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Common fixed point for generalized set valued contractions satisfying an implicit relation in partially ordered metric spaces

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Abstract. Let $(X, d, \preceq)$ be a partially ordered metric space. Let F, G be two set valued mappings on X. We obtained sufficient conditions for the existence of a common fixed point of F, G satisfying an implicit relation in X.

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1. Introduction and preliminaries

Let (X, d) be a metric space and $B(X)$ be the class of all nonempty bounded subsets of X. For $A, B \in B(X)$, let

$$\delta(A, B) := \sup\{d(a, b) : \ a \in A, \ b \in B\},$$

and

$$dist(a, B) := \inf\{d(a, b) : \ a \in A, \ b \in B\}.$$

If $A = \{a\}$, then we write $\delta(A, B) = \delta(a, B)$. Also in addition, if $B = \{b\}$, then $\delta(A, B) = d(a, b)$. Note that

$$\begin{aligned} \delta(A, B) &= 0 \text{ if and only if } A = B = \{x\}, \\ dist(A, B) &\leq \delta(A, B), \\ \delta(A, B) &= \delta(B, A), \\ \delta(A, A) &= diamA. \end{aligned}$$

Let $F : X \to X$ be a set valued mapping, i.e., $X \ni x \mapsto Fx$ is a subset of X. A point $x \in X$ is said to be a fixed point of the set valued mapping F if $x \in Fx$.

Definition 1. *A partially ordered set consists of a set X and a binary relation $\preceq$ on X which satisfies the following conditions:*

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1. $x \preceq x$ *(reflexivity),*

2. *if* $x \preceq y$ *and* $y \preceq x$*, then* $x = y$ *(antisymmetry),*

3. *if* $x \preceq y$ *and* $y \preceq z$*, then* $x \preceq z$ *(transitivity),*

for all x*,* y *and* z *in* X*. A set with a partial order* $\preceq$ *is called a partially ordered set. Let* $(X, \preceq)$ *be a partially ordered set and* $x, y \in X$*. Elements* x *and* y *are said to be comparable elements of* X *if either* $x \preceq y$ *or* $x \preceq y$*.*

Definition 2. *Let* A *and* B *be two nonempty subsets of* $(X, \preceq)$*, the relations between* A *and* B *are denoted and defined as follows:*

1. $A \prec_1 B$*: if for every* $a \in A$ *there exists* $b \in B$ *such that* $a \preceq b$*,*

2. $A \prec_2 B$*: if for every* $b \in B$ *there exists* $a \in A$ *such that* $a \preceq b$*,*

3. $A \prec_3 B$*: if* $A \prec_1 B$ *and* $A \prec_2 B$*.*

Implicit relations in metric spaces have been considered by several authors in connection with solving nonlinear functional equations (see for instance [2, 3, 4, 23] and reference cited therein).

Let R_+ be the set of non negative real numbers and $\mathcal{T}$ are set of continuous real valued functions $T : R_+^5 \to R$ satisfying the following conditions:

$\mathcal{T}_1$: $T(t_1, t_2, ..., t_5)$ is non-decreasing in t_1 and non-increasing in $t_2, ..., t_5$,

$\mathcal{T}_2$: there exists $h \in (0, 1)$ such that

$$T(u, v, v, u, v + u) \leq 0,$$

or

$$T(u, v, u, v, u + v) \leq 0,$$

implies

$$u \leq hv,$$

$\mathcal{T}_3$: $T(u, 0, 0, u, u) > 0$ and $T(u, 0, u, 0, u) > 0$, for all $u > 0$.

Next we give some examples for such T.

