MATHEMATICAL COMMUNICATIONS 37
Math. Commun., Vol. **15**, No. 1, pp. 37-55 (2010)

Characterizations of α-well-posedness for parametric quasivariational inequalities defined by bifunctions

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Received January 31, 2009; accepted October 17, 2009

Abstract. The purpose of this paper is to investigate the well-posedness issue of parametric quasivariational inequalities defined by bifunctions. We generalize the concept of α-well-posedness to parametric quasivariational inequalities having a unique solution and derive some characterizations of α-well-posedness. The corresponding concepts of α-well-posedness in the generalized sense are also introduced and investigated for the problems having more than one solution. Finally, we give some sufficient conditions for α-well-posedness of parametric quasivariational inequalities.

AMS subject classifications: 49J40, 49K40

Key words: Parametric quasivariational inequalities, α-well-posedness, metric characterizations, bifunctions

1. Introduction

The well-posedness issues, born for minimization problems, have been attracting much attention of many researchers in the fields of economics and mathematics in the last decade. Tykhonov [28] first considered the well-posedness of a minimization problem, already known as Tykhonov well-posedness, which means the existence and uniqueness of minimizers. Since then various concepts of well-posedness were introduced and studied for minimization problems. For details we refer readers to [3, 4, 9, 14, 29, 30] and the references therein. In recent years, the concept of well-posedness was generalized to variational inequalities. Perhaps the main motivation lies in the fact that a minimization problem is closely related to a variational inequality. The first notion of well-posedness for a classical variational inequality was due to Lucchetti and Patrone [23]. Parametric variational inequalities are problems where a parameter is allowed to vary in a certain subset of a metric space. It has been shown that the parametric variational inequality is a central ingredient in the class of Mathematical Programs with Equilibrium Constraints which appear in many applied contexts and have been studied by many authors. See e.g. [11, 21, 24]. So it is interesting and necessary to study well-posedness of parametric variational inequalities.

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Lignola and Morgan [21] introduced the concepts of the parametric well-posedness for a family of variational inequalities and investigated its links with the extended well-posedness [30] of the corresponding minimization problems. The definition of well-posedness [21] for a classical variational inequality is inspired by the corresponding one for a minimization problem in terms of the gap function introduced by Auslender [1], known as Auslender gap function, which allows transformation of a variational inequality into an equivalent minimization problem. The Auslender gap function is applied to numerical methods for variational inequalities when it is differentiable. But, the Auslender gap function is not always differentiable in general. To overcome this, Fukushima [12] introduced another gap function, known as Fukushima gap function, which is continuously differentiable and widely used in numerical methods for variational inequalities. Motivated by the numerical method in [12], Lignola and Morgan [8] introduced the concept of α-well-posedness for the classical variational inequality and proved that a variational inequality is α-well-posed if and only if a corresponding minimization problem with the Fukushima gap function being an objective function is Tykhonov well-posed (see Prop. 2.1 of [8]). A quasivariational inequality is an extension of the classical variational inequality in which the defining set of the problem varies with a variable. The quasivariational inequality was first considered by Bensoussan and Lions [5] and has wide applications (see [2, 11]). Recently, Lignola [20] further considered well-posedness of quasivariational inequalities.

Meanwhile, some new contributions have been given to the theory of variational inequalities. In terms of Dini directional derivative, Crespi et al. [6, 7] introduced a class of generalized Minty variational inequalities, which includes the class of classical Minty variational inequalities as a special case, and investigated its links with nondifferentiable minimization problems. Lalitha and Mehta [19] introduced a class of variational inequalities defined by bifunctions and discussed the relationship between minimization problems and the variational inequalities by using generalized monotonicity of bifunctions. Motivated and inspired by the above works, Fang and Hu [10] studied well-posedness of Stampacchia and Minty variational inequalities defined by bifunctions. The concepts of parametric well-posedness were introduced for the variational inequalities defined by bifunctions and some metric characterizations of parametric well-posedness were derived. In this paper, we further generalize the concept of α-well-posedness to parametric quasivariational inequalities defined by bifunctions. We establish some characterizations of parametric α-well-posedness. Our results generalize the corresponding results of [8, 10, 20].

2. Preliminaries and notations

Throughout this paper, unless otherwise specified, we always suppose that $\alpha \geq 0$ is a fixed number, K is a nonempty subset of a real Banach space X, P is a parametric norm space, $S : P \times K \to 2^K$ is a set-valued map and $h : P \times K \times X \to \bar{R}$ is a function, where 2^K denotes the family of all subsets of K and $\bar{R} := R \cup \{+\infty, -\infty\}$. We consider the following parametric Stampacchia and Minty quasivariational inequalities defined by (h, S):

$$SQVI_p(h,S) \ : \ \text{find } x \in K \text{ such that } x \in S(p,x),\ h(p,x,x-y) \leq 0,\ \forall y \in S(p,x),$$
$$MQVI_p(h,S) : \ \text{find } x \in K \text{ such that } x \in S(p,x),\ h(p,y,x-y) \leq 0,\ \forall y \in S(p,x).$$

Note that $SQVI_p(h,S)$ and $MQVI_p(h,S)$ provide very general formulations of variational inequalities, which include the classic Stampacchia and Minty variational inequalities as special cases (see e.g. [11, 13, 15]), quasivariational inequalities (see e.g. [2]), generalized Minty variational inequalities [6, 7] and variational inequalities defined by a bifunction [10, 19].

Here, we show that a parametric nondifferentiable minimization problem leads to a problem which can be incorporated in the $MQVI_p(h,S)$ model but not included in other models in [10, 20, 21, 23].

Consider the parametric minimization problem:

$$(MP)_p : \min_{x \in K(p)} I_p(x),$$

where $K(p) \subset K$ is a nonempty set and $I_p : K(p) \to R$ is a locally Lipschitz continuous function. $(MP)_p$ can be regarded as a parametric version of the standard minimization problem:

$$(MP) : \min_{x \in K} I(x),$$

where $I : K \to R$ is a locally Lipschitz continuous function. It is well-known that for each p, a point $x^* \in K(p)$ is a solution of $(MP)_p$ if and only if x^* solves the following variational inequality: find $x \in K(p)$ such that

$$(VI)_p : \ \langle \nabla I_p(x), x-y \rangle \leq 0,\ \forall y \in K(p)$$

whenever $K(p)$ is convex and I_p is convex and differentiable, where $\nabla I_p(x)$ denotes the derivative of I_p at x. In other words, a parametric differentiable minimization leads to a parametric variational inequality problem. When I_p is nondifferentiable and locally Lipshitz continuous, it shall be shown that $(MP)_p$ leads to a $SQVI_p(h,S)$ model but not a $(VI)_p$ model. Indeed, it is easy to see that every solution of $(MP)_p$ must be a solution of the following problem: for each p, find $x \in K(p)$ such that

$$(P_1) : \quad I_p^\circ(x, y-x) \geq 0, \forall y \in x + \mathcal{T}(x; K(p)),$$

where

$$I_p^\circ(x,v) := \limsup_{u \to x, t \to +0} \frac{I_p(u+tv) - I_p(u)}{t}$$

denotes the Clarke directional derivative of I_p at x in the direction v and $\mathcal{T}(x; K(p))$ denotes the tangent cone of $K(p)$ at $x \in K(p)$. For more details on the Clarke directional derivative and the tangent cone, we refer the readers to [25, 26].

