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The Euler characteristic of the symmetric product of a finite CW-complex*

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Abstract. In this paper it is first shown that if M is a 2-dimensional surface, then $M^{(m)}$ is orientable if and only if M is orientable. Using the Macdonald's result [5] the Betti numbers of $M^{(m)}$ are expressed explicitely, the Euler characteristic is found and it depends only on m and the Euler characteristic of M. Moreover, the formula for the Euler characteristic is proved alternatively also without using the Macdonalds's results.
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1. Some preliminaries about symmetric products of manifolds

Let M be an arbitrary set and m a positive integer. In the m-fold Cartesian product M^m we define a relation $\approx$ such that

$$(x_1, \cdots, x_m) \approx (y_1, \cdots, y_m) \Leftrightarrow$$

there exists a permutation $\theta : \{1, 2, \cdots, m\} \to \{1, 2, \cdots, m\}$ such that $y_i = x_{\theta(i)}$, $(1 \leq i \leq m)$.

This is a relation of equivalence and the class represented by $(x_1, \cdots, x_m)$ will be denoted by $[x_1, \cdots, x_m]$ and the set $M^m/\approx$ will be denoted by $M^{(m)}$. The set $M^{(m)}$ is called a symmetric product of M. Some authors call it a permutation product of M.

If M is a topological space, then $M^{(m)}$ is also a topological space. The space $M^{(m)}$ was introduced quite early [9], and it was studied in [8, 4, 7, 1, 10], and in the Ph.D. thesis of Wagner [17]. Some recent results are obtained in [16, 2, 3, 11, 12]. If M is an arbitrary connected manifold and $m > 1$, then it is proved in [9] that

$$\pi_1(M^{(m)}) \cong H_1(M, \mathbb{Z}). \tag{1}$$

Another important result [17] states that $(\mathbb{R}^n)^{(m)}$ is a manifold only for $n = 2$. If $n = 2$, then $(\mathbb{R}^2)^{(m)} = \mathbb{C}^{(m)}$ is homeomorphic to $\mathbb{C}^m$. Indeed, using that $\mathbb{C}$ is an algebraically closed field, it is obvious that the map $\varphi : \mathbb{C}^{(m)} \to \mathbb{C}^m$ defined by

$$\varphi[z_1, \cdots, z_m] = (\sigma_1(z_1, \cdots, z_m), \sigma_2(z_1, \cdots, z_m), \cdots, \sigma_m(z_1, \cdots, z_m))$$

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is a bijection, where σ_i $(1 \leq i \leq m)$ is the i-th symmetric function of $z_1, \cdots, z_m$, i.e.

$$\sigma_i(z_1, \cdots, z_m) = \sum_{1 \leq j_1 < j_2 < \ldots < j_i \leq m} z_{j_1} \cdot z_{j_2} \cdots z_{j_i}.$$

The map φ is also a homeomorphism. If M is a 1-dimensional complex manifold, then $M^{(m)}$ is a complex manifold, too. For example, if M is a 2-sphere, i.e. a complex manifold $\mathbb{C}P^1$, then $M^{(m)}$ is the projective complex space $\mathbb{C}P^m$. Using the symmetric products it is easy to see how $M^{(m)} = \mathbb{C}P^m$ decomposes into disjoint cells $\mathbb{C}^0, \mathbb{C}^1, \cdots, \mathbb{C}^m$. Let $\xi \in M$. Then we define $[x_1, \cdots, x_m] \in M_i$ if exactly i of the elements $x_1, \cdots, x_m$ are equal to ξ. Thus

$$\begin{aligned} M^{(m)} &= M_0 \cup M_1 \cup \cdots \cup M_m = (M \setminus \xi)^{(m)} \cup (M \setminus \xi)^{(m-1)} \cup \cdots \cup (M \setminus \xi)^{(0)} \\ &= \mathbb{C}^{(m)} \cup \mathbb{C}^{(m-1)} \cup \cdots \cup \mathbb{C}^{(0)} = \mathbb{C}^m \cup \mathbb{C}^{m-1} \cup \cdots \cup \mathbb{C}^0. \end{aligned}$$

This theory about symmetric products has an important role in the theory of topological commutative vector valued groups [14, 15, 13].

The Poincaré polynomial of a symmetric product of a compact polyhedron is given in [5]. If B_0, B_1, B_2, $\cdots$ are the Betti numbers of a space M, then the Poincaré polynomial of the mth symmetric product of M is the coefficient in front of t^m in the power series expansion of

$$\frac{(1+xt)^{B_1}(1+x^3t)^{B_3}\cdots}{(1-t)^{B_0}(1-x^2t)^{B_2}(1-x^4t)^{B_4}\cdots}. \tag{2}$$

Moreover, the homology of the symmetric products has been determined completely by J.Milgram [6].

