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Certain classes of multivalent functions with negative coefficients associated with a convolution structure

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Abstract. Making use of a convolution structure, we introduce a new class of analytic functions $\mathbb{T}_g^p(\lambda, \alpha, \beta,)$ defined in the open unit disc and investigate its various characteristics. Further we obtained distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathbb{T}_g^p(\lambda, \alpha, \beta)$.

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1. Introduction

Let $\mathcal{A}(p)$ denote a class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \ \ p \in N = \{1, 2, 3, \dots\}, \tag{1}$$

which are analytic in the open disc $U = \{z : z \in \mathcal{C}; \ |z| < 1\}$. For functions $f \in \mathcal{A}(p)$ given by (1) and $g \in \mathcal{A}(p)$ given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, \ \ p \in N = \{1, 2, 3, \dots\}$$

we define the Hadamard product (or convolution) of f and g by

$$f(z) * g(z) = (f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad z \in U. \tag{2}$$

In terms of the Hadamard product (or convolution), we choose g as a fixed function in $\mathcal{A}(p)$ such that $(f * g)(z)$ exists for any $f \in \mathcal{A}(p)$, and for various choices of g we get different linear operators which have been studied in the recent past. To illustrate some of these cases which arise from the convolution structure (2), we consider the following examples.

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1. For

$$g(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \dots (\alpha_r)_{n-p}}{(\beta_1)_{n-p} \dots (\beta_s)_{n-p}} \frac{z^n}{(n-p)!}, \qquad (3)$$

we get the Dziok–Srivastava operator

$$\Lambda(\alpha_1, \alpha_2, \cdots, \alpha_r; \beta_1, \beta_2, \cdots, \beta_s; z) f(z) \equiv H^p_{r,s} f(z) := (f * g)(z),$$

introduced by Dziok and Srivastava [6]; where $\alpha_1, \alpha_2, \cdots \alpha_r, \beta_1, \beta_2, \cdots \dots, \beta_s$ are complex parameters, $\beta_j \notin \{0, -1, -2, \cdots\}$ for $j = 1, 2, \cdots, s$, $r \leq s + 1, r, s \in \mathbb{N} \cup \{0\}$. Here $(a)_\nu$ denotes a well-known Pochhammer symbol (or shifted factorial).

Remark 1. *When $\alpha_1 = a, \beta_1 = c$ $s = 1$and $r = 2$, Dizok-Srivastava operator reduces to Carlson-Shaffer operator [3], further when $r = 1, s = 0; \alpha_1 = \nu + 1, \alpha_2 = 1; \beta_1 = 1$, then the above Dziok-Srivastava operator yields the Ruscheweyh derivative operator introduced by Patel and Cho [10].*

2. For

$$g(z) = \phi_p(a, c, z) := z^p + \sum_{n=p+1}^{\infty} \frac{(a)_{n-1}}{(c)_{n-1}} z^n \quad (c \neq 0, -1, -2, \cdots), \qquad (4)$$

we get the multivalent Carlson-Shaffer operator $L_p(a, c) f(z) := (f * g)(z)$. The operator

$$L(a, c) f(z) \equiv L_1(a, c) f(z) \equiv z F(a, 1; c; z) * f(z)$$

was introduced by Carlson-Shaffer [3] where $F(a, b; c; z)$ is the Gaussian hypergeometric function.

3. For $g(z) = \frac{z^p}{(1-z)^{\nu+p}}$ $(\nu \geq -p)$, we obtain the multivalent Ruscheweyh operator defined by

$$D^{\nu+p-1} f(z) := (f * g)(z) = z^p + \sum_{n=p+1}^{\infty} \binom{\nu + n - 1}{n - p} a_n z^n. \qquad (5)$$

The operator $D^{\nu+p-1} f$ was introduced by Patel and Cho [10]. In particular, $D^\nu f : \mathcal{A} \to \mathcal{A}$ for $p = 1$ and $\nu \geq -1$ was introduced by Ruschweyh [15]. Note that for $n \in \mathbb{N}_0$,

$$D^n f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!}.$$

4. For

$$g(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{n}{p}\right)^m z^n \quad (\, m \geq 0\,), \qquad (6)$$

we get the multivalent Sălăgean operator $D^m_p f(z)$ introduced by Shenan *et al.* [14]. In particular, the differential operator $\mathcal{D}^m \equiv \mathcal{D}^m_1$ was initially studied by Sălăgean [12].

