CODEN STJSAO
ZX470/1510

ISSN 0562-1887
UDK 621.833.1:519.216

Deterministic and Probabilistic Methods in Determination of Correct Mating Cylindrical Teeth Profiles

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Keywords

Correct mating profiles
Cylindrical gearing
Path of contact
Stochastic lattice

Ključne riječi

Čeličko ozubljenje
Pravilno spregnuti čeoni profili
Stohastička rešetka
Zahvatna crta

Received (primljeno): 2009-07-12
Accepted (prihvaćeno): 2011-03-30

Preliminary note

Currently design and clasification of gearing comes almost exclusively from a so-called "technological method". Its means that for a known form of one wheel (mostly a hob tool), a correctly mating form of a tooth flank of a mating wheel is determined. This commonly known and simple method positively describes also the path of contact of mating profiles. Such method defined already by Buckingham [1], when he wrote "The simplest way has been to define any definite conjugate gear-tooth system is to specify the form and size of its basic rack. There is a definite relation between a gear-tooth profile and its path of contact so that if either one is given, the other is fixed." In the article, the authors from a given shape of path of contact by design and classification of cylindrical gearing. There are two totally different methods introduced to solve this task, namely deterministic and stochastic one.

Deterministička i probabilistička metode u određivanju pravilno spregnutih čelnih profila zubi cilindričnih zupčanika

Prethodno priopćenje

Suvremeno oblikovanje i podjela ozubljenja temelje se gotovo isključivo na takozvanoj "tehnološkoj metodi". To znači da se za poznati oblik zuba jednog zupčanika (najčešće alata u obliku odvalnog pužnog glodala) određuje oblik zuba pravilno spregnutog profila drugog zupčanika. Ova opće poznata jednostavna procedura određuje istovremeno i zahvatnu crtu spregnutih profila. Istu metodu definirao je Buckingham [1], kada je napisao "Najjednostavniji način za definiranje proizvoljnog pravilno spregnutog zupčastog para je da se definira oblik i veličina njegove osnovne ozubnice. Postoji jednoznačan odnos između oblika profila zuba zupčanika i oblika njegove zahvatne crte, što znači da ako je jedan od njih zadan, drugi je jednoznačno određen." Autori u članku prikazuju način oblikovanja i podjelu ozubljenja cilindričnih zupčanika na osnovi poznatog oblika zahvatne crte. Predstavljene su dvije u potpunosti različite metode za rješavanje ovog zadatka, i to jedna deterministička, a druga stohastička.

1. Introduction

If the shape of path of contact is given, then the correct mating profiles $p1$, $p2$ of both conjugates wheel are defined by a geometrical transformation of path of contact from a fixed coordinate system connected with it into a rotating coordinate systems of both wheels. This conclusion can be used advantageously if the path of contact form is defined analytically. For a more complicated form of path of contact or for path of contact defined by nondifferentiable functions, by discrete points etc. one original stochastic method is here presented.

2. Determination of equations of correctly mating profile curves for analyticaly given path of contact

If a form of the path of contact (Figure 1) can be expressed through a vector parametric equation where a momentary pressure angle α is a parameter, i.e.

$$\vec{r} = r(\alpha), \tag{1}$$

then for determination of correct mating teeth profiles we need to know the turn angle value of the wheel corresponding to movement of the mesh point along the path of contact. It is possible to determine the relation $\varphi_r = f(\alpha)$ from a kinematic description of the mesh point motion along a path of contact as described by Pancuk [2], but it can be done also in an easier way using basic knowledge from differential geometry in a direct application of the fundamental law of gearing.

The main idea of this approach is obvious from figure 1 where the principle of the determination of the relation for the wheel turn $d\varphi_r(\alpha)$ at a mesh motion along the active tooth flank from point Z^* to point Z_1^* is presented. From the fundamental law of gearing the following relations result

$$|ZC| = |Z^*C^*| = r(\alpha),$$

$$\left|Z_1C\right| = \left|Z_1^*C_1^*\right| = r\left(\alpha + d\alpha\right),$$

$$\left|C^*C_1^*\right| = r_1 d\varphi_r\left(\alpha\right),$$

$$\left|MC^*\right| = r\left(\alpha + d\alpha\right) - r\left(\alpha\right) = r'\left(\alpha\right)d\alpha, \cdot$$

