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N-fractional calculus operator N^{μ} method to a modified hydrogen atom equation

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Abstract. By means of fractional calculus techniques, we find explicit solutions of the modified hydrogen atom equations. We use the N-fractional calculus operator N^{μ} method to derive the solutions of these equations.

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1. Introduction, definitions and preliminaries

Let

$$q(r) = \frac{\ell(\ell+1)}{r^2} - \frac{n}{r}, \quad (0 < r < \infty),$$

where ℓ is a positive integer or zero and $n > 0$. Then

$$\frac{d^2y}{dr^2} + \left[E + \frac{n}{r} - \frac{\ell(\ell+1)}{r^2}\right] y = 0. \tag{1}$$

In quantum mechanics the study of energy levels of the hydrogen atom leads to this equation [1]. For the problem having analogous singularity, some questions of spectral analysis are given in [7].

The differintegration operators and their generalizations [5, 6, 9, 10] have been used to solve some classes of differential equations and fractional differential equations.

Two of the most commonly encountered tools in the theory and applications of fractional calculus are provided by the Riemann-Liouville operator R_z^{υ} $(\upsilon \in \mathbb{C})$ and the Weyl operator W_z^{υ} $(\upsilon \in \mathbb{C})$, which are defined by [2, 3, 8, 11, 12]

$$R_z^{\upsilon} f(z) = \begin{cases} \frac{1}{\Gamma(\upsilon)} \int_0^z (z-t)^{\upsilon-1} f(t) dt : & Re(\upsilon) > 0, \\ \frac{d^n}{dz^n} R_z^{\upsilon+n} f(z) : & -n < Re(\upsilon) \leq 0; \quad n \in \mathbb{N}, \end{cases} \tag{2}$$

and

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$$W_z^{\upsilon} f(z) = \begin{cases} \frac{1}{\Gamma(\upsilon)} \int_z^{\infty} (t-z)^{\upsilon-1} f(t)dt : & Re(\upsilon) > 0, \\ \frac{d^n}{dz^n} W_z^{\upsilon+n} f(z) : & -n < Re(\upsilon) \leq 0; \quad n \in \mathbb{N}, \end{cases} \tag{3}$$

provided that the defining integrals in (2) and (3) exist, $\mathbb{N}$ being the set of positive integers.

Definition 1 (See [5, 4, 13]). *Let*

$$D = \{D^-, D^+\}, \quad C = \{C^-, C^+\},$$

C^- be a curve along the cut joining two points z and $-\infty + iIm(z)$, C^+ be a curve along the cut joining two points z and $\infty + iIm(z)$, D^- be a domain surrounded by C^-, and D^+ a domain surrounded by C^+. (Here D contains the points over the curve C).

Moreover, let $f = f(z)$ be a regular function in D $(z \in D)$,

$$f_\mu(z) = (f(z))_\mu = \frac{\Gamma(\mu+1)}{2\pi i} \int_C \frac{f(t)\,dt}{(t-z)^{\mu+1}}, \quad (\mu \in \mathbb{R} \setminus \mathbb{Z}^-; \mathbb{Z}^- = \{-1, -2, ...\})$$

and

$$f_{-n}(z) = \lim_{\mu \to -n} f_\mu(z), \quad (n \in \mathbb{Z}^+),$$

where $t \neq z$,

$$-\pi \leq \arg(t-z) \leq \pi \text{ for } C^-$$

and

$$0 \leq \arg(t-z) \leq 2\pi \text{ for } C^+,$$

then $f_\mu(z)$ $(\mu > 0)$ is said to be the fractional derivative of $f(z)$ of order μ and $f_\mu(z)$ $(\mu < 0)$ is said to be the fractional integral of (z) of order $-\mu$, provided (in each case) that $|f_\mu(z)| < \infty$ $(\mu \in \mathbb{R})$.

Finally, let the fractional calculus operator (Nishimoto's operator) N^μ be defined by (cf. [5])

$$N^\mu = \left(\frac{\Gamma(\mu+1)}{2\pi i} \int_C \frac{dt}{(t-z)^{\mu+1}} \right), \quad (\mu \notin \mathbb{Z}^-)$$

with

$$N^{-n} = \lim_{\mu \to -n} N^\mu, \quad (n \in \mathbb{Z}^+).$$

We find it worthwhile to recall here the following useful lemmas and properties associated with the fractional differintegration defined above (cf.e.g. [5, 4]).

