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The Existence of ϒ-fixed Point for the Multidimensional Nonlinear Mappings Satisfying (ψ, θ, φ)-weak Contractive Conditions

(Kewujudan Titik ϒ-tetap pada Pemetaan Tak-linear Multidimensi yang Ditakrifkan terhadap Ruang Metrik Bertertib Separa dan Memenuhi Syarat Penguncupan (ψ, θ, φ)-lemah)

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ABSTRACT

In this paper we prove the existence of ϒ-fixed point for a multidimensional nonlinear mappings $F : X^k \to X$ defined on the partially ordered metric spaces and satisfying (ψ, θ, φ)-weak contractive conditions. Moreover, we prove the uniqueness of that fixed point under extra conditions to (ψ, θ, φ)-weak contractive conditions.

Keywords: Contractive condition; fixed-point; partially ordered metric space

ABSTRAK

Dalam kertas ini, kami buktikan kewujudan titik ϒ-tetap pada pemetaan tak-linear multidimensi $F : X^k \to X$ yang ditakrifkan terhadap ruang metrik bertertib separa dan memenuhi syarat penguncupan (ψ, θ, φ)-lemah. Seterusnya, kami buktikan keunikan titik tetap tersebut dengan syarat tambahan kepada syarat (ψ, θ, φ)-lemah.

Kata kunci: Ruang metrik bertertib separa; syarat pengucupan; titik tetap

INTRODUCTION

Fixed point theory plays a crucial role in nonlinear analysis. Fixed point results are used to prove the existence (and also uniqueness) of the solutions of various type equations. One of initial results in this direction obtained by Banach (1922), which is well known as Banach contraction mapping principle. Due to its importance in nonlinear analysis, Banach contraction mapping principle has been generalized to couple, triple and quadruple fixed point by Berinde and Borcut (2011), Guo and Lakshmikantham (1987) and Karapinar (2011), respectively. Recently, Rold´an et al. (2012) have generalized these ideas by introducing the notion of ϒ-fixed point, that is to say, the multidimensional fixed point. We could say that their results present some of the first deep results in this direction. Following their idea, in this paper we prove the existence and uniqueness of ϒ- fixed point theorem for nonlinear mappings of any number of arguments under a *(*ψ, θ, φ*)*-weak contractive conditions (Theorem 3.2). This theorem improves and generalizes the main theorems of Bhaskar and Lakshmikantham (2006) and Berinde and Borcut (2011).

PRELIMINARIES

In order to fix the framework needed to state our main result, we recall the following notions.

Definition 2.1. (Aydi et al. 2012) An ordered metric space $(X, d, \prec)$ is called regular if it verifies the following.

- If $\{x_m\}$ is a nondecreasing sequence and $\{x_m\} \xrightarrow{d} x$, then $x_m \prec x$ for all m,
- If $\{y_m\}$ is a nondecreasing sequence and $\{y_m\} \xrightarrow{d} y$, then $y_m \succ y$ for all m.

Consider the set $\Lambda_k = \{1, 2, \ldots, k\}$ where $k \in$ •. Let $\{A, B\}$ be a partition of Λ_k that is $A \cup B = \Lambda_k$ and $A \cap B = \varnothing$. Henceforth we fix this partition. Using this partition we define a set of mappings:

$$\Omega_{A,B} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(A) \subseteq A, \sigma(B) \subseteq B\}$$

$$\Omega_{A,B} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}.$$

Let $(X, d, \prec)$ be a partially ordered metric space. Using this space we define k-dimensional partially ordered metric space $(X_k, d_k, \prec_k)$. From now on we denote $\underbrace{X \times X \times \ldots \times X}_{k}$ by X^k. Let $i \in A_k$, following (Rold´an et al. (2012)) we use following notation

$$x \prec_i y \Leftrightarrow \begin{cases} x \prec y, i \in A \\ x \succ y, i \in B. \end{cases} \tag{2.1}$$