From the infinitesimal triangle $C^*C_1^*M$ we obtain

$$r'\left(\alpha\right)d\alpha = \left(r_1 \cos\alpha\right)d\varphi_r\left(\alpha\right).$$

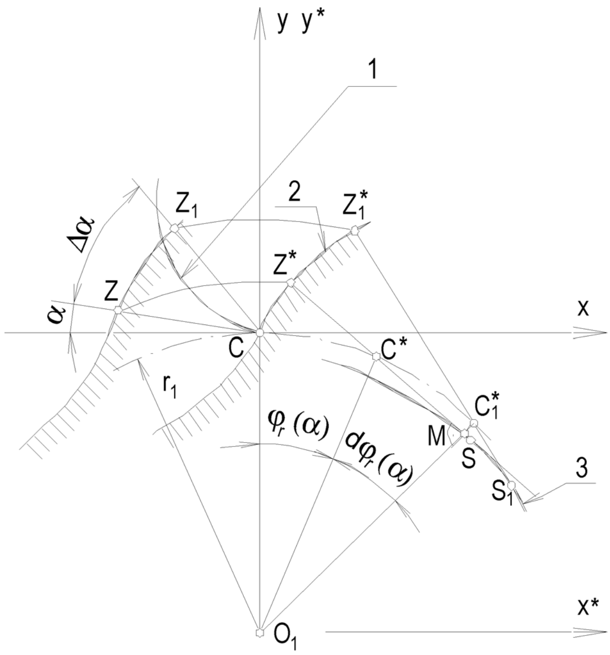

Figure 1. Derivation of teeth profile equations for a given path of contact: 1 –path of contact, 2 – tooth profile p1, 3 – tooth profile evolute

Slika 1. Izvođenje jednadžbi profila zuba za danu zahvatnu crtu 1 – zahvatna crta, 2 – profil zuba p1, 3 – evoluta profila zuba

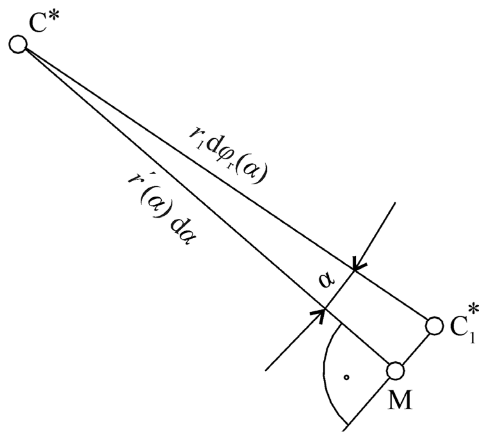

Figure 2. Detail of an infinitesimal triangle for a wheel turn angle determination

Slika 2. Detalj infinitezimalnog trokuta za određivanja kuta zakreta gonjenog zupčanika

and from this it is already easy to calculate the wheel turn angle φ_r in a dependence on the mesh point Z motion along the path of contact. It is obvious that for a motion between the pressure angles of two arbitrary points (e.g. Z, Z_1) α_1, α_2 we can write

$$\varphi_r = \frac{1}{r_1}\int_{\alpha_1}^{\alpha_2} \frac{r'}{\cos\alpha}\, d\alpha \cdot \quad (2)$$

Formula (1) for the path of contact can be rewritten into the components x, y of the absolute coordinate system Cxy where it has a form

$$x = \pm\, r\,(\alpha)\cos\alpha, \quad (3)$$

$$y = \pm\, r\,(\alpha)\sin\alpha. \quad (4)$$

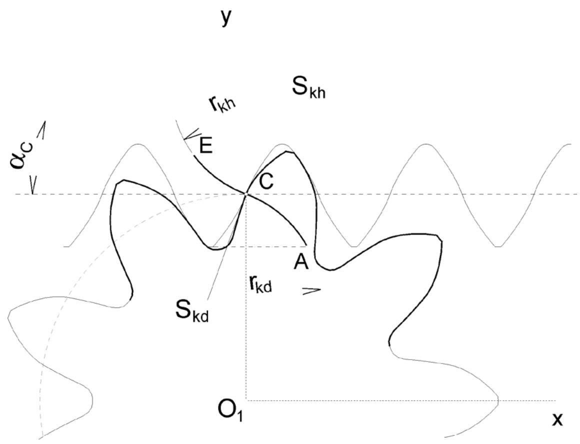

Figure 3. Cylindrical gearing defined by path of contact composed of two circular arcs

