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IONIZED GAS BOUNDARY LAYER ON BODIES OF REVOLUTION IN THE PRESENCE OF MAGNETIC FIELD

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Original scientific paper

This paper studies ionized gas flow in the boundary layer on bodies of revolution. The present magnetic field is normal to a nonporous contour of the body. The governing boundary layer equations are brought to a generalized form by general similarity method. The obtained equations are numerically solved by finite differences method. Based on the obtained solutions, diagrams of distributions of physical qualities in the boundary layer are given. Conclusions of behaviour of these quantities for the studied ionized gas problem are also drawn.

Keywords: *body of revolution, boundary layer, ionized gas, general similarity method, nonporous contour*

Granični sloj ioniziranog plina na rotacijskim tijelima pri postojanju magnetskog polja

Izvorni znanstveni članak

Istraživano je strujanje ioniziranog plina u graničnom sloju na rotacijskim tijelima. Prisutno magnetsko polje je okomito na neporoznu konturu tijela. Polazne jednadžbe graničnog sloja dovedene su na uopćeni oblik metodom uopćene sličnosti u verziji Saljnikova. Dobivene jednadžbe numerički su riješene metodom konačnih razlika. Na osnovu dobivenih rješenja prikazani su dijagrami raspodjela fizikalnih veličina u graničnom sloju. Izvedeni su zaključci o ponašanju ovih veličina kod razmatranog problema strujanja ioniziranog plina.

Ključne riječi: *granični sloj, ionizirani plin, metoda uopćene sličnosti, neporozna kontura, rotacijsko tijelo*

1 Introduction
Uvod

This paper studies ionized gas i.e. air flow in the boundary layer on bodies of revolution. The contour of the body within fluid is nonporous. The ionized gas flows in the conditions of the so-called equilibrium ionization.

The main objective of this investigation is to apply the general similarity method to the studied problem. The ultimate objective is to solve the obtained generalized equations in an appropriate approximation and to draw conclusions on behaviour of certain physical quantities and characteristics in the boundary layer.

The general similarity method was first introduced by L. G. Loitsianskii [1]. The method was later improved by V. N. Saljnikov [2]. The investigators of St Petersburg School and Belgrade School of boundary layer used this method to solve numerous important flow problems in the boundary layer (MHD and temperature boundary layer [1-4]). The original version of this method 1 was successfully used for planar boundary layer of the dissociated gas [5, 6]. Using Saljnikov's version of the general similarity method different planar problems of both dissociated [7, 8] and ionized gas flow [9] in the boundary layer were solved.

The paper gives the results of an investigation of the ionized gas (air) flow along a body of revolution. The results were obtained by application of Saljnikov's version of the general similarity method. Unlike other methods [10, 11], both versions of the general similarity method involve usage of the momentum equation and sets of corresponding parameters (form parameters, magnetic parameters...). These are the so-called similarity parameters.

2 Governing boundary layer equations
Polazne jednadžbe graničnog sloja

When gas flows at high velocities (e.g. supersonic flight of an aircraft through the Earth's atmosphere), the temperature in the viscous boundary layer increases significantly. At high temperatures ionization of gas (air) occurs together with dissociation. Because of this thermo-chemical reaction the gas becomes electroconductive (its electroconductivity is σ). Then the gas (air) consists of positively charged ions, electrons and atoms (of oxygen and nitrogen).

If the ionized gas flows in the magnetic field of the power $B_m = B_{my} = B_m(x)$, an electric current is formed in the gas, which causes appearance of the Lorentz force and the Joule's heat. Due to these effects, new terms, not found in the equations for homogenous unionized gas, appear in the equations of the ionized gas boundary layer [12].

For the case of ionized gas flow in the magnetic field, in the conditions of equilibrium ionization, the equation system of steady laminar boundary layer on bodies of revolution [7, 10, 12] has the following form:

$$\frac{\partial}{\partial x}(\rho u r^j) + \frac{\partial}{\partial y}(\rho v r)^j = 0, \quad (j=1)$$

$$\rho u \cdot \frac{\partial u}{\partial x} + \rho v \cdot \frac{\partial u}{\partial y} = -\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\partial}{\partial y} \cdot \left(\mu \cdot \frac{\partial u}{\partial y} \right) - \sigma \cdot B_m^2 \cdot u$$

$$\rho u \cdot \frac{\partial h}{\partial x} + \rho v \cdot \frac{\partial h}{\partial y} = \tag{1}$$

$$= u \cdot \frac{\mathrm{d}p}{\mathrm{d}x} + \mu \cdot \left(\frac{\partial u}{\partial y} \right)^2 + \frac{\partial}{\partial y} \cdot \left(\frac{\mu}{Pr} \cdot \frac{\partial h}{\partial y} \right) + \sigma \cdot B_m^2 \cdot u^2$$

$$u = 0, \quad v = 0, \quad h = h_w = \text{const.} \quad \text{for} \quad y = 0$$

$$u \to u_e(x), \quad h \to h_e(x) \quad \text{for} \quad y \to \infty.$$

In the mathematical model (1), the first equation represents a continuity equation of axisymmetrical flow (j=1) of the compressible fluid on bodies of revolution [12]. The second equation is dynamic and the third one is an equation of the ionized gas boundary layer. The term $\sigma \cdot B_m^2 \cdot u$ in the dynamic equation represents Lorentz force, and the term $\sigma \cdot B_m^2 \cdot u^2$ in the energy equation stands for Joule's heat [12].

In the system (1) the notation is standard for the boundary layer theory [13, 14]: x, y - are longitudinal and transversal coordinates, $u(x, y)$ - is a longitudinal projection of velocity in the boundary layer, $v(x, y)$ - transversal projection, ρ - ionized gas density, p - pressure, μ - coefficient of dynamic viscosity, h - enthalpy, $r(x)$ - radius of the cross-section of the body of revolution and $Pr = \mu \cdot c_p / \lambda$ - Prandtl number where λ - stands for coefficient of thermal conductivity and c_p - specific heat of ionized gas. The subscript e denotes physical values at the outer edge of the boundary layer and the subscript w stands for the values on a nonporous wall of the body of revolution within the fluid.

The radius $r(x)$ is normal to the axis of revolution (Fig. 1). The function $r(x)$ practically defines the contour of the body of revolution. The thickness of the boundary layer $\delta(x)$ is considerably smaller than the radius of the body of revolution ($\delta(x) \ll r(x)$), therefore, it can be ignored in relation to $r(x)$ 12. However, this cannot be applied to long thin bodies [12, 15].