5. For
$$g(z) = z^p + \sum_{n=p+1}^{\infty} n\left(\frac{n+l}{p+l}\right)^m z^n \quad (l \geq 0;\ m \in \mathbb{Z}), \tag{7}$$
we obtain the multiplier transformation $\mathcal{I}_p(l, m) := (f * g)(z)$ introduced by Ravichandran *et al.* [11]. In particular, $\mathcal{I}(l, m) \equiv \mathcal{I}_1(l, m)$ was studied by Cho and Kim [4] and Cho and Srivastava [5].

6. For
$$g(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{n+l}{p+l}\right)^m z^n \quad (l \geq 0;\ m \in \mathbb{Z}), \tag{8}$$
we get multiplier transformation $\mathcal{J}_p(l, m) := (f * g)(z)$. In particular $\mathcal{J}(l, m) \equiv \mathcal{J}_1(l, m)$ introduced by Cho and Srivastava [4].

Remark 2. *We note that for $l = 0$ the above operator reduces to the multivalent Sălăgean operator $D_p^m f(z)$ introduced by Shenan* et al. *[14].*

Motivated by the earlier works of [6, 8, 10, 14, 11, 16] we introduced a new subclass of multivalent functions with negative coefficients and discuss some interesting properties of this generalized function class.

For $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$, we let $\mathbb{P}_g(\lambda, \alpha, \beta)$ be the subclass of $\mathcal{A}(p)$ consisting of functions of the form (1) and satisfying the inequality
$$\left| \frac{\mathbb{J}_{g,\lambda}(z) - 1}{\mathbb{J}_{g,\lambda}(z) + (1 - 2\alpha)} \right| < \beta, \tag{9}$$
where
$$\mathbb{J}_{g,\lambda}(z) = (1 - \lambda)\frac{(f * g)}{z^p} + \frac{\lambda(f * g)'}{pz^{p-1}}, \tag{10}$$
$(f * g)(z)$ is given by (2) and g is fixed function for all $z \in U$. We further let $\mathbb{T}_g^p(\lambda, \alpha, \beta) = \mathbb{P}_g(\lambda, \alpha, \beta) \cap T(p)$, where
$$T(p) \ := \ \left\{ f \in \mathcal{A}(p) : f(z) = z - \sum_{n=p+1}^{\infty} |a_n| z^n, \quad z \in U \right\}. \tag{11}$$

We deem it proper to mention below some of the function classes which emerge from the function class $\mathbb{T}_g^p(\lambda, \alpha, \beta)$ defined above. Indeed, we observe that if we specialize the function $g(z)$ by means of (3) to (8), and denote the corresponding reducible classes of functions of $\mathbb{T}_g^p(\lambda, \alpha, \beta)$, as listed below.

When $g(z)$ is as defined in (3), the class $\mathbb{T}_g^p(\lambda, \alpha, \beta)$ reduces to the subclass $\mathcal{HT}(\lambda, \alpha, \beta)$ with (9) and
$$\mathbb{J}_{H,\lambda}(z) = (1 - \lambda)\frac{H_{s,r}^p[\alpha_1, \beta_1]f(z)}{z^p} + \lambda\frac{(H_{s,r}^p[\alpha_1, \beta_1]f(z))'}{pz^{p-1}}.$$

When $g(z)$ is as defined in (4), the class $\mathbb{T}_g^p(\lambda, \alpha, \beta)$ reduces to the subclass $\mathcal{LT}(\lambda, \alpha, \beta)$ with (9) and
$$\mathbb{J}_{L,\lambda}(z) = (1 - \lambda)\frac{L_p(a, c)f(z)}{z^p} + \frac{\lambda(L_p(a, c)f(z))'}{pz^{p-1}}.$$

