sw}(X)$ of X is the graph with vertex set

$$V(X) \times \{0, 1\}$$

in which (x, i) is adjacent to (y, i) if and only if $xy \in E(X)$, and (x, i) is adjacent to $(y, 1-i)$ if and only if $x \neq y$ and $xy \notin E(X)$. Note that $\mathrm{Sw}(X)$ is a regular graph on $2n$ vertices with valency $n-1$, and $\mathrm{Sw}(X)$ contains two copies of the graph X, namely the subgraphs induced by the vertices of $X \times \{i\}$, for $i = 0, 1$.

If $v \in V(X)$, then the vertices $(v, 0)$ and $(v, 1)$ have disjoint neighbourhoods of size $n-1$ in $\mathrm{Sw}(X)$, both of which are isomorphic to X_v. Hence $\mathrm{Sw}(X)$ is completely determined by any one of its neighbourhoods, and if it is connected, it has diameter three. Thus $\mathrm{Sw}(X)$ is determined only by the switching class of X, rather than by the particular choice of X, so two graphs X and Y are switching equivalent if and only if $\mathrm{Sw}(X)$ is isomorphic to $\mathrm{Sw}(Y)$. Also, its clear that $\mathrm{Sw}(\overline{X}) = \overline{\mathrm{Sw}(X)}$.

A switching graph is also called a *two-graph*. Lemma 2.6 and the previous discussion imply that there is a one-to-one correspondence between switching classes of graphs on n vertices and sets of n equiangular lines in Euclidean space. Since graphs in the same switching class give rise to isomorphic switching graphs, there is a one-to-one correspondence between sets of equiangular lines in Euclidean space and two-graphs. Sometimes two-graphs are defined as switching class of graphs, but when we refer to a two-graph on n vertices, we mean the switching graph associated to a switching class of graphs on n vertices.

Let X be a graph, let A denote the adjacency matrix of X, and let $\overline{A}$ denote the adjacency matrix of the complement of X. Then the adjacency matrix of $\mathrm{Sw}(X)$ is

$$\begin{pmatrix} A & \overline{A} \\ \overline{A} & A \end{pmatrix}$$

and $S(X) = A - \overline{A}$. Since

$$\begin{pmatrix} I & -I \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

and

$$\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} \begin{pmatrix} A & \overline{A} \\ \overline{A} & A \end{pmatrix} \begin{pmatrix} I & I \\ 0 & -I \end{pmatrix} = \begin{pmatrix} J-I & 0 \\ \overline{A} & S(X) \end{pmatrix},$$

it follows that the matrices

$$\begin{pmatrix} A & \overline{A} \\ \overline{A} & A \end{pmatrix}, \quad \begin{pmatrix} J-I & 0 \\ \overline{A} & S(X) \end{pmatrix}$$

are similar. Since $J - I$ is the adjacency matrix of the complete graph, this implies that the characteristic polynomial of $\mathrm{Sw}(X)$ is the product of the characteristic polynomial of K_n with the characteristic polynomial of $S(X)$. Thus $\mathrm{Sw}(X)$ has eigenvalues -1 and $n-1$ with multiplicities $n-1$ and 1, respectively, and the eigenvalues of $\mathrm{Sw}(X)$, which are the eigenvalues of the switching class of graphs associated to $\mathrm{Sw}(X)$. We call the eigenvalues of the switching class of X the *nontrivial eigenvalues* of the two-graph $\mathrm{Sw}(X)$.

A *regular two-graph* is a two-graph with only two nontrivial eigenvalues. Hence a two-graph is regular if the corresponding switching class of graphs has only two distinct eigenvalues. Lemma 2.5 implies that if a set of equiangular lines meets the absolute bound or Theorem 2.2 or the relative bound of Theorem 2.3, then the corresponding two-graph has only two nontrivial eigenvalues, and hence it is a regular two-graph. In fact, the converse is also true, as the next lemma shows.

2.7 Lemma. *A two-graph has only two nontrivial eigenvalues if and only if the set of equiangular lines it determines meets the relative bound of Theorem 2.3.*

Proof: If there is a set of equiangular lines which meets the relative bound, then by Theorem 2.3, I is in the span of the projections onto the lines, and so by Lemma 2.5, the Seidel matrix of any set of unit vectors spanning the lines has just two eigenvalues. These are the eigenvalues of the switching class of the associated graph, so the two-graph determined by this set of equiangular lines has just two nontrivial eigenvalues.

Conversely, suppose that Γ is a two-graph on n vertices with only two nontrivial eigenvalues, and that the set of equiangular lines that it determines lies in $\mathbb{R}^d$ and has mutual angle $\arccos(\alpha)$. Let S be the Seidel matrix for a set $\Omega = \{x_1, x_2, \ldots, x_n\}$ of unit vectors spanning these lines. We showed in Section 2.2 that the least eigenvalue of S is $-1/\alpha$ with multiplicity $n-d$. Let θ be the other eigenvalue. Since S is symmetric, it is similar to its diagonal matrix D of eigenvalues. Since $\mathrm{tr}(D) = \mathrm{tr}(S) = 0$, the sum of the eigenvalues of S is zero. Hence

$$-\frac{1}{\alpha}(n-d) + \theta d = 0,$$

which implies that

$$\theta = \frac{n-d}{\alpha d}.$$

Now suppose that $1/\alpha^2 > d$ and let $X_1, X_2, \ldots, X_n$ be the projections onto the set of n equiangular lines. Then $X_i = x_i x_i^T$, for each i. By Theorem 2.3,

$$n \leq \frac{d - d\alpha^2}{1 - d\alpha^2},$$

and equality holds if and only if

$$\sum_{i=1}^{n} X_i = \frac{n}{d} I.$$

Let U be the $d \times n$ matrix with $x_1, x_2, \ldots, x_n$ as its columns. Then

$$UU^T = \sum_{i=1}^{n} x_i x_i^T = \sum_{i=1}^{n} X_i.$$

Also,

$$U^T U = I + \alpha S$$

is the Gram matrix for the vectors in Ω. By Lemma 2.4, UU^T and $U^T U$ have the same nonzero eigenvalues with the same multiplicities. Since S has eigenvalues $-1/\alpha$ and $\frac{n-d}{d\alpha}$ with multiplicities $n-d$ and d, respectively, $U^T U$ has eigenvalues 0 and n/d with multiplicities $n-d$ and d, respectively. This implies that the $d \times d$ matrix UU^T has eigenvalue n/d with multiplicity d. We must have that

$$UU^T = \sum_{i=1}^{n} X_i = \frac{n}{d} I,$$

and so the relative bound of Theorem 2.3 holds with equality. □

The following result is immediate.

2.8 Theorem. *Let $0 < \alpha < 1$ and let d be an integer such that $d \geq 2$. Let*

$$n = \frac{d - d\alpha^2}{1 - d\alpha^2}.$$

There exists a set of n equiangular lines in $\mathbb{R}^d$ with mutual angle $\arccos(\alpha)$ if and only if there is a regular two-graph on n vertices with least eigenvalue $-1/\alpha$ of multiplicity $n-d$. □

In Chapter 3, several interesting properties of regular two-graphs will be established, and we will use these properties to generate feasible parameter sets for these structures.

Chapter 3

Regular Two-Graphs

3.1 Introduction

In Chapter 2 we showed shows that regular two-graphs determine sets of equiangular lines which meet the relative bound, so these structures can help us find large sets of equiangular lines. These graphs are also interesting combinatorially, as they have many interesting regularity properties. In this chapter, we discuss some of these properties, and use them to generate feasible parameter sets for these structures.

In Section 3.2, we define strongly regular graphs and develop some of their well known properties. In Section 3.3, we provide a proof due to Godsil and Royle [10] that all of the neighbourhoods of a regular two-graph are strongly regular with the same parameters, and that there is no regular two graph on n vertices if n is odd. In Section 3.4, we discuss antipodal distance-regular graphs and state some of their well known properties. In Section 3.5, we view regular two-graphs as double covers of the complete graph, and we show that a two-graph on n vertices is regular if and only if it is an antipodal distance-regular cover of K_n with diameter three. The proof of this result is due to Godsil and Hensel [9], and it will provide us with restrictions on the parameters of any regular two-graph. Finally, in Section 3.5, we use the properties of regular two-graphs established in the previous sections to generate feasible parameter sets.

3.2 Strongly Regular Graphs

A graph X which is neither complete nor empty is said to be *strongly regular* with parameters (n, k, a, c) if $V(X) = n$, X is k-regular, every pair of adjacent vertices has a common neighbours, and every pair of nonadjacent vertices has c common neighbours.

Example. The 5-cycle C_5 is a strongly regular graph with parameters $(5, 2, 0, 1)$.

Example. The Petersen graph, $\overline{L(K_5)}$, is strongly regular with parameters $(10, 3, 0, 1)$.

Example. Two families of strongly regular graphs are provided by the line graphs of K_n and $K_{n,n}$. The graph $L(K_n)$ has parameters

$$(n(n-1)/2, 2n-4, n-2, 4),$$

and the graph $L(K_{n,n})$ has parameters

$$(n^2, 2n-2, n-2, 2).$$

Example. The *Paley Graphs*: Let q be a prime power such that $q \equiv 0 \bmod 4$. The *Paley graph* $P(q)$ is the graph with vertex set $GF(q)$, the elements of the finite field of order q, in which two vertices are adjacent if and only if their difference is a nonzero square in $GF(q)$. The Paley graph $P(q)$ is strongly regular with parameters

$$(q, (q-1)/2, (q-5)/4, (q-1)/4).$$

The complement of a strongly regular graph with parameters (n, k, a, c) is strongly regular with parameters $(n, \overline{k}, \overline{a}, \overline{c})$, where

$$\begin{aligned}\overline{k} &= n-k-1,\\ \overline{a} &= n-2-2k+c,\\ \overline{c} &= n-2k+a.\end{aligned}$$

A strongly regular graph X is *primitive* if both X and its complement are connected. Otherwise X is *imprimitive*. There is only one class of imprimitive strongly regular graphs. These are the disconnected graphs, every component of which is complete, or their complements, the complete multipartite graphs.

3.1 Lemma. *Let X be strongly regular graph with parameters (n, k, a, c). Then the following statements are equivalent:*

(a) *X is not connected,*

(b) *$c = 0$,*

(c) *$a = k - 1$,*

(d) *X is isomorphic to mK_{k-1}, for some $m > 1$.*

Proof: Suppose X is not connected and let C be a component of X. If $v \in C$, then v has no common neighbour with any vertex not in C, so $c = 0$. If $c = 0$, then for any vertex v of X, any two neighbours of v must be adjacent, so $a = k - 1$. If $a = k - 1$, then any component must be K_{k+1}. Since X is not complete, there must be at least two components, so $X = mK_{k+1}$ for some $m > 1$, which implies that X is not connected. □

The next result from [10] shows that the parameters of a strongly regular graph are related.

3.2 Lemma. *Let X be a strongly regular graph with parameters (n, k, a, c). Then*

$$k(k - a - 1) = c(n - k - 1).$$

Proof: A vertex v in $V(X)$ has k neighbours and $n-1-k$ non-neighbours. We count the number of edges e between the neighbours and non-neighbours of v in two ways. Each of the k neighbours of v is adjacent to v and to a neighbours of v, and hence to $k - a - 1$ non-neighbours of v. Thus $e = k(k - a - 1)$. Also, each of the $n - k - 1$ non-neighbours of v is adjacent to c neighbours of v, so $e = c(n - k - 1)$. Hence

$$k(k - a - 1) = e = c(n - k - 1),$$

as required. □

Lemma 3.2 gives a simple feasibility condition that must be satisfied by the parameters of any strongly regular graph X. We can obtain some stronger feasibility conditions on the parameters by looking at the eigenvalues of $A(X)$ and their multiplicities. These values can be computed directly from the parameters of X. We will make use of the following two results in [10, page 165]. These are standard results in linear algebra, and are provided here without proof.

3.3 Lemma. *Let X be a graph with adjacency matrix A, and let u and v be vertices of X. The number of walks from u to v in X of length 2 is the uv-entry of A^2.* □

3.4 Lemma. *Let A be a real symmetric matrix. If x and y are eigenvectors of A with different eigenvalues, then $\langle x, y\rangle = 0$ (i.e. x and y are orthogonal).* □

For an eigenvalue λ of a matrix A, let m_λ denote the multiplicity of λ.

