MATHEMATICAL COMMUNICATIONS 25
Math. Commun., Vol. **15**, No. 1, pp. 25-35 (2010)

Contraction conditions with perturbed linear operators and applications*

Yicheng Liu[1,†] and Xiao Wang[1]

[1] *Department of Mathematics and Systems Science, College of Science, National University of Defense Technology, Changsha-410 073, P. R. China*

Received February 2, 2009; accepted July 5, 2009

Abstract. In this paper, we present some new fixed point theorems for both single-valued and multi-valued maps controlled by the contraction conditions with perturbed linear operators in continuous function spaces. Our results can be applied to various integral operators. Some previous results are generalized in this literature. As applications, the existence and uniqueness of solutions of impulsive periodic boundary value problems and functional differential inclusions are exhibited in the last section.

AMS subject classifications: 47H10, 34B37

Key words: fixed point theorem, multi-valued mapping, differential inclusions, periodic boundary value problem

1. Introduction

It is well-known that inversions of various perturbed differential operators yield the sum of a compact map and a contraction. Based on this observation, Krasnoselskii [6] established a famous fixed point theorem to combine the Banach contraction mapping principle and a Schauder fixed point theorem, and exhibited various applications and generalizations, see [1, 2, 3, 5, 8, 9, 13]. The Krasnoselskii's theorem can be seen as a compact operator having a fixed point property under a small perturbation. Symmetrically, the Krasnoselskii's theorem can also be understood as a contraction operator having a fixed point property under a small perturbation. In this paper, starting from this direction, we try to establish some new fixed point theorems controlled by the contraction conditions with perturbed linear operators.

Let $I \subset R^+$ be an interval and E a Banach space equipped with the norm $\|\cdot\|_E$. $BC(I,E)$ denotes the Banach space consisting of all bounded continuous mappings from I into E with norm $\|u\|_C = \max\{\|u(t)\|_E : t \in I\}$ for $u \in BC(I,E)$. In 1999, Lou [11] proved the fixed point theorem in continuous function spaces (see Corollary 2.1). Using the notion of K-normed spaces, de Pascale and de Pascale in [14] proved a similar fixed point theorem (see Corollary 2.3). Then, de Pascale and Zabreiko gave a generalization result in [15]. Recently, Suzuki [16] presented simple proofs for the above theorems.

*The second author is supported by the grant of Tianyuan Fund of Mathematics (10826058) from the National Natural Sciences Foundation of the P.R. China.
†Corresponding author. *Email addresses:* liuyc2001@hotmail.com (Y. Liu), wxiao2008@hotmail.com (X. Wang)

In this manuscript, we present some new fixed point theorems controlled by the contraction conditions with perturbed linear operators in continuous function spaces. Our results can be applied to various integral operators arising in nonlinear problems. Some previous results in [4, 11, 14] are generalized in this literature. As applications, the existence and uniqueness of solutions of impulsive periodic boundary value problems and differential inclusions are exhibited in the last section.

2. Some fixed point theorems

Let $T : C(I,R) \times I \to R$ be an operator. We say the operator $T(\cdot,t)$ is an increasing operator, if for any $y_1, y_2 \in C(I,R)$, $0 \le y_1(t) \le y_2(t)$ implies $T(y_1,t) \le T(y_2,t)$. Then we have the following result.

Theorem 1. *Let F be a nonempty closed subset of $BC(I,E)$, $A : F \to F$ an operator and $T(\cdot,t)$ a linear increasing operator. Suppose*

(H_1) *there exists $\beta \in [0,1)$ such that for any $u, v \in F$ and $t \in I$,*

$$\|Au(t) - Av(t)\|_E \le \beta \|u(t) - v(t)\|_E + T(\|u(\cdot) - v(\cdot)\|_E, t).$$

(H_2) *there exist an $\alpha \in [0, 1-\beta)$ and a positive bounded function $y \in C(\bar{I}, R)$ such that*

$$|T(y,t)| \le \alpha |y(t)| \text{ for all } t \in I.$$

Then A has a unique fixed point in F.

