THE STOKES CONJECTURE FOR WAVES WITH VORTICITY

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Abstract. We study stagnation points of two-dimensional steady gravity free-surface water waves with vorticity.

We obtain for example that, in the case where the free surface is an injective curve, the asymptotics at any stagnation point is given either by the "Stokes corner flow" where the free surface has a *corner of* $120°$, or the free surface ends in a *horizontal cusp*, or the free surface is *horizontally flat* at the stagnation point. The cusp case is a new feature in the case with vorticity, and it is not possible in the absence of vorticity.

In a second main result we exclude horizontally flat singularities in the case that the vorticity is 0 on the free surface. Here the vorticity may have infinitely many sign changes accumulating at the free surface, which makes this case particularly difficult and explains why it has been almost untouched by research so far.

Our results are based on calculations in the original variables and do not rely on structural assumptions needed in previous results such as isolated singularities, symmetry and monotonicity.

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1. Introduction

The classical hydrodynamical problem of traveling two-dimensional gravity water waves with vorticity can be described mathematically as a free-boundary problem for a semilinear elliptic equation: given an open connected set Ω in the (x, y) plane and a function γ of one variable, find a non-negative function ψ in Ω such that

$$\Delta\psi = -\gamma(\psi) \quad \text{in } \Omega \cap \{\psi > 0\}, \tag{1.1a}$$

$$|\nabla\psi(x, y)|^2 = -y \quad \text{on } \Omega \cap \partial\{\psi > 0\}. \tag{1.1b}$$

The present paper is an investigation by geometric methods of the singularities of the free boundary $\partial\{\psi > 0\}$.

Let us briefly describe, following [4], the connection between problem (1.1) and the nonlinear governing equations of fluid motion. Consider a wave of permanent form moving with constant speed on the free surface of an incompressible inviscid fluid, acted on by gravity. With respect to a frame of reference moving with the speed of the wave, the flow is steady and occupies a fixed region D in the plane. The boundary ∂D of the fluid region contains a part $\partial_a D$ which is free and in contact with an air region. Under the assumption that the fluid region D is simply connected, the incompressibility condition shows that the flow can be described by a *stream function* $\psi : D \to \mathbf{R}$, so that the relative fluid velocity is $(\psi_y, -\psi_x)$. The Euler equations imply that the *vorticity* $\omega := -\Delta\psi$ satisfies

$$\omega_x\psi_y = \omega_y\psi_x \quad \text{in } D. \tag{1.2}$$

It is easy to see that (1.2) is satisfied whenever

$$\omega = \gamma(\psi) \quad \text{in } D \tag{1.3}$$

for some (smooth) function γ of variable ψ, which will be referred to as a *vorticity function*. (Conversely, under additional assumptions, see [4], (1.2) implies the existence of such a function γ.) The kinematic boundary condition that the same particles always form the free surface $\partial_a D$ is equivalent to

$$\psi \text{ is locally constant on } \partial_a D.$$

Also, in the presence of (1.3), Bernoulli's Theorem and the fact that on the fluid-air interface $\partial_a D$ the pressure in the fluid equals the constant atmospheric pressure imply that

$$\frac{1}{2}|\nabla\psi|^2 + gy \text{ is locally constant on } \partial_a D,$$

where $g > 0$ is the gravitational constant. We therefore obtain, after some normalization, and at least in the case when $\partial_a D$ is connected, that the following equations and boundary conditions are satisfied:

$$-\Delta\psi = \gamma(\psi) \quad \text{in } D, \tag{1.4a}$$

$$\psi = 0 \quad \text{on } \partial_a D, \tag{1.4b}$$

$$|\nabla\psi|^2 + 2gy = 0 \quad \text{on } \partial_a D, \tag{1.4c}$$

Equations (1.4) are usually supplemented by suitable boundary conditions on the rest of the boundary of D, or some conditions on the flow at infinity if the fluid domain is unbounded. Classical types of waves which have received most attention in the literature are periodic and solitary waves of finite depth (in which the fluid domain D has a fixed flat bottom $y = -d$, at which ψ is constant), and periodic waves of infinite depth (in which the fluid domain extends to $y = -\infty$ and the condition $\lim_{y \to -\infty} \nabla\psi(x, y) = (0, -c)$ holds, where c is the speed of the wave). Conversely, for any vorticity function γ, any solution of (1.4) gives rise to a traveling free-surface gravity water wave, irrespective of whether D is simply connected or $\partial_a D$ is connected. Problem (1.1) is a local version of problem (1.4), under the additional assumption that $\psi > 0$ in the fluid region, and where ψ has been extended by the value 0 to the air region. In (1.1), the domain Ω is a neighborhood of a point of interest on the fluid-air interface, the fluid region D is identified with the set $\{(x, y) : \psi(x, y) > 0\}$ (in short $\{\psi > 0\}$) and the fluid-air interface $\partial_a D$ with $\partial\{\psi > 0\}$, while the gravitational constant g has been normalized by scaling. Note that problem (1.1) is also relevant for the description of more general steady flow configurations (for example, the fluid domain could have a non-flat bottom, and there could be some further external forcing acting at the boundary of the fluid region which is not in contact with the air region).

The theory of traveling water waves with vorticity has a long history, whose highlights include the pioneering paper of Gerstner [10], the first rigorous proof of existence of periodic waves of small amplitude by Dubreil-Jacotin [6], and the foundation [4] of Constantin and Strauss, which proved existence of smooth waves of large amplitude for the periodic finite-depth problem. The paper [4] has generated substantial interest and follow-up work on steady water waves with vorticity, see [20] for a survey of recent results.

In this paper we investigate the shape of the free boundary $\partial\{\psi > 0\}$ at *stagnation points*, which are points where the relative fluid velocity $(\psi_y, -\psi_x)$ is the zero vector. The Bernoulli condition (1.1b) shows that such points are on the real axis, while the rest of the free boundary is in the lower half-plane. Stokes [19] conjectured that, in the irrotational case $\gamma \equiv 0$, at any stagnation point the free surface has a (symmetric) corner of $120°$, and formal asymptotics suggest that the same result might be true also in the general case of waves with vorticity $\gamma \not\equiv 0$.

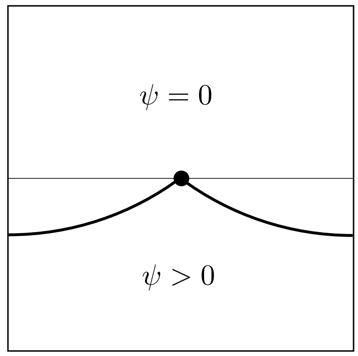

FIGURE 1. Stokes corner

In the irrotational case, the Stokes conjecture was first proved, under isolatedness, symmetry, and monotonicity assumptions, by Amick, Fraenkel and Toland [3] and Plotnikov [15] (see also [21] for a simplification of the proof in [3]), while a geometric proof has recently been given in [23] without any such structural assumptions.

In the case $\gamma \not\equiv 0$, the only rigorous results available on waves with stagnation points are very recent and require in an essential way symmetry and monotonicity of the free surface: In [22] it was proved that, at stagnation points, a symmetric monotone free boundary has either a corner of 120° or a horizontal tangent. Moreover, it was also shown there that, if $\gamma \geq 0$ close to the free surface, then the free surface necessarily has a corner of 120°. (On the other hand, if $\gamma(0) < 0$, there exist very simple examples where the free surface is the real axis, a line of stagnation points). The existence of waves, with non-zero vorticity, having stagnation points has been obtained in the setting of periodic waves of finite depth over a flat horizontal bottom, in the following cases in the paper [17] submitted simultaneously with the present paper: for any nonpositive vorticity function γ and any period of the wave, and under certain restrictions on the size of γ and the wave period (roughly speaking, the vorticity has to be sufficiently small and the period sufficiently large) if γ is positive somewhere. The extreme waves constructed in [17] are obtained as weak limits of large-amplitude smooth waves whose existence was proved by Constantin and Strauss [4], and they are symmetric and monotone. It was shown in [17] that the free surface of any symmetric monotone wave with stagnation points which is a limit of smooth waves cannot have a horizontal tangent at the stagnation points (in particular, the free surface cannot be horizontally flat), irrespective of the vorticity function γ, and therefore, as a a consequence of [22], the free surface of such a wave necessarily has corners of $120°$ at stagnation points.

The present paper is the first study of stagnation points of steady two-dimensional gravity water waves with vorticity in the absence of structural assumptions of isolatedness of stagnation points, symmetry and monotonicity of the free boundary, which have been essential assumptions in all previous works. We obtain for example that, in the case when the free surface is an injective curve, the asymptotics at any stagnation point is given either by the "Stokes corner flow" where the free surface has a *corner of* $120°$, or the free surface ends in a *horizontal cusp*,

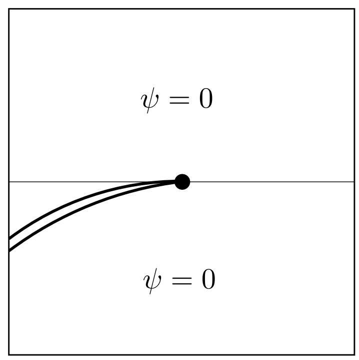

FIGURE 2. Cusp

or the free surface is *horizontally flat* at the stagnation point.

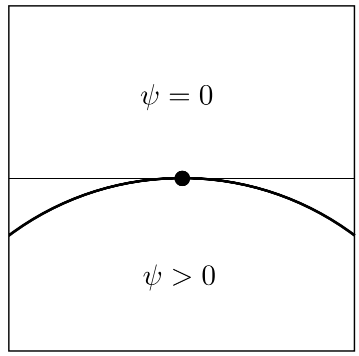

FIGURE 3. Horizontally flat stagnation point

The cusp case is a new feature in the case with vorticity, and it is not possible without the presence of vorticity [23]. It is interesting to point out that Gerstner [10] constructed an explicit example of a steady wave with vorticity whose free surface has a *vertical cusp* at a stagnation point. However, this vertical cusp is due to the fact that in his example the vorticity is infinite at the free surface, while in the present paper we only consider the case of vorticities which are smooth up to the free surface. We conjecture the cusps in our paper —the existence of which is still open— to be due to the break-down of the Rayleigh-Taylor condition in the presence of vorticity.

The second half of our paper is devoted to *excluding horizontally flat singularities* in the case that the vorticity is non-negative at the free surface. (Horizontally flat singularities are possible if the vorticity is negative at the free surface.) Of particular difficulty is the case when the vorticity is 0 at the free surface, and may have infinitely many sign changes accumulating there.

Let us briefly state our main result and give a plan of the paper:

Main Result. *Let ψ be a suitable weak solution of (1.1) (compare to Definition 3.2) satisfying*

$$|\nabla\psi(x,y)|^2 \leq C \max(-y,0) \quad \textit{locally in } \Omega,$$

let the free boundary $\partial\{\psi > 0\}$ be a continuous injective curve $\sigma = (\sigma_1, \sigma_2)$ such that $\sigma(0) = (x^0, 0)$, and assume that the vorticity function satisfies either $|\gamma(z)| \leq Cz$, or $\gamma(z) \geq 0$, for all z in a right neighborhood of 0.

(i) If the Lebesgue density of the set $\{\psi > 0\}$ at $(x^0, 0)$ is positive, then the free boundary is in a neighborhood of $(x^0, 0)$ the union of two C^1-graphs of functions

$\eta_1 : (x^0 - \delta, x^0] \to \mathbf{R}$ and $\eta_2 : [x^0, x^0 + \delta) \to \mathbf{R}$ which are both continuously differentiable up to x^0 and satisfy $\eta_1'(x^0) = 1/\sqrt{3}$ and $\eta_2'(x^0) = -1/\sqrt{3}$.