3.5 Theorem. *Let X be a connected strongly regular graph with parameters (n, k, a, c), such that $c < k$. Let*

$$\Delta = (a - c)^2 + 4(k - c).$$

Then X has three eigenvalues k, θ_1 and τ_1, where

$$\theta_1 = \frac{1}{2}\left((a-c) + \sqrt{\Delta}\right), \quad \tau_1 = \frac{1}{2}\left((a-c) - \sqrt{\Delta}\ \right),$$

with multiplicities $m_k = 1$,

$$m_{\theta_1} = \frac{(n-1)\tau_1 + k}{\tau_1 - \theta_1}, \quad m_{\tau_1} = \frac{(n-1)\theta_1 + k}{\theta_1 - \tau_1}.$$

Proof: Let A be the adjacency matrix of X. By Lemma 3.3, the uv-entry of A^2 is the number of walks of length 2 from u to v. Since X is strongly regular, this number is completely determined by whether u and v are equal, adjacent, or distinct and nonadjacent. We have

$$A^2_{uv} = \begin{cases} k, & \text{if } u = v, \\ a, & \text{if } uv \in E(X), \\ c, & \text{if } u \neq v \text{ and } uv \notin E(X). \end{cases}$$

Thus

$$A^2 = kI + aA + c(J - I - A),$$

which implies that

$$A^2 - (a - c)A - (k - c)I = cJ. \tag{3.1}$$

We will use this equation to find the eigenvalues of A. Now X is k-regular, so $\mathbf{1}$ is an eigenvector of A with eigenvalue k, and since X is connected, the eigenspace of A associated to the eigenvalue k is spanned by $\mathbf{1}$, so $m_k = 1$. Let z be an eigenvector of A with eigenvalue λ, where $\lambda \neq k$. Then by Lemma 3.4, $\mathbf{1}^T z = 0$, so $Jz = 0$. Hence

$$A^2 z - (a - c)Az - (k - c)Iz = cJz = 0,$$

and so

$$\lambda^2 - (a-c)\lambda - (k-c) = 0. \tag{3.2}$$

Thus the eigenvalues of A not equal to k are the zeroes of the quadratic in Equation 3.2. Since A is a real symmetric matrix, these zeroes are both real. Denote the two real zeroes of this quadratic by θ_1 and τ_1. Then

$$\theta_1 = \frac{1}{2}\left((a-c) + \sqrt{\Delta}\right), \quad \tau_1 = \frac{1}{2}\left((a-c) - \sqrt{\Delta}\right).$$

Now since $\theta_1\tau_1 = c-k$ and $c<k$, θ_1 and τ_1 are nonzero with opposite sign, and $\theta_1 > \tau_1$.

To find the multiplicities of the eigenvalues, observe that the sum of all the eigenvalues is equal to $\mathrm{tr}(A)$, which is 0. Thus

$$\theta_1 m_{\theta_1} + \tau_1 m_{\tau_1} + k = 0.$$

Also, since there are n eigenvalues, we have

$$m_{\theta_1} + m_{\tau_1} + 1 = n.$$

Solving this system of two linear equations gives the required values for m_{θ_1} and m_{τ_1}. □

We can also compute the multiplicities m_{θ_1} and m_{τ_1} directly from the parameters (n,k,a,c) of the strongly regular graph X. Since

$$(\theta_1 - \tau_1)^2 = (\theta_1 + \tau_1)^2 - 4\theta_1\tau_1 = (a-c)^2 + 4(k-c) = \Delta,$$

we can substitute the values for θ_1 and τ_1 in Theorem 3.5 into the expressions for m_{θ_1} and m_{τ_1} to obtain

$$m_{\theta_1} = \frac{1}{2}\left((n-1) - \frac{2k + (n-1)(a-c)}{\sqrt{\Delta}}\right)$$

and

$$m_{\tau_1} = \frac{1}{2}\left((n-1) + \frac{2k + (n-1)(a-c)}{\sqrt{\Delta}}\right).$$

Given a parameter set (n,k,a,c), we can compute m_{θ_1} and m_{τ_1} using the above equations. If the results are not positive integers, there can be no strongly regular graph with these parameters. This yields a powerful feasibility condition.

Theorem 3.5 also shows that if $k \neq c$, then a connected strongly regular graph X with parameters (n,k,a,c) has exactly three distinct eigenvalues. The following result in [10] shows that for regular connected graphs, the converse is also true.

3.6 Lemma. *A connected regular graph with exactly three distinct eigenvalues is strongly regular.*

Proof: Suppose X is connected and k-regular with eigenvalues k, θ_1 and τ_1. Let A be the adjacency matrix of X. Set

$$M := \frac{1}{(k-\theta_1)(k-\tau_1)}(A-\theta_1 I)(A-\tau_1 I).$$

Then if x is an eigenvector of A with eigenvalue θ_1 or τ_1, then Mx is the zero vector, and if x is a constant vector, $Mx = x$. Hence all of the eigenvalues of M are equal to 0 or 1, and any eigenvector of M with eigenvalue θ_1 or τ_1 lies in the kernel of M. Hence $\mathrm{rk}(M) = m_k$, which is equal to 1 since X is connected. Since M is symmetric, $\mathbf{1}^T M = M\mathbf{1} = \mathbf{1}$. We must have that $M = \frac{1}{n}J$.

Since M is a quadratic matrix polynomial in A, the fact that $M = \frac{1}{n}J$ implies that A^2 is a linear combination of the matrices I, J, and A. Thus we can write A^2 as a linear combination of the matrices I, A, and $J-I-A$. Hence the uv-entry of A^2 is determined only by whether the vertices u and v are equal, adjacent, or distinct and nonadjacent. By Lemma 3.3, the uv-entry of A^2 is the number of common neighbours of u and v. Hence X is strongly regular. □

We can obtain the parameters (n, k, a, c) of a strongly regular graph X directly from the eigenvalues of X, as the next result shows.

3.7 Lemma. *Let X be a connected strongly regular graph with parameters (n, k, a, c) and eigenvalues k, θ_1 and τ_1. Then*

(1) $n = \dfrac{(k-\theta_1)(k-\tau_1)}{k+\theta_1\tau_1}$,

(2) $a = k + \theta_1 + \tau_1 + \theta_1\tau_1$,

(3) $c = k + \theta_1\tau_1$.

Proof: Substituting the expressions for θ_1 and τ_1 given in Theorem 3.5, Equations (2) and (3) can be easily verified. These imply that

$$\begin{aligned}
\frac{(k-\theta_1)(k-\tau_1)}{k+\theta_1\tau_1} &= (1/c)\left[k^2-\theta_1 k-\tau_1 k+\theta_1\tau_1\right] \\
&= (1/c)\left[(c-\theta_1\tau_1)^2-\theta_1(c-\theta_1\tau_1)-\tau_1(c-\theta_1\tau_1)+\theta_1\tau_1\right] \\
&= (1/c)\left[(c-\theta_1\tau_1)(c-\theta_1\tau_1-\theta_1-\tau_1)+\theta_1\tau_1\right] \\
&= (1/c)\left[(c-(c-k))(c+(k-a))+(c-k)\right] \\
&= (k/c)(k-a-1)+k+1 \\
&= (n-k-1)+k+1 \qquad \text{(by Lemma 3.2)} \\
&= n.
\end{aligned}$$

This proves (1). □

We can get another feasibility condition on the parameters of a strongly regular graph from the inequalities known as the *Krein bounds*. The proof of this result can be found in [10, page 231], and it relies on the Cauchy-Schwarz inequality.

3.8 Theorem. *Let X be a primitive strongly regular graph with parameters (n,k,a,c) and eigenvalues k, θ_1 and τ_1. Then*

$$\theta_1{\tau_1}^2-2{\theta_1}^2\tau_1-{\theta_1}^2-k\theta_1+k{\tau_1}^2+2k\tau_1\geq 0,$$

and

$$\theta_1{}^2\tau_1-2\theta_1{\tau_1}^2-{\tau_1}^2-k\tau_1+k{\theta_1}^2+2k\theta_1\geq 0.$$

If the first inequality is tight, then $k\geq m_{\theta_1}$. If the second inequality is tight, then $k\geq m_{\tau_1}$. □

In the next section, we show that the neighbourhoods of a regular two-graph are all strongly regular with the same parameters. We will use the feasibility conditions on these strongly regular neighbourhoods obtained in this section to restrict the parameters of regular two-graphs.

3.3 Properties of Regular Two-Graphs

The switching graphs of the complete graph and the empty graph are regular two-graphs, and the neighbourhoods of these two-graphs are all complete or empty, respectively. We call these two-graphs trivial. In this section, we provide a proof due to Godsil and Royle in [10] that the neighbourhoods of nontrivial regular two-graphs are all strongly regular with the same parameters, and we show how these parameters are related to the parameters of the two-graph. First, we require the following standard result from linear algebra.

3.9 Lemma. *Let A be a real symmetric matrix, and let $p(x)$ be the characteristic polynomial of A. Then $p(A) = 0$.* □

3.10 Theorem. *Let Γ be a nontrivial two-graph on $n+1$ vertices. Then the following statements are equivalent:*

(a) *Γ is a regular two-graph,*

(b) *All of the neighbourhoods of Γ are regular,*

(c) *All of the neighbourhoods of Γ are (n,k,a,c)-strongly regular graphs with $k = 2c$,*

(d) *One neighbourhood of Γ is a (n,k,a,c)-strongly regular graph with $k = 2c$.*

Proof: (a)⇒(b): Suppose Γ is a regular two-graph. Let S be the Seidel matrix of some neighbourhood of Γ. Adjoining an isolated vertex v to this neighbourhood, we obtain a graph X such that $\Gamma = \mathrm{Sw}(X)$, and

$$T = \begin{pmatrix} 0 & -\mathbf{1}^T \\ -\mathbf{1} & S \end{pmatrix}$$

is the Seidel matrix of X. Since Γ is a regular two-graph, T has only two eigenvalues, and so its characteristic polynomial is quadratic with two real zeroes. Thus by Lemma 3.9, T satisfies an equation of the form

$$T^2 + aT + bI = 0,$$

for some constants a and b. Since

$$0 = T^2 + aT + bI = \begin{pmatrix} n+b & (-\mathbf{1}^T S - a\mathbf{1}^T) \\ -S\mathbf{1} - a\mathbf{1} & (J + S^2 + aS + bI) \end{pmatrix},$$

we see that $-S\mathbf{1} - a\mathbf{1} = 0$, and hence $S\mathbf{1} = -a\mathbf{1}$. This implies that S has constant row sum $-a$, and hence S is the Seidel matrix of a regular graph. Thus every neighbourhood of Γ is regular.

(b)⇒(c) Let X be a k-regular neighbourhood of Γ, and let $v \in V(X)$. Let $N(v)$ and $\overline{N}(v)$ denote the neighbourhood of v in X and its complement in X, respectively. Since Γ is nontrivial, both $N(v)$ and $\overline{N}(v)$ are nonempty, and

$$V(X) = \{v\} \cup N(v) \cup \overline{N}(v).$$

Then $(X \cup K_1)^{N(v)} = Y \cup K_1$, for some neighbourhood Y of Γ, and both X and Y are k-regular.

Let w be the isolated vertex in $X \cup K_1$. Then in $Y \cup K_1$, the vertex v is now isolated, w is adjacent to every vertex in $N(v)$, and the edges between $N(v)$ and $\overline{N}(v)$ have been complemented.

Consider a vertex in $N(v)$, and suppose it is adjacent to r vertices of $\overline{N}(v)$ in X. Then its valency in Y is

$$k + |\overline{N}(v)| - 2r = k,$$

so $r = |\overline{N}(v)|/2$. Thus every vertex in $N(v)$ is adjacent to the same number of vertices in $\overline{N}(v)$, and hence the same number of vertices in $N(v)$.

Now consider a vertex in $\overline{N}(v)$, and suppose it is adjacent to s vertices of $N(v)$ in X. Then its valency in Y is

$$k + |N(v)| - 2s = k,$$

so $|N(v)|/2 = k/2 = s$. Therefore, every vertex in $\overline{N}(v)$ is adjacent to $k/2$ vertices in $N(v)$.

Since v was an arbitrary vertex of X, we must have that X is strongly regular with the same parameters (n, k, a, c), with $k = 2c$, as required.

The fact that (c) implies (d) is obvious.