Proof. For any given $x_0 \in F$, let $x_n = Ax_{n-1}$, $(n = 1, 2, \dots)$. By (H_1), we have

$$\|Ax_{n+1}(t) - Ax_n(t)\|_E \le \beta \|x_{n+1}(t) - x_n(t)\|_E + T(\|x_{n+1}(\cdot) - x_n(\cdot)\|_E, t).$$

Set $a_n(t) = \|x_{n+1}(t) - x_n(t)\|_E$, then we get

$$a_{n+1}(t) \le \beta a_n(t) + T(a_n(\cdot), t). \tag{1}$$

□

In order to prove that the sequence $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_C$, we introduce an equivalent norm and show that $\{x_n\}$ is a Cauchy sequence with respect to the new one. Based on the condition (H_2), we see that there are two positive constants M and m such that $m \le |y(t)| \le M$ for all $t \in I$. Define the new norm $\|\cdot\|_1$ by

$$\|u\|_1 = \sup\{\frac{1}{|y(t)|}\|u(t)\|_E : t \in I\}, u \in BC(I,E).$$

Then

$$\frac{1}{M}\|u\|_C \le \|u\|_1 \le \frac{1}{m}\|u\|_C.$$

Thus the two norms $\|\cdot\|_1$ and $\|\cdot\|_C$ are equivalent.

Set $a_n = \|x_{n+1} - x_n\|_1$, then we have $a_n(t) \le y(t)a_n$ for $t \in I$. By (1) and the monotonicity of function y, we have

$$\begin{aligned}\frac{1}{y(t)}a_{n+1}(t) &\le \beta a_n + \frac{1}{y(t)}|T(a_n(\cdot),t)| \le \beta a_n + \frac{1}{y(t)}|T(y(\cdot)a_n,t)| \\ &= \beta a_n + \frac{a_n}{y(t)}|T(y(\cdot),t)| \le (\beta+\alpha)a_n.\end{aligned}$$

Thus

$$a_{n+1} \le (\beta+\alpha)a_n \le (\beta+\alpha)^2 a_{n-1} \le \cdots \le (\beta+\alpha)^{n+1}a_0.$$

This means $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_1$. Therefore $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_C$. Therefore, we see that $\{x_n\}$ has a limit point in F, say u. It is easy to prove that u is the fixed point of A in F.

Now we prove the uniqueness of the fixed point. Suppose both u and v ($u \ne v$) are fixed points of A, then $Au = u$, $Av = v$. Following (H_1), we have

$$\|Au(t) - Av(t)\|_E \le \beta\|u(t) - v(t)\|_E + T(\|u(\cdot) - v(\cdot)\|_E, t).$$

Based on the above arguments, it is easy to see that

$$\|u - v\|_1 = \|Au - Av\|_1 \le (\beta+\alpha)\|u - v\|_1.$$

This is impossible. Thus the fixed point of A is unique. This completes the proof of Theorem 1.

There are many integral operators satisfying the assumption (H_2). We show some popular examples for the inversions of various perturbed differential operators.

Example 1. *Let $I = [0, L]$ for some $L > 0$, $T : C(I, R) \times I \to R$ be given by*

$$T(u(\cdot),t) = \frac{K}{t^\eta}\int_0^t u(s)ds, \quad t \in (0, L],$$

where $\eta \in [0, 1)$, $K > 0$ is a constant. Then the linear operator T satisfies (H_2).

Proof. For given $\beta \in (0, 1)$, we choose a $k > 0$ such that

$$\beta + 2k^{1-\eta}K < 1.$$

Take $\alpha = 2k^{1-\eta}K \in [0, 1-\beta)$. Define function y from I into R by

$$y(t) = \begin{cases} 1, & \text{if } t \in [0,k], \\ e^{-1+\frac{t}{k}}, & \text{if } t \in (k, L]. \end{cases}$$

It is easy to see that y is a positive bounded increasing function. Furthermore, for $t \in [0, k]$,

$$T(y,t) = T(1,t) = Kt^{1-\eta} \le k^{1-\eta}K < \alpha y(t).$$

And for $t \in (k, L]$, $y(t) = e^{-1+\frac{t}{k}}$ and

$$
\begin{aligned}
T(y,t) &= \frac{K}{t^\eta}\int_0^t y(s)ds = \frac{K}{t^\eta}[\int_0^k y(s)ds + \int_k^t y(s)ds] \\
&\leq Kk^{1-\eta} + Kk^{-\eta}\int_k^t e^{-1+\frac{s}{k}}ds \leq 2k^{1-\eta}Ky(t) = \alpha y(t).
\end{aligned}
$$

Thus

$$|T(y,t)| \leq \alpha |y(t)| \text{ for all } \ t \in I.$$

□

Example 2. *Let $I = [0, L]$ for some $L > 0$, $T : C(I, R) \times I \to R$ be given by*

$$T(u(\cdot), t) = K\int_0^t e^{-\gamma(t-s)}u(s)ds, \quad t \in I$$

where $\gamma \in R$ and $K > 0$ are two constants. Then the operator T satisfies (H_2).

