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Modified double Szász-Mirakjan operators preserving $x^2 + y^2$

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Abstract. In this paper, we introduce a modification of the Szász-Mirakjan type operators of two variables which preserve $f_0(x,y) = 1$ and $f_3(x,y) = x^2 + y^2$. We prove that this type of operators enables a better error estimation on the interval $[0,\infty) \times [0,\infty)$ than the classical Szász-Mirakjan type operators of two variables. Moreover, we prove a Voronovskaya-type theorem and some differential properties for derivatives of these modified operators. Finally, we also study statistical convergence of the sequence of modified Szász-Mirakjan type operators.

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1. Introduction

Most of the approximating operators, L_n, preserve $f_i(x) = x^i$, $(i = 0, 1)$, $L_n(f_0; x) = f_0(x)$, $L_n(f_1; x) = f_1(x), n \in \mathbb{N}$, but $L_n(f_2; x) \neq f_2(x) = x^2$. Especially, these conditions hold for the Bernstein polynomials and the Szász-Mirakjan type operators [1, 2, 3, 14]. Recently, King [13] presented a non-trivial sequence of positive linear operators defined on the space of all real-valued continuous functions on $[0, 1]$ which preserves the functions f_0 and f_2. Duman and Orhan [4] have studied King's results using the concept of statistical convergence. Recently, Duman and Özarslan [5] have investigated some approximation results on the Szász-Mirakjan type operators preserving $f_2(x) = x^2$.

Functions $f_0(x,y) = 1$, $f_1(x,y) = x$ and $f_2(x,y) = y$ are preserved by most of approximating operators of two variables, L_n, i.e., $L_n(f_0; x,y) = f_0(x,y)$, $L_n(f_1; x,y) = f_1(x,y)$ and $L_n(f_2; x,y) = f_2(x,y)$, $n \in \mathbb{N}$, but $L_n(f_3; x,y) \neq f_3(x,y) = x^2 + y^2$. In this paper, we give a modification of the well-known Szász-Mirakjan type operators of two variables and show that this modification preserving $f_0(x,y)$ and $f_3(x,y)$ has a better estimation than the classical Szász-Mirakjan of two variables. Also, we obtain a Voronovskaya-type theorem and some differential properties of these modified operators. Finally, we study A-statistical convergence of this modification.

By $C(D)$ we denote the space of all continuous real valued functions on D where $D = [0,\infty) \times [0,\infty)$. By E_2 we denote the space of all functions $f : D \to R$ of

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exponential type where R is the disk with $|z| < R$, $R > 1$. More precisely, $f \in E_2$ if and only if there are three positive finite constants c, d and α with the property $|f(x,y)| \leq \alpha e^{cx+dy}$. Let L be a linear operator from $C(D) \cap E_2$ into $C(D) \cap E_2$. Then, as usual, we say that L is a positive linear operator provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ of a point $(x,y) \in D$ by $L(f;x,y)$.

Now fix $a, b > 0$. For proving our approximation results we use lattice homomorphism $H_{a,b}$ maps $C(D) \cap E_2$ into $C(E) \cap E_2$ defined by $H_{a,b}(f) = f\,|_E$ where $E = [0,a] \times [0,b]$ and $f\,|_E$ denote the restriction of the domain f to the interval E. $C(E)$ space is equipped with the supremum norm

$$\|f\| = \sup_{(x,y)\in E} |f(x,y)|\,, \ (f \in C(E))\,.$$

Following the paper by Erkuş and Duman [6], one can obtain the next Korovkin-type approximation result in a statistical sense (see the last for the basic properties of statistical convergence).

