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A RESULT IN ASYMPTOTIC ANALYSIS FOR THE FUNCTIONAL OF GINZBURG-LANDAU TYPE WITH EXTERNALLY IMPOSED MULTIPLE SMALL SCALES IN ONE DIMENSION

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Abstract. In this paper we present technical improvement of results in [19]. We study asymptotic behavior of the functional

$$\mathcal{J}^{\varepsilon}_{a,\beta,\gamma}(v) = \int_0^1 \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s, \varepsilon^{-\gamma}s)v^2(s)\Big)ds$$

as $\varepsilon \longrightarrow 0$, where a is 1×1-periodic. We determine (rescaled) minimal asymptotic energy associated to $\mathcal{J}^{\varepsilon}_{a,\beta,\gamma}$ as $\varepsilon \longrightarrow 0$ where $\beta, \gamma \geq 0$, $\beta+\gamma > 0$.

1. Introduction

We consider a variant of the energy in [1] which is perturbed by the highly oscillatory non-periodic term $a(\varepsilon^{-\beta}s, \varepsilon^{-\gamma}s)$, where $\beta, \gamma \geq 0$ are given parameters and $\beta \neq \gamma$. The functional $I^{\varepsilon}_{a,\beta}$ with periodic oscillatory term, studied in [19],

$$I^{\varepsilon}_{a,\beta}(v) := \int_0^1 \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s)v^2(s)\Big)ds, \tag{1.1}$$

is now replaced by

$$\mathcal{J}^{\varepsilon}_{a,\beta,\gamma}(v) := \int_0^1 \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s, \varepsilon^{-\gamma}s)v^2(s)\Big)ds, \tag{1.2}$$

where $v \in \mathrm{H}^2_{per}(\langle 0,1\rangle)$, $W \in \mathrm{C}(\mathbf{R}; [0,+\infty\rangle)$, $W(\xi) = 0$ if and only if $\xi \in \{-1,1\}$, W has superlinear growth in infinity, a is Carathéodory function on

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$\langle 0,1\rangle \times \langle 0,1\rangle$ which satisfies $a(\xi_1,\xi_2) \geq \alpha > 0$ (a.e. $(\xi_1,\xi_2) \in \langle 0,1\rangle \times \langle 0,1\rangle$), $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$. Typical choice for W is $W(\xi) := (\xi^2-1)^2$. In this paper we obtain formulas which show how rescaled energies associated to $\mathcal{J}^\varepsilon_{a,\beta,\gamma}$, namely,

$$\mathcal{E}^\varepsilon_a(\beta,\gamma) := \min_{v\in \mathrm{H}^2\langle 0,1\rangle} \varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a,\beta,\gamma}(v),\ \mathcal{E}^\varepsilon_{a,per}(\beta,\gamma) := \min_{v\in \mathrm{H}^2_{per}\langle 0,1\rangle} \varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a,\beta,\gamma}(v),$$

depend on a for various values of parameters $\beta,\gamma > 0$ as $\varepsilon \longrightarrow 0$. In particular, we generalize results in [19]. Organization of the paper is as follows: First, we fix the notation and quote some results which are the starting point for our considerations (section 2). Second, we consider the case $\gamma = 0$ (section 3). Finally, in section 4 we deal with the general case $\gamma > 0$. Due to highly technical nature of the proofs, we confine ourselves to presentation of the proofs in full detail only in the case $\beta \in \langle 0,1/3\rangle$ and $\gamma \in [0,1/3\rangle$. While in the case when $\beta > 1/3$ or $\gamma > 1/3$ proofs do not contain significant modifications in comparison to those already obtained in [19], the case $\beta = 1/3$ (or $\gamma = 1/3$) can be treated analogously as herein, with a few details more involved. The very basic result regarding oscillation on small scales is the well-known McShanne's Lemma:

LEMMA 1.1 (McShanne). *Consider Carathéodory function* $a \in \mathrm{L}^\infty_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$*,* $\beta,\gamma > 0$*,* $a^\varepsilon_0(s) := a(\varepsilon^{-\beta}s, s)$*,* $a^\varepsilon(s) := a(\varepsilon^{-\beta}s, \varepsilon^{-\gamma}s)$*,* $s \in \mathbf{R}$*. Then:*

- $a^\varepsilon_0 \xrightarrow{\ *\ } \overline{a}_0$ *in* $\mathrm{L}^\infty(\mathbf{R}^2)$*, where* $\overline{a}_0(s) := ⨍_0^1 a(\xi_1, s)d\xi_1$ *(a.e.* $s \in \mathbf{R}$*),*
- $a^\varepsilon \xrightarrow{\ *\ } \overline{a}$ *in* $\mathrm{L}^\infty(\mathbf{R}^2)$*, where* $\overline{a} := ⨍_0^1 ⨍_0^1 a(\xi_1,\xi_2)d\xi_1 d\xi_2$*.*

Functionals like (1.1) and (1.2) are examples of one-dimensional functionals of the Ginzburg-Landau type, which are common in modeling of physical systems where phase transition occurs. The literature on the subject is extensive. Here we only mention [1, 2] and [6–15]. Further list of references can be found in [1]. According to approach in [1], the relative impact of fine microstructures and small gradient perturbations can be captured by means of Γ-convergence of a family of suitably rescaled energies related to phase transition phenomena. Small parameter ε induces an internally created small scale which can be identified by approach in [1]. In a more general framework, we have to deal with mutually interacting and different small parameters. Due to competition of multiple small scales, tools like McShanne's Lemma above are not sufficient to capture actual asymptotic behavior of the system. In the case of functional (1.1) and (1.2) an interaction between internally created scale and externally imposed scales develops as $\varepsilon \longrightarrow 0$. Results related to functional (1.1) are obtained in [19]. In this paper we extend the analysis to the case of two different externally imposed small scales. In a number of other papers the authors were already considering the functionals of Ginzburg-Landau type with similar oscillation effect (for instance, see [4]).

An interested reader can find in [20] a more comprehensive list of references on multi-scale variational problems.

2. Some preliminaries

In this section we introduce the notation, and we quote some results which we will use in sections 3 and 4. Most of our notation is inherited from [1]: we work on the unit interval $\langle 0,1\rangle \subseteq \mathbf{R}$, but all the proofs can be carried out if we consider any bounded open interval $\Omega \subseteq \mathbf{R}$ endowed with Lebesgue measure (denoted by λ). As usual, $\mathrm{H}^2_{per}\langle 0,1\rangle$ denotes the set of all $\mathrm{H}^2_{loc}(\mathbf{R})$ functions, extended by periodicity out of $\langle 0,1\rangle$, while $\mathrm{C}^-\langle 0,1\rangle$ ($\mathrm{C}^+\langle 0,1\rangle$, resp.) denotes the set of all lower-semicontinuous (upper-semicontinuous, resp.) functions on $\langle 0,1\rangle$. As in [1], by Sx we denote a set of all discontinuities for some real function x, and by $|Sx|$ its cardinality. If $U \subseteq \mathbf{R}$ is open bounded interval, by $\mathcal{S}(U)$ we denote the set of all piecewise affine continuous functions $x : U \longrightarrow \mathbf{R}$ such that there holds $x'(\tau) \in \{-1,1\}$ (a.e. $\tau \in U$). By $b \otimes c$ we denote the tensor product of two real functions b and c, namely the mapping $(\xi_1,\xi_2) \mapsto b(\xi_1)c(\xi_2)$. If a is periodic function, $\overline{a}$ denotes average of a over its period. By $\lceil\sigma\rceil$ ($\lfloor\sigma\rfloor$, resp.) we denote the smallest integer greater or equal to $\sigma \in \mathbf{R}$ (the largest integer below $\sigma \in \mathbf{R}$, resp.). If $y \in K$, the L-periodic operator $\mathcal{P}_L : K \longrightarrow K$ is defined by $\mathcal{P}_L(y)(\tau) := y(\tau)$, if $\tau \in \langle -L, L\rangle$: otherwise $\mathcal{P}_L(y)$ is extended to $\mathbf{R}$ by L-periodicity.

Definition 2.1 (Γ-convergence). *Let X be a metric space. A sequence of functions $F^\varepsilon : X \longrightarrow [0,+\infty]$ Γ-converges to F on X, and we write $F^\varepsilon \xrightarrow{\Gamma} F$, if the following is fulfilled:*

(i) *Lower-bound inequality: for every $x \in X$ and a sequence (x^ε) in X such that $x^\varepsilon \longrightarrow x$ it holds $\liminf_\varepsilon F^\varepsilon(x^\varepsilon) \geq F(x)$.*
(ii) *Upper-bound inequality: For any y in X there exists a sequence (y^ε) in X such that $y^\varepsilon \longrightarrow y$ and $\limsup_\varepsilon F^\varepsilon(y^\varepsilon) \leq F(y)$.*

The proof of the following Proposition can be found in chapters 6 and 7 in [3]:

Proposition 2.2. *If $F^\varepsilon \xrightarrow{\Gamma} F$ and if the points x^ε minimize F^ε for every ε, then every cluster point x of the sequence (x^ε) minimizes F. In particular, there holds $\lim_{\varepsilon\longrightarrow 0} F^\varepsilon(x^\varepsilon) = F(x)$.*

If $\omega \subseteq \langle 0,1\rangle$, by χ^{per}_ω we denote 1-periodic extension to $\mathbf{R}$ of the characteristic function $\chi_\omega : \langle 0,1\rangle \longrightarrow \mathbf{R}$ defined by $\chi_\omega(s) := 1$ for $s \in \omega$, $\chi_\omega(s) := 0$ for $s \in \langle 0,1\rangle\backslash\omega$. We introduce the following abbreviations: $A_0 := 2\int_{-1}^1 \sqrt{W(\xi)}d\xi$, $C_0 := (3/4)^{2/3}$, $E_0 := C_0 A_0^{2/3}$. For a given bounded open interval $U \subseteq \mathbf{R}$ we also define $f^{\varepsilon,U}_s, f^U_s : \mathrm{L}^1(U) \longrightarrow [0,+\infty]$ by

$$f^{\varepsilon,U}_{s,a}(v) := \begin{cases} \fint_U \Big(\varepsilon^{2/3}v''^2 + \varepsilon^{-2/3}W(v') + a^\varepsilon_s v^2\Big), & \text{if } v \in \mathrm{H}^2(U), \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.1}$$

$$(2.2)\qquad f_{s,a}^{U}(x) := \begin{cases} \frac{A_0}{\lambda(U)}|S_U(x')| + a(s)\fint_U x^2, & \text{if } x \in \mathcal{S}(U), \\ +\infty, & \text{otherwise,} \end{cases}$$

where, for $U = \langle b_1, b_2\rangle$ we define $S_U(x') := Sx' \cap [b_1, b_2\rangle$ and $a_s^\varepsilon(\tau) := a(s + \varepsilon^{1/3-\beta}\tau)$, $\tau \in \mathbf{R}$. Then, by Proposition 3.4 in [1] we have $f_{s,a}^{\varepsilon,U} \xrightarrow{\Gamma} f_{s,a}^{U}$ on $\mathrm{L}^1(U)$ (a.e. $s \in \langle 0, 1\rangle$).

