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Explicit homomorphisms of hexagonal graphs to one vertex deleted Petersen graph*

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Abstract. The problem of deciding whether an arbitrary graph G has a homomorphism into a given graph H has been widely studied and has turned out to be very difficult. Hell and Nešetril proved that the decision problem is NP-complete unless H is bipartite. We consider a restricted problem where G is an arbitrary triangle-free hexagonal graph and H is a Kneser graph or its induced subgraph. We give an explicit construction which proves that any triangle-free hexagonal graph has a homomorphism into one-vertex deleted Petersen graph.

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Key words: homomorphism, H-coloring, triangle-free hexagonal graph

1. Introduction

Graph homomorphisms in the current sense were first studied by Sabidussi in the late fifties and early sixties. This was followed by much activity in different areas, see [3] and the references therein. One of the interesting questions is to study whether an arbitrary graph G has a homomorphism into a fixed graph H. Since an n-coloring of a graph G is a homomorphism of G to K_n, the term H-coloring of G has been employed to describe the existence of a homomorphism of a graph G into a graph H. In such a case the graph G is said to be H-colorable. Many authors have studied the complexity of the H-coloring problem. A key result was given by Hell and Nešetril in 1990 [7]. They proved that the H-coloring problem is NP-complete, if H is a non-bipartite graph and polynomial otherwise, assuming P$\neq$NP. Several restrictions of the H-coloring problem have been studied in [3]. The case when the input graph G is restricted to have a degree bounded by a small constant is investigated in [1].

An n-$[k]$ coloring of a graph G is a multicoloring of vertices of G, where every vertex $v \in G$ is assigned a subset of k different colors from the set of n colors, such that subsets assigned to any two adjacent vertices are disjoint. If there exists an n-$[k]$coloring of G, we say that G is n-$[k]$ colorable. In terms of homomorphisms,

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an n-$[k]$coloring is equivalent to a homomorphism to the Kneser graph $K(n,k)$ [3]. A motivation for interest in n-$[k]$colorings of hexagonal graphs comes from its relation to frequency assignment problems. A fundamental problem concerning cellular networks is to assign sets of frequencies (colors) to transmitters (vertices) in order to avoid unacceptable interferences [4]. The number of frequencies demanded at a transmitter may vary between transmitters. In a usual cellular model, transmitters are centers of hexagonal cells and the corresponding adjacency graph is a subgraph of the infinite triangular lattice. An integer $p(v)$ is assigned to each vertex of the triangular lattice and it is called the demand of the vertex v. If the demand is uniform, $p(v) = k$ and the set of available frequencies is n, then the problem is equivalent to the problem of the existence of an n-$[k]$coloring. For more details see [11]. Namely, the existence of an algorithm for n-$[k]$coloring of a hexagonal graph implies the existence of an algorithm for multicoloring of a weighted hexagonal graph with competitive ratio $\frac{n}{2k}$. For more details see [6]. Several results on n-$[k]$colorings of hexagonal graphs have been given recently. In [5] authors proved that every triangle-free hexagonal graph is 5-[2]colorable, indicating the existence of a distributed algorithm to find such a coloring. In [15] a 2-local algorithm for 5-[2]coloring was given. The 5-[2]colorability also follows from the fact that any triangle-free hexagonal graph is C_5-colorable [14]. The existence of 7-[3]coloring of any triangle-free hexagonal graph is proved in [2, 5]. Reference [13] gives an elegant idea that implies the existence of 14-[6]colorings. However, it is unclear how to design an algorithm for 7-[3]coloring from these proofs. McDiarmid and Reed conjectured that every triangle-free hexagonal graph is 9-[4]colorable [12], and the conjecture is still open. It is clear that the question for the existence of $(2k+1)$-$[k]$colorings has in general a negative answer for $k > 4$. On the other hand, planar graphs with large enough odd girth admit homomorphisms to Kneser graphs. More precisely, a planar graph with girth at least $10k - 7$ has a homomorphism to the Kneser graph $K(2k+1, k)$ [9]. Thus, for example, if the odd girth of a planar graph is at least 13, then this implies that the graph has a homomorphism to the Petersen graph $K(5,2)$. On the other hand, it is known that planar graphs with odd girth $2k+1$ that cannot be $(2k+1)$-$[k]$coloured exist for arbitrarily large k [16].

In this note we investigate a restricted H-coloring problem, where H is a Kneser graph $K(n,k)$ and G an arbitrary so-called hexagonal graph, which is an induced subgraph of a triangular lattice. We will consider only triangle-free hexagonal graphs, because a hexagonal graph which contains a triangle is obviously not $K(5,2)$-colorable, since graph $K(5,2)$ does not contain a triangle as a subgraph. The above mentioned results on n-$[k]$colorings of hexagonal graphs in terms of $K(n,k)$-colorability are the following. An arbitrary triangle-free hexagonal graph is $K(5,2)$-colorable by [6, 15], $K(7,3)$-colorable by [2, 5] and $K(14,6)$-colorable by [13]. Finally, McDiarmid and Reed conjectured that every triangle-free hexagonal graph is $K(9,4)$-colorable.