Theorem 1. *Let $A = (a_{nk})$ be a non-negative regular summability matrix. Let $\{L_n\}$ be a sequence of positive linear operators acting from $C(D) \cap E_2$ into itself. Assume that the following conditions hold:*

$$st_A - \lim_n L_n(f_i; x.y) = f_i(x,y)\,, \ \textit{uniformly on } E,\ i = 0,1,2,3,$$

where $f_0(x,y) = 1$, $f_1(x,y) = x$, $f_2(x,y) = y$ and $f_3(x,y) = x^2 + y^2$. Then, for all $f \in C(D) \cap E_2$, we have

$$st_A - \lim_n L_n(f; x.y) = f(x,y)\,, \ \textit{uniformly on } E.$$

2. Construction of operators

The double Szász-Mirakjan was introduced by Favard [8]:

$$S_n(f;x,y) = e^{-nx}e^{-ny}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} f\left(\frac{s}{n},\frac{t}{n}\right)\frac{(nx)^s}{s!}\frac{(ny)^t}{t!}, \tag{1}$$

where $(x,y) \in D$; $f \in C(D) \cap E_2$. It is clear that

$$\begin{aligned} S_n(f_0;x,y) &= f_0(x,y),\\ S_n(f_1;x,y) &= f_1(x,y),\\ S_n(f_2;x,y) &= f_2(x,y),\\ S_n(f_3;x,y) &= f_3(x,y) + \frac{x}{n} + \frac{y}{n}. \end{aligned}$$

Then, we observe that $S_n(f_i) \to f_i$ uniformly on E, where $i = 0,1,2,3$. If we replace matrix A by identity matrix in Theorem 1, then we immediately get classical result. Hence, for S_n operators given by (1), we have for all $f \in C(D) \cap E_2$,

$$\lim_n S_n(f;x,y) = f(x,y)\,, \quad \text{uniformly on } E.$$

Let $\{u_n(x)\}$ and $\{v_n(y)\}$ be two sequences of exponential-type continuous functions defined on interval $[0,\infty)$ with $0 \le u_n(x) < \infty$, $0 \le v_n(y) < \infty$. Let

$$\begin{aligned} H_n(f;x,y) &= S_n(f;u_n(x), v_n(y)) \\ &= e^{-nu_n(x)}e^{-nv_n(y)}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty} f\left(\frac{s}{n},\frac{t}{n}\right)\frac{(nu_n(x))^s}{s!}\frac{(nv_n(y))^t}{t!} \end{aligned} \tag{2}$$

for $f \in C(D) \cap E_2$. Hence, in the special case $u_n(x) = x$ and $v_n(y) = y$, $n = 1, 2, ...$ reduce to classical Szász-Mirakjan type operators given by (1).

It is clear that H_n are positive and linear. Also, we have

$$\begin{aligned} H_n(f_0;x,y) &= f_0(x,y), \\ H_n(f_1;x,y) &= u_n(x), \\ H_n(f_2;x,y) &= v_n(y), \\ H_n(f_3;x,y) &= u_n^2(x) + v_n^2(y) + \frac{u_n(x)}{n} + \frac{v_n(y)}{n}, \end{aligned} \tag{3}$$

Now, the following result follows immediately from Theorem 1 for the case $A = I$, the identity matrix.

Theorem 2. *Let H_n denote the sequence of positive linear operators given by (2). If*

$$\lim_n u_n(x) = x, \ \lim_n v_n(y) = y, \text{ uniformly on } E,$$

then, for all $f \in C(D) \cap E_2$,

$$\lim_n H_n(f;x,y) = f(x,y), \text{uniformly on } E.$$

Furthermore, we present the sequence $\{H_n\}$ of positive linear operators defined on $C(D) \cap E_2$ that preserve $f_0(x)$ and $f_3(x)$.