Lemma 1 (Linearity property)**.** *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(h_1 f(z) + h_2 g(z))_{\mu} = h_1 f_{\mu}(z) + h_2 g_{\mu}(z), \quad (\mu \in \mathbb{R}; z \in \Omega) \tag{4}$$

for any constants h_1 and h_2.

Lemma 2 (Index law)**.** *If the function $f(z)$ is single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(f_{\gamma}(z))_{\mu} = f_{\gamma+\mu}(z) = (f_{\mu}(z))_{\gamma}, \quad (f_{\gamma}(z) \neq 0; f_{\mu}(z) \neq 0; \gamma, \mu \in \mathbb{R}; z \in \Omega). \tag{5}$$

Lemma 3 (Generalized Leibniz rule)**.** *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(f(z).g(z))_{\mu} = \sum_{n=0}^{\infty} \binom{\mu}{n} f_{\mu-n}(z).g_n(z), \quad (\mu \in \mathbb{R}; z \in \Omega), \tag{6}$$

where $g_n(z)$ is the ordinary derivative of $g(z)$ of order n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$, it being tacitly assumed (for simplicity) that $g(z)$ is a polynomial part (if any) of the product $f(z)\,g(z)$.

Property 1. *For a constant λ,*

$$\left(e^{\lambda z}\right)_{\mu} = \lambda^{\mu} e^{\lambda z}, \quad (\lambda \neq 0; \mu \in \mathbb{R}; z \in \mathbb{C}). \tag{7}$$

Property 2. *For a constant λ,*

$$\left(e^{-\lambda z}\right)_{\mu} = e^{-i\pi\mu} \lambda^{\mu} e^{-\lambda z}, \quad (\lambda \neq 0; \mu \in \mathbb{R}; z \in \mathbb{C}). \tag{8}$$

Property 3. *For a constant λ,*

$$\left(z^{\lambda}\right)_{\mu} = e^{-i\pi\mu} \frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)} z^{\lambda-\mu}, \quad \left(\mu \in \mathbb{R}; z \in \mathbb{C}; \left|\frac{\Gamma(\mu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right). \tag{9}$$

2. The N^{μ} method applied to a modified hydrogen atom equation

Theorem 1. *Let $y \in \{y : 0 \neq |y_{\mu}| < \infty; \mu \in \mathbb{R}\}$ and $f \in \{f : 0 \neq |f_{\mu}| < \infty; \mu \in \mathbb{R}\}$. Then the non-homogeneous modified hydrogen atom equation (putting $E = k^2$ (k the corresponding wave number), $m = \ell + (1/2)$ in (1)):*

$$L[y, r, m, n] = y_2 + y\left[k^2 + \frac{n}{r} + \frac{(1/4) - m^2}{r^2}\right] = f, \quad (r \neq 0) \tag{10}$$

has particular solutions of the forms:

$$\begin{aligned} y &= r^{m+\frac{1}{2}} e^{-ikr} \left\{ \left[\left(f r^{\frac{1}{2}-m} e^{ikr} \right)_{-m-\frac{in}{2k}-\frac{1}{2}} e^{-2ikr} r^{m-\frac{in}{2k}-\frac{1}{2}} \right]_{-1} \right. \\ &\quad \left. \times\, e^{2ikr} r^{-m+\frac{in}{2k}-\frac{1}{2}} \right\}_{m+\frac{in}{2k}-\frac{1}{2}} \\ &\equiv y^{I} \end{aligned} \tag{11}$$

$$y = r^{m+\frac{1}{2}} e^{ikr} \left\{ \left[\left(f r^{\frac{1}{2}-m} e^{-ikr} \right)_{-m+\frac{in}{2k}-\frac{1}{2}} e^{2ikr} r^{m+\frac{in}{2k}-\frac{1}{2}} \right]_{-1} \times e^{-2ikr} r^{-m-\frac{in}{2k}-\frac{1}{2}} \right\}_{m-\frac{in}{2k}-\frac{1}{2}} \equiv y^{II}, \tag{12}$$