Proof. For any given $\beta \in (0, 1)$, we choose a constant $c > 0$ such that

$$\beta + \frac{K}{c+\gamma} < 1.$$

Take $\alpha = \frac{K}{c+\gamma}$ and $y(t) = e^{ct}$ for $t \in I$, then it is easy to see that

$$|T(y,t)| \leq \alpha |y(t)| \text{ for all } \ t \in I.$$

□

Example 3. *Let $I = [\eta, +\infty)$ for some $\eta > 0$, $T : C(I, R) \times I \to R$ be given by*

$$T(u(\cdot), t) = \frac{K}{t^\gamma}\int_\eta^t u(s)ds, \quad t \in I$$

where $\gamma \in (1, \infty)$, $K > 0$ is a constant. Then the linear operator T satisfies (H_2).

Proof. We choose two constants $\tau \geq \eta$ and $c > 0$ such that

$$\beta + \frac{K}{c\eta^\gamma} + K\tau^{1-\gamma} < 1.$$

Let $\alpha = \frac{K}{c\eta^\gamma} + K\tau^{1-\gamma}$ and define function y from I into R by

$$y(t) = \begin{cases} e^{ct}, & \text{if } \ t \in (\eta, \tau], \\ e^{c\tau}, & \text{if } \ t \in (\tau, \infty). \end{cases}$$

For $t \in [\eta, \tau]$,

$$T(y,t) = \frac{K}{t^\gamma}\int_\eta^t e^{cs}ds \leq \frac{K}{c\eta^\gamma}e^{ct} \leq \alpha y(t).$$

For $t \in (\tau, \infty)$,

$$\begin{aligned} T(y,t) &= \frac{K}{t^\gamma}\int_\eta^t y(s)ds = \frac{K}{t^\gamma}\int_\eta^\tau y(s)ds + \frac{K}{t^\gamma}\int_\tau^t y(s)ds \\ &\le \frac{K}{c\eta^\gamma}e^{c\tau} + \frac{K}{t^\gamma}\int_\tau^t e^{c\tau}ds \le (\frac{K}{c\eta^\gamma} + K\tau^{1-\gamma})e^{c\tau} \\ &= \alpha y(t). \end{aligned}$$

Thus

$$|T(y,t)| \le \alpha |y(t)| \text{ for all } \ t \in I.$$

□

Example 4. *Let $I = [0, L]$ for some $L > 0$, $T : C(I, R) \times I \to R$ be given by*

$$T(u(\cdot), t) = K\int_t^{t+L} \frac{e^{-\gamma(t-s)}}{e^{\gamma L} - 1} u(s)ds, \quad t \in I$$

where $\gamma > 0$ and $K > 0$ are two constants. Then the linear operator T satisfies (H_2) provided $\frac{K}{\gamma} < 1 - \beta$.

Proof. Since $\beta + \frac{K}{\gamma} < 1$, we choose a constant $c > 0$ such that

$$\beta + \frac{K}{c+\gamma} < 1.$$

Take $\alpha = \frac{K}{c+\gamma}$ and $y(t) = e^{ct}$ for $t \in I$, then it is easy to prove that

$$|T(y,t)| \le \alpha |y(t)| \text{ for all } \ t \in I.$$

□

Based on Theorem 1 and Examples 1-3, we get the following Corollaries.

Corollary 1 (see [11]). *Let $I = (0, L]$ and let F be a nonempty closed subset of $C(I, E)$ and $A : F \to F$ an operator. If there exist $\alpha, \beta \in [0, 1)$ and $K \ge 0$ such that for any $u, v \in F$ and $t \in I$,*

$$\|Au(t) - Av(t)\|_E \le \beta\|u(t) - v(t)\|_E + \frac{K}{t^\alpha}\int_0^t \|u(s) - v(s)\|_E ds.$$

Then A has a unique fixed point.