The product space X^k is endowed with the following natural partial order: for $\mathbf{x}, \mathbf{y} \in X^k$, $\mathbf{x} \prec_k \mathbf{y} \Leftrightarrow x_i \prec_i y_i$ for all $i \in \Lambda_k$. Obviously, $(X^k, \prec_k)$ is a partially ordered set. Now we define a metric. Consider the mapping $d_k : X^k \times X^k \to [0, +\infty)$ given by

$$d_k(\mathbf{x},\mathbf{y}) = \frac{1}{k}\sum_{i=1}^{k} d(x_i, y_i)$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$. It is easy to see that $(X^k, d_k, \prec_k)$ is a partially ordered metric space. Moreover

$$d_k(\mathbf{x}^n, \mathbf{x}) \to 0, n \to \infty \Leftrightarrow d(x_i^n, x_i) \to 0, n \to \infty$$

for all $i \in \Lambda_k$

where $\mathbf{x}^n = (x_1^n, x_2^n, \ldots, x_k^n), \mathbf{x} = (x_1, x_2, \ldots, x_k) \in X^k$. In the sequel, we consider the mapping $F : X^k \to X$.

Definition 2.2. (Rold´an et al. 2012) Let $(X, \prec)$ be a partially ordered space. We say that F has the mixed-monotone property with respect to (w.r.t) partition $\{A, B\}$ if

$$y \prec z \Rightarrow F(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k) \prec_i F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_k) \quad (2.2)$$

for all $x_1, x_2, \ldots, x_k, y, z, \in X$ and for all $i \in \Lambda_k$.

Henceforth, let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be k mappings from Λ_k into itself and let be the k-tuple.

Definition 2.3. (Rold´an et al. 2012) A point $\mathbf{x}^n = (x_1^n, x_2^n, \ldots, x_k^n) \in X^k$. is called a Υ-fixed point of the mapping F if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(k)}) = x_i \text{ for all } i \in \Lambda k. \quad (2.3)$$

Υ-Fixed Point Theorem

In this section we prove the existence of Υ-fixed points for a mapping $F : X^k \to X$ satisfying (ψ, θ, φ)-weak contractive condition in the setup of partially ordered metric spaces. Note that (ψ, θ, φ)-weak contraction condition was first appeared in (Choudhury 2012). To state our main theorem we have to introduce the notion of altering distance function which was introduced by Khan et al. (1984) as follows:

Definition 3.1. The function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function, if ψ is continuous, monotonically increasing and $\psi(x) = 0$ iff $x = 0$. Now, we are ready to formulate our main result.

Theorem 3.2. Let $(X, d, \prec)$ be a complete partially ordered metric space. Let $\Upsilon : \Lambda_k \to \Lambda_k$ be a k-tuple of mappings $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ which is verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. Let $F : X^k \to X$ be a mapping which obeys the following conditions:

(i) there exist an altering distance function ψ, an upper semi-continuous $\theta : [0, +\infty) \to [0, +\infty)$ and a lower semi-continuous $\varphi : [0, +\infty) \to [0, +\infty)$ such that $\psi\,(d(F(x_1, x_2, \ldots, x_k), F(y_1, y_2, \ldots, y_k)))$

$$\leq \frac{1}{k}\theta(d_k(\mathbf{x},\mathbf{y})) - \varphi\left(\frac{d_k(\mathbf{x},\mathbf{y})}{k}\right)$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ with $\mathbf{x} \prec_k \mathbf{y}$ where $\theta(0) = \varphi(0)$ and $\psi(x) - \theta(x) + \varphi(x) > 0$ for all $x > 0$;

(ii) $\psi(a + b) \leq \psi(a) + \psi(b)$ and $\varphi(c) \leq k\varphi\left(\frac{c}{k}\right)$ for all $a, b, c \geq 0$;

(iii) there exists $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in X^k$ such that $x_i^0 \prec_i F(x_{\sigma_i(1)}^0, x_{\sigma_i(2)}^0, \ldots, x_{\sigma_i(k)}^0)$ for all $i \in \Lambda_k$;

(iv) F has the mixed monotone property w.r.t $\{A, B\}$;

(v) F is continuous or $(X, d, \prec)$ is regular;

Then F has at least one Υ-fixed point. Moreover

(vi) if for any $\mathbf{x}, \mathbf{y} \in X^k$ there exists a $\mathbf{z} \in X^k$ such that $\mathbf{x} \prec_k \mathbf{z}$, $\mathbf{y} \prec_k \mathbf{z}$ then F has a unique Υ-fixed point $\mathbf{x}^* = (x_{\sigma_i(1)}^0, x_{\sigma_i(2)}^0, \ldots, x_{\sigma_i(k)}^0) \in X^k$.