$$y = r^{-m+\frac{1}{2}} e^{-ikr} \left\{ \left[\left(f r^{\frac{1}{2}+m} e^{ikr} \right)_{m-\frac{in}{2k}-\frac{1}{2}} e^{-2ikr} r^{-m-\frac{in}{2k}-\frac{1}{2}} \right]_{-1} \times e^{2ikr} r^{m+\frac{in}{2k}-\frac{1}{2}} \right\}_{-m+\frac{in}{2k}-\frac{1}{2}} \equiv y^{III}, \tag{13}$$

$$y = r^{-m+\frac{1}{2}} e^{ikr} \left\{ \left[\left(f r^{\frac{1}{2}+m} e^{-ikr} \right)_{m+\frac{in}{2k}-\frac{1}{2}} e^{2ikr} r^{-m+\frac{in}{2k}-\frac{1}{2}} \right]_{-1} \times e^{-2ikr} r^{m-\frac{in}{2k}-\frac{1}{2}} \right\}_{-m-\frac{in}{2k}-\frac{1}{2}} \equiv y^{IV}. \tag{14}$$

Here $y_2 = d^2y/dr^2$, $y = y(z)\,(z \in \mathbb{C})$, $f = f(z)$ *(an arbitrary given function) and* m, n *are given constants.*

Remark 1. *The cases* $m = 0$ *of* (13) *and* (14) *overlap with those in* (11) *and* (12), *respectively.*

Proof.

$$y = r^{\nu}\phi \quad \text{with } \phi = \phi(r) \quad (r \neq 0); \tag{15}$$

hence

$$y_1 = \nu r^{\nu-1}\phi + r^{\nu}\phi_1 \tag{16}$$

and

$$y_2 = \nu(\nu-1) r^{\nu-2}\phi + 2\nu r^{\nu-1}\phi_1 + r^{\nu}\phi_2. \tag{17}$$

Substituting the relations (15), (16) and (17) into (10), we have

$$\nu(\nu-1) r^{\nu-2}\phi + 2\nu r^{\nu-1}\phi_1 + r^{\nu}\phi_2 + r^{\nu}\phi\left[k^2 + \frac{n}{r} + \frac{(1/4)-m^2}{r^2}\right] = f. \tag{18}$$

With some rearrangement of the terms in (18), we have

$$\phi_2 r + 2\nu\phi_1 + \phi\left[\left(\nu^2 - \nu + \frac{1}{4} - m^2\right) r^{-1} + k^2 r + n\right] = f r^{1-\nu}. \tag{19}$$

Here we choose ν such that

$$\nu^2 - \nu + \frac{1}{4} - m^2 = 0,$$

that is

$$\nu = \frac{1}{2} \pm m. \tag{20}$$

(I): Let $\nu = m + (1/2)$. From (15) and (19) we have

$$y = r^{m+(1/2)} \phi \tag{21}$$

and

$$\phi_2 r + \phi_1 (2m+1) + \phi \left(n + k^2 r\right) = f r^{(1/2)-m}, \tag{22}$$

respectively.

Next, set

$$\phi = e^{\eta r} \psi \quad \text{with } \psi = \psi(r), \tag{23}$$

then equation (22) may be written in the form:

$$(e^{\eta r}\psi)_2 r + (e^{\eta r}\psi)_1 (2m+1) + e^{\eta r}\psi \left(n + k^2 r\right) = f r^{(1/2)-m}. \tag{24}$$

At this point, calculating the derivatives

$$(e^{\eta r}\psi)_1 = e^{\eta r}(\eta\psi + \psi_1) \tag{25}$$

and

$$(e^{\eta r}\psi)_2 = e^{\eta r}\left(\eta^2\psi + 2\eta\psi_1 + \psi_2\right), \tag{26}$$

and substituting from (23), (25) and (26) in (24), we can express (24) as

$$\psi_2 r + \psi_1 (2\eta r + 2m + 1) + \psi\left[\left(\eta^2 + k^2\right) r + (2m+1)\eta + n\right] = f r^{(1/2)-m} e^{-\eta r}. \tag{27}$$

Choose η such that

$$\eta^2 + k^2 = 0, \tag{28}$$

that is,

$$\eta = \pm k i. \tag{29}$$

(I) (i): In the case when $\eta = -ki$, we have

$$\phi = e^{-ikr}\psi \tag{30}$$

and

$$\psi_2 r + \psi_1 (-2ikr + 2m + 1) + \psi\left[-ik(2m+1) + n\right] = f r^{(1/2)-m} e^{ikr} \tag{31}$$

from (23) and (27) , respectively.