Proof. It follows immediately from Theorem 1 and Example 1. □

Corollary 2. *Let $I = [0, L]$ and let F be a nonempty closed subset of $C(I, E)$ and $A : F \to F$ an operator. If there exist $\gamma \in R, \beta \in [0, 1)$ and $K \ge 0$ such that for any $u, v \in F$ and $t \in I$,*

$$\|Au(t) - Av(t)\|_E \le \beta\|u(t) - v(t)\|_E + K\int_0^t e^{-\gamma(t-s)}\|u(s) - v(s)\|_E ds.$$

Then A has a unique fixed point.

Proof. It follows immediately from Theorem 1 and Example 2. □

Corollary 3 (see [14]). *Let $I = [\eta, \infty)$ and let F be a nonempty closed subset of $BC(I, E)$ and $A : F \to F$ an operator. If there exist $\alpha \in (1, \infty), \beta \in [0, 1)$ and $K \geq 0$ such that for any $u, v \in F$ and $t \in I$,*

$$\|Au(t) - Av(t)\|_E \leq \beta\|u(t) - v(t)\|_E + \frac{K}{t^\alpha}\int_\eta^t \|u(s) - v(s)\|_E ds.$$

Then A has a unique fixed point.

Proof. It follows immediately from Theorem 1 and Example 3. □

2.1. Multi-valued fixed point theorems

Let (X, d) be a metric space, for $x \in X$, $A \subset X$, define $D(x, A) = \inf\{d(x, y), y \in A\}$. We denote $CB(X)$ as the class of all nonempty bounded closed subsets of X. Let H be the Hausdorff metric with respect to d, that is, $H(A, B) = \max\{\sup_{x\in A} D(x, B), \sup_{y\in B} D(y, A)\}$, for every $A, B \in CB(X)$. Then, to some extent, we get the following multi-valued version of Theorem 1.

Theorem 2. *Let F be a nonempty closed subset of $BC(I, E)$, $A : F \to CB(F)$ a multi-valued operator and $T(\cdot, t)$ a linear nondecreasing operator for fixed $t \in I$. Suppose*

$(\tilde{H}_1)$ *there exists $\beta \in [0, 1)$ such that for any $u, v \in F$ and $t \in I$,*

$$H(Au, Av) \leq \beta\|u(t) - v(t)\|_E + T(\|u(\cdot) - v(\cdot)\|_E, t).$$

If (H_2) holds, then A has a fixed point in F.

Proof. By (H_2), there is an $\alpha > 0$ such that $\beta + \alpha < 1$. Set $\lambda = \beta + \alpha$. Following the method in [10], we construct a fixed point iteration sequence in F. For any given $x_0 \in F$, since Ax_0 is nonempty, there is an $x_1 \in F$ such that $x_1 \in Ax_0$. Noting that Ax_0 and Ax_1 are closed sets and $x_1 \in Ax_0$, we can find $x_2 \in Ax_1$ such that

$$\|x_1 - x_2\|_C \leq H(Ax_0, Ax_1) + \lambda.$$

For $x_2 \in F$, there exists an $x_3 \in Ax_2$ and

$$\|x_2 - x_3\|_C \leq H(Ax_1, Ax_2) + \lambda^2.$$

We continue this process to obtain a sequence $\{x_n\}$ in F such that

$$\|x_{n+1} - x_n\|_C \leq H(Ax_{n-1}, Ax_n) + \lambda^n, n = 1, 2, \ldots.$$

Similarly, we shall prove $\{x_n\}$ is a Cauchy sequence in F. Let $b_n = \|x_n - x_{n+1}\|_1$ and $b_n(t) = \|x_n(t) - x_{n+1}(t)\|_E$, where $\|\cdot\|_1$ has been defined in the proof of Theorem 1, then

$$\begin{aligned}\|x_{n+2} - x_{n+1}\|_C &\leq H(Ax_{n+1}, Ax_n) + \lambda^{n+1}\\ &\leq \beta b_n(t) + T(b_n(\cdot), t) + \lambda^{n+1}.\end{aligned}$$