Example 1. *Let* $T(t_1, ..., t_5) = t_1 - \alpha \ \max\{t_2, t_3, t_4\} - (1-\alpha)\beta \ t_5$*, where* $0 \leq \alpha < 1$*,* $0 \leq \beta < 1/2$*.*

$\mathcal{T}_1$*: It is obvious.*

$\mathcal{T}_2$*: Let* $u > 0$*,* $T(u, v, v, u, u + v) = u - \alpha \max\{u, v\} - (1 - \alpha)\beta(u + v) \leq 0$ *or* $u \leq \alpha \max\{u, v\} + (1 - \alpha)\beta(u + v)$*. If* $u \geq v$*, then* $u \leq \alpha u + (1 - \alpha)\beta(u + v)$ *and so* $(1 - \beta)u \leq \beta u$*, it implies that* $\beta \geq 1/2$*, a contradiction. Thus* $u < v$ *and* $u \leq \frac{\alpha+(1-\alpha)\beta}{1-(1-\alpha)\beta}v$*. Similarly, let* $u > 0$ *and* $T(u, v, u, v, u + v) \leq 0$*, then we have* $u \leq \frac{\alpha+(1-\alpha)\beta}{1-(1-\alpha)\beta}v$*. If* $u = 0$*, then* $u \leq \frac{\alpha+(1-\alpha)\beta}{1-(1-\alpha)\beta}v$*. Thus* $\mathcal{T}_2$ *is satisfied with* $h = \frac{\alpha+(1-\alpha)\beta}{1-(1-\alpha)\beta}v < 1$ *.*

$\mathcal{T}_3$: $T(u,0,0,u,u) = T(u,0,u,0,u) = (1-\alpha)(1-\beta)u > 0$, *for all* $u > 0$.

Therefore $T \in \mathcal{T}$.

Example 2. *Let* $T(t_1, ..., t_5) = t_1 - \alpha\ t_2 - \beta\ \max\{t_3, t_4\} - \gamma\ t_5$, *where* $\alpha,\ \beta,\ \gamma \geq 0$ *and* $\alpha + 2\beta + \gamma < 1$.

$\mathcal{T}_1$: *It is obvious.*

$\mathcal{T}_2$: *Let* $u > 0$, $T(u,v,v,u,u+v) = u - \alpha v - \beta \max\{u,v\} - \gamma(u+v) \leq 0$ *or* $u \leq \alpha v + \beta \max\{u,v\} + \gamma\ (u+v)$. *Thus* $u \leq \max\{(\alpha+\beta+\gamma)u + \beta v, (\alpha+\beta+\gamma)v + \beta u\}$. *If* $u \geq v$, *then* $u \leq (\alpha+\beta+\gamma)u + \beta v$, *it implies that* $\alpha + 2\beta + \gamma \geq 1$, *a contradiction. Thus* $u < v$ *and* $u \leq \frac{\alpha+\beta+\gamma}{1-\beta}v$. *Similarly, let* $u > 0$ *and* $T(u,v,u,v,u+v) \leq 0$, *then we have* $u \leq \frac{\alpha+\beta+\gamma}{1-\beta}v$. *If* $u = 0$, *then* $u \leq \frac{\alpha+\beta+\gamma}{1-\beta}v$. *Thus* $\mathcal{T}_2$ *is satisfied for* $h = \frac{\alpha+\beta+\gamma}{1-\beta} < 1$.

$\mathcal{T}_3$: $T(u,0,0,u,u) = T(u,0,u,0,u) = (1-\beta-\gamma)u > 0$, *for all* $u > 0$.

Therefore $T \in \mathcal{T}$.

Example 3. *Let* $T(t_1, ..., t_5) = t_1 - \alpha \max\{t_2, t_3, t_4, t_5/2\}$, *where* $0 \leq \alpha < 1$.

$\mathcal{T}_1$: *It is obvious.*

$\mathcal{T}_2$: *Let* $u > 0$, $T(u,v,v,u,u+v) = u - \alpha \max\{u,v\} \leq 0$ *or* $u \leq \alpha \max\{u,v\}$. *If* $u \geq v$, *then* $u \leq \alpha u$, *it implies that* $\alpha \geq 1$ *a contradiction. Thus* $u < v$ *and* $u \leq \alpha v$. *Similarly, let* $u > 0$ *and* $T(u,v,u,v,u+v) \leq 0$, *then we have* $u \leq \alpha v$. *If* $u = 0$, *then* $u \leq \alpha v$. *Thus* $\mathcal{T}_2$ *is satisfied with* $h = \alpha < 1$.