The asymptotic problem for the functional of Ginzburg-Landau type (1.1) was formulated in [1, p. 814]. Subsequently, it was studied in [19], where the following result was obtained:

Proposition 2.3. *Let*

$$\mathcal{E}_a(\beta) := \lim_{\varepsilon \longrightarrow 0} \min_{v\in \mathrm{H}^2\langle 0,1\rangle} \varepsilon^{-2/3} I_{a,\beta}^{\varepsilon}(v),\ \mathcal{E}_{a,per}(\beta) := \lim_{\varepsilon \longrightarrow 0} \min_{v\in \mathrm{H}^2_{per}\langle 0,1\rangle} \varepsilon^{-2/3} I_{a,\beta}^{\varepsilon}(v).$$

Then there holds:

$$(2.3)\qquad \mathcal{E}_a(\beta) = \mathcal{E}_{a,per}(\beta) = \begin{cases} E_0\overline{a^{1/3}}, & \text{if } \beta \in \langle 0, 1/3\rangle, \\ F_0(a), & \text{if } \beta = 1/3, \\ E_0\overline{a}^{1/3}, & \text{if } \beta > 1/3, \end{cases}$$

where $F_0(a) \approx E_0\overline{a^{1/3}}$ *when* $A_0 \approx 0$, $F_0(a) \approx E_0\overline{a}^{1/3}$ *when* $\frac{1}{A_0} \approx 0$.

As the following results show, we are able to compute rescaled asymptotic energy for more complex functionals. As an illustration for the situation where minimizers of the functional develop oscillations on multiple small scales, we are concerned with the generalization of the formula (2.3) to the case of functional (1.2). Our main result, Theorem 4.1, indeed proves that minimization problem associated to (1.2) is a multi-scale variational problem, although small scales of order ε^β and ε^γ are in fact externally triggered.

3. Case $\gamma = 0$

Consider the functional $\mathcal{J}_{a,\beta}^{\varepsilon} : \mathrm{H}^2\langle 0, 1\rangle \longrightarrow [0, +\infty\rangle$ defined by

$$(3.1)\qquad \mathcal{J}_{a,\beta}^{\varepsilon}(v) := \int_0^1 \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s, s)v^2(s)\Big)ds,$$

and associated energies

$$\mathcal{E}_{a,per}^{\varepsilon}(\beta, 0) := \min_{v\in \mathrm{H}^2_{per}\langle 0,1\rangle} \varepsilon^{-2/3}\mathcal{J}_{a,\beta}^{\varepsilon}(v), \quad \mathcal{E}_{a}^{\varepsilon}(\beta, 0) := \min_{v\in \mathrm{H}^2\langle 0,1\rangle} \varepsilon^{-2/3}\mathcal{J}_{a,\beta}^{\varepsilon}(v)$$

$$\mathcal{E}_{a,per}(\beta, 0) := \lim_{\varepsilon \longrightarrow 0} \mathcal{E}_{a,per}^{\varepsilon}(\beta, 0), \quad \mathcal{E}_{a}(\beta, 0) := \lim_{\varepsilon \longrightarrow 0} \mathcal{E}_{a}^{\varepsilon}(\beta, 0).$$

To begin with, we note that, bearing in mind results from [19], it is not difficult to check that the following holds:

THEOREM 3.1. *Let* $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ *satisfies* $a(\xi_1,\xi_2) \geq \alpha > 0$ *(a.e.* $(\xi_1,\xi) \in \langle 0,1\rangle \times \langle 0,1\rangle$*) and* $\beta \in \langle 1/3,+\infty\rangle$. *Then there holds*

$$\mathcal{E}_a(\beta,0) = \mathcal{E}_{a,per}(\beta,0) = E_0 \fint_0^1 \Big(\fint_0^1 a(\xi_1,\xi_2)d\xi_1\Big)^{1/3} d\xi_2. \tag{3.2}$$

If $\beta \in \langle 0,1/3\rangle$, we expect that there holds

$$\mathcal{E}_a(\beta,0) = \mathcal{E}_a(\beta,0) = E_0 \fint_0^1 \fint_0^1 a^{1/3}(\xi_1,\xi_2)d\xi_1 d\xi_2. \tag{3.3}$$

The proof of (3.3) requires some additional effort in comparison to the proof of (3.2). Indeed, $s \mapsto a(\varepsilon^{-\beta}s,s)$ no longer ε^{β}-periodic. Consequently, we can not compute minimal asymptotic energy associated to functional (1.2) as in [19] and a more careful comparison with minima of Γ-convergent functionals (2.1) is needed.

REMARK 3.2. In the case when function a equals $b \otimes c$, the condition $b \in \mathrm{L}^q_{per}\langle 0,1\rangle$, $c \in \mathrm{L}^p_{per}\langle 0,1\rangle$, where $1/p+1/q=1$, $p,q \in [1,+\infty]$, guarantees (by the Hölder inequality) integrability of the mapping $s \mapsto a(\varepsilon^{-\beta}s,s)$.

In results below we essentially require that $a = a(\xi_1,\xi_2)$ is a Carathéodory function, i.e., that $\xi_1 \mapsto a(\xi_1,\xi_2)$ is measurable for every $\xi_2 \in \mathbf{R}$ and that $\xi_2 \mapsto a(\xi_1,\xi_2)$ is continuous for almost every $\xi_1 \in \mathbf{R}$. We point out that the crucial ingredient in the proofs relies on some kind of "integer-property" of small parameter $\varepsilon > 0$. Roughly speaking, we show that arbitrary parameter $\varepsilon > 0$ can be changed in a satisfactory fashion so as to get new small parameter $\varepsilon_* > 0$ with the desired "integer-property". The proof of (3.3) is performed in several steps: in subsection 3.1 (subsection 3.2) we obtain the corresponding lower bound (upper bound, resp.) when a belongs to some natural classes of functions, and in subsection 3.3 we couple our results to get (3.3).

3.1. *Lower Bound.* First we deal with the lower bound associated to (3.3). Consider bounded open interval $\Omega \subseteq \mathbf{R}$. Set

$$J^{\varepsilon}_{\alpha,\omega}(w) = \int_{\omega} \Big(\varepsilon^{2-2\beta}w''^2(s) + W(w'(s)) + \alpha\varepsilon^{2\beta}w^2(s)\Big)ds, \tag{3.4}$$

$$J^{\varepsilon}_{a,\omega}(w) = \int_{\omega} \Big(\varepsilon^{2-2\beta}w''^2(s) + W(w'(s)) + a(s)\varepsilon^{2\beta}w^2(s)\Big)ds, \tag{3.5}$$

where $\omega \subseteq \Omega$ is measurable set, $\alpha > 0$, $a \in \mathrm{L}^1_{per}(\Omega)$. To begin with, we recall that there holds:

PROPOSITION 3.3. *Let* $\beta \in [0,1/3\rangle$. *If* $a^{\varepsilon} \in \mathrm{L}^1(\Omega)$ *satisfies* $a^{\varepsilon} \longrightarrow a$ *(a.e.* $s \in \Omega$*), where* $a \in \mathrm{L}^1(\Omega)$*, then there holds*

$$\liminf_{\varepsilon \longrightarrow 0} \min_{v \in \mathrm{H}^2(\Omega)} \varepsilon^{-2/3} J^{\varepsilon}_{a_{\varepsilon},\Omega}(v) \geq E_0 \int_{\Omega} a^{1/3}(\xi)d\xi. \tag{3.6}$$

Proof. Step 1. We assume that there exists $M > 0$ such that for every $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ there holds $\|a^\varepsilon\|_{L^\infty(\Omega)} \le M$. Hence $\|a\|_{L^\infty(\Omega)} \le M$. By the Egoroff theorem (cf. [5, p. 16]) for every $\eta \in \langle 0, 1 \rangle$ there exists a measurable set $\Omega_\eta \subseteq \Omega$ such that there holds $\lim_{\varepsilon \longrightarrow 0} \|a^\varepsilon - a\|_{L^\infty(\Omega_\eta)} = 0$. On the other hand there exists a sequence of simple functions (a_N),

$$a_N(s) = \sum_{m=1}^{N} \alpha_m^N \chi_{A_m^N}(s), \quad s \in \Omega,$$

such that $a \ge a_N$ for every $N \in \mathbf{N}$, $\lim_{N \longrightarrow +\infty} a_N = a$ almost everywhere. Consider $v_\varepsilon \in H^2(\Omega)$ such that

$$\inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a^\varepsilon}(v) = \varepsilon^{-2/3} J^\varepsilon_{a^\varepsilon}(v_\varepsilon).$$

Since $a^\varepsilon(s) \ge \alpha > 0$ (a.e $s \in \langle 0, 1 \rangle$), there exists $C = C(\alpha) > 0$ such that there holds $\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-2/3} \int_\Omega v_\varepsilon^2(s) ds \le C$. Thus, it results

$$\begin{aligned} \inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a^\varepsilon}(v) &\ge \varepsilon^{-2/3} J^\varepsilon_{a^\varepsilon, \Omega_\eta}(v_\varepsilon) \\ &\ge \inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a, \Omega_\eta}(v) - \|a^\varepsilon - a\|_{L^\infty(\Omega_\eta)} \varepsilon^{-2/3} \int_\Omega v_\varepsilon^2, \end{aligned}$$

$$\begin{aligned} \liminf_{\varepsilon \longrightarrow 0} \inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a^\varepsilon}(v) &\ge \liminf_{\varepsilon \longrightarrow 0} \inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a, \Omega_\eta}(v) \\ &- C \limsup_{\varepsilon \longrightarrow 0} \|a^\varepsilon - a\|_{L^\infty(\Omega_\eta)} \\ &= \liminf_{\varepsilon \longrightarrow 0} \inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a, \Omega_\eta}(v). \end{aligned}$$

Furthermore, there holds

$$\inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a, \Omega_\eta}(v) \ge \sum_{m=1}^{N} \inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{\alpha_m^N, A_m^n \cap \Omega_\eta}(v).$$

Therefore, by Corollary 5.7 in [16] we recover

$$\liminf_{\varepsilon \longrightarrow 0} \inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a, \Omega_\eta}(v) \ge \sum_{m=1}^{N} E_0 \int_{A_m^n \cap \Omega_\eta} (\alpha_m^N)^{1/3} ds = E_0 \int_{\Omega_\eta} a_N^{1/3}(s) ds.$$

By passing to the limit as $N \longrightarrow +\infty$, we obtain

$$\inf_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a, \Omega_\eta}(v) \ge E_0 \int_{\Omega_\eta} a^{1/3}(s) ds.$$

In effect, as $\eta \longrightarrow 0$, we get (3.6).

Step 2. Let $a \in L^1(\Omega)$. Set $a^\varepsilon_M(\xi) := \min\{a^\varepsilon(\xi), M\}$. Then $a^\varepsilon_M \longrightarrow a_M$ (a.e. $\xi \in \Omega$), where $a_M(\xi) := \min\{a(\xi), M\}$. By Step 1 there holds

$$\liminf_{\varepsilon \longrightarrow 0} \min_{v \in H^2(\Omega)} \varepsilon^{-2/3} J^\varepsilon_{a^\varepsilon_M, \Omega}(v) \ge E_0 \int_\Omega a_M^{1/3}(\xi) d\xi. \tag{3.7}$$

Finally, we pass to the limit as $M \longrightarrow +\infty$ by means of Fatou's Lemma to recover (3.6). ∎

In the first step, we prove the lower bound in the case when a is piecewise constant in ξ_2. The crucial feature of our proof is the fact that "pieces" of the domain where a takes constant values depend on ε.