It is known that I_p° is positively homogeneous with degree 1 and subadditive in the second variable, and that I_p is convex if and only if I_p° is monotone (defined below). By standard arguments (as in the proof of Theorem 2.1 of [19]), it can be proved that under suitable conditions, (P_1) is equivalent to the following problem: for each p, find $x \in K(p)$ such that

$$(P_2): \quad I_p^{\circ}(y, x-y) \leq 0, \forall y \in x + \mathcal{T}(x; K(p)).$$

Define

$$h(p, y, x-y) = I_p^{\circ}(y, x-y),\ p \in P, x \in K, y \in X$$

and

$$S(p, x) = x + \mathcal{T}(x; K(p)),\ \forall p \in P, x \in K.$$

Then a nondifferentiable $(MP)_p$ leads to (P_2) which can be incorporated in the $MQVI_p(h, S)$ model but not included in other models in [10, 20, 21, 23].

Some concepts of approximating solutions and of well-posedness have been introduced for the Stampacchia variational inequality, quasivariational inequality and variational inequality defined by a bifunction. For more details we refer readers to [8, 10, 20, 21, 22, 23] and the references therein. Inspired by the concept of α-well-posedness for variational inequalities [22], we further introduce the concepts of α-approximating solutions and of α-well-posedness for $SQVI_p(h, S)$ and $MQVI_p(h, S)$.

Definition 1. *Let $p \in P$ and $\{p_n\} \subset P$ with $p_n \to p$. A sequence $\{x_n\}$ is called an α-approximating sequence for $SQVI_p(h, S)$ [resp. $MQVI_p(h, S)$] corresponding to $\{p_n\}$ if:*

(i) $x_n \in K, \quad \forall n \in N$;

(ii) there exist sequences $\{\eta_n\}$ and $\{\epsilon_n\}$ with $\eta_n \downarrow 0$ and $\epsilon_n \downarrow 0$ such that

$$d(x_n, S(p_n, x_n)) \leq \eta_n,\ h(p_n, x_n, x_n - y) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n,\ \forall y \in S(p_n, x_n), \forall n \in N.$$

[resp. if:

(i) $x_n \in K, \quad \forall n \in N$;

(ii) there exist sequences $\{\eta_n\}$ and $\{\epsilon_n\}$ with $\eta_n \downarrow 0$ and $\epsilon_n \downarrow 0$ such that

$$d(x_n, S(p_n, x_n)) \leq \eta_n,\ h(p_n, y, x_n - y) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n,\ \forall y \in S(p_n, x_n), \forall n \in N].$$

Remark 1. *Definition 1 generalizes Definition 2.1 of Lignola and Morgan [21], Definition 3.1 of Del Prete et al. [8], Definition 2.3 of Lignola [20], and Definition 2.1 of Fang and Hu [10].*

Definition 2. *The family $\{SQVI_p(h, S) : p \in P\}$ [resp. $\{MQVI_p(h, S) : p \in P\}$] is said to be α-well-posed if for every $p \in P$, $SQVI_p(h, S)$ [resp. $MQVI_p(h, S)$] has a unique solution x_p and for every sequence $\{p_n\}$ converging to p, every α-approximating sequence for $SQVI_p(h, S)$ [resp. $MQVI_p(h, S)$] corresponding to $\{p_n\}$ norm-converges to x_p.*

Remark 2. *Definition 2 generalizes Definition 2.2 of Lignola and Morgan [21], Definition 3.2 of Del Prete et al. [8], Definition 2.4 of Lignola [20], and Definition 2.2 of Fang and Hu [10].*

When $SQVI_p(h,S)$ [resp. $MQVI_p(h,S)$] has more than one solution, we can introduce the corresponding concept of α-well-posedness in the generalized sense.

Definition 3. *The family $\{SQVI_p(h,S) : p \in P\}$ [resp. $\{MQVI_p(h,S) : p \in P\}$] is said to be α-well-posed in the generalized sense if for every $p \in P$, $SQVI_p(h,S)$ [resp. $MQVI_p(h,S)$] has a nonempty solution set S_p [resp. M_p] and for every sequence $\{p_n\}$ converging to p, every α-approximating sequence for $SQVI_p(h,S)$ [resp. $MQVI_p(h,S)$] corresponding to $\{p_n\}$ has a subsequence which norm-converges to some point of S_p [resp. M_p].*

To investigate the α-well-posedness of $SQVI_p(h,S)$ and $MQVI_p(h,S)$, we need the following concepts and results.

Definition 4 (see [18])**.** *Let A be a nonempty subset of X. The measure of noncompactness μ of the set A is defined by*

$$\mu(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i, diam A_i < \epsilon, i = 1, 2, \cdots, n\},$$

where diam means the diameter of a set.

Definition 5. *Given a set A and a sequence $\{A_n\}$ of nonempty subsets of X, the Kuratowski-Painlevé lower and upper limits are defined as follows:*

$$Liminf A_n = \{x \in X : x = \lim_{n\to\infty} x_n, x_n \in A_n, \text{ for all sufficiently large } n\},$$
$$Limsup\, A_n = \{x \in X : x = \lim_{s\to\infty} x_s, x_s \in A_{n_s}, \{n_s\} \text{ is a subsequence of } \{n\}\}.$$

We say that $\{A_n\}$ converges to A in the sense of Kuratowski-Painlevé iff

$$Limsup\, A_n \subset A \subset Liminf\, A_n.$$

Definition 6. *Let (E,d) be a metric space and let A, B be nonempty subsets of E. The Hausdorff metric $\mathcal{H}(\cdot,\cdot)$ between A and B is defined by*

$$\mathcal{H}(A,B) = \max\{e(A,B), e(B,A)\},$$

where $e(A,B) = \sup_{a\in A} d(a,B)$ with $d(a,B) = \inf_{b\in B} d(a,b)$. Let $\{A_n\}$ be a sequence of nonempty subsets of E. We say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic see e.g. [16, 18].

Definition 7 (see [18])**.** *Let (E,τ) and (F,σ) be two first countable topological spaces. A set-valued map $G : E \to 2^F$ is said to be*

(i) (τ,σ)-closed if for every $x \in E$, for every sequence $\{x_n\}$ τ-converging to x, and for every sequence $\{y_n\}$ σ-converging to y, such that $y_n \in G(x_n)$ for all $n \in N$, one has $y \in G(x)$, i.e., $G(x) \supset \limsup_n G(x_n)$;

(ii) (τ, σ)*-lower semicontinuous if for every* $x \in E$*, for every sequence* $\{x_n\}$ τ*-converging to* x*, and for every* $y \in G(x)$*, there exists a sequence* $\{y_n\}$ σ*-converging to* y*, such that* $y_n \in G(x_n)$ *for all sufficiently large* n*, i.e.,* $G(x) \subset \liminf_n G(x_n)$*;*

(iii) (τ, σ)*-subcontinuous if for every* $x \in E$*, for every sequence* $\{x_n\}$ τ*-converging to* x*, and every sequence* $\{y_n\}$ *with* $y_n \in G(x_n)$*,*$\{y_n\}$ *has a* σ*-convergent subsequence.*

Definition 8 (see [17, 19])**.** *A bifunction* $f : K \times X \to \bar{R}$ *is said to be*