Applying the operator N^{μ} to both members of (31) , we then obtain

$$(\psi_2 r)_\mu + [\psi_1 (-2ikr + 2m + 1)]_\mu + \{\psi [-ik (2m + 1) + n]\}_\mu = \left(f r^{(1/2)-m} e^{ikr}\right)_\mu . \quad (32)$$

Using (4), (5), (6) we have

$$(\psi_2 r)_\mu = \psi_{2+\mu} r + \psi_{1+\mu} \mu \quad (33)$$

and

$$[\psi_1 (-2ikr + 2m + 1)]_\mu = \psi_{1+\mu} (-2ikr + 2m + 1) - \psi_\mu 2ik\mu. \quad (34)$$

Making use of the relations (33) and (34), we may write (32) in the following form:

$$\psi_{2+\mu} r + \psi_{1+\mu} (-2ikr + 2m + 1 + \mu) + \psi_\mu [n - ik (2m + 1 + 2\mu)] = \left(f r^{(1/2)-m} e^{ikr}\right)_\mu . \quad (35)$$

Choose μ such that

$$\mu = -m - \frac{in}{2k} - \frac{1}{2}, \quad (36)$$

we have then

$$\psi_{2-m-\frac{in}{2k}-\frac{1}{2}} r + \psi_{1-m-\frac{in}{2k}-\frac{1}{2}} \left[-i\left(2kr + \frac{n}{2k}\right) + m + \frac{1}{2}\right] = \left(f r^{(1/2)-m} e^{ikr}\right)_{-m-\frac{in}{2k}-\frac{1}{2}} \quad (37)$$

from (35).

Next, writing

$$\psi_{1-m-\frac{in}{2k}-\frac{1}{2}} = u = u(r), \quad (38)$$

we obtain

$$u_1 + u\left[-2ik + \left(m + \frac{1}{2} - \frac{in}{2k}\right) r^{-1}\right] = \left(f r^{(1/2)-m} e^{ikr}\right)_{-m-\frac{in}{2k}-\frac{1}{2}} r^{-1} \quad (39)$$

from (37) . This is an ordinary differential equation of the first order which has a particular solution:

$$u = \left[\left(f r^{(1/2)-m} e^{ikr}\right)_{-m-\frac{in}{2k}-\frac{1}{2}} e^{-2ikr} r^{-1+m-\frac{in}{2k}+\frac{1}{2}}\right]_{-1} e^{2ikr} r^{-m+\frac{in}{2k}-\frac{1}{2}}. \quad (40)$$

Making use of the reverse process to obtain y^I, we finally obtain solution (11) from (40) , (38) , (30) and (21).

Inversely, (40) satisfies (39) clearly; then

$$\psi = u_{m+\frac{in}{2k}-\frac{1}{2}}, \tag{41}$$

satisfies (37). Therefore, (11) satisfies (10) because we have (21), (30), (40) and (41).

(I) (ii): In the case when $\eta = ik$, we have

$$\phi = e^{ikr}\psi \tag{42}$$

and

$$\psi_2 r + \psi_1 (2ikr + 2m + 1) + \psi [ik(2m+1) + n] = fr^{(1/2)-m}e^{-ikr} \tag{43}$$

from (23) and (27), respectively.

Applying the operator N^μ to both members of (43), we have

$$\begin{aligned}\psi_{2+\mu} r + \psi_{1+\mu}(2ikr + 2m + 1 + \mu) + \psi_\mu [n + ik(2m+1+2\mu)]\\ = \left(fr^{(1/2)-m}e^{-ikr}\right)_\mu.\end{aligned} \tag{44}$$

