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Subdivision of the spectra for factorable matrices on c and ℓ^p

Nuh Durna[1,*] and Mustafa Yildirim[1]

[1] *Department of Mathematics, Faculty of Science, Cumhuriyet University, Sivas 58 410, Turkey*

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Abstract. There are many different ways to subdivide the spectrum of a bounded linear operator; some of them are motivated by applications to physics (in particular, quantum mechanics). In a series of papers, B.E. Rhoades and M. Yildirim previously investigated the spectra and fine spectra for factorable matrices, considered as bounded operators over various sequence spaces. In the present paper, approximation point spectrum, defect spectrum and compression spectrum of factorable matrices are investigated.

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1. Introduction

Let w; c_0; c; ℓ^p; denote the set of all sequences; the space of all null sequences; convergent sequences; sequences such that $\sum_k \mid x_k \mid^p < \infty$; respectively.

An infinite matrix A is said to be conservative if it is a selfmap of c, the space of convergent sequences. Necessary and sufficient conditions for A to be conservative are the well-known Kojima-Schur conditions; i.e.,

(i) $\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$,

(ii) $\lim_n a_{nk} = \alpha_k$, exists for each k, and

(iii) $t = \lim_n \sum a_{nk}$ exists.

Associated with each conservative matrix A is a function χ defined by $\chi(A) = t - \sum \alpha_k$. If $\chi(A) \neq 0, A$ is called coregular, and if $\chi(A) = 0$, then A is called conull. A matrix $A = (a_{nk})$ is said to be regular if $\lim_A x = \lim x$ for each $x \in c$. If $\alpha_k = 0$ for each k and $t = 1$ in (iii), then the operator A is called regular.

A lower triangular matrix A is said to be factorable if $a_{nk} = a_n b_k$ for all $0 \leq k \leq n$.

A triangle is a triangular matrix with nonzero main diagonal entries.

The choices $a_n = 1/(n+1)$ and each $b_k = 1, a_n = (n+1)^{-p} (p > 1)$ and each $b_k = 1, a_n = a_n$ and each $b_k = 1$, and $a_n = P_n, b_k = p_k$, where $\{p_k\}$ is a nonnegative

[*]Corresponding author. *Email addresses:* `ndurna@cumhuriyet.edu.tr` (N. Durna), `yildirim@cumhuriyet.edu.tr` (M. Yildirim)

sequence with $p_0 > 0, P_n := \sum_{k=0}^{n} p_k$, generate C (the Cesáro matrix of order one), the p-Cesáro matrices and terraced matrices defined by Rhaly, and the weighted mean matrices, respectively.

In the past decades, B.E. Rhoades determined the fine spectra of certain classes of weighted mean matrices, considered as bounded linear operators over c, c_0, ℓ^p and bv_0 (see, e.g., [7, 19, 20, 21]). Some authors have considered spectral questions for certain classes of Rhaly matrices (see, e.g. [13, 18, 24, 25, 26, 27, 28]). The Spectrum of C, on various spaces, has been computed in [6, 10, 12, 15, 16, 17, 29]. B.E. Rhoades and M. Yildirim have calculated the spectrum and the fine spectrum of factorable matrices on c and ℓ^p in [22, 23]. The spectrum of linear operators defined by some particular limitation matrices over some sequence spaces has been considered by many authors, for example, R.B. Wenger [29], M. Gonzalez [10], A.M. Akhmedov and F. Başar [1], C. Coşkun [8], B. de Malafosse [14], and B. Altay and F. Başar [2], etc.

Motivated by various applications from mathematical physics, the spectrum of a bounded linear operator can be divided in very different ways, e.g. the point spectrum, continuous spectrum and residual spectrum. Again, from Goldberg [11], pp. 58–71), it follows that one can define the fine spectrum for a bounded operator T on a Banach space X, based on the possible behaviors of $R(T-\lambda)$ and $(T-\lambda)^{-1}$ with $\lambda \in \mathbb{C}$.

