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Estimation of Concentration Parameter for Simultaneous Circular Functional Relationship Model Assuming Unequal Error Variance

(Anggaran Parameter Kepekatan untuk Model Hubungan Fungsian Membulat Serentak dengan Andaian Ralat Varians tak Sama)

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ABSTRACT

In this study, we propose the estimation of the concentration parameter for simultaneous circular functional relationship model. In this case, the variances of the error term are not necessarily equal and the ratio of the concentration parameter $\lambda = \frac{v}{\kappa}$ *is not necessarily 1. The modified Bessel function was expended by using the asymptotic power series and it became a cubic equation of* κ*. From the cubic equation of* κ*, the roots were obtained by using the* `polyroot` *function in SPlus software. Simulation study was done to study the mean, estimated bias, absolute relative estimated bias, estimated standard error and estimated root mean square error of the estimation of the concentration parameter. From the simulation study, large concentration parameter and sample size show that the estimated concentration parameter has smaller bias. Also, an illustration to a real wind and wave data set is given to show its practical applicability.*

Keywords: Circular variables; concentration parameter; simultaneous circular functional relationship model; unequal error variance

ABSTRAK

Dalam kajian ini, kami ingin mencadangkan anggaran parameter kepekatan untuk model hubungan fungsian membulat serentak. Dalam kes ini, perbezaan tempoh ralat tidak semestinya sama dan nisbah parameter $\lambda = \frac{v}{\kappa}$ *kepekatan tidak semestinya 1. Fungsi Bessel yang diubah suai telah dimajukan dengan menggunakan siri kuasa asimptot dan ia menjadi satu persamaan kubik. Daripada persamaan kubik, punca diperoleh dengan menggunakan fungsi* `polyroot` *dalam perisian statistic* `SPlus`*. Kajian simulasi telah dilakukan untuk mengkaji purata, anggaran berat sebelah, mutlak relatif berat sebelah dianggarkan, anggaran ralat piawai dan dianggarkan punca purata ralat kuasa dua daripada anggaran parameter kepekatan. Daripada kajian simulasi, parameter kepekatan dan sampel saiz yang besar menunjukkan anggaran parameter kepekatan mempunyai berat sebelah yang lebih kecil. Ilustrasi menggunakan data sebenar angin dan gelombang diberikan untuk menunjukkan kesesuaian praktikal.*

Kata kunci: Model hubungan fungsian membulat serentak; parameter kepekatan; pemboleh ubah membulat; ralat varians tak sama

INTRODUCTION

A circular observation may be regarded as a unit vector in a plane, or as a point on a circle of unit radius. Each circular observation may be specified by the angle from the initial direction to the point on the circle corresponding to the observation, once an initial direction and an orientation of the circle have been chosen. Data are usually measured in degrees or in radians. The most popular example is the data of wind directions, which provides a natural source of circular data.

The distribution of the directions may arise either as a conditional distribution for a given speed, or as a marginal distribution of the wind speed and direction (Mardia & Jupp 2000). Besides, in physics, angular data is used in experiments with a bubble chamber, where points representing events are observed through circular window (Mardia 1972) meanwhile the geologists are studying in the direction of the earth's magnetic pole (Batschelet 1981). In medical application, Jammaladaka et al. (1986) have discussed about the angle of knee flexion to assess the recovery of ortheopaedic patients (Jammaladaka & Sengupta 2001).

Circular data refers to a set of observations measured by angles in the intervals of $[0, 2\pi)$ radians or $[0°, 360°)$. Circular data are not able to escape very far from each other and certainly not able to hide from view (Fisher 1993).

The von Mises distribution is said to be the most practicable distribution on the circle (Mardia & Jupp 2000). The distribution is analogous to the normal distribution because it has some similar characteristics with the normal distribution (Hassan et al. 2012).

The von Mises distribution $VM(\mu, \kappa)$ is first used by von Mises (1918) to study the deviations of atomic weights from integer values. Since then, it has been applied in studies of wildlife movement, for example, Kendall (1974) studied the bird navigation, where the von Mises variables are used for the distribution of angular errors when a bird is lost to sight over the horizon (Best & Fisher 1979).