It is obvious that if we replace $u_n(x)$ and $v_n(y)$ by $u_n^*(x)$ and $v_n^*(y)$ defined as

$$u_n^*(x) = \frac{-1+\sqrt{1+4n^2x^2}}{2n}, \ v_n^*(y) = \frac{-1+\sqrt{1+4n^2y^2}}{2n}, \ n = 1, 2, ..., \tag{4}$$

then we obtain

$$H_n(f_3;x,y) = f_3(x,y) = x^2 + y^2, \ \ n = 1, 2, ... \tag{5}$$

Simple calculations show that for $u_n^*(x)$ and $v_n^*(y)$ given by (4),

$$u_n^*(x) \ge 0, \ v_n^*(y) \ge 0, \ n = 1, 2, ..., \ x, y \in [0,\infty). \tag{6}$$

It is clear that

$$\lim_n u_n^*(x) = x, \ \lim_n v_n^*(y) = y, \quad \text{uniformly on } E.$$

3. Comparison with Szász-Mirakjan type operators

In this section, we compute the rates of convergence of operators $H_n(f;x,y)$ to $f(x,y)$ by means of the modulus of continuity. Thus, we show that our estimations are more powerful than the operators given by (1) on the interval D.

By $C_B(D)$ we denote the space of all continuous and bounded functions on D. For $f \in C_B(D) \cap E_2$, the modulus of continuity of f, denoted by $\omega(f;\delta)$, is defined to be

$$\omega(f;\delta) = \sup\left\{|f(u,v) - f(x,y)| : \sqrt{(u-x)^2 + (v-y)^2} < \delta, (u,v), (x,y) \in D\right\}.$$

Then it is clear that for any $\delta > 0$ and each $(x,y) \in D$

$$|f(u,v) - f(x,y)| \leq \omega(f;\delta)\left(\frac{\sqrt{(u-x)^2 + (v-y)^2}}{\delta} + 1\right).$$

After some simple calculations, for any sequence $\{L_n\}$ of positive linear operators on $C_B(D) \cap E_2$, for $f \in C_B(D) \cap E_2$, we can write

$$\begin{aligned}|L_n(f;x,y) - f(x,y)| \leq \omega(f;\delta)\Big\{1 &+ \frac{1}{\delta^2}L_n\left((u-x)^2 + (v-y)^2; x,y\right) \\ &+ |L_n(f_0;x,y) - f_0(x,y)|\Big\} \\ &+ |f(x,y)|\,|L_n(f_0;x,y) - f_0(x,y)|.\end{aligned} \tag{7}$$

Now we have the following:

Theorem 3. *If H_n is defined by (2), then for $(x,y) \in D$ and any $\delta > 0$, we have*

$$\begin{aligned}|H_n(f;x,y) - f(x,y)| \leq \omega(f,\delta)\Big\{1 + \frac{1}{\delta^2}\big(&2(x^2+y^2) - 2xH_n(f_1;x,y) \\ &-2yH_n(f_2;x,y)\big)\Big\}\end{aligned} \tag{8}$$

where $H_n(f_1;x,y) = u_n^*(x)$ and $H_n(f_2;x,y) = v_n^*(y)$ is given by (4).

Proof. Now, let $f \in C_B(D) \cap E_2$. Using linearity and monotonicity H_n and from (7), the proof is complete. □

Furthermore, when (8) holds,

$$2(x^2+y^2) - 2xH_n(f_1;x,y) - 2yH_n(f_2;x,y) \geq 0 \text{ for } (x,y) \in D.$$

Remark 1. *For the Szász-Mirakjan type operators given by (1), from (7) we may write that for every $f \in C_B(D) \cap E_2$, $n \in \mathbb{N}$,*

$$|S_n(f;x,y) - f(x,y)| \leq \omega(f,\delta)\left\{1 + \frac{1}{\delta^2}\left(\frac{x}{n} + \frac{y}{n}\right)\right\}. \tag{9}$$

Estimate (8) is better than estimate (9) if and only if

$$2(x^2+y^2)-2xH_n(f_1;x,y)-2yH_n(f_2;x,y)\le \frac{x}{n}+\frac{y}{n},\ (x,y)\in D. \tag{10}$$