PROPOSITION 3.4. *Let $\beta \in \langle 0, 1/3\rangle$. Consider $N \in \mathbf{N}$ and $\varepsilon \in \langle 0, \varepsilon_0(N)\rangle$. We define $\varepsilon_{N,**} := \lfloor \varepsilon^{-\beta} N^{-1} \rfloor^{-1/\beta}$, $\varepsilon_{**} := (\varepsilon_{N,**}^{-\beta} N)^{-1/\beta}$, $\rho_{\varepsilon,**} := \varepsilon_{**}^{\beta}\varepsilon^{-\beta} > 1$. Let $a^\varepsilon(\xi_1,\xi_2) = \sum_{k=1}^N a_k^\varepsilon(\xi_2)\chi_{\rho_{\varepsilon,**}^{-1} I_k}(\xi_1)$, $(\xi_1,\xi_2) \in \langle 0, \rho_{\varepsilon,**}^{-1}\rangle \times \langle 0,1\rangle$, where $I_k := \langle \frac{k-1}{N}, \frac{k}{N}\rangle$, $k = 1,\dots,N$. We set $a^\varepsilon(\xi_1,\xi_2) := 0$, $(\xi_1,\xi_2) \in \langle \rho_{\varepsilon,**}^{-1}, 1\rangle \times \langle 0,1\rangle$ and we extend a^ε by periodicity to $\mathbf{R}^2$. Let functions $a_k^\varepsilon \in \mathrm{L}^1_{per}\langle 0,1\rangle$ satisfy $a_k^\varepsilon(\xi_1) \longrightarrow a_k(\xi_1)$ as $\varepsilon \longrightarrow 0$ (a.e. $\xi_1 \in \langle 0,1\rangle$), $a_k(\xi_1) \geq \alpha > 0$ (a.e. $\xi_1 \in \langle 0,1\rangle$), $a_k \in \mathrm{L}^1_{per}\langle 0,1\rangle$. Then there holds*

$$\liminf_{\varepsilon\longrightarrow 0} \min_{v\in \mathrm{H}^2\langle 0,1\rangle} \varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a^\varepsilon,\beta}(v) \geq E_0 \int_0^1\!\!\!\!\!\!- \int_0^1\!\!\!\!\!\!- \; a^{1/3}(\xi_1,\xi_2)d\xi_1 d\xi_2, \tag{3.8}$$

where $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ is defined by $a(\xi_1,\xi_2) =: \sum_{k=1}^N a_k(\xi_1)\chi_{I_k}^{per}(\xi_2)$.

PROOF. We note that there holds

$$\begin{aligned}
&\varepsilon^{-2/3}\int_{\rho_{\varepsilon,**}^{-1} I_k} \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a_k^\varepsilon(\varepsilon^{-\beta}s)v^2(s)\Big)ds \\
&= \;\; \rho_{\varepsilon,**}\varepsilon^{-2/3}\int_{I_k}\Big(\varepsilon^2\rho_{\varepsilon,**}^2 \overline{v}''^2(\sigma) + W(\overline{v}'(\sigma)) + a_k^\varepsilon(\varepsilon_{**}^{-\beta}\sigma)\rho_{\varepsilon,**}^{-2}\overline{v}^2(\sigma)\Big)d\sigma.
\end{aligned}$$

Set $N_{**} := \varepsilon_{**}^{-\beta}$. We can write $I_k = \cup_{j=1}^{N_{**}}\langle \frac{k-1}{N} + \varepsilon_{**}^{\beta}\frac{j-1}{N}, \frac{k-1}{N} + \varepsilon_{**}^{\beta}\frac{j}{N}\rangle$. Consider $u_{**}(\sigma) := \varepsilon_{**}^{-\beta}u(\varepsilon_{**}^{\beta}\sigma)$. Consequently, since $\frac{\varepsilon_{**}^{-\beta}}{N} \in \mathbf{N}$, we get

$$\begin{aligned}
&\rho_{\varepsilon,**}^{-1}\varepsilon^{-2/3}\int_{I_k}\Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s)v^2(s)\Big)ds \\
&\geq \;\; \sum_{j=1}^{N_{**}} \varepsilon_{**}^{\beta} \min_{u_{**}\in \mathrm{H}^2\langle 0,\frac{1}{N}\rangle} \varepsilon^{-2/3}\int_0^{\frac{1}{N}}\Big(\varepsilon^{2-2\beta}u_{**}''^2 + W(u_{**}') + a_{j,k}^\varepsilon \varepsilon^{2\beta}u_{**}^2\Big),
\end{aligned}$$

where, for $j \in \mathbf{N}$, functions $a_{j,k}^\varepsilon : \mathbf{R} \longrightarrow \mathbf{R}$ are defined by

$$a_{j,k}^\varepsilon(\sigma) := \begin{cases} a_k^\varepsilon(\sigma + \frac{j-1}{N}), & \text{if } \sigma \in \langle 0, \frac{1}{N}\rangle \\ \text{by periodicity}, & \text{otherwise.}\end{cases} \tag{3.9}$$

At this point we note that the multi-set of functions $\{a_{j,k}^\varepsilon : j = 1,\dots,N_{**}\}$ (for fixed ε) contains at most N distinct functions. Indeed, by definition each of the functions $a_{1,k}^\varepsilon,\dots,a_{N,k}^\varepsilon$ appears exactly $N_{**}N^{-1} \in \mathbf{N}$ times in the mentioned multi-set. Thus, it results

$$\rho_{\varepsilon,**}^{-1}\varepsilon^{-2/3}\int_{I_k}\Big(\varepsilon^2 v''^2(s)+W(v'(s))+a(\varepsilon^{-\beta}s)v^2(s)\Big)ds$$
$$\geq \sum_{j=1}^{N}\frac{1}{N}\min_{u_{**}\in \mathrm{H}^2\langle 0,\frac{1}{N}\rangle}\varepsilon^{-2/3}\int_0^{\frac{1}{N}}\Big(\varepsilon^{2-2\beta}u_{**}''^2+W(u_{**}')+a_{j,k}^{\varepsilon}\varepsilon^{2\beta}u_{**}^2\Big).$$

By Proposition 5.9 in [16] we conclude that there holds

$$\liminf_{\varepsilon\longrightarrow 0}\min_{u_{**}\in \mathrm{H}^2\langle 0,\frac{1}{N}\rangle}\varepsilon^{-2/3}\int_0^{\frac{1}{N}}\Big(\varepsilon^{2-2\beta}u_{**}''^2+W(u_{**}')+a_{j,k}^{\varepsilon}\varepsilon^{2\beta}u_{**}^2\Big)$$
$$\geq E_0\int_0^{\frac{1}{N}}a_{j,k}^{1/3}(\sigma)d\sigma.$$

Set $A^\varepsilon(s):=a^\varepsilon(\varepsilon^{-\beta}s,s)$, $A_k^\varepsilon(s):=a_k^\varepsilon(\varepsilon^{-\beta}s)$. In effect, we have

$$\liminf_{\varepsilon\longrightarrow 0}\min_{v\in \mathrm{H}^2(\rho_{\varepsilon,**}^{-1}I_k)}\varepsilon^{-2/3}\int_{\rho_{\varepsilon,**}^{-1}I_k}\Big(\varepsilon^2 v''^2+W(v')+A_k^\varepsilon v^2\Big)$$
$$\geq \frac{1}{N}E_0 ⨍_0^1 a_k^{1/3}(\sigma)d\sigma.$$

At last, we compute

$$\liminf_{\varepsilon\longrightarrow 0}\min_{v\in \mathrm{H}^2\langle 0,1\rangle}\varepsilon^{-2/3}⨍_0^1\Big(\varepsilon^2 v''^2+W(v')+A^\varepsilon v^2\Big)$$
$$\geq \sum_{k=1}^{N}\liminf_{\varepsilon\longrightarrow 0}\min_{v\in \mathrm{H}^2(\rho_{\varepsilon,**}^{-1}I_k)}\varepsilon^{-2/3}\int_{\rho_{\varepsilon,**}^{-1}I_k}\Big(\varepsilon^2 v''^2+W(v')+A_k^\varepsilon v^2\Big)$$
$$\geq \sum_{k=1}^{N}\frac{1}{N}E_0 ⨍_0^1 a_k^{1/3}(\xi_1)d\xi_1=E_0 ⨍_0^1 ⨍_0^1 a^{1/3}(\xi_1,\xi_2)d\xi_1 d\xi_2.$$

∎

We can now address the case when a satisfy more general assumptions.

THEOREM 3.5. *Consider $a\in \mathrm{L}^1_{per}(\langle 0,1\rangle\times\langle 0,1\rangle)$ such that the mapping $\xi_2\mapsto a(\xi_1,\xi_2)$ is lower-semicontinuous for a.e. $\xi_1\in\langle 0,1\rangle$, $a(\xi_1,\xi_2)\geq\alpha>0$ (a.e. $(\xi_1,\xi_2)\in\langle 0,1\rangle\times\langle 0,1\rangle$). Then for $\beta\in\langle 0,1/3\rangle$ there holds*

$$\liminf_{\varepsilon\longrightarrow 0}\min_{v\in \mathrm{H}^2\langle 0,1\rangle}\varepsilon^{-2/3}\mathcal{J}_{a,\beta}^{\varepsilon}(v)\geq E_0 ⨍_0^1 ⨍_0^1 a^{1/3}(\xi_1,\xi_2)d\xi_1 d\xi_2, \tag{3.10}$$

PROOF. Set $b_k^\varepsilon(\xi_1):=\min_{\xi_2\in\rho_{\varepsilon,**}^{-1}I_k}a(\xi_1,\xi_2)$, $b_k(\xi_1):=\min_{\xi_2\in I_k}a(\xi_1,\xi_2)$, $a_N^\varepsilon(\xi_1,\xi_2):=\sum_{k=1}^N b_k^\varepsilon(\xi_1)\chi_{\rho_{\varepsilon,**}^{-1}I_k}(\xi_2)$, $a_N(\xi_1,\xi_2):=\sum_{k=1}^N b_k(\xi_1)\chi_{I_k}(\xi_2)$. Then there holds $a\geq a_N^\varepsilon$, $a\geq a_N$,

$$\lim_{\varepsilon\longrightarrow 0}a_N^\varepsilon(\xi_1,\xi_2)=a_N(\xi_1,\xi_2),\qquad \lim_{N\longrightarrow+\infty}a_N(\xi_1,\xi_2)=a(\xi_1,\xi_2).$$

By Proposition 3.4 we get

$$\begin{aligned}\liminf_{\varepsilon\longrightarrow 0}\min_{v\in \mathrm{H}^2\langle 0,1\rangle}\varepsilon^{-2/3}\mathcal{J}^{\varepsilon}_{a,\beta}(v) &\geq \liminf_{\varepsilon\longrightarrow 0}\min_{v\in \mathrm{H}^2\langle 0,1\rangle}\varepsilon^{-2/3}\mathcal{J}^{\varepsilon}_{a^{\varepsilon}_N,\beta}(v)\\ &\geq E_0\fint_0^1\fint_0^1 a_N^{1/3}(\xi_1,\xi_2)d\xi_1 d\xi_2 \qquad (3.11)\end{aligned}$$

To furnish the proof, we consider the limit as $N\longrightarrow+\infty$ in (3.11), which (by the dominated convergence theorem) yields (3.10). ∎

We immediately deduce:

COROLLARY 3.6. *Let* $a\in \mathrm{L}^p_{per}\langle 0,1\rangle\otimes\Big(\mathrm{C}^-\langle 0,1\rangle\cap \mathrm{L}^q_{per}\langle 0,1\rangle\Big)$, *where* $1/p+1/q=1$, $p,q\in[1,+\infty]$. *Then (3.10) holds.*

3.2. *Upper Bound.* It remains to establish the upper bound related to (3.3), namely

$$\limsup_{\varepsilon\longrightarrow 0}\min_{v\in \mathrm{H}^2_{per}\langle 0,1\rangle}\varepsilon^{-2/3}\mathcal{J}^{\varepsilon}_{a,\beta}(v)\leq E_0\fint_0^1\fint_0^1 a^{1/3}(\xi_1,\xi_2)d\xi_1 d\xi_2. \qquad (3.12)$$

The proof of (3.12) is more subtle than the proof of the lower bound obtained in Theorem 3.5. To begin with, we recall the following proposition:

PROPOSITION 3.7. *Consider open interval* $\omega\subset\mathbf{R}$ *and function* $c\in \mathrm{L}^1(\omega)$ *such that there holds* $c(s)\geq 0$ *(a.e.* $s\in\omega$*). Suppose that functions* $f^{\varepsilon,U_r}_{s,c}:=f^{\varepsilon}_{s,c}$ *(generated by* c *and* $U_r:=\langle -r,r\rangle$ *as in (2.1)) satisfy* $f^{\varepsilon}_{s,c}\xrightarrow{\Gamma}f_{s,c}$ *(where* $f_{s,c}:=f^{U_r}_{s,c}$*) on* $\mathrm{L}^1(U_r)$ *(a.e.* $s\in\omega$*). Then for every* $\eta>0$ *there exists* $\overline{M}_\eta>0$ *and a sequence of functions* $(\overline{v}^{\varepsilon}_*)$ *(which depends on* η*) such that* $\overline{v}^{\varepsilon}_*\in \mathrm{H}^2_{per}(U_r)$ *and with properties*

(3.13)

$$\limsup_{\varepsilon\longrightarrow 0}\int_\omega f^{\varepsilon}_{s,c}(R^{\varepsilon,*}_s\overline{v}^{\varepsilon}_*)ds\leq E_0\int_\omega c^{1/3}(s)ds+O(\overline{M}^2_\eta)\int_{F^M}c(s)ds+\eta\lambda(\omega),$$

where $F^M=\{s\in\omega: c(s)>M\}$,

$$|\overline{v}^{\varepsilon}_*(s)|\leq\overline{M}_\eta\varepsilon^{1/3-\beta},\quad s\in\omega. \qquad (3.14)$$

PROOF. See Proposition 4.11 and Theorem 4.13 in [16]. ∎

Next, we obtain the upper bound in the case when function $a=a(\xi_1,\xi_2)$ is piecewise constant in ξ_2:

PROPOSITION 3.8. *Consider a sequence of pairwise disjoint open intervals* (I_k), *such that* $\langle 0,1\rangle=\cup_{k=1}^{+\infty}I_k$. *Let* $a(\xi_1,\xi_2)=\sum_{k=1}^{+\infty}a_k(\xi_1)\chi^{per}_{I_k}(\xi_2)$. *If* $a_k\in \mathrm{L}^1_{per}\langle 0,1\rangle$ *satisfies* $a_k(\xi_1)\geq\alpha>0$ *(a.e.* $\xi_1\in\langle 0,1\rangle$*),* $k\in\mathbf{N}$, *then (3.12) holds.*

PROOF. First of all, notice that $\lambda_k := \lambda(I_k)$ can be assumed rational (by a standard density argument). We set $\lambda_k := \frac{p_k}{q_k}$, where $p_k, q_k \in \mathbf{N}$. We can also assume (without loss of generality) that there holds $p_k = 1$ for every $k \in \mathbf{N}$ (otherwise we divide each interval I_k into p_k pairwise disjoint intervals with measure $\frac{1}{q_k}$). Thus, without loss of generality $\lambda_k = \frac{1}{q_k}$. Consider arbitrary $\xi \in \langle 0, 1\rangle$, $\eta \in \langle 0, 1\rangle$ and $m \in \mathbf{N}$. In the following we often omit indexing of functions by ξ, η and m. Let $I_k := \langle t_{k-1}, t_k\rangle$. For simplicity we also assume that there holds $t_0 := 0$ (otherwise we relabel intervals I_k and functions a_k to make them well-ordered). Then for every $k \in \mathbf{N}$ we have $t_k = \sum_{i=1}^{k} \frac{1}{q_i}$. Set

$$E_k^m := I_k \backslash I_k^m, \quad I_k^m := \langle t_{k-1} + \frac{1}{2mq_k}, t_k - \frac{1}{2mq_k}\rangle, \ \ k \in \mathbf{N}.$$

Consider $\varepsilon_{k,m,*} := \lceil \varepsilon^{-\beta} m^{-1} \lambda_1 \cdots \lambda_k \rceil^{-1/\beta}$, $\varepsilon_{k,*} := (\varepsilon_{k,m,*}^{-\beta} m \lambda_1^{-1} \cdots \lambda_k^{-1})^{-1/\beta}$, $\rho_{\varepsilon,k,*} := \varepsilon_{k,*}^{\beta} \varepsilon^{-\beta} \in \langle 0, 1\rangle$, $N_{k,*} := \varepsilon_{k,*}^{-\beta}$. We define $a_{k,j} \in \mathrm{L}^1_{per}\langle 0, \frac{1}{q_k}\rangle$ by

$$a_{k,j}(\sigma) := \begin{cases} a_k(\sigma + (j-1)\lambda_k), & \text{if } \sigma \in I_k \\ \text{by periodicity}, & \text{otherwise}, \end{cases} \quad j \in \mathbf{N}, \ \ k \in \mathbf{N}. \tag{3.15}$$

We also define $f^{\varepsilon}_{s,k,j} := f^{\varepsilon}_{s,a_{k,j}}$. Since $f^{\varepsilon}_{s,k,j} \xrightarrow{\ \Gamma\ } f_{s,k,j}$ on $\mathrm{L}^1\langle -r, r\rangle$ (almost every $s \in \langle 0, \frac{1}{q_k}\rangle$), by Proposition 3.7 there exists a sequence $(\overline{v}^{\varepsilon}_{k,*,j})$ such that $\overline{v}^{\varepsilon}_{k,*,j} \in \mathrm{H}^2_{per}\langle 0, \frac{1}{q_k}\rangle$ and with properties

$$\limsup_{\varepsilon \longrightarrow 0} \int_0^{\frac{1}{q_k}} f^{\varepsilon}_{s,k,j}(R^{\varepsilon,*}_s \overline{v}^{\varepsilon}_{k,*,j}) ds \le \int_0^{\frac{1}{q_k}} a_{k,j}^{1/3} + O(\overline{M}_{\eta}^2) \int_{F^M_{k,j}} a_{k,j} + \frac{\eta}{q_k^2}, \tag{3.16}$$

$$\|\overline{v}^{\varepsilon}_{k,*,j}\|_{\mathrm{L}^\infty(\mathbf{R})} \le \overline{M}_{\eta} \varepsilon^{1/3-\beta}, \tag{3.17}$$

where $F^M_{k,j} = \{s \in I_k : a_{k,j}(\sigma) > M\}$. Consider $\overline{v}^{\varepsilon}_{k,*} \in \mathrm{H}^2_{per}\langle 0, 1\rangle$ defined by $\overline{v}^{\varepsilon}_{k,*}(s) := \overline{v}^{\varepsilon}_{k,*,j}(s)$, $s \in \langle \frac{j-1}{q_k}, \frac{j}{q_k}\rangle$, $j = 1, \ldots, q_k$. Set $\overline{v}^{\varepsilon}_k(s) := \varepsilon^{\beta}_{k,*} \overline{v}^{\varepsilon}_{k,*}(\varepsilon^{-\beta}_{k,*} s)$, $v^{\varepsilon}_k(s) := \rho^{-1}_{\varepsilon,k,*} \overline{v}^{\varepsilon}_k(\rho_{\varepsilon,k,*} s)$, $s \in \mathbf{R}$. Then $\overline{v}^{\varepsilon}_k \in \mathrm{H}^2_{per}\langle 0, \varepsilon^{\beta}_{k,*}\rangle$, $v^{\varepsilon}_k \in \mathrm{H}^2_{per}\langle 0, \varepsilon^{\beta}\rangle$. We consider the sequence

$$w^{\varepsilon}(s) := w^{\varepsilon}_k(s), \quad s \in I_k, \quad k \in \mathbf{N}, \tag{3.18}$$

where $w^{\varepsilon}_k : I_k \longrightarrow \mathbf{R}$ is defined by

$$w^{\varepsilon}_k(s) := \begin{cases} v^{\varepsilon}_k(s), & \text{if } s \in I^m_k, \\ \tilde{v}^{\varepsilon}(s), & \text{if } s \in I_k \backslash I^m_k, \end{cases} \tag{3.19}$$

where $\tilde{v}^{\varepsilon}_k : I_k \backslash I^m_k \longrightarrow \mathbf{R}$ is chosen in such a way that $w^{\varepsilon}_k \in \mathrm{H}^2_{per}\langle 0, \frac{1}{q_k}\rangle$, $w^{\varepsilon} \in \mathrm{H}^2_{per}\langle 0, 1\rangle$ and for every $k \in \mathbf{N}$ $\tilde{v}^{\varepsilon}_k$ on its domain has the following properties: derivative of $\tilde{v}^{\varepsilon}_k$ tales alternately the values 1 and -1 on consecutive intervals of order $\varepsilon^{1/3}$ (except the first and the last one, which have length of order $\overline{M}_{\eta}\varepsilon^{1/3}$), apart from transition layers of order ε at the end of each such interval, where the second derivative is of order ε^{-1}. The value of w^{ε}_k is of

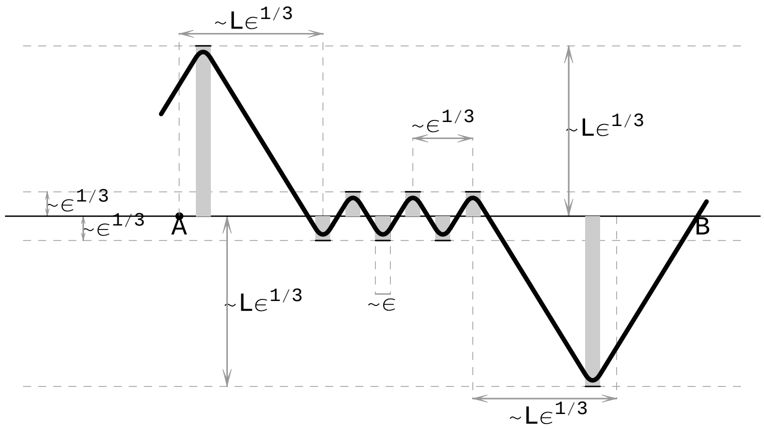

FIGURE 1. Construction of $\tilde{v}_k^\varepsilon$ on $\langle A,B\rangle$, where $A:=t_{k-1}$, $B:=t_{k-1}+\frac{1}{2mq_k}$, $L:=\overline{M}_\eta$.

order $\varepsilon^{1/3}$ (except in the first and the last interval, where it is of order $\overline{M}_\eta\varepsilon^{1/3}$ (cf. Figure 1)). In particular, by Proposition 3.7 and the construction above there holds $\|w_k^\varepsilon\|_{\mathrm{L}^\infty(\mathbf{R})}\le\overline{M}_\eta\varepsilon^{1/3}$. Note that we can write $I_k=\cup_{j=1}^{N_{k,*}}[t_{k-1}+\frac{j-1}{q_k}\varepsilon_{*,k}^\beta, t_{k-1}+\frac{j}{q_k}\varepsilon_{*,k}^\beta\rangle$. Moreover, we have $t_k=\sum_{i=1}^k\frac{1}{q_i}$ and $\varepsilon_{k,*}^{-\beta}t_{k-1}\in\mathbf{N}$. Since there exists $\varepsilon_0(m)>0$ such that for every $\varepsilon\in\langle 0,\varepsilon_0(m)]$ there holds $I_k^m\subset\rho_{\varepsilon,k,*}^{-1}I_k$, it results

$$
\begin{aligned}
&\varepsilon^{-2/3}\int_{I_k}\Big(\varepsilon^2(w_k^\varepsilon)''^2(s)+W((w_k^\varepsilon)'(s))+a_k(\varepsilon^{-\beta}s)(w_k^\varepsilon)^2(s)\Big)ds\\
\le\;&\varepsilon^{-2/3}\int_{\rho_{\varepsilon,k,*}I_k}\Big(\varepsilon^2(v_k^\varepsilon)''^2(s)+W((v_k^\varepsilon)'(s))+a_k(\varepsilon^{-\beta}s)(v_k^\varepsilon)^2(s)\Big)ds\\
&+\varepsilon^{-2/3}\int_{I_k\setminus I_k^m}\Big(\varepsilon^2(w_k^\varepsilon)''^2(s)+W((w_k^\varepsilon)'(s))+a_k(\varepsilon^{-\beta}s)(w_k^\varepsilon)^2(s)\Big)ds.
\end{aligned}
$$