Proof. The proof is divided into five steps. Existence. *Step 1*. In this step we define a sequence $\{\mathbf{x}^n\}$ of Υ-iteration of F and then we show that $\mathbf{x}^{n-1} \prec_k \mathbf{x}^n$ for all $n \geq 1$. By condition (iii) there exists $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in X^k$ such that $x_i^0 \prec_i F(x_{\sigma_i(1)}^0, x_{\sigma_i(2)}^0, \ldots, x_{\sigma_i(k)}^0)$ for all $i \in \Lambda_k$. Consider the following sequence $\mathbf{x}^n = (x_1^n, x_2^n, \ldots, x_k^n), n \geq 1$ where

$$\begin{aligned}
x_1^n &= F(x_{\sigma_1(1)}^{n-1}, x_{\sigma_1(2)}^{n-1}, \ldots, x_{\sigma_1(k)}^{n-1}) \\
x_2^n &= F(x_{\sigma_2(1)}^{n-1}, x_{\sigma_2(2)}^{n-1}, \ldots, x_{\sigma_2(k)}^{n-1}) \\
&\ldots. \\
&\ldots. \\
x_k^n &= F(x_{\sigma_k(1)}^{n-1}, x_{\sigma_k(2)}^{n-1}, \ldots, x_{\sigma_k(k)}^{n-1})
\end{aligned} \quad (3.1)$$

This sequence is called n-th Υ-iteration of F at $\mathbf{x}^0$. We claim that

$$\mathbf{x}^{n-1} \prec_k \mathbf{x}^n \text{ for all } n \geq 1. \quad (3.2)$$

To prove this claim we have to show $x_i^{n-1} \prec_i x_i^n$ for all $n \geq 1$ and $i \in \Lambda_k$. We prove this by induction. Note that by (ii) we have $\mathbf{x}^0 \prec_k \mathbf{x}^1$ that is $x_i^0 \prec_i x_i^1$ for all $i \in \Lambda_k$. Assume that (3.2) is true for some m that is

$$x_i^{m-1} \prec_i x_i^m \text{ for all } i \in \Lambda_k. \quad (3.3)$$

Take any $i \in \Lambda_k$ we fix it. Here it can be: either $i \in A$ or $i \in B$. We consider the case $i \in A$ and we show $x_i^m \prec x_i^{m+1}$. In the case of $i \in B$ the inequality $x_i^m \succ x_i^{m+1}$ can be proven analogously. Now we take $\sigma_i(1)$. Here also it can be: either $\sigma_i(1) \in A$ or $\sigma_i(1) \in B$. First we assume $\sigma_i(1) \in A$. By relation (3.3) we have $x^{m-1}_{\sigma_i(1)} \prec x^m_{\sigma_i(1)}$. Since F has the mixed monotone property w.r.t $\{A, B\}$ we have

$$F(x^{m-1}_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec F(x^m_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}).$$

Second we assume $\sigma_i(1) \in B$. By relation (3.3) we have $x^{m-1}_{\sigma_i(1)} \succ x^m_{\sigma_i(1)}$. Using the mixed monotone property of F w.r.t $\{A, B\}$ we get

$$F(x^{m-1}_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec F(x^m_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)})$$

Thus in both cases of $\sigma_i(1) \in A$ and $\sigma_i(1) \in B$ we have shown

$$F(x^{m-1}_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec F(x^m_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}). \qquad (3.4)$$

Next we take $\sigma_i(2)$. As above it can be: either $\sigma_i(2) \in A$ or $\sigma_i(2) \in B$. If $\sigma_i(2) \in A$ then by relation (3.3) we have $x^{m-1}_{\sigma_i(2)} \prec x^m_{\sigma_i(2)}$. By the mixed monotone property of F w.r.t $\{A, B\}$ we get

$$F(x^m_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec F(x^m_{\sigma_i(1)}, x^m_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}).$$

If $\sigma_i(2) \in B$ then again by relation (3.3) and the mixed monotone property of F w.r.t $\{A, B\}$ we obtain

$$F(x^m_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec F(x^m_{\sigma_i(1)}, x^m_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}).$$