(ii) Else $\sigma_1(t) \neq x^0$ in $(-t_1, t_1) \setminus \{0\}$, $\sigma_1 - x^0$ does not change its sign at $t = 0$, and

$$\lim_{t \to 0} \frac{\sigma_2(t)}{\sigma_1(t) - x^0} = 0.$$

If we assume in addition that either $\{\psi > 0\}$ is a subgraph of a function in the y-direction or that $\{\psi > 0\}$ is a Lipschitz set, then the set of stagnation points is locally in Ω a finite set, and at each stagnation point x^0 the statement in (i) holds.

Plan of the paper:

The flow of the paper follows [23] with new aspects and difficulties which we are going to point out:

After gathering some notation in section 2, in section 3 we introduce suitable weak solutions and prove a monotonicity formula. Consequences of the monotonicity formula (section 4) make a blow-up analysis of singularities possible. The general case (without the injective curve assumption) is stated in Theorem 4.5. Different from the zero vorticity case handled in [23], there appears a new case in which the Lebesgue density of the set $\{\psi > 0\}$ is 0. Assuming the free surface to be an injective curve in a neighborhood of the singularity we obtain in Theorem 4.6 a more precise description: In the new case the free surface forms *cusps* pointing in the x- or $-x$-direction. As in [23] we are able to show that Stokes corner singularities are isolated points (section 5).

Starting with section 6, the focus of our analysis is on points at which the set $\{\psi > 0\}$ has full Lebesgue density. In the case $\gamma(0) = 0$, an extension of the *frequency formula* (Theorem 6.7) introduced by the authors in [23] leads here to a *Bessel differential inequality* (see the proof of Theorem 6.12) which shows that the right-hand side of the frequency formula is integrable. This part is substantially different from [23]. It is then possible (sections 7-9) to do a blow-up analysis in order to exclude horizontally flat singularities (Theorem 10.1). All our results are based on calculations in the original variables.

2. Notation

We denote by χ_A the characteristic function of a set A. For any real number a, the notation a^+ stands for $\max(a, 0)$. We denote by $x \cdot y$ the Euclidean inner product in $\mathbf{R}^n \times \mathbf{R}^n$, by $|x|$ the Euclidean norm in $\mathbf{R}^n$ and by $B_r(x^0) := \{x \in \mathbf{R}^n : |x - x^0| < r\}$ the ball of center x^0 and radius r. We will use the notation B_r for $B_r(0)$, and denote by ω_n the n-dimensional volume of B_1. Also, $\mathcal{L}^n$ shall denote the n-dimensional Lebesgue measure and $\mathcal{H}^s$ the s-dimensional Hausdorff measure. By ν we will always refer to the outer normal on a given surface. We will use functions of bounded variation $BV(U)$, i.e. functions $f \in L^1(U)$ for which the distributional derivative is a vector-valued Radon measure. Here $|\nabla f|$ denotes the total variation measure (cf. [12]). Note that for a smooth open set $E \subset \mathbf{R}^n$, $|\nabla \chi_E|$ coincides with the surface measure on ∂E.

3. Notion of solution and monotonicity formula

In some sections of the paper we work with a n-dimensional generalization of the problem described in the Introduction. Let Ω be a bounded domain in $\mathbf{R}^n$ which has a non-empty intersection with the hyperplane $\{x_n = 0\}$, in which to consider the combined problem for fluid and air. We study solutions u, in a sense to be specified, of the problem

$$\begin{aligned} \Delta u &= -f(u) \quad \text{in } \Omega \cap \{u > 0\}, \\ |\nabla u|^2 &= x_n \quad \text{on } \Omega \cap \partial\{u > 0\}. \end{aligned} \tag{3.1}$$

Note that, compared to the Introduction, we have switched notation from ψ to u and from γ to f, and we have "reflected" the problem at the hyperplane $\{x_n = 0\}$. The nonlinearity f is assumed to be a continuous function with primitive $F(z) = \int_0^z f(t)\, dt$. Since our results are completely local, we do not specify boundary conditions on $\partial\Omega$. In view of the second equation in (3.1), it is natural to assume throughout the rest of the paper that $u \equiv 0$ in $\Omega \cap \{x_n \le 0\}$.

We begin by introducing our notion of a *variational solution* of problem (3.1).

Definition 3.1 (Variational Solution)**.** We define $u \in W^{1,2}_{\text{loc}}(\Omega)$ to be a *variational solution* of (3.1) if $u \in C^0(\Omega) \cap C^2(\Omega \cap \{u > 0\})$, $u \ge 0$ in Ω and $u \equiv 0$ in $\Omega \cap \{x_n \le 0\}$, and the first variation with respect to domain variations of the functional

$$J(v) := \int_\Omega \Big(|\nabla v|^2 - 2F(v) + x_n \chi_{\{v>0\}} \Big)\, dx$$

vanishes at $v = u$, i.e.

$$\begin{aligned} 0 &= -\frac{d}{d\epsilon} J(u(x + \epsilon\phi(x)))|_{\epsilon=0} \\ &= \int_\Omega \Big((|\nabla u|^2 - 2F(u))\text{div } \phi - 2\nabla u D\phi \nabla u + x_n \chi_{\{u>0\}} \text{div } \phi + \chi_{\{u>0\}}\phi_n \Big)\, dx \end{aligned}$$

for any $\phi \in C^1_0(\Omega; \mathbf{R}^n)$.

Note for future reference that for each open set $D \subset\subset \Omega$ there is $C_D < +\infty$ such that $\Delta u + C_D$ is a nonnegative Radon measure in D, the support of the singular part of which (with respect to the Lebesgue measure) is contained in the set $\partial\{u > 0\}$: by Sard's theorem $\{u = \delta\} \cap D$ is for almost every δ a smooth surface. It follows that for every non-negative $\zeta \in C_0^\infty(D)$

$$- \int_D (\nabla \max(u-\delta, 0) \cdot \nabla \zeta - C_D \zeta)\, dx$$

$$= \int_D \zeta(\chi_{\{u>\delta\}} \Delta u + C_D)\, dx - \int_{D\cap\partial\{u>\delta\}} \zeta \nabla u \cdot \nu \, d\mathcal{H}^{n-1} \geq 0,$$

provided that $|f(u)| \leq C_D$ in D. Letting $\delta \to 0$ and using that u is continuous and nonnegative in Ω, we obtain

$$- \int_D (\nabla u \cdot \nabla \zeta - C_D \zeta)\, dx \leq 0.$$

Thus $\Delta u + C_D$ is a nonnegative distribution in D, and the stated property follows.

Since we want to focus in the present paper on the analysis of stagnation points, we will assume that everything is smooth away from $x_n = 0$, however this assumption may be weakened considerably by using in $\{x_n > 0\}$ regularity theory for the Bernoulli free boundary problem (see [2] for regularity theory in the case $f = 0$ —which could effortlessly be perturbed to include the case of bounded f— and see [5] for another regularity approach which already includes the perturbation).

Definition 3.2 (Weak Solution)**.** We define $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to be a *weak solution* of (3.1) if the following are satisfied: u is a *variational solution* of (3.1) and the topological free boundary $\partial\{u > 0\} \cap \Omega \cap \{x_n > 0\}$ is locally a $C^{2,\alpha}$-surface.

Remark 3.3. (i) It follows that in $\{x_n > 0\}$ the solution is a classical solution of (3.1).

(ii) For any weak solution u of (3.1) such that

$$|\nabla u|^2 \leq C x_n^+ \quad \text{locally in } \Omega,$$

u is a variational solution of (3.1), $\chi_{\{u>0\}}$ is locally in $\{x_n > 0\}$ a function of bounded variation, and the total variation measure $|\nabla \chi_{\{u>0\}}|$ satisfies

$$r^{1/2-n} \int_{B_r(y)} \sqrt{x_n}\, d|\nabla \chi_{\{u>0\}}| \leq C_0$$

for all $B_r(y) \subset\subset \Omega$ such that $y_n = 0$ (see [23, Lemma 3.4]).

A first tool in our analysis is an extension of the monotonicity formula in [25], [24, Theorem 3.1] to the boundary case. The roots of those monotonicity formulas are harmonic mappings ([18], [16]) and blow-up ([14]).

Theorem 3.4 (Monotonicity Formula)**.** *Let u be a variational solution of (3.1), let $x^0 \in \Omega$ such that $x_n^0 = 0$, and let $\delta := \mathrm{dist}(x^0, \partial\Omega)/2$. Let, for any $r \in (0, \delta)$,*

$$I_{x^0,u}(r) = I(r) = r^{-n-1} \int_{B_r(x^0)} \left(|\nabla u|^2 - u f(u) + x_n \chi_{\{u>0\}}\right) dx, \tag{3.2}$$

$$J_{x^0,u}(r) = J(r) = r^{-n-2} \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}, \tag{3.3}$$

$$M_{x^0,u}(r) = M(r) = I(r) - \frac{3}{2}J(r) \tag{3.4}$$

and

$$K_{x^0,u}(r) = K(r) = r\int_{\partial B_r(x^0)} (2F(u) - uf(u))\, d\mathcal{H}^{n-1} + \int_{B_r(x^0)} ((n-2)uf(u) - 2nF(u))\, dx. \tag{3.5}$$

Then, for a.e. $r \in (0,\delta)$,

$$I'(r) = r^{-n-2}\left(2r\int_{\partial B_r(x^0)} (\nabla u\cdot\nu)^2\, d\mathcal{H}^{n-1} - 3\int_{\partial B_r(x^0)} u\nabla u\cdot\nu\, d\mathcal{H}^{n-1}\right) + r^{-n-2}K(r), \tag{3.6}$$

$$J'(r) = r^{-n-3}\left(2r\int_{\partial B_r(x^0)} u\nabla u\cdot\nu\, d\mathcal{H}^{n-1} - 3\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}\right) \tag{3.7}$$

and

$$M'(r) = 2r^{-n-1}\int_{\partial B_r(x^0)} \left(\nabla u\cdot\nu - \frac{3}{2}\frac{u}{r}\right)^2 d\mathcal{H}^{n-1} + r^{-n-2}K(r). \tag{3.8}$$

Proof. The identity (3.7) can be easily checked directly, being valid for any function $u \in W^{1,2}_{\text{loc}}(\Omega)$ (not necessarily a variational solution of (3.1)).

For small positive κ and $\eta_\kappa(t) := \max(0, \min(1, \frac{r-t}{\kappa}))$, we take after approximation $\phi_\kappa(x) := \eta_\kappa(|x-x^0|)(x-x^0)$ as a test function in the definition of a variational solution. We obtain

$$\begin{aligned}0 = &\int_\Omega \left(|\nabla u|^2 - 2F(u) + x_n\chi_{\{u>0\}}\right)\left(n\eta_\kappa(|x-x^0|) + \eta_\kappa'(|x-x^0|)|x-x^0|\right) dx\\ &- 2\int_\Omega \left(|\nabla u|^2\eta_\kappa(|x-x^0|) + \nabla u\cdot\frac{x-x^0}{|x-x^0|}\nabla u\cdot\frac{x-x^0}{|x-x^0|}\eta'(|x-x^0|)|x-x^0|\right) dx\\ &+ \int_\Omega \eta_\kappa(|x-x^0|)x_n\chi_{\{u>0\}}\, dx.\end{aligned}$$

Passing to the limit as $\kappa \to 0$, we obtain, for a.e. $r \in (0,\delta)$,

$$\begin{aligned}0 =\, &n\int_{B_r(x^0)} \left(|\nabla u|^2 - 2F(u) + x_n\chi_{\{u>0\}}\right) dx\\ &- r\int_{\partial B_r(x^0)} \left(|\nabla u|^2 - 2F(u) + x_n\chi_{\{u>0\}}\right) d\mathcal{H}^{n-1}\\ &+ 2r\int_{\partial B_r(x^0)} (\nabla u\cdot\nu)^2\ d\mathcal{H}^{n-1} - 2\int_{B_r(x^0)} |\nabla u|^2\, dx\\ &+ \int_{B_r(x^0)} x_n\chi_{\{u>0\}}\, dx.\end{aligned} \tag{3.9}$$

Also observe that letting $\epsilon \to 0$ in

$$\int_{B_r(x^0)} \nabla u \cdot \nabla \max(u-\epsilon,0)^{1+\epsilon}\,dx = \int_{B_r(x^0)} f(u)\max(u-\epsilon,0)^{1+\epsilon}\,dx + \int_{\partial B_r(x^0)} \max(u-\epsilon,0)^{1+\epsilon}\nabla u\cdot\nu\; d\mathcal{H}^{n-1}$$

for a.e. $r \in (0,\delta)$, we obtain the integration by parts formula

$$\int_{B_r(x^0)} \Big(|\nabla u|^2 - uf(u)\Big)\,dx = \int_{\partial B_r(x^0)} u\nabla u\cdot\nu\; d\mathcal{H}^{n-1} \tag{3.10}$$

for a.e. $r \in (0,\delta)$.