(i) monotone if $f(x, y - x) + f(y, x - y) \leq 0, \quad \forall x, y \in K.$

(ii) pseudomonotone if for any $x, y \in K$*,* $f(x, y - x) \geq 0 \Rightarrow f(y, x - y) \leq 0.$

Definition 9. *A bifunction* $f : K \times X \to \bar{R}$ *is said to be hemicontinuous if for every* $x, y \in K$ *and* $t \in [0, 1]$*, the function* $t \mapsto f(x + t(y - x), y - x)$ *is continuous at* 0^+*. Clearly, the continuity of* f *implies the hemicontinuity of* f*, but the converse is not true in general.*

Definition 10 (see [27])**.** *A bifunction* $f : K \times X \to \bar{R}$ *is said to be subodd if*

$$f(x, d) + f(x, -d) \geq 0, \quad \forall x \in K, d \in X.$$

Definition 11. *A function* $g : X \to \bar{R}$ *is said to be positively homogeneous with degree* $\rho > 0$ *if* $g(\lambda x) = \lambda^\rho g(x)$ *for all* $x \in X$ *and* $\lambda > 0$*.*

By the same arguments as in [19, Theorem 2.1], we have the following Minty type lemma.

Lemma 1. *Let* K *be convex, and* $f : K \times X \to \bar{R}$ *a hemicontinuous, subodd and pseudomonotone bifunction. If* f *is positively homogeneous with degree* $\rho > 0$ *in the second variable, then the following problems are equivalent:*

(i) find $x \in K$ *such that* $f(x, x - y) \leq 0, \quad \forall y \in K$*;*

(ii) find $x \in K$ *such that* $f(y, x - y) \leq 0, \quad \forall y \in K.$

Lemma 2. *Let* K *be convex,* $f : K \times X \to \bar{R}$ *positively homogeneous with degree* ρ *(*$0 < \rho < 2$*) in the second variable, and* $x \in K$ *a given point. Then*

$$f(x, x - y) \leq 0, \quad \forall y \in K$$

if and only if

$$f(x, x - y) \leq \frac{\alpha}{2} \|x - y\|^2, \quad \forall y \in K.$$

Proof. The necessity is obvious. For the sufficiency, suppose that

$$f(x, x - y) \leq \frac{\alpha}{2} \|x - y\|^2, \quad \forall y \in K.$$

For any $v \in K$, let $y(t) = x + t(v - x), \forall t \in [0,1]$. It follows that $y(t) \in K$ and so

$$f(x, x - y(t)) \leq \frac{\alpha}{2}\|x - y(t)\|^2.$$

Since f is positively homogeneous with degree ρ in the second variable, we get

$$f(x, x - v) \leq \frac{t^{2-\rho}\alpha}{2}\|x - v\|^2, \quad \forall v \in K.$$

Letting $t \to 0$ in the above inequality, we have

$$f(x, x - v) \leq 0, \quad \forall v \in K.$$

□

Lemma 3. *Let K be convex, and $f : K \times X \to \bar{R}$ a hemicontinuous, subodd and pseudomonotone bifunction. Assume that f is positively homogeneous with degree ρ ($0 < \rho < 2$) in the second variable and $x \in K$ is a given point. Then*

$$f(y, x - y) \leq 0, \quad \forall y \in K$$

if and only if

$$f(y, x - y) \leq \frac{\alpha}{2}\|x - y\|^2, \quad \forall y \in K.$$

Proof. The necessity is obvious. For the sufficiency, suppose that

$$f(y, x - y) \leq \frac{\alpha}{2}\|x - y\|^2, \quad \forall y \in K.$$

For any $v \in K$, let $y(t) = x + t(v - x), \forall t \in [0,1]$. It follows that $y(t) \in K$ and so

$$f(y(t), x - y(t)) \leq \frac{\alpha}{2}\|x - y(t)\|^2.$$

Since f is positively homogeneous with degree ρ in the second variable, we get

$$f(y(t), x - v) \leq \frac{t^{2-\rho}\alpha}{2}\|x - v\|^2, \quad \forall v \in K.$$

Letting $t \to 0$ in the above inequality, we have

$$f(x, x - v) \leq 0, \quad \forall v \in K.$$

It follows from Lemma 1 that

$$f(v, x - v) \leq 0, \quad \forall v \in K.$$

□

3. Metric characterizations of α-well-posedness

In this section we shall give some metric characterizations of α-well-posedness for parametric Stampacchia and Minty quasivariational inequalities defined by bifunctions. For $SQVI_p(h, S)$ and $MQVI_p(h, S)$, the sets of α-approximating solutions are defined by

$$T_p^S(\delta,\epsilon,\eta) = \cup_{p'\in B(p,\delta)}\{x \in K : d(x, S(p', x)) \\ \leq \eta, h(p', x, x-y) \leq \frac{\alpha}{2}\|x-y\|^2 + \epsilon, \forall y \in S(p', x)\}$$

and

$$T_p^M(\delta,\epsilon,\eta) = \cup_{p'\in B(p,\delta)}\{x \in K : d(x, S(p', x)) \\ \leq \eta, h(p', y, x-y) \leq \frac{\alpha}{2}\|x-y\|^2 + \epsilon, \forall y \in S(p', x)\}, \textit{ respectively},$$

where $B(p,\delta)$ denotes the closed ball centered at p with radius δ.

Theorem 1. *Let K be closed, $h : P\times K\times X \to \bar{R}$ positively homogeneous with degree ρ $(0 < \rho < 2)$ in the third variable and s-continuous, and let $S : P \times K \to 2^K$ be convex-valued, (s, w)-closed, (s, s)-lower semicontinuous and (s, w)-subcontinuous. Then, the family $\{SQVI_p(h, S) : p \in P\}$ is α-well-posed if and only if for every $p \in P$,*

$$T_p^S(\delta,\epsilon,\eta) \neq \emptyset, \forall\epsilon,\delta,\eta > 0, \textit{ and } diam\, T_p^S(\delta,\epsilon,\eta) \to 0 \textit{ as } (\delta,\epsilon,\eta) \to (0,0,0). \quad (1)$$

Proof. Suppose that for every $p \in P$,

$$T_p^S(\delta,\epsilon,\eta) \neq \emptyset, \forall\epsilon,\delta,\eta > 0, \text{ and } \operatorname{diam} T_p^S(\delta,\epsilon,\eta) \to 0 \text{ as } (\delta,\epsilon,\eta) \to (0,0,0).$$

We first prove that S_p has at most one solution. Let $u, v \in S_p$ with $u \neq v$. It follows that

$$u \in S(p, u), \quad h(p, u, u-y) \leq 0, \quad \forall y \in S(p, u)$$

and

$$v \in S(p, v), \quad h(p, v, v-y) \leq 0, \quad \forall y \in S(p, v).$$

It is easily seen that $u, v \in T_p^S(\delta,\epsilon,\eta)$ for all $\delta,\epsilon,\eta \geq 0$. Taking (1) into account, we have $u = v$, a contradiction.