Hence, in both cases of $\sigma_i(2) \in A$ and $\sigma_i(2) \in B$ we have obtained

$$F(x^m_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec F(x^m_{\sigma_i(1)}, x^m_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}). \qquad (3.5)$$

Continue this process until $\sigma_i(k)$ easily it can be shown that in both cases of $\sigma_i(k) \in A$ and $\sigma_i(k) \in B$ we have

$$F(x^m_{\sigma_i(1)}, x^m_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec F(x^m_{\sigma_i(1)}, x^m_{\sigma_i(2)}, ..., x^m_{\sigma_i(k)}). \qquad (3.6)$$

On the other hand, collecting relations (3.4)-(3.6) we get

$$x^m_i = F(x^{m-1}_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec F(x^m_{\sigma_i(1)}, x^{m-1}_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec ...$$

$$... \prec F(x^m_{\sigma_i(1)}, x^m_{\sigma_i(2)}, ..., x^{m-1}_{\sigma_i(k)}) \prec F(x^m_{\sigma_i(1)}, x^m_{\sigma_i(2)}, ..., x^m_{\sigma_i(k)}) = x^{m+1}_i \qquad (3.7)$$

Thus, for any $i \in A$ we get $x^m_i \prec_i x^{m+1}_i$. As we have mentioned above in the case of $i \in B$ the inequality $x^m_i \,{}_i\!\succ x^{m+1}_i$ can be proved analogously. Therefore we deduce

$$x^m_i \prec_i x^{m+1}_i \text{ for all } i \in \Lambda_k. \qquad (3.8)$$

This proves the claim (3.2).

Step 2. In this step we show that $\lim_{n\to\infty} \mathbf{d}_k(\mathbf{x}^{n-1}, \mathbf{x}^n) = 0$. We set

$$D^n_i = d(x^{n-1}_i, x^n_i) \text{ and } D^n = \frac{1}{k}\sum_{i=1}^k D^n_i \overset{def}{=} \mathbf{d}_k(\mathbf{x}^{n-1}, \mathbf{x}^n)$$

Since the set Λ_k is finite, there exists an index $i(\mathrm{n}) \in \Lambda_k$, such that $D^n_{i(\mathrm{n})} = \max_{1\le i\le k} D^k_i$. By $\mathbf{x}^{n-1} \prec_k \mathbf{x}^n$ and $\sigma_i(\Lambda_k) \subseteq \Lambda_k$ it implies

$$(x^{n-1}_{\sigma_i(1)}, x^{n-1}_{\sigma_i(2)}, ..., x^{n-1}_{\sigma_i(\mathrm{k})}) \prec_k (x^n_{\sigma_i(1)}, x^n_{\sigma_i(2)}, ..., x^n_{\sigma_i(\mathrm{k})})$$

for any $1 \le i \le k$ and $n \ge 1$. Thus using conditions (i) and (ii) we get

$$\begin{aligned}\psi(D^n_{i(\mathrm{n})}) &= \psi(d(F(x^{n-2}_{\sigma_{i(\mathrm{n})}(1)}, x^{n-2}_{\sigma_{i(\mathrm{n})}(2)}, ..., x^{n-2}_{\sigma_{i(\mathrm{n})}(\mathrm{k})}), \\ &\qquad F(x^{n-1}_{\sigma_{i(\mathrm{n})}(1)}, x^{n-1}_{\sigma_{i(\mathrm{n})}(2)}, ..., x^{n-1}_{\sigma_{i(\mathrm{n})}(\mathrm{k})}))) \\ &\le \frac{1}{k}\theta\left(\sum_{j=1}^k \frac{d(x^{n-2}_{\sigma_{i(\mathrm{n})}(\mathrm{j})}, x^{n-1}_{\sigma_{i(\mathrm{n})}(\mathrm{j})})}{k}\right) - \frac{1}{k}\varphi\left(\sum_{j=1}^k \frac{d(x^{n-2}_{\sigma_{i(\mathrm{n})}(\mathrm{j})}, x^{n-1}_{\sigma_{i(\mathrm{n})}(\mathrm{j})})}{k}\right) \\ &\le \frac{1}{k}\psi\left(D^{n-1}_{i(\mathrm{n}-1)} - \sum_{j=1}^k \frac{D^{n-1}_{i(\mathrm{n}-1)} - d(x^{n-2}_{\sigma_{i(\mathrm{n})}(\mathrm{j})}, x^{n-2}_{\sigma_{i(\mathrm{n})}(\mathrm{j})})}{k}\right).\end{aligned} \qquad (3.9)$$

