MATHEMATICAL COMMUNICATIONS
Math. Commun. **16**(2011), 227–234.

The upper semi-continuity of the solution map to the extended homogeneous complementarity problem with the R_0-condition*

Rong Hu[1,†]

[1] *Department of Mathematics, Chengdu University of Information Technology, Chengdu, Sichuan, P.R. China*

Received May 13, 2010; accepted September 27, 2010

Abstract. In this paper we introduce a concept of the R_0-condition for the extended homogeneous complementarity problem, and show that the upper semi-continuity of the solution map is equivalent to the R_0-condition in the extended homogeneous complementarity problem.

AMS subject classifications: 90C33

Key words: extended complementarity problem, homogeneity, solution map, upper semi-continuity, R_0-condition

1. Introduction

Various classes of complementarity problems have been studied intensively in the literature (see e.g. [1, 2, 3, 4, 9, 15, 12, 19]). Continuity properties of solution maps for various complementarity problems have been investigated in the past decades, for example, in [3, 10, 13, 16, 12, 7, 6, 8, 11, 17, 18, 14, 15]. Among the various complementarity problems, the classical linear complementarity problem has the simplest formulation and maybe the widest applications. The upper semi-continuity of the solution map to the classical linear complementarity problem has been considered in papers [3, 10, 13, 16]. There are many generalizations of the classical linear complementarity problem in the literature, see e.g. [3, 12, 2, 4]. The vertical linear complementarity problem due to Cottle and Dantzig [2] is a vertical generalization of the classical linear complementarity problem. In [7], Fang and Huang studied the upper semi-continuity of the solution map to the vertical linear complementarity problem with an R_0-condition. The horizontal linear complementarity problem is a horizontal generalization of the classical linear complementarity. In [6], Fang and Huang investigated the upper semi-continuity of the solution map in the horizontal linear complementarity problem with an R_0-condition. A more general form of the classical linear complementarity problem is the class of mixed linear complementarity problems which includes the class of horizontal linear complementarity problems as a special case. In [8], Fang and Huang introduced the concept of R_0-conditions for the mixed linear complementarity problem and studied the upper

*This work was supported by the National Natural Science Foundation of China (11001187).
†Corresponding author. *Email address:* `ronghumath@yahoo.com.cn` (R. Hu)

semi-continuity of the solution map in the mixed linear complementarity problem with the R_0-condition. For other related works, we can refer to [16, 11, 17, 18] and the references therein. De Schutter and De Moor [4] introduced the class of extended linear complementarity problems, which includes the classes of classical, horizontal, vertical, and mixed linear complementarity problems as special cases. Very recently, Fang and Huang [5] further considered the class of extended nonlinear complementarity problems. The purpose of this paper is to study the upper semi-continuity of the solution map to the extended homogeneous complementarity problem. A concept of the R_0-condition is introduced for the extended homogeneous complementarity problem. Such a condition covers those for the classical, horizontal, vertical, and mixed linear complementarity problems. For details, we refer the reader to [3, 16, 7, 6, 8] and the references therein. We establish the equivalence of the upper semi-continuity of the solution map and the R_0-condition in the extended homogeneous complementarity problem. Our results generalize the corresponding results presented in Oettli and Yen [16] and Fang and Huang [7, 6, 8].

2. Extended complementarity problem

Let $T : R^n \to R^p$ be a nonlinear function and let $\{\phi_j\}_{j=1}^m$ be subsets of the set $\{1, 2, \cdots, p\}$. Fang and Huang [5] considered the following extended nonlinear complementarity problem: find $x \in R^n$ such that

$$ENCP(T) \qquad w = T(x) \geq 0, \quad \min_{i \in \phi_j} w_i = 0, \quad j = 1, 2, \cdots, m.$$

When $T(x) = H(x) - b$, $ENCP(T)$ reduces to the following extended homogeneous complementarity problem:

$$EHCP(H, b) \quad \begin{cases} \text{find } x \in R^n \text{ such that} \\ H(x) - b \in R^p_+, \\ \prod_{i \in \phi_j} (H_i(x) - b_i) = 0, \quad j = 1, 2, \cdots, m, \end{cases}$$

where $H : R^n \to R^p$ is a function with $H(x) = (H_1(x), \cdots, H_p(x))$ and $H_i : R^n \to R$ is a positively homogeneous function with degree $\rho_i > 0$, and $b \in R^p$ is a given point with $b = (b_1, \cdots, b_p)$.

