

On the stability of solutions of nonlinear differential equations of fifth order with delay

CEMIL TUNÇ^{1,*}

¹ *Department of Mathematics, Faculty of Arts and Sciences, Yüzüncü Yıl University, Van-65 080, Turkey*

Received May 3, 2009; accepted March 5, 2010

Abstract. Criteria for global asymptotic stability of a null solution of a nonlinear differential equation of fifth order with delay

$$x^{(5)}(t) + \psi(x(t-r), x'(t-r), x''(t-r), x'''(t-r), x^{(4)}(t-r))x^{(4)}(t) \\ + f(x''(t-r), x'''(t-r)) + \alpha_3 x''(t) + \alpha_4 x'(t) + \alpha_5 x(t) = 0$$

are obtained by using Lyapunov's second method. By defining a Lyapunov functional, sufficient conditions are established, which guarantee the null solution of this equation is globally asymptotically stable. Our result consists of a new theorem on the subject.

AMS subject classifications: 34K20

Key words: stability, Lyapunov functional, differential equation, fifth order, delay

1. Introduction

This paper is concerned with the problem of the stability of the null solution of the nonlinear differential equation of fifth order with constant delay

$$x^{(5)}(t) + \psi(x(t-r), x'(t-r), x''(t-r), x'''(t-r), x^{(4)}(t-r))x^{(4)}(t) \\ + f(x''(t-r), x'''(t-r)) + \alpha_3 x''(t) + \alpha_4 x'(t) + \alpha_5 x(t) = 0 \quad (1)$$

or its equivalent system

$$\begin{aligned} x'(t) &= y(t) \\ y'(t) &= z(t) \\ z'(t) &= w(t) \\ w'(t) &= u(t) \\ u'(t) &= -\psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r))u(t) \\ &\quad - f(z(t), w(t)) - \alpha_3 z(t) - \alpha_4 y(t) - \alpha_5 x(t) \\ &\quad + \int_{t-r}^t f_z(z(s), w(s))w(s)ds + \int_{t-r}^t f_w(z(s), w(s))u(s)ds, \end{aligned} \quad (2)$$

*Corresponding author. *Email address:* cemtunc@yahoo.com (C. Tunç)

where ψ and f are continuous functions for the arguments displayed explicitly in Eq. (1); r is a positive constant, that is, r is fixed constant delay; α_3 , α_4 and α_5 are some positive constants. It is assumed that $f(z, 0) = 0$ and the derivatives $f_z(z, w) \equiv \frac{\partial}{\partial z} f(z, w)$ and $f_w(z, w) \equiv \frac{\partial}{\partial w} f(z, w)$ exist and are continuous for all z, w . All solutions considered are also assumed to be real valued.

To the best of our knowledge from literature, in the last three decades, much attention has been paid to investigation of stability of solutions of nonlinear differential equations of fifth order without delay:

$$x^{(5)}(t) + A_1 x^{(4)}(t) + A_2 x'''(t) + A_3 x''(t) + A_4 x'(t) + A_5 x(t) = 0,$$

in which $x \in \mathfrak{R}$, $t \in [0, \infty)$, A_1, A_2, A_3, A_4 and A_5 are not necessarily constants. For a comprehensive treatment of the subject we refer the reader to the papers of Abou-El Ela and Sadek [1], Burganskaja [5], Chukwu [7], Sinha [17], Tejumola and Afuwape [18], Tunç [19], Yu [21] and the references thereof for some works performed on the topic, which include some nonlinear differential equations of fifth order without delay. Throughout all aforementioned papers, Lyapunov's second (or direct) method [14] has been used as a basic tool to prove the problems established in these papers. Meanwhile, for some recent works, it is worth mentioning that in 2007 Adesina and Ukpera [2] investigated the convergence of solutions for the following nonlinear differential equation of fifth order without delay

$$x^{(5)} + ax^{(4)} + bx''' + f(x'') + g(x') + h(x) = p(t, x, x', x'', x''', x^{(4)}),$$

by using Lyapunov's second method, where a, b are positive constants, the functions f, g, h and p are real valued and continuous in their respective arguments. The authors presented some sufficient conditions for all solutions of the above equation to be convergence. Namely, two solutions convergent to each other if their difference and those of their derivatives up to order four approach zero as time approaches infinity. In this work, the authors showed that the nonlinear functions involved in the above equation are not necessarily differentiable, but satisfy certain increment ratios that lie in the closed sub-interval of the Routh-Hurwitz interval. Later, in 2008, Adesina and Ukpera [3] discussed boundedness, global exponential stability and periodicity of solutions of a class of nonlinear differential equations of fifth order with constant delay. Finally, in 2009, Adesina and Ukpera [4] also established some sufficient conditions in order for all solutions of a certain nonlinear differential equation of fifth order without delay to converge to a limiting regime under some boundedness restrictions and proved that this limiting regime is periodic or almost periodic. It should be noted that our equation, Eq. (1) and the assumptions that will be established here are different from those in the papers of Adesina and Ukpera [2-4].