Choosing μ such that

$$\mu = -m + \frac{in}{2k} - \frac{1}{2} \tag{45}$$

and replacing

$$\psi_{1-m+\frac{in}{2k}-\frac{1}{2}} = \omega = \omega(r), \tag{46}$$

we have then

$$\omega_1 + \omega\left[2ik + \left(m + \frac{1}{2} + \frac{in}{2k}\right)r^{-1}\right] = \left(fr^{(1/2)-m}e^{-ikr}\right)_{-m+\frac{in}{2k}-\frac{1}{2}} r^{-1} \tag{47}$$

from (44). A particular solution of equation (47) is given by

$$\omega = \left[\left(fr^{(1/2)-m}e^{-ikr}\right)_{-m+\frac{in}{2k}-\frac{1}{2}} e^{2ikr}r^{m+\frac{in}{2k}-\frac{1}{2}}\right]_{-1} e^{-2ikr}r^{-m-\frac{in}{2k}-\frac{1}{2}}. \tag{48}$$

Therefore, we have (12) from (48), (46), (42) and (21).

(II): Let $\nu = -m + (1/2)$. In the same way as the procedure in (I), replacing m by $-m$ in (I) (i) and (I) (ii), we have other solutions (13) and (14) different from (11) and (12), respectively, if $m \neq 0$. □

3. The operator N^μ method to a homogeneous modified hydrogen atom equation

Theorem 2. *If $y \in \overset{o}{\wp}$, just like in Theorem 1, then the homogeneous modified hydrogen atom equation:*

$$L[y, r, m, n] = 0 \quad (r \neq 0), \tag{49}$$

has solutions of the forms:

$$y = hr^{m+\frac{1}{2}}e^{-ikr}\left(e^{2ikr}r^{-m+\frac{in}{2k}-\frac{1}{2}}\right)_{m+(in/2k)-(1/2)} \equiv y^{(I)}, \tag{50}$$

$$y = hr^{m+\frac{1}{2}}e^{ikr}\left(e^{-2ikr}r^{-m-\frac{in}{2k}-\frac{1}{2}}\right)_{m-(in/2k)-(1/2)} \equiv y^{(II)}, \tag{51}$$

$$y = hr^{-m+\frac{1}{2}}e^{-ikr}\left(e^{2ikr}r^{m+\frac{in}{2k}-\frac{1}{2}}\right)_{-m+(in/2k)-(1/2)} \equiv y^{(III)}, \tag{52}$$

$$y = hr^{-m+\frac{1}{2}}e^{ikr}\left(e^{-2ikr}r^{m-\frac{in}{2k}-\frac{1}{2}}\right)_{-m-(in/2k)-(1/2)} \equiv y^{(IV)}, \tag{53}$$

for $m \neq 0$, *where* h *is an arbitrary constant.*

Remark 2. *In the case when* $m = 0$, (52) *and* (53) *overlap with* (50) *and* (51), *respectively.*

Proof. When $f = 0$ in Section 2, we have

$$u_1 + u\left[-2ik + \left(m + \frac{1}{2} - \frac{in}{2k}\right)r^{-1}\right] = 0 \tag{54}$$

and

$$\omega_1 + \omega\left[2ik + \left(m + \frac{1}{2} + \frac{in}{2k}\right)r^{-1}\right] = 0 \tag{55}$$

for $\eta = -ik$ and $\eta = ik$, instead of (39) and (47), respectively.

Therefore, we obtain (50) for (54) and (51) for (55).

And, for $\nu = -m + (1/2)$, replacing m by $-m$ in (54) and (55), we have (52) and (53), respectively. □

Theorem 3. *Let* $y \in \overset{o}{\wp}$ *and* $f \in \overset{o}{\wp}$, *just like in Theorem 1. Then the nonhomogeneous modified hydrogen atom equation* (10) *is satisfied by the fractional differintegrated functions (for example)*

$$y = y^{I} + y^{(I)}. \tag{56}$$

Proof. It is clear by Theorems 1 and 2. □

Example 1. *In the case when* $m = -\frac{1}{2}, n = 0$ *and* $k = \frac{1}{3}$, *we have then*

$$y_2 + \frac{1}{9}y = 0 \tag{57}$$

from (49). *Solutions of equation* (57) *are given as*

$$\begin{aligned} y = y^{(I)} &= he^{-ir/3}\left(e^{2ir/3}\right)_{-1} \\ &= -\frac{3h}{2}ie^{ir/3} \end{aligned} \tag{58}$$