Slika 3. Ozubljenje cilindričnog zupčanika definirano na temelju zahvatne crte sastavljene od dvaju kružnih lukova

The signs ± are for the form of the path of contact drawn in Figure 4, sign + is for a mesh in the upper part of the path of contact (above the x axis) and sign – is for a mesh in the path of contact below the x axis. As already explained, correctly meshing active tooth flank as a matter of fact a path of contact transformed from a fixed (absolute) coordinate system into a rotating coordinate system, which is connected with the appropriate wheel. This transformation can be carried out if an equation of the path of contact and the angle of the wheel turn in dependence on the same parameters are at disposal. It is clear that in our case these conditions are fulfilled, because they are expressed in formulas (2), (3) and (4). But it has to be taken into consideration (by selecting a transformation matrix) that the path of contact is expressed in the coordinate system with point C as a coordinate origin and the relevant wheel has a coordinate system with coordinate origin in point O_1. For this transformation from the fixed coordinate system Cxy into the rotating one $O_1x^*y^*$ in homogeneous coordinates we have

$$\begin{pmatrix} x^* & y^* & 1 \end{pmatrix} = \begin{pmatrix} \mp r(\alpha)\cos\alpha & \pm r(\alpha)\sin\alpha & 1 \end{pmatrix}\cdot$$
$$\cdot\begin{pmatrix} \cos\varphi_r & -\sin\varphi_r & 0 \\ \sin\varphi_r & \cos\varphi_r & 0 \\ r_1\sin\varphi_r & r_1\cos\varphi_r & 1 \end{pmatrix} \tag{5}$$

and after a matrix multiple from that

$$x^* = \pm\, r(\alpha)\cos(\alpha+\varphi_r) + r_1\sin\varphi_r, \tag{6}$$

$$y^* = \pm\, r(\alpha)\sin(\alpha+\varphi_r) + r_1\cos\varphi_r. \tag{7}$$

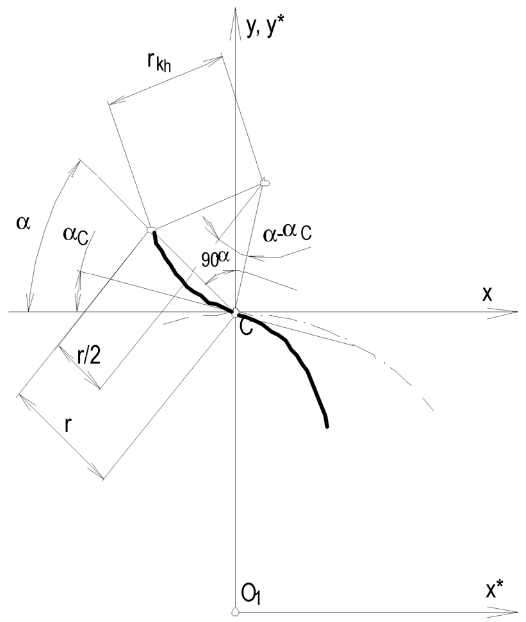

Figure 4. Geometrical dependencies for determination of a path of contact composed of two circular arcs

Slika 4. Geomterijski odnosi za određivanje zahvatne crte sastavljenje od dvaju kružnih lukova

It is useful to consider a path of contact composed of two circular arcs. A path of contact consisting of circular arcs can have various positions towards the central line: centres of the circular arcs can be on the central or apart it, radiuses of the circular arcs can be of any value, in an extreme case the curvature centroids of these circles may lie even in "infinity" and so on. It is obvious that these geometrical parameters of the path of contact will directly influence the form of the profiles of correctly meshing teeth which relate to it. This means that on the basis of the basic geometrical parameters, it is possible to classify particular kinds of plane cylindrical gearings and also the form of the profile curve of the tooth flank. In Figure 4 there is the most general case to define a plane cylindrical gearing by means of a path of contact composed of circular arcs according to Hlebanja [3] (it would be possible to consider also a combination of more arcs, but this case is practically identical to the drawn one – it would be solved in particular stages). The gearing set in this way defined the path of contact (and, of course, else by the centres of the wheel rotation) we can call *general plane convex-concave gearing*. Then according to Figure 4 it is