The above-mentioned articles, concerned with the decomposition of the spectrum defined by Goldberg. However, in [9] and [4, 5] approximate point spectrum, defect spectrum and compression spectrum of some limitation matrices over some sequence spaces were determined.

2. The spectrum

Let X and Y be the Banach spaces, and $L : X \to Y$ also a bounded linear operator. By $R(L)$, we denote the range of L, i.e.,

$$R(L) = \{y \in Y : y = Lx,\ x \in X\}.$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $L \in B(X)$, then the adjoint L^* of L is a bounded linear operator on the dual X^* of X defined by $(L^* f)(x) = f(Lx)$ for all $f \in X^*$ and $x \in X$.

Given an operator $L \in B(X)$, the set

$$\rho(L) := \{\lambda \in \mathbb{C} : \lambda I - L \text{ is a bijection}\} \tag{1}$$

is called the resolvent set of L and its complement with respect to the complex plain

$$\sigma(L) := \mathbb{C} \backslash \rho(L) \tag{2}$$

is called the spectrum of L. By the closed graph theorem, the inverse operator

$$R(\lambda; L) := (\lambda I - L)^{-1} \qquad (\lambda \in \rho(L)) \tag{3}$$

is always bounded; this operator is usually called a resolvent operator of L at λ.

2.1. Subdivision of the spectrum

In this section, we mention from the parts point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

2.1.1. The point spectrum, continuous spectrum and residual spectrum

Let X be a Banach space over $\mathbb{C}$ and $L \in B(X)$. Recall that a number $\lambda \in \mathbb{C}$ is called the eigenvalue of L if the equation

$$Lx = \lambda x \tag{4}$$

has a nontrivial solution $x \in X$. Any such x is then called the eigenvector, and the set of all eigenvectors is a subspace of X called eigenspace.

Throughout the following, we will call the set of eigenvalues

$$\sigma_p(L) := \{\lambda \in \mathbb{C} : Lx = \lambda x \text{ for some } x \neq 0\} \tag{5}$$

We say that $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(L)$ of L if the resolvent operator (3) is defined on a dense subspace of X and if it is unbounded. Furthermore, we say that $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(L)$ of L if the resolvent operator (3) exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not dense in X. In this case $R(\lambda; L)$ may be bounded or unbounded. Together with the point spectrum (5), these two subspectra form a disjoint subdivision

$$\sigma(L) = \sigma_p(L) \cup \sigma_c(L) \cup \sigma_r(L) \tag{6}$$

of the spectrum of L.

2.1.2. The approximate point spectrum, defect spectrum and compression spectrum

In this subsection, following Appell et al. [3], we give the definitions of three more subdivisions of the spectrum called the approximate point spectrum, defect spectrum and compression spectrum.

Given a bounded linear operator L in a Banach space X, we call a sequence $(x_k)_k$ in X a Weyl sequence for L if $\|x_k\| = 1$ and $\|Lx_k\| \to 0$ as $k \to \infty$.

In what follows, we call the set

$$\sigma_{ap}(L) := \{\lambda \in \mathbb{C} : \text{there exists aWeyl sequence for } \lambda I - L\} \tag{7}$$

the approximate point spectrum of L. Moreover, the subspectrum

$$\sigma_\delta(L) := \{\lambda \in \mathbb{C} : \lambda I - L \text{ is not surjective}\} \tag{8}$$

is called the defect spectrum of L.

The two subspectra (7) and (8) form a (not necessarily disjoint) subdivision

$$\sigma(L) = \sigma_{ap}(L) \cup \sigma_\delta(L) \tag{9}$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(L) = \{\lambda \in \mathbb{C} : \overline{R(\lambda I - L)} \neq X\} \tag{10}$$

which is often called the compression spectrum in the literature and which gives rise to another (not necessarily disjoint) decomposition

$$\sigma(L) = \sigma_{ap}(L) \cup \sigma_{co}(L) \tag{11}$$

of the spectrum. Clearly, $\sigma_p(L) \subseteq \sigma_{ap}(L)$ and $\sigma_{co}(L) \subseteq \sigma_\delta(L)$. Moreover, comparing these subspectra with those in (6) we note that

$$\sigma_r(L) = \sigma_{co}(L) \backslash \sigma_p(L) \tag{12}$$

and

$$\sigma_c(L) = \sigma(L) \backslash [\sigma_p(L) \cup \sigma_{co}(L)] \tag{13}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints.