When $g(z)$ is as defined in (5), the class $\mathbb{T}_g^p(\lambda,\alpha,\beta)$ reduces to the subclass $\mathcal{RT}^\nu(\lambda,\alpha,\beta)$ with (9) and

$$\mathbb{J}_{\nu,\lambda}(z) = (1-\lambda)\frac{D^{\nu+p-1}f(z)}{z^p} + \lambda\frac{(D^{\nu+p-1}f(z))'}{pz^{p-1}}.$$

When $g(z)$ is as defined in (7), the class $\mathbb{T}_g^p(\lambda,\alpha,\beta)$ reduces to the subclass $\mathcal{IT}(\lambda,\alpha,\beta)$ with (9) and

$$\mathbb{J}_{l,\lambda}(z) = (1-\lambda)\frac{\mathcal{I}_p(m,l)f(z)}{z^p} + \lambda\frac{(\mathcal{I}_p(m,l))'}{pz^{p-1}}.$$

When $g(z)$ is as defined in (8), the class $\mathbb{T}_g^p(\lambda,\alpha,\beta)$ reduces to the subclass $\mathcal{JT}(\lambda,\alpha,\beta)$ with (9) and

$$\mathbb{J}_{n,\lambda}(z) = (1-\lambda)\frac{\mathcal{J}_p(l,m)f(z)}{z^p} + \lambda\frac{(\mathcal{J}_p(l,m))'}{pz^{p-1}}.$$

When $g(z)$ is as defined in (6), the class $\mathbb{T}_g^p(\lambda,\alpha,\beta)$ reduces to the subclass $\mathcal{ST}(\lambda,\alpha,\beta)$ with (9) and

$$\mathbb{J}_{m,\lambda}(z) = (1-\lambda)\frac{\mathcal{D}_p^m f(z)}{z^p} + \lambda\frac{\mathcal{D}_p^{m+1}f(z)}{pz^{p-1}}.$$

The purpose of the present paper is to investigate the various properties and characteristics of functions belonging to the above defined subclass $\mathbb{T}_g^p(\lambda,\alpha,\beta)$ of multivalent functions in the open unit disk U. Apart from deriving a set of coefficient bounds for this function class, we also establish distortion bounds and several inclusion relationships involving the multivalent functions with negative coefficients belonging to this subclass.

2. Coefficient bounds

In this section we obtain coefficient estimates and extreme points for the class $\mathbb{T}_g^p(\lambda,\alpha,\beta)$.

Theorem 1. *Let the function f be defined by (11). Then $f \in \mathbb{T}_g^p(\lambda,\alpha,\beta)$ if and only if*

$$\sum_{n=p+1}^{\infty}(p+n\lambda)(1+\beta)a_n\, b_n \le 2p\beta(1-\alpha) \tag{12}$$

Proof. Suppose f satisfies (12). Then for $|z| = r < 1$,

$$\begin{aligned}
|\mathbb{J}_{g,\lambda}(z) - 1| - \beta\,|\mathbb{J}_{g,\lambda}(z) + (1-2\alpha)| &= \left| - \sum_{n=p+1}^{\infty} \frac{(p+n\lambda)}{p}(1+\beta)a_n b_n z^{n-p}\right| \\
&\quad - \beta\left|2(1-\alpha) - \sum_{n=p+1}^{\infty} \frac{(p+n\lambda)}{p} a_n b_n z^{n-p}\right| \\
&\le \sum_{n=p+1}^{\infty} \frac{(p+n\lambda)}{p} a_n b_n - 2\beta(1-\alpha) \\
&\quad + \sum_{n=p+1}^{\infty} \frac{(p+n\lambda)}{p} \beta a_n b_n \\
&= \sum_{n=p+1}^{\infty} \frac{(p+n\lambda)}{p}[1+\beta] a_n b_n - 2\beta(1-\alpha) \\
&\le 0, \qquad \text{by (12)}.
\end{aligned}$$

Hence, by maximum modulus theorem and (9), $f \in \mathbb{T}_g^p(\lambda,\alpha,\beta)$. To prove the converse, assume that

$$\left|\frac{\mathbb{J}_{g,\lambda}(z) - 1}{\mathbb{J}_{g,\lambda}(z) + (1-2\alpha)}\right| = \left|\frac{-\sum_{n=p+1}^{\infty} \frac{(p+n\lambda)}{p} a_n b_n z^{n-p}}{2(1-\alpha) - \sum_{n=p+1}^{\infty} \frac{(p+n\lambda)}{p} a_n b_n z^{n-p}}\right| \le \beta, \quad z \in U.$$