Since $b_n(t) \le y(t)b_n$, we have

$$\frac{1}{y(t)}\|x_{n+2} - x_{n+1}\|_C \le \frac{1}{y(t)}[\beta b_n(t) + |T(b_n(\cdot), t)| + \lambda^{n+1}] \\ \le (\beta + \alpha)b_n + \frac{1}{m}\lambda^{n+1},$$

where $m = \inf\{y(t) : t \in I\}$. This implies

$$b_{n+1} \le (\beta + \alpha)b_n + \frac{1}{m}\lambda^{n+1} = \lambda b_n + \frac{1}{m}\lambda^{n+1}.$$

So

$$b_n \le \lambda^n(b_0 + \frac{n}{m}) \quad \text{for} \quad n = 0, 1, 2, \ldots .$$

It is easy to show $\lim_{n\to\infty} b_n = 0$. Furthermore, it follows from $\sum_{n=0}^{+\infty} \lambda^n < \infty$ and $\sum_{n=0}^{+\infty} n\lambda^n < \infty$ that $\sum_{n=0}^{+\infty} b_n < \infty$. Thus $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_1$. Thus $\{x_n\}$ is a Cauchy sequence with respect to norm $\|\cdot\|_C$.

Therefore, there is a $u \in F$ such that $\lim_{n\to\infty} x_n = u$. We claim that $u \in Au$. Indeed, condition $(\tilde{H}_1)$ implies that

$$H(Au, Ax_n) \le \beta\|u(t) - x_n(t)\|_E + T(\|u(\cdot) - x_n(\cdot)\|_E, t) \\ \le \beta y(t)\|u - x_n\|_1 + |T(y, t)|\|u - x_n\|_1 \\ \le M[\beta + \alpha]\|u - x_n\|_1.$$

Since $x_n \in Ax_{n-1}$, we obtain

$$D(Au, x_{n+1}) \le H(Au, Ax_n) \le M[\beta + \alpha]\|u - x_n\|_1.$$

Thus $D(Au, u) = 0$. This means $u \in Au$. Hence A has a fixed point u. This completes the proof. □

In order to investigate the existence of solutions of functional differential inclusions by using the multi-valued fixed point theorems in continuous function spaces, we consider the Banach space $C[[-\sigma, 0], E]$ for some $\sigma > 0$ with supremum norm. For $I = [0, T]$ and $\varphi \in C[[-\sigma, 0], E]$, define a closed subset as $F = \{u \in C[I, E] : u(0) = \varphi(0)\}$. Then following similar arguments in Theorem 2, we obtain a useful result. Here we omit details of the proof.

Theorem 3. *Let F be given as above, $A : F \to CB(F)$ a multi-valued operator and $T(\cdot, t)$ a linear nondecreasing operator for fixed $t \in I$. For any $u, v \in F$ and $t \in [-\sigma, 0]$, define $u(t) = v(t) = \varphi(t)$, if (H_2) holds and there exists $\beta \in [0, 1)$ such that for $t \in I$,*

$$H(Au, Av) \le \beta \sup_{s\in[-\sigma,0]} \{\|u(t + s) - v(t + s)\|_E\} + T(\|u(\cdot) - v(\cdot)\|_E, t).$$

Then A has a fixed point in F.

Following Theorem 2, we obtain two fixed point theorems, which contain a generalization of the multivalued contraction principle established by Covitz and Nadler in [4].

Corollary 4. *Let $I = [0, L]$ and let F be a nonempty closed subset of $C(I, E)$ and $A : F \to CB(F)$ a multi-valued operator. If there exist $\alpha, \beta \in [0, 1)$ and $K \geq 0$ such that for any $u, v \in F$ and $t \in I \setminus \{0\}$,*

$$H(Au, Av) \leq \beta \|u(t) - v(t)\|_E + \frac{K}{t^\alpha} \int_0^t \|u(s) - v(s)\|_E ds.$$

Then A has a fixed point.

Corollary 5. *Let $I = [\eta, \infty)$ and let F be a nonempty closed subset of $BC(I, E)$ and $A : F \to CB(F)$ a multi-valued operator. If there exist $\alpha \in (1, \infty), \beta \in [0, 1)$ and $K \geq 0$ such that for any $u, v \in F$ and $t \in I$,*

$$H(Au, Av) \leq \beta \|u(t) - v(t)\|_E + \frac{K}{t^\alpha} \int_\eta^t \|u(s) - v(s)\|_E ds.$$

Then A has a fixed point in F.

3. Applications to nonlinear problems

Our results can be applied to various nonlinear phenomena. Here we present an illustrative example: a periodic boundary value problem with impulsive effects.