Let $p_n \to p$ and $\{x_n\} \subset K$ be an α-approximating sequence for $\{SQVI_p(h, S) : p \in P\}$ corresponding to $\{p_n\}$. Then there exist sequences $\{\eta_n\}$ and $\{\epsilon_n\}$ with $\eta_n \downarrow 0$ and $\epsilon_n \downarrow 0$ such that

$$d(x_n, S(p_n, x_n)) \leq \eta_n, h(p_n, x_n, x_n - y) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \forall y \in S(p_n, x_n), \forall n \in N.$$

This implies that $x_n \in T_p^S(\delta_n, \epsilon_n, \eta_n)$ with $\delta_n = \|p_n - p\|$. By condition (1), $\{x_n\}$ is a Cauchy sequence and it converges strongly to a point $x \in K$. We will prove x is the unique solution of $SQVI_p(h, S)$ by two steps.

Step I: Since $d(x_n, S(p_n, x_n)) \leq \eta_n < \eta_n + \frac{1}{n}$, there exists $y_n \in S(p_n, x_n)$ such that

$$\|x_n - y_n\| < \eta_n + \frac{1}{n}.$$

Further, since S is (s, w)-closed and (s, w)-subcontinuous, $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ which converges weakly to $y \in S(p, x)$. It follows that

$$d(x, S(p, x)) \leq \|x - y\| \leq \liminf_k \|x_{n_k} - y_{n_k}\| \leq \lim_k (\eta_{n_k} + \frac{1}{k}) \to 0.$$

Thus $x \in S(p, x)$.

Step II: Consider an arbitrary $z \in S(p, x)$. Since S is (s, s)-lower semicontinuous, there exists $z_n \in S(p_n, x_n)$ such that $z_n \to z$. It follows that

$$h(p_n, x_n, x_n - z_n) \leq \frac{\alpha}{2}\|x_n - z_n\|^2 + \epsilon_n, \forall n \in N.$$

Since h is s-continuous, we get

$$h(p, x, x - z) \leq \frac{\alpha}{2}\|x - z\|^2, \quad \forall z \in S(p, x).$$

It follows from Lemma 2 that

$$h(p, x, x - z) \leq 0, \quad \forall z \in S(p, x),$$

which together with $x \in S(p, x)$ implies that x is the unique solution of $SQVI_p(h, S)$.

Conversely, let $\{SQVI_p(h, S) : p \in P\}$ be α-well-posed. Clearly,

$$T_p^S(\delta, \epsilon, \eta) \supset S_p \neq \emptyset, \quad \forall \epsilon, \delta, \eta > 0.$$

Suppose by contradiction that there exists some $p \in P$ such that $\mathrm{diam T_p^S}(\delta, \epsilon, \eta) \not\to 0$ as $(\delta, \epsilon, \eta) \to (0, 0, 0)$. Then there exist a positive number l and sequences $\{\delta_n\}, \{\epsilon_n\}, \{\eta_n\}$ with $\delta_n \downarrow 0, \epsilon_n \downarrow 0, \eta_n \downarrow 0$, and $u_n, v_n \in K$ with $u_n \in T_p^S(\delta_n, \epsilon_n, \eta_n)$, $v_n \in T_p^S(\delta_n, \epsilon_n, \eta_n)$ such that

$$\|u_n - v_n\| > l, \quad \forall n \in N. \tag{2}$$

Since $u_n, v_n \in T_p^S(\delta_n, \epsilon_n, \eta_n)$, there exist $p_n, p_n' \in B(p, \delta_n, \eta_n)$ such that

$$d(u_n, S(p_n, u_n)) \leq \eta_n, h(p_n, u_n, u_n - y) \leq \frac{\alpha}{2}\|u_n - y\|^2 + \epsilon_n, \forall y \in S(p_n, u_n), \forall n \in N$$

and

$$d(v_n, S(p_n', v_n)) \leq \eta_n, h(p_n', v_n, v_n - y) \leq \frac{\alpha}{2}\|v_n - y\|^2 + \epsilon_n, \forall y \in S(p_n', v_n), \forall n \in N.$$

Therefore, $\{u_n\}$ and $\{v_n\}$ are α-approximating sequences for $SQVI_p(h, S)$ corresponding to $\{p_n\}$ and $\{p_n'\}$, respectively. Since $\{SQVI_p(h, S) : p \in P\}$ is α-well-posed, they have to converge to the unique solution of $SQVI_p(h, S)$. This gives a contradiction to (2). Thus condition (1) holds. □

When the condition $T_p^S(\delta,\epsilon,\eta)\neq\emptyset, \forall\epsilon,\delta,\eta>0$ is replaced by $S_p\neq\emptyset$, the continuity assumptions on h and S can be dropped.

Theorem 2. *Assume that $h: P\times K\times X\to\bar{R}$ is positively homogeneous with degree ρ $(0<\rho<2)$ in the third variable and $S: P\times K\to 2^K$ is convex-valued. Then, the family $\{SQVI_p(h,S): p\in P\}$ is α-well-posed if and only if for every $p\in P$, the solution set S_p of $SQVI_p(h,S)$ is nonempty and*

$$diam\, T_p^S(\delta,\epsilon,\eta)\to 0 \text{ as } (\delta,\epsilon,\eta)\to(0,0,0). \tag{3}$$

Proof. The necessity has been proved in Theorem 1. For the sufficiency, let $p_n\to p\in P$ and $\{x_n\}$ be an α-approximating sequence for $SQVI_p(h,S)$ corresponding to $\{p_n\}$. Then there exist sequences $\{\epsilon_n\}$ and $\{\eta_n\}$ with $\epsilon_n\downarrow 0$ and $\eta_n\downarrow 0$ such that

$$d(x_n,S(p_n,x_n))\le\eta_n, h(p_n,x_n,x_n-y)\le\frac{\alpha}{2}\|x_n-y\|^2+\epsilon_n,\quad \forall y\in S(p_n,x_n),\forall n\in N.$$

This means $x_n\in T_p^S(\delta_n,\epsilon_n,\eta_n)$ with $\delta_n=\|p_n-p\|$. Let x_p be the unique solution of $SQVI_p(h,S)$. Clearly, $x_p\in T_p^S(\delta_n,\epsilon_n,\eta_n)$ for all n. It follows from (3) that

$$\|x_n-x_p\|\le \operatorname{diam} T_p^S(\delta_n,\epsilon_n,\delta_n)\to 0.$$

Thus $\{SQVI_p(h,S): p\in P\}$ is α-well-posed. □

Remark 3. *Theorems 1 and 2 generalize Proposition 2.3 of Lignola and Morgan [21], Theorem 3.2 of Lignola [20] and Theorems 3.1 and 3.2 of Fang and Hu [10]. For the α-well-posedness of $MQVI_p(h,S)$, we have the following analogous metric characterizations.*

Theorem 3. *Let K be closed, $h: P\times K\times X\to\bar{R}$ positively homogeneous with degree ρ $(0<\rho<2)$ in the third variable and s-continuous, and let $S: P\times K\to 2^K$ be convex-valued, (s,w)-closed, (s,s)-lower semicontinuous and (s,w)-subcontinuous. Assume that $h(p,\cdot,\cdot)$ is subodd and pseudomonotone for all $p\in P$. Then, the family $\{MQVI_p(h,S): p\in P\}$ is α-well-posed if and only if for every $p\in P$,*

$$T_p^M(\delta,\epsilon,\eta)\neq\emptyset,\forall\epsilon,\delta,\eta>0, \text{ and } diam\, T_p^M(\delta,\epsilon,\eta)\to 0 \text{ as } (\delta,\epsilon,\eta)\to(0,0,0).$$

Proof. The proof follows similar arguments as in Theorem 1 with Lemma 2 being replaced by Lemma 3. □

When the condition $T_p^M(\delta,\epsilon,\eta)\neq\emptyset,\forall\epsilon,\delta,\eta>0$ is replaced by $M_p\neq\emptyset$, some assumptions in Theorem 3 can be dropped.