Thus,

$$\operatorname{Re}\left\{\frac{\sum_{n=p+1}^{\infty} \frac{(p+n\lambda)}{p} a_n b_n z^{n-p}}{2(1-\alpha) - \sum_{n=p+1}^{\infty} \frac{(p+n\lambda)}{p} a_n b_n z^{n-p}}\right\} < \beta. \tag{13}$$

Choose values of z on the real axis so that $\mathbb{J}_{g,\lambda}(z)$ is real. Upon clearing the denominator in (13) and letting $z \to 1^-$ through real values, we obtain the desired inequality (12). □

Corollary 1. *If $f(z)$ of the form (11) is in $\mathbb{T}_g^p(\lambda,\alpha,\beta)$, then*

$$a_n \le \frac{2p\beta(1-\alpha)}{(p+n\lambda)[1+\beta]b_n}, \quad n \ge p+1, \tag{14}$$

with the equality only for functions of the form

$$f(z) = z^p - \frac{2p\beta(1-\alpha)}{(p+n\lambda)[1+\beta]b_n} z^n, \quad n \ge p+1. \tag{15}$$

Corresponding to the various subclasses which arise from the function class $\mathbb{T}_g^p(\lambda,\alpha,\beta)$, by suitably choosing the function $g(z)$ as mentioned in (3) to (8), we arrive at the following corollaries giving the coefficient bound inequalities for these subclasses of functions.

Corollary 2. *A function $f \in \mathcal{HT}(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{n=p+1}^{\infty} (p+n\lambda)[1+\beta]a_n \ \Gamma_n \leq 2p\beta(1-\alpha).$$

where

$$\Gamma_n = \frac{1}{(n-p)!} \frac{(\alpha_1)_{n-p} \ldots (\alpha_l)_{n-p}}{(\beta_1)_{n-p} \ldots (\beta_m)_{n-p}} \tag{16}$$

Remark 3. *For specific choices of parameters α_1, β_1, s, r as stated in Remark 1, Corollary 2 would yield the coefficient bound inequalities for the subclasses of functions $\mathcal{LT}(\lambda, \alpha, \beta)$ and $\mathcal{RT}^{\nu}(\lambda, \alpha, \beta)$).*

Corollary 3. *A function $f \in \mathcal{IT}(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{n=p+1}^{\infty} (p+n\lambda)[1+\beta]n \left(\frac{n+l}{p+l}\right)^m a_n \ \leq 2p\beta(1-\alpha).$$

Corollary 4. *A function $f \in \mathcal{JT}(\lambda, \alpha, \beta)$ if and only if*

$$\sum_{n=p+1}^{\infty} (p+n\lambda)[1+\beta] \left(\frac{n+l}{p+l}\right)^m a_n \ \leq 2p\beta(1-\alpha).$$

Remark 4. *When $l = 0$, Corollary 4 would give the coefficient bound inequality for the subclass of functions$\mathcal{ST}(\lambda, \alpha, \beta)$.*

Theorem 2 (Extreme points). *Let*

$$f_p(z) = z^p$$

and

$$f_n(z) = z^p - \frac{2p\beta(1-\alpha)}{(p+n\lambda)[1+\beta]b_n} z^n, \ \ n \geq p+1, \tag{17}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $0 \leq \lambda \leq 1$. Then $f(z)$ is in the class $\mathbb{T}_g^p(\lambda, \alpha, \beta)$ if and only if it can be expressed in the form

$$f(z) = \mu_p z^p + \sum_{n=p+1}^{\infty} \mu_n f_n(z), \tag{18}$$

where $\mu_n \geq 0$ and $\sum_{n=p}^{\infty} \mu_n = 1$.