$$r = 2r_{kh}\cos(90^{o} - \alpha - \alpha_c)$$

which is

$$r = 2r_{kh}\sin(\alpha - \alpha_c). \tag{8}$$

It would be analogical also for the lower part of the path of contact. Then for the wheel turn φ_r in a mesh point motion from point C to point Z, which is determined through the pressure angle α (substituting the formula (8) into the equation (2)), we obtain

$$\varphi_r = \pm\frac{1}{r_1}\int_{\alpha_C}^{\alpha}\frac{2r_{kh,d}\cos(\alpha-\alpha_C)}{\cos\alpha}d\alpha\,. \tag{9}$$

After the integration the formula for the angle φ_r takes the form

$$\varphi_r = \pm\frac{2r_{kh,d}}{r_1}\left[(\alpha-\alpha_C)\cos\alpha_C + \sin\alpha_C\,\ln\frac{\alpha_C}{\alpha}\right] \tag{10}$$

and substituting formulas (10) and (8) into the equations (6), (7) we obtain the following parametric equations of the active tooth flank for calculation coordinates x^*, y^* of the searched form of the gearing profile

$$x^* = \pm\, 2r_{kh,d}\sin(\alpha - \alpha_c)\cos[\alpha+\varphi_r(\alpha)] + r_1\sin\varphi_r(\alpha) \tag{11}$$

$$y^* = \pm\, 2r_{kh,d}\sin(\alpha - \alpha_c)\sin[\alpha+\varphi_r(\alpha)] + r_1\cos\varphi_r(\alpha) \tag{12}$$

Equations of the rack can be very easily determined as an extreme case of the equations (11) and (12) for $r_1 = \infty$. Then for a rack form to the path of contact composed of two circular arcs we have

$$x^* = \mp 2r_{kh,d}\sin(\alpha-\alpha_C) \pm 2r_{kh,d}\cdot$$
$$\cdot\left[(\alpha-\alpha_C)\cos\alpha_C + lg\frac{\cos\alpha_C}{\cos\alpha}\sin\alpha_C\right] \tag{13}$$
$$y^* = \pm 2r_{kh,d}\sin(\alpha-\alpha_C)\sin\alpha\,.$$

3. Special cases of a path of contact composed of two circular arcs and related gearing to them

As already mentioned, a general convex-concave gearing defined by a path of contact composed of circular arcs, the centres of which do not be on a central line of the gears, represents the cycloidal, pin and involute ones as special gearing cases also.

4. Cycloidal gearing

According to the accepted definition of the plane gearings, the cycloidal gearing is defined by the path of contact composed of two circular arcs whose centres lie on the central O_1O_2 of the mating wheels. For determination of the parametric equations of the profile curves of both teeth of the mating wheels it is possible to use directly the equations (10), (11), or (12) where $\alpha_C = 0$. Then for a common case of the cycloidal gearing for active tooth flanks the parametric equations have the following forms

$$x^*_{cykl} = \mp 2r_{kh,d}\sin\alpha\cos\left[\alpha\left(1-\frac{2r_{kh,d}}{r_1}\right)\right] \pm r_1\sin\left(\frac{2r_{kh,d}}{r_1}\alpha\right) \tag{14}$$

$$y^*_{cykl} = \pm 2r_{kh,d}\sin\alpha\sin\left[\alpha\left(1-\frac{2r_{kh,d}}{r_1}\right)\right] \pm r_1\cos\left(\frac{2r_{kh,d}}{r_1}\alpha\right)$$

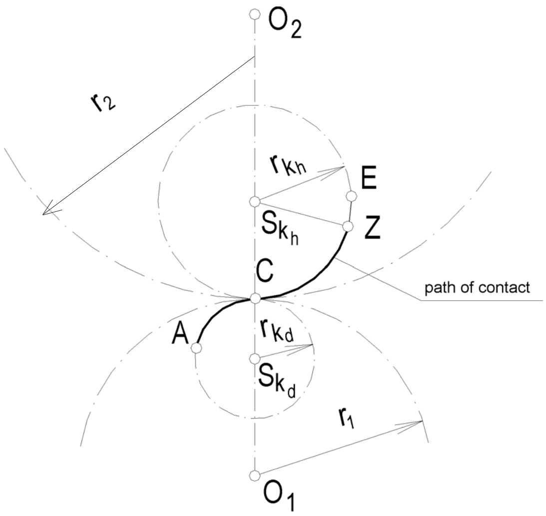

Figure 5. Cycloidal gearing

Slika 5. Cikloidno ozubljenje

Analogically the same procedure can be used for determination of the parametric equations of the rack with a cycloidal gearing. Here from the formulas (13) the following relations result

$$x^*_{cyklrack} = 2r_{kh,d}\,(\pm\sin\alpha\cos\alpha \pm \alpha),$$
$$y^*_{cyklrack} = \pm 2r_{kh,d}\sin^2\alpha\,. \tag{15}$$