$\mathcal{T}_3$: $T(u,0,0,u,u) = T(u,0,u,0,u) = (1-\alpha)u > 0$, *for all* $u > 0$.

Therefore $T \in \mathcal{T}$.

Example 4. *Let* $T(t_1, ..., t_5) = t_1 - \alpha\ \max\{t_2, t_3, t_4\} - \beta t_5$, *where* $\alpha, \beta > 0$ *and* $\alpha + 2\beta < 1$.

$\mathcal{T}_1$: *It is obvious.*

$\mathcal{T}_2$: *Let* $u > 0$, $T(u,v,v,u,u+v) = u - \alpha\ \max\{u,v\} - \beta(u+v) \leq 0$ *or* $u \leq \alpha \cdot \max\{u,v\} + \beta(u+v)$. *Thus* $u \leq \max\{(\alpha+\beta)u + \beta v, (\alpha+\beta)v + \beta u\}$. *If* $u \geq v$, *then* $u \leq (\alpha+\beta)u + \beta v \leq (\alpha + 2\beta)u$, *it implies that* $\alpha + 2\beta \geq 1$, *a contradiction. Thus* $u < v$ *and* $u \leq (\alpha+\beta)v + \beta u$ *and so* $u \leq \frac{\alpha+\beta}{1-\beta}v$. *Similarly, let* $u > 0$ *and* $T(u,v,u,v,u+v) \leq 0$, *then we have* $u \leq \frac{\alpha+\beta}{1-\beta}v$. *If* $u = 0$, *then* $u \leq \frac{\alpha+\beta}{1-\beta}v$. *Thus* $\mathcal{T}_2$ *is satisfied with* $h = \frac{\alpha+\beta}{1-\beta} < 1$.

$\mathcal{T}_3$: $T(u,0,0,u,u) = T(u,0,u,0,u) = (1-\alpha-\beta)u > 0$, *for all* $u > 0$.

Therefore $T \in \mathcal{T}$.

The existence of a fixed point in partially ordered metric spaces has been recently considered in [18, 19, 16, 15, 17, 22, 20, 8, 21, 1, 7, 10, 12, 14, 5, 6]. It is of interest to determine the existence of a fixed point in such a setting. The first result in this direction was given by Ran and Reurings in [22] where they extended the Banach contraction principle [13], in partially ordered sets with some application to linear and nonlinear matrix equations. Ran and Reurings [22] proved the following seminal result:

Theorem 1 (see [22])**.** *Let $(X, \preceq)$ be a partially ordered set such that for every pair $x, y \in X$ has an upper and lower bound. Let d be a metric on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous monotone (either order preserving or order reversing) mapping. Suppose that the following conditions hold:*

1. *there exists $\kappa \in (0, 1)$ with*
$$d(fx, fy) \leq \kappa d(x, y) \text{ for all } x \preceq y,$$
2. *there exists $x_0 \in X$ with $x_0 \preceq fx_0$ or $fx_0 \preceq x_0$.*

Then f is a Picard Operator (PO), that is, f has a unique fixed point $x^* \in X$ and for each $x \in X$,
$$\lim_{n\to\infty} f^n x = x^*.$$

Theorem 1 was further extended and refined in [18, 19, 16, 20, 21]. These results are hybrid of the two fundamental and classical theorems; Banach's fixed point theorem [13] and Tarski's fixed point theorem [24, 9, 11]. Motivated and inspired by [22], our aim in this paper is to give some new common fixed point results for set valued mappings satisfying an implicit relation in partially ordered metric spaces.

2. Main results

Let $(X, \preceq)$ be a partially ordered set and d a metric on X such that (X, d) is a complete metric space.

We begin this section with the following theorem that gives the existence of a fixed point (not necessarily unique) in a partially ordered metric space X for the set valued mappings satisfying an implicit relation.