For $k\in\mathbf{N}$ we calculate

$$
\begin{aligned}
&\rho_{\varepsilon,k,*}^{-1}\varepsilon^{-2/3}\int_{\rho_{\varepsilon,k,*}I_k}\Big(\varepsilon^2(v_k^\varepsilon)''^2(s)+W((v_k^\varepsilon)'(s))+a_k(\varepsilon^{-\beta}s)(v_k^\varepsilon)^2(s)\Big)ds\\
=\;&\varepsilon^{-2/3}\int_{I_k}\Big(\varepsilon^2\rho_{\varepsilon,k,*}^2(\overline{v}_k^\varepsilon)''^2(s)+W((\overline{v}_k^\varepsilon)'(s))+a_k(\varepsilon_{k,*}^{-\beta}s)\rho_{\varepsilon,k,*}^{-2}(\overline{v}_k^\varepsilon)^2(s)\Big)ds\\
=\;&\sum_{j=1}^{N_{k,*}}\varepsilon_{k,*}^\beta\varepsilon^{-2/3}\int_0^{\frac{1}{q_k}}\Big(\varepsilon^{2-2\beta}(\overline{v}_{k,*,j}^\varepsilon)''^2+W((\overline{v}_{k,*,j}^\varepsilon)')+a_{j,k}\varepsilon^{2\beta}(\overline{v}_{k,*,j}^\varepsilon)^2\Big).
\end{aligned}
$$

In particular, there are at most $\lambda_k^{-1} \in \mathbf{N}$ distinct λ_k-periodic functions $a_{k,j} : \mathbf{R} \longrightarrow \mathbf{R}$, $j = 1, \dots, N_{k,*}$. Furthermore, since by construction for every $k \in \mathbf{N}$ there exists at most λ_k^{-1} distinct λ_k-periodic functions $\overline{v}^{\varepsilon}_{k,*,j}$, for $j = 1, \dots, N_{k,*}$ by (3.18) there are at most λ_k^{-1} distinct values of the integral

$$\int_0^{\frac{1}{q_k}} \Big(\varepsilon^{2-2\beta}(\overline{v}^{\varepsilon}_{k,*,j})''^2 + W((\overline{v}^{\varepsilon}_{k,*,j})') + a_{j,k}\varepsilon^{2\beta}(\overline{v}^{\varepsilon}_{k,*,j})^2\Big).$$

Thus we infer:

$$\begin{aligned}
&\sum_{j=1}^{N_{k,*}} \varepsilon_*^{\beta}\varepsilon^{-2/3} \int_0^{\frac{1}{q_k}} \Big(\varepsilon^{2-2\beta}(\overline{v}^{\varepsilon})''^2_{k,*,j} + W((\overline{v}^{\varepsilon}_{k,*,j})') + a_{k,j}\varepsilon^{2\beta}(\overline{v}^{\varepsilon})^2_{k,*,j}\Big) \\
&= \sum_{j=1}^{q_k} \frac{1}{q_k}\varepsilon^{-2/3} \int_0^{\frac{1}{q_k}} \Big(\varepsilon^{2-2\beta}(\overline{v}^{\varepsilon})''^2_{k,*,j} + W((\overline{v}^{\varepsilon}_{k,*,j})') + a_{k,j}\varepsilon^{2\beta}(\overline{v}^{\varepsilon})^2_{k,*,j}\Big) \\
&= \sum_{j=1}^{q_k} \frac{1}{q_k} \int_0^{\frac{1}{q_k}} f^{\varepsilon}_{s,k,j}(R^{\varepsilon,*}_s \overline{v}^{\varepsilon}_{k,*,j})ds.
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
\varepsilon^{-2/3}\mathcal{J}^{\varepsilon}_{a,\beta}(w^{\varepsilon}) \le{}& \sum_{k=1}^{+\infty}\sum_{j=1}^{q_k} \frac{\rho_{\varepsilon,k,*}}{q_k} \int_0^{\frac{1}{q_k}} f^{\varepsilon}_{s,k,j}(R^{\varepsilon,*}_s \overline{v}^{\varepsilon}_{k,*,j})ds + \sum_{k=1}^{+\infty} e_{m,k}(w^{\varepsilon}_k) \\
&+ \sum_{k=1}^{+\infty} \varepsilon^{-2/3} \int_{I_k \setminus I_k^m} \Big(\varepsilon^2 (w^{\varepsilon}_k)''^2 + W((w^{\varepsilon}_k)')\Big)ds,
\end{aligned}$$

where $e_{m,k}(w^{\varepsilon}_k) := \varepsilon^{-2/3}\int_{I_k \setminus I_k^m} a_k(\varepsilon^{-\beta}s)(w^{\varepsilon}_k)^2(s)ds$. Set $E^m_k = E^m_{k,1} \cup E^m_{k,2}$, $E^m_{k,1} := [t_{k-1}, t_{k-1} + \frac{1}{2mq_k}]$, $E^m_{k,2} := [t_k - \frac{1}{2mq_k}, t_k\rangle$. Then there exists $\varepsilon_1(m) \ge \varepsilon_0(m)$ such that for every $\varepsilon \in \langle 0, \varepsilon_1(m)]$ there holds $E^m_{k,l} \subseteq \rho_{\varepsilon,k,*}\tilde{E}^m_{k,l}$, $l = 1, 2$, where $\tilde{E}^m_{k,1} := [t_{k-1}, t_{k-1} + \frac{1}{mq_k}]$, $\tilde{E}^m_{k,2} := [t_k - \frac{1}{mq_k}, t_k + \frac{1}{mq_k}\rangle$. Set $\tilde{E}^m_k := \tilde{E}^m_{k,1} \cup \tilde{E}^m_{k,2}$, $A^{\varepsilon}_k(s) := a_k(\varepsilon^{-\beta}s)$ $A^{\varepsilon,*}_k(s) := a_k(\varepsilon^{-\beta}_{k,*}s)$. Then $e_{m,k}(w^{\varepsilon}_k) \le \tilde{e}_{m,k}(w^{\varepsilon}_k)$, where $\tilde{e}_{m,k}(w^{\varepsilon}_k) := \varepsilon^{-2/3}\int_{\rho_{\varepsilon,k,*}\tilde{E}^m_k} A^{\varepsilon}_k (w^{\varepsilon}_k)^2$. Since $\frac{N_{k,*}}{m} \in \mathbf{N}$, we have $\tilde{E}^m_{k,1} = \cup_{j=1}^{\frac{N_*}{m}} [t_{k-1} + \frac{j-1}{q_k}\varepsilon^{\beta}_{k,*}, t_{k-1} + \frac{j}{q_k}\varepsilon^{\beta}_{k,*}\rangle$, and therefore

$$\begin{aligned}
\varepsilon^{-2/3}\int_{\rho_{\varepsilon,*}\tilde{E}^m_{k,1}} A^{\varepsilon}_k (w^{\varepsilon}_k)^2 &\le \rho_{\varepsilon,k,*}\varepsilon^{-2/3}\int_{\tilde{E}^m_{k,1}} A^{\varepsilon,*}_k \rho^{-2}_{\varepsilon,k,*}\overline{M}^2_{\eta}\varepsilon^{2/3} \\
&= \sum_{j=1}^{\frac{N_{k,*}}{m}} \rho^{-1}_{\varepsilon,k,*}\varepsilon^{\beta}_{k,*}\int_0^{\frac{1}{q_k}} \overline{M}^2_{\eta} a_{k,j}(\sigma)d\sigma.
\end{aligned}$$

Similarly as before, since $\frac{N_{k,*}}{mq_k} \in \mathbf{N}$, we conclude that in the sum above there exists at most q_k distinct integrals $\int_0^{\frac{1}{q_k}} a_{k,j}(\sigma)d\sigma$. Thus, it results

$$\begin{aligned}\varepsilon^{-2/3}\int_{\rho_{\varepsilon,k,*}\tilde{E}^m_{k,1}} a_k(\varepsilon^{-\beta}s)(w^\varepsilon_k)^2(s)ds &\leq \frac{1}{m}\sum_{j=1}^{q_k}\rho^{-1}_{\varepsilon,k,*}\frac{1}{q_k}\int_0^{\frac{1}{q_k}}\overline{M}^2_\eta a_{k,j}(\sigma)d\sigma \\ &\leq 2\overline{M}^2_\eta\frac{1}{mq_k}\|a_k\|_{\mathrm{L}^1\langle 0,1\rangle}.\end{aligned}$$

It is easy to verify that similar estimates hold on $\rho_{\varepsilon,k,*}\tilde{E}^m_{k,2}$. On the other hand, it can be checked that for every $k \in \mathbf{N}$ and $m \in \mathbf{N}$ there holds

$$\varepsilon^{-2/3}\int_{E^m_k}\Big(\varepsilon^2(w^\varepsilon_k)''^2(\sigma) + W((w^\varepsilon_k)'(\sigma))\Big)d\sigma = O(1)\varepsilon^{1/3}.$$

Hence, a careful application of the dominated convergence theorem yields

$$\begin{aligned}\limsup_{\varepsilon\longrightarrow 0}\varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a,\beta}(w^\varepsilon) &\leq \limsup_{\varepsilon\longrightarrow 0}\sum_{k=1}^{+\infty}\sum_{j=1}^{q_k}\frac{\rho_{\varepsilon,k,*}}{q_k}\int_0^{\frac{1}{q_k}} f^\varepsilon_{s,k,j}(R^{\varepsilon,*}_s\overline{v}^\varepsilon_{k,*,j})ds \\ &\quad+\sum_{k=1}^{+\infty}O(\overline{M}^2_\eta)\frac{1}{mq_k}\|a_k\|_{\mathrm{L}^1\langle 0,1\rangle} \\ &\leq \sum_{k=1}^{+\infty}\sum_{j=1}^{q_k}\left(E_0\frac{1}{q_k}\int_0^{\frac{1}{q_k}} a^{1/3}_{k,j} + O(\overline{M}^2_\eta)\frac{1}{q_k}\int_{F^M_{k,j}} a_{k,j}\right) \\ &\quad+O(\overline{M}^2_\eta)\|a\|_{\mathrm{L}^1(\langle 0,1\rangle\times\langle 0,1\rangle)}\frac{1}{m}+\eta.\end{aligned}$$

By passing to the limit as $M \longrightarrow +\infty$ and $m \longrightarrow +\infty$, we obtain

$$\begin{aligned}\limsup_{\varepsilon\longrightarrow 0}\min_{w\in\mathrm{H}^2_{per}\langle 0,1\rangle}\varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a,\beta}(w) &\leq \sum_{k=1}^{+\infty}\lambda(I_k)E_0 ⨍_0^1 a^{1/3}_k(\xi_1)d\xi_1 + \eta \\ &= E_0 ⨍_0^1 ⨍_0^1 a^{1/3}(\xi_1,\xi_2)d\xi_1 d\xi_2 + \eta.\end{aligned}$$

Arbitrariness of $\eta > 0$ completes the proof. □

Now we can derive the following:

Corollary 3.9. *Consider $p \in [1,+\infty]$. Let $a \in \mathrm{L}^p_{per}\langle 0,1\rangle \otimes \mathrm{L}^q_{per}\langle 0,1\rangle$, where $1/p+1/q=1$. Then upper bound (3.12) holds.*

Proof. Step 1. First we consider the case $p \in \langle 1,+\infty]$. Then $q \in [1,+\infty\rangle$. Set $a = b\otimes c$. Let $\kappa > 0$ be given. Since $C\langle 0,1\rangle \cap \mathrm{L}^\infty\langle 0,1\rangle$ is strongly dense in $\mathrm{L}^q\langle 0,1\rangle$, there exists $c^\kappa \in \mathrm{C}\langle 0,1\rangle \cap \mathrm{L}^\infty\langle 0,1\rangle$ such that $\|c-c^\kappa\|_{\mathrm{L}^q\langle 0,1\rangle} \leq \kappa$. Moreover, there exists a sequence of piecewise constant functions (c^κ_n), $c^\kappa_n(\xi_2) = \sum_{k=1}^{N^\kappa_n} c_{n,\kappa,k}\chi^{per}_{I^n_k}(\xi_2)$, $\xi_2 \in \mathbf{R}$, $I^n_k := \langle t^n_k, t^n_{k+1}\rangle$, $t^n_0 :=$

0, $t_k^n := t_0^n + k/N_n^\kappa$, $k = 1, \ldots, N_n^\kappa$, such that for every $\xi_2 \in \langle 0,1\rangle$ there holds $c_n^\kappa(\xi_2) \searrow c^\kappa(\xi_2)$ as $m \longrightarrow +\infty$. We define $a_{n,\kappa} := b \otimes c_n^\kappa$, $a_\kappa := b \otimes c^\kappa$. Furthermore, by considering $\varepsilon_{N_n^\kappa} := \lceil \varepsilon^{-\beta} \cdot (N_n^\kappa)^{-1} \cdot m^{-1} \rceil^{-1/\beta}$, $\varepsilon_* := (\varepsilon_{N_n^\kappa,*}^{-\beta} m N_n^\kappa)^{-1/\beta}$, $\rho_{\varepsilon,*} := \varepsilon_*^\beta \varepsilon^{-\beta} \in \langle 0,1\rangle$, we infer (quite in the same way as in the proof of Proposition 3.8) that for every $\eta \in \langle 0,1\rangle$, $M > 0$ and $m \in \mathbf{N}$, there exists $w^\varepsilon \in \mathrm{H}^2_{per}\langle 0,1\rangle$ (which depends on η, $M > 0$, $m \in \mathbf{N}$ and N_n^κ) and $\overline{M}_\eta > 0$ such that $\|w^\varepsilon\|_{\mathrm{L}^\infty(\mathbf{R})} \le \overline{M}_\eta \varepsilon^{1/3}$, and such that the following estimates hold:

$$\begin{aligned}\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-2/3} \mathcal{J}^\varepsilon_{a,\beta}(w^\varepsilon) &\le \limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-2/3} \mathcal{J}^\varepsilon_{a_n^\kappa,\beta}(w^\varepsilon) \\ &\quad + 2\overline{M}_\eta^2 \|c - c^\kappa\|_{\mathrm{L}^q\langle 0,1\rangle} \|b\|_{\mathrm{L}^p\langle 0,1\rangle},\end{aligned}$$

$$\begin{aligned}&\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-2/3} \mathcal{J}^\varepsilon_{a,\beta}(w^\varepsilon) \\ &\le \sum_{k=1}^{N_n^\kappa} \sum_{j=1}^{N_n^\kappa} \left(E_0 \frac{1}{N_n^\kappa} \int_0^{\frac{1}{N_n^\kappa}} a^{1/3}_{k,n,\kappa,j}(s)ds + \frac{1}{N_n^\kappa} O(\overline{M}_\eta^2) \int_{F^M_{k,n,\kappa,j}} a_{k,n,\kappa,j}(s)ds \right) \\ &\quad + O(\overline{M}_\eta^2)\frac{1}{m} + O(\overline{M}_\eta^2)\kappa + \eta,\end{aligned}$$

where $a_{k,n,\kappa,j} : \mathbf{R} \longrightarrow \mathbf{R}$ is defined by

$$a_{k,n,\kappa,j}(\sigma) := \begin{cases} a_{k,\kappa}(\sigma + \frac{j-1}{N_n^\kappa}), & \text{if } \sigma \in I_k^{n,\kappa} \\ \text{by periodicity}, & \text{otherwise}, \end{cases} \quad j \in \mathbf{N}, \tag{3.20}$$

$$F^M_{k,n,\kappa,j} := \{\sigma \in \langle 0,1\rangle : a_{k,n,\kappa,j}(\sigma) > M\}.$$

As $M \longrightarrow +\infty$, $m \longrightarrow +\infty$ we obtain

$$\begin{aligned}\limsup_{\varepsilon \longrightarrow 0} \min_{w \in \mathrm{H}^2_{per}\langle 0,1\rangle} \varepsilon^{-2/3} \mathcal{J}^\varepsilon_{a,\beta}(w) &\le \sum_{k=1}^{N_n^\kappa} \lambda(I_k) E_0 \int_0^1 b^{1/3}_{n,\kappa,k}(\xi_2) d\xi_2 + \eta \\ &= E_0 \int_0^1 \int_0^1 a^{1/3}_{n,\kappa}(\xi_1,\xi_2) d\xi_2 d\xi_1 + \eta.\end{aligned}$$

Finally we consider the limit as $n \longrightarrow +\infty$ and then as $\kappa \longrightarrow 0$, getting

$$\limsup_{\varepsilon \longrightarrow 0} \min_{w \in \mathrm{H}^2_{per}\langle 0,1\rangle} \varepsilon^{-2/3} \mathcal{J}^\varepsilon_{a,\beta}(w) \le E_0 \int_0^1 \int_0^1 a^{1/3}(\xi_1,\xi_2) d\xi_2 d\xi_1 + \eta. \tag{3.21}$$

By taking the limit as $\eta \longrightarrow 0$ in (3.21), we prove the assertion.

Step 2. Let $p = 1$ and $q = +\infty$. By the Luzin theorem (cf. [5, p. 15]) for every $\kappa > 0$ there exists compact set $\Omega_\kappa \subseteq \langle 0,1\rangle$ and $b^\kappa \in \mathrm{C}\langle 0,1\rangle$ such that $\lambda(s \in \langle 0,1\rangle : c^\kappa(s) \ne c(s)) \le \kappa$. Set $\overline{c}^\kappa := \min\{c^\kappa, \|b\|_{\mathrm{L}^\infty\langle 0,1\rangle}\}$, $\Omega_\kappa := \{s \in$

$\langle 0,1\rangle : \overline{c}^{\kappa}(s) = c(s)\}$. Then $\lambda(\langle 0,1\rangle \backslash \Omega_\kappa) \leq \kappa$. By using the notation from Step 1, we derive the following estimate:

$$\begin{aligned}\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-2/3} \mathcal{J}^{\varepsilon}_{a,\beta}(w^\varepsilon) &\leq \limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-2/3} \mathcal{J}^{\varepsilon}_{a_n^{\kappa},\beta}(w^\varepsilon) \\ &\quad + 2\overline{M}_\eta^2 \|c - \overline{c}^{\kappa}\|_{\mathrm{L}^\infty(\langle 0,1\rangle \backslash \Omega_\kappa)} \|b\|_{\mathrm{L}^1(\langle 0,1\rangle \backslash \Omega_\kappa)}.\end{aligned}$$

Thus we are able to finish the proof as in the Step 1. ∎

Remark 3.10. Note that sequence of functions (a_k) in the proof of Proposition 3.8 need not be dominated by some $\tilde{a} \in \mathrm{L}^1\langle 0,1\rangle$. In the following result, however, such a condition is essential.

Theorem 3.11. *Suppose that* $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ *satisfies: there exists* $\tilde{a} \in \mathrm{L}^1_{per}\langle 0,1\rangle$ *such that* $\operatorname{ess\,sup}_{\xi_2} a(\xi_1,\xi_2) \leq \tilde{a}(\xi_1)$ *(a.e.* $\xi_1 \in \langle 0,1\rangle$*), (in particular, if* $a \in \mathrm{L}^\infty_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$*), then upper bound (3.12) holds.*

Proof. Step 1. Let $a \in \mathrm{L}^\infty_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$. By outer Borel regularity of λ there exists a sequence of piecewise constant functions

$$a_n(\xi_1,\xi_2) = \sum_{k=1}^{+\infty} \alpha_k^n \chi^{per}_{I_k^n}(\xi_2)\chi^{per}_{\omega_k^n}(\xi_1), \quad (\xi_1,\xi_2) \in \mathbf{R}^2, \tag{3.22}$$

with the following properties:

- $a \leq a_n$, $n \in \mathbf{N}$, $\lim_{n \longrightarrow +\infty} a_n = a$,
- $I_k^n \subseteq \langle 0,1\rangle$ and $\omega_k^n \subseteq \langle 0,1\rangle$ are bounded open intervals,
- $a_n \in \mathrm{L}^\infty_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$.

Thus, if we define $a_k^n(\xi_1) := \alpha_k^n \chi^{per}_{\omega_k^n}(\xi_1)$, $a_k^n \in \mathrm{L}^1_{per}\langle 0,1\rangle$, by Proposition 3.8 the upper bound holds for a_n for every $n \in \mathbf{N}$. Then we pass to the limit as $n \longrightarrow +\infty$, and we get upper bound for $\mathcal{J}^{\varepsilon}_{a,\beta}$.

Step 2. Let $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ satisfies condition of the theorem. Consider $F^M_{\tilde{a},\varepsilon} := \{s \in \langle 0,\varepsilon^\beta\rangle : \tilde{a}(\varepsilon^{-\beta}s) > M\}$, $F^M_{\tilde{a}} := \{\sigma \in \langle 0,1\rangle : \tilde{a}(\sigma) > M\}$. Since $\tilde{a}(\varepsilon^{-\beta}s) \leq M$ implies $a(\varepsilon^{-\beta}s, s+(j-1)\varepsilon^\beta) \leq M$ (a.e. $s \in \langle 0,1\rangle$), $j \in \mathbf{N}$, for a.e. $s \in \langle 0,\varepsilon^\beta\rangle$ we get

$$\begin{aligned}a(\varepsilon^{-\beta}s, s+(j-1)\varepsilon^\beta) &\leq a_M(\varepsilon^{-\beta}s, s+(j-1)\varepsilon^\beta) \\ &\quad + a(\varepsilon^{-\beta}s, s+(j-1)\varepsilon^\beta)\chi_{F^M_{\tilde{a},\varepsilon}}(s),\end{aligned}$$

where $a_M(\xi_1,\xi_2) := \min\{a(\xi_1,\xi_2), M\}$. Then for every sequence (w_ε) such that $w_\varepsilon \in \mathrm{H}^2_{per}\langle 0,1\rangle$ and $\|w_\varepsilon\|_{\mathrm{L}^\infty(\mathbf{R})} \le \overline{M}_\eta \varepsilon^{1/3}$ there holds

$$\begin{aligned}
\varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a,\beta}(w_\varepsilon) &\le \fint_0^1 \Big(\varepsilon^2 w''^2_\varepsilon(s) + W(w'_\varepsilon(s))\Big)ds \\
&\quad + \sum_{j=1}^{N_*}\int_0^{\varepsilon^\beta} a(\varepsilon^{-\beta}s, s+(j-1)\varepsilon^\beta) w^2_{\varepsilon,j}(s)ds \\
&\le \varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a_M,\beta}(w_\varepsilon) + \int_1^{\rho^{-1}_{\varepsilon,*}} a_M(\varepsilon^{-\beta}s,s)w^2_\varepsilon(s)ds + \rho^{-1}_{\varepsilon,*}\overline{M}^2_\eta \int_{F^M_{\tilde a}} \tilde a(\sigma)d\sigma \\
&\le \varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a_M,\beta}(w_\varepsilon) + |1-\rho^{-1}_{\varepsilon,*}|M\overline{M}^2_\eta + \rho^{-1}_{\varepsilon,*}\overline{M}^2_\eta \int_{F^M_{\tilde a}} \tilde a(\sigma)d\sigma,
\end{aligned}$$

where $\rho_{\varepsilon,*} := N_*^{-1}\varepsilon^{-\beta} \in \langle 0,1\rangle$, $w_{\varepsilon,j}(s) := w(s+(j-1)\varepsilon^\beta)$, $s \in \langle 0,\varepsilon^\beta\rangle$, $j = 1,\dots,N_*$. As we pass to the limit as $\varepsilon \longrightarrow 0$, it results

$$\limsup_{\varepsilon\longrightarrow 0} \varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a,\beta}(w_\varepsilon) \quad \le \quad \limsup_{\varepsilon\longrightarrow 0} \varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a_M,\beta}(w_\varepsilon) + \overline{M}^2_\eta \int_{F^M_{\tilde a}} \tilde a(\sigma)d\sigma.$$