Thus, the order of approximation towards a function $f\in C_B(D)\cap E_2$ given by the sequence H_n will be at least as good as that of S_n whenever the following function $\phi_n(x,y)$ is non-negative:

$$\begin{aligned}\phi_n(x,y) &= \frac{x}{n}+\frac{y}{n}+2xH_n(f_1;x,y)+2yH_n(f_2;x,y)-2\left(x^2+y^2\right)\\ &= 2x\sqrt{x^2+\frac{1}{4n^2}}+2y\sqrt{y^2+\frac{1}{4n^2}}-2\left(x^2+y^2\right),\end{aligned}$$

where

$$H_n(f_1;x,y)=u_n^*(x)=\frac{-1+\sqrt{1+4n^2x^2}}{2n}$$

and

$$H_n(f_2;x,y)=v_n^*(y)=\frac{-1+\sqrt{1+4n^2y^2}}{2n}.$$

Since

$$\begin{aligned}2x\sqrt{x^2+\frac{1}{4n^2}} &\ge 2x^2,\ \text{for } x\ge 0,\\ 2y\sqrt{y^2+\frac{1}{4n^2}} &\ge 2y^2,\ \text{for } y\ge 0,\end{aligned}$$

(10) holds for every $x,y\ge 0$ and $n\in\mathbb{N}$. Therefore, our estimations are more powerful than the operators given by (1) on the interval D.

4. A Voronovskaya-type theorem

In this section, as in [5], we prove a Voronovskaya-type theorem for the operators H_n given by (2) with $\{u_n(x)\}$ and $\{v_n(y)\}$ replaced by $\{u_n^*(x)\}$ and $\{v_n^*(y)\}$,where $u_n^*(x)$ and $v_n^*(y)$ are defined by (4).

Lemma 1. *Let $x,y\in[0,\infty)$. Then, we get*

$$\lim_n n^2H_n\left((u-x)^4;x,y\right)=3x^2,\ \textit{uniformly on } E, \tag{11}$$

and

$$\lim_n n^2H_n\left((v-y)^4;x,y\right)=3y^2,\ \textit{uniformly on } E. \tag{12}$$

Proof. We shall prove only (11) because the proof of (12) is similar. After some simple calculations, we can write from (11) that

$$\begin{aligned}n^2H_n\left((u-x)^4;x,y\right) &= -\frac{4nx^3}{2nx+\sqrt{1+4n^2x^2}}+\frac{2x^2}{2nx+\sqrt{1+4n^2x^2}}\\ &\quad +2x\left(\frac{-1+\sqrt{1+4n^2x^2}}{n}\right)+\left(\frac{1-\sqrt{1+4n^2x^2}}{2n^2}\right).\end{aligned}$$

Now taking the limit as $n \to \infty$ on both sides of the above equality we get

$$\lim_n n^2 H_n \left((u-x)^4 ; x, y \right) = -x^2 + 0 + 4x^2 + 0 = 3x^2$$

unifomly with respect to $x \in [0, \infty)$. The proof is complete. □

Theorem 4. *For every $f \in C(D) \cap E_2$ such that f_x, f_y, f_{xx}, f_{xy}, $f_{yy} \in C(D) \cap E_2$, we have*

$$\lim_n n \{ H_n (f; x, y) - f(x, y) \} = \frac{1}{2} \{ x f_{xx}(x, y) + y f_{yy}(x, y) - f_x(x, y) - f_y(x, y) \},$$

uniformly on E.