When $T(x) = Mx - b$, where $M \in R^{p \times n}$ and $b \in R^p$, $ENCP(T)$ reduces to the extended linear complementarity problem due to De Schutter and De Moor [4]:

$$ELCP(M, b) \quad \begin{cases} \text{find } x \in R^n \text{ such that} \\ Mx - b \in R^p_+, \\ \prod_{i \in \phi_j} (Mx - b)_i = 0, \quad j = 1, 2, \cdots, m. \end{cases}$$

Remark 1. *The extended nonlinear complementarity problem provides a unifying framework for the classical, vertical, horizontal, and other linear complementarity problems. For details, we refer to [12, 4, 5] and the references therein.*

In this paper we consider the upper semicontinuity of the solution maps to $EHCP(H, b)$ and $ELCP(M, b)$. In the sequel we define $\mathcal{H}$ by

$$\mathcal{H} = \{H : H : R^n \to R^p \text{ is a continuous function with } H(x) = (H_1(x), \cdots, H_p(x)) \text{ and for any fixed } i, H_i(\lambda x) = \lambda^{\rho_i} H_i(x) \text{ for all } \lambda > 0 \text{ and } x \in R^n\},$$

where $\rho_i > 0$ is constant for all i.

Endow $\mathcal{H}$ with a norm by

$$\|H\| = \max_{\|x\|=1} \|H(x)\|, \quad \forall H \in \mathcal{H}.$$

Definition 1. *Given $H \in \mathcal{H}$, $M \in R^{p\times n}$, vector $b \in R^p$, and subsets $\{\phi_j\}_{j=1}^m$. Let $\Psi(H, b)$ and $\Phi(M, b)$ be the solution sets of $EHCP(H, b)$ and $ELCP(M, b)$ respectively. Set*

$$\mathcal{H}_0 = \{H \in \mathcal{H} : \Psi(H, 0) = \{0\}\}$$

and

$$\mathcal{M}_0 = \{M \in R^{p\times n} : \Phi(M, 0) = \{0\}\}.$$

We say that H (resp. M) satisfies the R_0-condition if and only if $H \in \mathcal{H}_0$ (resp. $M \in \mathcal{M}_0$). In the following, we always consider Ψ and Φ as set-valued maps.

Remark 2. *R_0-condition for the extended homogeneous complementarity problem covers those for the classical, horizontal, vertical, and mixed linear complementarity problems. For details, one can refer to [3, 16, 7, 6, 8] and the references therein.*

Definition 2. *Let X and Y be Hausdorff topological spaces. A set-valued map $G : X \to 2^Y$ is said to be upper semi-continuous at $x \in X$ if, for any open set $\Omega \subset Y$ with $G(x) \subset \Omega$, there exists a neighborhood V of x such that $G(x') \subset \Omega$ for all $x' \in V$. We say that G is upper semi-continuous if G is upper semi-continuous at every point x of X.*

Definition 3. *We say that a set-valued map $G : X \to 2^Y$ has a closed graph if, for any $\{x_\alpha\} \subset X$ and $\{y_\alpha\} \subset Y$ with $y_\alpha \in G(x_\alpha)$, $x_\alpha \to x$ and $y_\alpha \to y$ imply that $y \in G(x)$.*

3. Main results

In this section, we investigate the upper semi-continuity of the solution maps in the extended homogeneous complementarity problems with R_0-conditions.