Note also that

$$I'(r) = -(n+1)r^{-n-2}\int_{B_r(x^0)} (|\nabla u|^2 - uf(u) + x_n\chi_{\{u>0\}})\,dx + r^{-n-1}\int_{\partial B_r(x^0)} (|\nabla u|^2 - uf(u) + x_n\chi_{\{u>0\}})\,d\mathcal{H}^{n-1}. \tag{3.11}$$

Using (3.9) and (3.10) in (3.11), we obtain (3.6). Finally, (3.8) follows immediately by combining (3.6) and (3.7). □

4. Densities

From now on we assume

Assumption 4.1. *Let u satisfy*

$$|\nabla u|^2 \le Cx_n^+ \quad \textit{locally in } \Omega.$$

Remark 4.2. Note that Assumption 4.1 implies that

$$u(x) \le C_1 x_n^{3/2}$$

and that in the case $x_n^0 = 0$,

$$r^{-n-2}|K(r)| \le C_2\frac{1}{\sqrt{r}},$$

where C_2 depends on x^0 but is locally uniformly bounded.

Remark 4.3. Unfortunately the combination of vorticity and gravity makes it hard to obtain the estimate

$$|\nabla u|^2 + 2F(u) - x_n^+ \le 0 \tag{4.1}$$

related to the Rayleigh-Taylor condition in the time-dependent problem, but the weaker estimate Assumption 4.1 has been verified under certain assumptions in [22].

We first show that the function $M_{x^0,u}$ has a right limit $M_{x^0,u}(0+)$, of which we derive structural properties.

Lemma 4.4. *Let u be a variational solution of (3.1) satisfying Assumption 4.1. Then:*

(i) Let $x^0 \in \Omega$ be such that $x^0_n = 0$. Then the limit $M_{x^0,u}(0+)$ exists and is finite. (Note that $u = 0$ in $\{x_n = 0\}$ by assumption.)

(ii) Let $x^0 \in \Omega$ be such that $x^0_n = 0$, and let $0 < r_m \to 0+$ as $m \to \infty$ be a sequence such that the blow-up sequence

$$u_m(x) := \frac{u(x^0 + r_m x)}{r_m^{3/2}} \tag{4.2}$$

converges weakly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$ to a blow-up limit u_0. Then u_0 is a homogeneous function of degree $3/2$, i.e.

$$u_0(\lambda x) = \lambda^{3/2} u_0(x) \quad \text{for any } x \in \mathbf{R}^n \text{ and } \lambda > 0.$$

(iii) Let u_m be a converging sequence of (ii). Then u_m converges strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$.

(iv) Let $x^0 \in \Omega$ be such that $x^0_n = 0$. Then

$$M_{x^0,u}(0+) = \lim_{r\to 0+} r^{-n-1} \int_{B_r(x^0)} x_n^+ \chi_{\{u>0\}}\, dx,$$

and in particular $M_{x^0,u}(0+) \in [0, +\infty)$. Moreover, $M_{x^0,u}(0+) = 0$ implies that $u_0 = 0$ in $\mathbf{R}^n$ for each blow-up limit u_0 of (ii).

(v) The function $x \mapsto M_{x,u}(0+)$ is upper semicontinuous in $\{x_n = 0\}$.

(vi) Let u_m be a sequence of variational solutions of (3.1) with nonlinearity f_m in a domain Ω_m, where

$$\Omega_1 \subset \Omega_2 \subset ... \subset \Omega_m \subset \Omega_{m+1} \subset ... \quad \text{and} \quad \bigcup_{m=1}^{\infty} \Omega_m = \mathbf{R}^n,$$

such that u_m converges strongly to u_0 in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^n)$, $\chi_{\{u_m>0\}}$ converges weakly in $L^2_{\mathrm{loc}}(\mathbf{R}^n)$ to χ_0, and $f_m(u_m)$ converges to 0 locally uniformly in $\mathbf{R}^n$. Then u_0 is a variational solution of (3.1) with nonlinearity $f = 0$ in $\mathbf{R}^n$ and satisfies the Monotonicity Formula (with $f = 0$), but with $\chi_{\{u_0>0\}}$ replaced by χ_0. Moreover, for each $x^0 \in \mathbf{R}^n$ such that $x^0_n = 0$, and all instances of $\chi_{\{u_0>0\}}$ replaced by χ_0,

$$M_{x^0,u_0}(0+) \geq \limsup_{m\to\infty} M_{x^0,u_m}(0+).$$

Proof. (i): By Remark 4.2,

$$u(x) \leq C_2 x_n^{3/2} \quad \text{locally in } \Omega$$

and

$$|r^{-n-2} K(r)| \leq C_3 r^{-1/2} \text{ for each } x^0 \in \Omega \text{ satisfying } x^0_n = 0. \tag{4.3}$$

Thus $r \mapsto r^{-n-2}K(r)$ is integrable at such points x^0, and from Theorem 3.4 we infer that the function $M_{x^0,u}$ has a finite right limit $M_{x^0,u}(0+)$.

(ii): For each $0 < \sigma < \infty$ the sequence u_m is by assumption bounded in $C^{0,1}(B_\sigma)$. For any $0 < \varrho < \sigma < \infty$, we write the identity (3.8) in integral form as

$$2\int_\varrho^\sigma r^{-n-1} \int_{\partial B_r(x^0)} \left(\nabla u \cdot \nu - \frac{3}{2}\frac{u}{r}\right)^2 \, d\mathcal{H}^{n-1} dr \\ = M(\sigma) - M(\varrho) - \int_\varrho^\sigma r^{-n-2} K(r)\, dr. \tag{4.4}$$

It follows by rescaling in (4.4) that

$$2\int_{B_\sigma(0)\setminus B_\varrho(0)} |x|^{-n-3} \left(\nabla u_m(x) \cdot x - \frac{3}{2} u_m(x)\right)^2 \, dx \\ \leq M(r_m\sigma) - M(r_m\varrho) + \int_{r_m\varrho}^{r_m\sigma} r^{-n-2}|K(r)|\, dr \to 0 \quad \text{as } m \to \infty,$$

which yields the desired homogeneity of u_0.

(iii): In order to show strong convergence of u_m in $W^{1,2}_{\text{loc}}(\mathbf{R}^n)$, it is sufficient, in view of the weak L^2-convergence of ∇u_m, to show that

$$\limsup_{m\to\infty} \int_{\mathbf{R}^n} |\nabla u_m|^2 \eta\, dx \leq \int_{\mathbf{R}^n} |\nabla u_0|^2 \eta\, dx$$

for each $\eta \in C^1_0(\mathbf{R}^n)$. Let $\delta := \text{dist}(x^0, \partial\Omega)/2$. Then, for each m, u_m is a variational solution of

$$\Delta u_m = -r_m^{1/2} f(r_m^{3/2} u_m) \quad \text{in } B_{\delta/r_m} \cap \{u_m > 0\}, \\ |\nabla u_m|^2 = x_n \quad \text{on } B_{\delta/r_m} \cap \partial\{u_m > 0\}. \tag{4.5}$$

Since u_m converges to u_0 locally uniformly, it follows from (4.5) that u_0 is harmonic in $\{u_0 > 0\}$. Also, using the uniform convergence, the continuity of u_0 and its harmonicity in $\{u_0 > 0\}$ we obtain as in the proof of (3.10) that

$$\int_{\mathbf{R}^n} |\nabla u_m|^2 \eta\, dx = -\int_{\mathbf{R}^n} u_m \left(\nabla u_m \cdot \nabla\eta - r_m^{1/2} f(r_m^{3/2} u_m)\eta\right) \, dx \\ \to -\int_{\mathbf{R}^n} u_0 \nabla u_0 \cdot \nabla\eta\, dx = \int_{\mathbf{R}^n} |\nabla u_0|^2 \eta\, dx$$

as $m \to \infty$. It therefore follows that u_m converges to u_0 strongly in $W^{1,2}_{\text{loc}}(\mathbf{R}^n)$ as $m \to \infty$.

(iv): Let us take a sequence $r_m \to 0+$ such that u_m defined in (4.2) converges weakly in $W^{1,2}_{\text{loc}}(\mathbf{R}^n)$ to a function u_0. Note that by the definition of a variational solution, u_m and u_0 are identically zero in $x_n \leq 0$. Using (iii) and the homogeneity of u_0, we obtain that

$$\lim_{m\to\infty} M_{x^0,u}(r_m) = \int_{B_1} |\nabla u_0|^2\,dx - \frac{3}{2}\int_{\partial B_1} u_0^2\,d\mathcal{H}^{n-1} + \lim_{r\to 0+} r^{-n-1}\int_{B_r(x^0)} x_n^+\chi_{\{u>0\}}\,dx = \lim_{r\to 0+} r^{-n-1}\int_{B_r(x^0)} x_n^+\chi_{\{u>0\}}\,dx.$$

Thus $M_{x^0,u}(0+)\geq 0$, and equality implies that for each $\tau>0$, u_m converges to 0 in measure in the set $\{x_n>\tau\}$ as $m\to\infty$, and consequently $u_0=0$ in $\mathbf{R}^n$.

(v): For each $\delta>0$ we obtain from the Monotonicity Formula (Theorem 3.4), Remark 4.2 as well as the fact that $\lim_{x\to x^0} M_{x,u}(r)=M_{x^0,u}(r)$ for $r>0$, that

$$M_{x,u}(0+)\leq M_{x,u}(r)+C\sqrt{r}\leq M_{x^0,u}(r)+\frac{\delta}{2}\leq M_{x^0,u}(0+)+\delta,$$

if we choose for fixed x^0 first $r>0$ and then $|x-x^0|$ small enough.

(vi) The fact that u_0 is a variational solution of (3.1) and satisfies the Monotonicity Formula in the sense indicated follows directly from the convergence assumption. The proof for the rest of the claim follows by the same argument as in (v).

□

In the two-dimensional case, we identify the possible values of $M_{x^0,u}(0+)$, and classify the blow-up limits at x^0 in terms of the value of $M_{x^0,u}(0+)$, which leads to a proof of asymptotic homogeneity of the solution.

Theorem 4.5 (Two-dimensional Case)**.** *Let $n=2$, let u be a variational solution of (3.1) satisfying Assumption 4.1, let $x^0\in\Omega$ be such that $x_2^0=0$, and suppose that*

$$r^{-3/2}\int_{B_r(x^0)} \sqrt{x_2}\,d|\nabla\chi_{\{u>0\}}|\leq C_0$$

for all $r>0$ such that $B_r(x^0)\subset\subset\Omega$. Then the following hold:

(i)

$$M(0+)\in\left\{0, \int_{B_1} x_2^+\chi_{\{x:\pi/6<\theta<5\pi/6\}}\,dx, \int_{B_1} x_2^+\,dx\right\}.$$

(ii) If $M(0+)=\int_{B_1} x_2^+\chi_{\{x:\pi/6<\theta<5\pi/6\}}\,dx$, then

$$\frac{u(x^0+rx)}{r^{3/2}}\to\frac{\sqrt{2}}{3}\rho^{3/2}\cos\left(\frac{3}{2}\left(\min\left(\max\left(\theta,\frac{\pi}{6}\right),\frac{5\pi}{6}\right)-\frac{\pi}{2}\right)\right)\quad \textit{as } r\to 0+$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ and locally uniformly on $\mathbf{R}^2$, where $x=(\rho\cos\theta,\rho\sin\theta)$.