In particular, estimates above show that computation of upper bound for $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ (which satisfies boundedness condition as above) can be reduced to computation of upper bound for $a_M \in \mathrm{L}^\infty_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$. Therefore, by Proposition 3.8 for a suitable choice of w_ε (as we finally pass to the limit as $M \longrightarrow +\infty$) we obtain the desired upper bound. ∎

Remark 3.12. Thanks to Proposition 3.8, it is easy to verify that (3.12) also holds if $a \in \mathrm{C}^+(\langle 0,1\rangle \times \langle 0,1\rangle) \cap \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ (or if $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ such that the mapping $\xi_2 \mapsto a(\xi_1,\xi_2)$ is upper semicontinuous for a.e. $\xi_1 \in \langle 0,1\rangle$).

3.3. *Computation of Macroscopic Energy.* We combine Theorem 3.5 and Remark 3.12 to get the following two results:

Theorem 3.13. *If $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ is Carathéodory function on $\langle 0,1\rangle \times \langle 0,1\rangle$, then (3.3) holds.*

Corollary 3.14. *Consider $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ such that the mapping $\xi_2 \mapsto a(\xi_1,\xi_2)$ is lower-semicontinuous for a.e. $\xi_1 \in \langle 0,1\rangle$. If there exists $\tilde a \in \mathrm{L}^1\langle 0,1\rangle$ with the property $\operatorname{ess\,sup}_{\xi_2} a(\xi_1,\xi_2) \le \tilde a(\xi_1)$ (a.e. $\xi_1 \in \langle 0,1\rangle$), then (3.3) holds.*

We mention here two more subsets of $\mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ closely resembling the Carathéodory class for which it is possible to compute energies $\mathcal{E}_{a,per}(\beta,0)$ and $\mathcal{E}_a(\beta,0)$. If $p,q \in [1,+\infty]$, $p_k,q_k \in [1,+\infty]$, $k \in \mathbf{N}$, we set

$$X := \operatorname{span}\Big[\mathrm{L}^p_{per}\langle 0,1\rangle \otimes \Big(\mathrm{C}^-\langle 0,1\rangle \cap \mathrm{L}^q_{per}\langle 0,1\rangle\Big)\Big], \text{where } 1/p + 1/q = 1,$$

$Y := \mathrm{conv}\Big[\mathrm{L}^{p_k}_{per}\langle 0,1\rangle \otimes \Big(\mathrm{C}^-\langle 0,1\rangle \cap \mathrm{L}^{q_k}_{per}\langle 0,1\rangle\Big) : k \in \mathbf{N}\Big]$, where $1/p_k + 1/q_k = 1$.

COROLLARY 3.15. *If $a \in X$ ($a \in Y$, resp.), then (3.3) holds.*

PROOF. The claim follows since it is easy to verify that the proof of Corollary 3.6 (Corollary 3.9, resp.) actually can be completed for a which belongs to the linear hull of $\mathrm{L}^p_{per}\langle 0,1\rangle \otimes \Big(\mathrm{C}^-\langle 0,1\rangle \cap \mathrm{L}^q_{per}\langle 0,1\rangle\Big)$ (see also [18, Corollary 4.10]). Similar conclusion is valid for the convex hull Y. ∎

4. CASE $\gamma > 0$

Consider $\beta, \gamma \geq 0$ and the functional $\mathcal{J}^\varepsilon_{a,\beta,\gamma} : \mathrm{H}^2\langle 0,1\rangle \longrightarrow [0,+\infty\rangle$ defined by

$$\mathcal{J}^\varepsilon_{a,\beta,\gamma}(v) := \fint_0^1 \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s, \varepsilon^{-\gamma}s)v^2(s)\Big)ds. \tag{4.1}$$

We expect that the minimizers of $\mathcal{J}^\varepsilon_{a,\beta,\gamma}$ develop fine hierarchy of small scales (roughly of size $\varepsilon^{1/3}$, ε^β and ε^γ). To justify this, we determine which small scale is relevant to computation of minimal asymptotic energy of $\mathcal{J}^\varepsilon_{a,\beta,\gamma}$. By the formulas below we can extract desired information. In particular, formulas (4.2)-(4.5) show that characteristic scale is $\varepsilon^{1/3}$ and that all shorter scales can be eliminated, i.e. replaced with the corresponding limits (in our case, the average of a). Oscillations on longer scales do not change the value in the limit as $\varepsilon \longrightarrow 0$, which means that the latter scales are not relevant.

THEOREM 4.1. *Let us assume that $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle \times \langle 0,1\rangle)$ is Carathéodory function on $\langle 0,1\rangle \times \langle 0,1\rangle$. Set*

$$\mathcal{E}_a(\beta,\gamma) := \lim_{\varepsilon \longrightarrow 0} \mathcal{E}^\varepsilon_a(\beta,\gamma), \quad \mathcal{E}_{a,per}(\beta,\gamma) := \lim_{\varepsilon \longrightarrow 0} \mathcal{E}^\varepsilon_{a,per}(\beta,\gamma).$$

Then there holds:

- *If $0 < \gamma < \beta < 1/3$ or $0 < \beta < \gamma < 1/3$, then*

$$\mathcal{E}_{a,per}(\beta,\gamma) = \mathcal{E}_a(\beta,\gamma) = E_0 \fint_0^1 \fint_0^1 a^{1/3}(\xi_1,\xi_2)d\xi_1 d\xi_2. \tag{4.2}$$

- *If $0 < \beta < 1/3$, $\gamma > 1/3$, then*

$$\mathcal{E}_{a,per}(\beta,\gamma) = \mathcal{E}_a(\beta,\gamma) = E_0 \fint_0^1 \Big(\fint_0^1 a(\xi_1,\xi_2)d\xi_2\Big)^{1/3} d\xi_1. \tag{4.3}$$

- *If $0 < \gamma < 1/3$, $\beta > 1/3$, then*

$$\mathcal{E}_{a,per}(\beta,\gamma) = \mathcal{E}_a(\beta,\gamma) = E_0 \fint_0^1 \Big(\fint_0^1 a(\xi_1,\xi_2)d\xi_1\Big)^{1/3} d\xi_2. \tag{4.4}$$

- *If* $\gamma > 1/3$, $\beta > 1/3$, *then*

(4.5)

$$\mathcal{E}_{a,per}(\beta,\gamma) = \mathcal{E}_a(\beta,\gamma) = E_0 \overline{a}^{1/3}, \quad where \quad \overline{a} := \fint_0^1 \fint_0^1 a(\xi_1,\xi_2)d\xi_1 d\xi_2.$$

Proof. We only sketch main points in the proof of (4.2) (the remaining claims (4.3), (4.4) and (4.5) are in fact consequence of the McShanne Lemma and we leave the details to the interested reader). Essential ingredients are already contained in the proof of lower and upper bound when $\gamma = 0$. Let $0 < \gamma < \beta < \frac{1}{3}$.

Step 1. (the lower bound) We set $\varepsilon_{**,\gamma} := \lfloor \varepsilon^{-\gamma} \rfloor^{-\frac{1}{\gamma}}$, $\rho_{\varepsilon,**,\gamma} := \varepsilon_{**,\gamma}^{\gamma}\varepsilon^{-\gamma} > 1$. Set $A_1^{\varepsilon}(\sigma) := a(\varepsilon^{\gamma-\beta}\sigma,\sigma)$ $A_1^{\varepsilon,N}(\sigma) := a_N^{\varepsilon}(\varepsilon^{\gamma-\beta}\sigma,\sigma)$, where for $N \in \mathbf{N}$ we define $a_N^{\varepsilon} \in \mathrm{L}^1(\langle 0, \varepsilon_{**,\gamma}^{-\gamma}\rangle \times \langle 0,1\rangle)$ by

$$a_N^{\varepsilon}(\xi_1,\xi_2) := \sum_{i=1}^{\varepsilon_{**,\gamma}^{-\gamma}} \sum_{k=1}^{N} b_{k,i}^{\varepsilon,N}(\xi_1) \overline{\chi}_{\rho_{\varepsilon,**}^{-1}(I_k+i-1)}(\xi_2), \tag{4.6}$$

$$b_{k,i}^{\varepsilon,N}(\xi_2) := \min_{\xi_2 \in \rho_{\varepsilon,**}^{-1}(I_k+i-1)} a(\xi_1,\xi_2), \quad \xi_1 \in \langle 0,1\rangle, \tag{4.7}$$

$b_{k,i}^{\varepsilon,N} \in \mathrm{L}_{per}^1\langle 0,1\rangle$, $i = 1,\ldots,\varepsilon_{**,\gamma}^{-\gamma}$, $\overline{\chi}_{\rho_{\varepsilon,**}^{-1}(I_k+i-1)}(\xi_2) := \chi_{\rho_{\varepsilon,**}^{-1}(I_k+i-1)}(\xi_2)$, $i = 1,\ldots,\varepsilon_{**,\gamma}^{-\gamma} - 1$, and

$$\overline{\chi}_{\rho_{\varepsilon,**}^{-1}(I_k+\varepsilon_{**,\gamma}^{-\gamma}-1)}(\xi_2) := \begin{cases} 1, & \text{if } \xi_2 \in \langle \varepsilon_{**,\gamma}^{-\gamma}-1, \rho_{\varepsilon,**}^{-1}\varepsilon_{**,\gamma}^{-\gamma}\rangle \\ 0, & \text{if } \xi_2 \in \langle \rho_{\varepsilon,**}^{-1}\varepsilon_{**,\gamma}^{-\gamma}, \varepsilon_{**,\gamma}^{-\gamma}\rangle. \end{cases}$$

Let $v_{**}(\sigma) := \varepsilon^{-\gamma}v(\varepsilon^{\gamma}\sigma)$, $\sigma \in \langle 0,\varepsilon^{-\gamma}\rangle$. Since $\varepsilon^{-\gamma} \geq \varepsilon_{**,\gamma}^{-\gamma}$, we estimate

$$\begin{aligned} &\varepsilon^{-2/3}\fint_0^1 \Big(\varepsilon^2 v''^2(s) + W(v'(s)) + a(\varepsilon^{-\beta}s,\varepsilon^{-\gamma}s,)v^2(s)\Big)ds \\ &\quad\geq \varepsilon^{\gamma}\varepsilon_{*,\gamma}^{-\gamma}\varepsilon^{-2/3}\fint_0^{\varepsilon_{**,\gamma}^{-\gamma}} \Big(\varepsilon^{2-2\gamma}v_{**}''^2 + W(v_{**}') + A_1^{\varepsilon}\varepsilon^{2\gamma}v_{**}^2\Big) \\ &\quad\geq \rho_{\varepsilon,**,\gamma}^{-1}\varepsilon^{-2/3}\fint_0^{\varepsilon_{**,\gamma}^{-\gamma}} \Big(\varepsilon^{2-2\gamma}v_{**}''^2 + W(v_{**}') + A_1^{\varepsilon,N}\varepsilon^{2\gamma}v_{**}^2\Big). \end{aligned}$$

At this point we consider $\varepsilon_{N,**} := \lfloor \varepsilon^{\gamma-\beta}N^{-1}\rfloor^{-\frac{1}{\beta-\gamma}}$, $\varepsilon_{**} := (\varepsilon_{N,**}^{\gamma-\beta}N)^{-\frac{1}{\beta-\gamma}}$, $\rho_{\varepsilon,**} := \varepsilon_{**}^{\beta-\gamma}\varepsilon^{\gamma-\beta} > 1$. Then $\varepsilon_{**}^{\gamma-\beta}N^{-1} \in \mathbf{N}$, and $a_N^{\varepsilon} \nearrow a$ as in Proposition 3.4 and Theorem 3.5. Now we pass to the limit as $\varepsilon \longrightarrow 0$ and as $N \longrightarrow +\infty$, which gives the lower bound.