Theorem 2. *Let $F, G : X \to B(X)$ be such that the following conditions are satisfied:*

1. *there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Fx_0$,*
2. *if $x, y \in X$ is such that $x \preceq y$, then $Gy \prec_3 Fx$,*
3. *if $x_n \to x$ is any sequence in X whose consecutive terms are comparable then $x_n \preceq x$, for all n,*
4. *$T(\delta(Fx, Gy), d(x, y), dist(x, Fx), dist(y, Gy),\ dist(x, Gy) + dist(y, Fx)) \leq 0$, for all distinct comparable elements x, y of X and for some $T \in \mathcal{T}$.*

Then there exists $x \in X$ *with* $\{x\} = Fx = Gx$.

Proof. Let $x_0 \in X$, then from assumption 1, there exists $x_1 \in Fx_0$ such that $x_0 \preceq x_1$. Now by using assumptions 2, $Gx_1 \prec_3 Fx_0$ which implies $Gx_1 \prec_2 Fx_0$. From this we get the existence of $x_2 \in Gx_1$ such that $x_2 \preceq x_1$. Since $x_0 \preceq x_1$, therefore by using assumption 4, we have

$$T\,(\,\delta(Fx_0, Gx_1), d(x_0, x_1), dist(x_0, Fx_0), dist(x_1, Gx_1), dist(x_0, Gx_1) \\ +dist(x_1, Fx_0)) \leq 0.$$

Using the facts

$$\begin{aligned} d(x_1, x_2) &\leq \delta(Fx_0, Gx_1), \\ dist(x_0, Fx_0) &\leq d(x_0, x_1), \\ dist(x_1, Gx_1) &\leq d(x_1, x_2), \\ dist(x_0, Gx_1) + dist(x_1, Fx_0) &\leq d(x_0, x_2) \end{aligned}$$

and by $\mathcal{T}_1$ we have

$$T(d(x_1, x_2), d(x_0, x_1), d(x_0, x_1), d(x_1, x_2), d(x_0, x_1) + d(x_1, x_2)) \leq 0,$$

that is

$$T(u, v, v, u, u + v) \leq 0,$$

where $u = d(x_1, x_2)$, $v = d(x_0, x_1)$.
Next, by $\mathcal{T}_2$ there exists $h \in (0, 1)$ such that

$$d(x_1, x_2) \leq hd(x_0, x_1). \tag{1}$$

Again since $x_2 \preceq x_1$, therefore by assumption 2, $Gx_1 \prec_3 Fx_2$ which implies $Gx_1 \prec_1 Fx_2$. From this we get the existence of $x_3 \in Fx_2$ such that $x_2 \preceq x_3$. Now by assumption 4,

$$T\,(\,\delta(Fx_2, Gx_1), d(x_2, x_1), dist(x_2, Fx_2), dist(x_1, Gx_1), dist(x_2, Gx_1) \\ +dist(x_1, Fx_2)) \leq 0.$$

By $\mathcal{T}_1$ we have

$$T(d(x_3, x_2), d(x_1, x_2), d(x_2, x_3), d(x_1, x_2), d(x_1, x_2) + d(x_2, x_3)) \leq 0,$$

that is,

$$T(u, v, u, v, u + v) \leq 0,$$

where $u = d(x_2, x_3)$, $v = d(x_1, x_2)$. Now by $\mathcal{T}_2$ there exists $h \in (0, 1)$ such that

$$d(x_2, x_3) \leq hd(x_1, x_2), \tag{2}$$

and from (1) and (2), we get

$$d(x_2, x_3) \leq h^2 d(x_0, x_1). \tag{3}$$

Continuing in this manner we can define a sequence $\{x_n\}$ whose consecutive terms are comparable such that $x_{2n+1} \in Fx_{2n}$ and $x_{2n+2} \in Gx_{2n+1}$, for $n = 0, 1, 2....$ Again by assumption 4,

$$T\,(\,\delta(Fx_{2n}, Gx_{2n+1}), d(x_{2n}, x_{2n+1}), dist(x_{2n}, Fx_{2n}), dist(x_{2n+1}, Gx_{2n+1}),$$
$$dist(x_{2n}, Gx_{2n+1}) + dist(x_{2n+1}, Fx_{2n})) \leq 0,$$

which implies that

$$T\,(\,d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}),$$
$$d(x_{2n}, x_{2n+1}) + dt(x_{2n+1}, x_{2n+2})) \leq 0,$$