Theorem 4. *Assume that $h: P\times K\times X\to\bar{R}$ is positively homogeneous with degree ρ $(0<\rho<2)$ in the third variable and $S: P\times K\to 2^K$ is convex-valued. Then, the family $\{MQVI_p(h,S): p\in P\}$ is α-well-posed if and only if for every $p\in P$, the solution set M_p of $MQVI_p(h,S)$ is nonempty and*

$$diam\, T_p^M(\delta,\epsilon,\eta)\to 0 \text{ as } (\delta,\epsilon,\eta)\to(0,0,0).$$

Proof. The conclusion follows from similar arguments as in Theorem 2. □

Remark 4. *Theorems 3 and 4 generalize Theorems 3.3 and 3.4 of Fang and Hu [10].*

Next we give two applications of metric characterization of α-well-posedness.

Example 1. *Let $X = K = l^2$, $S(u) = \{x \in K : \|x\| \leq \|u\|\}$ and $h(p,u,v) = \|u\|^2 \cdot \|v\|$ for all $p \in P, u \in K, v \in X$. Then h is positively homogeneous with degree $\rho = 1$ in the third variable and continuous, and S is convex-valued, closed, lower semicontinuous and subcontinuous. It follows that*

$$\begin{aligned} T_p^S(\delta,\epsilon,\eta) &= \{x \in K : d(x,S(x)) \leq \eta, \|x\|^2 \cdot \|x-y\| \leq \frac{\alpha}{2}\|x-y\|^2 + \epsilon, \forall y \in S(x)\} \\ &= \{x \in l^2 : d(x,S(x)) \leq \eta, \|x\|^2 \leq \frac{\alpha}{2}\|x-y\| + \frac{\epsilon}{\|x-y\|}, \forall y \in S(x)\} \\ &\subset \{x \in l^2 : \|x\|^2 \leq \sqrt{2\alpha\epsilon}\} \end{aligned}$$

for sufficiently small $\epsilon > 0$. It follows that $diam T_p^S(\delta,\epsilon,\eta) \to 0$ as $(\delta,\epsilon,\eta) \to (0,0,0)$. By Theorem 1, $\{SQVI_p(h,S) : p \in P\}$ is α-well-posed.

Example 2. *Let K be the closed unit ball of a real Banach space X and $P = R$. Let $S(u) = \{x \in K : \|x\| \leq 2\|u\|\}$ and $h(p,u,v) = p\|u\| \cdot \sqrt{\|v\|}$ for all $p \in P, u \in K, v \in X$. Then h is positively homogeneous with degree $\rho = \frac{1}{2}$ in the third variable and continuous, and S is convex-valued, closed, lower semicontinuous and subcontinuous. It is easily seen that when $T_p^S(\delta,\epsilon,\eta) = K$ for all $p \leq 0$ and so $diam T_p^S(\delta,\epsilon,\eta) = 2 \not\to 0$ as $(\delta,\epsilon,\eta) \to (0,0,0)$. By Theorem 1, $\{SQVI_p(h,S) : p \in P\}$ is not α-well-posed.*

4. Metric characterizations of α-well-posedness in the generalized sense

In this section we shall give some metric characterizations of α-well-posedness in the generalized sense for parametric quasivariational inequalities having more than one solution. We first establish a metric characterization by considering the measure of noncompactness of α-approximating solution sets.

Theorem 5. *Assume that K is closed and P is finite dimensional. Let $h : P \times K \times X \to \bar{R}$ be positively homogeneous with degree ρ $(0 < \rho < 2)$ in the third variable and s-continuous, and let $S : P \times K \to 2^K$ be convex-valued, (s,w)-closed, (s,s)-lower semicontinuous and (s,w)-subcontinuous. Then, the family $\{SQVI_p(h,S) : p \in P\}$ is α-well-posed in the generalized sense if and only if for every $p \in P$,*

$$T_p^S(\delta,\epsilon,\eta) \neq \emptyset, \forall \delta,\epsilon,\eta > 0, \text{ and } \mu(T_p^S(\delta,\epsilon,\eta)) \to 0 \text{ as } (\delta,\epsilon,\eta) \to (0,0,0). \quad (4)$$

Proof. Suppose that $\{SQVI_p(h,S) : p \in P\}$ is α-well-posed in the generalized sense. Then $T_p^S(\delta,\epsilon,\eta) \neq \emptyset$ for all $p \in P, \delta,\epsilon,\eta > 0$ since $T_p^S(\delta,\epsilon,\eta) \supset S_p \neq \emptyset$. We first show that S_p is compact. Let $\{x_n\}$ be a sequence in S_p. Clearly $\{x_n\}$ is an α-approximating sequence for $SQVI_p(h,S)$. Since $\{SQVI_p(h,S) : p \in P\}$ is α-well-posed in the generalized sense, $\{x_n\}$ has a subsequence converging strongly to some point of S_p. Thus S_p is compact. Observe that for every $\delta,\epsilon,\eta > 0$,

$$\mathcal{H}(T_p^S(\delta,\epsilon,\eta), S_p) = \max\{e(T_p^S(\delta,\epsilon,\eta), S_p), e(S_p, T_p^S(\delta,\epsilon,\eta))\} = e(T_p^S(\delta,\epsilon,\eta), S_p).$$

By using the compactness of S_p, we get

$$\mu(T_p^S(\delta,\epsilon,\eta)) \leq 2\mathcal{H}(T_p^S(\delta,\epsilon,\eta),S_p)+\mu(S_p)=2e(T_p^S(\delta,\epsilon,\eta),S_p).$$

To prove (4), it is sufficient to show

$$e(T_p^S(\delta,\epsilon,\eta),S_p)\to 0 \text{ as } (\delta,\epsilon,\eta)\to(0,0,0).$$

If $e(T_p^S(\delta,\epsilon,\eta),S_p)\not\to 0$ as $(\delta,\epsilon,\eta)\to(0,0,0)$. Then there exist $\tau>0$ and $\{\delta_n\}$, $\{\epsilon_n\}$, $\{\eta_n\}$ with $\delta_n\downarrow 0,\epsilon_n\downarrow 0,\eta_n\downarrow 0$, and $x_n\in K$ with $x_n\in T_p^S(\delta_n,\epsilon_n,\eta_n)$ such that

$$x_n\notin S_p+B(0,\tau),\quad \forall n\in N. \tag{5}$$

Being $x_n\in T_p^S(\delta_n,\epsilon_n,\eta_n)$, $\{x_n\}$ is an α-approximating sequence for $SQVI_p(h,S)$. Since $\{SQVI_p(h,K):p\in P\}$ is α-well-posed in the generalized sense, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging strongly to some point of S_p. This contradicts (5) and so

$$e(T_p^S(\delta,\epsilon,\eta),S_p)\to 0 \text{ as } (\delta,\epsilon,\eta)\to(0,0,0).$$

Conversely, assume that (4) holds. We first show that $T_p^S(\delta,\epsilon,\eta)$ is closed for all $\delta,\epsilon,\eta>0$. Let $x_n\in T_p^S(\delta,\epsilon,\eta)$ with $x_n\to x$. Then there exists $p_n\in B(p,\delta)$ such that

$$d(x_n,S(p_n,x_n))\leq\eta, h(p_n,x_n,x_n-y)\leq\frac{\alpha}{2}\|x_n-y\|^2+\epsilon,\quad \forall y\in S(p_n,x_n). \tag{6}$$

Since P is finite dimensional, without loss of generality, we may suppose that $p_n\to p'\in B(p,\delta)$.

