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A CURVATURE EFFECT ON THE CRITICAL RICHARDSON NUMBER

Utjecaj zakrivljenosti na kritični Richardsonov broj

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Abstract – Transitions from laminar to turbulent flow associated with a critical level in the stratified shear flow are related to the critical (gradient) Richardson number, Ri_c, the minimum value of which is $Ri_c \approx 1/4$. Based on the linear theory and a modified Taylor-Goldstein equation, here it is plausibly shown that Ri_c may slightly vary from this value if the mean wind speed profile contains a small curvature (parameterized here) around the critical level. The modified Ri_c that depends on the dimensionless curvature $\alpha \equiv (U''\zeta/U')$ is

$Ri_c(\alpha) = (1 + \alpha / 2)[\alpha + (1 + \alpha / 2) / 4]$.

Key word index: Richardson number, critical level, Taylor-Goldstein equation, parameterization.

Sažetak – Kritični nivo stratificiranoga smicajućega strujanja združen je s prijelazom laminarnoga u turbulentno strujanje koje ovisi o kritičnom (gradijentnom) Richardsonovu broju. Taj broj ima minimalnu vrijednost $Ri_c \approx 1/4$. Na temelju linearne teorije i modificirane Taylor-Goldstein-ove jednadžbe, plauzibilno je pokazano da Ri_c može odstupati od 1/4 ako srednje strujanje sadrži zakrivljenost (koja je ovdje parametrizirana) oko toga kritičnoga nivoa. Modificirani Ri_c ovisi o bezdimenzionalnoj zakrivljenosti $\alpha \equiv (U''\zeta/U')$ kao

$Ri_c(\alpha) = (1 + \alpha / 2)[\alpha + (1 + \alpha / 2) / 4]$.

Ključne riječi: Richardsonov broj, kritični nivo, Taylor-Goldsteinova jednadžba, parametrizacija.

1. INTRODUCTION

The (gradient) Richardson number, Ri, is a principal and widely used parameter which describes the stability of shear flows. It is defined as

$$Ri_c \equiv (g / \Theta_0)(\partial\Theta / \partial z) / (|\partial U / \partial z|^2) \qquad (1)$$

where g is acceleration due to gravity, Θ_0 and Θ are the reference and mean potential temperature, respectively, U is the background wind speed, and z is height (e.g. Gossard and Hooke, 1975). A critical, minimum value of Ri, henceforth called Ri_c, determines a sufficient condition for the linear stability of an inviscid, laminar, parallel, stratified flow (Miles[1], 1961, 1990). This Ri_c ($\approx 1/4$) is justified by theory (Howard, 1961), experiments (Thorpe, 1969, 1971; although some experiments reveal a value as low as 0.21), and atmospheric measurements (Browning, 1971; Emmanuel et al. 1972; Metcalf, 1975; King et al. 1987).

Kelly and Maslowe (1970) argue in their nonlinear analysis that a single Richardson number parameter is inadequate to describe wave-turbulence transitions near a critical level. The same conclusion follows from Vinnichenko et al. (1980). Stull (1988, p. 176-177) discussed a hysteresis phenomenon identified by Ri_{c1} (≈ 0.21 or 0.25) and Ri_{c2} (≈ 1). A qualitative explanation is given in terms of two conditions needed for turbulence: instability, which may occur whenever $Ri < Ri_{c2}$, where Ri_{c2} is the upper value of Ri_c, and some trigger mechanism such as Kelvin-Helmholtz waves ($Ri \approx Ri_{c1}$, Ri_{c1} is the lower value of Ri_c). The values of 0.25 and 0.5 are sometimes referred to as the criterion for the onset of instability and its maintenance, respectively (e.g., Turner, 1973).