To the best of our knowledge, so far, nonlinear delay differential Eq. (1) has not been the subject of investigation for stability of solutions. Our motivation comes from the papers mentioned above, which were carried out on the nonlinear differential equations of fifth order without delay and with delay. Throughout this paper, we also use Lyapunov's second (or direct) method [14] to investigate the stability of the null solution of delay differential Eq. (1) by defining a Lyapunov functional.

2. Preliminaries

In order to reach the main result of this paper, we will give some important basic information for the general autonomous delay differential system (see also Èl'sgol'ts [8], Èl'sgol'ts and Norkin [9], Hale [11], Kolmanovskii and Myshkis [12], Krasovskii [13], Razumikhin ([15, 16]) and Yoshizawa [20]).

We consider a general autonomous delay differential system

$$x' = f(x_t), \quad x_t(\theta) = x(t + \theta), \quad -r \leq \theta \leq 0, \quad t \geq 0, \tag{3}$$

where $f : C_H \rightarrow \mathfrak{R}^n$ is a continuous mapping, $f(0) = 0$, and we suppose that f takes closed bounded sets into bounded sets of \mathfrak{R}^n . Here $(C, \|\cdot\|)$ is the Banach space of continuous function $\phi : [-r, 0] \rightarrow \mathfrak{R}^n$ with supremum norm, $r > 0$; C_H is the open H -ball in C ; $C_H := \{\phi \in (C[-r, 0], \mathfrak{R}^n) : \|\phi\| < H\}$. Standard existence theory, see Burton [6], shows that if $\phi \in C_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \phi)$ such that on $[t_0, t_0 + \alpha]$ satisfying Eq. (3) for $t > t_0$, $x_t(t, \phi) = \phi$ and α is a positive constant. If there is a closed subset $B \subset C_H$ such that the solution remains in B , then $\alpha = \infty$. Further, the symbol $|\cdot|$ will denote the norm in \mathfrak{R}^n with $|x| = \max_{1 \leq i \leq n} |x_i|$.

Definition 1 (see [6]). *Let $f(0) = 0$. The zero solution of Eq. (3) is:*

- (i) *stable if for each $t_1 \geq t_0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $[\|\phi\| \leq \delta, t \geq t_1]$ imply that $|x(t, t_1, \phi)| < \varepsilon$.*
- (ii) *asymptotically stable if it is stable and if for each $t_1 \geq t_0$ there is an η such that $\|\phi\| \leq \eta$ implies that $x(t, t_0, \phi) \rightarrow 0$ as $t \rightarrow \infty$.*

Definition 2 (see [6]). *A continuous positive definite function $W : \mathfrak{R}^n \rightarrow [0, \infty)$ is called a wedge.*

Definition 3 (see [6]). *A continuous function $W : [0, \infty) \rightarrow [0, \infty)$ with $W(0) = 0$, $W(s) > 0$ if $s > 0$, and W strictly increasing is a wedge. (We denote wedges by W or W_i , where i is an integer).*

Definition 4 (see [6]). *Let D be an open set in \mathfrak{R}^n with $0 \in D$. A function $V : D \rightarrow [0, \infty)$ is called*

- (i) *positive definite if $V(0) = 0$ and if there is a wedge W_1 with $V(x) \geq W_1(|x|)$,*
- (ii) *decesent if there is a wedge W_2 with $V(x) \leq W_2(|x|)$.*

Definition 5 (see [17]). *If V is a continuous scalar function in C_H , we define the derivative of V along the solutions of Eq. (3) by the following relation*

$$\dot{V}(\phi) = \limsup_{h \rightarrow 0^+} \frac{V(x_h(\phi)) - V(\phi)}{h}. \tag{4}$$

Proposition 1 (see [10]). *Suppose $f(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \leq s < \infty$, $u(s) \rightarrow \infty$ as $s \rightarrow \infty$ such that for all $\phi \in C$*

$$(i) \quad u(|\phi(0)|) \leq V(\phi), \quad V(\phi) \geq 0,$$

$$(ii) \quad \dot{V}(\phi) < 0 \text{ for } \phi \neq 0.$$

Then all solutions of Eq. (3) approach zero as $t \rightarrow \infty$ and the origin is globally asymptotically stable.