Recall that the existence of an explicit homomorphism gives rise to an efficient algorithm, see [14]. While explicit homomorphisms and algorithms are known for $K(5,2)$ [6, 15, 14], there are only existence proofs without construction for $K(7,3)$ [5, 13].

We will focus on the case where G is a triangle-free hexagonal graph and H

is the one-vertex deleted induced subgraph of $K(5,2)$, also known as the Petersen graph. As it is well known that the Petersen graph is vertex transitive, all one-vertex deleted subgraphs are isomorphic. We will use the notation $K_x(5,2)$ for the one vertex deleted subgraphs of the Petersen graph.

In this paper we prove

Theorem 1. *Any triangle-free hexagonal graph is $K_x(5,2)$-colorable. There is a homomorphism that only depends on the embedding of the graph into the infinite lattice and can be computed in linear time.*

More precisely, assuming that we are given the coordinates of the vertices in the triangular lattice, the homomorphism is given as a closed expression. We sketch an idea how to evaluate the expressions for all vertices in linear time. This result is of theoretical interest because it is an improvement of [14]; however, a true challenge is to construct an explicit homomorphism for the $K(7,3)$ case.

A straightforward corollary of Theorem 1 is $K(5,2)$-colorability, a result which also follows directly from [5, 6, 15].

The rest of the paper is organized as follows. In Section 2 some basic definitions and results are recalled. In Subsection 3.1 a construction for finding a homomorphism of any triangle-free hexagonal graph to Kneser graph $K_x(5,2)$ is given, and in Subsection 3.2 the correctness of one construction is proved. In Section 4 some remarks and open problems are given.

2. Preliminaries

Let G and H be graphs. A function $\varphi : G \to H$ is a homomorphism from G to H if it is an adjacency preserving mapping from $V(G)$ to $V(H)$, namely a mapping for which $[\varphi u, \varphi v] \in E(H)$ whenever $[u,v] \in E(G)$. We write simply $\varphi : G \to H$ and $u \mapsto \varphi(u)$.

The path of length n will be denoted by $P_n = (v_0, v_1, v_2, ..., v_n)$. Let u and v be two vertices of a graph G. The distance between u and v in G is the length of the shortest path from u to v in G which will be denoted as $dist_G(u,v)$.

Given two positive integers n and k, Kneser graph $K(n,k)$ is the graph whose vertices represent the k-subsets of the set $\{1,...,n\}$, and where two vertices are connected if and only if they correspond to disjoint subsets. Graph $K(n,k)$ therefore has $\binom{n}{k}$ vertices and is regular of degree $\binom{n-k}{k}$. It is obvious that Kneser graph $K(n,k)$ is connected for $n > 2k$. Note that the well known Petersen graph is isomorphic to Kneser graph $K(5,2)$, see Figure 1. Later we will give a homomorphism to the one-vertex deleted subgraph in which the vertex corresponding to the set $\{1,2\}$ is deleted.

Recall that hexagonal graphs are induced subgraphs of a triangular lattice. Vertices of a triangular lattice may be described as follows. The position of each vertex is an integer linear combination $x\vec{p} + y\vec{q}$ of two vectors $\vec{p} = (1,0)$ and $\vec{q} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Thus, we may identify the vertices of a triangular grid with pairs (x,y) of integers. Two vertices are adjacent when the Euclidean distance between them is one. Therefore each vertex (x,y) has six neighbors $(x \pm 1, y)$, $(x, y \pm 1)$, $(x+1, y-1)$ and $(x-1, y+1)$. For simplicity, we will refer to the neighbors as R (right), L (left), UR (up-right), DL (down-left), DR (down-right) and UL (up-left), respectively. There is an obvious 3-coloring of the infinite triangular lattice which gives rise to a partition

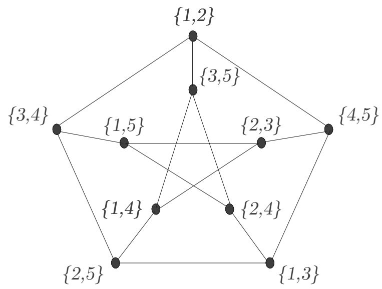

Figure 1. *Petersen graph* $P \cong K(5,2)$

of a vertex set of any triangular lattice graph into three independent sets I_0, I_1 and I_2. The partition can be decided from the coordinates by the rule: a vertex with coordinates (x, y) is in the independent set I_i where $i(x, y) = x + 2y \pmod 3$. According to the partition I_0, I_1 and I_2, vertices are assigned their base colors, which are denoted by R (red), B (blue) and G (green), respectively. Figure 2 shows an example of a triangle-free hexagonal graph with 3-coloring.