Proposition 1. *The map $\Psi : \mathcal{H} \times R^p \to 2^{R^n}$ has a closed graph.*

Proof. Let $\{H^k\} \subset \mathcal{H}$, $\{b^k\} \subset R^p$, and $\{x^k\} \subset R^n$ such that $H^k \to \bar{H} \in \mathcal{H}$, $b^k \to \bar{b} \in R^p$, $x^k \to \bar{x}$ and $x^k \in \Psi(H^k, b^k)$. It follows that

$$\begin{cases} x^k \in R^n, \\ H^k(x^k) - b^k \in R^p_+, \\ \prod_{i\in\phi_j}(H^k_i(x^k) - b^k_i) = 0, \quad j = 1, 2, \cdots, m. \end{cases}$$

Since $H^k \to \bar{H}$, $b^k \to \bar{b}$ and $x^k \to \bar{x}$, we get

$$\begin{cases} \bar{x} \in R^n, \\ \bar{H}(\bar{x}) - \bar{b} \in R^p_+, \\ \prod_{i\in\phi_j}(\bar{H}_i(\bar{x}) - \bar{b}_i) = 0, \quad j = 1, 2, \cdots, m. \end{cases}$$

This yields that $\bar{x} \in \Psi(\bar{H}, \bar{b})$ and so Ψ has a closed graph. □

By similar arguments we have:

Proposition 2. *The map* $\Phi : R^{p\times n} \times R^p \to 2^{R^n}$ *has a closed graph.*

Theorem 1. *Given* $H \in \mathcal{H}$ *and subsets* $\{\phi_j\}_{j=1}^m$. *If* H *satisfies the* R_0*-condition, then* Ψ *is upper semi-continuous at* (H, b) *for all* $b \in R^p$. *Conversely, if there exists* $\bar{b} \in R^p$ *such that* $\Psi(H, \bar{b})$ *is bounded and* $\Psi(\cdot, \bar{b})$ *is upper semi-continuous at* H, *then* H *satisfies the* R_0*-condition.*

Proof. Let H satisfy the R_0-condition. Suppose on the contrary that $\Psi(\cdot,\cdot)$ is not upper semi-continuous at (H, b) for some $b \in R^p$. Then there exists an open set $\Omega \subset R^n$ with $\Psi(H, b) \subset \Omega$, and there exist sequences $\{H^k\} \subset \mathcal{H}$, $\{b^k\} \subset R^p$ and $\{x^k\} \subset R^n$ such that $H^k \to H$, $b^k \to b$, $x^k \in \Psi(H^k, b^k)$, but $x^k \notin \Omega$ for all k. It follows that

$$\begin{cases} x^k \in R^n, \\ H^k(x^k) - b^k \in R^p_+, \\ \prod_{i\in\phi_j}(H^k_i(x^k) - b^k_i) = 0, \quad j = 1, 2, \cdots, m. \end{cases} \tag{1}$$

We claim that $\{x^k\}$ has no bounded subsequences. Indeed, if $\{x^k\}$ has a bounded subsequence, then by Proposition 1 its accumulation point x^* belongs to $\Psi(H, b)$, thus, $x^* \in \Omega$, a contradiction. Hence

$$\|x^k\| \to \infty.$$

Without loss of generality, we may assume that

$$\frac{x^k}{\|x^k\|} \to \hat{x} \neq 0.$$

By (1) and the definition of $\mathcal{H}$, it is easy to see that

$$\frac{x^k}{\|x^k\|} \in \Psi(H^k, c^k), \tag{2}$$

where

$$c^k = (\frac{b^k_1}{\|x^k\|^{\rho_1}}, \cdots, \frac{b^k_p}{\|x^k\|^{\rho_p}}).$$

Letting $k \to \infty$ in (2), we obtain $\hat{x} \in \Psi(H, 0)$ from Proposition 1. This arrives at a contradiction since H satisfies the R_0-condition. Thus Ψ is upper semi-continuous at (H, b) for all $b \in R^p$.

Conversely, suppose that there exists $\bar{b} \in R^p$ such that $\Psi(H, \bar{b})$ is bounded and $\Psi(\cdot, \bar{b})$ is upper semi-continuous at H. If H does not satisfy the R_0-condition, then there exists $\bar{x} \neq 0$ such that

$$\begin{cases} \bar{x} \in R^n, \\ H(\bar{x}) \in R^p_+, \\ \prod_{i \in \phi_j} H_i(\bar{x}) = 0, \quad j = 1, 2, \cdots, m. \end{cases} \tag{3}$$

For any given $t > 0$, define $x^t = \bar{x}/t$ and H^t as follows:

$$H^t = (H^t_1, \cdots, H^t_p), \quad H^t_i(x) = H_i(x) + \frac{|t\langle z, x\rangle|^{\rho_i}}{|\langle z, \bar{x}\rangle|^{\rho_i}} \bar{b}_i \quad i = 1, \cdots, p, \tag{4}$$

where $z \in R^n$ is a fixed vector with $\langle z, \bar{x}\rangle \neq 0$.