that is,

$$T(u, v, v, u, u + v) \leq 0,$$

where $u = d(x_{2n+1}, x_{2n+2})$, $v = d(x_{2n}, x_{2n+1})$. Next, by $\mathcal{T}_2$ there exists $h \in (0, 1)$ such that

$$d(x_{2n+1}, x_{2n+2}) \leq hd(x_{2n}, x_{2n+1}). \tag{4}$$

Therefore, we have

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) \leq h^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq h^n d(x_0, x_1)$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence in X. Let $m > n$. Then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \\ &\leq [h^n + h^{n+1} + h^{n+2} + \cdots + h^{m-1}]d(x_0, x_1) \\ &= h^n[1 + h + h^2 \cdots + h^{m-n-1}]d(x_0, x_1) \\ &= h^n \, \frac{1 - h^{m-n}}{1 - h} \, d(x_0, x_1) \\ &< \frac{h^n}{1 - h} \, d(x_0, x_1), \end{aligned}$$

because $h \in (0, 1), 1 - h^{m-n} < 1$. Therefore $d(x_n, x_m) \to 0$ as $n \to \infty$ implies that $\{x_n\}$ is a Cauchy sequence and thus there exists some point (say) x in the complete metric space X such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = x \in \lim_{n\to\infty} Fx_{2n},$$
$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+2} = x \in \lim_{n\to\infty} Gx_{2n+1},$$

and by assumption 2, $x_n \preceq x$ for all n. Next, by assumption 4,

$$T\,(\,\delta(Fx_{2n}, Gx), d(x_{2n}, x), dist(x_{2n}, Fx_{2n}), dist(x, Gx),$$
$$dist(x_{2n}, Gx) + dist(x, Fx_{2n})) \leq 0,$$

which gives

$$T\,(\,\delta(x_{2n+1}, Gx), d(x_{2n}, x), d(x_{2n}, x_{2n+1}), dist(x, Gx),$$
$$dist(x, Gx) + d(x, x_{2n+1})) \leq 0.$$

Letting $n \to \infty$ and using $\mathcal{T}_1$ we get

$$T(\delta(x, Gx), 0, 0, \delta(x, Gx), \delta(x, Gx)) \leq 0,$$

that is

$$T(u, 0, 0, u, u) \leq 0,$$

and from $\mathcal{T}_3$ we have $u = \delta(x, Gx) = 0$, which gives $Gx = \{x\}$. Similarly,

$$T\,(\,\delta(Fx, Gx_{2n+1}), d(x, x_{2n+1}), dist(x, Fx), dist(x_{2n+1}, Gx_{2n+1}),$$
$$dist(x, Gx_{2n+1}) + dist(x_{2n+1}, Fx)) \leq 0,$$

which implies

$$T\,(\,\delta(Fx, x_{2n+2}), d(x, x_{2n+1}), dist(x, Fx), d(x_{2n+1}, x_{2n+2}),$$
$$d(x, x_{2n+2}) + dist(x_{2n+1}, Fx)) \leq 0.$$

Letting $n \to \infty$ and using $\mathcal{T}_1$ we get

$$T(\delta(Fx, x), 0, \delta(Fx, x), 0, \delta(Fx, x)) \leq 0,$$

that is,

$$T(u, 0, u, 0, u) \leq 0,$$

and from $\mathcal{T}_3$ we have $u = \delta(Fx, x) = 0$, which gives $Fx = \{x\}$. Hence $x = F(x) = G(x)$. □

Theorem 3. *Let $F, G : X \to B(X)$ be such that the following conditions are satisfied:*

1. *there exists $x_0 \in X$ such that $Fx_0 \prec_2 \{x_0\}$,*
2. *if $x, y \in X$ is such that $x \preceq y$, then $Fy \prec_3 Gx$,*
3. *if $x_n \to x$ is any sequence in X whose consecutive terms are comparable, then $x_n \preceq x$, for all n,*
4. *$T(\delta(Fx, Gy), d(x, y), dist(x, Fx), dist(y, Gy), dist(x, Gy) + dist(y, Fx)) \leq 0$, for all distinct comparable elements x, y of X and for some $T \in \mathcal{T}$.*

Then there exists $x \in X$ with $\{x\} = Fx = Gx$.