Note that $C_H = C$ when $H = \infty$; and that the set R of ϕ in C for which $\dot{V}_{(3)}(\phi) = 0$ has the largest invariant set $M = \{0\}$ by the condition $\dot{V}(\phi) < 0$ for $\phi \neq 0$.

3. Main result

Our main result is the following theorem.

Theorem 1. *In addition to basic assumptions imposed on the functions ψ and f appeared in Eq. (1), we assume that there are positive constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \varepsilon, \varepsilon_0, \delta, \lambda, \rho, M$ and L such that the following conditions hold for every x, y, z, w and u :*

(i)

$$\begin{aligned} \alpha_1 &> 0, \alpha_1\alpha_2 - \alpha_3 > 0, (\alpha_1\alpha_2 - \alpha_3)\alpha_3 - (\alpha_1\alpha_4 - \alpha_5)\alpha_1 > 0, \\ \delta_0 &:= (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3) - (\alpha_1\alpha_4 - \alpha_5)^2 > 0, \alpha_5 > 0, \\ \Delta_1 &:= \frac{(\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - (\alpha_1\alpha_4 - \alpha_5) > 2\varepsilon\alpha_2, \\ \Delta_2 &:= \frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} - \frac{\alpha_1\alpha_4 - \alpha_5}{\alpha_1\alpha_2 - \alpha_3} - \frac{\varepsilon}{\alpha_1} > 0. \end{aligned}$$

(ii)

$$\begin{aligned} 2\varepsilon_0 &\leq \psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1 \\ &\leq \min \left\{ \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \frac{\varepsilon}{4\alpha_1^2}, \frac{\varepsilon\alpha_4}{4\delta^2} \right\}. \end{aligned}$$

(iii)

$$\frac{f(z, w)}{w} \geq \alpha_2, \quad w \neq 0$$

and

$$\begin{aligned} \left(\frac{f(z, w)}{w} - \alpha_2 \right)^2 &\leq \min \left[\frac{\varepsilon^2\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, \frac{\varepsilon^2\alpha_2}{4\delta^2} \right], \\ |f_z(z, w)| &\leq M \quad \text{and} \quad |f_w(z, w)| \leq L. \end{aligned}$$

Then, the trivial null solution of Eq. (1) is globally asymptotically stable provided that

$$r < \min \left\{ \frac{\varepsilon\alpha_4}{\delta(L+M)}, \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)}{\alpha_4(L+M)(\alpha_1\alpha_2 - \alpha_3)}, \frac{\varepsilon}{2\alpha_1(L+M) + 4\rho}, \frac{\varepsilon_0}{L+M+2\lambda} \right\}.$$

Proof. We define the Lyapunov functional $V = V(x_t, y_t, z_t, w_t, u_t)$ as:

$$\begin{aligned}
 2V = & u^2 + 2\alpha_1uw + \frac{2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}uz + 2\delta wy + 2 \int_0^w f(z, \xi)d\xi \\
 & + \left[\alpha_1^2 - \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right] w^2 + 2 \left[\alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right] wz \\
 & + 2\alpha_1\delta wy + 2\alpha_4wy + 2\alpha_5wx + \alpha_1\alpha_3z^2 + \left[\frac{\alpha_2\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_4 - \alpha_1\delta \right] z^2 \\
 & + 2\delta\alpha_2yz + 2\alpha_1\alpha_4zy - 2\alpha_5zy + 2\alpha_1\alpha_5zx + \frac{\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}y^2 \\
 & + (\delta\alpha_3 - \alpha_1\alpha_5)y^2 + \frac{2\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}yx + \delta\alpha_5x^2 \\
 & + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta)d\theta ds + 2\lambda \int_{-r}^0 \int_{t+s}^t u^2(\theta)d\theta ds,
 \end{aligned} \tag{5}$$

where s is a real variable such that the integrals

$$\int_{-r}^0 \int_{t+s}^t w^2(\theta)d\theta ds$$

and

$$\int_{-r}^0 \int_{t+s}^t u^2(\theta)d\theta ds$$

are non-negative, ρ and λ are some positive constants which will be determined later in the proof and δ is a positive constant defined by

$$\delta := \frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} + \varepsilon. \tag{6}$$