It is obvious

$$\sum_{j=1}^k \frac{D^{n-1}_{i(\mathrm{n}-1)} - d(x^{n-2}_{\sigma_{i(\mathrm{n})}(\mathrm{j})}, x^{n-1}_{\sigma_{i(\mathrm{n})}(\mathrm{j})})}{k} \ge 0.$$

By monotonicity of ψ it implies

$$\psi\left(D^{n-1}_{i(\mathrm{n}-1)} - \sum_{j=1}^k \frac{D^{n-1}_{i(\mathrm{n}-1)} - d(x^{n-2}_{\sigma_{i(\mathrm{n})}(\mathrm{j})}, x^{n-1}_{\sigma_{i(\mathrm{n})}(\mathrm{j})})}{k}\right) \le \psi\left(D^{n-1}_{i(\mathrm{n}-1)}\right).$$

This and inequality (3.9) implies

$$\psi\left(D^n_{i(\mathrm{n})}\right) \le \frac{1}{k}\psi\left(D^{n-1}_{i(\mathrm{n}-1)}\right). \qquad (3.10)$$

Iterating into it $\psi\left(D^n_{i(n)}\right) \le \frac{1}{k^{n-1}}\psi\left(D^1_{i(1)}\right)$. Taking the limit we get $\lim_{n\to\infty} D^n_{i(\mathrm{n})} = 0$. On the other hand

$$\mathbf{d}_k(\mathbf{x}^{n-1}, \mathbf{x}^n) = \frac{1}{k}\sum_{i=1}^k D^n_i \le D^n_{i(\mathrm{n})}.$$

Taking the limit from both side of this inequality when $n \to \infty$ we get

$$\lim_{n\to\infty} \mathbf{d}_k(\mathbf{x}^{n-1}, \mathbf{x}^n) \to 0.$$

Step 3. In this step we show that the sequence $\{\mathbf{x}^n = (x^n_1, x^n_2, ..., x^n_k)\} \in X^k$ is a Cauchy sequence in $(X^k, \mathbf{d}_k)$. Suppose $\{\mathbf{x}^n\}$ is not a Cauchy sequence. Then there exist an $\varepsilon > 0$ and subsequences $\{\mathbf{x}^{n_s}\}$ and $\{\mathbf{x}^{m_s}\}$ with $n_s > m_s > s$ such that

$$\mathbf{d}_k(\mathbf{x}^{m_s}, \mathbf{x}^{n_s}) = \frac{1}{k}\sum_{i=1}^{k} d(\mathrm{x}_i^{m_s}, \mathrm{x}_i^{n_s}) \geq \varepsilon. \tag{3.11}$$

Denote $\mathsf{A}_i^s = d(\mathrm{x}_i^{m_s}, \mathrm{x}_i^{n_s}), i \in \Lambda_k$. It is clear that for any s there exists an index $i(\mathrm{s}) \in \Lambda_k$ such that $\mathsf{A}_{i(\mathrm{s})}^s = \max_{1\leq i\leq k} d(\mathrm{x}_i^{m_s}, \mathrm{x}_i^{n_s})$. The same manner as in Step 2 we get

$$\psi\left(\mathsf{A}_{i(\mathrm{s})}^s\right) \leq \frac{1}{k}\psi\left(\mathsf{A}_{i(\mathrm{s}-1)}^{s-1}\right).$$

Iterating into it

$$\psi\left(\mathsf{A}_{i(\mathrm{s})}^s\right) \leq \frac{1}{k^{s-1}}\psi\left(\mathsf{A}_{i(1)}^{1}\right).$$

Taking the limit we get

$$\lim_{s\to\infty} \mathsf{A}_{i(\mathrm{s})}^s = 0. \tag{3.12}$$

On the other hand by (3.11)

$$\varepsilon \leq \mathbf{d}_k(x^{m_s}, x^{n_s}) = \frac{1}{k}\sum_{i=1}^{k} \mathsf{A}_i^s \leq \mathsf{A}_{i(\mathrm{s})}^s$$

This contradicts to (3.12). *Step 3* is proven.