by (50). *The function shown by* (58) *satisfies* (57) *clearly.*

Example 2. *In the case when* $m = -\frac{1}{2}, n = 0,\ k = \frac{1}{2}$, *and* $f = ir$, *we have*

$$y_2 + \frac{1}{4} y = ir \tag{59}$$

and

$$\begin{aligned} y = y^I &= e^{-ir/2}\left\{\left[\left(irre^{ir/2}\right)_0 e^{-ir} r^{-1}\right]_{-1} e^{ir}\right\}_{-1} \\ &= e^{-ir/2}\left\{\left[ire^{-ir/2}\right]_{-1} e^{ir}\right\}_{-1} \\ &= e^{-ir/2}\left\{2ie^{ir/2}\,(ir+2)\right\}_{-1} \\ &= e^{-ir/2}\left(4ire^{ir/2}\right) \\ &= 4ir \end{aligned} \tag{60}$$

from (10) *and* (11)*. The function shown by* (60) *satisfies* (59) *clearly.*

4. Two further cases of a modified hydrogen atom equation

In the same way as in the preceding sections, we can solve the following nonhomogeneous modified hydrogen atom equation:

$$y_2 + y\left[k^2 + \frac{n}{r} + \frac{(1/4) + m^2}{r^2}\right] = f \tag{61}$$

and

$$y_2 + y\left[-k^2 + \frac{n}{r} + \frac{(1/4) + m^2}{r^2}\right] = f, \tag{62}$$

which are obtained by replacing m by im ($-k^2$ instead of k^2) in (10), that is,

$$y_2 + y\left[k^2 + \frac{n}{r} + \frac{(1/4) - (im)^2}{r^2}\right] = f \tag{63}$$

and

$$y_2 + y\left[-k^2 + \frac{n}{r} + \frac{(1/4) - (im)^2}{r^2}\right] = f. \tag{64}$$

i) Therefore, the solutions for (63) are given by replacing m by im in (11), (12), (13) and (14), as follows:

$$\begin{aligned} y^I &= r^{im+\frac{1}{2}} e^{-ikr}\left\{\left[\left(fr^{\frac{1}{2}-im} e^{ikr}\right)_{-i\left(m+\frac{n}{2k}\right)-\frac{1}{2}} e^{-2ikr} r^{i\left(m-\frac{n}{2k}\right)-\frac{1}{2}}\right]_{-1}\right. \\ &\quad \left.\times\, e^{2ikr} r^{-i\left(m-\frac{n}{2k}\right)-\frac{1}{2}}\right\}_{i\left(m+\frac{n}{2k}\right)-\frac{1}{2}}, \end{aligned} \tag{65}$$

$$y^{II} = r^{im+\frac{1}{2}} e^{ikr} \left\{ \left[\left(f r^{\frac{1}{2}-im} e^{-ikr} \right)_{-i\left(m-\frac{n}{2k}\right)-\frac{1}{2}} e^{2ikr} r^{i\left(m+\frac{n}{2k}\right)-\frac{1}{2}} \right]_{-1} \right.$$
$$\left. \times\, e^{-2ikr} r^{-i\left(m+\frac{in}{2k}\right)-\frac{1}{2}} \right\}_{i\left(m-\frac{n}{2k}\right)-\frac{1}{2}}, \tag{66}$$

$$y^{III} = r^{-im+\frac{1}{2}} e^{-ikr} \left\{ \left[\left(f r^{\frac{1}{2}+im} e^{ikr} \right)_{i\left(m-\frac{n}{2k}\right)-\frac{1}{2}} e^{-2ikr} r^{-i\left(m+\frac{n}{2k}\right)-\frac{1}{2}} \right]_{-1} \right.$$
$$\left. \times\, e^{2ikr} r^{i\left(m+\frac{n}{2k}\right)-\frac{1}{2}} \right\}_{-i\left(m-\frac{n}{2k}\right)-\frac{1}{2}}, \tag{67}$$

$$y^{IV} = r^{-im+\frac{1}{2}} e^{ikr} \left\{ \left[\left(f r^{\frac{1}{2}+im} e^{-ikr} \right)_{i\left(m+\frac{n}{2k}\right)-\frac{1}{2}} e^{2ikr} r^{-i\left(m-\frac{in}{2k}\right)-\frac{1}{2}} \right]_{-1} \right.$$
$$\left. \times\, e^{-2ikr} r^{i\left(m-\frac{n}{2k}\right)-\frac{1}{2}} \right\}_{-i\left(m+\frac{n}{2k}\right)-\frac{1}{2}}. \tag{68}$$