(iii) If $M(0+)\in\{0,\int_{B_1} x_2^+\,dx\}$, then

$$\frac{u(x^0+rx)}{r^{3/2}}\to 0\quad \textit{as } r\to 0+,$$

strongly in $W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ and locally uniformly on $\mathbf{R}^2$.

Proof. Consider a blow-up sequence u_m as in Lemma 4.4 (ii), where $r_m \to 0+$, with blow-up limit u_0. Because of the strong convergence of u_m to u_0 in $W^{1,2}_{\text{loc}}(\mathbf{R}^2)$ and the compact embedding from BV into L^1, u_0 is a homogeneous solution of

$$0 = \int_{\mathbf{R}^2} \Big(|\nabla u_0|^2 \text{div } \phi - 2\nabla u_0 D\phi \nabla u_0 \Big)\, dx + \int_{\mathbf{R}^2} \Big(x_2 \chi_0 \text{div } \phi + \chi_0 \phi_2 \Big)\, dx \tag{4.6}$$

for any $\phi \in C^1_0(\mathbf{R}^2; \mathbf{R}^2)$, where χ_0 is the strong L^1_{loc}-limit of $\chi_{\{u_m>0\}}$ along a subsequence. The values of the function χ_0 are almost everywhere in $\{0, 1\}$, and the locally uniform convergence of u_m to u_0 implies that $\chi_0 = 1$ in $\{u_0 > 0\}$. The homogeneity of u_0 and its harmonicity in $\{u_0 > 0\}$ show that each connected component of $\{u_0 > 0\}$ is a cone with vertex at the origin and of opening angle $120°$. Since $u = 0$ in $\{x_2 \le 0\}$, this shows that $\{u_0 > 0\}$ has at most one connected component. Note also that (4.6) implies that χ_0 is constant in each open connected set $G \subset \{u_0 = 0\}^\circ$ that does not intersect $\{x_2 = 0\}$.

Consider first the case when $\{u_0 > 0\}$ is non-empty, and is therefore a cone as described above. Let z be an arbitrary point in $\partial\{u_0 > 0\} \setminus \{0\}$. Note that the normal to $\partial\{u_0 > 0\}$ has the constant value $\nu(z)$ in $B_\delta(z) \cap \partial\{u_0 > 0\}$ for some $\delta > 0$. Plugging in $\phi(x) := \eta(x)\nu(z)$ into (4.6), where $\eta \in C^1_0(B_\delta(z))$ is arbitrary, and integrating by parts, it follows that

$$0 = \int_{\partial\{u_0>0\}} \left(-|\nabla u_0|^2 + x_2(1 - \bar{\chi}_0) \right) \eta \, d\mathcal{H}^1. \tag{4.7}$$

Here $\bar{\chi}_0$ denotes the constant value of χ_0 in the respective connected component of $\{u_0 = 0\}^\circ \cap \{x_2 \neq 0\}$. Note that by Hopf's principle, $\nabla u_0 \cdot \nu \neq 0$ on $B_\delta(z) \cap \partial\{u_0 > 0\}$. It follows therefore that $\bar{\chi}_0 \neq 1$, and hence necessarily $\bar{\chi}_0 = 0$. We deduce from (4.7) that $|\nabla u_0|^2 = x_2$ on $\partial\{u_0 > 0\}$. Computing the solution u_0 of the corresponding ordinary differential equation on ∂B_1 yields that

$$u_0(x) = \frac{\sqrt{2}}{3} \rho^{3/2} \cos\Big(\frac{3}{2}\big(\min(\max(\theta, \frac{\pi}{6}), \frac{5\pi}{6}) - \frac{\pi}{2}\big)\Big), \quad \text{where } x = (\rho\cos\theta, \rho\sin\theta),$$

and that $M(0+) = \int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\, dx$ in the case under consideration.

Consider now the case $u_0 = 0$. It follows from (4.6) that χ_0 is constant in $\{x_2 > 0\}$. Its value may be either 0 in which case $M(0+) = 0$, or 1 in which case $M(0+) = \int_{B_1} x_2^+\, dx$.

Since the limit $M(0+)$ exists, the above proof shows that it can only take one of the three distinct values $\Big\{0, \int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\, dx, \int_{B_1} x_2^+\, dx\Big\}$. The above proof also yields, for each possible value of $M(0+)$, the existence of a *unique* blow-up limit, as claimed in the statement of the Theorem. □

Under the assumption that the free boundary is locally an injective curve, we now derive its asymptotic behavior as it approaches a stagnation point.

Theorem 4.6 (Curve Case)**.** *Let $n = 2$, let u be a weak solution of (3.1) satisfying Assumption 4.1, and let $x^0 \in \Omega$ be such that $x^0_2 = 0$. Suppose in addition that*

$\partial\{u > 0\}$ is in a neighborhood of x^0 a continuous injective curve $\sigma : (-t_0, t_0) \to \mathbf{R}^2$ such that $\sigma = (\sigma_1, \sigma_2)$ and $\sigma(0) = x^0$. Then the following hold:

(i) If $M(0+) = \int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\, dx$, then (cf. Figure 4) $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$ and, depending on the parametrization, either

$$\lim_{t\to 0+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}} \text{ and } \lim_{t\to 0-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}},$$

or

$$\lim_{t\to 0+} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = -\frac{1}{\sqrt{3}} \text{ and } \lim_{t\to 0-} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = \frac{1}{\sqrt{3}}.$$

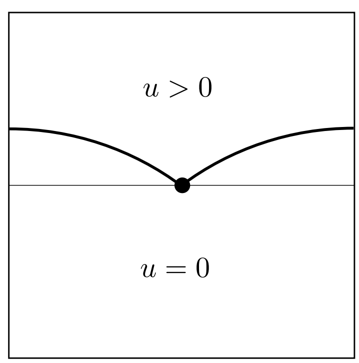

FIGURE 4. Stokes corner

(ii) If $M(0+) = \int_{B_1} x_2^+\, dx$, then (cf. Figure 5) $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$, $\sigma_1 - x_1^0$ changes sign at $t = 0$ and

$$\lim_{t\to 0} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = 0.$$

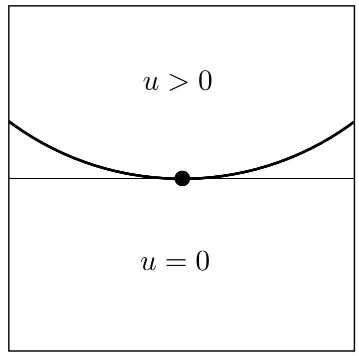

FIGURE 5. Full density singularity

(iii) If $M(0+) = 0$, then (cf. Figure 6 and Figure 7) $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$, $\sigma_1 - x_1^0$ does not change its sign at $t = 0$, and

$$\lim_{t \to 0} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = 0.$$

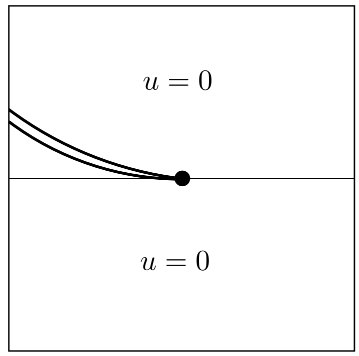

FIGURE 6. Left cusp

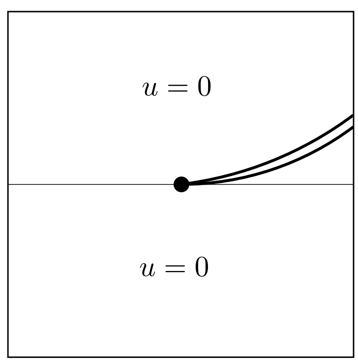

FIGURE 7. Right cusp

Proof. We may assume that $x_1^0 = 0$. Moreover, for each $y \in \mathbf{R}^2$ we define arg y as the complex argument of y, and we define the sets

$$\mathcal{L}_\pm := \{\theta_0 \in [0,\pi] \ : \ \text{there is } t_m \to 0\pm \text{ such that } \arg \sigma(t_m) \to \theta_0 \text{ as } m \to \infty\}.$$

Step 1: *Both* $\mathcal{L}_+$ *and* $\mathcal{L}_-$ *are subsets of* $\{0, \pi/6, 5\pi/6, \pi\}$.

Indeed, suppose towards a contradiction that a sequence $0 \neq t_m \to 0, m \to \infty$ exists such that $\arg \sigma(t_m) \to \theta_0 \in (\mathcal{L}_+ \cup \mathcal{L}_-) \setminus \{0, \pi/6, 5\pi/6, \pi\}$, let $r_m := |\sigma(t_m)|$ and let

$$u_m(x) := \frac{u(r_m x)}{r_m^{3/2}}.$$

For each $\rho > 0$ such that $\tilde{B} := B_\rho(\cos\theta_0, \sin\theta_0)$ satisfies

$$\emptyset = \tilde{B} \cap \Big(\{(x, 0) \,:\, x \in \mathbf{R}\} \cup \{(x, |x|/\sqrt{3}) \,:\, x \in \mathbf{R}\} \Big),$$

we infer from the formula for the unique blow-up limit u_0 (see Theorem 4.5) that the signed measure

$$\Delta u_m(\tilde{B}) \to \Delta u_0(\tilde{B}) = 0 \text{ as } m \to \infty.$$

On the other hand,

$$\Delta u_m = -r_m^{1/2} f(r_m^{3/2} u_m) + \sqrt{x_2}\mathcal{H}^1 \lfloor \partial_{\{u_m > 0\}},$$

which implies, since $\tilde{B} \cap \partial\{u_m > 0\}$ contains a curve of length at least $2\rho - o(1)$, that

$$0 \leftarrow \Delta u_m(\tilde{B}) \geq c(\theta_0, \rho) - C_1 r_m^{1/2} \text{ as } m \to \infty,$$

where $c(\theta_0, \rho) > 0$, a contradiction. Thus the property claimed in Step 1 holds.

Step 2: It follows that $\sigma_1(t) \neq 0$ for all sufficiently small $t \neq 0$. Now a continuity argument yields that both $\mathcal{L}_+$ and $\mathcal{L}_-$ are connected sets. Consequently

$$\ell_+ := \lim_{t \to 0+} \arg \sigma(t)$$

exists and is contained in the set $\{0, \pi/6, 5\pi/6, \pi\}$, and

$$\ell_- := \lim_{t \to 0-} \arg \sigma(t)$$

exists and is contained in the set $\{0, \pi/6, 5\pi/6, \pi\}$.

Step 3: In the case $M(0+) = \int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\, dx$, we know now from the formula for u_0 that $\Delta u_0(B_{1/10}(\sqrt{3}/2, 1/2)) > 0$ and that $\Delta u_0(B_{1/10}(-\sqrt{3}/2, 1/2)) > 0$. It follows that the set $\{\ell_+, \ell_-\}$ contains both $\pi/6$ and $5\pi/6$. But then the sets $\{\ell_+, \ell_-\}$ and $\{\pi/6, 5\pi/6\}$ must be equal, and the fact that $u = 0$ on $x_2 = 0$ implies case (i) of the Theorem.

Step 4: In the case $M(0+) \in \{0, \int_{B_1} x_2^+\, dx\}$, we have that $\Delta u_0(B_{1/10}(\sqrt{3}/2, 1/2)) = 0$, which implies that $\ell_+, \ell_- \notin \{\pi/6, 5\pi/6\}$. Thus $\ell_+, \ell_- \in \{0, \pi\}$. Using the fact that $u = 0$ on $x_2 = 0$, we obtain in the case $\ell_+ \neq \ell_-$ that $M(0+) = \int_{B_1} x_2^+\, dx$ and in the case $\ell_+ = \ell_-$ that $M(0+) = 0$. Together, the last two properties prove case (ii) and case (iii) of the Theorem. □

Remark 4.7. In [23] we used a strong version of the Rayleigh-Taylor condition (which is always valid in the case of zero vorticity) in order to prove that the cusps of case (iii) are not possible. Unfortunately we do not have the Rayleigh-Taylor condition (4.1) in the case with nonzero vorticity, and the method of [23] breaks down here. Still we *conjecture* that the cusps in case (iii) are *not* possible when assuming the Rayleigh-Taylor condition.