Example 5. *Consider the following periodic boundary value problem with impulsive effects:*

$$\begin{cases} u'(t) + \gamma u(t) = f(t, u(t)), & t \in (0, 1) \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), & k = 1, 2, \cdots, m, \\ u(0) = u(1), \end{cases} \tag{2}$$

where $f \in C([0, 1] \times R^n, R^n)$, $\gamma > 0$, $I_k \in C(R^n, R^n)$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = 1$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively.

Many authors have investigated various impulsive periodic boundary value problems, see for example [7, 12]. Here we assume:

(A_1) there exists constant l satisfying

$$|f(t, x) - f(t, y)| \leq l|x - y|,$$

for $x, y \in R^n$.

(A_2) there exist constants c_k satisfying

$$|I_k(x) - I_k(y)| \leq c_k|x - y|,$$

for each $k = 1, \cdots, m$ and all $x, y \in R^n$.

In order to define the solution of (2), we introduce the space $\Omega = \{u : [0,1] \to R^n : u \in C((t_k, t_{k+1}), R^n),\ u(t_k^-)$ and $u(t_k^+)$ exist and $u(t_k) = u(t_k^-),\ k = 1, \cdots, m\}$. Then Ω is a Banach space with the norm $\|u\|_\Omega := \sup\{|u(t)| : t \in [0,1]\}$. A function $u \in \Omega$ is said to be a solution of (2) if u satisfies the impulsive differential equations (2) on $[0,1] \setminus \{t_1, t_2, \cdots, t_m\}$ and the boundary conditions.

Theorem 4. *Assume* (A_1) *and* (A_2) *hold. Then the impulsive periodic boundary value problem (2) has a unique solution on* $[0,1]$ *provided* $\frac{l}{\gamma} + \sum_{k=1}^m c_k < 1$.

Proof. Noting that $u(t)$ is a solution of problem (2) if and only if $u(t)$ is a 1-periodic solution of equation

$$\begin{cases} u'(t) + \gamma u(t) = f(t, u(t)), & t \in (0,1) \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta u|_{t=t_k} = I_k(u(t_k^-)), & k = 1, 2, \cdots, m, \end{cases} \tag{3}$$

where $f(t+1, u) = f(t, u)$ for $t \in R$ and $u \in C_1 := \{u \in C(R, R^n) : u(t+1) = u(t), t \in R\}$, we transform problem (3) into a fixed point problem. Consider the map $A : C_1 \to C_1$ defined by

$$Au(t) = \int_t^{t+1} \frac{e^{-\gamma(t-s)}}{e^\gamma - 1} f(s, u(s)) ds + \sum_{0 \le t_k < t \ mod(1)} I_k(u(t_k^-)), \quad t \in R.$$

For any $u, v \in C_1$, following (A_1) and (A_2), we have $(t \in [0,1])$

$$|Au(t) - Av(t)| \le l \int_t^{t+1} \frac{e^{-\gamma(t-s)}}{e^\gamma - 1} |u(s) - v(s)| ds + \sum_{k=1}^m c_k \|u - v\|_{C_1}.$$

Let $T(x(\cdot), t) = l \int_t^{t+1} \frac{e^{-\gamma(t-s)}}{e^\gamma - 1} x(s) ds$ for $x \in C([0,1], R)$. By example 4, $\frac{l}{\gamma} + \sum_{k=1}^m c_k < 1$ and Theorem 1, we conclude that A has a unique fixed point u in C_1. Thus $u(t)$ is the unique solution of boundary value problem (2).

Example 6. *Consider the existence of solutions of the following differential inclusions with distributed delay:*

$$[x(t) - \int_{-\tau}^0 k(s) x(t+s) ds]' \in G(t, \int_0^t h(t,s) x(s) ds), \tag{4}$$

where $t \in I = [0, T]$ *for* $T > 0$, $\tau > 0$, $G \in C[I \times R^n, CB(R^n)]$, $h \in C[\Omega, R^{n \times n}]$, $\Omega = \{(t,s) \in I^2 : 0 \le s \le t \le T\}$ *and* $k \in C[I, R^{n \times n}]$ *with* $\int_{-\tau}^0 |k(s)| ds = k < 1$. $x(t) = \varphi(t)$ *for* $t \in [-\tau, 0]$ *and* $\varphi \in C[[-\tau, 0], R^n]$.