In general, the ionized gas electroconductivity is a variable quantity that depends on the temperature, i.e. the enthalpy. In our investigations, by analogy with power B_m of the magnetic field [12], the electroconductivity is assumed to be the function only of the longitudinal variable x, i.e.

$$\sigma = \sigma(x). \tag{2}$$

Taking the boundary conditions at the outer edge of the boundary layer into consideration, the pressure $p(x)$ can be eliminated from the system (1).

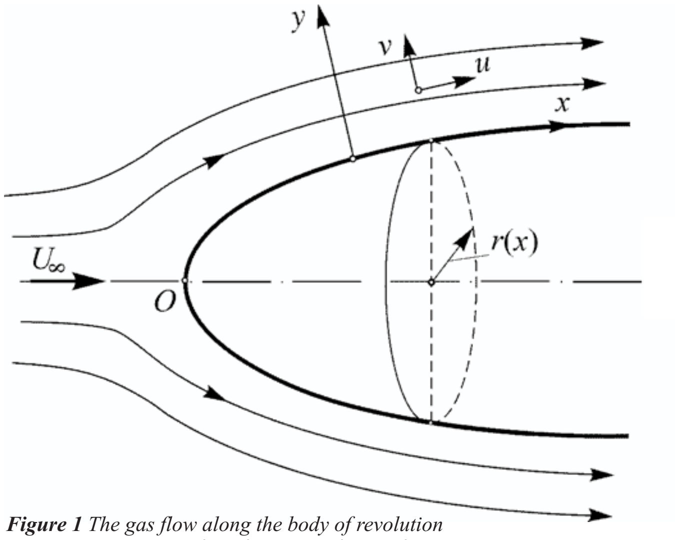

***Figure 1** The gas flow along the body of revolution*
***Slika 1.** Strujanje plina duž rotacijskog tijela*

Then the governing equation system can be brought to the following form:

$$\begin{aligned}
&\frac{\partial}{\partial x}\left[\rho \cdot u \cdot \left(\frac{r}{L}\right)^j\right] + \frac{\partial}{\partial y}\left[\rho \cdot v \cdot \left(\frac{r}{L}\right)^j\right] = 0, \quad (L = \text{const.},\ j = 1)\\
&\rho u \cdot \frac{\partial u}{\partial x} + \rho v \cdot \frac{\partial u}{\partial y} = \rho_e u_e \cdot \frac{\mathrm{d}u_e}{\mathrm{d}x} + \frac{\partial}{\partial y}\left(\mu \cdot \frac{\partial u}{\partial y}\right) + \sigma \cdot B_m^2 \cdot (u_e - u)\\
&\rho u \cdot \frac{\partial h}{\partial x} + \rho v \cdot \frac{\partial h}{\partial y} = -u \cdot \rho_e \cdot u_e \cdot \frac{\mathrm{d}u_e}{\mathrm{d}x} + \mu \cdot \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\frac{\mu}{Pr} \cdot \frac{\partial h}{\partial y}\right) + \sigma \cdot B_m^2 \cdot (u^2 - u \cdot u_e)\\
&u = 0,\ v = 0,\ h = h_w = \text{const.} \quad \text{for} \quad y = 0\\
&u \to u_e(x),\ h \to h_e(x) \quad \text{for} \quad y \to \infty.
\end{aligned} \tag{3}$$

In the system (3), the continuity equation is written in the form that is more suitable for the obtained momentum equation. Here, L is a constant characteristic length (and for the numerical calculation, it is taken that L=1).

By the procedure [16] (by analogy with incompressible fluid flow 1), from the first two equations of the system (3) and by integration transversal to the boundary layer (from y=0 to $y \to \infty$), the following equation is obtained:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[\int_0^\infty \rho \cdot u \cdot \left(\frac{r}{L}\right)^j \cdot (u_e - u)\mathrm{d}y\right] =\\
&= \left(\frac{r}{L}\right)^j \cdot \frac{\mathrm{d}u_e}{\mathrm{d}x} \cdot \int_0^\infty \rho \cdot u - \rho_e \cdot u_e)\mathrm{d}y + \left(\frac{r}{L}\right)^j \cdot \left(\mu \cdot \frac{\partial u}{\partial y}\right)_{y=0} -\\
&-\left(\frac{r}{L}\right)^j \cdot \sigma \cdot B_m^2 \cdot \int_0^\infty (u_e - u)\mathrm{d}y.
\end{aligned}$$

For solution of this integral, new variables are introduced:

$$\begin{aligned}
&s(x) = \frac{1}{\rho_0 \cdot \mu_0} \cdot \int_0^x \rho_w \cdot \mu_w \cdot \left(\frac{r}{L}\right)^{2j} \mathrm{d}x\\
&z(x, y) = \left(\frac{r}{L}\right)^j \cdot \int_0^y \frac{\rho}{\rho_0} \cdot \mathrm{d}y, \quad (j = 1).
\end{aligned} \tag{4}$$

In the transformations (4) and further in the paper, the values ρ_0 and μ_0=$\rho_0 \cdot \nu_0$ denote the known values of the density and dynamic, i.e. kinematic viscosity at a certain point of the ionized gas boundary layer. Here, ρ_w and μ_w denote the known values of these quantities on the wall of the body of revolution. Note that transformations (4) were used for j=0 in papers [5, 17].

Having changed the variables, it is relatively easy to obtain the momentum equation. This equation can be found in its three forms:

$$\frac{\mathrm{d}Z^{**}}{\mathrm{d}s}=\frac{F_m}{u_e},\quad \frac{\mathrm{d}f}{\mathrm{d}s}=\frac{u_e'}{u_e}\cdot F_m+\frac{u_e''}{u_e'}\cdot f,\quad \frac{\Delta^{**'}}{\Delta^{**}}=\frac{u_e'}{u_e}\cdot\frac{F_m}{2f},\qquad (5)$$

Where prime (′) denotes a derivation per the longitudinal variable s.