5. Pin gearing

If we are looking for the equations of the active tooth flanks of the pin (point) gearing, we can extract from the equations (14)÷(15). Moreover in this case $r_{kh} = r_1$, or $r_{kd} = r_1$. Then, for example, for the case of the pin gearing defined in Figure 6, i.e. $r_{kh} = r_{w2}$, the equations of the profile p_2 to the lower part of the path of contact (towards the position of the point O_2 – see the Figure 6 – and above-mentioned note, and formally considering $r_{kd} = r_{w2}$) will have the form

$$x_{pinP2} = 2r_{w2}\sin\alpha\cos(-\alpha) + r_{w2}\sin(-2\alpha),$$
$$y^*_{pinP2} = 2r_{w2}\sin\alpha\sin(-\alpha) + r_{w2}\cos(-2\alpha)$$

and after modification

$$y^*_{pinP2} = r_{w2}\,[2\sin^2\alpha + \cos^2\alpha - \sin^2\alpha] = r_{w2}. \tag{16}$$

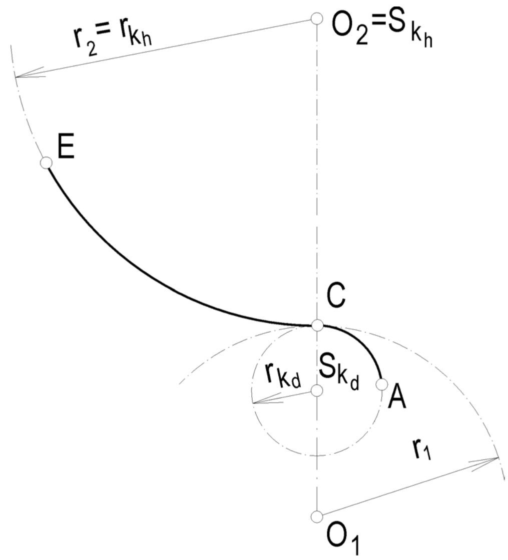

Figure 6. Pin gearing

Slika 6. Ozubljenje s valjcima

6. Involute gearing

Involute gearing can be understood as a special case of convex-concave gearing determined by the circle form of a path of contact in such a way that for their arc radiuses $r_{kh} = r_{kd} = \infty$, $\alpha_C = \alpha =$ const applies. Therefore, for the development of equations of correctly meshing profiles it is possible to apply the procedure stated as a limitary case of equation (11), (12).

7. Stochastic lattice method

Qualitatively, another method to determine correctly meshing teeth profiles for a given path of contact is the stochastic lattice method. It uses a probabilistic procedure to obtain a deterministic result. This method is especially suitable to be used in great CAD systems for gearing designing. A form of the profiles is obtained with an appropriate in advance specified accuracy and a sequence of this procedure is the following one.

Let $y = \varphi(x)$ be a continuous, piecewise smooth function describing a line of action in an orthogonal coordinate system, let r be a radius of a pitch circle. Let us assume the plane area Ω , so that the following conditions are satisfied:

For each point $[x, y] \in \Omega$ there exists an uniquely determined point $[u, v] \in E_2$ so that

$v = \varphi(u)$,

$x^2 + (r + y)^2 = u^2 + (r + v)^2$.

Due to the above mentioned conditions, it is possible to define a two-dimensional transformation

$\Psi: \Omega \rightarrow E_2,\ [x, y] \rightarrow \Psi([x, y]) = [\xi(x, y), \eta(x, y)] = [u, v]$.