(d)⇒(a): Let X be a strongly regular neighbourhood of Γ with parameters (n, k, a, c), where $k = 2c$. Let A be the adjacency matrix of X and suppose that A has eigenvalues k, θ_1 and τ_1. Then $S = I + 2A - J$ is the Seidel matrix of X. Now adjoin an isolated vertex to X to form the graph $X \cup K_1$. This graph has Seidel matrix

$$T = \begin{pmatrix} 0 & -\mathbf{1}^T \\ -\mathbf{1} & S \end{pmatrix},$$

and $\mathrm{Sw}(X \cup K_1) = \Gamma$. If z is an eigenvalue of S orthogonal to $\mathbf{1}$, with eigenvalue λ, then

$$Tz = \begin{pmatrix} 0 & -\mathbf{1}^T \\ -\mathbf{1} & S \end{pmatrix} \begin{pmatrix} 0 \\ z \end{pmatrix} = \begin{pmatrix} -\mathbf{1}^T z \\ Sz \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ z \end{pmatrix},$$

and thus $(0, z^T)^T$ is an eigenvalue of T with the same eigenvalue. There are at least $n-1$ eigenvectors of A which are orthogonal to $\mathbf{1}$, and they all correspond to eigenvalue θ_1 or τ_1. Hence there are at least $n-1$ eigenvectors of $S = I+2A-J$ which are orthogonal to $\mathbf{1}$. Therefore, T has at least $n-1$ eigenvectors with eigenvalues $\theta = 1+2\theta_1$ or $\tau = 1+2\tau_1$.

The above partition of the matrix T is equitable, with quotient matrix

$$Q = \begin{pmatrix} 0 & -n \\ -1 & -(n-1-2k) \end{pmatrix}.$$

Therefore, any eigenvector of Q yields an eigenvector of T which is constant on the two cells of the partition. The $n-1$ eigenvectors we have already found are all of the form $(0, z^T)^T$ where z is orthogonal to $\mathbf{1}$, so none of these are constant on this partition of T. Hence any eigenvector of Q is not among them. Thus the remaining two eigenvalues of T are precisely the two eigenvalues of Q. Using $k = 2c$ and Lemma 3.7, we have $k-c = -\theta_1\tau_1$, $a-c = \theta_1+\tau_1$. Hence

$$\begin{aligned} n &= -(2\theta_1+1)(2\tau_1+1), \\ k &= -2\theta_1\tau_1, \\ a &= \theta_1+\tau_1-\theta_1\tau_1, \\ c &= -\theta_1\tau_1. \end{aligned}$$

Thus

$$Q = \begin{pmatrix} 0 & (2\theta_1+1)(2\tau_1+1) \\ -1 & 2(\theta_1+\tau_1+1) \end{pmatrix},$$

which has eigenvalues $2\theta_1+1 = \theta$ and $2\tau_1+1 = \tau$. Thus T has precisely two eigenvalues. Since the eigenvalues of T are the nontrivial eigenvalues of Γ, we must have that Γ is a regular two-graph. $\square$

3.11 Corollary. *A nontrivial regular two-graph has an even number of vertices.*

Proof: Let Γ be a nontrivial regular two-graph on $n+1$ vertices with nontrivial eigenvalues θ and τ. The previous proof shows that $\theta = 2\theta_1+1$ and $\tau = 2\tau_1+1$, where θ_1 and τ_1 are the two eigenvalues of any strongly regular neighbourhood X of Γ which are not equal to the valency of X. Also,

$$n = -(2\theta_1+1)(2\tau_1+1) = -(4\theta_1\tau_1+2(\theta_1+\tau_1)+1).$$

Since the entries of $A(X)$ are integers, θ_1 and τ_1 are algebraic integers. Hence they are either integers or they are algebraically conjugate. In either case, both $\theta_1\tau_1$ and $\theta_1+\tau_1$ are integers, and hence n is odd. Thus $n+1$ is even. $\square$

Theorem 3.10 shows that we can construct a regular two-graph from any strongly regular graph X with parameters (n, k, a, c) with $k = 2c$. Simply adjoin an isolated vertex to X, then $\mathrm{Sw}(X \cup K_1)$ is a regular two-graph.

Example. The 5-cycle is strongly regular with parameters $(5, 2, 0, 1)$. Hence $\mathrm{Sw}(C_5 \cup K_1)$ is a regular two-graph on 6 vertices.

Example. The Paley graphs $P(q)$ described in Section 3.2 are strongly regular with parameters

$$(n, k, a, c) = (q, (q-1)/2, (q-5)/4, (q-1)/4),$$

Hence $\mathrm{Sw}(P(q) \cup K_1)$ is a regular two-graph on $q+1$ vertices, for any prime power q with $q \equiv 1 \bmod 4$. Using the equations of Lemma 3.5, we see that the eigenvalues of the Paley graph $P(q)$ are $k = (q-1)/2$, with multiplicity 1, and $\theta_1 = (-1+\sqrt{q})/2$ and $\tau_1 = (-1-\sqrt{q})/2$ with equal multiplicities $(q-1)/2$. Thus the two eigenvalues of $\mathrm{Sw}(PG(q))$ are $\theta = \sqrt{q}$ and $\tau = -\sqrt{q}$ with equal multiplicities $(q+1)/2$. Note that C_5 is the Paley graph $P(5)$.

The proof of Theorem 3.10 also shows how to find the eigenvalues of the strongly regular neighbourhoods of a regular two-graph from its nontrivial eigenvalues θ and τ. If the eigenvalues of the strongly regular neighbourhoods are k, θ_1 and τ_1, where k is the valency, then

$$\theta_1 = \frac{1}{2}(\theta - 1), \quad \tau_1 = \frac{1}{2}(\tau - 1), \quad k = -2\theta_1\tau_1. \tag{3.3}$$

Now all of the parameters of the strongly regular neighbourhoods can be computed using Lemma 3.7.

In Section 3.5 we view a regular two-graph as an antipodal distance-regular graph of diameter three. This approach will help us generate feasible parameter sets for regular two-graphs in Section 3.6. First, we will look at properties of general antipodal distance-regular graphs.

3.4 Antipodal Distance-Regular Graphs

A connected graph X is *distance-regular* if for any two vertices u and v, the number of vertices at distance i from u and j from v depends only on i, j, and the distance between u and v. Since u may equal v, X is necessarily regular. Distance-regular graphs have important connections to other areas of combinatorics, including finite geometry and coding theory. A distance-regular graph of diameter two is strongly regular.

A graph X of diameter d is *antipodal* if the vertices at distance d from a given vertex are all at distance d from each other. The antipodal distance-regular graphs are covering graphs. The antipodal distance-regular graphs of diameter one are the complete graphs, and those of diameter two are the complete bipartite graphs. Thus the antipodal distance-regular graphs of diameter three are the first nontrivial case. These graphs cover the complete graph.

We will be interested in two-fold coverings of the complete graph, as any two-graph on n vertices is a two-fold cover of K_n of diameter three. We will show that the two-graph is regular if and only if it is a distance-regular double cover of K_n, and develop conditions that the parameters of such a cover must satisfy. First, we establish some basic properties of antipodal distance-regular graphs and covering graphs. The exposition of these properties is based on a survey of this topic in [9].

Let X be a distance-regular graph of diameter d. If u and v are two vertices of G at distance i, let $p^i_{j,k}$ denote the number of vertices at distance j from u and k from v. The numbers $p^i_{j,k}$ (for $0 \leq i, j, k \leq d$) are called the *intersection numbers* of X. Let c_i, a_i, and b_i denote the number of neighbours of v at distance $i-1$, i, and $i+1$ from u, respectively (these are the intersection numbers $p^i_{i-1,1}$, $p^i_{i,1}$, and $p^i_{i+1,1}$). These numbers determine all of the intersection numbers of X. Since $a_i + b_i + c_i = b_0$, the valency of X, we only need the numbers in the set

$$\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\},$$

called the *intersection array* of X. If X is antipodal, then "being at distance d" induces an equivalence relation on the vertices of X, and the equivalence classes are called *fibres*.

Suppose X is an antipodal distance-regular graph with diameter $d > 2$ and intersection array

$$\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\}.$$

The following list of fundamental properties of antipodal distance-regular graphs summarize results in [1] and [8].

(a) If there is an edge between two given fibres, then each vertex in one fibre has a unique neighbour in the other.

(b) If the distance between two fibres is i, then each vertex in the first fibre is at distance i from a vertex in the second fibre, and at distance $d - i$ from every other vertex in the second fibre.

(c) Let Q be the graph which has the fibres of X as vertices, with two fibres adjacent if and only if there is an edge between them in X. Then Q is a distance regular graph with intersection array

$$\{b_0, b_1, \ldots, b_{m-1}; 1, c_2, \ldots, \gamma c_m\},$$

where m satisfies $d = 2m + 1$ or $d = 2m$, and γ equals the size of a fibre if $d = 2m$, and $\gamma = 1$ if $d = 2m + 1$.

(d) Every eigenvalue of Q is also an eigenvalue of X with the same multiplicity.

The graph Q is called the *antipodal quotient* of X.

We now define covering graphs. Let X be a graph, and suppose that there is a partition Π of $V(X)$ into cells satisfying the following conditions:

(a) each cell is an independent set, and

(b) between any two cells either there are no edges, or there is an induced matching.

Let X/Π be the graph with the cells of Π as vertices, in which two vertices are adjacent if and only if there is an induced matching between them. Then we say that X is a *covering graph* of X/Π. The map sending each vertex of X to its corresponding cell in X/Π is called the *covering map*, and the cells are called *fibres*. If X/Π is connected, then each fibre has the same size, which is called the *index* of the covering. If the index is r, we call X an *r-fold covering graph* of X/Π.

An antipodal distance-regular graph X is an example of a covering graph, as it covers Q, the antipodal quotient of X. The switching graph of any graph on n vertices is a two-fold cover of K_n, since there is an induced matching between every fibre in this graph. If it is connected, it has diameter three. In the next section we show that a two-graph is an antipodal distance-regular graph if and only if it is a regular two-graph.

3.5 Distance-Regular Double Covers of K_n

The antipodal distance-regular graphs of diameter three are of interest to us because they cover the complete graph, and if they have index two then they are regular two-graphs. We shall prove this, and show that the intersection arrays of distance-regular double covers are determined by three parameters, including the index. Thus the intersection arrays for regular two-graphs on

n vertices are completely determined by two parameters. We will derive some feasibility conditions that these parameters must satisfy. First, we discuss antipodal distance-regular r-fold covers of K_n in general.

Let X be an antipodal distance-regular graph of diameter three. Then X is an r-fold cover of a graph of diameter one, namely K_n, for some r and n. Hence X has valency $n-1$, and $|V(X)| = rn$. Let $v \in V(X)$, and let $X_i(v)$ denote the set of vertices of X at distance i from v. Since X is antipodal of diameter three, the vertices of $X_3(v)$ must be at distance three from each other. Thus

$$c_3 = n-1, \quad b_2 = 1.$$

By counting the edges between $X_2(v)$ and $X_3(v)$ in two ways, we find that there are $(r-1)(n-1)$ vertices in $X_2(v)$. Now counting the edges between $X_1(v)$ and $X_2(v)$ in two ways, we find that

$$(n-1)b_1 = (r-1)(n-1)c_2,$$

so

$$b_1 = (r-1)c_2.$$

Also, b_0 is the valency of X, so $b_0 = n-1$. Thus the intersection array of X is

$$\{n-1, (r-1)c_2, 1; 1, c_2, n-1\},$$

and hence is completely determined by the parameter set (n, r, c_2). Since distance-regular two-graphs are antipodal distance-regular two-fold covers of K_n, their intersection arrays are completely determined by the parameters (n, c_2).

Now c_2 is the number of common neighbours of two vertices at distance two in X. Also,

$$a_1 = n-2-(r-1)c_2 = n-2-c_2$$

is the number of common neighbours of two adjacent vertices of X. The next result due to Godsil and Hensel in [9] gives a useful characterization of antipodal distance-regular two-fold covers of K_n.

3.12 Lemma. *Let X be a two-fold covering graph of K_n, and let c_2 be a positive integer. Then X is an antipodal distance-regular cover with parameters (n, c_2) if and only if two nonadjacent vertices from different fibres always have c_2 common neighbours.*

Proof: Suppose X is an antipodal distance regular two-fold cover of K_n with parameters (n, c_2). Let u and v be two nonadjacent vertices from different fibres, and let U and V be the fibres containing u and v, respectively. Then u is adjacent to a vertex of V and v is adjacent to a vertex of U, and hence u must be at distance two from v. Thus they have c_2 neighbours in common.

Conversely, let X be a two-fold cover of K_n, and suppose that any two nonadjacent vertices from different fibres have c_2 neighbours in common. Let $F = \{u, u'\}$ be a fibre of X. Since the fibres are independent sets, u and u' cannot be adjacent. Since there is an induced matching between any two fibres, they cannot be at distance two. Now let v be a neighbour of u. Then u lies in a different fibre, and is not adjacent to $u_\prime$. Thus v has c_2 common neighbours with u'. Hence U and u' are at distance three.