While obtaining the momentum equation, the common quantities are defined: conditional displacement thickness $\Delta^{*}(s)$, conditional momentum loss thickness $\Delta^{**}(s)$, conditional thickness $\Delta_1^{**}(s)$, nondimensional friction function $\zeta(s)$, form parameter $f(s)$, magnetic parameter $g(s)$ as well as a characteristic function of the boundary layer F_{m}. These, as well as other quantities of the boundary layer are defined using the expressions:

$$\Delta^{*}(s)=\int_0^\infty\left(\frac{\rho_e}{\rho}-\frac{u}{u_e}\right)\mathrm{d}z,\ \Delta^{**}(s)=\int_0^\infty\frac{u}{u_e}\cdot\left(1-\frac{u}{u_e}\right)\mathrm{d}z,\ \Delta_1^{*}(s)=\int_0^\infty\frac{\rho_e}{\rho}\cdot\left(1-\frac{u}{u_e}\right)\mathrm{d}z$$

$$\zeta=\left[\frac{\partial(u/u_e)}{\partial(z/\Delta^{**})}\right]_{z=0},\ H=\frac{\Delta^{*}}{\Delta^{**}},\ H_1=\frac{\Delta_1^{*}}{\Delta^{**}},\ Z^{**}=\frac{\Delta^{**2}}{\nu_0}$$

$$f(s)=\frac{u_e'\cdot\Delta^{**2}}{\nu_0}=u_e'\cdot Z^{**}=f_1(s);\quad g(s)=N_\sigma\cdot Z^{**}=g_1(s)\qquad (6)$$

$$N_\sigma=\frac{1}{(r/L)^{2j}}\cdot\frac{\rho_0\cdot\mu_0}{\rho_w\cdot\mu_w}\cdot\frac{\sigma\cdot B_m^2}{\rho_e},\ F_m=2\cdot\left[\zeta-(2+H)\cdot f\right]-2\cdot g\cdot H_1.$$

For j=0 all the expressions (4), (5) and (6) completely come down to the corresponding expressions for planar ionized gas flow in the boundary layer [7].

3 Transformations of the boundary layer equations
Transformacija jednadžba graničnog sloja

In order to apply the general similarity method, the friction function $\psi(s,z)$ is introduced using the relations

$$u=\frac{\partial\psi}{\partial z},\ \tilde{v}=\frac{1}{(r/L)^{2j}}\cdot\frac{\rho_0\cdot\mu_0}{\rho_w\cdot\mu_w}\cdot\left[u\cdot\frac{\partial z}{\partial x}+v\cdot\frac{\rho}{\rho_0}\cdot\left(\frac{r}{L}\right)^j\right]=-\frac{\partial\psi}{\partial s},\quad (j=1)\qquad (7)$$

that come from the continuity equation. For j=0, the relations (7) come down to the corresponding expressions used in e.g. papers [5, 7] for planar flow of dissociated i.e., ionized gas.

Applying (4) and (7), the governing equation system (3) is brought to this form:

$$\frac{\partial\psi}{\partial z}\cdot\frac{\partial^2\psi}{\partial s\partial z}-\frac{\partial\psi}{\partial s}\cdot\frac{\partial^2\psi}{\partial z^2}=\frac{\rho_e}{\rho}\cdot u_e\cdot\frac{\mathrm{d}u_e}{\mathrm{d}s}+\nu_0\cdot\frac{\partial}{\partial z}\cdot\left(Q\cdot\frac{\partial^2\psi}{\partial z^2}\right)+N_\sigma\cdot\frac{\rho_e}{\rho}\cdot\left(u_e-\frac{\partial\psi}{\partial z}\right)$$

$$\frac{\partial\psi}{\partial z}\cdot\frac{\partial h}{\partial s}-\frac{\partial\psi}{\partial s}\cdot\frac{\partial h}{\partial z}=-\frac{\rho_e}{\rho}\cdot u_e\cdot\frac{\mathrm{d}u_e}{\mathrm{d}s}\cdot\frac{\partial\psi}{\partial z}+\nu_0\cdot Q\cdot\left(\frac{\partial^2\psi}{\partial z^2}\right)^2+\nu_0\cdot\frac{\partial}{\partial z}\cdot\left(\frac{Q}{Pr}\cdot\frac{\partial h}{\partial z}\right)+N_\sigma\cdot\frac{\rho_e}{\rho}\cdot\frac{\partial\psi}{\partial z}\cdot\left(\frac{\partial\psi}{\partial z}-u_e\right)\qquad (8)$$

$$\psi=0,\ \frac{\partial\psi}{\partial z}=0,\quad h=h_w=\text{const.}\quad\text{for}\quad z=0$$

$$\frac{\partial\psi}{\partial z}\to u_e(s),\quad h\to h_e(s)\quad\text{for}\quad z\to\infty.$$

In the system (8), the nondimensional function Q is determined with the expression

$$Q=\frac{\rho\cdot\mu}{\rho_w\cdot\mu_w}=Q(s,z),\quad (Q=1\ \text{ for }\ z=0,\quad Q=\frac{\rho_e\cdot\mu_e}{\rho_w\cdot\mu_w}=Q(s)\ \text{ for }\ z\to\infty).\qquad (9)$$

The obtained equations (8) are formally the same as the corresponding equations in [7], while for j=0 they are completely the same.

4 Generalized equations of the boundary layer
Poopćene jednadžbe graničnog sloja

In order to apply Saljnikov's version of the generalized similarity method, the procedure already used for solution of boundary layer fluid flow problems was carried out. For that reason, a new change of variables is performed:

$$s = s,\ \eta(s,z) = \frac{u_e^{b/2}}{K(s)} z,\ \psi(s,z) = u_e^{1-b/2} \cdot K(s) \cdot \Phi(s,\eta)$$
$$h(s,z) = h_1 \cdot \bar{h} \cdot (s,\eta);\ \ K(s) = \left(a \cdot \nu_0 \cdot \int_0^s u_e^{b-1} \mathrm{d}s \right)^{1/2} \quad (10)$$
$$a, b = \text{const.} \quad h_1 = \text{const.};$$

here $\eta(s, z)$ - stands for a newly introduced transversal variable, $\Phi(s, \eta)$ - new stream function, $\bar{h}(s,\eta)$ - nondimensional enthalpy, h_1 - total enthalpy in the outer flow of the body of revolution within the fluid, while a, b - stand for arbitrary constants.

