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On certain Durrmeyer type operators*

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Abstract. Deo [5] introduced n-th Durrmeyer operators defined for functions integrable in some interval I. There are gaps and mistakes in some of his lemmas and theorems. Further, in his paper [4] he did not give results on simultaneous approximation as the title reveals. The purpose of this paper is to correct those mistakes.
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1. Introduction

Deo proposed the following operators defined for functions integrable on I as

$$(V_n f)(x) = (n-c) \sum^{\otimes} p_{n,k}(x) \int_I p_{n,k}(t) f(t)\, dt \quad x \in I,$$

$p_{n,k}(x) = \frac{(-x)^k \phi_n^{(k)}(x)}{k!}$ whenever the right-hand side makes sense and $\phi_n(x), I, c, \sum^{\otimes}$ are given as:

$$\phi_n(x) = \begin{cases} (1-x)^n, & I=[0,1], \quad c=-1, \\ e^{-nx}, & I=[0,\infty), \quad c=0 \\ (1+cx)^{-\frac{n}{c}}, & I=[0,\infty), \quad c>0, \end{cases}$$

$\sum^{\otimes} = \sum_{k=0}^{\infty}$ for $c \geqslant 0$ and $\sum^{\otimes} = \sum_{k=0}^{n}$ when $c=-1$.

We introduce the class $\mathcal{H}$ defined by

$$\mathcal{H} \overset{\text{def}}{\equiv} \left\{ f |\int_I \frac{|f(t)|}{\beta_n(t)}\, dt < \infty, \text{ for some } n \in N,\ t \in I = [0,\infty) \right\},$$

where the function β_n is defined as:

$$\beta_n(t) = \begin{cases} (1+ct)^{n/c}, \ c>0 \\ e^{nt}, \qquad c=0. \end{cases}$$

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Clearly the class H contains the class $\mathcal{L}$ of all Lebesgue integrable functions on $[0,\infty)$. Further, we assume that

$$\phi_\alpha(t) = \begin{cases} t^\alpha, & c > 0 \\ e^{\alpha t}, & c = 0, \end{cases}$$

where $\alpha > 0$.

We define the norm $\|\cdot\|_{C_\alpha}$ in the space H by $\|f\|_{C_\alpha} = \sup\limits_{0\leqslant t<\infty} \dfrac{|f(t)|}{\phi_\alpha(t)}$.

Let $d_0, d_1, ...d_k$ be $(k+1)$ arbitrary but fixed distinct positive integers. We define the linear combination $V_n(f,k,x)$ of the operators V_n, as follows:

$$V_n(f,k,x) = \sum_{j=0}^{k} C(j,k) V_{d_j n}(f,x),$$

where

$$C(j,k) = \prod_{i=0, i\neq j}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \text{ and } C(0,0) = 1.$$

The aim of this paper is to correct and improve the results given in [5]. Further, we extend the results proved in [4] to the case of simultaneous approximation.

2. Auxiliary results

Lemma 1 (see [5]). *Let $r, m \in N \cup \{0\}$ and $n > cr$, we define the functions $\mu_{r,n,m}(x)$ as follows*

$$\mu_{r,n,m}(x) = [n - c(r+1)] \sum_{k=0}^{\infty} p_{n+cr,k}(x) \int_I p_{n-cr,k+r}(t)(t-x)^m\, dt, \; x \in I.$$

Then, there holds the recurrence relation

$$\begin{aligned}[n - c(r+m+2)]\mu_{r,n,m+1}(x) &= x(1+cx)\{\mu'_{r,n,m}(x) + 2m\mu_{r,n,m-1}(x)\} \\ &\quad + (r+m+1)(1+2cx)\mu_{r,n,m}(x), \\ & n > c(r+m+2).\end{aligned}$$

Consequently,

$$\mu_{r,n,0}(x) = 1, \; \mu_{r,n,1}(x) = \frac{(r+1)(1+2cx)}{n - c(r+2)}$$

and

$$\mu_{r,n,2}(x) = \frac{2(n-c)(x(1+cx)) + (r+1)(r+2)(1+2cx)^2}{(n-c(r+2))(n-c(r+3))}.$$

For all $x \in I$, $\mu_{r,n,m}(x) = \left(n^{-[(m+1)/2]}\right)$, where $[\alpha]$ denotes the integer part of α.