Step 2. (the upper bound) For simplicity we assume that a is continuous and bounded. Consider a sequence (a_N) of piecewise constant functions with N pieces (length of every piece equals exactly $\frac{1}{N}$) such that $a \leq a_N$, $a_N \longrightarrow a$. For a given $N, m \in \mathbf{N}$ we set $\varepsilon_{*,N,m} := \lceil \varepsilon^{\gamma-\beta}N^{-1}m^{-1}\rceil^{-\frac{1}{\beta-\gamma}}$, $\varepsilon_* :=$

$(\varepsilon_{*,N,m}^{\gamma-\beta}Nm)^{-\frac{1}{\beta-\gamma}}$, $\rho_{\varepsilon,*} := \varepsilon_*^{\beta-\gamma}\varepsilon^{\gamma-\beta} \in \langle 0,1\rangle$, $\varepsilon_{*,\gamma} := [\rho_{\varepsilon,*}\varepsilon^{-\gamma}]^{-\frac{1}{\gamma}}$. Then there holds $\varepsilon_{*,\gamma}^{-\gamma} \in \mathbf{N}$ and $\varepsilon_{*,\gamma}^{-\gamma} \geq \varepsilon^{-\gamma}$. We define $a^\varepsilon : \mathbf{R} \longrightarrow \mathbf{R}$ by

$$a^\varepsilon(\xi_1,\xi_2) := \begin{cases} a(\rho_{\varepsilon,*}\xi_1,\xi_2), & \text{if } \xi_2 \in \langle 0, \varepsilon_{*,\gamma}^{-\gamma}\rangle \\ \text{by periodicity}, & \text{otherwise}, \end{cases}$$

so that there holds $a^\varepsilon(\rho_{\varepsilon,*}^{-1}\xi_1,\xi_2) = a(\xi_1,\xi_2)$ (a.e. $\xi_2 \in \langle 0, \varepsilon_{*,\gamma}^{-\gamma}\rangle$), $\xi_1 \in \mathbf{R}$. Moreover, continuity of a with respect to ξ_2 implies

$$a^\varepsilon(\xi_1,\xi_2) \longrightarrow a(\xi_1,\xi_2) \quad (\text{a.e. } \xi_1). \tag{4.8}$$

(4.8) is due to the fact that a^ε can be represented as composition of $\mathcal{P}_{\varepsilon_{*,\gamma}^{-\gamma}}$ periodic operator and affine transformation in ξ_2. Both of these operators converge to identic operator as $\varepsilon \longrightarrow 0$, hence (4.8) holds. By the dominated convergence theorem it results $a_\varepsilon \longrightarrow a$ in $\mathrm{L}^1\langle 0,1\rangle$. In the first step we prove the upper bound for $\mathcal{J}^\varepsilon_{a_\varepsilon,\beta,\gamma}$. Consider $v \in \mathrm{H}^2_{per}\langle 0,1\rangle$. Set $\overline{v}_*(s) := \varepsilon^{-\gamma}\rho_{\varepsilon,*}v(\varepsilon^\gamma\rho_{\varepsilon,*}^{-1}s)$, $s \in \mathbf{R}$. Since $\varepsilon_*^{\gamma-\beta} \in \mathbf{N}$, we calculate

$$\begin{aligned} &\varepsilon^{-2/3} \fint_0^1 \Big(\varepsilon^2 v''^2 + W(v') + A_2^\varepsilon v^2\Big) ds \\ &\leq \quad \theta_1(\varepsilon) \int_0^{\varepsilon_{*,\gamma}^{-\gamma}} \Big(\varepsilon^{2-2\gamma}\rho_{\varepsilon,*}^2 \overline{v}_*''^2 + W(\overline{v}_*') + A_3^\varepsilon \varepsilon^{2\gamma}\rho_{\varepsilon,*}^{-2}\overline{v}_*^2\Big) \\ &= \quad \theta_2(\varepsilon) \int_0^1 \Big(\varepsilon^{2-2\gamma}\rho_{\varepsilon,*}^2 \overline{v}_*''^2 + W(\overline{v}_*') + A_4^\varepsilon \varepsilon^{2\gamma}\rho_{\varepsilon,*}^{-2}\overline{v}_*^2\Big), \end{aligned}$$

where $\theta_1(\varepsilon) := \rho_{\varepsilon,*}^{-1}\varepsilon^\gamma\varepsilon^{-2/3}$, $\theta_2(\varepsilon) := \varepsilon_{*,\gamma}^{-\gamma}\rho_{\varepsilon,*}^{-1}\varepsilon^\gamma\varepsilon^{-2/3}$, $A_2^\varepsilon(s) := a^\varepsilon(\varepsilon^{-\beta}s, \varepsilon^{-\gamma}s)$, $A_3^\varepsilon(\sigma) := a^\varepsilon(\varepsilon_*^{\gamma-\beta}\sigma, \rho_{\varepsilon,*}^{-1}\sigma)$ $A_4^\varepsilon(\sigma) := a(\varepsilon_*^{\gamma-\beta}\sigma,\sigma)$. Let us approximate 1-periodic continuous function $\xi_2 \mapsto a(\xi_1,\xi_2)$ by a piecewise constant 1-periodic function a_N as in Proposition 3.8. Hence, it follows

$$\begin{aligned} &\varepsilon^{-2/3} \fint_0^1 \Big(\varepsilon^2 v''^2 + W(v') + A_2^\varepsilon v^2\Big) \\ &\leq \quad \theta_3(\varepsilon) \fint_0^1 \Big(\varepsilon^{2-2\gamma}\rho_{\varepsilon,*}^2 \overline{v}_*''^2 + W(\overline{v}_*') + A_5^{\varepsilon,N} \varepsilon^{2\gamma}\rho_{\varepsilon,*}^{-2}\overline{v}_*^2\Big), \end{aligned}$$

where $\theta_3(\varepsilon) := \varepsilon_{*,\gamma}^{-\gamma}\rho_{\varepsilon,*}^{-1}\varepsilon^\gamma\varepsilon^{-2/3}$, $A_5^{\varepsilon,N}(\sigma) := a_N(\varepsilon_*^{\gamma-\beta}\sigma,\sigma)$. By using the fact that $\varepsilon_*^{\gamma-\beta}N^{-1} \in \mathbf{N}$, we continue as in the proof of Proposition 3.8 (with almost no modification: roughly speaking, the only distinction in the proof comes from the fact that the scale of order ε^β is now replaced by the scale of order $\varepsilon^{\beta-\gamma}$). In the second step we compare minimal values of $\mathcal{J}^\varepsilon_{a,\beta,\gamma}$ and $\mathcal{J}^\varepsilon_{a_\varepsilon,\beta,\gamma}$ as $\varepsilon \longrightarrow 0$. If we consider a sequence (w_ε) such that $w_\varepsilon \in \mathrm{H}^2_{per}\langle 0,1\rangle$, $\|w_\varepsilon\|_{\mathrm{L}^\infty(\mathbf{R})} \leq \overline{M}_\eta\varepsilon^{1/3}$,

$$\limsup_{\varepsilon \longrightarrow 0} \varepsilon^{-2/3}\mathcal{J}^\varepsilon_{a_\varepsilon,\beta,\gamma}(w_\varepsilon) \leq E_0\overline{a^{1/3}} + \overline{M}_\eta^2 \int_{F_M} a + \eta,$$

we eventually arrive at the conclusion that there holds

$$\begin{aligned}\limsup_{\varepsilon\longrightarrow 0}\varepsilon^{-2/3}\mathcal{J}^{\varepsilon}_{a,\beta,\gamma}(w_\varepsilon) &\le \limsup_{\varepsilon\longrightarrow 0}\varepsilon^{-2/3}\mathcal{J}^{\varepsilon}_{a_\varepsilon,\beta,\gamma}(w_\varepsilon)+O(\overline{M}^2_\eta)\int_{F_M} a\\ &\quad+\limsup_{\varepsilon\longrightarrow 0} O(\overline{M}^2_\eta)\fint_0^1\fint_0^1 |a-a_\varepsilon|\\ &\le E_0\overline{a^{1/3}}+\overline{M}^2_\eta\int_{F_M} a+\eta.\end{aligned}$$

□

At last, we state (without the proof) the result which corresponds to the case $\beta = 1/3$ and $\gamma > 0$:

COROLLARY 4.2. *Set* $C_0 := (3/4)^{2/3}$. *If* $a \in \mathrm{L}^1_{per}(\langle 0,1\rangle\times\langle 0,1\rangle)$ *is Carathéodory function on* $\langle 0,1\rangle\times\langle 0,1\rangle$*, then there holds:*

- *If* $0 < \gamma < 1/3$*, then*

$$\lim_{A_0\longrightarrow 0} A_0^{-2/3}\mathcal{E}_{a,per}(1/3,\gamma) = C_0\fint_0^1\fint_0^1 a^{1/3}(\xi_1,\xi_2)d\xi_1 d\xi_2,$$

$$\lim_{A_0\longrightarrow +\infty} A_0^{-2/3}\mathcal{E}_{a,per}(1/3,\gamma) = C_0\fint_0^1\Big(\fint_0^1 a(\xi_1,\xi_2)d\xi_1\Big)^{1/3} d\xi_2.$$

- *If* $\gamma > 1/3$*, then*

$$\lim_{A_0\longrightarrow 0} A_0^{-2/3}\mathcal{E}_{a,per}(1/3,\gamma) = C_0\fint_0^1\Big(\fint_0^1 a(\xi_1,\xi_2)d\xi_2\Big)^{1/3} d\xi_1,$$

$$\lim_{A_0\longrightarrow +\infty} A_0^{-2/3}\mathcal{E}_{a,per}(1/3,\gamma) = C_0\overline{a}^{1/3},$$

Besides, there holds

$$\lim_{A_0\longrightarrow 0} A_0^{-2/3}\mathcal{E}_{a,per}(1/3,\gamma) = \lim_{A_0\longrightarrow 0} A_0^{-2/3}\mathcal{E}_{a}(1/3,\gamma),$$

$$\lim_{A_0\longrightarrow +\infty} A_0^{-2/3}\mathcal{E}_{a,per}(1/3,\gamma) = \lim_{A_0\longrightarrow +\infty} A_0^{-2/3}\mathcal{E}_{a}(1/3,\gamma).$$

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