The von Mises distribution has the probability density function

$$g(\theta;\mu,\kappa)=\frac{1}{2\pi I_0(\kappa)}e^{\kappa\cos(\theta-\mu)}, \tag{1}$$

where $I_0(\kappa)$ is the modified Bessel function of the first kind; and order zero, which can be defined by :

$$I_0(\kappa)=\frac{1}{2\pi}\int_0^{2\pi}e^{\kappa\cos\theta}d\theta, \tag{2}$$

where μ is the mean direction; and κ is the concentration parameter.

The distribution is influenced by the concentration parameter κ inversely as the Normal distribution $N(\mu, \sigma^2)$ is influenced by σ^2. Thus, a concentrated von Mises distribution will have high concentration parameter, and a dispersed von Mises distribution will have a low concentration parameter (Caires & Wyatt 2003).

MATERIALS AND METHODS

SIMULTANEOUS CIRCULAR FUNCTIONAL RELATIONSHIP MODEL

The error-in-variables model differs from the ordinary linear regression model in that the true independent variables or the explanatory variables are not observed directly, but are masked by measurements error (Hussin et al. 2009). In this paper, we are considering a type of error-in-variables model, namely a functional relationship model for circular variables that is developed to be simultaneous. Suppose the variables $Y_{ji}(j = 1,\dots, q$: $i = 1,\dots,n)$ and $X_i(i = 1,\dots,n)$ are related by the linear functional relationship of $Y_j = \alpha_j + \beta_j X (\text{mod } 2\pi)$. Let (x_i, y_{ji}) be the observed values and these observations correspond to measurement of the actual values of (X_i, Y_{ji}) made with some random errors of $(\delta_i, \varepsilon_{ji})$. The random errors δ_i and ε_{ji} are assumed to be distributed independently with von Mises distribution $\delta_i \sim VM(0, \kappa)$ and $\varepsilon_{ji} \sim VM(0, v_j)$, respectively.

The model of simultaneous linear functional are can be written as:

$$x_i = X_i + \delta_i \text{ and } y_{ji} = Y_{ji} + \varepsilon_{ji},$$

where $Y_j = \alpha_j + \beta_j X (\text{mod } 2\pi)$ for $j = 1,\dots, q; 1,\dots,n$.

PARAMETER ESTIMATION

In this case, the variances of the error term are not necessarily equal and the ratio of the concentration parameter $\lambda = \frac{v}{\kappa}$ is not necessarily 1. The log likelihood function of the von Mises distribution is given in (3).

$$\log L=-2n\log(2\pi)-n\log I_0(\kappa)-n\sum_{j=1}^{q}\log I_0(\lambda_j\kappa)+\kappa\sum_{i=1}^{n}\cos(x_i-X_i)+\kappa\sum_{j=1}^{q}\lambda_j\sum_{i=1}^{n}\cos(y_{ji}-\alpha_j-X_i) \tag{3}$$

MAXIMUM LIKELIHOOD ESTIMATION OF α_j

Differentiating (3) with respect to α_j we get

$$\frac{\partial\log L}{\partial\alpha_j}=\kappa\sum_{j=1}^{q}\lambda_j\sum_{i=1}^{n}\sin(y_{ji}-\alpha_j-X_i), \tag{4}$$

and by setting $\frac{\partial\log L}{\partial\alpha_j}=0$,

we obtain

$$\kappa\sum_{j=1}^{q}\lambda_j\sum_{i=1}^{n}\sin(y_{ji}-\alpha_j-X_i)=0. \tag{5}$$