One may ask whether most variations of Ri_c come from nonlinear effects, measurement errors, etc., or

[1] Prof. Miles points out that G. I. Taylor was among the first to do the analysis in 1915; Rayleigh, Prandtl, Haurwitz and Richardson were also involved in similar stability analyses.

whether the linear theory can account for some of Ri_c departures from its well-established value of 1/4. The aim of this paper is to analytically estimate a range of values for the critical Richardson number, Ri_c, in atmospheric shear flows. The validity of the Taylor-Goldstein equation (TGE) is assumed and also that the curvature and shear of the mean wind speed can be related in the TGE.

2. A MODIFIED TAYLOR-GOLDSTEIN EQUATION

a) no critical-layer curvature included

The Miles theorem and its extension are given in Miles (1961) and Howard (1961) where the derived value is Ri_c=1/4. The form of the TGE that is used here is:

$$w'' + m^2 w = 0, \tag{2}$$

where m^2 is defined as

$$m^2 \equiv [N^2/(c-U)^2 + U''/(c-U) - k^2], \tag{3}$$

and w is the vertical complex amplitude of a monochromatic wave, N is the Brunt-Väisälä frequency ($N^2 \equiv Ri\, U'^2$, typically $N^2 \sim 10^{-3}s^{-2}$), the apostrophes denote spatial derivatives with respect to z, c is the horizontal phase speed, and k is the horizontal wavenumber ($|w| << |U|$ by hypothesis). As c approaches the background speed U, the vertical wavenumber function $m(z)$ becomes arbitrarily large; hence, (2) should be locally modified. Based on a scale analysis, the modified $m(z)$ near the critical level is:

$$m^2 \rightarrow m_1^2 \approx N^2/(c-U)^2 \tag{4}$$

It is assumed that the critical level is approached smoothly. This means that U changes smoothly and gradually in the vicinity of the critical level, i.e.,

$$U(\zeta) = U_0 + (dU/d\zeta)\,\zeta,$$

where $\zeta \equiv z - z_c$, z_c is the critical level height, and $U_0 = c$ (for details see Booker and Bretherton, 1967; Mobbs and Darby, 1989). The analysis presented here is concerned only with Ri. After substituting (4) with the assumed $U(\zeta)$, (2) becomes

$$d^2w\,/\,d\zeta^2 + N^2/(dU/d\zeta)^2\,\zeta^{-2}\,w = 0\,. \tag{5}$$

Solutions to (5) are given by the method of Frobenius (e.g., Bender and Orszag, 1978; Simmons, 1991). Ri_c can be obtained by assuming that $w \propto \zeta^{\,n}$; hence,

$$n(n-1) + Ri = 0 \tag{6a}$$

which yields two independent solutions:

$$n_{1,2} = 0.5 \pm \mathrm{i}\,(Ri - 1/4)^{1/2}\,. \tag{6b}$$

Therefore,

$$Ri_c = 1/4 \tag{7}$$

as a necessary condition for the onset of instability. Derivations of Ri_c based on an energy argument may be seen, for example, in Drazin and Reid (1981), and Miles (1990). If Ri is less than Ri_c, a nonlinear analysis is desirable in order to describe the flow.

b) a parameterized critical-layer curvature

The critical level is stretched to a critical layer in order to examine Ri_c variations due to the wind profile curvature. The curvature is presumably small and it does not dominate but slightly modifies the wind profile. As a refinement to the previous, brief derivation, one may try to locally replace (3) so to include the curvature of $U(z)$. The expansion of $U(\zeta)$ now includes the quadratic term, i.e., locally

$$U(\zeta) \approx U_0 + (dU/d\zeta)\,\zeta + (d^2U/d\zeta^2)\,\zeta^2/2.$$

With $U_0 = c$, a scale analysis for $m(z)$ analogous to (4) yields

$$m^2 \rightarrow m_2^2 \approx N^2 / [(dU/d\zeta)\zeta + (d^2U/d\zeta^2)\zeta^2/2]^2 - (d^2U/d\zeta^2) / [(dU/d\zeta)\zeta + (d^2U/d\zeta^2)\zeta^2/2] \tag{8}$$

The assumed m_2^2 comes from the expectation that the critical layer as a whole affects Ri_c. Hence, the following modification can be viewed as a quasi-nonlocal one because the curvature, which includes the second derivative, is a non-local quantity while the overall analysis presented here is a local analysis of (2).