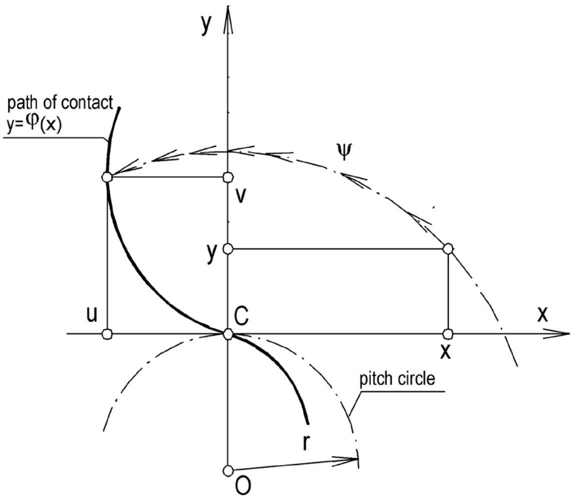

Figure 7. Two dimensional transformation

Slika 7. Dvodimenzijska transformacija

Typical representatives of such a path of contact are, for example a straight line corresponding to the involute type of gearing or a couple of circular arcs defining the cycloidal or convex-concave gearing. In the last mentioned situations, it is possible to obtain values $\xi(x, y), \eta(x, y)$ using fundamental trigonometric methods. In a general case the use of appropriate numerical methods is required. The situation is demonstrated in a transparently symbolical way in Figure 8.

For the sake of simplicity let us assume such a part of the stochastic lattice where $x_i, y_j \geq 0$. The case $x_i, y_j \leq 0$ is an analogical one.

Let

$$[0,0] \xrightarrow{\pi_0} [\alpha_1, \beta_1] \xrightarrow{\pi_1} \ldots [\alpha_k, \beta_k] \xrightarrow{\pi_k}$$

$$\xrightarrow{\pi_k} [\alpha_{k+1}, \beta_{k+1}] \xrightarrow{\pi_{k+1}} \ldots \xrightarrow{\pi_{n-1}} [\alpha_n, \beta_n]$$

be a stochastic process on Ξ . Let us define the transition probabilities π_k in the following way:

If $[\alpha_k, \beta_k] = [x_i, y_j]$, then

$$[x_i, y_j] \xrightarrow{\pi_x(i,j)} [x_{i+1}, y_j]$$

$$[x_i, y_j] \xrightarrow{\pi_y(i,j)} [x_i, y_{j+1}]$$

where, $\pi_x = \dfrac{\sin(\lambda_{ij})}{\sin(\lambda_{ij}) + \cos(\lambda_{ij})}$, $\pi_y = 1 - \pi_x$.

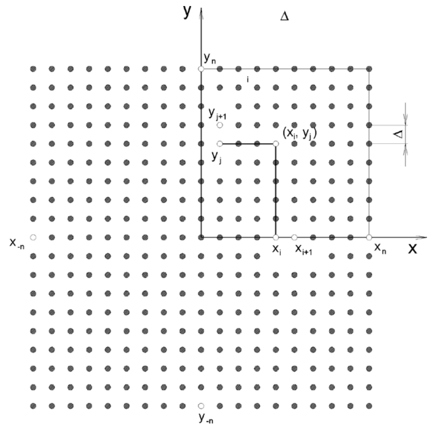

Figure 8. Stochastic lattice

Slika 8. Stohastička rešetka

In the last formulas, the key role is played by λ_{ij} which is an angle between a path of contact to an active tooth flank in an actual mesh point and an abscissa joining the mesh point with the pitch circle centre. The required values of the goniometric functions can be obtained using coordinates of point $\Psi([x_i, y_j])$. The situation is, in a transparent way, shown in Figure 9.

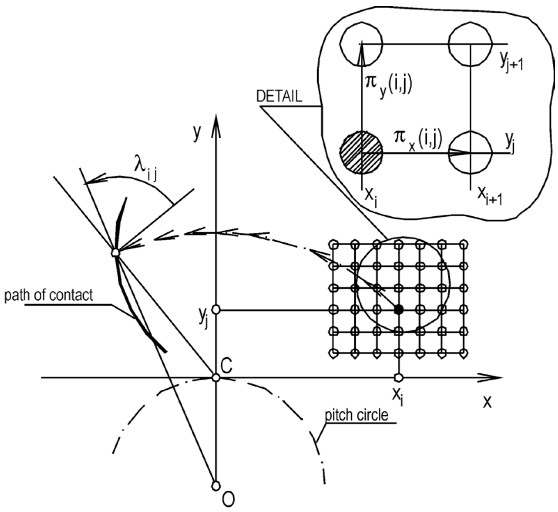

Figure 9. Realization of the stochastic process

Slika 9. Realizacija stohastičkog procesa

An open polygon joining the neighbouring points of each realization of the above determined stochastic process can be understood as a stochastic approximation of the active tooth flank which corresponds to a path of contact defined by a function dependency $y = \varphi(x)$.