Rearranging the terms in (5), we get

$$\kappa\sum_{j=1}^{q}\lambda_j\sum_{i=1}^{n}\left[\sin(y_{ji}-X_i)\cos(\alpha_j)-\cos(y_{ji}-X_i)\sin(\alpha_j)\right]=0, \tag{6}$$

$$\sum_{i=1}^{n}\sin(y_{ji}-X_i)\cos(\alpha_j)=\sum_{i=1}^{n}\cos(y_{ji}-X_i)\sin(\alpha_j). \tag{7}$$

$$\frac{\sin(\alpha_j)}{\cos(\alpha_j)}=\frac{\sum_{i=1}^{n}\sin(y_{ji}-X_i)}{\sum_{i=1}^{n}\cos(y_{ji}-X_i)}=\frac{S}{C} \tag{8}$$

or

$$\tan(\alpha_j)=\frac{S}{C}. \tag{9}$$

Therefore,

$$\hat{\alpha}_j=\tan^{-1}\left(\frac{S}{C}\right) \tag{10}$$

where $\tan^{-1}\left(\frac{S}{C}\right)$ is the arc tangent of $\left(\frac{S}{C}\right)$ in the range $[0, 2\pi)$, that is

$$\hat{\alpha}_j=\begin{cases}\tan^{-1}\left(\frac{S}{C}\right) & \text{when } S>0,\ C>0\\ \tan^{-1}\left(\frac{S}{C}\right)+\pi & \text{when } C<0\\ \tan^{-1}\left(\frac{S}{C}\right)+2\pi & \text{when } S<0,\ C>0\end{cases}. \tag{11}$$

MAXIMUM LIKELIHOOD ESTIMATION OF X_i

Differentiating (3) with respect to X_i, we obtain

$$\frac{\partial \log L}{\partial X_i} = \kappa \sin(x_i - X_i) + \kappa \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \sin(y_{ji} - \alpha_j - X_i). \quad (12)$$

Setting $\frac{\partial \log L}{\partial X_i} = 0$, we get

$$\kappa \sin(x_i - X_i) + \kappa \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \sin(y_{ji} - \alpha_j - X_i) = 0. \quad (13)$$

X_i may be solved iteratively with some "initial guess", suppose $\hat{X}_{i0}$ is an initial estimate of $\hat{X}_i$. Then,

$$x_i - \hat{X}_i = x_i - \hat{X}_{i0} + \hat{X}_{i0} - \hat{X}_i = \left(x_i - \hat{X}_{i0}\right) + \Delta_i$$
$$\text{where} \quad \Delta_i = \hat{X}_{i0} - \hat{X}_i \quad (14)$$

We may also have $y_{ji} - \hat{\alpha}_j - \hat{X}_i = (y_{ji} - \hat{\alpha}_j - \hat{X}_{i0}) + \Delta_i$.

Thus, the partial derivative equation above becomes:

$$\sin\left(x_i - \hat{X}_{i0} + \Delta_i\right) + \kappa \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \sin(y_{ji} - \hat{\alpha}_j - \hat{X}_{i0} + \Delta_i) = 0. \quad (15)$$

When Δ_i is small, then $\cos\Delta_i \approx 1$ and $\sin \Delta_i \approx \Delta_i$. Hence the equation is simplified (approximately) become:

$$\hat{X}_{i1} \approx \hat{X}_{10} + \frac{\sin\left(x_i - \hat{X}_{i0}\right) + \kappa \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \sin\left(y_{ji} - \hat{\alpha}_j - \hat{X}_{i0}\right)}{\cos\left(x_i - \hat{X}_{i0}\right) + \kappa \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \cos\left(y_{ji} - \hat{\alpha}_j - \hat{X}_{i0}\right)}. \quad (16)$$

where $\hat{X}_{i1}$ is an improvement of $\hat{X}_{i0}$.

MAXIMUM LIKELIHOOD ESTIMATION OF K

From the log likelihood function of the von Mises distribution in (3), the equation is differentiated with respect to κ and we get

$$\frac{\partial \log L}{\partial \kappa} = -nA(\kappa) - n\lambda_j A(\lambda_j \kappa) + \sum_{i=1}^{n} \cos\left(x_i - X_i\right) + \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \cos\left(y_{ji} - \alpha_j - X_i\right) \quad (17)$$

Setting $\frac{\partial \log L}{\partial \kappa} = 0$, then obtain

$$-nA(\kappa) - n\lambda_j A(\lambda_j \kappa) + \sum_{i=1}^{n} \cos\left(x_i - X_i\right) + \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \cos\left(y_{ji} - \alpha_j - X_i\right) = 0, \quad (18)$$