It follows from (4) that

$$H^t \in \mathcal{H}, \quad H^t \to H \text{ as } t \to 0, \quad H^t_i(x^t) - \bar{b}_i = \frac{1}{t^{\rho_i}} H_i(\bar{x}), \quad i = 1, \cdots, p. \tag{5}$$

From (3) and (5), we have $x^t \in \Psi(H^t, \bar{b})$. Since $\Psi(H, \bar{b})$ is bounded, there exists a bounded open neighborhood Ω such that $\Psi(H, \bar{b}) \subset \Omega$. By the upper semi-continuity of $\Psi(\cdot, \bar{b})$ at H, one has $x^t \in \Omega$ for all sufficiently small t. It is impossible since $\|x^t\| \to \infty$ as $t \to 0$. Thus H satisfies R_0-condition. □

As a particular case of Theorem 1 we obtain the following result:

Theorem 2. *Given matrix M and subsets $\{\phi_j\}_{j=1}^m$. If M satisfies the R_0-condition, then Φ is upper semi-continuous at (M, b) for all $b \in R^p$. Conversely, if there exists $\bar{b} \in R^p$ such that $\Phi(M, \bar{b})$ is bounded and $\Phi(\cdot, \bar{b})$ is upper semi-continuous at M, then M satisfies the R_0-condition.*

The following example can show that M does not satisfy the R_0-condition and Φ is not upper semi-continuous at (M, b) for some $b \in R^p$.

Example 1. *Let*

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \in R^{4\times 2},$$

$\phi_1 = \{1, 2\}$ and $\phi_2 = \{3, 4\}$. It is easy to see that $\Phi(M, 0) = \{(0, x_2) : x_2 \geq 0\} \neq \{0\}$. Then M does not satisfy the R_0-condition. Next, we show that Φ is not upper semi-continuous at (M, b) with $b = 0$. Indeed, let

$$M^k = \begin{pmatrix} 1 & \frac{1}{k} \\ \frac{1}{k} & 1 \\ -1 & -\frac{1}{k} \\ \frac{1}{k} & 1 \end{pmatrix},$$

$b^k \equiv 0$, and $\Omega = \{(x_1, x_2) \in R^2 : |x_1| < 1\} \supset \Phi(M, 0)$. Clearly Ω is open, $M^k \to M$ and $b^k \to b$ as $k \to \infty$. Then there exists sequence $x^k = (-1, k) \in \Phi(M^k, b^k)$, but $x^k \notin \Omega$. Thus, Φ is not upper semi-continuous at $(M, 0)$.

Note that the assumption that there exists $\bar{b} \in R^p$ such that $\Psi(H, \bar{b})$ (resp. $\Phi(M, \bar{b})$) is bounded plays an important role in the proof of Theorem 1 (resp. Theorem 2). It is interesting to know whether or not there exists $\bar{b}$ such that $\Psi(H, \bar{b})$ (resp. $\Phi(M, \bar{b})$) is bounded. The following theorem shows that under suitable conditions, such a $\bar{b}$ exists for the extended linear complementarity problem.

Theorem 3. *Given matrix M and subsets $\{\phi_j\}_{j=1}^m$. Assume that $\phi_j \cap \phi_i \neq \emptyset$ for all i, j with $i \neq j$ and $n \leq m$. Then there exists $\bar{b} \in R^p$ such that $\Phi(M, \bar{b})$ is bounded.*

Proof. Let $b \in R^p$ be such that $\Phi(M, b)$ is unbounded. Then there exists a sequence $\{x^k\} \subset \Phi(M, b)$ such that $\|x^k\| \to \infty$. It follows that

$$\begin{cases} x^k \in R^n, \\ Mx^k - b \in R^p_+, \\ \prod_{i\in\phi_j} (Mx^k - b)_i = 0, \quad j = 1, 2, \cdots, m. \end{cases} \tag{6}$$