Let $[0,0],[\alpha_1^k,\beta_1^k],[\alpha_2^k,\beta_2^k],...,[\alpha_n^k,\beta_n^k]$ for $k = 1, 2, ...m$ be m independent realizations of this stochastic process. Due to its definition, it is easy to see, that an arbitrary point can be represented merely by a couple $[x_i, y_j]$, for which $i + j = q$. For all fixed $q = 1, 2,...,n$ let the values be defined

$$\Gamma_q(i) = card\left\{[x_i, y_{q-i}] : \left(\begin{array}{l}\exists k \in \{1,2,...,m\} : [x_i, y_{q-i}] = \\ = [\alpha_q^k, \beta_q^k]\end{array}\right)\right\},$$

$i = 0, 2, ...q$

Let

$\tau(q) = Me\ \{\Gamma_q(0), \Gamma_q(1), \Gamma_q(2),..., \Gamma_q(q)\}$

denote a median of the presented numerical set. An open polygon joining the neighbouring points of a sequence

$\{[0,0],[x_{\tau(1)}, y_{1-\tau(1)}],[x_{\tau(2)}, y_{2-\tau(2)}], [x_{\tau(3)}, y_{3-\tau(3)}],...,[x_{\tau(n)}, y_{n-\tau(n)}]\}$

represents an approximate model of the active tooth flank which corresponds to the above mentioned path of contact.

It is possible to achieve an arbitrary degree of precision, in dependence on a number of realization m and the grade of slightness of stochastic lattice determined by the constant Δ. The presented method is not sensitive to the complexity of the form of the path of contact. The class of appropriate curves is an essential extension of the common generally known types.

In order to demonstrate the efficiency of the described approach, some concrete examples in the Figures 10 and 11 are displayed. Figure 10 shows a classical case related to the involute type of gearing. It is possible to see the path of contact, a part of a pitch circle and the resulting form of the active tooth flank obtained by the use of the above mentioned method.

The "corridor" in the upper right side corner of the figure represents the area in the exact centre of which would lay the analytically computed evolvent connected to the actual gearing. The strong coincidence between the obtained result and expected reality is clearly seen.

The situation in Figure 11 is principally different from the previous one. The path of contact consists of two independent parts, which have the following analytical representations:

- the upper (left) part : $\varphi(x) = 0{,}5\ (-0{,}01x^3 - 0{,}4x)$
- the lower (right) part: $\varphi(x) = 0{,}5\ (-0{,}01x^2 - 0{,}4x)$ (17)

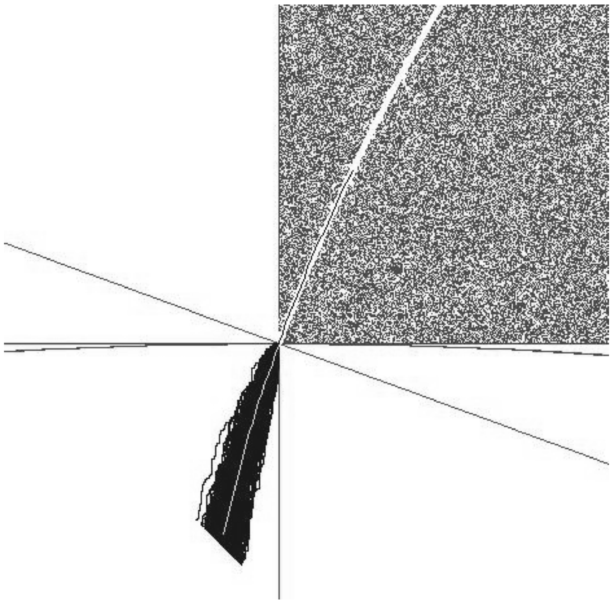

Figure 10. Stochastic lattice method used for a line type of the path of contact (involute gearing)

Slika 10. Metode stohastičke rešetke na zahvatnu crtu u obliku pravca (evolventno 10 Primjena ozubljenje)

In both, Figure 10 and Figure 11, we can see some realizations of individual stochastic processes, which look like spidery ragged lines copying the used stochastic lattice. On account of completeness let us note that in the last example $\Delta = 0{,}04$, $n = 200$ and 2000 stochastic realizations have been executed.