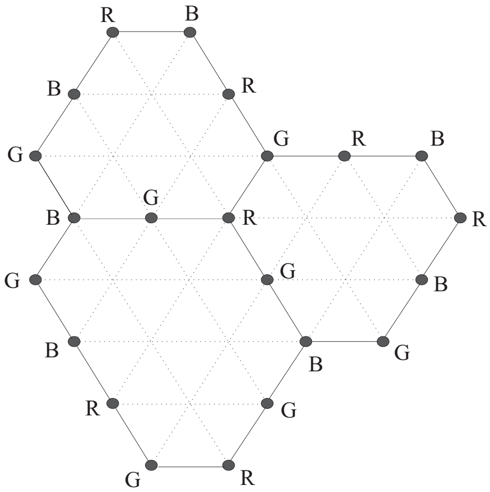

Figure 2. *RBG-coloring of a triangle-free hexagonal graph*

Let us define two integer functions, $B(x, y) = 5 - i(x, y)$ and $f_1(v) = B(x(v), y(v))$, defined on the triangular grid and on the vertices of a hexagonal graph, respectively. The image of both functions is the set $\{3, 4, 5\}$. More precisely,
- if the base color of v is red, then $f_1(v) = 5$,
- if the base color of v is blue, then $f_1(v) = 4$,
- if the base color of v is green, then $f_1(v) = 3$.

3. The main result

In this section an explicit construction of a homomorphism from an arbitrary triangle-free hexagonal graph to one vertex deleted Petersen graph $K_x(5,2)$ is given.

Let G be a hexagonal graph without triangles. Recall that for each vertex v, its position in the triangular grid is given by coordinates denoted by $(x(v), y(v))$.

A vertex $v \in G$ is said to be suitable if it has neither L, nor RD nor RU neighbor in the triangular lattice. All eight possible suitable vertices are presented in Figure 3, where suitable vertices are colored black. It is clear that the set of suitable vertices is independent. In the sequel the set of all suitable vertices of a graph G will be denoted as S_G. The intuition behind the partition of the vertex set to suitable and not suitable vertices is to break odd cycles. It is essential that unsuitable vertices induce bipartite subgraph whose connected components have a special structure, see the definition of tristars below.

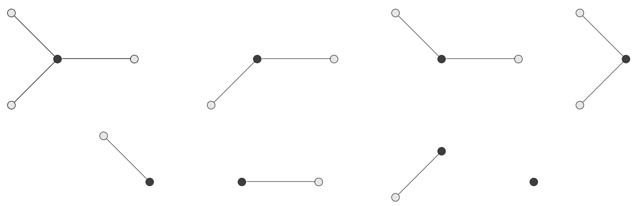

Figure 3. *All possible suitable vertices*

A path $(x_0, x_1, ..., x_n)$ is left (resp. rightup, rightdown) if x_{i+1} is the L (resp. RU, RD) neighbor of x_i, for $i = 0, 1, ..., n-1$, in a triangular lattice. A tristar is the union of one left, one rightup and one rightdown path emerging from a common origin x, named a center of a tristar or shortly a center. We allow that one, two or even all three paths of a tristar are of length zero. In the second and third case the tristar is isomorphic to a path and to an isolated vertex, respectively. An example of a tristar with center x is depicted in Figure 4.

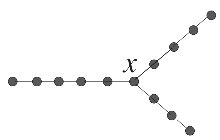

Figure 4. *A tristar*

3.1. The construction

The input of the algorithm is a triangle-free hexagonal graph where each vertex knows its base color. The output of the algorithm is a homomorphism $F : V(G) \to V(K_x(5,2))$. Homomorphism F is the assignment $F(v) = \{f_1(v), f_2(v)\}$, where $f_1(v)$ is defined at the end of Section 2 and $f_2(v) : V(G) \to \{1,2\}$ is given by the following rules:

1. if v is a suitable vertex, then

$$f_2(v) = B(x(v), y(v) + 1).$$

2. if v is not a suitable vertex, i.e. $v \in G \backslash S_G$, then

$$f_2(v) = dist_G(u, x) \mod 2 + 1,$$

where $dist_G(u, x)$ is the distance of v from the center x of the tristar.

Note that the only connected components of a graph $G \backslash S_G$ are tristars. Therefore, every vertex $v \in V(G) \backslash S_G$ belongs to exactly one tristar T_v of a graph $G \backslash S_G$, with exactly one center $x \in T_v$, which means that the distance $dist_G(u, x)$ is well defined.