Proposition 1 (see [3], Proposition 1.3)**.** *The spectra and subspectra of an operator $L \in B(X)$ and its adjoint $L^* \in B(X^*)$ are related by the following relations:*

(a) $\sigma(L^*) = \sigma(L)$.

(b) $\sigma_c(L^*) \subseteq \sigma_{ap}(L)$.

(c) $\sigma_{ap}(L^*) = \sigma_\delta(L)$.

(d) $\sigma_\delta(L^*) = \sigma_{ap}(L)$.

(e) $\sigma_p(L^*) = \sigma_{co}(L)$.

(f) $\sigma_{co}(L^*) \supseteq \sigma_p(L)$.

(g) $\sigma(L) = \sigma_{ap}(L) \cup \sigma_p(L^*) = \sigma_p(L) \cup \sigma_{ap}(L^*)$.

2.1.3. Goldberg's classification of spectrum

If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$, the range of T:

(I) $R(T) = X$

(II) $\overline{R(T)} = X$, but $R(T) \neq X$,

(III) $\overline{R(T)} \neq X$

and three possibilities for T^{-1}:

(1) T^{-1} exists and is continuous,

(2) T^{-1} exists but is discontinuous,

(3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3. If an operator is in state III_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but it is discontinuous (see [11]).

If λ is a complex number such that $T = \lambda I - L \in I_1$ or $T = \lambda I - L \in II_1$, then $\lambda \in \rho(L, X)$. All scalar values of λ not in $\rho(L, X)$ comprise the spectrum of L. A further classification of $\sigma(L, X)$ gives rise to the fine spectrum of L. That is, $\sigma(L, X)$ can be divided into the subsets $I_2\sigma(L, X) = \emptyset$, $I_3\sigma(L, X)$, $II_2\sigma(L, X)$, $II_3\sigma(L, X)$, $III_1\sigma(L, X)$, $III_2\sigma(L, X)$, $III_3\sigma(L, X)$. For example, if $T = \lambda I - L$ is in a given state, III_2 (say), then we write $\lambda \in III_2\sigma(L, X)$.

By the definitions given above, we can write the following table

		1	2	3
		$R(\lambda; L)$ exists and is bounded	$R(\lambda; L)$ exists and is unbounded	$R(\lambda; L)$ does not exists
I	$R(\lambda I - L) = X$	$\lambda \in \rho(L)$	–	$\lambda \in \sigma_p(L)$ $\lambda \in \sigma_{ap}(L)$
II	$\overline{R(\lambda I - L)} = X$	$\lambda \in \rho(L)$	$\lambda \in \sigma_c(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$	$\lambda \in \sigma_p(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$
III	$\overline{R(\lambda I - L)} \neq X$	$\lambda \in \sigma_r(L)$ $\lambda \in \sigma_\delta(L)$ $\lambda \in \sigma_{co}(L)$	$\lambda \in \sigma_r(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$ $\lambda \in \sigma_{co}(L)$	$\lambda \in \sigma_p(L)$ $\lambda \in \sigma_{ap}(L)$ $\lambda \in \sigma_\delta(L)$ $\lambda \in \sigma_{co}(L)$

Table 1.

3. The approximate point spectrum, defect spectrum and compression spectrum of a factorable operator

For many of our results we shall consider factorable matrices which belong to $\mathcal{F} := \{A : A$ is a factorable lower triangular matrix with nonnegative entries and $0 \leq a_n b_n \leq 1$ diagonal entries and with at most a finite number of zeros on the main diagonal$\}$. Define $\gamma = \lim a_n b_n$ and $c_n := a_n b_n$.