For the value of κ near to 1, Dobson (1978) gives the approximation as in (19)

$$A^{-1}(\kappa) \approx \frac{9 - 8\kappa + 3\kappa^2}{8(1-\kappa)}, \quad (19)$$

whereas when the value of $\lambda_j = 1$, Fisher (1993) gives the approximation

$$A^{-1}(\kappa) = \begin{cases} 2\kappa + \kappa^3 + \frac{5}{6}\kappa^3 & \kappa < 0.53 \\ -0.4 + 1.39\kappa + \frac{0.43}{(1-\kappa)} & 0.53 \le \kappa < 0.85 \\ \frac{1}{\kappa^3 - 4\kappa^2 + 3\kappa} & \kappa \ge 0.85 \end{cases} . \quad (20)$$

However, in this case, the value of λ_j is not necessarily 1. Hence, we rearrange (18) and get

$$n\left[A(\kappa) + \lambda_j A(\lambda_j \kappa)\right] = \sum_{i=1}^{n} \cos\left(x_i - X_i\right) + \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \cos\left(y_{ji} - \alpha_j - X_i\right), \quad (21)$$

$$A(\kappa) + \lambda_j A(\lambda_j \kappa) = \frac{1}{n}\left[\sum_{i=1}^{n} \cos\left(x_i - X_i\right) + \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \cos\left(y_{ji} - \alpha_j - X_i\right)\right], \quad (22)$$

Let $\lambda_j = \lambda$;

$$A(\kappa) + q\lambda A(\lambda \kappa) = \frac{1}{n}\left[\sum_{i=1}^{n} \cos\left(x_i - X_i\right) + \lambda \sum_{j=1}^{q} \sum_{i=1}^{n} \cos\left(y_{ji} - \alpha_j - X_i\right)\right], \quad (23)$$

where

$$A(\kappa) = \frac{I_0'(\kappa)}{I_0(\kappa)} = \frac{I_1(\kappa)}{I_0(\kappa)} = 1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3} + 0\kappa^{-4}. \quad (24)$$

and

$$q\lambda A(\lambda\kappa) = q\lambda\left(1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3} + 0\kappa^{-4}\right). \quad (25)$$

Therefore

$$A(\kappa) + q\lambda A(\lambda\kappa) = c,$$
$$1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3} + 0\kappa^{-4} + q\lambda\left(1 - \frac{1}{2\kappa} - \frac{1}{8\kappa^2} - \frac{1}{8\kappa^3} + 0\kappa^{-4}\right) = c, \quad (26)$$

Equation (26) is then simplified and it becomes a cubic expression as in (27)

$$8(\lambda^3 + q\lambda^4 - c\lambda^3)\kappa^3 - 4(\lambda^3 + q\lambda^3)\kappa^2 - (\lambda^3 + q\lambda^2)\kappa - (\lambda^3 + q\lambda) = 0,$$
$$8(1 + q\lambda - c)\kappa^3 - 4(1+q)\kappa^2 - (1 + \frac{q}{\lambda})\kappa - (1 + \frac{q}{\lambda^2}) = 0. \quad (27)$$

Equation (27) is then being solved by using the polyroot function available in Splus software to find the estimation of κ. The function gives three values root where one of them is a real root and another two values are complex roots. The real root was chosen to be $\hat{\kappa}$ and $\hat{\kappa}$ was corrected by multiplying with the correction factor of $\frac{q}{(q+1)}$ to be the estimation of the concentration parameter. This is because in circular case, the estimation of a concentration parameter needs to be corrected by multiplying it by $\frac{q}{(q+1)}$ where q is the number of simultaneous equation (Hussin et al. 2010). In this case, the correction factor becomes $\frac{2}{3}$ since q=2. Thus, $\tilde{\kappa} = \frac{2\hat{\kappa}}{3}$ indeed gives a better approximation to the value of κ.