If $w \notin F$, then since X covers K_n, it must have a neighbour in F. Thus the other vertex of F must be at distance two from w. Hence the vertices at distance two from u are precisely the vertices adjacent to the vertex in $F \setminus \{u\}$. Since v is adjacent to u, and is not adjacent to u', it has exactly c_2 neighbours in common with u', and hence v has exactly c_2 neigbours at distance two from u. Thus it must have exactly $(n-1) - 1 - c_2 = a_1$ neighbours adjacent to u.

We conclude that X is an antipodal distance-regular graph with intersection array

$$\{n-1, c_2, 1; 1; c_2, n-1\},$$

as required. □

Notice that Lemma 3.12 implies that any distance-regular two-fold cover of K_n with diameter three must be antipodal.

We can compute the eigenvalues of any distance-regular two-fold cover X of K_n and their multiplicities directly from the parameters n and c_2, as the next lemma shows. Since K_n is the antipodal quotient of X, the eigenvalues of K_n are also eigenvalues of X with the same multiplicities. The computations used to find the other eigenvalues of X are similar to those used to find the eigenvalues of a strongly regular graph in Lemma 3.5, and so they are omitted here.

3.13 Lemma. *Let X be a distance-regular two-fold cover of K_n with parameters (n, c_2). Let $a_1 = n-2-c_2$, let $\delta = a_1 - c_2$, and let $\Delta = -\delta^2 + 4(n-1)$. Then -1 and $n-1$ are eigenvalues of X with multiplicities 1 and $n-1$, respectively, and X has only two other eigenvalues, θ and τ, given by*

$$\theta = \frac{1}{2}\left(\delta + \sqrt{\Delta}\right), \quad \tau = \frac{1}{2}\left(\delta - \sqrt{\Delta}\right),$$

with

$$m_\theta = \frac{n\tau}{\tau - \theta}, \quad m_\tau = \frac{n\theta}{\theta - \tau}.$$

□

The next result shows that regular two-graphs are antipodal distance-regular two-fold covers of K_n.

3.14 Theorem. *A graph Γ is a regular two-graph on n vertices if and only if it is an antipodal distance-regular two-fold cover of K_n.*

Proof: If Γ is an antipodal distance-regular two-fold cover of K_n, then there is an induced matching between every two fibres, and so it is the switching graph of some graph X on n vertices. Lemma 3.13 shows that Γ has only two nontrivial eigenvalues, and hence it is a regular two-graph.

Conversely, suppose that Γ is a regular two-graph on n vertices. Then it is the switching graph of some graph X on n vertices, so it is a two-fold cover of K_n with n fibres of size two, each of the form $\{(v,0),(v,1)\}$ for some vertex v in $V(X)$. Since Γ is regular, Theorem 3.10 implies that all of its neighbourhoods are strongly regular with the same parameters $(n-1,k,a,c)$. Let (v,i) and (w,j) be nonadjacent vertices of Γ from different fibres. We show that (v,i) and (w,j) have $c_2 := n-k-2$ common neighbours.

Since there is an induced matching between every two fibres, we must have that $\{(w,j),(v,1-i)\} \in E(\Gamma)$. The neighbourhoods of (v,i) and $(v,1-i)$, are disjoint $(n-1,k,a,c)$-strongly regular graphs, so (w,j) is adjacent to $(v,1-i)$ and k neighbours of $(v,1-i)$. The other $(n-1)-(k+1)$ neighbours of (w,j) must be in the neighbourhood of (v,i), and hence (w,j) and (v,i) have $n-k-2 = c_2$ common neighbours.

Hence every two nonadjacent vertices of Γ from different fibres have c_2 common neighbours, so by Lemma 3.12, Γ is an antipodal distance-regular two-fold cover of K_n. □

Theorem 3.14 shows that the distance-regular two-fold covers of K_n are all of the regular two-graphs. In the next section, we generate parameter sets for regular two-graphs by looking at several feasibility conditions on the parameters of distance-regular two-fold covers of the complete graph.

3.6 Parameter Sets

In this section we will generate feasible parameter sets for regular two-graphs, by viewing them as distance-regular two-fold covers of the complete graph.

We now develop conditions that the parameters of any distance-regular two-fold cover Γ of K_n must satisfy. Requiring that the multiplicities m_θ and m_τ of the two nontrivial eigenvalues θ and τ of Γ are integers gives a strong feasibility condition on the possible values of n and c_2. The formulas for these multiplicities given in Lemma 3.13 imply that $\theta - \tau$ must divide both $n\theta$ and $n\tau$. The eigenvalues θ and τ are also constrained, as the next lemma shows.

3.15 Lemma. *Let Γ be a distance-regular two-fold cover of K_n with parameters (n, c_2) and nontrivial eigenvalues θ and τ, with $\theta > \tau$. Let $a_1 = n-2-c_2$ and let $\delta = a_1 - c_2$. If $\delta = 0$, then $\theta = -\tau = \sqrt{n-1}$. If $\delta \neq 0$, then θ and τ are odd integers.*

Proof: Using the formulas for θ and τ given in Lemma 3.13, we verify that

$$\theta\tau = 1 - n, \quad \theta + \tau = \delta.$$

Hence, if $\delta = 0$, $\theta = -\tau$, so $\theta\tau = -\theta^2 = 1 - n$, and thus $\theta = \sqrt{n-1} = -\tau$, as required. If $\delta \neq 0$, then if m_θ and m_τ are integers, we must have that θ and τ are integers. By Corollary 3.11, n is even, and thus $n - 1$ is odd. Hence θ and τ are divisors of an odd integer, so θ and τ must be odd. □

The following result due to Taylor [21] eliminates several possible parameter sets for regular two-graphs with non-integral eigenvalues.

3.16 Lemma. *Let Γ be a regular two-graph with nontrivial eigenvalues θ and τ. If θ and τ are not rational, then $n - 1$ is a sum of two squares.* □

The subgraph Γ_v induced by the neighbourhood of any vertex v in an (n, c_2) distance-regular two-fold cover Γ of K_n is regular with valency $a_1 = n - 2 - c_2$. In fact, Theorem 3.10 implies that X is strongly regular with parameters $(n-1, a_1, a, a_1/2)$, for some nonnegative integer a. Since X cannot be complete, we must have $0 \leq a_1 \leq n - 3$. If $a_1 = 0$, then the neighbourhood Γ_v is an empty graph, for every vertex v of Γ. In this case Γ is the complete bipartite graph $K_{n,n}$, which is a trivial regular two-graph. Hence we require that $1 \leq a_1 \leq n - 3$. Since the sum of the degrees of the vertices of Γ_v must be even, $(n-1)a_1$ is even, so since $n-1$ is odd, a_1 must be even. Thus

$$2 \leq a_1 \leq n - 4.$$

Since $c_2 = n - 2 - a_1$, c_2 must also be even, and

$$2 \leq c_2 \leq n - 4.$$

In Section 3.2, we developed several conditions on the parameters of any strongly regular graph. We can apply these to the strongly regular neighbourhoods of Γ. In particular, strong feasibility conditions on the eigenvalues a_1, θ_1, τ_1 of these strongly regular neighbourhoods and their multiplicities arise from the Krein bounds of Lemma 3.8. The eigenvalues θ_1 and τ_1 of any strongly regular neighbourhood of Γ which are not equal to the valency a_1 can be computed from the nontrivial eigenvalues of Γ. They are given by

$$\theta_1 = \frac{1}{2}(\theta - 1), \quad \tau_1 = \frac{1}{2}(\tau - 1).$$

The rest of the parameters of the strongly regular neighbourhoods can now be computed using the formulas of Lemma 3.7, and the multiplicities m_{θ_1} and m_{τ_1} can be computed using the formulas in Lemma 3.5. The parameters $(n-1, a_1, a, a_1/2)$ of the strongly regular neighbourhoods must also satisfy the relation of Lemma 3.2, namely

$$a_1(a_1 - a - 1) = \frac{a_1}{2}((n-1) - a_1 - 1).$$

Let Γ be a regular two-graph with parameters $(n, a_1, c_2; \theta, m_\theta, \tau, m_\tau)$, and let $(n, a_1, a, a_1/2; \theta_1, m_{\theta_1}, \tau_1, m_{\tau_1})$ be the parameters of the strongly regular neighbourhoods of Γ. The feasibility conditions on the parameters of a regular two-graph are summarized below.

(a) n is even,

(b) $2 \leq a_1 \leq n-4$ and a_1 is even,

(c) $2 \leq c_2 \leq n-4$ and c_2 is even,

(d) θ and τ are odd divisors of $1-n$, or $\theta = -\tau = \sqrt{n-1}$,

(e) If $\theta = -\tau = \sqrt{n-1}$, then $n-1$ is the sum of two squares.

(f) $(\theta - \tau)$ is a divisor of both $n\theta$ and $n\tau$,

(g) $a_1(a_1 - a - 1) = (a_1/2)((n-1) - a_1 - 1)$,

(h) ${\theta_1}{\tau_1}^2 - 2{\theta_1}^2\tau_1 - {\theta_1}^2 - a_1\theta_1 + a_1{\tau_1}^2 + 2a_1\tau_1 \geq 0$. If this inequality is tight, then $a_1 \geq m_{\theta_1}$.

(i) ${\theta_1}^2\tau_1 - 2\theta_1{\tau_1}^2 - {\tau_1}^2 - a_1\tau_1 + a_1{\theta_1}^2 + 2a_1\theta_1 \geq 0$. If this inequality is tight, then $a_1 \geq m_{\tau_1}$.

Conditions (h) and (i) are the Krein bounds of Lemma 3.8 applied to the eigenvalues of the strongly regular neighbourhoods of Γ.

Given n and θ, every parameter in the parameter set

$$(n, a_1, c_2; \theta, m_\theta, \tau, m_\tau)$$

for a regular two-graph Γ can be computed, along with every parameter in the corresponding parameter set

$$(n, a_1, a, a_1/2; \theta_1, m_{\theta_1}, \tau_1, m_{\tau_1})$$

of the strongly regular neighbourhoods of Γ. We can compute these values using formulas derived in the previous sections as follows: Let n and θ be given. Then

$$\begin{aligned}
\tau &= (1-n)/\theta, \\
\tau_1 &= (\tau - 1)/2, \\
\theta_1 &= (\theta - 1)/2, \\
a_1 &= -2\theta_1\tau_1, \\
c_2 &= n - 2 - a_1, \\
a &= \theta_1 + \tau_1 - \theta_1\tau_1, \\
m_\theta &= \frac{n\tau}{\tau - \theta}, \\
m_\tau &= \frac{n\theta}{\theta - \tau}, \\
m_{\theta_1} &= \frac{(n-1)\tau_1 + a_1}{\tau_1 - \theta_1}, \\
m_{\tau_1} &= \frac{(n-1)\theta_1 + a_1}{\theta_1 - \tau_1}.
\end{aligned}$$

To generate the feasible parameter sets for regular two-graphs on n vertices for $n \le N$, for each even integer $n \le N$, and each odd divisor θ of n, compute the rest of the parameters using the formulas listed above. If $\sqrt{n-1}$ is not an integer, also compute a parameter set for $\theta = \sqrt{n-1}$. This will generate all of the possible parameter sets for regular two-graphs which satisfy feasibility conditions (a) and (d), along with the parameter sets of their strongly regular neighbourhoods. Test each of these parameter sets to see if they satisfy each of the other feasibility conditions. For $n \le 100$, the feasible parameter sets for regular two-graphs on n vertices are listed in Tables 3.1 and 3.2. Feasible parameter sets for regular two-graph with integer eigenvalues are listed in Table 3.1, and those with irrational eigenvalues are listed in Table 3.2.