Proof. Suppose $f(z)$ can be written as in (18). Then

$$\begin{aligned} f(z) &= \mu_p z^p - \sum_{n=p+1}^{\infty} \mu_n \left[z^p - \frac{2p\beta(1-\alpha)}{(p+n\lambda)[1+\beta]b_n} z^n \right] \\ &= z^p - \sum_{n=p+1}^{\infty} \mu_n \frac{2p\beta(1-\alpha)}{(p+n\lambda)[1+\beta]b_n} z^n. \end{aligned}$$

Now,

$$\sum_{n=p+1}^{\infty} \frac{(p+n\lambda)[1+\beta]b_n}{2p\beta(1-\alpha)} \mu_n \frac{2p\beta(1-\alpha)}{(p+n\lambda)[1+\beta]b_n} = \sum_{n=p+1}^{\infty} \mu_n = 1 - \mu_1 \leq 1.$$

Thus $f \in \mathbb{T}_g^p(\lambda, \alpha, \beta)$. Conversely, let us have $f \in \mathbb{T}_g^p(\lambda, \alpha, \beta)$. Then by using (14), we set

$$\mu_n = \frac{(p+n\lambda)[1+\beta]b_n}{2p\beta(1-\alpha)} a_n, \quad n \geq p+1$$

and $\mu_p = 1 - \sum_{n=p+1}^{\infty} \mu_n$. Then we have $f(z) = \sum_{n=p+1}^{\infty} \mu_n f_n(z)$ and hence this completes the proof of Theorem 2. □

Corollary 5. *Let*

$$f_p(z) = z^p$$

and

$$f_n(z) = z^p - \frac{2p\beta(1-\alpha)}{\Gamma_n(p+n\lambda)[1+\beta]} z^n, \ \ n \geq p+1, \tag{19}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\lambda \geq 0$. *Then* $f(z)$ *is in the class* $\mathcal{HT}(\lambda, \alpha, \beta)$ *if and only if it can be expressed in the form*

$$f(z) = \mu_p z^p + \sum_{n=p+1}^{\infty} \mu_n f_n(z),$$

where

$$\mu_n \geq 0, \quad \sum_{n=p}^{\infty} \mu_n = 1$$

and Γ_n *is given by (16).*

Remark 5. *For specific choices of parameters* r, s, α_1, β_1 *(as mentioned in the Remarks 1 and 2), Corollary 5 we can prove analogous results for the subclasses of functions* $\mathcal{LT}(\lambda, \alpha, \beta)$ *and* $\mathcal{RT}^{\nu}(\lambda, \alpha, \beta)$. *Further on lines similar to the above theorem one can easily prove the extreme points results for the classes* $\mathcal{IT}(\lambda, \alpha, \beta)$ *and* $\mathcal{JT}(\lambda, \alpha, \beta)$.

3. Distortion bounds

In this section we obtain distortion bounds for the class $\mathbb{T}_g^p(\lambda, \alpha, \beta)$.

Theorem 3. *If* $f \in \mathbb{T}_g^p(\lambda, \alpha, \beta)$, *then*

$$r^p - \frac{2p\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]b_{p+1}} r^{p+1} \leq |f(z)| \ \leq r^p + \frac{2p\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]b_{p+1}} r^{p+1} \tag{20}$$

holds if the sequence $\{\sigma_n\}$ *is non-increasing for* $n > p$ *and*

$$pr^{p-1} - \frac{2p(p+1)\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]b_{p+1}}r^p \leq |f'(z)| \leq pr^{p-1} + \frac{2p(p+1)\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]b_{p+1}}r^p \tag{21}$$

holds if the sequence $\{\sigma_n/n\}$ *is non-increasing for* $n > p$*, where* $\sigma_n = (p+n\lambda)b_n, (n > p)$.

The bounds in (20) and (21) are sharp, since the equalities are attained by the function

$$f(z) = z^p - \frac{2p\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]b_{p+1}}z^{p+1} \quad |z| = \pm r. \tag{22}$$

Proof. In view of Theorem 1, we have

$$\sum_{n=p+1}^{\infty} a_n \leq \frac{2p\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]b_{p+1}} \tag{23}$$

Using (11) and (23), we obtain

$$|z|^p - |z|^{p+1}\sum_{n=p+1}^{\infty} a_n \leq |f(z)| \leq |z|^p + |z|^{p+1}\sum_{n=p+1}^{\infty} a_n$$

$$r^p - r^{p+1}\frac{2p\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]b_{p+1}} \leq |f(z)| \leq r^p + r^{p+1}\frac{2p\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]b_{p+1}}. \tag{24}$$