Results and Discussion

Simulation study was carried to assess the accuracy and the biasness of the estimation of concentration parameter κ obtained by using the proposed method. The number of simulation is set to be s, meanwhile the values of n and κ for the error terms have been generated. The following are the bias measures of $\tilde{\kappa}$. The measures of biasness of $\tilde{\kappa}$ are mean, absolute relative estimate bias, estimated root mean square error and estimated standard error. The following are the measures.

(a) Mean of $\tilde{\kappa}$, $\bar{\tilde{\kappa}} = \frac{1}{s}\sum_{p=1}^{s}\tilde{\kappa}_p$,

(b) Absolute relative estimate bias, (AREB) (%) = $\left(\frac{\left|\bar{\tilde{\kappa}} - \kappa\right|}{\kappa}\right)$ × 100%,

TABLE 1. Simulation result of $\tilde{\kappa}$ when λ =1

	n	Mean	AREB	ERMSE	ESE
	30	4.9285	1.4305	0.9155	0.9476
κ =5	70	4.7034	5.9318	0.8022	0.8023
	100	4.6580	6.8399	0.7189	0.7190
	150	4.6075	7.8498	0.6570	0.6571
	300	4.5708	8.5834	0.5662	0.5663
	30	8.3180	3.9756	1.6151	1.6153
κ =8	70	8.0344	0.4306	1.0461	1.0463
	100	7.9210	0.9880	0.8510	0.8511
	150	7.8753	1.5589	0.6970	0.6970
	300	7.8043	2.4459	0.5273	0.5274
	30	10.5161	5.1607	2.0550	2.0552
κ =10	70	10.1039	1.0386	1.2476	1.2477
	100	9.9544	0.4563	1.0185	1.0186
	150	9.9467	0.5329	0.8308	0.8309
	300	9.8725	1.2754	0.5920	0.5921
	30	15.9395	6.2634	3.1889	3.1893
κ =15	70	15.2308	1.5384	1.8321	1.8322
	100	15.1155	0.7697	1.5403	1.5404
	150	15.0172	0.1149	1.2029	1.2031
	300	14.8990	0.6736	0.8584	0.8585
	30	21.1655	5.8275	4.1110	4.1114
κ =20	70	20.3897	1.9485	2.5240	2.5243
	100	20.2212	1.1062	2.0064	2.0066
	150	20.1007	0.5036	1.6214	1.6215
	300	19.9618	0.1911	1.1464	1.1465
	30	31.9505	6.5017	6.4762	6.4769
κ =30	70	30.7258	2.4192	3.8029	3.8033
	100	30.4081	1.3605	3.0526	3.0529
	150	30.2020	0.6732	2.4904	2.4906
	300	30.0101	0.0337	1.7036	1.7038
	30	53.6494	7.2988	11.0290	11.0301
κ =50	70	51.2429	2.4859	6.4386	6.4392
	100	50.7666	1.5332	5.1864	5.1869
	150	50.4858	0.9717	4.1691	4.1695
	300	50.1733	0.3466	2.8529	2.8531

(c) Estimated root mean square errors, (ERMSE) = $\sqrt{\frac{1}{s}\sum_{p=1}^{s}\left(\tilde{\kappa}_p-\kappa\right)^2}$

(d) Estimated standard errors, (ESE) = $\sqrt{\frac{1}{s-1}\sum_{p=1}^{s}\left(\tilde{\kappa}_p-\kappa\right)^2}$

In this study, the value of s is set to be 5000 for each simulation and $p = 2$. The values of X have been generated from the von Mises distribution of VM $\left(\frac{\pi}{4},3\right)$ and the true value of α_1 and α_2 is $\frac{\pi}{4} \approx 0{,}7854$. In the simulation, the values of λ are varied to be 1, 1.5 and 2. The actual κ values considered for the error term δ_i are $\kappa = 5, 8, 10, 15, 20, 30, 50$ and the sample size are $n = 30, 70, 100, 150$ and 300.

Tables 1, 2 and 3 represent the result of the simulation when the ratio of concentration parameter, λ =1, λ =1.5 and λ =2, respectively.