(A_3) There exists a bounded function l such that, for any $u, v \in R^n$,

$$H(G(t,u), G(t,v)) \le l(t) |u - v|.$$

Theorem 5. *Suppose that* (A_3) *holds, then (4) has at least one solution on* I.

Proof. We transfer the existence of solution of (4) into a fixed point problem. Let $F = \{u \in C[I, R^n] : u(0) = \varphi(0)\}$, define $A : F \to CB(F)$ by

$$Au(t) = \varphi(0) - \int_{-\tau}^{0} k(s)\varphi(s)ds + \int_{-\tau}^{0} k(s)u(t+s)ds + \int_0^t G(s, \int_0^s h(s,r)u(r)dr)ds,$$

where $u(t) = \varphi(t)$ for $t \in [-\tau, 0]$. Then (4) has the solution

$$S(t) = \begin{cases} u(t), & \text{if } t \in I, \\ \varphi(t), & \text{if } t \in [-\tau, 0] \end{cases}$$

if and only if $u(t)$ is the fixed point of A. Furthermore, by direct computation, we have

$$H(Au, Av) \le k \sup_{s\in[-\tau,0]} \{|u(t+s) - v(t+s)|\} + Lh \int_0^t |u(s) - v(s)|ds,$$

where $L = \max\{|l(t)| : t \in I\}$ and $h = \max\{|h(t,s)| : (t,s) \in \Omega\}$. Taking $\alpha = 0$, $\beta = k$ and $K = Lh$, then it follows from Theorem 2.3 that A has a fixed point u in $C[I, R^n]$. Thus (4) has a unique solution $S(t)$. This completes the proof. □

References

[1] T. A. Burton, *Integral equations, implicit functions, and fixed points*, Proc. Amer. Math. Soc. **124**(1996), 2383–2390.

[2] T. A. Burton, *A fixed point theorem of Krasnoselskii*, Appl. Math. Lett. **11**(1998), 85–88.

[3] T. A. Burton, T. Furumochi, *Krasnoselskiis fixed point theorem and stability*, Nonlinear Anal. **49**(2002), 445–454.

[4] H. Covitz, S. B. Nadler, *Multivalued contraction mappings in generalized metric spaces*, Israel J. Math. **8**(1970), 5–11.

[5] B. C. Dhage, *Remarks on two fixed-point theorems involving the sum and the product of two operators*, Comput. Math. Appl. **46**(2003), 1779–1785.

[6] M. A. Krasnoselskii, *Some problems of nonlinear analysis.* Amer. Math. Soc. Trans. **10**(1958), 345–409

[7] J. Li, J. J. Nieto, J. Shen, *Impulsive periodic boundary value problems of first-order differential eqautions*, J. Math. Anal. Appl. **325**(2007), 226–236.

[8] Y. C. Liu, Z. X. Li, *Schaefer Type Theorem and Periodic Solutions of Evolution Equations*, J. Math. Anal. Appl. **216**(2006), 237–255.

[9] Y. C. Liu, Z. X. Li, *Krasnoselskii type fixed point theorems and applications*, Proc. Amer. Math. Soc. **136**(2008), 1213–1220.

[10] Y. C. Liu, J. Wu, Z. X. Li, *Common fixed points of singlevalued and multivalued maps*, Int. J. Math. Math. Sci. **19**(2005), 3045–3055.

[11] B. Lou, *Fixed points for operators in a space of continuous functions and applications*, Proc. Amer. Math. Soc. **127**(1999), 2259–2264.

[12] J. J. Nieto, *Periodic boundary value problems for first-order impulsive ordinary differential eqautions*, Nonlinear Anal. **51**(2002), 1223–1232.

[13] S. Park, *Generalizations of the Krasnoselskii fixed point theorem*, Nonlinear Anal. **67**(2007), 3401–3410.

[14] E. de Pascale, L. de Pascale, *Fixed points for some non-obviously contractive operators*, Proc. Amer. Math. Soc. **130**(2002), 3249–3254.

[15] E. de Pascale, P. P. Zabreiko, *Fixed point theorems for operators in spaces of continuous functions*, Fixed Point Theory **5**(2004), 117–129.

[16] T. Suzuki, *Lous fixed point theorem in a space of continuous mappings*. J. Math. Soc. Japan. **58**(2006), 669–774.