Based on (10), important quantities and characteristics of the boundary layer (6) can be expressed in the form of more suitable relations as:

$$\frac{u}{u_e} = \frac{\partial \Phi}{\partial \eta},\quad \Delta^{**}(s) = \frac{K(s)}{u_e^{b/2}} \cdot B(s)$$
$$B(s) = \int_0^\infty \frac{\partial \Phi}{\partial \eta} \cdot \left(1 - \frac{\partial \Phi}{\partial \eta} \right) \mathrm{d}\eta$$
$$\frac{\Delta^*(s)}{\Delta^{**}(s)} = H = \frac{A(s)}{B(s)}$$
$$\frac{\Delta_1^*(s)}{\Delta^{**}(s)} = H_1 = \frac{A_1(s)}{B(s)},\quad \zeta = B \cdot \left(\frac{\partial^2 \Phi}{\partial \eta^2} \right)_{\eta=0} \quad (11)$$
$$A(s) = \int_0^\infty \left(\frac{\rho_e}{\rho} - \frac{\partial \Phi}{\partial \eta} \right) \mathrm{d}\eta$$
$$A_1(s) = \int_0^\infty \frac{\rho_e}{\rho} \cdot \left(1 - \frac{\partial \Phi}{\partial \eta} \right) \mathrm{d}\eta;$$

where the quantities A, A_1 and B are considered to continual functions of the variable s.

The new transversal variable $\eta(s, z)$ and the stream function can also be written in a more suitable form as:

$$\eta(s,z) = \frac{B(s)}{\Delta^{**}(s)} z,\quad \psi(s,z) = \frac{u_e(s) \cdot \Delta^{**}(s)}{B(s)} \cdot \Phi(s,\eta). \quad (12)$$

Using the newly introduced variables (10), i.e. (12), the equation system (8) after a comprehensive transformation is brought to a form suitable for further analysis. Thus transformed equation system is written as:

$$\frac{\partial}{\partial \eta}\left(Q \cdot \frac{\partial^2 \Phi}{\partial \eta^2} \right) + \frac{a \cdot B^2 + (2-b) \cdot f}{2B^2} \cdot \Phi \cdot \frac{\partial^2 \Phi}{\partial \eta^2} + \frac{f}{B^2} \cdot \left[\frac{\rho_e}{\rho} - \left(\frac{\partial \Phi}{\partial \eta} \right)^2 \right] + \frac{g}{B^2} \frac{\rho_e}{\rho} \left(1 - \frac{\partial \Phi}{\partial \eta} \right) = \frac{u_e}{u_e'} \cdot \frac{f}{B^2} \cdot \left(\frac{\partial \Phi}{\partial \eta} \cdot \frac{\partial^2 \Phi}{\partial s \partial \eta} - \frac{\partial \Phi}{\partial s} \cdot \frac{\partial^2 \Phi}{\partial \eta^2} \right)$$
$$\frac{\partial}{\partial \eta}\left[\frac{Q}{\Pr} \cdot \frac{\partial \bar{h}}{\partial \eta} \right] + \frac{a \cdot B^2 + (2-b) \cdot f}{2B^2} \cdot \Phi \cdot \frac{\partial \bar{h}}{\partial \eta} - \frac{2\kappa \cdot f}{B^2} \cdot \frac{\rho_e}{\rho} \cdot \frac{\partial \Phi}{\partial \eta} + 2 \cdot \kappa \cdot Q \cdot \left(\frac{\partial^2 \Phi}{\partial \eta^2} \right)^2 + \frac{2\kappa \cdot g}{B^2} \cdot \frac{\rho_e}{\rho} \cdot \frac{\partial \Phi}{\partial \eta} \left(\frac{\partial \Phi}{\partial \eta} - 1 \right) =$$
$$= \frac{u_e}{u_e'} \cdot \frac{f}{B^2} \cdot \left(\frac{\partial \Phi}{\partial \eta} \cdot \frac{\partial \bar{h}}{\partial s} - \frac{\partial \Phi}{\partial s} \cdot \frac{\partial \bar{h}}{\partial \eta} \right) \quad (13)$$
$$\Phi(s,\eta) = 0,\ \frac{\partial \Phi}{\partial \eta} = 0,\ \bar{h} = \bar{h}_w \quad \text{for} \quad \eta = 0$$
$$\frac{\partial \Phi}{\partial \eta} \to 1,\ \bar{h} \to \bar{h}_e = 1 - \kappa \quad \text{for} \quad \eta \to \infty.$$

The quantity $\kappa = f_0 = u_e^2/2h_1$ appears in the energy equation of the system (13) and in the boundary conditions, and just like with dissociated gas [5], it is called the local compressibility parameter. This parameter represents an in advance given function of s.

Furthermore, both equations of the system (13) contain the quotient u_e/u_e' in the corresponding terms; hence, the solution of the system will depend on each concrete form of the given velocity $u_e(s)$ on the outer edge of the boundary layer. Therefore, the obtained system is not generalized in terms of Loitsianskii [1], and it is not possible to obtain the so-called generalized boundary layer equations using the functions $\Phi(s,\eta)$ and $\bar{h}(s,\eta)$.

In order to bring the system (8) to a generalized form, it is necessary, from the very beginning, to introduce the corresponding sets of parameters in the transformations (10), i.e., (12). Therefore, we introduce the stream function Φ and the nondimensional enthalpy $\bar{h}$ by the expressions:

$$\psi(s,z) = \frac{u_e \cdot \Delta^{**}}{B} \cdot \Phi(\eta,\ \kappa,\ f_1, f_2, f_3,\ ...,\ g_1,\ g_2,\ g_3,\ ...)$$
$$h(s,z) = h_1 \cdot \bar{h}(\eta,\ \kappa,\ f_1, f_2, f_3,\ ...,\ g_1,\ g_2,\ g_3,\ ...); \quad (14)$$

in which (f_k) stands for a set of the form parameters of Loitsianskii's type [1], while (g_k) denotes a set of magnetic parameters. In the literature, the relations as (14) are called "the general similarity transformations" and the sets of parameters are called "the similarity parameters". These parameters are independent variables (instead of the variable s).