Hence (20) follows from (24).
Further, since

$$\sum_{n=p+1}^{\infty} na_n \leq \frac{2p(p+1)\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]b_{p+1}}.$$

Hence (21) follows from

$$pr^{p-1} - (p+1)r^p\sum_{n=p+1}^{\infty} a_n \leq |f'(z)| \leq pr^{p-1} + (p+1)r^p\sum_{n=p+1}^{\infty} a_n.$$

□

Corollary 6. *If* $f \in \mathcal{HT}(\lambda,\alpha,\beta)$*, then*

$$r^p - \frac{2p\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]\Gamma_{p+1}}r^{p+1} \leq |f(z)| \leq r^p + \frac{2p\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]\Gamma_{p+1}}r^{p+1} \tag{25}$$

holds if the sequence $\{\sigma_n\}$ *is non-increasing for* $n > p$ *and*

$$pr^{p-1} - \frac{2p(p+1)\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]\Gamma_{p+1}}r^p \leq |f'(z)| \leq pr^{p-1} + \frac{2p(p+1)\beta(1-\alpha)}{(p+p\lambda+\lambda)[1+\beta]\Gamma_{p+1}}r^p \tag{26}$$

holds if the sequence $\{\sigma_n/n\}$ *is non-increasing for* $n > p$, *where* $\sigma_n = (p + n\lambda)\Gamma_n$, $(n > p)$.

The bounds in (25) and (26) are sharp for

$$f(z) = z^p - \frac{2p\beta(1-\alpha)}{(p + p\lambda + \lambda)[1+\beta]\Gamma_{p+1}} z^{p+1} \tag{27}$$

where Γ_{p+1} is given by (16).

Remark 6. *For specific choices of parameters* r, s, α_1, β_1 *(as mentioned in Remarks 1 and 2), Corollary 6 we can deduce analogous results for the subclasses of functions* $\mathcal{LT}(\lambda, \alpha, \beta)$ *and* $\mathcal{RT}^\nu(\lambda, \alpha, \beta)$. *Further on lines similar to the distortion theorem one can easily prove the distortion bounds for functions in the classes* $\mathcal{IT}(\lambda, \alpha, \beta)$ *and* $\mathcal{JT}(\lambda, \alpha, \beta)$.

Corollary 7. *If* $f \in \mathcal{ST}(\lambda, \alpha, \beta)$, *then*

$$r^p - \frac{p\beta(1-\alpha)}{2^{m-1}(p + p\lambda + \lambda)[1+\beta]} r^{p+1} \le |f(z)| \ \le r^p + \frac{p\beta(1-\alpha)}{2^{m-1}(p + p\lambda + \lambda)[1+\beta]} r^{p+1} \tag{28}$$

$$pr^{p-1} - \frac{pp+1\beta(1-\alpha)}{2^{m-1}(p + p\lambda + \lambda)[1+\beta]} r^p \le |f'(z)| \ \le r^p + \frac{p(p+1)\beta(1-\alpha)}{2^{m-1}(p + p\lambda + \lambda)[1+\beta]} r^p. \tag{29}$$

The bounds in (28) and (29) are sharp for

$$f(z) = z^p - \frac{p\beta(1-\alpha)}{2^{m-1}(p + p\lambda + \lambda)[1+\beta]} z^{p+1}. \tag{30}$$

4. Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness and convexity for the class $\mathbb{T}_g^p(\lambda, \alpha, \beta)$ are given in this section.