5. Partial regularity at non-degenerate points

Definition 5.1 (Stagnation Points)**.** Let u be a variational solution of (3.1). We call $S^u := \{x \in \Omega : x_n = 0 \text{ and } x \in \partial\{u > 0\}\}$ the set of *stagnation points*.

Throughout the rest of this section we assume that $n = 2$.

Definition 5.2 (Non-degeneracy)**.** Let u be a variational solution of (3.1).

We say that a point $x^0 \in \Omega \cap \partial\{u > 0\} \cap \{x_2 = 0\}$ is *degenerate* if

$$\frac{u(x^0 + rx)}{r^{3/2}} \to 0 \quad \text{as } r \to 0+,$$

strongly in $W^{1,2}_{\text{loc}}(\mathbf{R}^2)$. Otherwise we call $x^0 \in \Omega \cap \partial\{u > 0\} \cap \{x_2 = 0\}$ *non-degenerate.*

Remark 5.3. Note that Theorem 4.5 gives alternative characterizations of degeneracy/non-degeneracy in terms of the blow-up limit or the density.

Proposition 5.4 (Partial regularity in two dimensions)**.** *Let $n = 2$, let u be a variational solution of (3.1) satisfying Assumption 4.1 and suppose that*

$$r^{-3/2} \int_{B_r(y)} \sqrt{x_2}\, d|\nabla \chi_{\{u>0\}}| \le C_0$$

for all $B_r(y) \subset\subset \Omega$ such that $y_2 = 0$. Let $x^0 \in S^u$ be a non-degenerate point. Then in some open neighborhood, x^0 is the only non-degenerate stagnation point.

Proof. Suppose towards a contradiction that there exists a sequence x^m of non-degenerate stagnation points converging to x^0, with $x^m \ne x^0$ for all m. Choosing $r_m := |x^m - x^0|$, there is no loss of generality in assuming that the sequence $(x^m - x^0)/r_m$ is constant, with value $z \in \{(-1, 0), (1, 0)\}$. Consider the blow-up sequence

$$u_m(x) = \frac{u(x^0 + r_m x)}{r_m^{3/2}}.$$

Since x^m is a non-degenerate point for u, it follows that z is a non-degenerate point for u_m, and therefore Theorem 4.5 shows that

$$M_{z,u^m}(0+) = \int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\, dx.$$

By the proof of Theorem 4.5(ii), the sequence u_m converges strongly in $W^{1,2}_{\rm loc}(\mathbf{R}^2)$ to the homogeneous solution

$$u_0(\rho,\theta) = \frac{\sqrt{2}}{3}\rho^{3/2}\cos(\frac{3}{2}(\min(\max(\theta,\frac{\pi}{6}),\frac{5\pi}{6}) - \frac{\pi}{2})),$$

while $\chi_{\{u_m>0\}}$ converges strongly in $L^1_{\rm loc}(\mathbf{R}^2)$ to $\chi_{\{u_0>0\}}$. It follows from Lemma 4.4 (vi) that

$$M_{z,u_0}(0+) \geq \limsup_{m\to\infty} M_{z,u_m}(0+) = \int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\, dx,$$

contradicting the fact that $M_{z,u_0}(0+) = 0$. □

Remark 5.5. It follows that in two dimensions S^u can be decomposed into a countable set of "Stokes points" with the asymptotics as in Theorem 4.5 (ii), accumulating (if at all) only at degenerate stagnation points, and a set of degenerate stagnation points which will be analyzed in the following sections.

6. Degenerate points and frequency formula

Definition 6.1. Let u be a variational solution of (3.1). We define

$$\Sigma^u := \{x^0 \in S^u : M_{x^0,u}(0+) = \int_{B_1} x_n^+\, dx\}.$$

Remark 6.2. The set Σ^u is closed, as a consequence of the upper semicontinuity Lemma 4.4 (v) as well as the characterization of the set of values of $M_{x^0,u}(0+)$ in Theorem 4.5.

Remark 6.3. In the case of two dimensions and $\{u > 0\}$ being a supergraph or a Lipschitz set (each of the latter assumptions excluding the case $M_{x^0,u}(0+) = 0$), we infer from Theorem 4.5 that the set $S^u \setminus \Sigma^u$ equals the set of non-degenerate stagnation points and is according to Proposition 5.4 a finite or countable set.

Remark 6.4. (i) In the case $-f \equiv c > 0$, the function $u(x) = \frac{c}{2}x_n^2$ is a weak solution of (3.1). In this example, $\Sigma^u = \{x_n = 0\}$. Similarly, one may prove that for any f such that $f(0) < 0$, there exists an explicit solution $u(x) = u(x_n)$ such that $\Sigma^u = \{x_n = 0\}$. Thus degenerate points may exist in the case $f(0) < 0$.

(ii) The following Proposition shows that $\Sigma^u = \emptyset$ in the case when $n = 2$ and $f \geq 0$ in a right neighborhood of 0 (in particular this is satisfied when $f(0) > 0$).

Proposition 6.5. *Let $n = 2$, let u be a weak solution of (3.1) satisfying Assumption 4.1, and let $x^0 \in S^u$. Suppose in addition that $\partial\{u > 0\}$ is an injective curve in a neighborhood of x^0, and the nonlinearity f satisfies $f \geq 0$ in a right neighborhood of 0. Then $M(0+) \neq \int_{B_1} x_2^+\, dx$.*

Proof. For the sake of completeness we repeat the proof of [22, Proposition 5.12] which in turn is based on the following particular case of a result of Oddson [13], which we quote for easy reference.

Lemma 6.6. *Let $r_0 > 0$ and $\mu > 1$. Let*

$$G := \{(\rho\cos\theta, \rho\sin\theta) : 0 < \rho < r_0, |\theta| < \pi/(2\mu)\}.$$

Let $w \in C^2(G) \cap C(\overline{G})$ be a superharmonic function in G, such that $w(0,0) = 0$ and $w > 0$ in $\overline{G} \setminus \{(0,0)\}$. Then there exists $\kappa > 0$ such that

$$w(\rho\cos\theta, \rho\sin\theta) \geq \kappa\rho^\mu \cos\mu\theta \quad \text{in } \overline{G},$$

and in particular

$$w(\rho, 0) \geq \kappa\rho^\mu \quad \text{for all } \rho \in (0, r_0).$$

Suppose for a contradiction that $M(0+) = \int_{B_1} x_2^+\,dx$. Then, the assumption on f and Theorem 4.6 yield the existence of $r_0 > 0$ and $\alpha \in (0, \pi/6)$, such that u is superharmonic in $\{u > 0\} \cap B_{r_0}$ and $\overline{G} \setminus \{(0,0)\} \subset \{u > 0\} \cap B_{r_0}$, where $G := \{(\rho\cos\theta, \rho\sin\theta) : 0 < \rho < r_0, \alpha < \theta < \pi - \alpha\}$. After a suitable rotation, we may apply Lemma 6.6, obtaining the existence $\kappa > 0$ such that

$$u(0, x_2) \geq \kappa x_2^\mu \quad \text{for all } x_2 \in (0, r_0),$$

where $\mu := \pi/(\pi - 2\alpha)$, so that $\mu < 3/2$. But this contradicts the estimate

$$u(0, x_2) \leq C x_2^{3/2},$$

which is a consequence of the Bernstein estimate assumption 4.1. □

Motivated by Remark 6.4, we will focus in the present paper on the case $f(0) = 0$.

Theorem 6.7 (Frequency Formula)**.** *Let u be a variational solution of (3.1) satisfying Assumption 4.1, let x^0 be a stagnation point, and let $\delta := \mathrm{dist}(x^0, \partial\Omega)/2$. Let*

$$D_{x^0,u}(r) = D(r) = \frac{r\int_{B_r(x^0)}(|\nabla u|^2 - uf(u))\,dx}{\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1}}$$

and

$$V_{x^0,u}(r) = V(r) = \frac{r\int_{B_r(x^0)} x_n^+(1 - \chi_{\{u>0\}})\,dx}{\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1}}.$$

Then the "frequency"

$$\begin{aligned} H_{x^0,u}(r) &= H(r) = D(r) - V(r) \\ &= \frac{r\int_{B_r(x^0)}\Big(|\nabla u|^2 - uf(u) + x_n^+(\chi_{\{u>0\}} - 1)\Big)\,dx}{\int_{\partial B_r(x^0)} u^2\,d\mathcal{H}^{n-1}} \end{aligned}$$

satisfies for a.e. $r \in (0,\delta)$ the identities

$$
\begin{aligned}
&H'(r)\\
&= \frac{2}{r}\int_{\partial B_r(x^0)} \left[\frac{r(\nabla u\cdot\nu)}{\left(\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}\right)^{1/2}} - D(r)\frac{u}{\left(\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}\right)^{1/2}}\right]^2 d\mathcal{H}^{n-1}\\
&\qquad + \frac{2}{r}V^2(r) + \frac{2}{r}V(r)\left(H(r)-\frac{3}{2}\right) + \frac{K(r)}{\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}}
\end{aligned}
\tag{6.1}
$$

and

$$
\begin{aligned}
&H'(r)\\
&= \frac{2}{r}\int_{\partial B_r(x^0)} \left[\frac{r(\nabla u\cdot\nu)}{\left(\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}\right)^{1/2}} - H(r)\frac{u}{\left(\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}\right)^{1/2}}\right]^2 d\mathcal{H}^{n-1}\\
&\qquad + \frac{2}{r}V(r)\left(H(r)-\frac{3}{2}\right) + \frac{K(r)}{\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}};
\end{aligned}
\tag{6.2}
$$

here

$$
\begin{aligned}
K(r) = r&\int_{\partial B_r(x^0)} (2F(u) - uf(u))\, d\mathcal{H}^{n-1}\\
&+ \int_{B_r(x^0)} ((n-2)uf(u) - 2nF(u))\, dx
\end{aligned}
$$

is the function defined in Theorem 3.4.

Remark 6.8. The root of this formula is the classical frequency formula of F. Almgren for Q-valued harmonic functions [1]. Almgren's formula has subsequently been extended to various perturbations. Note however that while our formula may look like a perturbation of the "linear" formula for Q-valued harmonic functions, it is in fact a truly nonlinear formula.

Proof. Note that, for all $r \in (0,\delta)$,

$$H(r) = \frac{I(r) - \int_{B_1} x_n^+\, dx}{J(r)}.$$

Hence, for a.e. $r \in (0,\delta)$,

$$H'(r) = \frac{I'(r)}{J(r)} - \frac{(I(r) - \int_{B_1} x_n^+\, dx)}{J(r)}\frac{J'(r)}{J(r)},$$

Using the identities (3.6) and (3.7), we therefore obtain that, for a.e. $r \in (0, \delta)$,

$$\begin{aligned} H'(r) = {} & \frac{\left(2r \int_{\partial B_r(x^0)} (\nabla u \cdot \nu)^2 \, d\mathcal{H}^{n-1} - 3 \int_{\partial B_r(x^0)} u \nabla u \cdot \nu \, d\mathcal{H}^{n-1}\right) + K(r)}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} \\ & - (D(r) - V(r)) \frac{1}{r} \frac{\left(2r \int_{\partial B_r(x^0)} u \nabla u \cdot \nu \, d\mathcal{H}^{n-1} - 3 \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}\right)}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} \\ = {} & \frac{2}{r} \left(\frac{r^2 \int_{\partial B_r(x^0)} (\nabla u \cdot \nu)^2 \, d\mathcal{H}^{n-1}}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} - \frac{3}{2} D(r) \right) \\ & - \frac{2}{r} (D(r) - V(r)) \left(D(r) - \frac{3}{2} \right) + \frac{K(r)}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}}, \end{aligned} \tag{6.3}$$

where we have also used the fact, which follows from (3.10), that

$$D(r) = \frac{r \int_{\partial B_r(x^0)} u \nabla u \cdot \nu \, d\mathcal{H}^{n-1}}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}}. \tag{6.4}$$

Identity (6.1) now follows by merely rearranging (6.3), making use again of (6.4) and the fact that $D(r) = V(r) + H(r)$.