The right hand side of (8) suggests that these terms are of the same order (otherwise (8) does not hold) even though the first term ($\sim |N^2|$) should be larger than the second one ($\sim |d^2U/d\zeta^2|$). Therefore,

$$N^2 \sim U''U'\zeta\,[1 + U''\zeta/(2U')]\,,$$

where the apostrophes denote derivatives with respect to ζ. Based on a heuristic assumption that U'' can be parameterized ($U'' \sim U'/\zeta$), define

$$\alpha \equiv (U''\zeta/U') \tag{10}$$

Parameter is a dimensionless measure of the relative importance for the $U(\zeta)$ curvature. The local wind speed $U(\zeta)$ can be easily found from (10). A preliminary restriction on the range of values for α comes from the requirement that U may only slightly depart from a straight-line profile at the critical layer. Hence, that gives a modification for Ri_c, i.e., a small interval for Ri_c values. Figure 1 shows a possible and idealized range of shapes for U at a critical layer. The realizability of various U-profiles in the atmosphere is the ultimate restriction for α and this defined $U(\zeta)$, $|\alpha|$ ought not to be larger than, roughly, 0.3. A further analysis of the parameterization (10) and its realizability has been left for another study.

So far, the only difference between this approach and that in subsection 2a is the introduction of the small parameter due to the critical-level stretching into

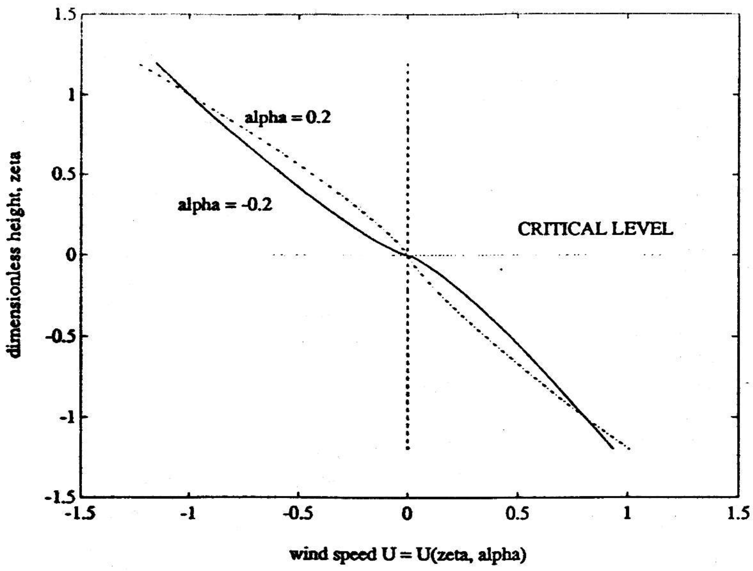

Figure 1. An idealized set of wind profiles in the vicinity of a critical level. They represent small variations from the straight-line profile. The small and parameterized wind profile curvature affects the critical Richardson number Ri_c as described by Equation (14).

Slika 1. Idelalizirani profili vjetra u blizini kritičnoga nivoa predstavljeni kao male varijacije od pravocrtnog profila. Jednadžba (14) opisuje utjecaj male i parametrizirane zakrivljenosti profila vjetra na kritični Richardsonov broj Ri_c.

the critical layer. Hence, it is expected that the TGE holds, and using (8), (2) becomes:

$$d^2w/d\zeta^2+\{N^2/[(dU/d\zeta)\zeta+(d^2U/d\zeta^2)\zeta^2/2]^2 -$$

$$-(d^2U/d\zeta^2)/[(dU/d\zeta)\zeta+(d^2U/d\zeta^2)\zeta^2/2]\}\,w=0 \quad (11a)$$

or in a more compact form:

$$d^2w/d\zeta^2 + \{Ri\,(1+\alpha/2)^{-2}-\alpha(1+\alpha/2)^{-1}\}\,\zeta^{-2}w = 0 \quad (11b)$$