3.1. Subdivision of the spectrum of A on c

Theorem 1. *Let $A \in \mathcal{F}$ be regular. Then $1 \in III_3\sigma(A, c)$.*

Proof. Since $a_0 b_0 = 1$, $(I - A)\, e = 0$, $I - A$ is not invertible where $e = (1, 1, \cdots) \in c$. Therefore $I - A \in 3$. Let $z \in c$ such that $z_0 \neq 0$. Then

$$\|(I - A)\, x - z\| \geq |z_0| > \frac{|z_0|}{2}.$$

Hence $z \notin \overline{R\,(I - A)}$; so $\overline{R\,(I - A)} \neq c$, i.e, $I - A \in III$. □

Theorem 2. *Let $A \in \mathcal{F}$ be regular such that $\gamma = \lim c_n$ exists and is less than 1 and $c_n \geq \gamma$ for all n sufficiently large, then*

(a)

$$\sigma_{ap}(A, c) = \left\{ \lambda : \left| \lambda - \frac{1}{2 - \gamma} \right| = \frac{1 - \gamma}{2 - \gamma} \right\} \cup \overline{E},$$

(b)

$$\sigma_{\delta}(A, c) = \left\{ \lambda : \left| \lambda - \frac{1}{2 - \gamma} \right| \leq \frac{1 - \gamma}{2 - \gamma} \right\} \cup S,$$

(c)

$$\sigma_{co}(A, c) = \left\{ \lambda : \left| \lambda - \frac{1}{2 - \gamma} \right| < \frac{1 - \gamma}{2 - \gamma} \right\} \cup S,$$

where $E := \left\{ \lambda = c_n : 0 \leq \lambda \leq \dfrac{\gamma}{2 - \gamma},\ n \geq 0 \right\}$ *and* $S := \overline{\{c_n : n \geq 0\}}$.

Proof. If $A \in \mathcal{F}$ is regular such that $\gamma = \lim c_n$ exists and is less than 1 and $c_n \geq \gamma$ for all n sufficiently large, then $I_3\sigma(A, c) = \emptyset$ and $III_2\sigma(A, c) = \emptyset$ follow from [22] Corollary 1, Corollary 4 and Theorems 5-7 and our result from Theorem 1.

(a) Since $\sigma_{ap}(A, c) = \sigma(A, c) \backslash III_1\sigma(A, c)$,

$$\begin{aligned} \sigma_{ap}(A, c) &= \left[\left\{ \lambda : \left| \lambda - \frac{1}{2 - \gamma} \right| \leq \frac{1 - \gamma}{2 - \gamma} \right\} \cup S \right] \\ &\quad \backslash \left[\left(\left\{ \lambda : \left| \lambda - \frac{1}{2 - \gamma} \right| < \frac{1 - \gamma}{2 - \gamma} \right\} \backslash S \right) \right. \\ &\quad \left. \cup \left\{ \lambda = c_n : \frac{\gamma}{2 - \gamma} < \lambda < 1 \right\} \right] \\ &= \left\{ \lambda : \left| \lambda - \frac{1}{2 - \gamma} \right| = \frac{1 - \gamma}{2 - \gamma} \right\} \cup \overline{E}, \end{aligned}$$

is obvious from [22] Corollary 1, Corollary 4 and Theorem 5.

(b) Since $\sigma_\delta(A,c) = \sigma(A,c) \backslash I_3\sigma(A,c)$ and $I_3\sigma(A,c) = \emptyset$, the equality

$$\sigma_\delta(A,c) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| \leq \frac{1-\gamma}{2-\gamma}\right\} \cup S,$$

is true.