Theorem 4. *Let the function* $f(z)$ *defined by (11) belong to the class* $\mathbb{T}_g^p(\lambda, \alpha, \beta)$. *Then* $f(z)$ $p-$*valently is close-to-convex of order* δ $(0 \le \delta < p)$ *in the disc* $|z| < r_1$, *where*

$$r_1 := \inf_{n \ge p+1} \left[\frac{(p-\delta)(p + n\lambda)[1+\beta]\ b_n}{2pn\beta(1-\alpha)} \right]^{\frac{1}{n-p}} \quad . \tag{31}$$

Proof. Given $f \in T$ and f is close-to-convex of order δ, we have

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p - \delta. \tag{32}$$

For the left-hand side of (32) we have

$$|\frac{f'(z)}{z^{p-1}} - p| \le \sum_{n=p+1}^{\infty} n a_n |z|^{n-p}.$$

The last expression is less than $p-\delta$ if

$$\sum_{n=p+1}^{\infty} \frac{n}{p-\delta} a_n |z|^{n-p} < 1.$$

Using the fact, that $f \in \mathbb{T}_g^p(\lambda, \alpha, \beta)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{(p+n\lambda)[1+\beta]a_n b_n}{2p\beta(1-\alpha)} \leq 1.$$

We can say (32) is true if

$$\frac{n}{p-\delta}|z|^{n-p} \leq \frac{(p+n\lambda)[1+\beta]b_n}{2p\beta(1-\alpha)}.$$

Or, equivalently,

$$|z|^{n-p} = \left[\frac{(p-\delta)(p+n\lambda)[1+\beta]\ b_n}{2pn\beta(1-\alpha)}\right]$$

which completes the proof. □

Theorem 5. *Let $f \in \mathbb{T}_g^p(\lambda, \alpha, \beta)$. Then*

1. *f is $p-$valently starlike of order $\delta (0 \leq \delta < p)$ in the disc $|z| < r_2$; that is, $Re\ \left\{\frac{zf'(z)}{f(z)}\right\} > \delta$, $(|z| < r_2\ ;\ 0 \leq \delta < p)$, where*

$$r_2 = \inf_{n \geq p+1} \left\{\frac{(p-\delta)(p+n\lambda)[1+\beta]\ b_n}{2p\beta(1-\alpha)(k+p-\delta)}\right\}^{\frac{1}{n}}.$$

2. *f is $p-$valently convex of order δ $(0 \leq \delta < p)$ in the disc $|z| < r_3$, that is $Re\ \left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta$, $(|z| < r_3; 0 \leq \delta < p)$, where*

$$r_3 = \inf_{n \geq p+1} \left\{\frac{(p-\delta)(p+n\lambda)[1+\beta]\ b_n}{2n\beta(1-\alpha)(n-\delta)}\right\}^{\frac{1}{n}}.$$

Each of these results are sharp for the extremal function $f(z)$ given by (17).

Proof. Given $f \in T$ and f is $p-$valently starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| < p-\delta. \tag{33}$$

For the left-hand side of (33) we have

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum_{n=p+1}^{\infty}(n-p)a_n\ |z|^n}{1-\sum_{n=p+1}^{\infty} a_n\ |z|^n}.$$

The last expression is less than $p-\delta$ if

$$\sum_{n=p+1}^{\infty}\frac{n-\delta}{p-\delta}a_n\;|z|^n<1.$$

Using the fact, that $f\in\mathbb{T}_g^p(\lambda,\alpha,\beta)$ if and only if

$$\sum_{n=p+1}^{\infty}\frac{(p+n\lambda)[1+\beta]a_nb_n}{2p\beta(1-\alpha)}\;<1.$$

We can say (33) is true if

$$\frac{n-\delta}{p-\delta}|z|^n<\frac{(p+n\lambda)[1+\beta]\;b_n}{2p\beta(1-\alpha)}.$$

Or, equivalently,

$$|z|^n<\frac{(p-\delta)(p+n\lambda)[1+\beta]\;b_n}{2p\beta(1-\alpha)(n-\delta)}$$

which yields the starlikeness of the family.
(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (2), on lines similar the proof of (1). □

Concluding remarks: Corresponding to the various subclasses which arise from the function class $\mathbb{T}_g^p(\lambda,\alpha,\beta)$, by suitably choosing the function $g(z)$ as mentioned in (3) to (8), and for specific choices of parameters α_1,β_1,r,s we can deduce analogous results for the subclasses of functions introduced in this paper. Furthermore, by suitably choosing the values of g, α, β,λ, and $p=1$ the class $\mathbb{T}_g^p(\lambda,\alpha,\beta)$ and the above subclasses reduce to the various subclasses introduced and studied in the literature (see [1, 2, 7, 9, 13, 17]).

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