Regular Two-Graph $(n, a_1, c_2; \theta, m_\theta, \tau, m_\tau)$	**Strongly Regular Neighbourhhood** $(n-1, a_1, a, \frac{a_1}{2}; \theta_1, m_{\theta_1}, \tau_1, m_{\tau_1})$
$(10, 4, 4; 3, 5, -3, 5)$	$(9, 4, 1, 2; 1, 4, -2, 4)$
$(16, 6, 8; 3, 10, -5, 6)$	$(15, 6, 1, 3; 1, 9, -3, 5)$
$(16, 8, 6; 5, 6, -3, 10)$	$(15, 8, 4, 4; 2, 5, -2, 9)$
$(26, 12, 12; 5, 13, -5, 13)$	$(25, 12, 5, 6; 2, 12, -3, 12)$
$(28, 10, 16; 3, 21, -9, 7)$	$(27, 10, 1, 5; 1, 20, -5, 6)$
$(28, 16, 10; 9, 7, -3, 21)$	$(27, 16, 10, 8; 4, 6, -2, 20)$
$(36, 16, 18; 5, 21, -7, 15)$	$(35, 16, 6, 8; 2, 20, -4, 14)$
$(36, 18, 16; 7, 15, -5, 21)$	$(35, 18, 9, 9; 3, 14, -3, 20)$
$(50, 24, 24; 7, 25, -7, 25)$	$(49, 24, 11, 12; 3, 24, -4, 24)$
$(64, 30, 32; 7, 36, -9, 28)$	$(63, 30, 13, 15; 3, 35, -5, 27)$
$(64, 32, 30; 9, 28, -7, 36)$	$(63, 32, 16, 16; 4, 27, -4, 35)$
$(76, 32, 42; 5, 57, -15, 19)$	$(75, 32, 10, 16; 2, 56, -8, 18)$
$(76, 42, 32; 15, 19, -5, 57)$	$(75, 42, 25, 21; 7, 18, -3, 56)$
$(82, 40, 40; 9, 41, -9, 41)$	$(81, 40, 19, 20; 4, 40, -5, 40)$
$(96, 40, 54; 5, 76, -19, 20)$	$(95, 40, 12, 20; 2, 75, -10, 19)$
$(96, 54, 40; 19, 20, -5, 76)$	$(95, 54, 33, 27; 9, 19, -3, 75)$
$(100, 48, 50; 9, 55, -11, 45)$	$(99, 48, 22, 24; 4, 54, -6, 44)$
$(100, 50, 48; 11, 45, -9, 55)$	$(99, 50, 25, 25; 5, 44, -5, 54)$

Table 3.1: Parameter Sets for Regular Two-Graphs with Integer Eigenvalues (for $n \leq 100$)

Regular Two-Graph $(n, a_1, c_2; \theta, m_\theta, \tau, m_\tau)$	**Strongly Regular Neighbourhood** $(n-1, a_1, a, \frac{a_1}{2}; \theta_1, m_{\theta_1}, \tau_1, m_{\tau_1})$ $F(x) := (-1+\sqrt{x})/2$ $G(x) := (-1-\sqrt{x})/2$
$(6, 2, 2; \sqrt{5}, 3, -\sqrt{5}, 3)$	$(5, 2, 0, 1; F(5), 2, G(5), 2)$
$(14, 6, 6; \sqrt{13}, 7, -\sqrt{13}, 7)$	$(13, 6, 2, 3; F(13), 6, G(13), 6)$
$(18, 8, 8; \sqrt{17}, 9, -\sqrt{17}, 9)$	$(17, 8, 3, 4; F(17), 8, G(17), 8)$
$(30, 14, 14; \sqrt{29}, 15, -\sqrt{29}, 15)$	$(29, 14, 6, 7; F(29), 14, G(29), 14)$
$(34, 16, 16; \sqrt{33}, 17, -\sqrt{33}, 17)$	$(33, 16, 7, 8; F(33), 16, G(33), 16)$
$(38, 18, 18; \sqrt{37}, 19, -\sqrt{37}, 19)$	$(37, 18, 8, 9; F(37), 18, G(37), 18)$
$(42, 20, 20; \sqrt{41}, 21, -\sqrt{41}, 21)$	$(41, 20, 9, 10; F(41), 20, G(41), 20)$
$(46, 22, 22; \sqrt{45}, 23, -\sqrt{45}, 23)$	$(45, 22, 10, 11; F(45), 22, G(45), 22)$
$(54, 26, 26; \sqrt{53}, 27, -\sqrt{53}, 27)$	$(53, 26, 12, 13; F(53), 26, G(53), 26)$
$(58, 28, 28; \sqrt{57}, 29, -\sqrt{57}, 29)$	$(57, 28, 13, 14; F(57), 28, G(57), 28)$
$(62, 30, 30; \sqrt{61}, 31, -\sqrt{61}, 30)$	$(61, 30, 14, 15; F(61), 30, G(61), 30)$
$(66, 32, 32; \sqrt{65}, 33, -\sqrt{65}, 33)$	$(65, 32, 15, 16; F(65), 32, G(65), 32)$
$(74, 36, 36; \sqrt{73}, 37, -\sqrt{73}, 37)$	$(73, 36, 17, 18; F(73), 36, G(73), 36)$
$(86, 42, 42; \sqrt{85}, 43, -\sqrt{85}, 43)$	$(85, 42, 20, 21; F(85), 42, G(85), 42)$
$(90, 44, 44; \sqrt{89}, 45, -\sqrt{89}, 45)$	$(89, 44, 21, 22; F(89), 44, G(89), 44)$
$(98, 48, 48; \sqrt{97}, 49, -\sqrt{97}, 49)$	$(97, 48, 23, 24; F(97), 48, G(97), 48)$

Table 3.2: Parameter Sets for Regular Two-Graphs with Irrational Eigenvalues (for $n \leq 100$)

Chapter 4

Constructing Regular Two-Graphs

4.1 Introduction

In Chapter 2 it was shown that any set L of n equiangular lines in $\mathbb{R}^d$ corresponds to a two-graph on n vertices, and it Chapter 3 we saw that this two-graph can be viewed as a double cover of the complete graph on n vertices. By Theorem 2.8, the two-graph is regular and the double cover is distance-regular if and only if n meets the relative bound of Theorem 2.3. A clique of order d in this double cover corresponds to a d-subset of a set Ω of unit vectors spanning the lines in L, in which the inner product of any two distinct vectors is positive. Such a d-subset is called a *positive basis* of Ω. In Section 4.2 we will see that a positive basis forms a basis of R^d, and then use linear algebra to characterize the vectors corresponding to vertices of the double cover which are outside of the clique. Using this characterization, we obtain a construction for regular two-graphs with cliques of specified order. We will prove that the existence of such a structure depends on the existence of an incidence structure with special properties. Quasi-symmetric designs provide examples of these incidence structures.

In Section 4.3, the set of all feasible parameter sets for regular two-graphs on n vertices which are candidates for this construction is generated, for $n \leq 300$, and these parameter sets are listed in Table 4.1. Several examples of regular two-graphs with large cliques are constructed in Section 4.4, and some arise from quasi-symmetric designs. We discuss some existence results regarding quasi-symmetric designs due to Calderbank in [3] and [4]. We conclude by posing an open problem regarding when these incidence structures form 2-designs.

The concept of using a positive basis to characterize the structure of a regular two-graph containing a clique of specified order is due to Chris Godsil.

4.2 The Construction

Let $0 < \alpha < 1$ and let d and n be integers such that $d \geq 2$ and $n \geq 2$. Let Ω be a set of n unit vectors spanning a set of n equiangular lines in $\mathbb{R}^d$ with mutual angle $\arccos(\alpha)$. Then for any two distinct vectors x and y of Ω, $\langle x, y\rangle = \pm\alpha$. A *positive basis* of Ω is a d-subset of Ω in which the inner product of any two vectors is positive. As the name suggests, a positive basis of Ω is a basis of $\mathbb{R}^d$. The Gram matrix for a positive basis has a very simple form, as the next lemma shows.

4.1 Lemma. *Let $0 < \alpha < 1$ and let $n \geq 2$ and $d \geq 2$ be integers. Let Ω be a set of n unit vectors spanning a set of n equiangular lines in $\mathbb{R}^d$ with mutual angle $\arccos(\alpha)$. Then a positive basis of Ω is a basis of $\mathbb{R}^d$, and the Gram matrix for any positive basis of Ω is the $d \times d$ matrix $G = (1-\alpha)I + \alpha J$. Furthermore, G is invertible and*

$$G^{-1} = (\tfrac{1}{1-\alpha})I + \tfrac{\alpha}{(\alpha-1)(1+(d-1)\alpha)}J.$$

Proof: Suppose that $B = \{x_1, x_2, \ldots, x_d\}$ is a positive basis for Ω. Let L be the matrix whose columns are the vectors in B. The Gram matrix for B is $G = L^T L$, in which the ij-th entry is $\langle x_i, x_j\rangle$. Since this value is equal to 1 if $i = j$, and α otherwise, G has the required form. Since $1 - \alpha > 0$, the matrix $(1-\alpha)I$ is positive definite, and since $\alpha > 0$, the matrix αJ is positive semidefinite. Thus G is positive definite, and hence invertible. This implies that L is invertible, so its columns are linearly independent, and thus the vectors in B form a basis of $\mathbb{R}^d$. Finally, one checks that $G((\frac{1}{1-\alpha})I + \frac{\alpha}{(\alpha-1)(1+(d-1)\alpha)}J) = I$. This completes the proof. □

In Chapter 2 it was shown that the set Ω corresponds to a two-graph. This two-graph is the switching graph of the graph X with vertex set Ω, in which two vectors x and y of Ω are adjacent if $\langle x, y\rangle = \alpha$. If $B = \{x_1, x_2, \ldots, x_d\}$ is a positive basis of Ω, it forms a clique of order d in the graph X, and hence it forms two disjoint cliques of order d in the corresponding two-graph. Each vector y in $\Omega \setminus B$ satisfies the following three properties:

(a) $\langle y, x_i\rangle = \pm\alpha$ (for $1 \leq i \leq d$),

(b) $\langle y, y\rangle = 1$, and

(c) $\langle y, z\rangle = \pm\alpha$, for every z in $\Omega \setminus B$ such that $z \neq y$.

A vector y in $\mathbb{R}^d$ which satisfies property (a) is said to be in *good position* with respect to the positive basis B. Property (c) says that $\Omega \setminus B$ spans an equiangular set of lines.

If Ω spans a set of equiangular lines which meets the relative bound of Theorem 2.3, then

$$|\Omega| = n = \frac{d - d\alpha^2}{1 - d\alpha^2},$$

and by Theorem 2.8, the two-graph corresponding to Ω is regular and has least eigenvalue $-1/\alpha$ with multiplicity $n-d$. Hence, given d and α, we can construct a regular two-graph with a clique of order d by finding a set Ω of $n = \frac{d-d\alpha^2}{1-d\alpha^2}$ equiangular unit vectors in $\mathbb{R}^d$ with mutual angle $\arccos(\alpha)$, which contains a positive basis B.

To construct such a set Ω, assume that $B = \{x_1, x_2, \ldots, x_d\}$ is a set of unit vectors in $\mathbb{R}^d$ such that $\langle x_i, x_j\rangle = \alpha$ (for $i \neq j$), and find a set C of $n-d$ vectors in $\mathbb{R}^d \setminus B$ which satisfy properties (i), (ii) and (iii). Then $B \cup C$ is the required set Ω. Note that it is always possible to find such a set B. The next two lemmas characterize the feasible sets of $n-d$ vectors in $C = \Omega \setminus B$ and their relationship to one another, and Theorem 4.4 shows that these vectors form an incidence structure with certain properties.

Before looking at these results, we define a useful function f_B on the vectors in $\mathbb{R}^d$ outside of a positive basis B. For a positive basis B of a set of unit vectors spanning a set of equiangular lines in $\mathbb{R}^d$ with mutual angle $\arccos(\alpha)$, let

$$f_B : \mathbb{R}^d \setminus B \mapsto \mathbb{R}^d$$

be defined by

$$f_B(y) := \alpha^{-1} L^T y,$$

in which L is the matrix whose columns are the vectors in B. Since B is a basis of $\mathbb{R}^d$, the columns of L are linearly independent, and hence L is invertible. Thus f_B is invertible, and

$$f_B^{-1}(y) = \alpha L^{-T} y.$$

4.2 Lemma. *Let Ω be a set of unit vectors spanning a set of equiangular lines in $\mathbb{R}^d$ with mutual angle* $\arccos(\alpha)$. *Let B be a positive basis of Ω, and let G be the Gram matrix of the vectors in B. If $y \in \Omega \setminus B$, then $f_B(y)$ is a ± 1-vector in $\mathbb{R}^d$ with*

$$m = \frac{1}{2}\left(d - \sqrt{d(d-1) + (2d-1)\alpha^{-1} - (d-2)\alpha^{-2} - \alpha^{-3}}\right)$$

or $d - m$ entries equal to -1.