Since (6.1) holds, it follows by inspection that (6.2) holds if and only if

$$\begin{aligned} \int_{\partial B_r(x^0)} \left[r(\nabla u \cdot \nu) - D(r) u \right]^2 \, d\mathcal{H}^{n-1} + V^2(r) \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1} \\ = \int_{\partial B_r(x^0)} \left[r(\nabla u \cdot \nu) - H(r) u \right]^2 \, d\mathcal{H}^{n-1}. \end{aligned} \tag{6.5}$$

However, (6.5) is easily verified as a consequence of (6.4) and the fact that $D(r) = H(r) + V(r)$. In conclusion, identity (6.2) also holds. □

The following lemma is motivated by [9, (4.11)].

Lemma 6.9. *Let u be a variational solution of* (3.1). *Then*

$$r \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1} = \int_{B_r(x^0)} \left(n u^2 + (|\nabla u|^2 - u f(u))(r^2 - |x|^2) \right) dx. \tag{6.6}$$

Proof. As

$$\int_{B_r(x^0)} 2 n u^2 \, dx = - \int_{B_r(x^0)} u^2 \Delta (r^2 - |x|^2) \, dx,$$

a proof can be obtained integrating by parts twice. □

From now on we make the following assumption concerning the growth of f:

Assumption 6.10. *There exists a constant $C < +\infty$ such that*

$$|f(z)| \le C z \quad \textit{for all } z \in (0, z_0). \tag{6.7}$$

Note that when f is a C^1 function, the above is a consequence of $f(0) = 0$. Assumption 6.10 also implies that

$$|F(z)| \le Cz^2/2 \quad \text{for all } z \in (0, z_0).$$

As a corollary of Lemma 6.9 we obtain thus:

Corollary 6.11. *Let u be a variational solution of* (3.1) *such that Assumption 4.1 and Assumption 6.10 hold. Then there exists $r_0 > 0$ such that*

$$r \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1} \ge \int_{B_r(x^0)} u^2 \quad \textit{for all } r \in (0, r_0)$$

and

$$|K(r)| \le C_0 r \int_{\partial B_r(x^0)} u^2 \quad \textit{for all } r \in (0, r_0). \tag{6.8}$$

Theorem 6.12. *Let u be a variational solution of* (3.1) *such that Assumption 4.1 and Assumption 6.10 hold, let $x^0 \in \Sigma^u$, and let $\delta := \operatorname{dist}(x^0, \partial\Omega)/2$. Then the following hold, for some $r_0 \in (0, \delta)$ sufficiently small:*

(i) There exists a positive constant C_1 such that

$$H(r) - \frac{3}{2} \ge -C_1 r^2 \quad \textit{for all } r \in (0, r_0).$$

(ii) There exists a positive constant β such that

$$r \mapsto e^{\beta r^2} J(r) \quad \textit{is nondecreasing on } (0, r_0).$$

(iii) $r \mapsto \frac{1}{r} V^2(r) \in L^1(0, r_0)$.

(iv) The function H has a right limit $H(0+)$, where $H(0+) \ge 3/2$.

(v) The function

$$H'(r) - \frac{2}{r} \int_{\partial B_r(x^0)} \left[\frac{r(\nabla u \cdot \nu)}{\left(\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}\right)^{1/2}} - H(r) \frac{u}{\left(\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}\right)^{1/2}} \right]^2 d\mathcal{H}^{n-1}$$

is bounded from below by a function in $L^1(0, r_0)$.

Proof. Since assumption (6.7) holds, we deduce from (3.8) using (6.8) that, for all r sufficiently small,

$$I(r) - \frac{3}{2} J(r) - \int_{B_1} x_n^+ \, dx \ge -C_0 \int_0^r t^{-n-1} \int_{\partial B_t(x^0)} u^2 \, d\mathcal{H}^{n-1} dt. \tag{6.9}$$

This implies that, for all $r \in (0, r_0)$,

$$r^{-n-1} \int_{B_r} (|\nabla u|^2 - u f(u)) \, dx - \frac{3}{2} r^{-n-2} \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1} \ge -C_0 \int_0^r t^{-n-1} \int_{\partial B_t(x^0)} u^2 \, d\mathcal{H}^{n-1} dt. \tag{6.10}$$

Let $Y:(0,r_0)\to\mathbf{R}$ be given by

$$Y(r)=\int_0^r t^{-n-1}\int_{\partial B_t(x^0)} u^2\,d\mathcal{H}^{n-1}dt.$$

We deduce from (3.7) and (6.10) that

$$\frac{d}{dr}\left(\frac{Y'(r)}{r}\right)\geq -\alpha\frac{Y(r)}{r}, \tag{6.11}$$

for some positive constant $\alpha<+\infty$. Observe now that, as a consequence of the *Bessel type differential inequality* (6.11),

$$\frac{d^2}{dr^2}\left(\frac{Y(r)}{r^{1/2}}\right)\geq \frac{\frac{3}{4}-\alpha r^2}{r^{5/2}}Y(r)\geq 0\quad\text{for all } r\in(0,r_0), \tag{6.12}$$

for some r_0 sufficiently small. Thus $r\mapsto Y(r)/r^{1/2}$ is a convex function on $(0,r_0)$, and since

$$\lim_{r\to 0+}\frac{Y(r)}{r^{1/2}}=0,$$

it follows that

$$\frac{Y(r)}{r^{1/2}}-0\leq (r-0)\frac{d}{dr}\left(\frac{Y(r)}{r^{1/2}}\right)\quad\text{for all } r\in(0,r_0),$$

and therefore

$$\frac{3}{2}\frac{Y(r)}{r}\leq Y'(r)\quad\text{for all } r\in(0,r_0).$$

This implies, together with (6.9), that

$$r^{-n-1}\int_{B_r}(|\nabla u|^2-uf(u))-x_n^+(1-\chi_{\{u>0\}})\,dx-\frac{3}{2}r^{-n-2}\int_{\partial B_r}u^2\,d\mathcal{H}^{n-1}$$
$$\geq -\frac{2}{3}C_0r^{-n}\int_{\partial B_r(x^0)}u^2\,d\mathcal{H}^{n-1}, \tag{6.13}$$

which is equivalent to (i).

Taking also into account (3.7), (6.13) also implies that, for a.e. r sufficiently small,

$$J'(r)\geq -2\beta rJ(r),$$

for some constant $\beta>0$, which is equivalent to (ii).

Now, using (6.8) and part (i) in (6.1), we obtain that, for a.e. $r\in(0,r_0)$,

$$H'(r)\geq \frac{2}{r}V^2(r)-2C_1rV(r)-C_0r. \tag{6.14}$$

As

$$2C_1rV(r)\leq \frac{1}{r}V^2(r)+C_1^2r^3, \tag{6.15}$$

we obtain from (6.14) that, for a.e. $r\in(0,r_0)$,

$$H'(r)\geq \frac{1}{r}V^2(r)-C_1^2r^3-C_0r. \tag{6.16}$$

Since, by part (ii), $r \mapsto H(r)$ is bounded below as $r \to 0$, we obtain (iii). We also deduce from (6.16) and part (i) that $H(r)$ has a limit as $r \to 0+$, and that $H(0+) \geq 3/2$, thus proving (iv).

We now consider (6.2), and deduce from part (i) using (6.15) that, for a.e. $r \in (0, r_0)$,

$$H'(r) - \frac{2}{r}\int_{\partial B_r(x^0)} \left[\frac{r(\nabla u \cdot \nu)}{\left(\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}\right)^{1/2}} \right. \tag{6.17}$$

$$\left. - H(r) \frac{u}{\left(\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}\right)^{1/2}} \right]^2 d\mathcal{H}^{n-1}$$

$$\geq -2C_0 r V(r) - C_2 r$$

$$\geq -\frac{1}{r} V^2(r) - C_1^2 r^3 - C_0 r, \tag{6.18}$$

which, together with part (iii), proves (v). □

7. Blow-up limits

The Frequency Formula allows passing to blow-up limits.

Proposition 7.1. *Let u be a variational solution of* (3.1), *and let $x^0 \in \Sigma^u$. Then:*
(i) There exist $\lim_{r \to 0+} V(r) = 0$ and $\lim_{r \to 0+} D(r) = H_{x^0,u}(0+)$.
(ii) For any sequence $r_m \to 0+$ as $m \to \infty$, the sequence

$$v_m(x) := \frac{u(x^0 + r_m x)}{\sqrt{r_m^{1-n} \int_{\partial B_{r_m}(x^0)} u^2 \, d\mathcal{H}^{n-1}}} \tag{7.1}$$

is bounded in $W^{1,2}(B_1)$.

(iii) For any sequence $r_m \to 0+$ as $m \to \infty$ such that the sequence v_m in (7.1) *converges weakly in $W^{1,2}(B_1)$ to a blow-up limit v_0, the function v_0 is homogeneous of degree $H_{x^0,u}(0+)$ in B_1, and satisfies*

$$v_0 \geq 0 \text{ in } B_1, \; v_0 \equiv 0 \text{ in } B_1 \cap \{x_n \leq 0\} \text{ and } \int_{\partial B_1} v_0^2 \, d\mathcal{H}^{n-1} = 1.$$

Proof. We first prove that, for any sequence $r_m \to 0+$, the sequence v_m defined in (7.1) satisfies, for every $0 < \varrho < \sigma < 1$,

$$\int_{B_\sigma \setminus B_\varrho} |x|^{-n-3} \left[\nabla v_m(x) \cdot x - H_{x^0,u}(0+) v_m(x)\right]^2 dx \to 0 \quad \text{as } m \to \infty. \tag{7.2}$$

Indeed, for any such ϱ and σ, it follows by scaling from (6.18) that, for every m such that $r_m < \delta$,

$$\int_\varrho^\sigma \frac{2}{r} \int_{\partial B_r} \left[\frac{r(\nabla v_m \cdot \nu)}{\left(\int_{\partial B_r} v_m^2 \, d\mathcal{H}^{n-1}\right)^{1/2}} - H(r_m r) \frac{v_m}{\left(\int_{\partial B_r} v_m^2 \, d\mathcal{H}^{n-1}\right)^{1/2}} \right]^2 d\mathcal{H}^{n-1} \, dr$$
$$\leq H(r_m\sigma) - H(r_m\varrho) + \int_{r_m\varrho}^{r_m\sigma} \frac{1}{r} V^2(r) + C_1^2 r^3 + C_0 r dr \to 0 \quad \text{as } m \to \infty,$$

as a consequence of Theorem 6.12 (iv)-(v). The above implies that

$$\int_\varrho^\sigma \frac{2}{r} \int_{\partial B_r} \left[\frac{r(\nabla v_m \cdot \nu)}{\left(\int_{\partial B_r} v_m^2 \, d\mathcal{H}^{n-1}\right)^{1/2}} - H(0+) \frac{v_m}{\left(\int_{\partial B_r} v_m^2 \, d\mathcal{H}^{n-1}\right)^{1/2}} \right]^2 d\mathcal{H}^{n-1} \, dr$$
$$\to 0 \quad \text{as } m \to \infty. \tag{7.3}$$

Now note that, for every $r \in (\varrho, \sigma) \subset (0, 1)$ and all m as before, it follows by using Theorem 6.12 (ii), that

$$\int_{\partial B_r} v_m^2 \, d\mathcal{H}^{n-1} = \frac{\int_{\partial B_{r_m r}(x_0)} u^2 \, d\mathcal{H}^{n-1}}{\int_{\partial B_{r_m}(x_0)} u^2 \, d\mathcal{H}^{n-1}} \leq e^{\beta r_m^2 (1-r^2)} r^{n+2} \to r^{n+2}, m \to \infty.$$

Therefore (7.2) follows from (7.3), which proves our claim.