In this paper some consequences of the results of Macdonald are given and an alternative proof for the expression of the Euler characteristic of the symmetric product of M is given, too.

2. Some conclusions about the symmetric products

First, we prove the following theorem which gives a necessary and sufficient condition for the orientability of a 2-dimensional surface.

Theorem 1. *Let M be a 2-dimensional manifold. The manifold $M^{(m)}$ is orientable if and only if M is orientable.*

Proof. If M is orientable, then M is a complex manifold. Thus, $M^{(m)}$ is also a complex manifold and hence $M^{(m)}$ is orientable.

Assume that M is non-orientable. Then there exist a non-orientable open subset T of M and an open subset U which is homeomorphic to $\mathbb{R}^2$, such that $T \cap U = \emptyset$. Then $M^{(m)}$ contains the non-orientable submanifold $T \times U^{(m-1)} \cong T \times \mathbb{R}^{2m-2}$ with the same dimension and hence $M^{(m)}$ is not orientable. □

For example, it is known that $(\mathbb{R}P^2)^{(m)} = \mathbb{R}P^{2m}$, and both spaces $\mathbb{R}P^2$ and $\mathbb{R}P^{2m}$ are not orientable.

Let M be a finite CW complex and let us denote by

$$B_i = dim(H_i(M,\mathbb{Z})), \quad i = 0,1,2,\cdots,n = dimM$$

and let us denote the Betti numbers of the CW complex $M^{(m)}$ by

$$B_i^{(m)} = dim(H_i(M^{(m)},\mathbb{Z})), \quad i = 0,1,2,\cdots,nm.$$

As a direct consequence of the result of Macdonald [5] we have the following two theorems.

Theorem 2. *The Betti numbers* $B_i^{(m)}$ $\quad i = 0,1,2,\cdots,nm$, *are given explicitly by*

$$B_i^{(m)} = (-1)^i \sum_{\alpha_0,\cdots,\alpha_n} \binom{\alpha_0 - 1 + B_0}{\alpha_0} \cdot \binom{\alpha_1 - 1 - B_1}{\alpha_1} \times \binom{\alpha_2 - 1 + B_2}{\alpha_2} \cdots \binom{\alpha_n - 1 + (-1)^n B_n}{\alpha_n}, \tag{3}$$

where the summation over $\alpha_0,\cdots,\alpha_n$ *is under the following restrictions*

$$0 \le \alpha_0,\cdots,\alpha_n \le m, \ \alpha_0 + \cdots + \alpha_n = m, \ \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = i. \tag{4}$$

Proof. Using the equality

$$\binom{\alpha - 1 + B}{\alpha} = (-1)^\alpha \binom{-B}{\alpha},$$

for a positive integer α and integer B, we should prove that

$$B_i^{(m)} = (-1)^{i+m} \sum_{\alpha_0,\cdots,\alpha_n} \binom{-B_0}{\alpha_0} \cdot \binom{B_1}{\alpha_1} \cdot \binom{-B_2}{\alpha_2} \cdots \binom{(-1)^{n+1} B_n}{\alpha_n},$$

where the summation over $\alpha_1,\cdots,\alpha_n$ is given by (4). Since

$$\frac{(1+xt)^{B_1}(1+x^3t)^{B_3}\cdots}{(1-t)^{B_0}(1-x^2t)^{B_2}(1-x^4t)^{B_4}\cdots} = \sum_{m=0}^{\infty}\sum_{i=0}^{mn} B_i^{(m)} x^i t^m,$$

it is sufficient to prove that

$$(1-t)^{-B_0}(1+xt)^{B_1}(1-x^2t)^{-B_2}(1+x^3t)^{B_3}(1-x^4t)^{-B_4}\cdots$$
$$= \sum_{m=0}^{\infty}\sum_{i=0}^{mn}\sum_{\alpha_0,\cdots,\alpha_n} (-1)^{i+m} x^i t^m \sum_{\alpha_0,\cdots,\alpha_n} \binom{-B_0}{\alpha_0} \cdot \binom{B_1}{\alpha_1} \times \binom{-B_2}{\alpha_2} \cdots \binom{(-1)^{n+1} B_n}{\alpha_n},$$

where the summation over $\alpha_1,\cdots,\alpha_n$ is given by (4). Notice that $(-1)^{i+m} = (-1)^{\alpha_0+\alpha_2+\alpha_4+\cdots}$. Applying the binomial formula for the powers of the left-hand side and choosing the summand $(-1)^{\alpha_0}\binom{-B_0}{\alpha_0}t^{\alpha_0}$ from $(1-t)^{-B_0}$, $\binom{B_1}{\alpha_1}t^{\alpha_1}x^{\alpha_1}$ from $(1+xt)^{B_1}$, $(-1)^{\alpha_2}\binom{-B_2}{\alpha_2}t^{\alpha_2}x^{2\alpha_2}$ from $(1-x^2t)^{-B_2}$ and so on, and sum over $\alpha_0,\alpha_1,\alpha_2,\cdots$, according to (4), we obtain that the previous equality is true. $\square$