(c) From Table 1, $\sigma_{co}(A,c) = III_1\sigma(A,c) \cup III_2\sigma(A,c) \cup III_3\sigma(A,c)$. Since $III_2\sigma(A,c) = \emptyset$, then from [22] Corollary 1, Corollary 4 and Theorems 5-6 and our result from Theorem 1, we get

$$\begin{aligned}\sigma_{co}(A,c) = & \left[\left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| < \frac{1-\gamma}{2-\gamma}\right\} \backslash S\right] \\ & \cup \left\{\lambda = c_n : \frac{\gamma}{2-\gamma} < \lambda < 1\right\} \cup \left\{\lambda = c_n : 0 \leq \lambda \leq \frac{\gamma}{2-\gamma}\right\} \\ = & \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| < \frac{1-\gamma}{2-\gamma}\right\} \cup S.\end{aligned}$$

□

The following corollaries can be obtained by Proposition 1.

Corollary 1. *The following equalities are true;*

(a)

$$\sigma_{ap}(A^*, \ell^1) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| \leq \frac{1-\gamma}{2-\gamma}\right\} \cup S,$$

(b)

$$\sigma_\delta(A^*, \ell^1) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| = \frac{1-\gamma}{2-\gamma}\right\} \cup \overline{E},$$

(c)

$$\sigma_p(A^*, \ell^1) = \left\{\lambda : \left|\lambda - \frac{1}{2-\gamma}\right| < \frac{1-\gamma}{2-\gamma}\right\} \cup S,$$

where A^* *denotes the adjoint of* A, $E := \left\{\lambda = c_n : 0 \leq \lambda \leq \frac{\gamma}{2-\gamma},\ n \geq 0\right\}$ *and* $S := \overline{\{c_n : n \geq 0\}}$.

3.2. Subdivision of the spectrum of A on ℓ^p $(1 < p < \infty)$

Theorem 3. *Let* A *be a regular factorable matrix with* $\theta = \liminf c_n$. *If there exist values of* n *such that* $0 < c_n \leq \theta/(2-\theta)$, *then* $\lambda = c_n$ *implies* $\lambda \in III_3\sigma(A, \ell^p)$.

Proof. Let c_k be any diagonal entry satisfying $0 < c_k \le \theta/(2-\theta)$. Let j be the smallest integer such that $c_j = c_k$.

Let $T_\lambda := \lambda I - A$. Consider the system $T_\lambda^* x = 0$. Suppose that $\lambda = a_{mm}$ for some m. Then $(T_\lambda^*)_{m+1} x = 0$ becomes

$$\begin{aligned} a_{mm} x_{m+1} - \sum_{k=0}^{\infty} a^*_{m+1,k} x_k &= 0, \\ a_{mm} x_{m+1} - \sum_{k=0}^{\infty} a_{k,m+1} x_k &= 0, \\ a_{mm} x_{m+1} - b_{m+1} \sum_{k=0}^{\infty} a_k x_k &= 0, \end{aligned}$$

which implies that

$$\frac{a_{mm} x_{m+1}}{b_{m+1}} = \sum_{k=0}^{\infty} a_k x_k.$$

Solving the system $(T_\lambda^*)_{m+2} x = 0$ yields

$$\begin{aligned} a_{mm} x_{m+2} - \sum_{k=0}^{\infty} a^*_{m+2,k} x_k &= 0, \\ a_{mm} x_{m+2} - \sum_{k=0}^{\infty} a_{k,m+2} x_k &= 0, \\ a_{mm} x_{m+2} - b_{m+2} \sum_{k=0}^{\infty} a_k x_k &= 0, \end{aligned}$$

which implies that

$$\frac{a_{mm} x_{m+2}}{b_{m+2}} = \sum_{k=0}^{\infty} a_k x_k.$$

Thus we have

$$\begin{aligned} \frac{a_{mm} x_{m+2}}{b_{m+2}} &= \frac{a_{mm} x_{m+1}}{b_{m+1}}. \\ x_{m+2} &= \frac{b_{m+2} x_{m+1}}{b_{m+1}}, \quad \text{for each } n > m \end{aligned}$$

and hence

$$x_n = \frac{b_n x_{m+1}}{b_{m+1}},$$

which, since $x \in \ell^p$, forces $x_n = 0$ for each $n > m$.