Proof. Let $(x, y) \in D$ and f_x, f_y, f_{xx}, f_{xy}, $f_{yy} \in C(D) \cap E_2$. We define the function ϕ: if $(u, v) \neq (x, y)$, then

$$\begin{aligned} \phi_{(x,y)}(u, v) = & \frac{1}{\sqrt{(u-x)^4 + (v-y)^4}} \left\{ f(u, v) - \sum_{i=0}^{2} \frac{1}{i!} \left(f_x(x, y)(u-x) \right. \right. \\ & \left. \left. + f_y(x, y)(v-y) \right)^{(i)} \right\}, \end{aligned}$$

else $\phi_{(x,y)}(u, v) = 0$. $g^{(i)}$ is a derivative of function g for $i = 0, 1, 2$. It is not hard to see that $\phi_{(x,y)}(.,.) \in C(D) \cap E_2$. By the Taylor formula for $f \in C(D) \cap E_2$, we have

$$\begin{aligned} f(u, v) = & f(x, y) + f_x(x, y)(u-x) + f_y(x, y)(v-y) + \frac{1}{2} \left\{ f_{xx}(x, y)(u-x)^2 \right. \\ & \left. + 2 f_{xy}(x, y)(u-x)(v-y) + f_y(x, y)(v-y)^2 \right\} \\ & + \phi_{(x,y)}(u, v) \sqrt{(u-x)^4 + (v-y)^4}. \end{aligned}$$

Since the operator H_n is linear, we obtain

$$\begin{aligned} n \{ H_n(f; x, y) - f(x, y) \} = & f_x(x, y) n (u_n^*(x) - x) + f_y(x, y) n (v_n^*(y) - y) \\ & + \frac{1}{2} \left\{ f_{xx}(x, y) n \left(2x^2 - 2x u_n^*(x) \right) \right. \\ & + 2 f_{xy}(x, y) n (x - u_n^*(x)) (y - v_n^*(y)) \\ & \left. + f_{yy}(x, y) n \left(2y^2 - 2y v_n^*(y) \right) \right\} \\ & + n H_n \left(\phi_{(x,y)}(u, v) \sqrt{(u-x)^4 + (v-y)^4}; x, y \right). \quad (13) \end{aligned}$$

Applying the Cauchy-Schwarz inequality for the last term on the right-hand side of

(13), we get

$$\left|nH_n\left(\phi_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y\right)\right|$$
$$\leq\left(H_n\left(\phi_{(x,y)}^2(u,v);x,y\right)\right)^{1/2}\left(H_n\left((u-x)^4+(v-y)^4;x,y\right)\right)^{1/2}$$
$$=\left(H_n\left(\phi_{(x,y)}^2(u,v);x,y\right)\right)^{1/2}\left(H_n\left((u-x)^4;x,y\right)\right.$$
$$\left.+H_n\left((v-y)^4\right);x,y\right)^{1/2}. \tag{14}$$

Let $\eta_{(x,y)}(u,v)=\phi_{(x,y)}^2(u,v)$. In this case, observe that $\eta_{(x,y)}(x,y)=0$ and $\eta_{(x,y)}(.,.)\in C(D)\cap E_2$. From Theorem 1 for $A=I$, which is the identity matrix,

$$\lim_n H_n\left(\phi_{(x,y)}^2(u,v);x,y\right)=\lim_n H_n\left(\eta_{(x,y)}(u,v);x,y\right)$$
$$=\eta_{(x,y)}(x,y)=0, \tag{15}$$

uniformly on E. Using (15) and Lemma 1, from (14) we obtain

$$\lim_n nH_n\left(\phi_{(x,y)}(u,v)\sqrt{(u-x)^4+(v-y)^4};x,y\right)=0, \tag{16}$$

uniformly on E. Also, observe that by (4)

$$\lim_n n\left(u_n^*(x)-x\right)=-\frac{1}{2},$$
$$\lim_n n\left(v_n^*(y)-y\right)=-\frac{1}{2},$$
$$\lim_n n\left(2x^2-2xu_n^*(x)\right)=x,$$
$$\lim_n n\left(2y^2-2yv_n^*(y)\right)=y.$$
$$\lim_n n\left(u_n^*(x)-x\right)\left(v_n^*(y)-y\right)=0. \tag{17}$$