ii) Similarly, the solutions for (64), substituting the relations $(15), (16)$ and (17) into (64), we have

$$\phi_2 r + 2\nu\phi_1 + \phi \left[\left(\nu^2 - \nu + \frac{1}{4} - (im)^2 \right) r^{-1} - k^2 r + n \right] = f r^{1-\nu}. \tag{69}$$

Here we choose ν such that

$$\nu^2 - \nu + \frac{1}{4} + m^2 = 0,$$

that is

$$\nu = \frac{1}{2} \pm im. \tag{70}$$

Let $\nu = im + (1/2)$. From (15) and (69) we have

$$y = r^{im+(1/2)} \phi \tag{71}$$

and

$$\phi_2 r + \phi_1 (2im + 1) + \phi \left(n - k^2 r \right) = f r^{(1/2)-im} \tag{72}$$

Next, set (23). Then equation (72) may be written in the form:

$$\left(e^{\eta r}\psi\right)_2 r + \left(e^{\eta r}\psi\right)_1 (2im + 1) + e^{\eta r}\psi \left(n - k^2 r \right) = f r^{(1/2)-im} \tag{73}$$

Substituting the relations $(23), (25)$ and (26) into (73), we have

$$\psi_2 r + \psi_1 (2\eta r + 2im + 1) + \psi \left[\left(\eta^2 - k^2 \right) r + (2im + 1)\eta + n \right]$$
$$= f r^{(1/2)-im} e^{-\eta r}. \tag{74}$$

Choose η such that

$$\eta^2 - k^2 = 0,$$

that is,

$$\eta = \pm k. \tag{75}$$

ii.1) In the case when $\eta = -k$, we have

$$\phi = e^{-kr}\psi \tag{76}$$

and

$$\psi_2 r + \psi_1 (-2kr + 2im + 1) + \psi [-k (2im + 1) + n] = fr^{(1/2)-im} e^{kr} \tag{77}$$

from (23) and (74).

Applying the operator N^{μ} to both members of (77), we then obtain

$$(\psi_2 r)_\mu + [\psi_1 (-2kr + 2im + 1)]_\mu + \{\psi [-k (2im + 1) + n]\}_\mu = \left(fr^{(1/2)-im} e^{kr}\right)_\mu . \tag{78}$$

Using (4), (5), (6) we have

$$\psi_{2+\mu} r + \psi_{1+\mu} (-2kr + 2im + 1 + \mu) + \psi_\mu [n - k (2im + 1 + 2\mu)] = \left(fr^{(1/2)-im} e^{kr}\right)_\mu . \tag{79}$$

Choose μ such that

$$\mu = -im + \frac{n}{2k} - \frac{1}{2}, \tag{80}$$

we have then

$$\psi_{2-im+\frac{n}{2k}-\frac{1}{2}} r + \psi_{1-im+\frac{n}{2k}-\frac{1}{2}} \left[-2kr + im + \frac{n}{2k} + \frac{1}{2}\right] = \left(fr^{(1/2)-im} e^{kr}\right)_{-im+\frac{n}{2k}-\frac{1}{2}} \tag{81}$$

from (79).

Next, writing

$$\psi_{1-im+\frac{n}{2k}-\frac{1}{2}} = u = u(r), \tag{82}$$

we obtain

$$u_1 + u\left[-2k + \left(im + \frac{n}{2k} + \frac{1}{2}\right) r^{-1}\right] = \left(fr^{(1/2)-im} e^{kr}\right)_{-im+\frac{n}{2k}-\frac{1}{2}} r^{-1} \tag{83}$$

from (81). This is an ordinary differential equation of the first order which has a particular solution:

$$u = \left[\left(f r^{(1/2)-im} e^{kr} \right)_{-im+\frac{n}{2k}-\frac{1}{2}} e^{-2kr} r^{im+\frac{n}{2k}-\frac{1}{2}} \right]_{-1} e^{2kr} r^{-im-\frac{n}{2k}-\frac{1}{2}}. \tag{84}$$