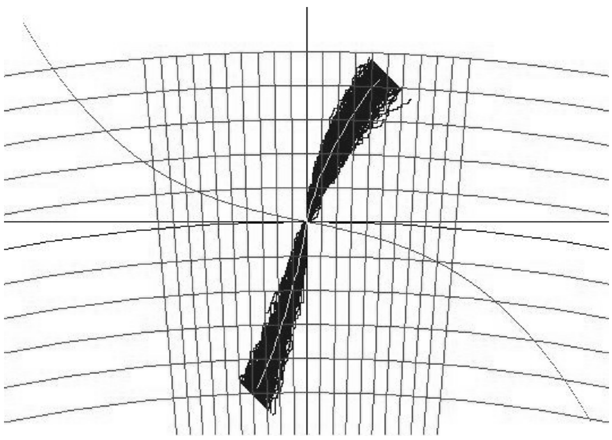

Figure 11. Stochastic lattice method used for a path of contact defined by two curves acc. to expressions (17)

Slika 11. Primjena metode stohastičke rešetke na zahvatnu crtu definiranu pomoću dvije krivulje na temelju izraza (17)

8. Conclusions

As the gearing is defined by the form of the path of contact, it is appropriate to define the so called path of contact form coefficient χ and according to its rating we can determine the gearing type. Coefficient χ is defined through the expression

$$\chi = \frac{\pi m_n}{2 r_k \cos\beta \cos\alpha_C}$$

where also the case of gearing with helical teeth (the angle β) is considered and m_n is a standard gearing module.

It is obvious that for the involute gearing $\chi = 0$, for cycloidal gearing

$$\chi = \frac{\pi m_n}{2 r_k \cos\beta}$$

and e. g. for pin gearing for which $r_k = r_2$

$$\chi = \frac{\pi}{z_2 \cos\beta}.$$

The advantage of this general definition of the plane gearing is:

1. The possibility of the targeted selection of gearing properties during its design by means of selection of the appropriate path of contact form
2. Consistent relations for all geometric gearing characteristics not considering its type (simplification of CAD/CAM systems creation for gearing design and smooth transition when considering characteristics of different gearing types)
3. Simple comparison of different gearing type characteristics
4. Standard possibility of gearing model creation in case of strength computation by FEM, etc.

The practical application of the described stochastic approach of the design of correctly fitting profiles of plane gearing was realized in the CATIA v5 system (see Figure 12). The whole application procedure is quite simple and can be easily implemented for other systems of computer aided design or possibly be even used independently. The authors have also expanded the described procedure with interference of gearing control and that with the stochastic approach as well. This way it is possible to design operational gearing for basically any given shape of the path of contact.

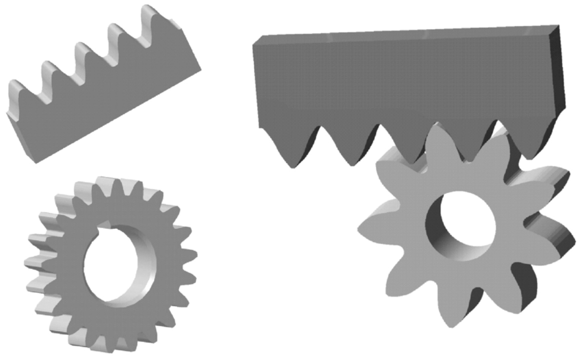

Figure 12. Practical results of application of described methods for design the correct mating profiles when the path of contact is given made in CATIA v5 environment

Slika 12. Praktični rezultati primjene opisanih metoda u konstruiranju pravilno spregnutih profila za zahvatnu krivulju modeliranu u CATIA v5 okruženju

Acknowledgments

This work was supported by the Slovak APVV agency within the solution of the project APVT 20-007602 and was made under the agreement about partnership between Mechanical Engineering Faculty of Slovak University of Technology Bratislava and Faculty of Mechanical Engineering and Naval Architecture of University of Zagreb.

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