Then, taking limit as $n\to\infty$ in (13) and using (16) and (17), we have

$$\lim_n n\left\{H_n(f;x,y)-f(x,y)\right\}=\frac{1}{2}\left\{xf_{xx}(x,y)+yf_{yy}(x,y)\right.$$
$$\left.-f_x(x,y)-f_y(x,y)\right\},$$

uniformly on E. □

Theorem 5. *For every $f\in C(D)\cap E_2$ such that f_x, $f_y\in C(D)\cap E_2$, we have*

$$\lim_n\frac{\partial}{\partial x}H_n(f;x,y)=\frac{\partial f}{\partial x}(x,y),\ x\neq 0,\textit{uniformly on } E, \tag{18}$$

$$\lim_n\frac{\partial}{\partial y}H_n(f;x,y)=\frac{\partial f}{\partial y}(x,y),\ y\neq 0,\ \textit{uniformly on } E. \tag{19}$$

Proof. We shall prove only (18) because the proof of (19) is identical. Let $(x,y)\in D$ and $f_x,\ f_y\in C(D)\cap E_2$. From (2) with $\{u_n(x)\}$ and $\{v_n(y)\}$ replaced by $\{u_n^*(x)\}$ and $\{v_n^*(y)\}$, where $u_n^*(x)$ and $v_n^*(y)$ are defined by (4), we obtain

$$\begin{aligned}\frac{\partial}{\partial x}H_n(f;x,y) &= -\frac{2n^2x}{\sqrt{1+4n^2x^2}}e^{-nu_n^*(x)}e^{-nv_n^*(y)}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}f\left(\frac{s}{n},\frac{t}{n}\right)\\ &\quad\times\frac{(nu_n^*(x))^s}{s!}\frac{(nv_n^*(y))^t}{t!}+\frac{4n^3x}{1+4n^2x^2-\sqrt{1+4n^2x^2}}e^{-nu_n^*(x)}\\ &\quad\times e^{-nv_n^*(y)}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\frac{s}{n}f\left(\frac{s}{n},\frac{t}{n}\right)\frac{(nu_n^*(x))^s}{s!}\frac{(nv_n^*(y))^t}{t!}\\ &= -\frac{2n^2x}{\sqrt{1+4n^2x^2}}H_n(f(u,v);x,y)+\frac{4n^3x}{1+4n^2x^2-\sqrt{1+4n^2x^2}}\\ &\quad\times H_n(uf(u,v);x,y). \qquad (20)\end{aligned}$$

Define the function η by

$$\eta_{(x,y)}(u,v)=\begin{cases}\frac{f(u,v)-f(x,y)-f_x(x,y)(u-x)-f_y(x,y)(v-y)}{\sqrt{(u-x)^2+(v-y)^2}} &, (u,v)\neq(x,y),\\ 0 &, (u,v)=(x,y).\end{cases}$$

Then by assumption we get $\eta_{(x,y)}(x,y)=0$ and $\eta_{(x,y)}(.,.)\in C(D)\cap E_2$. By the Taylor formula for $f\in C(D)\cap E_2$, we have

$$\begin{aligned}f(u,v) &= f(x,y)+f_x(x,y)(u-x)+f_y(x,y)(v-y)\\ &\quad+\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2}.\end{aligned}$$

Since the operator H_n is linear, we obtain

$$\begin{aligned}\frac{\partial}{\partial x}H_n(f;x,y) &= f_x(x,y)(x-u_n^*(x))\frac{2n^2x+n\sqrt{1+4n^2x^2}+n}{\sqrt{1+4n^2x^2}}\\ &\quad-\frac{2n^2x}{\sqrt{1+4n^2x^2}}H_n\left(\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right)\\ &\quad+\frac{4n^3x}{1+4n^2x^2-\sqrt{1+4n^2x^2}}\\ &\quad\times H_n\left(u\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right)\\ &= f_x(x,y)(x-u_n^*(x))\frac{2n^2x+n\sqrt{1+4n^2x^2}+n}{\sqrt{1+4n^2x^2}}\\ &\quad+\frac{4n^3x}{1+4n^2x^2-\sqrt{1+4n^2x^2}} \qquad (21)\\ &\quad\times H_n\left((u-u_n^*(x))\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right).\end{aligned}$$