It is clear that $V(0, 0, 0, 0, 0) = 0$. Since $f(z, 0) = 0$ and $\frac{f(z,w)}{w} \geq \alpha_2$, ($w \neq 0$), we have $2 \int_0^w f(z, \xi)d\xi = 2 \int_0^w \frac{f(z,\xi)}{\xi}\xi d\xi \geq 2 \int_0^w \alpha_2\xi d\xi = \alpha_2w^2$. Hence, the Lyapunov functional $V = V(x_t, y_t, z_t, w_t, u_t)$ defined by (5) can be recast as:

$$\begin{aligned}
 2V \geq & \left[u + \alpha_1w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}z + \delta y \right]^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2} \left(z + \frac{\alpha_5}{\alpha_4}y \right)^2 \\
 & + \frac{(\alpha_1\alpha_4 - \alpha_5)}{(\alpha_1\alpha_2 - \alpha_3)} \left[\frac{\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}x + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{(\alpha_1\alpha_4 - \alpha_5)}y + \alpha_1z + w \right]^2 \\
 & + \Delta_2 (w + \alpha_1z)^2 + \frac{\varepsilon}{\alpha_1}w^2 + 2\varepsilon \left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) yz \\
 & + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta)d\theta ds + 2\lambda \int_{-r}^0 \int_{t+s}^t u^2(\theta)d\theta ds + \sum_{i=1}^2 V_i,
 \end{aligned} \tag{7}$$

where

$$V_1 := \delta\alpha_5x^2 - \frac{\alpha_5^2(\alpha_1\alpha_2 - \alpha_3)}{(\alpha_1\alpha_4 - \alpha_5)}x^2$$

and

$$V_2 := \left[\delta\alpha_3 - \alpha_1\alpha_5 - \frac{\alpha_5^2\delta_0}{\alpha_4(\alpha_1\alpha_4 - \alpha_5)^2} - \delta^2 \right] y^2.$$

By noting (6), it is clear that

$$V_1 = \varepsilon\alpha_5x^2$$

and

$$V_2 \geq \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)}y^2$$

provided that

$$\frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)} \geq \varepsilon \left[\varepsilon + \frac{2\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \alpha_3 \right],$$

which we now assume.

Summing up the equality and inequality obtained for V_1 and V_2 into (7), we have

$$\begin{aligned} 2V \geq & \left[u + \alpha_1w + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5}z + \delta y \right]^2 + \frac{\alpha_4\delta_0}{(\alpha_1\alpha_4 - \alpha_5)^2} \left(z + \frac{\alpha_5}{\alpha_4}y \right)^2 \\ & + \Delta_2(w + \alpha_1z)^2 + \varepsilon\alpha_5x^2 + \frac{\alpha_5\delta_0}{4\alpha_4(\alpha_1\alpha_4 - \alpha_5)}y^2 + \frac{\varepsilon}{\alpha_1}w^2 \\ & + 2\varepsilon \left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) yz + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds + 2\lambda \int_{-r}^0 \int_{t+s}^t u^2(\theta) d\theta ds. \end{aligned} \quad (8)$$

Clearly, it follows from the first six terms included in (8) that there exist sufficiently small positive constants D_i , ($i = 1, 2, 3, 4, 5$), such that

$$\begin{aligned} 2V \geq & D_1x^2 + D_2y^2 + D_3z^2 + D_4w^2 + D_5u^2 + 2\varepsilon \left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) yz \\ & + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds + 2\lambda \int_{-r}^0 \int_{t+s}^t u^2(\theta) d\theta ds. \end{aligned} \quad (9)$$

Now, we consider the terms

$$V_3 =: \frac{D_2}{2}y^2 + 2\varepsilon \left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) yz + \frac{D_3}{2}z^2,$$

which are contained in (9).