In the studied flow case, the introduced sets of parameters are determined with the expressions:

$$\kappa = f_0(s) = \frac{u_e^2}{2h_1}, \quad f_k(s) = u_e^{k-1} \cdot u_e^{(k)} \cdot Z^{**^k}$$
$$g_k(s) = u_e^{k-1} \cdot N_\sigma^{(k-1)} \cdot Z^{**^k} \tag{15}$$

where they satisfy the following recurrent simple differential equations (16):

$$\frac{u_e}{u_e'} \cdot f_1 \cdot \frac{\mathrm{d}\kappa}{\mathrm{d}s} = 2\kappa \cdot f_1 = \theta_0$$
$$\frac{u_e}{u_e'} \cdot f_1 \cdot \frac{\mathrm{d}f_k}{\mathrm{d}s} = \left[(\text{k-1}) \cdot f_1 + k \cdot F_m\right] \cdot f_k + f_{k+1} = \theta_k \tag{16}$$
$$\frac{u_e}{u_e'} \cdot f_1 \cdot \frac{\mathrm{d}g_k}{\mathrm{d}s} = \left[(k-1) \cdot f_1 + k \cdot F_m\right] \cdot g_k + g_{k+1} = \gamma_k$$
$$(k = 1,\ 2,\ 3,\ ...).$$

Note that the introduced sets of parameters and equations (16) have formally the same structure as the corresponding expressions with incompressible fluid flow [1]. Based on (15), the first parameters of the sets (f_k) and (g_k), $(f_1 = u_e' \cdot Z^{**},\ g_1 = N_\sigma \cdot Z^{**})$ represent the already defined form parameter f and magnetic parameter g (6).

Having applied the similarity transformations (12) and (14), the boundary layer equation system (8) is finally transformed into:

$$\frac{\partial}{\partial\eta}\left(Q \cdot \frac{\partial^2\Phi}{\partial\eta^2}\right) + \frac{a \cdot B^2 + (2-b) \cdot f_1}{2B^2} \cdot \Phi \cdot \frac{\partial^2\Phi}{\partial\eta^2} + \frac{f_1}{B^2}\left[\frac{\rho_e}{\rho} - \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] + \frac{g_1}{B^2} \cdot \frac{\rho_e}{\rho} \cdot \left(1 - \frac{\partial\Phi}{\partial\eta}\right) =$$
$$= \frac{1}{B^2} \cdot \left[\sum_{k=0}^{\infty}\theta_k \cdot \left(\frac{\partial\Phi}{\partial\eta} \cdot \frac{\partial^2\Phi}{\partial\eta\partial f_k} - \frac{\partial\Phi}{\partial f_k} \cdot \frac{\partial^2\Phi}{\partial\eta^2}\right) + \sum_{k=1}^{\infty}\gamma_k \cdot \left(\frac{\partial\Phi}{\partial\eta} \cdot \frac{\partial^2\Phi}{\partial\eta\partial g_k} - \frac{\partial\Phi}{\partial g_k} \cdot \frac{\partial^2\Phi}{\partial\eta^2}\right)\right]$$
$$\frac{\partial}{\partial\eta}\left[\frac{Q}{\mathrm{Pr}} \cdot \frac{\partial\bar{h}}{\partial\eta}\right] + \frac{a \cdot B^2 + (2-b) \cdot f_1}{2B^2} \cdot \Phi \cdot \frac{\partial\bar{h}}{\partial\eta} - \frac{2\kappa f_1}{B^2} \cdot \frac{\rho_e}{\rho} \cdot \frac{\partial\Phi}{\partial\eta} + 2\kappa Q \cdot \left(\frac{\partial^2\Phi}{\partial\eta^2}\right)^2 - \frac{2\kappa g_1}{B^2} \cdot \frac{\rho_e}{\rho} \cdot \left(1 - \frac{\partial\Phi}{\partial\eta}\right) \cdot \frac{\partial\Phi}{\partial\eta} = \tag{17}$$
$$= \frac{1}{B^2} \cdot \left[\sum_{k=0}^{\infty}\theta_k \cdot \left(\frac{\partial\Phi}{\partial\eta} \cdot \frac{\partial\bar{h}}{\partial f_k} - \frac{\partial\Phi}{\partial f_k} \cdot \frac{\partial\bar{h}}{\partial\eta}\right) + \sum_{k=1}^{\infty}\gamma_k \cdot \left(\frac{\partial\Phi}{\partial\eta} \cdot \frac{\partial\bar{h}}{\partial g_k} - \frac{\partial\Phi}{\partial g_k} \cdot \frac{\partial\bar{h}}{\partial\eta}\right)\right]$$
$$\Phi = 0,\ \frac{\partial\Phi}{\partial\eta} = 0,\ \bar{h} = \bar{h}_w = \text{const.} \quad \text{for} \quad \eta = 0$$
$$\frac{\partial\Phi}{\partial\eta} \to 1,\ \bar{h} \to \bar{h}_e(s) = 1 - \kappa \quad \text{for} \quad \eta \to \infty\ .$$

Since in (17) the distribution of the outer velocity $u_e(s)$, does not figure explicitly, this system is generalized. The system of generalized equations (17) represents a general mathematical model of the ionized gas flow in the boundary layer on bodies of revolution. Once again note that the obtained system (17) has the same form as the equation system of the corresponding ionized gas planar flow [7]. For j=0, these equation systems are the same because in that case the transformations of the variables and parameters have the same form.

In three-parametric twice localized approximation ($\kappa = f_0 \neq 0$, $f_1 = f \neq 0$, $g_1 = g \neq 0$, $f_2 = f_3 = ... = g_2 = g_3 = ... = 0$ and $\partial/\partial\kappa = 0$, $\partial/\partial g = 0$) the obtained equation system is significantly simplified. In this approximation the generalized equation system (17) comes down to:

$$\frac{\partial}{\partial\eta}\left(Q \cdot \frac{\partial^2\Phi}{\partial\eta^2}\right) + \frac{a \cdot B^2 + (2-b) \cdot f_1}{2B^2} \cdot \Phi \cdot \frac{\partial^2\Phi}{\partial\eta^2} + \frac{f_1}{B^2} \cdot \left[\frac{\rho_e}{\rho} - \left(\frac{\partial\Phi}{\partial\eta}\right)^2\right] + \frac{g_1}{B^2} \cdot \frac{\rho_e}{\rho} \cdot \left(1 - \frac{\partial\Phi}{\partial\eta}\right) = \frac{F_m \cdot f_1}{B^2} \cdot \left(\frac{\partial\Phi}{\partial\eta} \cdot \frac{\partial^2\Phi}{\partial\eta\partial f_1} - \frac{\partial\Phi}{\partial f_1} \cdot \frac{\partial^2\Phi}{\partial\eta^2}\right)$$
$$\frac{\partial}{\partial\eta}\left[\frac{Q}{Pr} \cdot \frac{\partial\bar{h}}{\partial\eta}\right] + \frac{a \cdot B^2 + (2-b) \cdot f_1}{2B^2} \cdot \Phi \cdot \frac{\partial\bar{h}}{\partial\eta} - \frac{2\kappa \cdot f_1}{B^2} \cdot \frac{\rho_e}{\rho} \cdot \frac{\partial\Phi}{\partial\eta} + 2\kappa \cdot Q \cdot \left(\frac{\partial^2\Phi}{\partial\eta^2}\right)^2 - \frac{2\kappa \cdot g_1}{B^2} \cdot \frac{\rho_e}{\rho} \cdot \left(1 - \frac{\partial\Phi}{\partial\eta}\right) \cdot \frac{\partial\Phi}{\partial\eta} =$$
$$= \frac{F_m \cdot f_1}{B^2} \cdot \left(\frac{\partial\Phi}{\partial\eta} \cdot \frac{\partial\bar{h}}{\partial f_1} - \frac{\partial\Phi}{\partial f_1} \cdot \frac{\partial\bar{h}}{\partial\eta}\right) \tag{18}$$
$$\Phi = 0,\ \frac{\partial\Phi}{\partial\eta} = 0,\ \bar{h} = \bar{h}_w = \text{const.} \quad \text{for} \quad \eta = 0$$
$$\frac{\partial\Phi}{\partial\eta} \to 1,\ \bar{h} \to \bar{h}_e(s) = 1 - \kappa \quad \text{for} \quad \eta \to \infty.$$
$$(\Phi = \Phi^1 = \Phi^1(\eta,\ \kappa,\ f_1,\ g_1\ ), \quad \bar{h} = \bar{h}^1 = \bar{h}^1(\eta,\ \kappa,\ f_1,\ g_1));$$

where

$$F_m = 2\cdot\left[\zeta - \left(2+H\right)\cdot f_1\right] - 2g_1\cdot H_1.$$

The system of approximate generalized equations (18) represents a mathematical model of this problem of ionized gas flow in the boundary layer on bodies of revolution.

5 Numerical solution
Numeričko rješenje

For numerical solution of the system (18), the order of the dynamic equation is first decreased using the change

$$\frac{u}{u_e} = \frac{\partial \Phi}{\partial \eta} = \varphi(\eta,\kappa,f_1,g_1). \tag{19}$$

Taking into consideration the change (19), the equation system for numerical integration is finally brought to the form (20):

$$\begin{aligned}
&\frac{\partial}{\partial\eta}\left(Q\cdot\frac{\partial\varphi}{\partial\eta}\right)+\frac{a\cdot B^2+(2-b)\cdot f_1}{2B^2}\cdot\Phi\cdot\frac{\partial\varphi}{\partial\eta}+\frac{f_1}{B^2}\left[\frac{\rho_e}{\rho}-\varphi^2\right]+\frac{g_1}{B^2}\cdot\frac{\rho_e}{\rho}\cdot\left(1-\varphi\right)=\frac{F_m\cdot f_1}{B^2}\cdot\left(\varphi\cdot\frac{\partial\varphi}{\partial f_1}-\frac{\partial\Phi}{\partial f_1}\cdot\frac{\partial\varphi}{\partial\eta}\right)\\
&\frac{\partial}{\partial\eta}\left[\frac{Q}{\mathrm{Pr}}\cdot\frac{\partial\bar h}{\partial\eta}\right]+\frac{a\cdot B^2+(2-b)\cdot f_1}{2B^2}\cdot\Phi\cdot\frac{\partial\bar h}{\partial\eta}-\frac{2\kappa f_1}{B^2}\cdot\frac{\rho_e}{\rho}\cdot\varphi+2\kappa Q\cdot\left(\frac{\partial\varphi}{\partial\eta}\right)^2-\frac{2\kappa\cdot g_1}{B^2}\cdot\frac{\rho_e}{\rho}\cdot\left(1-\varphi\right)\cdot\varphi=\frac{F_m\cdot f_1}{B^2}\cdot\left(\varphi\cdot\frac{\partial\bar h}{\partial f_1}-\frac{\partial\Phi}{\partial f_1}\cdot\frac{\partial\bar h}{\partial\eta}\right)\\
&\varphi=0,\ \Phi=0,\ \bar h=\bar h_w=\text{const.}\ \text{ for }\ \eta=0\\
&\varphi\to1,\ \bar h\to\bar h_e=1-\kappa\quad\text{for}\quad\eta\to\infty.
\end{aligned} \tag{20}$$

Furthermore, by analogy with the dissociated gas [5], for the function Q and the density ratio ρ_e/ρ approximate values are adopted in the form of the expressions:

$$Q = Q(\bar h)\approx\left(\frac{\bar h_w}{\bar h}\right)^{1/3},\quad \frac{\rho_e}{\rho}\approx\frac{\bar h}{1-\kappa}. \tag{21}$$

Note that a detailed investigation with thermodynamic tables for dissociated i.e., ionized gas is needed for finding the exact laws on distributions of these quantities, which is not the objective of our investigation.

Numerical solution of the system of conjugated partial differential equations (20) is performed by means of finite differences method using the passage method. For that reason, the boundary layer area is replaced with a planar integration grid that contains cells with values $\Delta\eta$ = 0,05 and Δf_1= 0,001 [7, 17, 18]. Some derivatives in the equations (20) are replaced with the corresponding finite differences at discrete points of the grid. The values of the functions φ, Φ and $\bar h$ are calculated at these points of each calculating layer. The number of discrete points for each calculating layer is $M = N = 401$.

Due to localization per the compressibility and magnetic parameter, these parameters play the role of simple parameters. Therefore the equation system (20) is solved for each in advance given value of these parameters. Since Prandtl number practically does not depend on the temperature, the equations are solved for Pr = 0,712 (for air). The usual values are taken for the constants a and b [2]: a = 0,4408 and b = 5,7140.

For a concrete solution of the generalized equation system (20), i.e., the corresponding system of algebraic equations expressed by means of the finite differences method, a suitable program in FORTRAN programming language was written. It is based on the program used in the paper [2]. The accepted values for the characteristic functions B, Q and F_m at zero iteration are B^0_{K+1} = 0,469, $Q^0_{1,K+1}$ = 1 and $F^0_{m,K+1}$ = 0,4411. The equation system (20) is nonlinear, therefore it is solved using an iterative procedure.