By the Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
& n\left|H_n\left(\left(u-u_n^*(x)\right)\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right)\right| \\
& \leq \left(H_n\left(\eta^2_{(x,y)}(u,v);x,y\right)\right)^{1/2}\cdot\left(n^2H_n\left(\left(u-u_n^*(x)\right)^2(u-x)^2\right.\right. \\
& \quad \left.\left.+\left(u-u_n^*(x)\right)^2(v-y)^2;x,y\right)\right)^{1/2} \\
& = \left(H_n\left(\eta^2_{(x,y)}(u,v);x,y\right)\right)^{1/2}\cdot\left\{n^2H_n\left(\left(u-u_n^*(x)\right)^2(u-x)^2;x,y\right)\right. \\
& \quad \left.+H_n\left(\left(u-u_n^*(x)\right)^2(v-y)^2;x,y\right)\right\}^{1/2}. \qquad (22)
\end{aligned}
$$

Let $\phi_{(x,y)}(u,v) = \eta^2_{(x,y)}(u,v)$. In this case, observe that $\phi_{(x,y)}(x,y) = 0$ and $\phi_{(x,y)}(.,.) \in C(D)\cap E_2$. From Theorem 1, we have

$$
\begin{aligned}
\lim_n H_n\left(\eta^2_{(x,y)}(u,v);x,y\right) &= \lim_n H_n\left(\phi_{(x,y)}(u,v)\right) \\
&= \phi_{(x,y)}(x,y) = 0, \qquad (23)
\end{aligned}
$$

uniformly on E. We also obtain

$$
\begin{aligned}
&\lim_n n^2H_n\left(\left(u-u_n^*(x)\right)^2(v-y)^2;x,y\right) = xy, \\
&\lim_n n^2H_n\left(\left(u-u_n^*(x)\right)^2(u-x)^2;x,y\right) = 4x^4-2x^3-2x^2. \qquad (24)
\end{aligned}
$$

Using (23) and (24), from (22) we obtain

$$
\lim_n n\left|H_n\left(\left(u-u_n^*(x)\right)\eta_{(x,y)}(u,v)\sqrt{(u-x)^2+(v-y)^2};x,y\right)\right| = 0, \qquad (25)
$$

uniformly on E. Since

$$
\lim_n\left(x-u_n^*(x)\right)\frac{2n^2x+n\sqrt{1+4n^2x^2}+n}{\sqrt{1+4n^2x^2}} = 1,
$$

considering (25) in (22), we have

$$
\lim_n \frac{\partial}{\partial x}H_n(f;x,y) = \frac{\partial f}{\partial x}(x,y),\ x\neq 0,
$$

uniformly on E. So the proof is completed. □

5. A-statistical convergence

Gadjiev and Orhan [11] have investigated the Korovkin-type approximation theory via statistical convergence. In this section, using the concept of A-statistical convergence, we give the Korovkin-type approximation theorem for H_n operators given by (2).

Now, we first recall the concept of A-statistical convergence.

Let $A=(a_{nk})$ be an infinite summability matrix. For a given sequence $x:=(x_k)$, the A-transform of x, denoted by $Ax:=((Ax)_n)$, is given by

$$(Ax)_n=\sum_{k=1}^{\infty}a_{nk}x_k,$$

provided the series converges for each $n\in\mathbb{N}$. We say that A is regular if $\lim_n (Ax)_n = L$ whenever $\lim_n x_n = L$ [12]. Assume that A is a non-negative regular summability matrix. Then $x=(x_n)$ is said to be A-statistically convergent to L if, for every $\varepsilon>0$, $\lim\limits_n \sum\limits_{k\in\mathbb{N}:|x_k-L|\geq\varepsilon} a_{nk}x_k=0$, which is denoted by $st_A-\lim\limits_n x_n=L$ [9] (see also [15]). We note that by taking $A=C_1$, the Cesáro matrix, A-statistical convergence reduces to the concept of statistical convergence (see [7, 10, 16] for details). If A is the identity matrix, then A-statistical convergence coincides with the ordinary convergence. It is not hard to see that every convergent sequence is A-statistically convergent.