Set

$$w^k = Mx^k - b.$$

Then

$$b = Mx^k - Iw^k, \tag{7}$$

where I denotes the unit matrix of $R^{p\times p}$. Since $\|x^k\| \to \infty$, without loss of generality, we can suppose that

$$\|(x^k, w^k)\| \to \infty \quad \text{and} \quad \frac{(x^k, w^k)}{\|(x^k, w^k)\|} \to (\bar{x}, \bar{w}) \neq (0, 0). \tag{8}$$

It follows from (6)-(8) that

$$\prod_{i\in\phi_j} (\bar{w})_i = 0, \quad j = 1, 2, \cdots, m \tag{9}$$

and

$$0 = M\bar{x} - I\bar{w}. \tag{10}$$

By (6), there exist $J \subset \{1, \cdots, p\}$, $\Lambda \subset \{1, \cdots, n\}$, $\{w^{k_l}\} \subset \{w^k\}$ and $\{x^{k_l}\} \subset \{x^k\}$ such that J has m elements and for all l, $(w^{k_l})_j = 0$ whenever $j \in J$, and $(x^{k_l})_i = 0$ whenever $i \in \Lambda$. From (8) and (9), we further have $\bar{w}_j = 0$ whenever $j \in J$ and $\bar{x}_i = 0$ whenever $i \in \Lambda$. Then, from (7), we know that the p-dimension vector b is a linear combination of $r(\leq n + p - m)$ vectors from the following:

$$M_{\cdot 1}, \cdots, M_{\cdot n}, \cdots, I_{\cdot 1}, \cdots, I_{\cdot p}, \tag{11}$$

where $M_{\cdot i}$ and $I_{\cdot i}$ denote the i-th column of M and I, respectively. By (10), these r vectors are linearly dependent and so b can be represented as a linear combination of $r-1$ vectors out of the vectors stated in (11). Since $n \le m$, we get $r \le p$. Summarizing, b is contained in a proper linear subspace of R^p, which is spanned by $r-1$ vectors out of the vectors stated in (11). So the set of all $b \in R^p$ such that $\Phi(M,b)$ is unbounded is contained in the union of finitely many proper linear subspaces of R^p. Since $r \le p$, the union cannot equal the whole space R^p. Hence there exists some $\bar{b} \in R^p$ such that $\Phi(M,\bar{b})$ is bounded. □

From Theorems 2 and 3, we obtain the following result:

Theorem 4. *Given matrix M and subsets $\{\phi_j\}_{j=1}^m$. Assume that $\phi_j \cap \phi_i \ne \emptyset$ for all i,j with $i \ne j$ and $n \le m$. Then $\Phi : R^{p\times n} \times R^p \to R^n$ is upper semi-continuous at (M,b) for all $b \in R^p$ if and only if M satisfies the R_0-condition.*

Theorem 5. *Given matrix M and subsets $\{\phi_j\}_{j=1}^m$. Then M satisfies R_0-condition if and only if $\Phi(M,b)$ is bounded for all $b \in R^p$.*

Proof. Let $\Phi(M,b)$ be bounded for all $b \in R^p$. Then $\Phi(M,0)$ is bounded. Suppose on the contrary that M does not satisfy the R_0-condition. Then there exists $\bar{x} \ne 0$ such that $\bar{x} \in \Phi(M,0)$. By simple arguments, we have $\lambda\bar{x} \in \Phi(M,0)$ for all $\lambda > 0$. This contradicts the fact that $\Phi(M,0)$ is bounded. Thus M satisfies the R_0-condition.

Conversely, let M satisfy the R_0-condition. If there exists $\bar{b} \in R^p$ such that $\Phi(M,\bar{b})$ is unbounded. Without loss of generality, choose $x^k \in \Phi(M,\bar{b})$ such that $\|x^k\| \to \infty$ and $x^k/\|x^k\| \to \hat{x} \ne 0$. By proceeding similarly to Theorem 1, we have

$$\frac{x^k}{\|x^k\|} \in \Phi(M, \frac{\bar{b}}{\|x_k\|}).$$

Since Φ has a closed graph, by Proposition 2, we obtain $\hat{x} \in \Phi(M,0)$. This arrives a contradiction since M satisfies the R_0-condition. Thus $\Phi(M,b)$ is bounded for all $b \in R^p$. □