6 Results
Rezultati

Numerical solution of the system of generalized equations (20) is obtained for each cross-section of the boundary layer (starting from the cross-section f_1 = 0,00) in the form of tables. Only some of the results in the form of diagrams are shown in this paper. The subscript 1 is omitted in the parameters f_1 and g_1.

Figure 2 shows the diagram of the nondimensional velocity $u/u_e = \partial\Phi/\partial\eta$ for three cross-sections of the boundary layer $f_1 = f$ = –0,18; f = 0,00 and f = 0,18), while the value of the compressibility parameter is $\kappa = f_0$ = 0,02. Figure 3 gives the diagram of the distribution of the nondimensional enthalpy $\bar h$ at these cross-sections, where the compressibility parameter is $\kappa = f_0$ = 0,03. Figure 4 represents the distribution of the nondimensional enthalpy $\bar h$ at one cross section of the boundary layer (f = 0,08) but for three different values of the porosity parameter. Figure 5 shows the distribution of the nondimensional friction function ζ (f) in the boundary layer for three values of the magnetic parameter g. Finally, Figure 6 gives the diagram of the distribution of the characteristic function $F_m(f)$ of the boundary layer.

7 Conclusions
Zaključci

The general similarity method was successfully applied to the ionized gas flow in the boundary layer. However, there are some difficulties in application of this method,

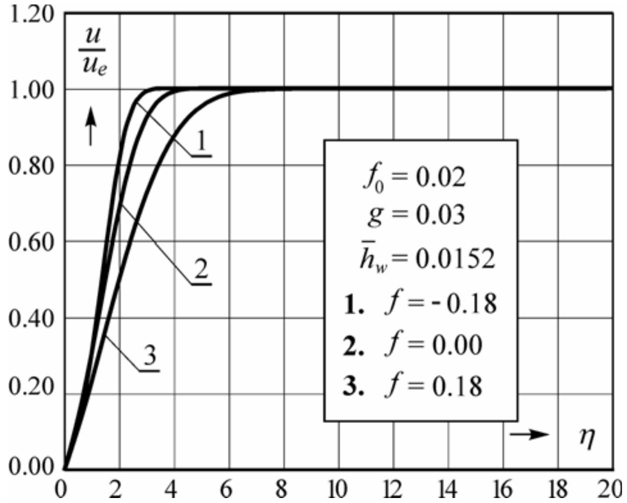

Figure 2 Diagram of the nondimensional velocity u/u_e
Slika 2. Dijagram bezdimenzijske brzine u/u_e

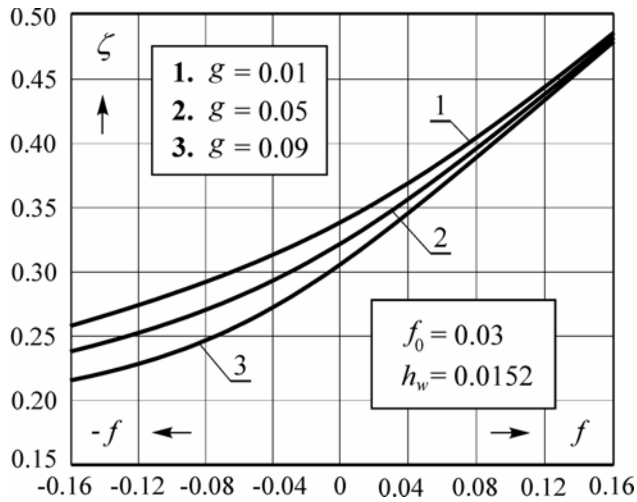

Figure 5 Distribution of the nondimensional friction function $\zeta(f)$
Slika 5. Raspodjela bezdimenzijske funkcije trenja $\zeta(f)$

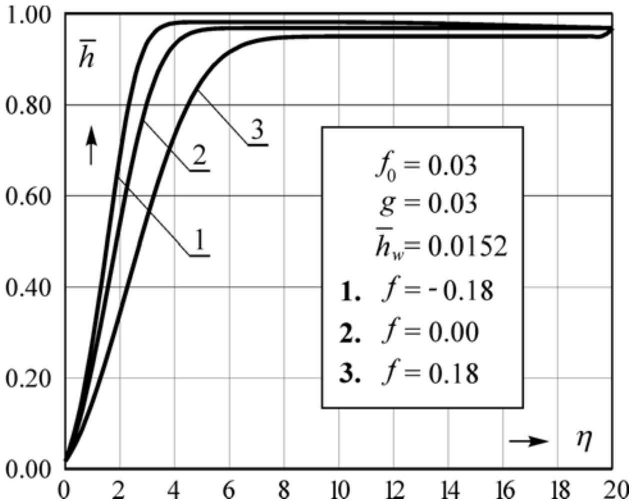

Figure 3 Diagram of the nondimensional enthalpy $\overline{h}$
Slika 3. Dijagram bezdimenzijske entalpije $\overline{h}$

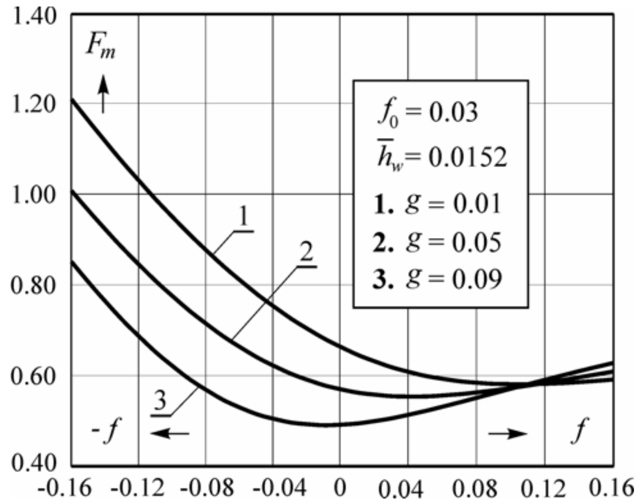

Figure 6 Distribution of the characteristic function $F_m(f)$
Slika 6. Raspodjela karakteristične funkcije $F_m(f)$