Step 4. In this step we prove the existence of Υ-fixed point. By assumption (X, d) is a complete metric space. It can be easily checked that $(X^k, \mathbf{d}_k)$ is also a complete metric space. Therefore, there exists $\mathbf{x}^* \in X^k$ such that $\lim_{n\to\infty} \mathbf{x}^n = \mathbf{x}^*$ that is,

$$\begin{aligned}
&\lim_{n\to\infty} x_1^n = \lim_{n\to\infty} F(x_{\sigma_1(1)}^{n-1}, x_{\sigma_1(2)}^{n-1}, \ldots, x_{\sigma_1(k)}^{n-1}) = x_1^* \\
&\lim_{n\to\infty} x_2^n = \lim_{n\to\infty} F(x_{\sigma_2(1)}^{n-1}, x_{\sigma_2(2)}^{n-1}, \ldots, x_{\sigma_2(k)}^{n-1}) = x_2^* \\
&\qquad\qquad .... \\
&\qquad\qquad .... \\
&\lim_{n\to\infty} x_k^n = \lim_{n\to\infty} F(x_{\sigma_k(1)}^{n-1}, x_{\sigma_k(2)}^{n-1}, \ldots, x_{\sigma_k(k)}^{n-1}) = x_k^*
\end{aligned} \tag{3.13}$$

Now we show that the point $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_k^*) \in X^k$ is a Υ-fixed point of F. Indeed, suppose that the assertion of the condition (v) holds, i.e., F is continuous. Then

$$\begin{aligned}
&\lim_{n\to\infty} F(x_{\sigma_1(1)}^{n-1}, x_{\sigma_1(2)}^{n-1}, \ldots, x_{\sigma_1(k)}^{n-1}) = F(x_{\sigma_1(1)}^*, x_{\sigma_1(2)}^*, \ldots, x_{\sigma_1(k)}^*) \\
&\lim_{n\to\infty} F(x_{\sigma_2(1)}^{n-1}, x_{\sigma_2(2)}^{n-1}, \ldots, x_{\sigma_2(k)}^{n-1}) = F(x_{\sigma_2(1)}^*, x_{\sigma_2(2)}^*, \ldots, x_{\sigma_2(k)}^*) \\
&\qquad\qquad\qquad\qquad .... \\
&\qquad\qquad\qquad\qquad .... \\
&\lim_{n\to\infty} F(x_{\sigma_k(1)}^{n-1}, x_{\sigma_k(2)}^{n-1}, \ldots, x_{\sigma_k(k)}^{n-1}) = F(x_{\sigma_k(1)}^*, x_{\sigma_k(2)}^*, \ldots, x_{\sigma_k(k)}^*)
\end{aligned} \tag{3.14}$$

Relations (3.13) and (3.14) imply

$$F(x_{\sigma_i(1)}^*, x_{\sigma_i(2)}^*, \ldots, x_{\sigma_i(k)}^*) = x_i^*, 1 \leq i \leq k \tag{3.15}$$

So the point $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_k^*) \in X^k$ is a Υ-fixed point of F. Next we suppose that the part of condition (v) holds, i.e., $(X, d, \prec)$ is regular. Then relation (3.13) implies $\mathbf{x}^n = (x_1^n, x_2^n, \ldots, x_k^n) \prec_k \mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_k^*)$. Since $\sigma_i(\Lambda_k) \subseteq \Lambda_k$ for any $1 \leq i \leq k$, we have $(x_{\sigma_i(1)}^n, x_{\sigma_i(2)}^n, \ldots, x_{\sigma_i(k)}^n) \prec_k (x_{\sigma_i(1)}^*, x_{\sigma_i(2)}^*, \ldots, x_{\sigma_i(k)}^*)$. Therefore, the same manner as in Steps 2 and 3 can be shown

$$\begin{aligned}
&\lim_{n\to\infty} \psi(d(F(x_{\sigma_i(1)}^*, x_{\sigma_i(2)}^*, \ldots, x_{\sigma_i(k)}^*), \\
&F(x_{\sigma_i(1)}^n, x_{\sigma_i(2)}^n, \ldots, x_{\sigma_i(k)}^n))) = 0.
\end{aligned}$$