Equation (11) is a modified TGE near the critical level. A resemblance between (5) and (11) is provided through being sufficiently small; the former can be seen as (11) when $\alpha \rightarrow 0$ and this plausibly justifies the modified TGE[2]. If one assumes that $w \propto \zeta^n$ as before, then

$$n(n-1)+Ri\,(1+\alpha/2)^{-2}-\alpha\,(1+\alpha/2)^{-1}=0 \quad (12a)$$

[2] The same name, the modified Taylor-Goldstein equation, may have been used elsewhere and with different meaning. For instance, in Grisogono (1994) the same name applies to a fourth-order equation including eddy viscosity for momentum; however, the equation is always associated with wave-like motion.

and

$$n_{1,2} = 1/2\pm i[Ri(1+\alpha/2)^{-2}-\alpha\,(1+\alpha/2)^{-1}-1/4]^{1/2} \quad (12b)$$

It appears that "simple" waves may dominate if

$$Ri(\alpha) > (1+\alpha/2)[\alpha+(1+\alpha/2)/4]; \quad (13)$$

therefore,

$$Ri_c(\alpha) = (1+\alpha/2)[\alpha+(1+\alpha/2)/4]\,. \quad (14)$$

Equation (14) gives a linear-parabolic dependence of $Ri_c(\alpha)$ on the parameterized wind speed curvature and is the main result here. Also note that $Ri_c(0) = 1/4$ as in (7).

3. DISCUSSION AND CONCLUDING REMARKS

In reality, it is difficult to obtain any constant value of α associated with a critical layer. Therefore, an overall curvature effect on $Ri_c(\alpha)$ is provided here by an integration over assumed values. Define average Ri_c, Ri_c, $<Ri_c>$, as

$$<Ri_c> = (b-a)^{-1}\int_a^b Ri_c(\alpha)\,d\alpha \quad (15)$$

for an expected range $a \le \alpha \le b$. Table 1 shows a set of $<Ri_c>$ values when $-0.20 \le a < 0$ and $0 < b \le 0.20$.

From Table 1:
when $a \geq -b$, then $<Ri_c> < 1/4$;
when $a < b$, then $<Ri_c> < 1/4$.

Even though a and b are assigned as an anti-symmetric set of values around zero in Table 1, the average $<Ri_c>$ is somewhat shifted above $Ri_c(0) = 1/4$ [see also (14)].

Abrupt changes in the stable ABL could be sometimes caused by initially unnoticeable variations in the wind profile that still contains appreciable changes in its curvature (e.g., in the upper part of the stable ABL). This may modify Ri_c so that Ri_c becomes locally greater than 1/4 and permits shear instabilities to develop as the beginning of a sudden flow collapse from a laminar to a turbulent regime. The way Ri_c is formulated here suggest an additional form for Ri given in (1), e.g., $Ri_{NEW} = Ri\,(1+\alpha/2)^{-2}$. Thus, some non-local effects on flow stability would be taken into account. However, the applicability of the modified TGE must be carefully reconsidered before actual employment.

In conclusion, Ri_c may slightly depart from its well-defined theoretical value of 1/4 if the vertical wind profile possesses a curvature. This curvature is likely to occur in atmospheric flows and it contributes to deviations of the actual critical gradient Richardson number $Ri_c = 1/4$. Since the curvature effect is presumably small, its parameterization via the small parameter in the TGE is probably acceptable.

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Table 1. The averaged critical Richardson number, $<Ri_c>$, from (15)

Tablica 1. Srednji kritični Richardsonov broj, $<Ri_c>$ iz (15)

b / a	0.05	0.10	0.15	0.20
–0.20	0.162	0.193	0.225	0.258
–0.15	0.191	0.222	0.254	0.287
–0.10	0.220	0.252	0.285	0.318
–0.05	0.250	0.283	0.316	0.350

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