Then the system $(T_\lambda^*)_r x = 0$ becomes

$$a_{mm}x_r - \sum_{k=0}^{\infty} a_{rk}^* x_k = 0, \qquad r = 0, 1, \ldots, m,$$

which has nontrivial solutions. Thus T_{c_j} is not surjective, hence $T_{c_j} \in III$.

If $c_j = \theta/(2-\theta)$, then clearly $T_{c_j} \in 3$. Assume that $0 < c_j < \theta/(2-\theta)$ and let r denote the largest integer such that $c_r = c_k$. Solving $T_{c_r}x = 0$ leads to the equality

$$x_{r+m} = x_r \left[\prod_{i=1}^{m} \left(\xi_{i+r} + \left(1 - \frac{1}{a_r b_r}\right) a_{r+i-1} b_{r+i} \right) \right]^{-1}, \tag{14}$$

where $\xi_n = \dfrac{a_{n-1}}{a_n} - \dfrac{a_{n-1}}{a_n} t_n + t_{n-1}$.

For $m \geq n$, from equality (14),

$$\begin{aligned} \frac{|x_{j+m+1}|}{|x_{m+j}|} &= \frac{1}{\xi_{m+1+j} + \left(1 - \frac{1}{c_j}\right) a_{j+m+1} b_{j+m+1}} \\ &\to \frac{1}{1 + \left(1 - \frac{1}{c_j}\right)\theta} < 1, \quad as \quad n \to \infty. \end{aligned}$$

Consequently, $\{x_n\} \in \ell^1$, hence $\{x_n\} \in \ell^p$ and T_{c_r} is not injective. Therefore $\lambda \in III_3\sigma(A, \ell^p)$.

Suppose that A has a zero on the main diagonal and $\theta > 0$. Let j denote the smallest positive integer for which $c_j = 0$. Let e^j denote the coordinate sequence with a 1 in the jth position and all other entries zero. Then $Ae^j = 0$, and $T_{c_j} = -A$ is not 1-1. By setting $x_0 = 0, x_n = 0$ for $n > j+1$, the system $T_{c_j}^* x = 0$ reduces to a homogeneous linear system of j equations in $j+1$ unknowns. □

Remark 1. *Theorem 4 of [21] is a special case of Theorem 3.*

Theorem 4. *Let $A \in \mathcal{F}$ be regular. Then $1 \in III_3\sigma(A, \ell^p)$.*

Proof. Since $a_0 b_0 = 1$, $(I - A)\, e_0 = 0$, where $e = (1, 0, 0, \cdots) \in \ell^p$, $I - A$ is 1-1. Therefore $I - A \in 3$. Also $R(I-A) \subseteq \ell^p \backslash \{e_0\}$. Hence $\overline{R(I-A)} \neq \ell^p$; i.e., $I - A \in III$. □

Theorem 5. *Let $A \in \mathcal{F}$ be regular such that $\gamma = \lim c_n$ exists and is less than 1 and $c_n \geq \gamma$ for all n sufficiently large, then*

(a)

$$\sigma_{ap}(A, \ell^p) = \left\{ \lambda : \left| \lambda - \frac{1}{2-\gamma} \right| = \frac{1-\gamma}{2-\gamma} \right\} \cup \overline{E},$$

(b)

$$\sigma_\delta(A, \ell^p) = \left\{ \lambda : \left| \lambda - \frac{1}{2-\gamma} \right| \leq \frac{1-\gamma}{2-\gamma} \right\} \cup S,$$

(c)

$$\sigma_{co}(A,\ell^p)=\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|<\frac{1-\gamma}{2-\gamma}\right\}\cup S,$$

where $E:=\left\{\lambda=c_n:0\le\lambda\le\frac{\gamma}{2-\gamma},\ n\ge 0\right\}$ *and* $S:=\overline{\{c_n:n\ge 0\}}$.

Proof. If $A\in\mathcal{F}$ is regular such that $\gamma=\lim c_n$ exists and is less than 1 and $c_n\ge\gamma$ for all n sufficiently large, then $I_3\sigma(A,\ell^p)=\emptyset$ and $III_2\sigma(A,\ell^p)=\emptyset$ are taken by Corollary 1, Theorems 6-8 in [23] and our results from Theorems 3-4.