Proof. The proof follows along the similar lines as in Theorem 2. □

Next, we have the following analogue of Theorem 2 and Theorem 3.

Theorem 4. *Let $F, G : X \to B(X)$ be such that the following conditions are satisfied:*

1. *there exists $x_0 \in X$ such that $\{x_0\} \prec_1 Gx_0$,*
2. *if $x, y \in X$ is such that $x \preceq y$, then $Fy \prec_3 Gx$,*

3. *if* $x_n \to x$ *is any sequence in* X *whose consecutive terms are comparable, then* $x_n \preceq x$*, for all* n*,*

4. $T(\delta(Fx, Gy), d(x, y), dist(x, Fx), dist(y, Gy), dist(x, Gy) + dist(y, Fx)) \leq 0$*, for all distinct comparable elements* x, y *of* X *and for some* $T \in \mathcal{T}$*.*

Then there exists $x \in X$ *with* $\{x\} = Fx = Gx$*.*

Theorem 5. *Let* $F, G : X \to B(X)$ *be such that the following conditions are satisfied:*

1. *there exists* $x_0 \in X$ *such that* $Gx_0 \prec_2 \{x_0\}$*,*

2. *if* $x, y \in X$ *is such that* $x \preceq y$*, then* $Gy \prec_3 Fx$*,*

3. *if* $x_n \to x$ *is any sequence in* X *whose consecutive terms are comparable, then* $x_n \preceq x$*, for all* n*,*

4. $T(\delta(Fx, Gy), d(x, y), dist(x, Fx), dist(y, Gy), dist(x, Gy) + dist(y, Fx)) \leq 0$*, for all distinct comparable elements* x, y *of* X *and for some* $T \in \mathcal{T}$*.*

Then there exists $x \in X$ *with* $\{x\} = Fx = Gx$*.*

By taking $F = G$ in Theorems 2 - 5 we obtain the following consequence:

Corollary 1. *Let* $F : X \to B(X)$ *be such that the following conditions are satisfied:*

1. *there exists* $x_0 \in X$ *such that* $\{x_0\} \prec_1 Fx_0$ *or* $Fx_0 \prec_2 \{x_0\}$*,*

2. *if* $x, y \in X$ *is such, that* $x \preceq y$ *then* $Fy \prec_3 Fx$*,*

3. *if* $x_n \to x$ *is any sequence in* X *whose consecutive terms are comparable, then* $x_n \preceq x$*, for all* n*,*

4. $T(\delta(Fx, Fy), d(x, y), dist(x, Fx), dist(y, Fy), dist(x, Fy) + dist(y, Fx)) \leq 0$*, for all distinct comparable elements* x, y *of* X *and for some* $T \in \mathcal{T}$*.*

Then there exists $x \in X$ *with* $\{x\} = Fx$*.*

Example 5. *Let* $X = \{(0, 0), (0, -1/2), (-1/8, 1/8)\}$ *be a subset of* R^2 *with a usual order defined as: for* (u, v)*,* $(x, y) \in X$*,* $(u, v) \leq (x, y)$ *if and only if* $u \leq x, y \leq v$*. Let* d *be a metric on* X *defined as:*

$$d(\underline{x}, \underline{y}) = d((x_1, y_1), (x_2, y_2)) := \max\{| x_1 - x_2 |, | y_1 - y_2 |\}, \text{ for all } \underline{x}, \underline{y} \in X,$$

so that (X, d) *is a complete metric space. Mapping* $F : X \to B(X)$ *is defined as:*

$$F(\underline{x}) = \begin{cases} \{(-\frac{1}{8}, \frac{1}{8})\}, & \text{if } \underline{x} = (-\frac{1}{8}, \frac{1}{8}) \\ \{(0, 0), (-\frac{1}{8}, \frac{1}{8})\}, & \text{if } \underline{x} \in \{(0, 0), (0, -\frac{1}{2})\} \end{cases}.$$