The trend of the bias measures may not be too obvious when κ is small; for example, for λ =1, we can see that when κ = 5, the mean value of $\tilde{\kappa}$ is closer to the real value only when n is small. This is also true for the AREB values. However, when the values of κ are larger, say, κ value is above 15, for any fixed κ, as n increases, the mean gets closer to the real value of κ and the AREB decreases. The ESE and ERMSE both show a clear decreasing trend as n increases for any fixed value of κ. The same can be made for other values of λ, namely for λ = 1.5 and λ = 2, respectively.

In summary, we note that for any fixed κ, all the bias measures, namely AREB, ERMSE and ESE decrease as n increases.

TABLE 2. Simulation result of $\tilde{\kappa}$ when λ =1.5

	n	Mean	AREB	ERMSE	ESE
κ =5	30	4.4282	11.4354	1.5272	1.5808
	70	4.5575	8.8503	1.0693	1.0694
	100	4.4724	10.5511	0.9831	0.9832
	150	4.3888	12.2237	0.9239	0.9240
	300	4.3093	13.8148	0.8610	0.8610
κ =8	30	8.3740	4.6750	1.7163	1.7165
	70	7.9686	0.3931	1.0926	1.0927
	100	7.8976	1.2802	0.9793	0.9794
	150	7.8435	1.9565	0.8253	0.8253
	300	7.7702	2.8722	0.6519	0.6519
κ =10	30	10.5708	5.7077	2.1370	2.1372
	70	10.0949	0.9488	1.2700	1.2701
	100	10.0370	0.3697	1.0538	1.0539
	150	9.9567	0.4332	0.8606	0.8607
	300	9.8901	1.0992	0.6493	0.6493
κ =15	30	15.8952	5.9678	3.1332	3.1335
	70	15.2767	1.8447	1.8417	1.8419
	100	15.1379	0.9191	1.5109	1.5111
	150	15.0690	0.4602	1.2195	1.2196
	300	14.9649	0.2339	0.8638	0.8639
κ =20	30	21.3108	6.5541	4.2212	4.2216
	70	20.4757	2.3786	2.5370	2.5372
	100	20.2443	1.2214	2.0868	2.0870
	150	20.1092	0.5461	1.6502	1.6503
	300	19.9776	0.1122	1.1604	1.1605
κ =30	30	32.0462	6.8208	6.5131	6.5138
	70	30.6850	2.2835	3.8102	3.8106
	100	30.4162	1.3875	3.1103	3.1106
	150	30.2902	0.9674	2.5149	2.5151
	300	30.0836	0.2787	1.7332	1.7334
κ =50	30	53.2998	6.5995	10.6477	10.6487
	70	51.2610	2.5220	6.4067	6.4074
	100	50.9770	1.9540	5.2483	5.2488
	150	50.6286	1.2572	4.2258	4.2262
	300	50.1871	0.3743	2.9040	2.9042

TABLE 3. Simulation result of $\tilde{\kappa}$ when λ =2

	n	Mean	AREB	ERMSE	ESE
κ =5	30	4.7930	4.1400	1.5099	1.5101
	70	4.3812	12.3766	1.2940	1.2941
	100	4.2664	14.6715	1.2425	1.2426
	150	4.1673	16.6545	1.1812	1.1813
	300	4.0564	18.8714	1.1214	1.1215
κ =8	30	8.3550	4.4371	1.7346	1.7348
	70	7.9622	0.4721	1.2213	1.2214
	100	7.8868	1.4145	1.1001	1.1002
	150	7.7880	2.6497	0.9574	0.9574
	300	7.6929	3.8388	0.8030	0.8030
κ =10	30	10.5265	5.2650	2.1121	2.1123
	70	10.1102	1.1024	1.2918	1.2919
	100	10.0033	0.0333	1.1511	1.1512
	150	9.9022	0.9783	0.9705	0.9705
	300	9.8647	1.3528	0.7224	0.7225
κ =15	30	15.9215	6.1430	3.1677	3.1680
	70	15.2812	1.8746	1.8500	1.8502
	100	15.1981	1.3209	1.5409	1.5411
	150	15.0782	0.5212	1.2540	1.2541
	300	14.9745	0.1701	0.8595	0.8595
κ =20	30	21.3126	6.5629	4.3133	4.3137
	70	20.5244	2.6220	2.5780	2.5783
	100	20.2431	1.2157	2.0522	2.0524
	150	20.1741	0.8704	1.6664	1.6665
	300	20.0196	0.0978	1.1536	1.1537
κ =30	30	31.9239	6.4128	6.3716	6.3722
	70	30.7176	2.3919	3.7646	3.7650
	100	30.6140	2.0465	3.1809	3.1813
	150	30.2813	0.9376	2.5010	2.5012
	300	30.0875	0.2917	1.7421	1.7423
κ =50	30	53.6717	7.3434	10.9905	10.9916
	70	51.3578	2.7156	6.3455	6.3461
	100	50.8465	1.6930	5.2845	5.2850
	150	50.5695	1.1390	4.1725	4.1729
	300	50.1407	0.2813	2.9091	2.9093