Clearly, V_3 represents a quadratic expression. So, it can be easily seen that V_3 is positive semi-definite if the symmetric matrix

$$\begin{bmatrix} \frac{D_2}{2} & \varepsilon \left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) \\ \varepsilon \left(\frac{\alpha_3\alpha_4 - \alpha_2\alpha_5}{\alpha_1\alpha_4 - \alpha_5} \right) & \frac{D_3}{2} \end{bmatrix}$$

is positive semi-definite. That is, $V_3 \geq 0$, provided that

$$\varepsilon^2 \leq \left(\frac{\alpha_1\alpha_4 - \alpha_5}{\alpha_3\alpha_4 - \alpha_2\alpha_5} \right)^2 \frac{D_2 D_3}{4} = D_6, \quad D_6 > 0.$$

By using this fact, we get from (9) that

$$\begin{aligned} 2V \geq & D_1 x^2 + \frac{D_2}{2} y^2 + \frac{D_3}{2} z^2 + D_4 w^2 + D_5 u^2 \\ & + 2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds + 2\lambda \int_{-r}^0 \int_{t+s}^t u^2(\theta) d\theta ds. \end{aligned}$$

As a result, since the integrals

$$2\rho \int_{-r}^0 \int_{t+s}^t w^2(\theta) d\theta ds$$

and

$$\lambda \int_{-r}^0 \int_{t+s}^t u^2(\theta) d\theta ds$$

are non-negative, it is obvious that there exists a positive constant D_7 which satisfies the inequality

$$D_7(x^2(t) + y^2(t) + z^2(t) + w^2(t) + u^2(t)) \leq V(x_t, y_t, z_t, w_t, u_t), \quad (10)$$

where $D_7 = \frac{1}{2} \min \{D_1, \frac{D_2}{2}, \frac{D_3}{2}, D_4, D_5\}$.

Now, by a direct calculation from (5) and (2) one finds

$$\begin{aligned} \frac{d}{dt} V(x_t, y_t, z_t, w_t, u_t) = & -[\psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1] u^2 \\ & - \left[\alpha_1 \frac{f(z, w)}{w} - \left\{ \alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right\} \right] w^2 \\ & - \left[\frac{\alpha_3\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \{\delta\alpha_2 + (\alpha_1\alpha_4 - \alpha_5)\} \right] z^2 \\ & - \left[\delta\alpha_4 - \frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right] y^2 \\ & - \alpha_1[\psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) \\ & - \alpha_1] wu \end{aligned}$$

$$\begin{aligned}
& -\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} [\psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1] zu \\
& -\delta [\psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1] yu \\
& -\frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \left[\frac{f(z, w)}{w} - \alpha_2 \right] wz - \delta \left[\frac{f(z, w)}{w} - \alpha_2 \right] wy \\
& +u \int_{t-r}^t f_z(z(s), w(s))w(s)ds + \alpha_1 w \int_{t-r}^t f_z(z(s), w(s))w(s)ds \\
& +\delta y \int_{t-r}^t f_z(z(s), w(s))w(s)ds + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \int_{t-r}^t f_z(z(s), w(s))w(s)ds \\
& +u \int_{t-r}^t f_w(z(s), w(s))u(s)ds + \alpha_1 w \int_{t-r}^t f_w(z(s), w(s))u(s)ds \\
& +\delta y \int_{t-r}^t f_w(z(s), w(s))u(s)ds + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \int_{t-r}^t f_w(z(s), w(s))u(s)ds \\
& +\rho w^2 r - \rho \int_{t-r}^t w^2(s)ds + \lambda u^2 r - \lambda \int_{t-r}^t u^2(s)ds.
\end{aligned} \tag{11}$$

Making use of the assumptions (i)-(iii) and (6), we get

$$\begin{aligned}
& [\psi(x(t-r), y(t-r), z(t-r), w(t-r), u(t-r)) - \alpha_1] \geq 2\varepsilon_0, \\
& \left[\alpha_1 \frac{f(z, w)}{w} - \left\{ \alpha_3 + \frac{\alpha_1\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \delta \right\} \right] \geq \varepsilon, \\
& \left[\frac{\alpha_3\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} - \{ \delta\alpha_2 + (\alpha_1\alpha_4 - \alpha_5) \} \right] > \varepsilon\alpha_2
\end{aligned}$$

and

$$\left[\delta\alpha_4 - \frac{\alpha_4\alpha_5(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \right] = \varepsilon\alpha_4.$$