Proof: Suppose that $B = \{x_1, x_2, \ldots, x_d\}$ is a positive basis for Ω. Then by Lemma 4.1, B is a basis of $\mathbb{R}^d$. In particular, each vector y in Ω can be written as a linear combination of vectors in B, say

$$y = \sum_{i=1}^{d} a_i x_i$$

for unique real scalars $a_1, a_2, \ldots, a_d$. Furthermore, if $y \in \Omega \setminus B$, y is in good position with respect to B, and hence for each i ($1 \leq i \leq d$) we have

$$\pm\alpha = \langle x_i, y\rangle = x_i^T \sum_{j=1}^{d} a_j x_j = \sum_{j=1}^{d} a_j \langle x_i, x_j\rangle. \tag{4.1}$$

Let G be the Gram matrix for B. Then the ij-th entry of G is $\langle x_i, x_j\rangle$, so Equation 4.1 implies that

$$[\pm\alpha, \pm\alpha, \ldots, \pm\alpha]^T = [\langle x_1, y\rangle, \langle x_2, y\rangle, \ldots, \langle x_d, y\rangle]^T = G[a_1, a_2, \ldots, a_d]^T,$$

and thus

$$[a_1, a_2, \ldots, a_d]^T = \alpha G^{-1}\hat{y},$$

in which $\hat{y}$ is the ± 1-vector in $\mathbb{R}^d$ whose i-th entry is equal to 1 if $\langle x_i, y\rangle = \alpha$, and -1 if $\langle x_i, y\rangle = -\alpha$. If L is the matrix whose columns are the vectors in B, we have

$$\begin{aligned} y &= L[a_1, a_2, \ldots, a_d]^T \\ &= \alpha L G^{-1}\hat{y} \\ &= \alpha L(L^T L)^{-1}\hat{y} \\ &= \alpha L^{-T}\hat{y}, \end{aligned}$$

and thus

$$\hat{y} = \alpha^{-1} L^T y = f_B(y).$$

Hence $f_B(y)$ is a ± 1-vector in $\mathbb{R}^d$, and

$$y = \alpha L^{-T} f_B(y). \tag{4.2}$$

Now suppose $y \in \Omega \setminus B$ and let m denote the number of negative entries in $f_B(y)$. Since y is a unit vector, Equation 4.2 implies that

$$1 = \langle y, y \rangle = y^T y = [\alpha(f_B(y))^T L^{-1}][\alpha L^{-T} f_B(y)],$$

and hence

$$1 = \alpha^2 (f_B(y))^T (L^T L)^{-1} f_B(y) = \alpha^2 (f_B(y))^T G^{-1} f_B(y).$$

Now substituting the formula for G^{-1} given in Lemma 4.1 into the above equation gives

$$\begin{aligned} 1 &= \alpha^2 (f_B(y))^T \left[\frac{1}{1-\alpha} I + \frac{\alpha}{(\alpha-1)(1+(d-1)\alpha)} J \right] f_B(y) \\ &= \frac{\alpha^2}{1-\alpha} (f_B(y))^T f_B(y) + \frac{\alpha^3}{(\alpha-1)(1+(d-1)\alpha)} (\mathbf{1}^T f_B(y))^2 \\ &= \frac{\alpha^2}{1-\alpha} d + \frac{\alpha^3}{(\alpha-1)(1+(d-1)\alpha)} (d-2m)^2. \end{aligned} \tag{4.3}$$

Solving for m in Equation 4.3, we obtain

$$m = \frac{1}{2} \left(d \pm \sqrt{d(d-1) + (2d-1)\alpha^{-1} - (d-2)\alpha^{-2} - \alpha^{-3}} \right)$$

.

This completes the proof. □

In Lemma 4.2, if m is not a positive integer, then there are no unit vectors in $\mathbb{R}^d$ which are in good position with respect to B, so in this case the maximum size of a set of equiangular unit vectors in $\mathbb{R}^d$ with mutual angle $\arccos(\alpha)$ which contains a positive basis is d. (This corresponds to the trivial regular two-graph, $\mathrm{Sw}(K_d)$, the switching graph of the complete graph on d vertices.) Thus given any pair (d, α) which yields a non-integral value for m in the formula of Lemma 4.2, we can conclude that there exists no set of equiangular unit vectors with these parameters meeting the relative bound of Theorem 2.3, and hence there exists no associated regular two-graph containing a clique of order d. In the next section, many of the feasible parameter sets for regular two-graphs in Tables 3.1 and 3.2 will be eliminated as candidates for this construction technique based on this conclusion.

If m is a positive integer in Lemma 4.2, then each unit vector $y \in \mathbb{R}^d \setminus B$ which is in good position with respect to B can be written in the form

$$y = \alpha L^{-T} f_B(y),$$

in which $f_B(y)$ is a ± 1-vector in $\mathbb{R}^d$ with exactly m or $d-m$ entries equal to -1, and L is the matrix whose columns are the vectors in B. Let Y be the set of unit vectors in $\mathbb{R}^d \setminus B$ which are in good position with respect to B. If $d = 2m$, then $|Y| = \binom{d}{m}$. Otherwise, there are $\binom{d}{m} + \binom{d}{d-m} = 2\binom{d}{m}$ vectors in the set Y. In this case, the set Y can be partitioned into two sets Y_1 and Y_2 of equal order, such that for each vector y in Y_1, $f_B(y)$ contains m negative entries, and for each vector z in Y_2, $f_B(z)$ contains $d-m$ negative entries. Thus $Y_2 = \{-y \ : \ y \in Y_1\}$.

To construct the desired regular two-graph, let $n = \frac{d-d\alpha^2}{1-d\alpha^2}$ and find an $(n-d)$-subset C of Y in which the inner product of any two distinct vectors is equal to $\pm\alpha$. Then $\Omega = B \cup C$ is a set of equiangular unit vectors which meets the relative bound and contains a positive basis, so it corresponds to a regular two-graph with a clique of order d. Even for rather small values of d, finding the set C by brute force can be a formidable task. If $d \neq 2m$, we can cut the search space in half by observing that any suitable subset C of $Y = Y_1 \cup Y_2$ is switching equivalent to a subset C' of Y_1. To see this, suppose that C is a subset of $Y = Y_1 \cup Y_2$ which contains some elements of Y_2, say $C = C_1 \cup C_2$ where $C_1 \subseteq Y_1$ and $C_2 \subseteq Y_2$. Let $C_2' = \{-y : y \in C_2\}$ and let $C' = C_1 \cup C_2'$. Then $C' \subseteq Y_1$ and C' is switching equivalent to C, so the graph corresponding to the vectors in $B \cup C$ is switching equivalent to the graph corresponding to the vectors in $B \cup C'$. Thus the regular two-graphs corresponding to the sets of vectors $B \cup C$ and $B \cup C'$ are isomorphic. Hence we need only search the $\binom{d}{m}$ vectors in Y_1 to find the required $(n-d)$-subset C of Y. This is still quite a large search space. The next lemma gives us more information about the relationship between vectors in $C = \Omega \setminus B$.

4.3 Lemma. *Let $B = \{x_1, x_2, \ldots, x_d\}$ be a positive basis for a set Ω of unit vectors spanning a set of equiangular lines in $\mathbb{R}^d$ with mutual angle arccos(α), and let G be the Gram matrix for B. Let y and z be distinct vectors in $\Omega \setminus B$. Suppose that $f_B(y)$ and $f_B(z)$ both contain m negative entries. If $\langle y, z\rangle = \alpha$, then $f_B(y)$ and $f_B(z)$ differ in sign in*

$$s_1 = \frac{1}{2}\left[d - \left(\frac{(1-\alpha)(1+(d-1)\alpha) + (d-2m)^2\alpha^2}{\alpha(1+(d-1)\alpha)}\right)\right] \tag{4.4}$$

positions, and if $\langle y, z\rangle = -\alpha$*, then* $f_B(y)$ *and* $f_B(z)$ *differ in sign in*

$$s_2 = \frac{1}{2}\left[d + \left(\frac{(1-\alpha)(1+(d-1)\alpha) - (d-2m)^2\alpha^2}{\alpha(1+(d-1)\alpha)}\right)\right] \tag{4.5}$$

positions.

Proof: Since $y, z \in \Omega \setminus B$, $\langle y, z\rangle = \pm\alpha$. If L is the matrix whose columns are the vectors in B, then

$$\begin{aligned}\pm\alpha = y^T z &= [\alpha(f_B(y))^T L^{-1}][\alpha L^{-T} f_B(z)]\\ &= \alpha^2 (f_B(y))^T (L^T L)^{-1} f_B(z)\\ &= \alpha^2 (f_B(y))^T G^{-1} f_B(z).\end{aligned}$$

Now substituting the formula for G^{-1} given in Lemma 4.1 into the above equation gives

$$\begin{aligned}\pm\alpha &= \alpha^2 (f_B(y))^T \left[\frac{1}{1-\alpha} I + \frac{\alpha}{(\alpha-1)(1+(d-1)\alpha)} J\right] f_B(z)\\ &= \frac{\alpha^2}{1-\alpha}(f_B(y))^T f_B(z) + \frac{\alpha^3}{(\alpha-1)(1+(d-1)\alpha)}(\mathbf{1}^T f_B(y))(\mathbf{1}^T f_B(z)),\end{aligned}$$

in which $\mathbf{1}$ is the vector of $\mathbb{R}^d$ with every entry equal to 1. Let s' be the number of positions in which $f_B(y)$ and $f_B(z)$ differ in sign. Then

$$\pm\alpha = \frac{\alpha^2}{1-\alpha}(d - 2s') + \frac{\alpha^3}{(\alpha-1)(1+(d-1)\alpha)}(d-2m)^2. \tag{4.6}$$

If $\langle y, z\rangle = \alpha$, then solving for s' in Equation 4.6, we obtain

$$s' = \frac{1}{2}\left[d - \left(\frac{(1-\alpha)(1+(d-1)\alpha) + (d-2m)^2\alpha^2}{\alpha(1+(d-1)\alpha)}\right)\right] = s_1.$$

If $\langle y, z\rangle = -\alpha$, then

$$s' = \frac{1}{2}\left[d + \left(\frac{(1-\alpha)(1+(d-1)\alpha) - (d-2m)^2\alpha^2}{\alpha(1+(d-1)\alpha)}\right)\right] = s_2.$$

□

In Section 3.3 it will be shown that if m and $-1/\alpha$ are integers, then s_1 and s_2 are even integers. We will also see that there is only one parameter set for a regular two-graph which yields an integral value for m where s_1 and s_2 are not both integers.

The obvious relationship between ± 1-vectors in $\mathbb{R}^d$ and subsets of

$$\{1, 2, \ldots d\}$$

suggests that Lemmas 4.2 and 4.3 can be restated in the language of incidence structures. We obtain the following characterization of a regular two-graphs with a clique of specified order.

4.4 Theorem. *Let $0 < \alpha < 1$ and let d be an integer such that $d \geq 2$. Let $n = \frac{d-d\alpha^2}{1-d\alpha^2}$, let*

$$m = \frac{1}{2}\left(d - \sqrt{d(d-1) + (2d-1)\alpha^{-1} - (d-2)\alpha^{-2} - \alpha^{-3}}\right),$$

let

$$s_1 = \frac{1}{2}\left[d - \left(\frac{(1-\alpha)(1+(d-1)\alpha) + (d-2m)^2\alpha^2}{\alpha(1+(d-1)\alpha)}\right)\right],$$

and let

$$s_2 = \frac{1}{2}\left[d + \left(\frac{(1-\alpha)(1+(d-1)\alpha) - (d-2m)^2\alpha^2}{\alpha(1+(d-1)\alpha)}\right)\right].$$

Let $\ell_1 = m - s_1/2$ and $\ell_2 = m - s_2/2$. There exists a regular two-graph on n vertices with a clique of order d with least eigenvalue $-1/\alpha$ if and only if there exists an incidence structure on d points with $n - d$ blocks of size m, such that any two distinct blocks β and δ satisfy $|\beta \cap \delta| = \ell_1$ or $|\beta \cap \delta| = \ell_2$.

Proof: Suppose that there exists a regular two-graph with the stated properties. Then by Theorem 2.8, there exists a set of n equiangular lines in $\mathbb{R}^d$ with mutual angle $\arccos(\alpha)$. Since there is a clique of order d in this two-graph, there exists a set Ω of unit vectors spanning these lines which contains a positive basis B.

Let G be the Gram matrix for B, and let L be the matrix whose columns are the vectors in B. There are $n - d$ vectors in $\Omega \setminus B$, and by Lemma 4.2, for each vector y in this set,

$$f_B(y) = \alpha^{-1} L^T y \tag{4.7}$$

is a ± 1-vector in $\mathbb{R}^d$ which contains exactly m or $d - m$ entries equal to -1. Switching on some of the vectors in $\Omega \setminus B$ if necessary, we can assume, without loss of generality, that for every vector y in $\Omega \setminus B$, $f_B(y)$ has exactly m negative entries.