For example, for $A=C_1$, the Cesáro matrix and the sequence $x=(x_n)$ defined as

$$x_n=\begin{cases}1, \text{ if } n \text{ is square,}\\ 0, \quad \text{otherwise,}\end{cases}$$

it is easy to see that $st_{C_1}-\lim\limits_n x_n=0$.

The Korovkin-type approximation theorem is given by Theorem 1 as follows:

Theorem 6. *Let $A=(a_{nk})$ be a non-negative regular summability matrix. Let H_n denote the sequence of positive linear operators given by (2). If*

$$st_A-\lim_n u_n(x)=x,\ st_A-\lim_n v_n(y)=y, \text{ uniformly on } E,$$

then, for all $f\in C(D)\cap E_2$,

$$st_A-\lim_n H_n(f;x,y)=f(x,y)\,, \text{ uniformly on } E.$$

Now, we choose a subset K of $\mathbb{N}$ such that $\delta_A(K)=1$. Define the function sequence $\{p_n^*\}$ and $\{q_n^*\}$ by

$$p_n^*(x)=\begin{cases}0, & n\notin K\\ u_n^*(x)\,, & n\in K\end{cases},\quad q_n^*(y)=\begin{cases}0, & n\notin K\\ v_n^*(y)\,, & n\in K\end{cases} \tag{26}$$

where $u_n^*(x)$ and $v_n^*(y)$ is given by (4).

It is clear that p_n^* and q_n^* are continuous and exponential-type on $[0,\infty)$ and

$$st_A-\lim_n u_n^*(x)=x,\ st_A-\lim_n v_n^*(y)=y \tag{27}$$

uniformly on E.

We turn to $\{H_n\}$ given by (2) with $\{u_n(x)\}$ and $\{v_n(y)\}$ replaced by $\{p_n^*(x)\}$ and $\{q_n^*(y)\}$, where $p_n^*(x)$ and $q_n^*(y)$ are defined by (26). Show that $\{H_n\}$ are positive linear operators and

$$\begin{aligned} H_n(f_1;x,y) &= p_n^*(x) \\ H_n(f_2;x,y) &= q_n^*(x) \end{aligned} \tag{28}$$

and

$$H_n(f_3;x,y) = \begin{cases} f_3(x,y), & n \in K, \\ 0, & \text{otherwise}, \end{cases} \tag{29}$$

where K is any subset of $\mathbb{N}$ such that $\delta_A(K) = 1$.

Since $\delta_A(K) = 1$, it is clear that

$$st_A - \lim_n H_n(f_3;x,y) = f_3(x,y), \tag{30}$$

uniformly on E.

Relations (3), (27), (28) and (29) and Theorem 1 yield the following:

Theorem 7. *Let $A = (a_{nk})$ be a non-negative regular summability matrix. $\{H_n\}$ denotes the sequence of positive linear operators given by (2) with $\{u_n(x)\}$ and $\{v_n(y)\}$ replaced by $\{p_n^*(x)\}$ and $\{q_n^*(y)\}$, where $p_n^*(x)$ and $q_n^*(y)$ are defined by (26). Then*

$$st_A - \lim_n H_n(f;x,y) = f(x,y),$$

uniformly on E.

We note that $\{H_n\}$ is the sequence of positive linear operators given by (2) with $\{u_n(x)\}$ and $\{v_n(y)\}$ replaced by $\{p_n^*(x)\}$ and $\{q_n^*(y)\}$, where $p_n^*(x)$ and $q_n^*(y)$ are defined by (26) which does not satisfy the condition of the Theorem 2.

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