By using the assumptions $|f_z(z, w)| \leq M$ and $|f_w(z, w)| \leq L$ of the theorem and the inequality $2|ab| \leq a^2 + b^2$, we obtain the following inequalities:

$$u \int_{t-r}^t f_z(z(s), w(s))w(s)ds \leq \frac{M}{2} ru^2(t) + \frac{M}{2} \int_{t-r}^t w^2(s)ds,$$

$$\begin{aligned} \alpha_1 w \int_{t-r}^t f_z(z(s), w(s))w(s)ds &\leq \frac{\alpha_1 M}{2} r w^2(t) + \frac{\alpha_1 M}{2} \int_{t-r}^t w^2(s)ds, \\ \delta y \int_{t-r}^t f_z(z(s), w(s))w(s)ds &\leq \frac{\delta M}{2} r y^2(t) + \frac{\delta M}{2} \int_{t-r}^t w^2(s)ds, \\ \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \int_{t-r}^t f_z(z(s), w(s))w(s)ds &\leq \frac{\alpha_4 M(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} r z^2(t) \\ &\quad + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)M}{2(\alpha_1\alpha_4 - \alpha_5)} \int_{t-r}^t w^2(s)ds, \\ u \int_{t-r}^t f_w(z(s), w(s))u(s)ds &\leq \frac{L}{2} r u^2(t) + \frac{L}{2} \int_{t-r}^t u^2(s)ds, \\ \alpha_1 w \int_{t-r}^t f_w(z(s), w(s))u(s)ds &\leq \frac{\alpha_1 L}{2} r w^2(t) + \frac{\alpha_1 L}{2} \int_{t-r}^t u^2(s)ds, \\ \delta y \int_{t-r}^t f_w(z(s), w(s))u(s)ds &\leq \frac{\delta L}{2} r y^2(t) + \frac{\delta L}{2} \int_{t-r}^t u^2(s)ds \end{aligned}$$

and

$$\begin{aligned} \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} z \int_{t-r}^t f_w(z(s), w(s))u(s)ds &\leq \frac{\alpha_4 L(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} r z^2(t) \\ &\quad + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}{2(\alpha_1\alpha_4 - \alpha_5)} \int_{t-r}^t u^2(s)ds. \end{aligned}$$

Replacing the last equality and the preceding inequalities into (11), we obtain

$$\begin{aligned} \frac{dV}{dt} &\leq - \left[\frac{\varepsilon\alpha_4}{2} - \left(\frac{\delta(L+M)}{2} \right) r \right] y^2 - \left\{ \frac{\varepsilon\alpha_2}{2} - \left(\frac{\alpha_4(L+M)(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} \right) r \right\} z^2 \\ &\quad - \left[\frac{\varepsilon}{4} - \left(\frac{\alpha_1(L+M)}{2} + \rho \right) r \right] w^2 - \left[\frac{\varepsilon_0}{2} - \left(\frac{L+M}{2} + \lambda \right) r \right] u^2 \\ &\quad - \left[\rho - \left(\frac{M}{2} + \frac{\alpha_1 M}{2} + \frac{\delta M}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)M}{2(\alpha_1\alpha_4 - \alpha_5)} \right) \right] \int_{t-r}^t w^2(s)ds \\ &\quad - \left[\lambda - \left(\frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}{2(\alpha_1\alpha_4 - \alpha_5)} \right) \right] \int_{t-r}^t u^2(s)ds - \sum_{k=4}^8 V_k, \end{aligned} \tag{12}$$

where

$$\begin{aligned} V_4 &= \frac{1}{4} [\psi - \alpha_1] u^2 + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} [\psi - \alpha_1] zu + \frac{\varepsilon\alpha_2}{4} z^2, \\ V_5 &= \frac{1}{4} [\psi - \alpha_1] u^2 + \alpha_1 [\psi - \alpha_1] wu + \frac{\varepsilon}{4} w^2, \\ V_6 &= \frac{1}{4} [\psi - \alpha_1] u^2 + \delta [\psi - \alpha_1] yu + \frac{\varepsilon\alpha_4}{4} y^2, \\ V_7 &= \frac{\varepsilon}{4} w^2 + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)}{\alpha_1\alpha_4 - \alpha_5} \left[\frac{f}{w} - \alpha_2 \right] wz + \frac{\varepsilon\alpha_2}{4} z^2, \\ V_8 &= \frac{\varepsilon}{4} w^2 + \delta \left[\frac{f}{w} - \alpha_2 \right] wy + \frac{\varepsilon\alpha_4}{4} y^2. \end{aligned}$$