TABLE 4. Mean and variance for parameter estimates with AIC values

λ	$\hat{\alpha}_1$ (variance)	$\hat{\alpha}_2$ (variance)	$\tilde{\kappa}$ (variance)	AIC
0.75	0.2084 (0.00072)	0.3011 (0.00070)	15.9116 (0.09796)	-1139.28
1.0	0.2083 (0.00083)	0.3008 (0.00085)	13.6884 (0.12138)	-1310.66
1.5	0.2082 (0.00103)	0.3004 (0.00104)	11.1891 (0.18883)	-812.16
2.0	0.2081 (0.00127)	0.3003 (0.00132)	9.7083 (0.20485)	-176.66

APPLICATION TO REAL DATA

The model is illustrated by the wind and wave direction data collected from the Humberside coast of the North Sea, United Kingdom (Mokhtar 2016). With the sample size of 49, the wave direction data measured by the HF radar system, developed by UK Rutherford and Appleton Laboratories, is addressed as the variable x. Variable y_1 is the wind data measured by anchored wave buoy and the variable y_2 is for the data of the wind direction which was measured by HF radar system.

Table 4 shows the parameter estimates and the variance of parameter estimates for the real data set with the Akaike Information Criterion (AIC). The relationship between the variables x, y_1 and y_2 can be expressed for example for $\lambda = 1$, y_1, $0.2083 + X(\text{mod } 2\pi)$ and $y_2 = 0.3008 + X(\text{mod } 2\pi)$ where $\tilde{\kappa} = 13.6884$.

The variance of the parameter estimates were calculated by using the bootstrapping method. Meanwhile AIC was used to identify the optimum model in a class of competing models. It is based on the maximum likelihood function, which is unbiased. The best approximating model is the one with minimum AIC in the class of competing models (Mutua 1994). The formula of calculating AIC is:

AIC = -2log (maximized likelihood for model) +2 (number of fitted parameter)

In the application of the proposed method to the real data set, it is shown that the model with $\lambda = 1$ or equal error variances has the smallest value of AIC. This indicates that the best model for the real data set is $Y_1 = 0.2083 + X(\text{mod } 2\pi)$ and $Y_2 = 0,3008 + X(\text{mod } 2\pi)$ with $\tilde{\kappa} = 13.6884$.

The variances of $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\tilde{\kappa}$ are relatively small, which are 0.00083, 0.00085 and 0.12138, respectively.

CONCLUSION

This study proposes the estimation of the concentration parameter for simultaneous circular functional relationship model when the variances of the error term are not necessarily equal. The modified Bessel function was extended by considering the asymptotic power series that is of a cubic equation of $\varkappa$. Roots of the cubic equation of $\varkappa$ were obtained by using the polyroot function in SPlus statistical software. Simulation was done to study the biasness of the estimation of the concentration parameter. From the simulation study, large concentration parameter and sample size show that the estimated concentration parameter has smaller bias measures. An illustration to a real wind and wave data set is given to show its practical applicability and we can say that the absolute relative estimate bias of the data is small which is less than 2 percents.

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