This and relation (3.13) implies $\psi(d(F(x_{\sigma_i(1)}^*, x_{\sigma_i(2)}^*, \ldots, x_{\sigma_i(k)}^*), x_i^*)) = 0$. Hence

$$F(x_{\sigma_i(1)}^*, x_{\sigma_i(2)}^*, \ldots, x_{\sigma_i(k)}^*) = x_i^*, 1 \leq i \leq k. \tag{3.16}$$

Therefore, relations (3.15) and (3.16) prove the existence of Υ-fixed point of F in both cases of the condition (v). This proves Step 4.

Uniqueness. *Step 5*. Now we prove the uniqueness of Υ-fixed point of F. Suppose $\mathbf{y}^* = (y_1^*, y_2^*, \ldots, y_k^*) \in X^k$ is another Υ-fixed point of F. By condition (vi) there exists $\mathbf{z} = (z_1, z_2, \ldots, z_k) \in X^k$ such that $\mathbf{x}^* \prec_k \mathbf{z}$ and $\mathbf{y}^* \prec_k \mathbf{z}$. Put $\mathbf{z}^0 = \mathbf{z} \in X^k$ define n-th Υ-iteration of F at $\mathbf{z}^0$ as follows:

$$\begin{aligned}
&z_1^n = F(z_{\sigma_1(1)}^{n-1}, z_{\sigma_1(2)}^{n-1}, \ldots, z_{\sigma_1(k)}^{n-1}) \\
&z_2^n = F(z_{\sigma_2(1)}^{n-1}, z_{\sigma_2(2)}^{n-1}, \ldots, z_{\sigma_2(k)}^{n-1}) \\
&\qquad .... \\
&\qquad .... \\
&z_k^n = F(z_{\sigma_k(1)}^{n-1}, z_{\sigma_k(2)}^{n-1}, \ldots, z_{\sigma_k(k)}^{n-1}).
\end{aligned} \tag{3.17}$$

Next we claim,

$$\mathbf{x}^* \prec_k \mathbf{z}^n \text{ and } \mathbf{y}^* \prec_k \mathbf{z}^n, \tag{3.18}$$

for all $n \geq 0$. By induction we prove only the first relation of (3.18), the second one can be proved analogously. Note that by condition (v) we have $\mathbf{x}^* \prec_k \mathbf{z}^0$. We assume for some $n = m$ the relation $\mathbf{x}^* \prec_k \mathbf{z}^m$ holds. Utilizing this and the manner as in proof of relations (3.4)-(3.8) it can be shown that

$$x_i^* = F(x_{\sigma_i(1)}^*, x_{\sigma_i(2)}^*, \ldots, x_{\sigma_i(k)}^*) \prec_i F(z_{\sigma_i(1)}^m, z_{\sigma_i(2)}^m, \ldots, z_{\sigma_i(k)}^m) = z_i^{m+1} \tag{3.19}$$

for all $i \in \Lambda_k$ that is $\mathbf{x}^* \prec_k \mathbf{z}^{m+1}$. The claim (3.18) is proven. From this claim it follows that

$$(x_{\sigma_i(1)}^*, x_{\sigma_i(2)}^*, \ldots, x_{\sigma_i(k)}^*) \prec_i (z_{\sigma_i(1)}^n, z_{\sigma_i(2)}^n, \ldots, z_{\sigma_i(k)}^n)$$

for all $i \in \Lambda_k$. Therefore, the same manner as in (3.9) and (3.10) we get

$$\psi(\max_{1\le i\le k} d(x_i^*, z_i^n)) \le \frac{1}{k}\psi(\max_{1\le i\le k} d(x_i^*, z_i^{n-1})). \tag{3.20}$$

Iterating into $\psi(\max_{1\le i\le k} d(x_i^*, z_i^n)) \le \frac{1}{k^n}\psi(\max_{1\le i\le k} d(x_i^*, z_i^0))$. Taking the limit when $n \to \infty$ we have $\lim_{n\to\infty}\psi(\max_{1\le i\le k} d(x_i^*, z_i^n)) = 0$. On the one hand $\mathbf{d}_k(\mathbf{x}^*, \mathbf{z}^n) \le \psi(\max_{1\le i\le k} d(x_i^*, z_i^n))$.