3.2. Correctness of the construction

Recall that the only connected components of $V(G) \backslash S_G$ are tristars (where paths and isolated vertices are special cases of a tristar).

Remark 1. *Note that by the definition of base colors (see page 394) the base colors of neighbors of suitable vertices are fixed. Therefore, we have only three different possibilities, which are shown in Figure 4.*

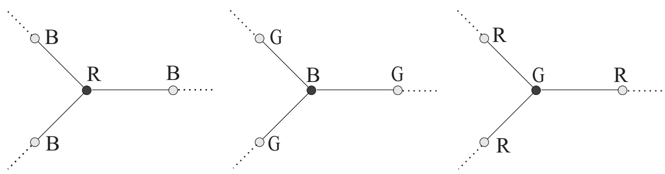

Figure 4. *Base colors of a suitable vertex and its neighbors*

We have to show that for an arbitrary triangle-free hexagonal graph G the mapping $F : G \to V(K_x(5, 2))$ is a homomorphism. Therefore, we have to show that for an arbitrary edge $uv \in E(G)$ it holds $F(u)F(v) \in E(K_x(5, 2))$.

Let uv be an edge in a graph G. Since suitable vertices are independent, we have only two different cases. One of the vertices is suitable or none of the vertices is suitable.

Case 1: Without loss of generality, let $u \in S_G$ and $v \notin S_G$. By Remark 1 and Figure 4 we have three different possibilities.

- If the base color of u is red, then the base color of v is blue and u and v are mapped as follows: $F(u) = \{5, 3\}$, $F(v) = \{4, f_2(v)\}$.
- If the base color of u is blue, then the base color of v is green and u and v are mapped as follows: $F(u) = \{4, 5\}$, $F(v) = \{3, f_2(v)\}$.
- If the base color of u is green, then the base color of v is red and u and v are mapped as follows: $F(u) = \{3, 4\}$, $F(v) = \{5, f_2(v)\}$.

Since $f_2(v) \in \{1, 2\}$, it holds $F(u) \cap F(v) = \emptyset$ for all three possibilities, which is equivalent to $F(u)F(v) \in E(K(5,2))$.

Case 2: Let $u, v \notin S_G$. Vertices u and v have different base colors, since they are neighbors. This yields $f_1(u) \neq f_1(v)$. Let $T_v = T_u$ be a tristar containing u and v. Let x be a center of a tristar T_v. Two different cases are possible.
- One of the vertices u and v is a center, let $u = x$. In this case we have $dist_G(u, x) = 0$, which implies $f_2(u) = 1$ and $dist_G(v, x) = 1$, which implies $f_2(v) = 2$.
- None of the vertices u and v is a center (i.e. $u \neq x$ and $v \neq x$). In this case it holds $|dist_G(v, x) - dist_G(u, x)| = 1$ and $f_2(u) \neq f_2(v)$.
So, we have again $F(u) \cap F(v) = \emptyset$, which implies $F(u)F(v) \in E(K(5,2))$.

Finally, note that $F(v) = \{1, 2\}$ never occurs in the construction, which means $F(u)F(v) \in E(K_x(5,2))$ in both cases. Hence, we have

Proposition 1. *Let G be a triangle-free hexagonal graph. Then there exists a homomorphism $F : V(G) \to V(K_x(5,2))$.*

4. Towards an algorithm

A straightforward implementation of the construction runs in linear time. Roughly speaking, in the first round one can compute $f_1(v)$ for all the vertices and decide for each vertex whether it is suitable or not and whether it is a center of tristar or not. In the second round, one can assign the colors $f_2(v)$ easily if the vertices are visited in the following order: suitable vertices and centers first, then the other vertices starting from the tristar centers.

Hence the construction given can be implemented in linear time as stated in Theorem 1.

Unfortunately, the construction sketched above does not give a distributed algorithm, since there may be arbitrary long segments of tristars in the graph $G \backslash S_G$, where vertices have to know the distance to the center of a tristar. In practice, it is very important that an algorithm is distributed. In particular, it is desirable that the communication paths are short. An algorithm is called k-local if the computation at a vertex depends only on information at vertices of distance at most k [8]. Reference [15] gives a 2-local distributed algorithm for 5-[2]coloring that implies the existence of a 2-local distributed algorithm for $K(5,2)$-coloring. Hence it would be interesting to see if for reasonably small k a k-local algorithm for $K_x(5,2)$-coloring of triangle-free hexagonal graphs is possible to design.

Moreover, the existence of an algorithm for $K(7,3)$-coloring and $K(9,4)$-coloring of an arbitrary triangle-free hexagonal graph and its best locality is still left as an open problem.

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