(a) Since $\sigma_{ap}(A,c)=\sigma(A,c)\backslash III_1\sigma(A,c)$, then from Corollary 1, Theorem 6 and Theorem 8 in [23] we get the following equality;

$$\begin{aligned}\sigma_{ap}(A,\ell^p)&=\left[\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|\le\frac{1-\gamma}{2-\gamma}\right\}\cup S\right]\\&\quad\backslash\left[\left(\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|<\frac{1-\gamma}{2-\gamma}\right\}\backslash S\right)\right.\\&\quad\left.\cup\left\{\lambda=c_n:\frac{\gamma}{2-\gamma}<\lambda<1\right\}\right]\\&=\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|=\frac{1-\gamma}{2-\gamma}\right\}\cup\overline{E},\end{aligned}$$

(b) Since $\sigma_\delta(A,\ell^p)=\sigma(A,\ell^p)\backslash I_3\sigma(A,\ell^p)$ and $\sigma I_3(A,\ell^p)=\emptyset$, we have

$$\sigma_\delta(A,\ell^p)=\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|\le\frac{1-\gamma}{2-\gamma}\right\}\cup S,$$

(c) From Table 1, $\sigma_{co}(A,\ell^p)=III_1\sigma(A,\ell^p)\cup III_2\sigma(A,\ell^p)\cup III_3\sigma(A,\ell^p)$. Since $III_2\sigma(A,\ell^p)=\emptyset$, then from [23] Corollary 1, Theorem 6, Theorem 8 and our results from Theorems 3-4, we get

$$\begin{aligned}\sigma_{co}(A,\ell^p)&=\left[\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|<\frac{1-\gamma}{2-\gamma}\right\}\backslash S\right]\\&\quad\cup\left\{\lambda=c_n:\frac{\gamma}{2-\gamma}<\lambda<1\right\}\cup\left\{\lambda=c_n:0\le\lambda\le\frac{\gamma}{2-\gamma}\right\}\\&=\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|<\frac{1-\gamma}{2-\gamma}\right\}\cup S.\end{aligned}$$

□

The following corollaries can be obtained by Proposition 1.

Corollary 2. *The following equalities are true;*

(a)

$$\sigma_{ap}(A^*,\ell^q)=\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|\le\frac{1-\gamma}{2-\gamma}\right\}\cup S,$$

(b)

$$\sigma_{\delta}(A^*,\ell^q)=\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|=\frac{1-\gamma}{2-\gamma}\right\}\cup\overline{E},$$

(c)

$$\sigma_{p}(A^*,\ell^q)=\left\{\lambda:\left|\lambda-\frac{1}{2-\gamma}\right|<\frac{1-\gamma}{2-\gamma}\right\}\cup S,$$

where A^* *denotes the adjoint of* A, $\frac{1}{p}+\frac{1}{q}=1$, $E:=\left\{\lambda=c_n:0\le\lambda\le\frac{\gamma}{2-\gamma},\ n\ge 0\right\}$ *and* $S:=\overline{\{c_n:n\ge 0\}}$.

4. Conclusion

There is wide literature related to the spectrum and fine spectrum of certain linear operators represented by particular limitation matrices over some sequence spaces. Although the fine spectrum with respect to the Goldberg's classification of the factorable operator over the sequence spaces c and ℓ^p with $(1<p<\infty)$ were studied by B.E. Rhoades and M. Yildirim [22, 23], respectively, the present paper introduces the concepts of the approximate point spectrum, defect spectrum and compression spectrum, and gives the subdivisions of the spectrum of the factorable operator over the sequence space ℓ^p as new subdivisions of spectrum. This is a new development of the spectrum of an infinite matrix over a sequence space. Following the same way, it is natural that one can derive some new results on subdivisions of the spectrum of factorable matrices or other particular limitation matrices over the spaces which are not considered here, from the known results via Table 1, in the usual sense.

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