For each vector y in $\Omega \setminus B$, let β_y denote the m-subset of $\{1, 2, \ldots, d\}$ which contains i if and only if the i-th position of $f_B(y)$ is equal to -1. Let

$$\Phi := \{\beta_y : y \in \Omega \setminus B\}.$$

If y and z are distinct vectors in $\Omega \setminus B$, then by Lemma 4.3, $f_B(y)$ and $f_B(z)$ differ in sign in s_1 or s_2 position. This implies that $|\beta_y \cap \beta_z| = m - s_1/2 = \ell_1$ or $|\beta_y \cap \beta_z| = m - s_2/2 = \ell_2$. Hence $\{1, 2, \ldots, d\}$ is the point set and Φ is the block set of an incidence structure with the required properties.

Conversely, suppose that there exists an incidence structure on point set $\{1, 2, \ldots, d\}$ and block set Φ which satisfies the stated properties. Let $B = \{x_1, x_2, \ldots, x_d\}$ be a set of unit vectors in $\mathbb{R}^d$ such that $\langle x_i, x_j \rangle = \alpha$, for $i \neq j$. Note that it is possible to find such a set B. Since B is a positive basis of itself, by Lemma 2.1, B is a basis of $\mathbb{R}^d$. Let L be the matrix whose columns are the vectors in B, and let $G = L^T L$, the Gram matrix for the vectors in B.

For each block β in Φ, let x_β denote the ± 1-vector in $\mathbb{R}^d$ with i-th entry equal to -1 if and only if $i \in \beta$, and let

$$y_\beta := f^{-1}(x_\beta) = \alpha L^{-T}(x_\beta).$$

Now for each β in Φ, x_β has exactly m negative entries. Since m is a solution to Equation 4.3 in the proof of Lemma 4.2, we must have that $\langle y_\beta, y_\beta \rangle = 1$, for all β in Φ.

Now suppose that β and δ are distinct blocks of Φ. If $|\beta \cap \delta| = \ell_1$, then x_β and x_δ differ in sign in $2(m - \ell_1) = 2(m - (m - s_1/2)) = s_1$ positions. Similarly, if $|\beta \cap \delta| = \ell_2$, then x_β and x_δ differ in sign in s_2 positions. Since s_1 and s_2 satisfy Equation 4.6 of Lemma 4.3, we have that $\langle y_\beta, y_\delta \rangle = \alpha$ whenever $|\beta \cap \delta| = \ell_1$, and $\langle y_\beta, y_\delta \rangle = -\alpha$ whenever $|\beta \cap \delta| = \ell_2$. Thus the set

$$C := \{y_\beta : \beta \in \Phi\}$$

is a set of unit vectors spanning a set of equiangular lines in $\mathbb{R}^d$, with mutual angle $\arccos(\alpha)$.

Finally, we show that any vector in C is in good position with respect to B. Since B is a basis of $\mathbb{R}^d$, each vector y_β in C can be written as a linear combination of vectors in B. Hence there exist unique real scalars $a_1, a_2, \ldots, a_d$ such that

$$y_\beta = \sum_{j=1}^{d} a_j x_j.$$

Thus

$$\langle x_i, y_\beta \rangle = x_i^T \sum_{j=1}^{d} a_j x_j = \sum_{j=1}^{d} a_j \langle x_i, x_j \rangle.$$

Since the ij-th entry of G is $\langle x_i, x_j \rangle$, we have

$$[\langle x_1, y_\beta \rangle, \langle x_2, y_\beta \rangle, \ldots, \langle x_d, y_\beta \rangle]^T = G[a_1, a_2, \ldots, a_d]^T. \tag{4.8}$$

To show that y_β is in good position with respect to B, we need only show that $G[a_1, a_2, \ldots, a_d]^T$ is a $\pm\alpha$ vector. Now

$$L[a_1, a_2, \ldots, a_d]^T = y_\beta = \alpha L^{-T} x_\beta,$$

and thus

$$\begin{aligned} \alpha x_\beta &= L^T L[a_1, a_2, \ldots, a_d]^T \\ &= G[a_1, a_2, \ldots, a_d]^T. \end{aligned}$$

Thus $G[a_1, a_2, \ldots, a_d]^T$ is a $\pm\alpha$-vector, as required.

Hence $B \cup C$ is a set of n unit vectors which span a set of n equiangular lines in $\mathbb{R}^d$ with mutual angle $\arccos(\alpha)$, which contains the positive basis B. Now Theorem 2.8 implies that there exists a regular two-graph with the stated parameters, and the basis B corresponds to a clique of order d in this graph. □

The proof of Theorem 4.4 gives a method for constructing a regular two-graph which contains a clique of order d with the given parameters from an incidence structure with the stated properties. Suppose we are given such an incidence structure with point set $\{1, 2, \ldots, d\}$ and block set Φ. Start with an ordered set of d vertices, say $V = \{v_1, v_2, \ldots, v_d\}$. Form the graph X with vertex set $V(X) := V \cup \Phi$ and edge set

$$E(X) := \{v_i\beta : \beta \in \Phi,\ i \notin \beta\} \cup \{\beta\delta : \beta, \delta \in \Phi,\ |\beta \cap \delta| = \ell_1\}.$$

The vertices of V correspond to the vectors of a positive basis

$$B = \{x_1, x_2, \ldots, x_d\},$$

and they form a clique in this graph. The vertices β of Φ correspond to the vectors y_β in $C = \Omega \setminus B$, where Ω is a set of unit vectors spanning a set of equiangular lines with mutual angle $\arccos(\alpha)$, which meets the relative bound, and which contains the positive basis B. Now for any i in $\{1, 2, \ldots, d\}$ and any β in Φ, $\langle x_i, y_\beta \rangle = \alpha$ if and only if $i \notin \beta$. Also, if β and

δ are distinct elements of Φ, then $|\beta \cap \delta| = \ell_1$ if and only if $\langle y_\beta, y_\delta \rangle = \alpha$. Hence two vertices are adjacent in X if and only if their associated unit vectors have inner product equal to α. Thus X is the graph defined by the Seidel matrix $S(X)$ corresponding to the Gram matrix $G = I + \alpha S(X)$ of the vectors in Ω. Since Ω meets the relative bound, Theorem 2.8 guarantees that the switching graph of X is a regular two-graph with least eigenvalue $-1/\alpha$ of multiplicity $n-d$, and by construction, it contains a clique of order d. Its complement is a regular two-graph with an independent set of order d.

4.3 Feasible Parameter Sets

Before applying the construction of Section 4.2, it would be useful to know which of the feasible parameter sets for regular two-graphs listed in Tables 3.1 and 3.2 could describe graphs which contain a clique of the required size. Given an integer $d \geq 2$ and a real number α such that $0 < \alpha < 1$, if the value for m in Lemma 4.2 is not a positive integer, there are no unit vectors outside of a positive basis B in R^d which are in good position with respect to B. Hence there does not exist a non-trivial regular two-graph which contains a clique of order d having least eigenvalue $\tau = -1/\alpha$ with multiplicity $m_\tau = n - d$, where $n = \frac{d - d\alpha^2}{1 - d\alpha^2}$. Thus, any feasible parameter set $(n, a_1, c_2; \theta, m_\theta, \tau, m_\tau)$ for a regular two-graph in Tables 3.1 and 3.2 for which $\alpha = -1/\tau$ and $d = n - m_\tau$ yield an non-integral value for m can be eliminated as a candidate for this construction technique. Lemma 4.5 shows that we can eliminate all of the parameter sets with $m_\theta = m_\tau$, except for the case when $n = 6$.

4.5 Lemma. *Suppose that Γ is a nontrivial regular two-graph with parameters*

$$(n, a_1, c_2; \theta, m_\theta, \tau, m_\tau).$$

Let $\alpha = -1/\tau$, let $d = n - m_\tau$, and let

$$m = \frac{1}{2}\left(d - \sqrt{d(d-1) + (2d-1)\alpha^{-1} - (d-2)\alpha^{-2} - \alpha^{-3}}\right).$$

If $n > 6$ and $m_\theta = m_\tau$, then m is not an integer, and consequently Γ does not contain a clique of order d.

Proof: If $m_\theta = m_\tau = n/2$, then the equations of Lemma 3.13 show that $\theta = -\tau$, and hence by Lemma 3.15, $\tau = -\sqrt{n-1}$. Thus $\alpha = 1/\sqrt{n-1}$, and

$d = n/2$. Substituting these values into the formula for m yields

$$m = \frac{1}{4}\left(n - \sqrt{-n^2 + 8n - 8}\right).$$

Thus m is not an integer unless

$$-n^2 + 8n - 8 \geq 0,$$

which implies that

$$4 - 2\sqrt{2} \leq n \leq 4 + 2\sqrt{2}.$$

Since n must be even, if $n > 6$, m is not an integer. □

Lemma 4.5 shows that we can eliminate all of the parameter sets in Tables 3.1 and 3.2 with $m_\tau = m_\theta$ as candidates for the construction of Theorem 4.4. In particular, this eliminates all of the parameter sets in Table 3.2 except for the one on 6 vertices.

The next result verifies that if m is an integer, the intersection sizes ℓ_1 and ℓ_2 of the blocks of the incidence structure of Theorem 4.4 are integers.

4.6 Lemma. *Let Γ be a regular two-graph with parameter set*

$$(n, a_1, c_2; \theta, m_\theta, \tau, m_\tau)$$

such that $n > 6$. Let $\alpha = -1/\tau$ and let $d = n - m_\tau$. Let m, s_1, s_2 be defined as in Lemmas 4.2 and 4.3. If m is an integer, then s_1 and s_2 are even integers.

Proof: If m is an integer, then by Lemma 4.5, $m_\theta \neq m_\tau$, so by Lemma 3.15, the eigenvalues θ and τ are integers. Equation 4.3 of Lemma 4.2 shows that

$$(d - 2m)^2 = \frac{(1 - \alpha - d\alpha^2)(1 + (d-1)\alpha)}{-\alpha^3}.$$

Substituting $\alpha = -1/\tau$ into the above equation gives

$$(d - 2m)^2 = (\tau^2 + \tau - d)(\tau - d + 1). \tag{4.9}$$

By Lemma 4.3,

$$s_1 = \frac{1}{2}\left[d - \left(\frac{1-\alpha}{\alpha}\right) - \frac{(d-2m)^2\alpha}{(1 + (d-1)\alpha)}\right].$$

Substituting $\alpha = -1/\tau$ and the formula for $(d-2m)^2$ given in Equation 4.9 into this equation yields

$$s_1 = \frac{1}{2}(\tau + 1)^2,$$

Parameter Set $(n, a_1, c_2; \theta, m_\theta, \tau, m_\tau)$	d	α	m	s_1	s_2	ℓ_1	ℓ_2
$(6, 2, 2; \sqrt{5}, 3, -\sqrt{5}, 3)$	3	$1/\sqrt{5}$	1	–	–	–	–
$(16, 8, 6; 5, 6, -3, 10)$	6	1/3	3	2	4	2	1
$(28, 16, 10; 9, 7, -3, 21)$	7	1/3	2	2	4	1	0
$(96, 54, 40; 19, 20, -5, 76)$	20	1/5	10	8	12	6	4
$(126, 72, 52; 25, 21, -5, 105)$	21	1/5	8	8	12	4	2
$(276, 162, 112; 55, 23, -5, 253)$	23	1/5	7	8	12	3	1
$(288, 160, 126; 41, 42, -7, 246)$	42	1/7	21	18	24	12	9

Table 4.1: Feasible Parameter Sets for Regular Two-Graphs With Large Cliques

which is an even integer since τ is odd. Similarly, substituting $\alpha = -1/\tau$ and the above expression for $(d-2m)^2$ into the the equation for s_2 in Lemma 4.3 yields

$$s_2 = \frac{1}{2}(\tau - 1)(\tau + 1),$$

which implies that s_2 is also an even integer. □

Each of the feasible parameter sets $(n, a_1, c_2; \theta, m_\theta, \tau, m_\tau)$ for regular two-graphs with integer eigenvalues in Table 3.1 was tested to check whether the values $d = n - m_\tau$ and $\alpha = -1/\tau$ yield an integral value for m in the formula of Lemma 4.2. For $n \leq 300$, the parameter sets for regular two-graphs on n vertices for which m is an integer are provided in Table 4.1, along with their corresponding values for d, α, m, s_1, s_2, ℓ_1, and ℓ_2. These are all of the feasible parameter sets for our construction, for $n \leq 300$.