In the proof of [5, Lemma 2.2], from the step

$$(V_n^{(r)}f)(x) = (n-c)\sum_{k=0}^{\infty}\frac{(-1)^r(-x)^k\phi_n^{(k+r)}(x)}{k!}$$
$$\times\int_I\sum_{i=0}^{r}\binom{r}{i}\left\{\frac{(-1)^{r-i}(-t)^{k+i}\phi_n^{(k+i)}(t)}{(k+i)!}\right\}f(t)\,dt$$

the lemma is proved using integration by parts r times. But for this the expressions of the type $p_{n-cr,k+r}^{(r-j)}(t)f^{(j-1)}(t)|_I, j=1,2,...,r$ must be zero; and in order to claim this we must have $f^{(r-1)}(t)=O(\phi_\alpha(t))$ for some $\alpha>0$ as $t\to\infty$ and $n>\alpha+cr$, $r=1,2,3....$ Hence [5, Lemma 2.2] should be stated as follows (the proof remains the same).

Lemma 2. *Let f be r times differentiable on $[0,\infty)$ such that $f^{(r-1)}(t)=O(\phi_\alpha(t))$ for some $\alpha>0$ as $t\to\infty$. Then for $r=1,2,..$ and $n>\alpha+cr$, we have*

$$(V_n^{(r)}f)(x) = (n-c)\beta(n,r)\sum_{k=0}^{\infty}p_{n+cr,k}(x)\int_I p_{n-cr,k+r}(t)f^{(r)}(t)\,dt,$$

where $\beta(n,r)=\displaystyle\prod_{j=0}^{r-1}\frac{n+cj}{n-c(j+1)}$.

3. Main result

In [5] Deo has stated the following theorem:

Theorem 1. *If $f^{(r)}(t), r\geqslant 0$ is bounded and integrable in I and if admits the $(r+2)$-th derivative at a point $x\in I$, and $f^{(r)}(t)=O(t^\alpha)$ as $t\to\infty$ for some $\alpha>0$, then we get*

$$\lim_{x\to\infty} n\left\{\frac{n-c(r+1)}{(n-c)\beta(n,r)}(V_n^{(r)}f)(x)-f^{(r)}(x)\right\} = (r+1)(1+2cx)f^{(r+1)}(x)$$
$$+\phi^2(x)f^{(r+2)}(x).$$

We wish to make the following comment regarding Theorem 1:

(i) in the hypothesis of the theorem, the existence of the r-th derivative of f is assumed globally while the conclusion is obtained locally.

So Theorem 1 should be stated as follows:

Theorem 2. *Let $f\in\mathcal{H}$ be bounded on every finite sub-interval of $[0,\infty)$ admitting a derivative of order $(r+2)$ at a fixed point $x\in(0,\infty)$. Let $f(t)=O(\phi_\alpha(t))$ as $t\to\infty$ for some $\alpha>0$, then we have*

$$\lim_{x\to\infty} n\left\{\frac{n-c(r+1)}{(n-c)\beta(n,r)}(V_n^{(r)}f)(x)-f^{(r)}(x)\right\} = (r+1)(1+2cx)f^{(r+1)}(x)$$
$$+\phi^2(x)f^{(r+2)}(x).$$

Proof. By Taylor's expansion of f, we write

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\epsilon(t,x)(t-x)^{r+2},$$

where $\epsilon(t,x)\to 0$ as $t\to x$.