We finally obtain the solution

$$\begin{aligned} y^{I} &= r^{im+\frac{1}{2}} e^{-kr} \left\{ \left[\left(f r^{\frac{1}{2}-im} e^{kr} \right)_{-im+\frac{n}{2k}-\frac{1}{2}} e^{-2kr} r^{im+\frac{n}{2k}-\frac{1}{2}} \right]_{-1} \right. \\ &\quad \left. \times\, e^{2kr} r^{-im-\frac{n}{2k}-\frac{1}{2}} \right\}_{im-\frac{n}{2k}-\frac{1}{2}} \end{aligned} \tag{85}$$

from (84), (82), (76) and (71).

ii.2) Similarly, in the case when $\eta = k$, we obtain

$$\begin{aligned} y^{II} &= r^{im+\frac{1}{2}} e^{kr} \left\{ \left[\left(f r^{\frac{1}{2}-im} e^{-kr} \right)_{-im-\frac{n}{2k}-\frac{1}{2}} e^{2kr} r^{im-\frac{n}{2k}-\frac{1}{2}} \right]_{-1} \right. \\ &\quad \left. \times\, e^{-2kr} r^{-im+\frac{n}{2k}-\frac{1}{2}} \right\}_{im+\frac{n}{2k}-\frac{1}{2}}. \end{aligned} \tag{86}$$

Let $\nu = -im + (1/2)$. In the same way as the procedure in (ii), replacing im by $-im$ (ii.1) and (ii.2), we have

$$\begin{aligned} y^{III} &= r^{-im+\frac{1}{2}} e^{-kr} \left\{ \left[\left(f r^{\frac{1}{2}+im} e^{kr} \right)_{im+\frac{n}{2k}-\frac{1}{2}} e^{-2kr} r^{-im+\frac{n}{2k}-\frac{1}{2}} \right]_{-1} \right. \\ &\quad \left. \times\, e^{2kr} r^{im-\frac{n}{2k}-\frac{1}{2}} \right\}_{-im-\frac{n}{2k}-\frac{1}{2}}, \end{aligned} \tag{87}$$

$$\begin{aligned} y^{IV} &= r^{-im+\frac{1}{2}} e^{kr} \left\{ \left[\left(f r^{\frac{1}{2}+im} e^{-kr} \right)_{im-\frac{n}{2k}-\frac{1}{2}} e^{2kr} r^{-im-\frac{n}{2k}-\frac{1}{2}} \right]_{-1} \right. \\ &\quad \left. \times\, e^{-2kr} r^{im+\frac{n}{2k}-\frac{1}{2}} \right\}_{-im+\frac{n}{2k}-\frac{1}{2}}. \end{aligned} \tag{88}$$

iii) In the homogeneous case for equation (63) with $f = 0$, using the results (50), (51), (52) and (53), and replacing m by im, we obtain

$$y^{(I)} = h r^{im+\frac{1}{2}} e^{-ikr} \left(e^{2ikr} r^{-i\left(m-\frac{n}{2k}\right)-\frac{1}{2}} \right)_{i\left(m+\frac{n}{2k}\right)-\frac{1}{2}}, \tag{89}$$

$$y^{(II)} = h r^{im+\frac{1}{2}} e^{ikr} \left(e^{-2ikr} r^{-i\left(m+\frac{in}{2k}\right)-\frac{1}{2}} \right)_{i\left(m-\frac{n}{2k}\right)-\frac{1}{2}}, \tag{90}$$

$$y^{(III)} = h r^{-im+\frac{1}{2}} e^{-ikr} \left(e^{2ikr} r^{i\left(m+\frac{n}{2k}\right)-\frac{1}{2}} \right)_{-i\left(m-\frac{n}{2k}\right)-\frac{1}{2}}, \tag{91}$$

$$y^{(IV)} = h r^{-im+\frac{1}{2}} e^{ikr} \left(e^{-2ikr} r^{i\left(m-\frac{n}{2k}\right)-\frac{1}{2}} \right)_{-i\left(m+\frac{n}{2k}\right)-\frac{1}{2}}. \tag{92}$$

for $m \neq 0$, where h is an arbitrary constant.

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