Theorem 3. *Euler characteristics of M and the symmetric product $M^{(m)}$ are related by*

$$\chi(M^{(m)}) = \binom{m+\chi(M)-1}{m}. \tag{5}$$

Proof.

$$\sum_{i=0}^{nm}(-1)^i B_i^{(m)} = \sum_{i=0}^{nm}\sum_{\alpha_0,\cdots,\alpha_n}(-1)^{\alpha_0}\binom{-B_0}{\alpha_0}(-1)^{\alpha_1}\binom{B_1}{\alpha_1}\cdots(-1)^{\alpha_n}\binom{(-1)^{n+1}B_n}{\alpha_n},$$

where the summation over $\alpha_0,\cdots,\alpha_n$ satisfies conditions (4). Since $\alpha_1+2\alpha_2+\cdots+n\alpha_n = i$ and we sum over i, using that $\alpha_0+\cdots+\alpha_n = m$ we obtain

$$\begin{aligned}\chi(M^{(m)}) &= \sum_{i=0}^{nm}(-1)^i B_i^{(m)} = \sum_{\alpha_0+\cdots+\alpha_n=m}(-1)^m\binom{-B_0}{\alpha_0}\binom{B_1}{\alpha_1}\cdots\binom{(-1)^{n+1}B_n}{\alpha_n}\\ &= (-1)^m\binom{-B_0+B_1-B_2+\cdots+(-1)^{n+1}B_n}{m}\\ &= (-1)^m\binom{-\chi(M)}{m} = \binom{m+\chi(M)-1}{m}.\end{aligned}$$

□

Notice that Theorem 3 also follows from (2). Indeed if we put $x=-1$, then the Euler characteristic of $M^{(m)}$ is the coefficient in front of t^m. Indeed, the expression from (2) for $x=-1$ becomes

$$\begin{aligned}(1-t)^{-B_0}(1-t)^{B_1}(1-t)^{-B_2}(1-t)^{B_3}\cdots = (1-t)^{-\chi(M)} &= \sum_{m=0}^{\infty}(-1)^m t^m\binom{-\chi(M)}{m}\\ &= \sum_{m=0}^{\infty} t^m\binom{m+\chi(M)-1}{m},\end{aligned}$$

and hence the proof of Theorem 3 follows.

Theorem 4. *If M is a compact CW complex such that $\chi(M)=0$, then $M^{(m)}$ cannot be decomposed into cells of type $\mathbb{C}^i$.*

Proof. If $\chi(M)=0$, then $\chi(M^{(m)})=0$. If $M^{(m)}$ can be decomposed into cells of type $\mathbb{C}^i$, then its Euler characteristic should be a positive number, which contradicts to $\chi(M^{(m)})=0$. □

Notice that if M is a torus, then it is a complex manifold, but its symmetric product cannot be decomposed into complex cells $\mathbb{C}^0, \mathbb{C}, \mathbb{C}^2, \cdots$

3. Direct proof of the Theorem 3

In this section we present a direct proof of Theorem 3, without using the Macdonald's results.

Here Euler characteristic χ will denote the number of even dimensional cells of type $\mathbb{R}^i$ minus the number of odd dimensional cells of the same type where all of the cells are disjoint.

Proof. Notice that (5) is trivially satisfied for $m = 1$, because $M^{(1)} = M$. So, without loss of generality, in the proof we assume that $m \geq 2$. First, let us prove formula (5) for the cells $C = \mathbb{R}^n$. The proof is by induction of n. If $n = 0$, i.e. C is a point, then $C^{(m)}$ is also a point, and (5) is true because $\chi(C) = \chi(C^{(m)}) = 1$. Further, if $n = 1$, i.e. $C = \mathbb{R}$, then according to [17], $C^{(m)}$ is homeomorphic to the half space H^m and (5) holds because $\chi(C) = -1$ and $\chi(C^{(m)}) = 0$. If $C = \mathbb{R}^2$, then $C^{(m)} = \mathbb{R}^{2m}$ and (5) is true, because $\chi(C) = \chi(C^{(m)}) = 1$.

Let us assume that $n \geq 3$. We should prove that $\chi((\mathbb{R}^n)^{(m)})$ is equal to 0 if n is an odd number and it is 1 if n is an even number, for $m > 1$. In all calculations we will neglect all sets whose Euler characteristic is 0 according to the inductive assumption.