It is clear that the expressions given by V_4 , V_5 , V_6 , V_7 and V_8 represent certain specific quadratic forms, respectively. Making use of the basic information on the positive semi-definite of a quadratic form, one can easily conclude that $V_4 \geq 0$, $V_5 \geq 0$, $V_6 \geq 0$, $V_7 \geq 0$ and $V_8 \geq 0$ provided that

$$\begin{aligned} (\psi - \alpha_1) &\leq \frac{\varepsilon\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2}, (\psi - \alpha_1) \leq \frac{\varepsilon}{4\alpha_1^2}, (\psi - \alpha_1) \leq \frac{\varepsilon\alpha_4}{4\delta^2}, \\ \left(\frac{f}{w} - \alpha_2 \right)^2 &\leq \frac{\varepsilon^2\alpha_2(\alpha_1\alpha_4 - \alpha_5)^2}{4\alpha_4^2(\alpha_1\alpha_2 - \alpha_3)^2} \end{aligned}$$

and

$$\left(\frac{f}{w} - \alpha_2 \right)^2 \leq \frac{\varepsilon^2\alpha_4}{4\delta^2},$$

respectively.

Thus, in view of the above discussion and inequality (12), it follows that

$$\begin{aligned} \frac{dV}{dt} &\leq - \left[\frac{\varepsilon\alpha_4}{2} - \left(\frac{\delta(L+M)}{2} \right) r \right] y^2 - \left\{ \frac{\varepsilon\alpha_2}{2} - \left(\frac{\alpha_4(L+M)(\alpha_1\alpha_2 - \alpha_3)}{2(\alpha_1\alpha_4 - \alpha_5)} \right) r \right\} z^2 \\ &\quad - \left[\frac{\varepsilon}{4} - \left(\frac{\alpha_1(L+M)}{2} + \rho \right) r \right] w^2 - \left[\frac{\varepsilon_0}{2} - \left(\frac{L+M}{2} + \lambda \right) r \right] u^2 \\ &\quad - \left[\rho - \left(\frac{M}{2} + \frac{\alpha_1 M}{2} + \frac{\delta M}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)M}{2(\alpha_1\alpha_4 - \alpha_5)} \right) \right] \int_{t-r}^t w^2(s) ds \\ &\quad - \left[\lambda - \left(\frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)L}{2(\alpha_1\alpha_4 - \alpha_5)} \right) \right] \int_{t-r}^t u^2(s) ds. \end{aligned} \tag{13}$$

If we now choose the constants ρ and λ as

$$\rho = \left(\frac{M}{2} + \frac{\alpha_1 M}{2} + \frac{\delta M}{2} + \frac{\alpha_4(\alpha_1\alpha_2 - \alpha_3)M}{2(\alpha_1\alpha_4 - \alpha_5)} \right)$$

and

$$\lambda = \left(\frac{L}{2} + \frac{\alpha_1 L}{2} + \frac{\delta L}{2} + \frac{\alpha_4(\alpha_1 \alpha_2 - \alpha_3)L}{2(\alpha_1 \alpha_4 - \alpha_5)} \right),$$

then the inequality in (13) implies that

$$\begin{aligned} \frac{dV}{dt} \leq & - \left[\frac{\varepsilon \alpha_4}{2} - \left(\frac{\delta(L+M)}{2} \right) r \right] y^2 - \left\{ \frac{\varepsilon \alpha_2}{2} - \left(\frac{\alpha_4(L+M)(\alpha_1 \alpha_2 - \alpha_3)}{2(\alpha_1 \alpha_4 - \alpha_5)} \right) r \right\} z^2 \\ & - \left[\frac{\varepsilon}{4} - \left(\frac{\alpha_1(L+M)}{2} + \rho \right) r \right] w^2 - \left[\frac{\varepsilon_0}{2} - \left(\frac{L+M}{2} + \lambda \right) r \right] u^2. \end{aligned} \quad (14)$$

Hence, one can easily get from (14) that

$$\frac{d}{dt} V(x_t, y_t, z_t, w_t, u_t) \leq -D_8 y^2 - D_9 z^2 - D_{10} w^2 - D_{11} u^2 \leq 0$$

for some positive constants D_i , ($i = 8, 9, 10, 11$,) provided that

$$r < \min \left\{ \frac{\varepsilon \alpha_4}{\delta(L+M)}, \frac{\varepsilon \alpha_2(\alpha_1 \alpha_4 - \alpha_5)}{\alpha_4(L+M)(\alpha_1 \alpha_2 - \alpha_3)}, \frac{\varepsilon}{2\alpha_1(L+M) + 4\rho}, \frac{\varepsilon_0}{L+M+2\lambda} \right\}.$$