Using [5, Lemma 2.2], we can write

$$\begin{aligned}n\Big[\frac{n-c(r+1)}{(n-c)\beta(n,r)}(V_n^{(r)}f)(x)-f^{(r)}(x)\Big]&=n[n-c(r+1)]\Big[\sum_{i=r+1}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{k=0}^{\infty}p_{n+cr,k}(x)\\&\times\int_I p_{n-cr,k-r}(t)\frac{d^r}{dx^r}(t-x)^i\,dt\\&+n\frac{[n-c(r+1)]}{\beta(n,r)}\sum_{k=0}^{\infty}p_{n,k}^{(r)}(x)\\&\times\int_I p_{n,k}(t)\epsilon(t,x)(t-x)^{r+2}\,dt\Big]\\&=n\Big[f^{(r+1)}(x)\mu_{r,n,1}(x)+\frac{1}{2}f^{(r+1)}\mu_{r,n,2}(x)\Big]+I_n,\end{aligned}$$

where

$$I_n=\frac{n[n-c(r+1)]}{\beta(n,r)}\sum_{k=0}^{\infty}p_{n,k}^{(r)}(x)\int_I p_{n,k}(t)\epsilon(t,x)(t-x)^{r+2}\,dt.$$

In order to prove the theorem it is sufficient to show that $I_n\to 0$ as $n\to\infty$. Using Lorentz type lemma, we get

$$\begin{aligned}|I_n|&\leqslant\frac{n[n-c(r+1)]}{\beta(n,r)}\sum^{\otimes}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i|k-nx|^j\frac{|q_{i,j,r}(x)|}{\big(x(1+cx)\big)^r}p_{n,k}(x)\\&\quad\times\int_I p_{n,k}(t)|\epsilon(t,x)||(t-x)|^{r+2}\,dt\\&\leqslant C\frac{n[n-c(r+1)]}{\beta(n,r)}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\sum^{\otimes}p_{n,k}(x)|k-nx|^j\\&\quad\times\int_I p_{n,k}(t)|\epsilon(t,x)||(t-x)|^{r+2}\,dt\\&\leqslant C\frac{n[n-c(r+1)]}{\beta(n,r)}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\Big(\sum^{\otimes}p_{n,k}(x)(k-nx)^{2j}\Big)^{1/2}\\&\quad\times\Big(\sum^{\otimes}p_{n,k}(x)\Big(\int_I p_{n,k}(t)|\epsilon(t,x)||(t-x)|^{r+2}\,dt\Big)^2\Big)^{1/2}\end{aligned}$$

i.e.

$$\leqslant C\frac{n[n-c(r+1)]}{\beta(n,r)}n^{r/2}\Big(\sum^{\otimes}p_{n,k}(x)$$
$$\times\Big(\int_I p_{n,k}(t)|\epsilon(t,x)||(t-x)|^{r+2}\,dt\Big)^2\Big)^{1/2},$$

where $C=C(x)=\sup\limits_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}\dfrac{|q_{i,j,r}(x)|}{\big(x(1+cx)\big)^r}$.

For a given $\varepsilon>0$, there exists a $\delta>0$ such that $|\epsilon(t,x)|<\varepsilon$ whenever $0<|t-x|<\delta$. For $|t-x|\geqslant\delta$, we have $|\epsilon(t,x)|\leqslant K|t-x|^{2s}$, for any $s\geqslant 0$. Therefore, we have

$$\begin{aligned}\Big(\int_I p_{n,k}(t)|\epsilon(t,x)||t-x|^{r+2}\,dt\Big)^2 &\leqslant \Big(\int_I p_{n,k}(t)\,dt\Big)\Big(\int_I p_{n,k}(t)\big(\epsilon(t,x)\big)^2(t-x)^{2r+4}\,dt\Big)\\ &= \frac{1}{(n-c)}\Big(\int_{|t-x|<\delta}+\int_{|t-x|\geqslant\delta}\Big)p_{n,k}(t)\big(\epsilon(t,x)\big)^2\\ &\quad\times(t-x)^{2r+4}\,dt\\ &= \frac{1}{(n-c)}\Big(\int_{|t-x|<\delta}p_{n,k}(t)\varepsilon^2(t-x)^{2r+4}\,dt\\ &\quad+\int_{|t-x|\geqslant\delta}p_{n,k}(t)K^2(t-x)^{2r+2s+4}\,dt\Big).\end{aligned}$$