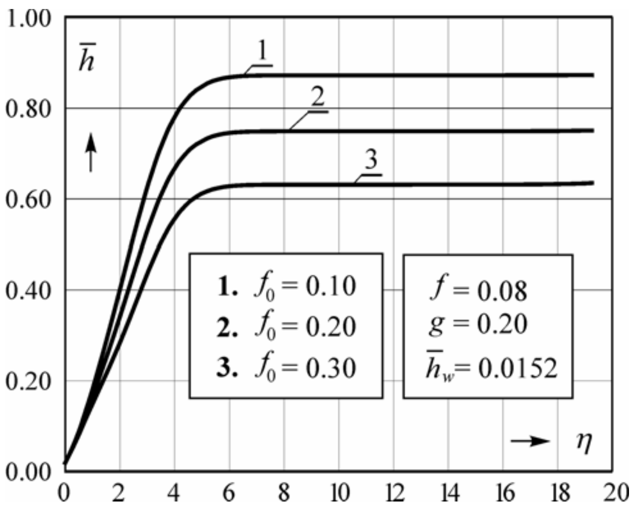

Figure 4 Diagram of the nondimensional enthalpy $\overline{h}$ for different values of the parameter $\kappa = f_0$
Slika 4. Dijagram bezdimenzijske entalpije $\overline{h}$ za razne vrijednosti parametra $\kappa = f_0$

mainly of mathematical nature.

Application of this method has given important quality results that illustrate the behaviour of the distributions of the physical and characteristic quantities at certain cross-sections of the boundary layer.

Note that the system of generalized equations (17) has the same form as the corresponding equation system for the planar ionized gas flow along a nonporous contour. It, obviously, enables the transformations (4) that contain the terms $(r/L)^2$ and r/L.

Based on the numerical results a general conclusion can be drawn: the distributions of the obtained solutions of the boundary layer equations have the same behaviour as with other problems of dissociated gas flow.

Based on the diagrams here presented and others not shown, the following concrete conclusions can be drawn:

- The nondimensional flow velocity u/u_e at different cross-sections of the boundary layer on bodies of revolution converges very fast towards unity (Fig. 2).
- The nondimensional enthalpy converges relatively fast towards the value $\overline{h} = \overline{h}_e = 1 - \kappa$ at the outer edge of the boundary layer (Fig. 3).
- The compressibility parameter has a significant influence on distribution of the nondimensional enthalpy at any cross-section of the boundary layer (Fig. 4).
- The diagram in Figure 5 shows that the magnetic parameter has a significant influence on the nondimensional friction function ζ, and hence on the boundary layer separation point. A decrease in the value of the magnetic parameter postpones the separation of the boundary layer.

- The magnetic parameter has a significant influence on the distribution of the characteristic function F_m of the ionized gas boundary layer on bodies of revolution (Fig. 6).

In order to obtain more exact quantitative solutions, the system (17) should be solved in three-parametric approximation but without localization per corresponding parameters, in particular without localization per the compressibility parameter. This is very important because the compressibility parameter has a great influence on the change of enthalpy in the boundary layer. Furthermore, it has been noted to change even the general character of the behaviour of the distribution of the enthalpy [5], which could be also expected in this case. However, solution of these equations without localization would be brought with both mathematical and software difficulties.

8
References
Literatura

[1] Loitsianskii, L. G. Liquid and gas mechanics. Nauka, Moscow, 1978.
[2] Saljnikov, V.; Dallmann, U. Verallgemeinerte Ähnlichkeitslösungen für dreidimensionale, laminare, stationäre, kompressible Grenzschichtströmungen an schiebenden profilierten Zylindern. Institut für Theoretische Strömungsmechanik, DLR-FB 89-34, Göttingen, 1989.
[3] Boričić, Z.; Nikodijević D.; Milenković, D. Unsteady MHD Boundary Layer on a Porous Surface. // Facta Universitatis, Series Mechanics, Automatic Control and Robotics, 1, 5 (1995), pp. 631-643.
[4] Pavlović, M. Temperature boundary layer on a rotating surface - the problem of the constant temperature wall. // Theoret. Appl. Mech. 33, 2 (2006), pp. 91-106.
[5] Krivtsova, N. V. Parameter method of solving of the laminar boundary layer equations with axial pressure gradient in the conditions of equilibrium dissociation of the gas. // Engineering-Physical Journal, 10 (1966), pp. 143-153.
[6] Krivtsova, N. V. Laminar boundary layer in the equilibrium dissociated gas at an arbitrary distribution of the outer velocity. // Liquid and gas mechanics, 10 (1966), pp. 106-112.
[7] Obrović, B.; Boundary layer of dissociated gas. Monograph, University of Kragujevac, Faculty of Mechanical Engineering, Kragujevac, 1994.
[8] Obrović, B.; Savić, S. Dissociated gas flow in the boundary layer in the case of a porous contour of the body within fluid. // Facta Universitatis, Series Mechanics, Automatic Control and Robotics 3, 15 (2003), pp. 989-1000.
[9] Savić, S.; Obrović, B. The influence of variation of electroconductivity on ionized gas flow in the boundary layer along a porous wall. // Theoret. Appl. Mech. 33, 2 (2006), pp. 149-179.
[10] Schlichting, H. Grenzschicht-theorie. Verlag G. Braun, Karlsruhe, 1974.
[11] Watanabe, T.; Pop, I. Thermal boundary layer in magnetohydrodynamic flow over a flat plate in the presence of a transverse magnetic field. // Acta Mech. 105 (1994), pp. 233–238.
[12] Loitsianskii, L. G. Laminar boundary layer. FML, Moscow, 1962.
[13] Dorrance, W. H. Viscous hypersonic flow. Theory of reacting and hypersonic boundary layers, Mir, Moscow, 1966.
[14] Anderson, John D. Jr. Hypersonic and high temperature gas dynamics. New York, St. Louis, San Francisco: McGraw-Hill Book Company, 1989.
[15] Kostić, S. T. Multiparametric boundary layer method on thin rotating bodies, Liquid and gas mechanics, 6 (1991), pp. 22-27.
[16] Obrović, B.; Savić, S. Momentum Equation of the Boundary Layer on Rotating Bodies for the Case of Dissociated Gas Flow Along a Porous Contour. // FME Transactions, 35, 2 (2007), pp. 63-68.
[17] Obrović, B.; Savić, S. Ionized gas boundary layer on a porous wall of the body within the electroconductive fluid. // Theoret. Appl. Mech. 31, 1 (2004), pp. 47-71.
[18] Obrović, B.; Nikodijević, D.; Savić, S. Boundary Layer of Dissociated Gas on Bodies of Revolution of a Porous Contour. // Strojniški vestnik - Journal of Mechanical Engineering, 55, 4 (2009), pp. 244-253.

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