Let us also note that, as a consequence of Corollary 6.11, for each r sufficiently small

$$\left| D(r) - \frac{r \int_{B_r(x^0)} |\nabla u|^2 \, dx}{\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}} \right| \leq C r^2. \tag{7.4}$$

This implies that, for any sequence $r_m \to 0+$, the sequence v_m defined in (7.1) satisfies

$$|D(r_m) - \int_{B_1} |\nabla v_m|^2 \, dx| \leq C r_m^2. \tag{7.5}$$

We can now prove all parts of the Proposition.

(i) Suppose towards a contradiction that (i) is not true. Let $s_m \to 0$ be such that the sequence $V(s_m)$ is bounded away from 0. From the integrability of $r \mapsto \frac{2}{r} V^2(r)$ we obtain that

$$\min_{r \in [s_m, 2s_m]} V(r) \to 0 \quad \text{as } m \to \infty.$$

Let $t_m \in [s_m, 2s_m]$ be such that $V(t_m) \to 0$ as $m \to \infty$. For the choice $r_m := t_m$ for every m, the sequence v_m given by (7.1) satisfies (7.2). The fact that $V(r_m) \to 0$ implies that $D(r_m)$ is bounded, and hence, using (7.5), that v_m is bounded in $W^{1,2}(B_1)$. Let v_0 be any weak limit of v_m along a subsequence. Note that by the compact embedding $W^{1,2}(B_1) \hookrightarrow L^2(\partial B_1)$, v_0 has norm 1 on $L^2(\partial B_1)$, since this is true for v_m for all m. It follows from (7.2) that v_0 is homogeneous of degree $H_{x^0,u}(0+)$. Note that, by using Theorem 6.12 (ii),

$$\begin{aligned} V(s_m) &= \frac{s_m^{-n-1}\int_{B_{s_m}(x^0)} x_n^+(1-\chi_{\{u>0\}})\,dx}{s_m^{-n-2}\int_{\partial B_{s_m}(x^0)} u^2\,d\mathcal{H}^{n-1}} \\ &\le \frac{s_m^{-n-1}\int_{B_{r_m}(x^0)} x_n^+(1-\chi_{\{u>0\}})\,dx}{e^{\beta[(r_m^2/4)-s_m^2]}(r_m/2)^{-n-2}\int_{\partial B_{r_m/2}(x^0)} u^2\,d\mathcal{H}^{n-1}} \\ &\le \frac{e^{3\beta r_m^2/4}}{2}\frac{\int_{\partial B_{r_m}(x^0)} u^2\,d\mathcal{H}^{n-1}}{\int_{\partial B_{r_m/2}(x^0)} u^2\,d\mathcal{H}^{n-1}} V(r_m) \\ &= \frac{e^{3\beta r_m^2/4}}{2\int_{\partial B_{1/2}} v_m^2\,d\mathcal{H}^{n-1}} V(r_m). \end{aligned} \tag{7.6}$$

Since, at least along a subsequence,

$$\int_{\partial B_{1/2}} v_m^2\,d\mathcal{H}^{n-1} \to \int_{\partial B_{1/2}} v_0^2\,d\mathcal{H}^{n-1} > 0,$$

(7.6) leads to a contradiction. It follows that indeed $V(r) \to 0$ as $r \to 0+$. This implies that $D(r) \to H_{x^0,u}(0+)$.

(ii) Let r_m be an arbitrary sequence with $r_m \to 0+$. In view of (7.5), the boundedness of the sequence v_m in $W^{1,2}(B_1)$ is equivalent to the boundedness of $D(r_m)$, which is true by (i).

(iii) Let $r_m \to 0+$ be an arbitrary sequence such that v_m converges weakly to v_0. The homogeneity degree $H_{x^0,u}(0+)$ of v_0 follows directly from (7.2). The fact that $\int_{\partial B_1} v_0^2\,d\mathcal{H}^{n-1} = 1$ is a consequence of $\int_{\partial B_1} v_m^2\,d\mathcal{H}^{n-1} = 1$ for all m, and the remaining claims of the Proposition are obvious. □

8. Concentration compactness in two dimensions

In the two-dimensional case we prove concentration compactness which allows us to preserve variational solutions in the blow-up limit at degenerate points and excludes concentration. In order to do so we combine the concentration compactness result of Evans and Müller [7] with information gained by our Frequency Formula. In addition, we obtain strong convergence of our blow-up sequence which is necessary in order to prove our main theorems. The question whether the following Theorem holds in any dimension seems to be a hard one.

Theorem 8.1. *Let $n = 2$, let the nonlinearity satisfy Assumption 6.10 and let u be a variational solution of* (3.1)*, and let $x^0 \in \Sigma^u$. Let $r_m \to 0+$ be such that the sequence v_m given by* (7.1) *converges weakly to v_0 in $W^{1,2}(B_1)$. Then v_m converges to v_0 strongly in $W^{1,2}_{\mathrm{loc}}(B_1 \setminus \{0\})$, v_0 is continuous on B_1 and Δv_0 is a nonnegative Radon measure satisfying $v_0 \Delta v_0 = 0$ in the sense of Radon measures in B_1.*

Proof. The proof is similar to that in [23, Theorem 9.1], but there are some subtle changes so that we will supply the whole proof for the sake of completeness.

Note first that the homogeneity of v_0 given by Proposition 7.1, together with the fact that v_0 belongs to $W^{1,2}(B_1)$, imply that v_0 is continuous. As

$$\begin{aligned}\Delta v_m(x) &= \frac{r_m^2 \Delta u(x^0+r_m x)}{\sqrt{r_m^{-1}\int_{\partial B_{r_m}(x^0)} u^2\, d\mathcal{H}^{n-1}}} = \frac{-r_m^2 f(u(x^0+r_m x))}{\sqrt{r_m^{-1}\int_{\partial B_{r_m}(x^0)} u^2\, d\mathcal{H}^{n-1}}} \\ &\geq -C_1 \frac{-r_m^2 u(x^0+r_m x)}{\sqrt{r_m^{-1}\int_{\partial B_{r_m}(x^0)} u^2\, d\mathcal{H}^{n-1}}} = -C_1 r_m^2 v_m(x) \text{ for } v_m(x)>0,\end{aligned} \tag{8.1}$$

we obtain from the sign of the singular part of Δv_m with respect to the Lebesgue measure that $\Delta v_m \geq -C_1 r_m^2 v_m$ in B_1 in the sense of measures. From [11, Theorem 8.17] we infer therefore that

$$\sup_{B_\sigma} v_m \leq C_2(\sigma)\int_{B_1} v_m\, dx$$

for each $\sigma \in (0,1)$. Consequently

$$\Delta v_m \geq -C_3(\sigma) r_m^2 \text{ in } B_\sigma \tag{8.2}$$

in the sense of measures. It follows that for each nonnegative $\eta \in C_0^\infty(B_1)$ such that $\eta = 1$ in $B_{(\sigma+1)/2}$

$$\begin{aligned}\int_{B_{(\sigma+1)/2}} d\Delta v_m &= \int_{B_{(\sigma+1)/2}} \eta\, d\Delta v_m \leq \int_{B_1} \eta\, d\Delta v_m + C_1 r_m^2 \int_{B_1\setminus B_{(\sigma+1)/2}} v_m \\ &= \int_{B_1} v_m \Delta\eta + C_1 r_m^2 \int_{B_1\setminus B_{(\sigma+1)/2}} v_m \leq C_4 \text{ for all } m\in\mathbf{N}.\end{aligned} \tag{8.3}$$

From (8.1) and the fact that v_m is bounded in $L^1(B_1)$, we obtain also that Δv_0 is a nonnegative Radon measure on B_1. The continuity of v_0 implies therefore that $v_0 \Delta v_0$ is well defined as a nonnegative Radon measure on B_1.

In order to apply the concentrated compactness result [7], we modify each v_m to

$$\tilde v_m := (v_m + C_3(\sigma) r_m^2 |x|^2) * \phi_m \in C^\infty(B_1),$$

where ϕ_m is a standard mollifier such that

$$\Delta \tilde v_m \geq 0, \int_{B_\sigma} d\Delta\tilde v_m \leq C_2 < +\infty \quad \text{for all } m,$$

and

$$\|v_m - \tilde v_m\|_{W^{1,2}(B_\sigma)} \to 0 \quad \text{as } m\to\infty.$$

From [8, Chapter 4, Theorem 3] we know that $\nabla \tilde v_m$ converges a.e. to the weak limit ∇v_0, and the only possible problem is concentration of $|\nabla \tilde v_m|^2$. By [7, Theorem 1.1] and [7, Theorem 3.1] we obtain that

$$\partial_1 \tilde v_m \partial_2 \tilde v_m \to \partial_1 v_0 \partial_2 v_0$$

and

$$(\partial_1 \tilde{v}_m)^2 - (\partial_2 \tilde{v}_m)^2 \to (\partial_1 v_0)^2 - (\partial_2 v_0)^2$$

in the sense of distributions on B_σ as $m \to \infty$. It follows that

$$\partial_1 v_m \partial_2 v_m \to \partial_1 v_0 \partial_2 v_0 \tag{8.4}$$

and

$$(\partial_1 v_m)^2 - (\partial_2 v_m)^2 \to (\partial_1 v_0)^2 - (\partial_2 v_0)^2$$

in the sense of distributions on B_σ as $m \to \infty$. Let us remark that this alone would allow us to pass to the limit in the domain variation formula for v_m in the set $\{x_2 > 0\}$.

Observe now that (7.2) shows that for each $0 < \varrho < \sigma$

$$\nabla v_m(x) \cdot x - H_{x^0,u}(0+) v_m(x) \to 0$$

strongly in $L^2(B_\sigma \setminus B_\varrho)$ as $m \to \infty$. It follows that

$$\partial_1 v_m x_1 + \partial_2 v_m x_2 \to \partial_1 v_0 x_1 + \partial_2 v_0 x_2$$

strongly in $L^2(B_\sigma \setminus B_\varrho)$ as $m \to \infty$. But then

$$\int_{B_\sigma \setminus B_\varrho} (\partial_1 v_m \partial_1 v_m x_1 + \partial_1 v_m \partial_2 v_m x_2)\eta \, dx$$
$$\to \int_{B_\sigma \setminus B_\varrho} (\partial_1 v_0 \partial_1 v_0 x_1 + \partial_1 v_0 \partial_2 v_0 x_2)\eta \, dx$$

for each $\eta \in C^0_0(B_\sigma \setminus \overline{B}_\varrho)$ as $m \to \infty$. Using (8.4), we obtain that

$$\int_{B_\sigma \setminus B_\varrho} (\partial_1 v_m)^2 x_1 \eta \, dx \to \int_{B_\sigma \setminus B_\varrho} (\partial_1 v_0)^2 x_1 \eta \, dx$$

for each $0 \le \eta \in C^0_0((B_\sigma \setminus \overline{B}_\varrho) \cap \{x_1 > 0\})$ and for each $0 \ge \eta \in C^0_0((B_\sigma \setminus \overline{B}_\varrho) \cap \{x_1 < 0\})$ as $m \to \infty$. Repeating the above procedure three times for rotated sequences of solutions (by 45 degrees) yields that ∇v_m converges strongly in $L^2_{\text{loc}}(B_\sigma \setminus \overline{B}_\varrho)$. Since σ and ϱ with $0 < \varrho < \sigma < 1$ were arbitrary, it follows that ∇v_m converges to ∇v_0 strongly in $L^2_{\text{loc}}(B_1 \setminus \{0\})$.