Remark 3. *It is well-known that $C^\infty = \{0\}$ if and only if C is bounded, where C^∞ denotes the asymptotic cone of the set $C \subset R^n$. By Theorem 5, M satisfies the R_0-condition if and only if $(\Phi(M,b))^\infty = \{0\}$ for all $b \in R^p$.*

As a consequence of Theorems 4 and 5, we have:

Theorem 6. *Given matrix M and subsets $\{\phi_j\}_{j=1}^m$. Assume that $\phi_j \cap \phi_i \ne \emptyset$ for all i,j with $i \ne j$ and $n \le m$. Then Φ is upper semi-continuous at (M,b) for all $b \in R^p$ if and only if $\Phi(M,b)$ is bounded for all $b \in R^p$.*

Acknowledgement

The author would like to thank one anonymous referee for his/her helpful comments and suggestions which lead to improvements of this paper.

References

[1] S. S. Chang, H. W. J. Lee, D. P. Wu, *A class of random complementarity problems in Hilbert spaces*, Math. Commun. **10**(2005), 95–100.

[2] R. W. Cottle, G. B. Dantzig, *A generalization of the linear complementarity problem*, J. Comb. Theory **8**(1970), 79–90.

[3] R. W. Cottle, J. S. Pang, R. E. Stone, *The Linear Complementarity Problems*, Academic Press, New York, 1992.

[4] B. De Schutter, B. De Moor, *The extended linear complementarity problem*, Math. Program. Ser. A **71**(1995), 289–325.

[5] Y. P. Fang, N. J. Huang, *A characterization of an acceptable solution of the extended nonlinear complementarity problem*, Z. Angew. Math. Mech. **84**(2004), 564–567.

[6] Y. P. Fang, N. J. Huang, *A characterization of upper semi-continuity of the solution map to the horizontal linear complementarity problem of type R_0*, Z. Angew. Math. Mech. **85**(2005), 904–907.

[7] Y. P. Fang, N. J. Huang, *On the upper semi-continuity of the solution map to the vertical implicit homogeneous complementarity problem of type R_0*, Positivity **10**(2006), 95–104.

[8] Y. P. Fang, N. J. Huang, *The equivalence of upper semi-continuity of the solution map and R_0-condition in the mixed linear complementarity Problem*, Appl. Math. Lett. **19**(2006), 667–672.

[9] F. Flores-Bazán, R. López, *Asymptotic analysis, existence and sensitivity results for a class of multivalued complementarity problems*, ESAIM Control Optim. Cal. Var. **12**(2006), 271–293.

[10] M. S. Gowda, *On the continuity of the solution map in linear complementarity problems*, SIAM J. Optim. **2**(1992), 619–634.

[11] N. J. Huang, Y. P. Fang, *The upper semicontinuity of the solution maps in vector implicit quasicomplementarity problems of type R_0*, Appl. Math. Lett. **16**(2003), 1151–1156.

[12] G. Isac, *Topological Methods in Complementarity Theory*, Kluwer Academic Publishers, Dordrecht, 2000.

[13] M. J. M. Jansen, S. H. Tijs, *Robustness and nondegenerateness for linear complementarity problems*, Math. Program. **37**(1987), 293–308.

[14] C. Jones, M. Gowda, *On the connectedness of solution sets in linear complementarity problems*, Linear Algebra Appl. **272**(1998), 33–44.

[15] K. G. Murty, *On the number of solutions to the complementarity problem and spanning properties of complementary cones*, Linear Algebra Appl. **5**(1972), 65–108.

[16] W. Oettli, N. D. Yen, *Continuity of the solution set of homogeneous equilibrium problems and linear complementarity problems*, in: *Variational Inequalities and Network Equilibrium Problems*, (F. Giannessi and A. Maugeri, Eds.), Plenum Press, New York, 1995.

[17] W. Oettli, N. D. Yen, *Quasicomplementarity problems of type R_0*, J. Optim. Theory Appl. **89**(1996), 467–474.

[18] S. M. Robinson, *A characterization of stability in linear programming*, Operations Res. **25**(1977), 435–447.

[19] Y. Zhao, *Existence of a solution to nonlinear variational inequality under generalized positive homogeneity*, Oper. Res. Lett. **25**(1999), 231–239.