For $(0, -\frac{1}{2}) \leq (0, 0)$*;*

$$\delta(F(0, -\frac{1}{2}), F(0, 0)) = \frac{1}{8} \leq \frac{1}{3}.\frac{1}{2} = \frac{1}{3} d((0, -1/2), (0, 0)).$$

Thus for all $\underline{x} \leq \underline{y}$ *we have*

$$\delta(F\underline{x}, F\underline{y}) \leq \frac{1}{3}\, d(\underline{x}, \underline{y}) \leq \frac{1}{3} \max\{\, d(x,y), dist(x,Fx), dist(y,Fy), \frac{dist(x,Fy)+dist(y,Fx)}{2}\}.$$

Also note that $(0,0) \in X$ *is such that* $\{(0,0)\} \prec_1 F(0,0)$ *and for all* $\underline{x} \preceq \underline{y}$*, then* $F\underline{y} \prec F\underline{x}$*. Consequently, all conditions of Corollary 1 are satisfied and* $\{(-\frac{1}{8}, \frac{1}{8})\} = F(-\frac{1}{8}, \frac{1}{8})$*.*

Remark 1. *The conclusion of Corollary 1 still holds if we replace assumption 2 by:*

- *if* $x, y \in X$ *is such that* $x \preceq y$*, then* $Fy \prec_3 Fx$ *or* $Fx \prec_3 Fy$*.*

Corollary 1 is further extended as:

Corollary 2. *Let* $F : X \to B(X)$ *be such that the following conditions are satisfied:*

1. *there exists* $x_0 \in X$ *such that* $\{x_0\} \prec_1 Fx_0$*,*
2. *if* $x, y \in X$ *is such that* $x \preceq y$*, then* $Fx \prec_1 Fy$*,*
3. *if* $x_n \to x$ *is any sequence in* X *whose consecutive terms are comparable, then* $x_n \preceq x$*, for all* n*,*
4. $T(\delta(Fx,Fy), d(x,y), dist(x,Fx), dist(y,Fy), dist(x,Fy) + dist(y,Fx)) \leq 0$*, for all distinct comparable elements* x, y *of* X *and for some* $T \in \mathcal{T}$*.*

Then there exists $x \in X$ *with* $\{x\} = Fx$*.*

Proof. Let $x_0 \in X$, then from assumption 1 there exists $x_1 \in Fx_0$ such that $x_0 \preceq x_1$. Now by using assumptions 2, $Fx_0 \prec_1 Fx_1$. From this we get the existence of $x_2 \in Fx_1$ such that $x_1 \preceq x_2$. Since $x_0 \preceq x_1$, therefore by using assumption 4 we have

$$T\,(\,\delta(Fx_0, Fx_1), d(x_0,x_1), dist(x_0,Fx_0), dist(x_1,Fx_1), dist(x_0,Fx_1) + dist(x_1,Fx_0)) \leq 0.$$

Using $\mathcal{T}_1$ we have

$$T(d(x_1,x_2), d(x_0,x_1), d(x_0,x_1), d(x_1,x_2), d(x_0,x_1) + d(x_1,x_2)) \leq 0,$$

that is,

$$T(u, v, v, u, u+v) \leq 0,$$

where $u = d(x_1,x_2)$, $v = d(x_0,x_1)$. Next, by using $\mathcal{T}_2$, there exists $h \in (0,1)$ such that

$$d(x_1,x_2) \leq h d(x_0,x_1). \tag{5}$$

Again since $x_1 \preceq x_2$, therefore by assumption 2, $Fx_1 \prec_1 Fx_2$. From this we get the existence of $x_3 \in Fx_2$ such that $x_2 \preceq x_3$. By assumption 4,

$$T\,(\,\delta(Fx_1, Fx_2), d(x_1, x_2), dist(x_1, Fx_1), dist(x_2, Fx_2), \\ dist(x_1, Fx_2)) + dist(x_2, Fx_1) \leq 0.$$