Since $d(x_n,S(p_n,x_n))\leq\eta<\eta+\frac{1}{n}$, there exists $y_n\in S(p_n,x_n)$ such that

$$\|x_n-y_n\|<\eta+\frac{1}{n}.$$

Further, since S is (s,w)-closed and (s,w)-subcontinuous, $\{y_n\}$ has a subsequence $\{y_{n_k}\}$ which converges weakly to $y\in S(p',x)$. It follows that

$$d(x,S(p',x))\leq\|x-y\|\leq\liminf_k\|x_{n_k}-y_{n_k}\|\leq\liminf_k(\eta+\frac{1}{k})=\eta. \tag{7}$$

Consider an arbitrary $z\in S(p',x)$. Since S is (s,s)-lower semicontinuous, there exists $z_n\in S(p_n,x_n)$ such that $z_n\to z$. It follows from (6) that

$$h(p_n,x_n,x_n-z_n)\leq\frac{\alpha}{2}\|x_n-z_n\|^2+\epsilon,\forall n\in N.$$

Since h is s-continuous, we get

$$h(p',x,x-z)\leq\frac{\alpha}{2}\|x-z\|^2+\epsilon,\quad\forall z\in S(p',x). \tag{8}$$

From (7) and (8) we get $x\in T_p^S(\delta,\epsilon,\eta)$, and so $T_p^S(\delta,\epsilon,\eta)$ is nonempty closed. Observe that

$$S_p=\cap_{\delta>0,\epsilon>0,\eta>0}T_p^S(\delta,\epsilon,\eta).$$

Since

$$\mu(T_p^S(\delta,\epsilon,\eta)) \to 0,$$

Theorem on p.412 in [18] can be applied and one concludes that S_p is nonempty, compact, and

$$e(T_p^S(\delta,\epsilon,\eta),S_p) = \mathcal{H}(T_p^S(\delta,\epsilon,\eta),S_p) \to 0 \text{ as } (\delta,\epsilon,\eta) \to (0,0,0).$$

Let $p_n \to p \in P$ and $\{x_n\}$ be an α-approximating sequence for $SQVI_p(h,S)$ corresponding to $\{p_n\}$. Then there exist $\{\epsilon_n\}$ and $\{\eta_n\}$ with $\epsilon_n \downarrow 0$ and $\eta_n \downarrow 0$ such that

$$d(x_n,S(p_n,x_n)) \le \eta_n, h(p_n,x_n,x_n-y) \le \frac{\alpha}{2}\|x_n-y\|^2+\epsilon_n, \quad \forall y \in S(p_n,x_n), \forall n \in N.$$

Taking $\delta_n = \|p_n - p\|$, we have $x_n \in T_p^S(\delta_n,\epsilon_n,\eta_n)$. Then there exists a sequence $\{\bar{x}_n\}$ in S_p such that

$$\|x_n - \bar{x}_n\| = d(x_n,S_p) \le e(T_p^S(\delta_n,\epsilon_n,\eta_n),S_p) \to 0.$$

Since S_p is compact, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging to $\bar{x} \in S_p$. Hence the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges strongly to $\bar{x}$. Thus

$$\{SQVI_p(h,S) : p \in P\}$$

is α-well-posed in the generalized sense. □

Example 3. *Let K be the closed unit ball of a real Banach space X, and let $S(u) = \{x \in K : \|x\| \le \|u\|\}$ and $h(p,u,v) = -\|u\| \cdot \sqrt[3]{\|v\|}$ for all $p \in P, u \in K, v \in X$. Then h is positively homogeneous with degree $\rho = \frac{1}{3}$ in the third variable and continuous, and S is convex-valued, closed, lower semicontinuous and subcontinuous. It is easily seen that $T_p^S(\delta,\epsilon,\eta) = K$. Clearly,* $\mathrm{diam}\mathrm{T}_\mathrm{p}^\mathrm{S}(\delta,\epsilon,\eta) = 2 \not\to 0$, *but $\mu(T_p^S(\delta,\epsilon,\eta)) = 0$ as $(\delta,\epsilon,\eta) \to (0,0,0)$. By Theorems 1 and 5, $\{SQVI_p(h,S) : p \in P\}$ is not α-well-posed, but α-well-posed in the generalized sense.*

When the condition $T_p^S(\delta,\epsilon,\eta) \ne \emptyset, \forall \epsilon,\delta,\eta > 0$ is replaced by that S_p is nonempty compact, some assumptions on h and S can be dropped.

Theorem 6. *The family $\{SQVI_p(h,S) : p \in P\}$ is α-well-posed in the generalized sense if and only if for every $p \in P$, the solution set S_p of $SQVI_p(h,S)$ is nonempty compact and*

$$e(T_p^S(\delta,\epsilon,\eta),S_p) \to 0 \text{ as } (\delta,\epsilon,\eta) \to (0,0,0). \tag{9}$$

Proof. The necessity has been proved in Theorem 5. For the sufficiency, assume that S_p is nonempty compact for all $p \in P$ and condition (9) holds. Let $p_n \to p \in P$ and $\{x_n\}$ be an α-approximating sequence for $SQVI_p(h,S)$ corresponding to $\{p_n\}$. Then there exist sequences $\{\epsilon_n\}$ and $\{\eta_n\}$ with $\eta_n \downarrow 0$ and $\epsilon_n \downarrow 0$ such that

$$d(x_n,S(p_n,x_n)) \le \eta_n, h(p_n,x_n,x_n-y) \le \frac{\alpha}{2}\|x_n-y\|^2+\epsilon_n, \quad \forall y \in S(p_n,x_n), \forall n \in N.$$

Taking $\delta_n = \|p_n - p\|$, we have $x_n \in T_p^S(\delta_n, \epsilon_n, \eta_n)$. Then there exists a sequence $\{\bar{x}_n\}$ in S_p such that

$$\|x_n - \bar{x}_n\| = d(x_n, S_p) \le e(T_p^S(\delta_n, \epsilon_n), S_p) \to 0.$$

Since S_p is compact, there exists a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ converging to $\bar{x} \in S_p$. Hence the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges strongly to $\bar{x}$. Thus $\{SQVI_p(h,S) : p \in P\}$ is α-well-posed in the generalized sense. □

Remark 5. *Let us mention:*

(i) Theorem 5 and 6 give a metric characterization of α-well-posedness in the generalized sense by using the noncompact measure μ and Hausdorff metric, respectively. When it is easy to compute the solution set S_p, it is better to use Theorem 6 because it is difficult to compute the noncompact measure of a set.

(ii) Theorem 5 and 6 generalize Theorem 4.1 of Lignola [20] and Theorems 4.1 and 4.2 of Fang and Hu [10].

The following example shows that compactness of S_p is essential in Theorem 6.