Finally, it follows that $\frac{d}{dt} V(x_t, y_t, z_t, w_t, u_t) \equiv 0$ if and only if $y_t = z_t = w_t = u_t = 0$, $\frac{d}{dt} V(\phi) < 0$ for $\phi \neq 0$ and $V(\phi) \geq u(|\phi(0)|) \geq 0$. Thus all the conditions of the above Proposition are satisfied. This shows that the null solution of Eq. (1) is globally asymptotically stable. \square

References

- [1] A. M. A. ABOU-EL ELA, A. I. SADEK, *A stability result for the solutions of certain fifth-order differential equations*, Bull. Fac. Sci. Assiut. Univ. C. **24**(1995), 1–11.
- [2] O. A. ADESINA, A. S. UKPERA, *Convergence of solutions for a fifth-order nonlinear differential equation*, Electron. J. Differential Equations **2007**(2007), 1–11.
- [3] O. A. ADESINA, A. S. UKPERA, *On the qualitative behaviour of solutions of some fifth order nonlinear delay differential equations*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N. S.) **54**(2008), 135–146.
- [4] O. A. ADESINA, A. S. UKPERA, *On the existence of a limiting regime in the sense of Demidovich for a certain fifth order nonlinear differential equation*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **16**(2009), 193–207.
- [5] L. I. BURGANSKAJA, *The stability in the large of the zero solution of certain fifth order nonlinear differential equations*, Differentsial'nye Uravnenija **7**(1971), 1752–1764, in Russian.
- [6] T. A. BURTON, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, Orlando, 1985.
- [7] E. N. CHUKWU, *On the boundedness and stability of solutions of some differential equations of the fifth order*, SIAM J. Math. Anal. **7**(1976), 176–194.
- [8] L. È. ÈL'SGOL'TS, *Introduction to the theory of differential equations with deviating arguments*, Holden-Day Inc., San Francisco, 1966.

- [9] L. È. ÈL'SGOL'TS, S. B. NORKIN, *Introduction to the theory and application of differential equations with deviating arguments*, Academic Press, New York, 1973.
- [10] J. K. HALE, *Sufficient conditions for stability and instability of autonomous functional-differential equations*. J. Differential Equations **1**(1965), 452–482.
- [11] J. K. HALE, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.
- [12] V. KOLMANOVSKII, A. MYSHKIS, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic Publishers, Dordrecht, 1999.
- [13] N. N. KRASOVSKII, *Stability of motion. Applications of Lyapunov's second method to differential systems and equations with delay*, Stanford University Press, Stanford CA, 1963.
- [14] A. M. LYAPUNOV, *Stability of Motion*, Academic Press, London, 1966.
- [15] B. S. RAZUMIKHIN, *Stability in first approximation of systems with lag*. J. Appl. Math. Mech. **22**(1958), 215–229.
- [16] B. S. RAZUMIKHIN, *The application of Lyapunov's method to problems in the stability of systems with delay*, (in Russian), Avtomat. i Telemekh. **21**(1960), 740–748; English transl. in Autom. Remote Control **21**(1960), 515–520.
- [17] A. S. C. SINHA, *On stability of a fifth-order nonlinear differential equation*, Proceedings of the IEEE **59**(1971), 1382–1383.
- [18] H. O. TEJUMOLA, A. U. AFUWAPE, *Resonant and non-resonant oscillations for some fifth order nonlinear differential equations with delay*, Discovery and Innovation **2**(1990), 27–36.
- [19] C. TUNÇ, *A study of the stability and boundedness of the solutions of nonlinear differential equations of the fifth order*, Indian J. Pure Appl. Math. **33**(2002), 519–529.
- [20] T. YOSHIZAWA, *Stability theory by Liapunov's second method*, The Mathematical Society of Japan, Tokyo, 1966.
- [21] Y. H. YU, *Stability and boundedness of solutions to nonlinear differential equations of the fifth order*, J. Central China Normal Univ. Natur. Sci. **24**(1990), 267–273, in Chinese.