By using $\mathcal{T}_1$ we have

$$T(d(x_2, x_3), d(x_1, x_2), d(x_1, x_2), d(x_2, x_3), d(x_1, x_2) + d(x_2, x_3)) \leq 0,$$

that is,

$$T(u, v, u, v, u + v) \leq 0,$$

where $u = d(x_2, x_3)$, $v = d(x_1, x_2)$. Next, by using $\mathcal{T}_2$ there exists $h \in (0, 1)$ such that

$$d(x_2, x_3) \leq hd(x_1, x_2), \tag{6}$$

and from (5) and (6) we get

$$d(x_2, x_3) \leq h^2 d(x_0, x_1). \tag{7}$$

Continuing in this manner we can define a non-decreasing sequence $\{x_n\}$ such that $x_{n+1} \in F(x_n)$, for $n = 0, 1, 2 \ldots$. By using induction, we have

$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) \leq h^2 d(x_{n-2}, x_{n-1}) \cdots \leq h^n d(x_0, x_1)$$

Next we will show that $\{x_n\}$ is a Cauchy sequence in X. Let $m > n$. Then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + \cdots + d(x_{m-1}, x_m) \\ &\leq [h^n + h^{n+1} + h^{n+2} + \cdots + h^{m-1}]d(x_0, x_1) \\ &= h^n[1 + h + h^2 + \cdots + h^{m-n-1}]d(x_0, x_1) \\ &= h^n \ \frac{1 - h^{m-n}}{1 - h} \ d(x_0, x_1) \\ &< \frac{h^n}{1 - h} \ d(x_0, x_1), \end{aligned}$$

because $h \in (0, 1), 1 - h^{m-n} < 1$. Therefore $d(x_n, x_m) \to 0$ as $n \to \infty$ implies that $\{x_n\}$ is a Cauchy sequence and thus there exists some point (say) x in the complete metric space X such that

$$\lim_{n \to \infty} x_n = x$$

and by assumption 3, $x_n \preceq x$ for all n. Next by assumption 4,

$$T(\delta(Fx_n, Fx), d(x_n, x), dist(x_n, Fx_n), dist(x, Fx), dist(x_n, Fx) + dist(x, Fx_n)) \leq 0,$$

which gives

$$T(\delta(x_{n+1}, Fx), d(x_n, x), d(x_n, x_{n+1}), dist(x, Fx), dist(x_n, Fx) + d(x, x_{n+1})) \leq 0.$$

Letting $n \to \infty$ and using $\mathcal{T}_1$ we get

$$T(\delta(x, Fx), 0, 0, \delta(x, Fx), \delta(x, Fx)) \leq 0,$$

that is,

$$T(u, 0, 0, u, u) \leq 0,$$

and from $\mathcal{T}_3$, we have $u = \delta(x, Fx) = 0$, which gives $Fx = \{x\}$. □

Remark 2. *If we replace assumption 4; for all comparable elements x, y of Corollary 2 by: for all $x \preceq y$ of a partially ordered set X, then the conclusion still holds.*

Corollary 3. *Let $F : X \to B(X)$ be such that the following conditions are satisfied:*

1. *there exists $x_0 \in X$ such that $Fx_0 \prec_2 \{x_0\}$,*
2. *if $x, y \in X$ is such that $x \preceq y$, then $Fx \prec_2 Fy$,*
3. *if $x_n \to x$ is any sequence in X whose consecutive terms are comparable, then $x_n \preceq x$, for all n,*
4. *$T(\delta(Fx, Fy), d(x, y), dist(x, Fx), dist(y, Fy), dist(x, Fy) + dist(y, Fx)) \leq 0$, for all distinct comparable elements x, y of X and for some $T \in \mathcal{T}$.*

Then there exists $x \in X$ with $\{x\} = Fx$.

Remark 3. *If we replace assumption 3 and 4 of Corollary 1, respectively, by:*

- *if $x_n \to x$ is any sequence in X whose consecutive terms are comparable, then $x \preceq x_n$, for all n.*
- *$T(\delta(Fx, Fy), d(x, y), dist(x, Fx), dist(y, Fy), dist(x, Fy) + dist(y, Fx)) \leq 0$, for all $y \preceq x$ and for some $T \in \mathcal{T}$.*

Then the conclusion still holds.

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