As a consequence of the strong convergence and Assumption 6.10, we obtain now, using the fact that the singular part of Δv_m lives on a subset of $\{v_m = 0\}$, that

$$\left| \int_{B_1} \nabla(\eta v_0) \cdot \nabla v_0 \, dx \right| \leftarrow \left| \int_{B_1} \nabla(\eta v_m) \cdot \nabla v_m \, dx \right|$$
$$\le C_1 r_m^2 \int_{B_1} \eta v_m^2 \, dx \to 0, m \to \infty \qquad \text{for all } \eta \in C^1_0(B_1 \setminus \{0\}).$$

Combined with the fact that $v_0 = 0$ in $B_1 \cap \{x_2 \le 0\}$ and the fact that the singular part of Δv_0 lives on a subset of $\{v_0 = 0\} \cup \{x_2 = 0\}$, this proves that $v_0 \Delta v_0 = 0$ in the sense of Radon measures on B_1. □

9. Degenerate points in two dimensions

Theorem 9.1. *Let $n = 2$, let the nonlinearity satisfy Assumption 6.10 and let u be a variational solution of* (3.1). *Then at each point x^0 of the set Σ^u there exists an integer $N(x^0) \geq 2$ such that*

$$H_{x^0,u}(0+) = N(x^0)$$

and

$$\frac{u(x^0 + rx)}{\sqrt{r^{-1}\int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^1}} \to \frac{\rho^{N(x^0)}|\sin(N(x^0)\min(\max(\theta,0),\pi))|}{\sqrt{\int_0^\pi \sin^2(N(x^0)\theta)d\theta}} \quad \text{as } r \to 0+,$$

strongly in $W^{1,2}_{\text{loc}}(B_1 \setminus \{0\})$ and weakly in $W^{1,2}(B_1)$, where $x = (\rho\cos\theta, \rho\sin\theta)$.

Proof. Let $r_m \to 0+$ be an arbitrary sequence such that the sequence v_m given by (7.1) converges weakly in $W^{1,2}(B_1)$ to a limit v_0. By Proposition 7.1 (iii) and Theorem 8.1, $v_0 \not\equiv 0$, v_0 is homogeneous of degree $H_{x^0,u}(0+) \geq 3/2$, v_0 is continuous, $v_0 \geq 0$ and $v_0 \equiv 0$ in $\{x_2 \leq 0\}$, $v_0 \Delta v_0 = 0$ in B_1 as a Radon measure, and the convergence of v_m to v_0 is strong in $W^{1,2}_{\text{loc}}(B_1 \setminus \{0\})$. Moreover, the strong convergence of v_m and the fact proved in Proposition 7.1 (i) that $V(r_m) \to 0$ as $m \to \infty$ imply that

$$0 = \int_{B_1} \left(|\nabla v_0|^2 \text{div } \phi - 2\nabla v_0 D\phi \nabla v_0 \right) dx$$

for every $\phi \in C^1_0(B_1 \cap \{x_2 > 0\}; \mathbf{R}^2)$. It follows that at each polar coordinate point $(1, \theta) \in \partial B_1 \cap \partial\{v_0 > 0\}$,

$$\lim_{\tau \to \theta+} \partial_\theta v_0(1, \tau) = -\lim_{\tau \to \theta-} \partial_\theta v_0(1, \tau).$$

Computing the solution of the ODE on ∂B_1, using the homogeneity of degree $H_{x^0,u}(0+)$ of v_0 and the fact that $\int_{\partial B_1} v_0^2 \, d\mathcal{H}^1 = 1$, yields that $H_{x^0,u}(0+)$ must be an *integer* $N(x^0) \geq 2$ and that

$$v_0(\rho, \theta) = \frac{\rho^{N(x^0)}|\sin(N(x^0)\min(\max(\theta,0),\pi))|}{\sqrt{\int_0^\pi \sin^2(N(x^0)\theta)d\theta}}. \tag{9.1}$$

The desired conclusion follows from Proposition 7.1 (ii). □

Theorem 9.2. *Let $n = 2$, let the nonlinearity satisfy Assumption 6.10 and let u be a variational solution of* (3.1). *Then the set Σ^u is locally in Ω a finite set.*

Proof. Suppose towards a contradiction that there is a sequence of points $x^m \in \Sigma^u$ converging to $x^0 \in \Omega$, with $x^m \neq x^0$ for all m. From the upper semicontinuity Lemma 4.4 (v) we infer that $x^0 \in \Sigma^u$. Choosing $r_m := 2|x^m - x^0|$, there is no loss of generality in assuming that the sequence $(x^m - x^0)/r_m$ is constant, with value

$z \in \{(-1/2, 0), (1/2, 0)\}$. Consider the blow-up sequence v_m given by (7.1), and also the sequence

$$u_m(x) = \frac{u(x^0 + r_m x)}{r_m^{3/2}}.$$

Note that each u_m is a variational solution of (4.5), and v_m is a scalar multiple of u_m. Since $x_m \in \Sigma^u$, it follows that $z \in \Sigma^{u_m}$. Therefore, Theorem 6.12 (i) shows that, for each m,

$$r \int_{B_r(z)} |\nabla v_m|^2\, dx \geq \left(\frac{3}{2} - C_1 r^2\right) \int_{\partial B_r(z)} v_m^2\, d\mathcal{H}^1 \quad \text{for all } r \in (0, 1/2).$$

It is a consequence of Theorem 9.1 that the sequence v_m converges strongly in $W^{1,2}(B_{1/4}(z))$ to v_0 given by (9.1), hence

$$r \int_{B_r(z)} |\nabla v_0|^2\, dx \geq \left(\frac{3}{2} - C_1 r^2\right) \int_{\partial B_r(z)} v_0^2\, d\mathcal{H}^1 \quad \text{for all } r \in (0, 1/4).$$

But this contradicts the fact (which can be checked directly) that

$$\lim_{r \to 0+} \frac{r \int_{B_r(z)} |\nabla v_0|^2\, dx}{\int_{\partial B_r(z)} v_0^2\, d\mathcal{H}^1} = 1.$$

□

10. Conclusion

Theorem 10.1. *Let $n = 2$, let u be a weak solution of (3.1) satisfying Assumption 4.1, let the free boundary $\partial\{u > 0\}$ be a continuous injective curve $\sigma = (\sigma_1, \sigma_2)$ such that $\sigma(0) = x^0 = (x_1^0, 0)$, and assume that the nonlinearity f satisfies either Assumption 6.10, or $f \geq 0$ in a right neighborhood of 0.*

(i) If $M_{x^0,u}(0+) = \int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\, dx$, then the free boundary is in a neighborhood of x^0 the union of two C^1-graphs of functions $\eta_1 : (x_1^0 - \delta, x_1^0] \to \mathbf{R}$ and $\eta_2 : [x_1^0, x_1^0 + \delta) \to \mathbf{R}$ which are both continuously differentiable up to x_1^0 and satisfy $\eta_1'(x_1^0) = -1/\sqrt{3}$ and $\eta_2'(x_1^0) = 1/\sqrt{3}$.

(ii) Else $M_{x^0,u}(0+) = 0$, and $\sigma_1(t) \neq x_1^0$ in $(-t_1, t_1) \setminus \{0\}$, and $\sigma_1 - x_1^0$ does not change its sign at $t = 0$, and

$$\lim_{t \to 0} \frac{\sigma_2(t)}{\sigma_1(t) - x_1^0} = 0.$$

If we assume in addition that either $\{u > 0\}$ is a supergraph of a function in the x_2-direction or that $\{u > 0\}$ is a Lipschitz set, then the set S^u of stagnation points is locally in Ω a finite set, and at each stagnation point x^0 the statement in (i) holds.

Proof. We first show that the set Σ^u is empty. Suppose towards a contradiction that there exists $x^0 \in \Sigma^u$. From Theorem 9.1 we infer that there exists an integer

$N(x^0) \geq 2$ such that

$$v_r(x) := \frac{u(x^0+rx)}{\sqrt{r^{-1}\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^1}} \tag{10.1}$$
$$\to \frac{\rho^{N(x^0)}|\sin(N(x^0)\min(\max(\theta,0),\pi))|}{\sqrt{\int_0^\pi \sin^2(N(x^0)\theta)d\theta}} \quad \text{as } r\to 0+,$$

strongly in $W^{1,2}_{\text{loc}}(B_1\setminus\{0\})$ and weakly in $W^{1,2}(B_1)$, where $x=(\rho\cos\theta,\rho\sin\theta)$. On the other hand, Theorem 4.6 (ii) implies that for any ball $\tilde B \subset\subset B_1\cap\{x_2>0\}$, $v_r>0$ in $\tilde B$. Consequently (see (8.1))

$$|\Delta v_r| \leq C_1 r^2 v_r \text{ in } \tilde B$$

for sufficiently small r. It follows that v_0 is harmonic in $\tilde B$, contradicting (10.1) in view of $N(x^0)\geq 2$. Hence Σ^u is indeed empty.

Let us consider the case $M_{x^0,u}(0+) = \int_{B_1} x_2^+ \chi_{\{x:\pi/6<\theta<5\pi/6\}}\, dx$. From Theorem 4.6 and Proposition 5.4 we infer that

$$\frac{u(x^0+rx)}{r^{3/2}} \to \frac{\sqrt{2}}{3}\rho^{3/2}\cos(\frac{3}{2}(\min(\max(\theta,\frac{\pi}{6}),\frac{5\pi}{6})-\frac{\pi}{2})) \quad \text{as } r\to 0+, \tag{10.2}$$

strongly in $W^{1,2}_{\text{loc}}(\mathbf{R}^2)$ and locally uniformly on $\mathbf{R}^2$, where $x=(\rho\cos\theta,\rho\sin\theta)$.

We assume for simplicity that $x^0=0$. We will show that in a neighborhood of 0 the free boundary is the union of two C^1-graphs $\eta_1:(-\delta,0]\to\mathbf{R}$ and $\eta_2:[0,\delta)\to\mathbf{R}$ which are both continuously differentiable up to 0 and satisfy $\eta_1'(0)=-1/\sqrt{3}$ and $\eta_2'(0)=1/\sqrt{3}$: as the proofs for $x_1>0$ and $x_1<0$ are similar, we will give only the proof for $x_1>0$.

For

$$v(x) := \frac{u(\rho x)}{\rho^{3/2}}$$

we have that

$$\Delta v(x) = -\sqrt{\rho} f(u(\rho x)) \text{ for } v(x)>0,$$
$$|\nabla v(x)|^2 = x_2 \text{ for } x\in\partial\{v>0\}.$$

Scaling once more for $\xi\in\partial B_1\cap\partial\{v>0\}$, which implies that for ρ small enough, $\xi_2\geq\frac{1}{10}$, we obtain for

$$w(x) := \frac{v(\xi+rx)}{\xi_2 r}$$

that

$$\Delta w(x) = -\frac{\sqrt{\rho}r}{\xi_2} f(\rho\xi+r\rho x) \text{ for } w(x)>0,$$
$$|\nabla w(x)|^2 = 1+\frac{rx_2}{\xi_2} \text{ for } x\in\partial\{w>0\}.$$

We are going to use a flatness-implies-regularity result of [5]. Note that although not stated in [5], [5, Lemma 4.1] yields as in the proof of [5, Theorem 1.1] that for each $\epsilon \in (0, \epsilon_0)$

$$\max(x \cdot \bar\nu - \epsilon, 0) \le w \le \max(x \cdot \bar\nu + \epsilon, 0) \text{ in } B_1 \tag{10.3}$$

implies that the outward unit normal ν^w on the free boundary $\partial\{w > 0\}$ satisfies

$$|\nu^w(0) - \bar\nu| \le C\epsilon^2.$$

Note that $\nu^w(0) = \nu(\rho\xi)$. Since (10.3) is by (10.2) satisfied for $\bar\nu = (1/2, -\sqrt{3}/2)$, $r = r(\epsilon)$ and every sufficiently small $\rho > 0$, we obtain that the outward unit normal $\nu(x)$ on $\partial\{u > 0\}$ converges to $\bar\nu$ as $x \to 0, x_1 > 0$. It follows that the present curve component is the graph of a C^1-function (up to $x_1 = x_1^0$) in the x_2-direction.

The remaining statements of the Theorem follow from Theorem 4.6. □

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