Example 4. *Let $X = R^m$, $K = R^m_+$, $S(u) = \{x \in R^m_+ : \|x\| \le \|u\|\}$, and $h(p,u,v) = -\max_{1\le i\le m}|v_i|$ for all $p \in P, u \in K, v = (v_1, v_2, \cdots, v_m) \in X$. It is easily seen that $S_p = T_p^S(\delta,\epsilon,\eta) = R^m_+$. It follows that $e(T_p^S(\delta,\epsilon), S_p) \to 0$ as $(\delta,\epsilon,\eta) \to (0,0,0)$. However, $\{SQVI_p(h,S) : p \in P\}$ is not α-well-posed in the generalized sense since the diverging sequence $x_n = ne$ with $e = (1,1,\cdots,1)$ is an α-approximating sequence.*

Under pseudomonotonicity assumption, we have the following analogous metric characterizations of α-well-posedness in the generalized sense of $MQVI_p(h,S)$,

Theorem 7. *Assume that K is closed and P is finite dimensional. Let $h : P \times K \times X \to \bar{R}$ be positively homogeneous with degree ρ ($0 < \rho < 2$) in the third variable, s-continuous, and $h(p,\cdot,\cdot)$ subodd and pseudomonotone for all $p \in P$, and let $S : P \times K \to 2^K$ be convex-valued, (s,w)-closed, (s,s)-lower semicontinuous and (s,w)-subcontinuous. Then, the family $\{MQVI_p(h,S) : p \in P\}$ is α-well-posed in the generalized sense if and only if for every $p \in P$,*

$$T_p^M(\delta,\epsilon,\eta) \ne \emptyset, \forall \epsilon,\delta,\eta > 0, \text{ and } \mu(T_p^M(\delta,\epsilon,\eta)) \to 0 \text{ as } (\delta,\epsilon,\eta) \to (0,0,0).$$

Proof. Observe that

$$\cap_{\delta>0,\epsilon>0,\eta>0} T_p^M(\delta,\epsilon,\eta) = \{x \in K : x \in S(p,x), h(p,y,x-y) \le \frac{\alpha}{2}\|x-y\|^2, \forall y \in S(p,x)\}.$$

From Lemma 3 we get

$$M_p = \cap_{\delta>0,\epsilon>0,\eta>0} T_p^M(\delta,\epsilon,\eta). \tag{10}$$

Taking (10) into account, we can prove the conclusion by similar arguments as in Theorem 5. □

When the condition $T_p^M(\delta,\epsilon,\eta) \neq \emptyset, \forall \epsilon,\delta,\eta > 0$ is replaced by that M_p is nonempty compact, some assumptions on h and S can be dropped.

Theorem 8. *Assume that $h : P \times K \times X \to \bar{R}$ is positively homogeneous with degree ρ ($0 < \rho < 2$) in the third variable and $S : P \times K \to 2^K$ is convex-valued. Then, the family $\{MQVI_p(h,S) : p \in P\}$ is α-well-posed in the generalized sense if and only if for every $p \in P$, the solution set M_p of $MVI_p(h,S)$ is nonempty and*

$$e(T_p^M(\delta,\epsilon,\eta), M_p) \to 0 \quad \text{as} \quad (\delta,\epsilon,\eta) \to (0,0,0).$$

Proof. The conclusion follows from similar arguments as in Theorem 6. □

5. Conditions for α-well-posedness

Concerning well-posedness of the Stampacchia variational inequality in a finite dimensional space, a classic result is that under suitable conditions, well-posedness is equivalent to the existence and uniqueness of solutions, and well-posedness in the generalized sense is equivalent to the existence of solutions. In this section we shall derive some analogous results for α-well-posedness of $SQVI_p(h,S)$ and $MQVI_p(h,S)$ under suitable conditions.

Theorem 9. *Let K be a nonempty closed subset of an Euclidean space X and $h : P \times K \times X \to \bar{R}$ positively homogeneous with degree ρ ($0 < \rho < 2$) in the third variable and s-continuous, and let $S : P \times K \to 2^K$ be convex-valued, (s,w)-closed, (s,s)-lower semicontinuous and (s,w)-subcontinuous. Assume that $T_p^S(\epsilon,\epsilon,\epsilon)$ is nonempty bounded for some $\epsilon > 0$. Then, the family $\{SQVI_p(h,S) : p \in P\}$ is α-well-posed if and only if $SQVI_p(h,S)$ has a unique solution for all $p \in P$.*

Proof. The necessity is obvious. For the sufficiency, suppose that $SQVI_p(h,S)$ has a unique solution x_p for all $p \in P$. Let $p_n \to p$ and $\{x_n\}$ be an α-approximating sequence for $SQVI_p(h,S)$ corresponding to $\{p_n\}$. Then there exist sequences $\{\epsilon_n\}$ and $\{\eta_n\}$ with $\epsilon_n \downarrow 0$ and $\eta_n \downarrow 0$ such that

$$d(x_n, S(p_n,x_n)) \le \eta_n, h(p_n,x_n,x_n-y) \le \frac{\alpha}{2}\|x_n-y\|^2 + \epsilon_n, \forall y \in S(p_n,x_n), \forall n \in N.$$

Setting $\delta_n = \|p_n - p\|$, we have $x_n \in T_p^S(\delta_n,\epsilon_n,\eta_n)$ for all $n \in N$. Let $\epsilon > 0$ be such that $T_p^S(\epsilon,\epsilon,\epsilon)$ is nonempty bounded. Then there exists $n_0 \in N$ such that $x_n \in T_p^S(\delta_n,\epsilon_n,\eta_n) \subset T_p^S(\epsilon,\epsilon,\epsilon)$ for all $n \ge n_0$. Hence, $\{x_n\}$ is bounded. Let $\{x_{n_k}\}$ be any convergent subsequence of $\{x_n\}$ with limit x. As proved in Theorem 1, x solves $SQVI_p(h,S)$. Since $SQVI_p(h,S)$ has a unique solution x_p, $x_n \to x_p$. Therefore, $\{SQVI_p(h,S) : p \in P\}$ is α-well-posed. □

The following example shows that the assumption that $T_p^S(\epsilon,\epsilon,\epsilon)$ is nonempty bounded for some $\epsilon > 0$ is essential in Theorem 9.

Example 5 (see Example 3.1 of [20]). *Let $P = X = R$ and $K = [0,+\infty)$, $h(p,x,y) = -ye^{-x}$ for all $p \in P, x \in K, y \in X$, and*

$$S(u) = \begin{cases} \{x : u \le x \le 1\}, & \text{if } u \le 1,\\ \{x : 1 \le x \le 2u-1\}, & \text{if } u > 1.\end{cases}$$

Then h is positively homogeneous with degree $\rho = 1$ in the third variable, continuous, and S is closed, convex-valued, lower semicontinuous and subcontinuous. It is easily seen that $S_p = \{1\}$. When $\alpha = 0$,

$$T_p^S(\epsilon, \epsilon, \epsilon) = \{u \in [0,1] : e^{-u}(1-u) \le \epsilon\} \cup \{u \in (1, +\infty) : e^{-u}(u-1) \le \epsilon\}$$

is unbounded for all $\epsilon > 0$ and $\{SQVI_p(h, S) : p \in P\}$ is not 0-well-posed since every diverging sequence $\{u_n\}$ is 0-approximating.