Therefore,

$$\lim_{n\to\infty}\mathbf{d}_k(\mathbf{x}^*, \mathbf{z}^n) = 0. \tag{3.21}$$

Similarly it can be shown

$$\lim_{n\to\infty}\mathbf{d}_k(\mathbf{y}^*, \mathbf{z}^n) = 0. \tag{3.22}$$

On the other hand, we have $\mathbf{d}_k(\mathbf{x}^*, \mathbf{y}^*) \le \mathbf{d}_k(\mathbf{x}^*, \mathbf{z}^n) + \mathbf{d}_k(\mathbf{z}^n, \mathbf{y}^*)$. Relations (3.21) and (3.22) imply $\mathbf{d}_k(\mathbf{x}^*, \mathbf{y}^*) = 0$. Therefore $\mathbf{x}^* = \mathbf{y}^*$. This proves the uniqueness of Υ-fixed point of F. Therefore, Theorem 3.2 is completely proven.

In the sequel, we present some consequences of Theorem 3.2.

Remark 3.3. Note that Theorem 3.7 in Shaddad et al. (2014) is a consequence of Theorem 3.2. Indeed, in the case $k = 2$ we get Theorem 3.7 in Shaddad et al. (2014).

Remark 3.4. Notice that Theorems 2.1 and 2.2 in Bhaskar and Lakshmikantham (2006) is a consequence of Theorem 3.2.

Indeed, let us consider the case $k = 2$. Consider a partition A = {1} and B = {2} of Λ_2. If we define $\Upsilon = (\sigma_1, \sigma_2)$ as follows:

$$\Upsilon = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) \\ \sigma_2(1) & \sigma_2(2) \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{3.23}$$

then we get the definition of coupled fixed point of a mapping $F : X^2 \to X$. Note that the election in (3.23) is called Bhaskar and Lakshmikantham's election (Bhaskar & Lakshmikantham 2006). Moreover, the contractive condition appearing in Bhaskar and Lakshmikantham's theorem is:

$$d(F(x_1,x_2), F(y_1,y_2)) \le \frac{\delta}{2}(d(x_1,y_1) + d(x_2,y_2)) = \delta\mathbf{d}_2(\mathbf{x},\mathbf{y}),$$

for any $\mathbf{x}, \mathbf{y} \in X^2$ such that $\mathbf{x} \prec_2 \mathbf{y}$ and for some $\delta \in (0,1)$. If we choose $\psi(x) = x$, $\theta(x) = 2\delta x$, $\delta < \frac{1}{2}$ and $\varphi(x) = 0$ in Theorem 3.2 we get the desired result.

Remark 3.5. Theorem 3.2 generalizes the main tripled fixed point result in Berinde and Borcut (2011).

Actually, in Berinde and Borcut (2011), for the case of $k = 3$ the partition of Λ_3 chosen as A = {1,3}, B = {2} and $\Upsilon = (\sigma_1, \sigma_2, \sigma_3)$ is given as the following form:

$$\Upsilon = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \sigma_1(3) \\ \sigma_2(1) & \sigma_2(2) & \sigma_2(3) \\ \sigma_3(1) & \sigma_3(2) & \sigma_3(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}. \tag{3.24}$$

The contractive condition in Berinde and Borcut's theorem is:

$$d(F(x_1, x_2, x_3), F(y_1, y_2, y_3)) \le \delta_1 d(x_1, y_1) + \delta_2 d(x_2, y_2) + \delta_3 d(x_3, y_3)$$

for any $\mathbf{x}, \mathbf{y} \in X^3$ such that $\mathbf{x} \prec_3 \mathbf{y}$ where $\delta_1, \delta_2, \delta_3 \ge 0$ and $\delta_1 + \delta_2 + \delta_3 < 1$. It is obvious

$$d(F(x_1, x_2, x_3), F(y_1, y_2, y_3)) \le 3 \max_{1\le i\le 3}(\delta_i).\ \mathbf{d}_3(\mathbf{x}, \mathbf{y}).$$

Therefore, applying Theorem 3.2 and this remark we obtain the desired result.

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