Assume that $n = 2k + 1$ and the statement is true for $2k$. We consider the space $\mathbb{R}^n$ as a disjoint union of a hyperplane Σ and the corresponding two open half spaces Σ_1 and Σ_2. We prove that $\chi((\mathbb{R}^{2k+1})^{(m)}) = 0$ by induction of m. Assume that it is true for $2, 3, \cdots, m - 1$. According to the inductive assumption, it is sufficient to consider only those cells where only one point, m points or no points are chosen in Σ_1 (also Σ_2). But if m points are chosen in Σ_1 (resp. Σ_2), then in Σ_2 (resp. Σ_1) there is no one chosen point. Hence there we should consider only these six cases: 1) all points are chosen from Σ_1, 2) all points are chosen from Σ_2, 3) all points are chosen from Σ, 4) one point is chosen from Σ_1, one point from Σ_2 and the rest $m-2$ points are chosen from Σ, 5) one point is chosen from Σ_1 and the rest $m - 1$ points are chosen from Σ, 6) one point is chosen from Σ_2 and the rest $m - 1$ points are chosen from Σ. Using the inductive assumption for $n - 1 = dim\Sigma$, for the Euler characteristic of $(\mathbb{R}^n)^{(m)}$ we obtain the equality

$$\begin{aligned}\chi((\mathbb{R}^n)^{(m)}) &= \chi(\Sigma_1^{(m)}) + \chi(\Sigma_2^{(m)}) + \chi(\Sigma^{(m)}) + \chi(\Sigma_1) \times \chi(\Sigma_2) \times \chi(\Sigma^{(m-2)}) \\ &\quad + \chi(\Sigma_1) \times \chi(\Sigma^{(m-1)}) + \chi(\Sigma_2) \times \chi(\Sigma^{(m-1)}) \\ &= 2\chi((\mathbb{R}^n)^{(m)}) + 1 + 1 - 1 - 1\end{aligned}$$

and hence we obtain $\chi((\mathbb{R}^n)^{(m)}) = 0$.

Assume that $n = 2k$ and the statement is true for $2k - 1$. We consider the space $\mathbb{R}^n$ as a disjoint union of a hyperplane Σ and two open half spaces Σ_1 and Σ_2 as in the previous case. We prove that $\chi((\mathbb{R}^{2k})^{(m)}) = 1$ by induction of m. Assume that it is true for $2, 3, \cdots, m - 1$. According to the inductive assumption, it is sufficient to consider only those cells where one point or no points are chosen in Σ. Hence we obtain the following equality

$$\begin{aligned}\chi((\mathbb{R}^n)^{(m)}) &= \sum_{i=0}^{m}\chi(\Sigma_1^{(i)})\cdot\chi(\Sigma_2^{(m-i)}) + \sum_{i=0}^{m-1}\chi(\Sigma_1^{(i)})\cdot\chi(\Sigma_2^{(m-1-i)})\cdot\chi(\Sigma)\\ &= 2\chi((\mathbb{R}^n)^{(m)}) + (m-1) + m(-1)^{2k-1} = 2\chi((\mathbb{R}^n)^{(m)}) - 1,\end{aligned}$$

and hence $\chi((\mathbb{R}^n)^{(m)}) = 1$.

To complete the proof it is sufficient to prove that if (5) is true for arbitrary m and disjoint sets X_1 and X_2 of cells of the CW complex M, then formula (5) is true for $X_1 \cup X_2$. Let $\chi(X_1) = p$, $\chi(X_2) = q$ and assume that

$$\begin{aligned}\chi(X_1^{(i)}) &= \binom{i+p-1}{i} \quad \text{for} \quad i = 0,1,2,\cdots,\\ \chi(X_2^{(i)}) &= \binom{i+q-1}{i} \quad \text{for} \quad i = 0,1,2,\cdots.\end{aligned}$$

Then

$$\begin{aligned}\chi\Big((X_1\cup X_2)^{(m)}\Big) &= \chi\Big[X_1^{(m)} \cup (X_1^{(m-1)}\times X_2)\cup(X_1^{(m-2)}\times X_2^{(2)})\cup\cdots\cup X_2^{(m)}\Big]\\ &= \sum_{s=0}^{m}\chi(X_1^{(s)})\cdot\chi(X_2^{(m-s)})\\ &= \sum_{s=0}^{m}\binom{s+p-1}{s}\binom{m-s+q-1}{m-s}\\ &= \sum_{s=0}^{m}(-1)^s\binom{-p}{s}(-1)^{m-s}\binom{-q}{m-s}\\ &= (-1)^m\sum_{s=0}^{m}\binom{-p}{s}\binom{-q}{m-s}\\ &= (-1)^m\binom{-p-q}{m}\\ &= \binom{m+p+q-1}{m}.\end{aligned}$$

□

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