For α-well-posedness of $MQVI_p(h, S)$ we have the analogous result:

Theorem 10. *Let K be a nonempty closed subset of an Euclidean space X, $h : P \times K \times X \to \bar{R}$ positively homogeneous with degree ρ $(0 < \rho < 2)$ in the third variable, s-continuous, and $h(p, \cdot, \cdot)$ subodd and pseudomonotone for all $p \in P$, and let $S : P \times K \to 2^K$ be convex-valued, (s, w)-closed, (s, s)-lower semicontinuous and (s, w)-subcontinuous. Assume that $T_p^M(\epsilon, \epsilon, \epsilon)$ is nonempty bounded for some $\epsilon > 0$. Then, the family $\{MQVI_p(h, S) : p \in P\}$ is α-well-posed if and only if $MQVI_p(h, S)$ has a unique solution for all $p \in P$.*

Proof. The necessity is obvious. For the sufficiency, suppose that $MQVI_p(h, S)$ has a unique solution x_p for all $p \in P$. Let $p_n \to p$ and $\{x_n\}$ be an α-approximating sequence for $MQVI_p(h, S)$ corresponding to $\{p_n\}$. Then there exist sequences $\{\epsilon_n\}$ and $\{\eta_n\}$ with $\epsilon_n \downarrow 0$ and $\eta_n \downarrow 0$ such that

$$d(x_n, S(p_n, x_n)) \le \eta_n, h(p_n, y, x_n - y) \le \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \forall y \in S(p_n, x_n), \forall n \in N.$$

Setting $\delta_n = \|p_n - p\|$, we have $x_n \in T_p^S(\delta_n, \epsilon_n, \eta_n)$ for all $n \in N$. Let $\epsilon > 0$ be such that $T_p^M(\epsilon, \epsilon, \epsilon)$ is nonempty bounded. Then there exists $n_0 \in N$ such that $x_n \in T_p^M(\delta_n, \epsilon_n, \eta_n) \subset T_p^M(\epsilon, \epsilon, \epsilon)$ for all $n \ge n_0$. So $\{x_n\}$ is bounded. Let $\{x_{n_k}\}$ be any convergent subsequence of $\{x_n\}$ with limit x. As proved in Theorem 3, x solves $MQVI_p(h, S)$. Since $MQVI_p(h, S)$ has a unique solution x_p, $x_n \to x_p$. Therefore, $\{MQVI_p(h, S) : p \in P\}$ is α-well-posed. □

The following results show that under suitable conditions α-well-posedness in the generalized sense is equivalent to the existence of solutions.

Theorem 11. *Let K be a nonempty closed subset of an Euclidean space X and $h : P \times K \times X \to \bar{R}$ positively homogeneous with degree ρ $(0 < \rho < 2)$ in the third variable and s-continuous, and let $S : P \times K \to 2^K$ be convex-valued, (s, w)-closed, (s, s)-lower semicontinuous and (s, w)-subcontinuous. Assume that $T_p^S(\epsilon, \epsilon, \epsilon)$ is nonempty bounded for some $\epsilon > 0$. Then the family $\{SQVI_p(h, S) : p \in P\}$ is α-well-posed in the generalized sense.*

Proof. Let $p_n \to p$ and $\{x_n\}$ be an α-approximating sequence for $SQVI_p(h, S)$ corresponding to $\{p_n\}$. Then there exist sequences $\{\epsilon_n\}$ and $\{\eta_n\}$ with $\eta_n \downarrow 0$ and $\epsilon_n \downarrow 0$ such that

$$d(x_n, S(p_n, x_n)) \le \eta_n, h(p_n, x_n, x_n - y) \le \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \forall y \in S(p_n, x_n), \forall n \in N.$$

As proved in Theorem 9, $\{x_n\}$ is bounded. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ as $k \to \infty$. As proved in Theorem 1, x solves $SQVI_p(h, S)$. Therefore, $\{SQVI_p(h, S) : p \in P\}$ is α-well-posed in the generalized sense. □

For α-well-posedness in the generalized sense of $MQVI_p(h, S)$ we have the analogous result:

Theorem 12. *Let K be a nonempty closed subset of an Euclidean space X and $h : P \times K \times X \to \bar{R}$ positively homogeneous with degree ρ ($0 < \rho < 2$) in the third variable, s-continuous, and $h(p, \cdot, \cdot)$ subodd and pseudomonotone for all $p \in P$, and let $S : P \times K \to 2^K$ be convex-valued, (s, w)-closed, (s, s)-lower semicontinuous and (s, w)-subcontinuous. Assume that $T_p^M(\epsilon, \epsilon, \epsilon)$ is bounded for some $\epsilon > 0$. Then the family $\{MQVI_p(h, S) : p \in P\}$ is α-well-posed in the generalized sense.*

Proof. Let $p_n \to p$ and $\{x_n\}$ be an α-approximating sequence for $MQVI_p(h, S)$ corresponding to $\{p_n\}$. Then there exist sequences $\{\epsilon_n\}$ and $\{\eta_n\}$ with $\eta_n \downarrow 0$ and $\epsilon_n \downarrow 0$ such that

$$d(x_n, S(p_n, x_n)) \leq \eta_n, h(p_n, y, x_n - y) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \forall y \in S(p_n, x_n), \forall n \in N.$$

As proved in Theorem 10, $\{x_n\}$ is bounded. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$ as $k \to \infty$. As proved in Theorem 3, x solves $MQVI_p(h, S)$. Therefore, $\{MQVI_p(h, S) : p \in P\}$ is α-well-posed in the generalized sense. □

The following example shows that the assumption that $T_p^S(\epsilon, \epsilon, \epsilon)$ (resp. $T_p^M(\epsilon, \epsilon, \epsilon)$) is nonempty bounded for some $\epsilon > 0$ is essential in Theorem 11 (resp. Theorem 12).

Example 6. *Let $X = R^m, K = R_+^m$, and $P = R$, let $S(u) = \{x = (x_1, x_2, \cdots, x_m) \in R_+^m : x_i \leq u_i, i = 1, 2, \cdots, m\}$, and $h(p, u, v) = 0$ for all $p \in P, u = (u_1, u_2, \cdots, u_m) \in K, v \in X$. It is easy to see that h is positively homogeneous with degree ρ ($0 < \rho < 2$) in the third variable, continuous, $h(p, \cdot, \cdot)$ is subodd and pseudomonotone for all $p \in P$, and S is closed, convex-valued, lower semicontinuous and subcontinuous. It is easily seen that $T_p^S(\epsilon, \epsilon, \epsilon) = T_p^M(\epsilon, \epsilon, \epsilon) = R_+^m$ for all $\epsilon > 0$. Let $p_n \to p$. Clearly, $\{ne\}_{n \in N}$ with $e = (1, 1, \cdots, 1)$ is an α-approximating sequence for $SQVI_p(h, S)$ and $SMVI_p(h, S)$ corresponding to $\{p_n\}$, but it has no convergent subsequences.*

Acknowledgment

The authors would like to thank two anonymous referees for their helpful suggestions and comments. This work was partially supported by the National Natural Science Foundation (10826064 and 10671135), the Specialized Research Fund for the Doctoral Program of Higher Education (20060610005) and the Youth Science Foundation of Sichuan University (07069).

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