In the next two examples, we construct regular two-graphs with cliques of order d for the parameter sets with $n = 6$ and $n = 16$. In Section 4.4, we will use quasi-symmetric designs to construct regular two-graphs for the parameter sets with $n = 28$ and $n = 276$. It will be shown that for the parameter set with $n = 126$, the associated incidence structure of Theorem 4.4 is not a 2-design. It is still not know whether regular two-graphs with cliques of order d exist for the parameter sets with $n = 96, 126, 288$.

Example. In Section 3.3 we saw that the switching graph Γ of $C_5 \cup K_1$ is a regular two graph with nontrivial eigenvalues $\sqrt{5}$ and $-\sqrt{5}$, with equal multiplicity 3. Switching on the isolated vertex of $C_5 \cup K_1$ yields a graph Y

which contains a clique of order 3. Since $\mathrm{Sw}(Y)$ is isomorphic to

$$\Gamma = \mathrm{Sw}(X \cup K_1),$$

Γ contains two disjoint cliques of order 3. One can verify that Γ is the unique regular two-graph with parameters $(6, 2, 2; \sqrt{5}, 3, -\sqrt{5}, 3)$.

Example. To construct a regular two-graph with parameter set

$$(16, 8, 6; 5, 6, -3, 10)$$

containing a clique of order $d = 6$ using Theorem 4.4, we require an incidence structure on 6 points, with $n - d = 10$ blocks of size $m = 3$, such that any two blocks meet in $\ell_1 = 2$ or $\ell_2 = 1$ points. The incidence structure with block set

$$\{123, 124, 125, 126, 134, 135, 136, 145, 146, 156\}$$

satisfies these conditions.

In the next section we look at a class of designs satisfying the intersection properties of Theorem 4.4.

4.4 Quasi-Symmetric Designs

Quasi-symmetric designs provide a class of examples of incidence structures which satisfy the properties of Theorem 4.4.

A t-(v, k, λ_t) design is a set P of v *points*, together with a collection Φ of k-subsets of P, called *blocks*, such that every t-subset of points in P lies in exactly λ_t blocks. A 2-design is *quasi-symmetric* if there are constants ℓ_1 and ℓ_2, with $\ell_1 > \ell_2$, such that any two distinct blocks of D have exactly ℓ_1 or ℓ_2 points in common.

If D is a t-(v, k, λ_t) design, and S is an s-subset of points with $s < t$, we can count the number of blocks λ_s of D containing S. We count the pairs (T, B), where T is a t-set containing S and B is a block containing T, in two ways. First, S lies in $\binom{v-s}{t-s}$ t-subsets T, each of which lies in λ_t blocks. Also, for each block containing S, there are $\binom{k-s}{t-s}$ possible choices for T. Hence

$$\lambda_s \binom{k-s}{t-s} = \lambda_t \binom{v-s}{t-s}. \tag{4.10}$$

Since the number of blocks does not depend on the choice of S, D is also a s-(v, k, λ_s) design, for each $s < t$. Hence if such a t-design exists, then the values of λ_s must be integers for all $s < t$. The parameter λ_0 is the total number of blocks in the design, and is usually denoted by b. Setting $s = 0$ in Equation 4.10, we have

$$b\binom{k}{t} = \lambda_t\binom{v}{t}. \tag{4.11}$$

Thus for a 2-design,

$$b = \frac{\lambda_2(v^2 - v)}{k^2 - k}.$$

The number of blocks containing each point is λ_1. It is called the *replication number* of the design, and is usually denoted by r. Setting $t = 1$ in Equation 4.11 yields

$$bk = vr.$$

Hence in a 2-design,

$$r = \frac{bk}{v} = \frac{\lambda_2(v-1)}{k-1}.$$

Example. Every 2-$(v, k, 1)$ design is quasi-symmetric with block intersection sizes $(\ell_1, \ell_2) = (1, 0)$. To construct a regular two-graph on the parameter set $(28, 16, 10; 9, 7, -3, 21)$, with a clique of order 7, by Theorem 4.4 we require an incidence structure on $d = 7$ points with $n - d = 21$ blocks of size $m = 2$, in which every two blocks intersect in $\ell_1 = 1$ or $\ell_2 = 0$ points. The set of 21 subsets of size 2 of any set of 7 points forms a simple 2-(7,2,1) design with the desired properties.

Example. *The Witt Design on 23 Points*:
The Witt design is a 4-$(23, 7, 1)$ design, and it is one of the most interesting combinatorial structures. We describe a construction for this design given in [10] and show that it is quasi-symmetric with intersection sizes $(\ell_1, \ell_2) = (3, 1)$.

Over $GF(2)$, the polynomial $x^{23} - 1$ factors as

$$(x-1)g(x)h(x),$$

in which

$$g(x) = x^{11} + x^9 + x^7 + x^6 + x^5 + x + 1$$

and

$$h(x) = x^{11}g(x^{-1}) = x^{11} + x^{10} + x^6 + x^5 + x^4 + x^2.$$

Both $h(x)$ and $g(x)$ are irreducible polynomial of degree 11. Let R denote the ring of polynomials over $GF(2)$, modulo $x^{23} - 1$, and let C be the ideal of this ring generated by $g(x)$. Then C is the set of all polynomials in this ring divisible by $g(x)$. The powers of $g(x)$ modulo $x^{23} - 1$ form the ring C. Each element in R can be represented by a polynomial over $GF(2)$ with degree at most 22, so we can represent each element f of R by a binary vector of length 23, with the i-th coordinate equal to the coefficient of x^i in f (for $0 \leq i \leq 22$). The ideal C of R contains 2048 polynomials, and hence can be represented by a set C' of 2048 binary vectors of length 23 (these 2048 vectors form the binary Golay code). Exactly 253 of these vectors have precisely 7 nonzero entries. The set of nonzero positions of a binary vector is called its *support*, and the supports of these 253 vectors form a 4-$(23, 7, 1)$ design, known as the *Witt design* on 23 points. It is known that this is the unique design with these parameters. This design has $b = 253$ blocks, and each point occurs in $r = 77$ blocks. Every pair of points occurs in exactly $\lambda_2 = 21$ blocks, and every 3-set of points lies in $\lambda_3 = 5$ blocks, so this is also a 2-$(23, 7, 21)$ design and a 3-$(23, 7, 5)$ design.

We now show that the Witt design is quasi-symmetric. Let B be a block of this design. Relabeling the points if necessary, we can assume that $B = \{1, 2, 3, 4, 5, 6, 7\}$. For each i and j in $\{0, 1, 2, 3, 4, 5, 6, 7\}$ with $i+j \leq 7$, let $\lambda_{i,j}$ be the number of blocks of the Witt design that contain the first $i-j$ points of B, but none of the next j points. Then since $\lambda_{i,0} = \lambda_i$, for $i \leq 3$, we have

$$\lambda_{0,0} = 253, \quad \lambda_{1,0} = 77, \quad \lambda_{2,0} = 21, \quad \lambda_{3,0} = 5.$$

Also $\lambda_{i,0} = 1$ if $4 \leq i \leq 7$. One can verify that if $i, j \geq 1$, then

$$\lambda_{i,j} = \lambda_{i-1,j-1} - \lambda_{i,j-1}.$$

Thus the lower triangular portion of the 8×8 matrix M with $M_{ij} = \lambda_{i,j}$ is given by

$$M = \begin{pmatrix} 253 & & & & & & & \\ 77 & 176 & & & & & & \\ 21 & 56 & 120 & & & & & \\ 5 & 16 & 40 & 80 & & & & \\ 1 & 4 & 12 & 28 & 52 & & & \\ 1 & 0 & 4 & 8 & 20 & 32 & & \\ 1 & 0 & 0 & 4 & 4 & 16 & 16 & \\ 1 & 0 & 0 & 0 & 4 & 0 & 16 & 0 \end{pmatrix}.$$

The last row of M implies that exactly 1 block contains every point of B, 4 blocks contain the first 3 points of B and none of the other 4 points, and 16 blocks contain the first point of B and no other point of B. For this design, the values $\lambda_{i,j}$ are independent of the choice of B. Thus any two blocks of the Witt design meet in either 1 or 3 points.

Example. *The Unique Regular Two-Graph on 276 Vertices*:
To construct a regular two-graph with parameters

$$(276, 162, 112; 55, 23, -5, 253)$$

with a clique of order 23, we require an incidence structure on $d = 23$ points with $n - d = 253$ blocks of size $m = 7$, such that any two blocks meet in $\ell_1 = 3$ or $\ell_2 = 1$ points. The Witt design has the desired properties. In [12], Goethals and Seidel proved that this is the unique regular two-graph (up to complementing) on 276 vertices. Their proof reduces to the uniqueness of the ternary Golay code. This construction shows that this regular two-graph contains a clique of order 23.

In [3], Calderbank derived the following necessary conditions for the existence of a 2-(v, k, λ) design in which the block intersection sizes are all congruent modulo 2.

4.7 Theorem. *Let p be a prime. Let D be a 2-(v, k, λ) design with block intersection sizes $\ell_1, \ell_2, \ldots, \ell_t$, such that $\ell_1 \equiv \ell_2 \equiv \ldots \equiv \ell_t \equiv \ell \bmod 2$. Then one of the following conditions hold:*

(a) $r \equiv \lambda \bmod 4$,

(b) $\ell \equiv 0 \bmod 2$, $k \equiv 0 \bmod 4$, *and* $v \equiv \pm 1 \bmod 8$,

(c) $\ell \equiv 1 \bmod 2$, $k \equiv v \bmod 4$, *and* $v \equiv \pm 1 \bmod 8$.

□

Example. To construct a regular two-graph with parameters

$$(126, 72, 52; 25, 21, -5, 105)$$

which contains a clique of order 21, by Theorem 4.4 we require an incidence structure on $d = 21$ points, with $n - d = 105$ blocks of size $m = 8$, such that any two blocks meet in exactly $\ell_1 = 4$ or $\ell_2 = 2$ points. If there were a quasi-symmetric design with these properties, it would be a 2-$(21, 8, 14)$ design with intersection numbers congruent to 0 modulo 2. Theorem 4.7 shows that a quasi-symmetric design with these parameters does not exist. Hence, we cannot use a quasi-symmetric design to construct such a regular two-graph.

Of course, the last example does not imply that there does not exist a regular two-graph with parameters $(126, 72, 52; 25, 21, -5, 105)$ containing a 21-clique. The incidence structure required by Theorem 4.4 need not be a 2-design, as the regular two-graph constructed on 16 vertices in the last section shows.

Example. If there exists a 2-design satisfying the properties of the incidence structure required by Theorem 4.4 for the parameter set

$$(96, 54, 40; 19, 20, -5, 76),$$

it would be a 2-(20,10,18) design with intersection numbers $(\ell_1, \ell_2) =$ (6,4). The parameters of this design satisfy the necessary conditions of Theorem 4.7, but it is still not known if such a design exists. Using a greedy algorithm to search the set of subsets of size 10 of the set $\{1, 2, \ldots, 20\}$, we can find 70 blocks, every two of which meet 4 or 6 points. By Theorem 4.4 we need 76 such blocks to construct the desired regular two-graph. With these 70 blocks, using the method of Theorem 4.4 we can find a set of 90 vectors in $\mathbb{R}^{20}$ which span a set of 90 equiangular lines, with mutual angle arccos(1/5), which contains a positive basis. (The largest set of equiangular lines in $\mathbb{R}^{20}$ known to exist has size 90. Taylor found such a set in [21].) This corresponds to a graph on 90 vertices which has least eigenvalue -5 with multiplicity 76, which contains a clique of order 20.

Example. If there exists a 2-design satisfying the properties of the incidence structure required by Theorem 4.4 for the parameter set

$$(288, 160, 126; 41, 42, -7, 246),$$

it would be a 2-(42,21,60) design with intersection numbers $(\ell_1, \ell_2) = (12, 9)$. It is not known if such a design exists.

Neumaier developed a classification of quasi-symmetric 2-(v, k, λ) designs, in which he distinguishes four classes of designs and lists the parameter sets of the 23 exceptional designs with $v \leq 40$ that do not belong to any of the four classes. Calderbank updated this list in [3] and [4] to include exceptional design parameters with $v \leq 70$. It is interesting to note that all of the parameter sets for quasi-symmetric designs which satisfy the requirements of Theorem 4.4 for some parameter set in Table 4.1 (with $n > 28$) are on Neumaier's exceptional list of parameters.

Theorem 4.4 raises an interesting question: given a regular two-graph with a clique of order d, when is the incidence structure of Theorem 4.4 a 2-design? The only examples of these regular two-graphs which we were able to construct from 2-designs correspond to sets of equiangular lines which meet the absolute bound of Lemma 2.2. Is this a necessary condition?

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