In view of [5, Lemma 2.1],

$$\begin{aligned}\sum^{\otimes}p_{n,k}(x)\Big(\int_I p_{n,k}(t)|\epsilon(t,x)||(t-x)|^{r+2}\,dt\Big)^2 &\leqslant \frac{(n-c)}{(n-c)^2}\sum^{\otimes}p_{n,k}(x)\\ &\quad\times\int_I p_{n,k}(t)\varepsilon^2(t-x)^{2r+4}\,dt\\ &\quad+\frac{K^2(n-c)}{(n-c)^2}\sum^{\otimes}p_{n,k}(x)\\ &\quad\times\int_{|t-x|\geqslant\delta}p_{n,k}(t)(t-x)^{2r+2s+4}\,dt\\ &=\varepsilon^2O\big(n^{-(r+4)}\big)+K^2O\big(n^{-(r+s+4)}\big)\\ &=\varepsilon^2O\big(n^{-(r+4)}\big)+O\big(n^{-(r+s+4)}\big).\end{aligned}$$

This in view of [5, Lemma 2.1] gives

$$\begin{aligned}|I_n| &\leqslant C\,\frac{n[n-c(r+1)]}{\beta(n,r)}n^{r/2}\times\varepsilon^2O\big(n^{-(r+4)}\big)^{1/2}+o(1)\\ &\leqslant \varepsilon+o(1),\quad\text{choosing } s>0.\end{aligned}$$

Since ε is arbitrary, this implies that $I_n\to 0$ as $n\to\infty$.

Finally, taking the limit $n\to\infty$ and using the values of $\mu_{r,n,1}(x)$ and $\mu_{r,n,2}(x)$ the theorem is proved. □

Remark 1. *If $f^{(r)}(t) = O(t^\alpha)$ (as $t \to \infty$), then $f(t)$ will be of order $t^{\alpha+r}$. Moreover, $\sin e^t$ is of order $O(1)$ while its r-th derivative ($r \geqslant 1$) is not of $O(t^\alpha)$. So the hypothesis of Theorem 2 is certainly weaker than the hypothesis of Theorem 1.*

Further, in [5] the author gave another theorem as:

Theorem 3. *Let $f^{(r+1)} \in C[0,\infty)$ and $[0,\lambda] \subseteq [0,\infty)$ and let $\omega(f^{(r+1)};.)$ be the modulus of continuity of $f^{(r+1)}$, then for $r = 0, 1, 2, \ldots$*

$$\left\| \frac{n-c(r+1)}{(n-c)\beta(n,r)} (V_n^{(r)} f)(x) - f^{(r)}(x) \right\|_{C[0,\lambda]} \leqslant \frac{(r+1)(1+2c\lambda)}{[n-c(r+2)]} \|f^{(r+1)}\| + C(n,r)\left(\sqrt{\eta} + \frac{\eta}{2}\right) \times \omega\big(f^{(r+1)}; C(n,r)\big),$$

where the norm is sup-norm over $[o,\lambda]$,

$$\eta = 2\lambda^2\{c^2(2r^2+6r+3)+cn\} + 2\lambda\{2c(r^2+3r+1)+n\} + (r^2+3r+2)$$

and

$$C(n,r) = \frac{1}{(n-c(r+2))(n-c(r+3))}.$$

Regarding this theorem, we wish to make the following comments:

(i) in the hypothesis of the theorem the existence of the $(r+1)$th derivative of f is assumed globally while the conclusion is obtained locally.

(ii) in the proof of the theorem, the property $\omega(f^{(r+1)};\delta) \to 0$ as $\delta \to 0$ is used which need not be true unless one assumes that $f^{(r+1)}$ is uniformly continuous on $[0,\infty)$.

For example, consider the function $g(x) = \cos \pi x^2$, $x \in [0,\infty)$.

Clearly, this function is bounded and continuous on $[0,\infty)$. But, $|g(\sqrt{n+1}) - g(\sqrt{n})| = 2$, while $|\sqrt{n+1} - \sqrt{n}| \to 0$ as $n \to \infty$, so the function is not uniformly continuous. Hence $\omega(g;\delta)$ does not tend to zero as δ tends to zero.

In the light of above comments, Theorem 3 should be stated as follows:

Theorem 4. *Let $f \in \mathcal{H}$ be bounded on every finite subinterval of $[0,\infty)$ and $f(t) = O(\phi_\alpha(t))$ as $t \to \infty$ for some $\alpha > 0$. If $f^{(r+1)}$ exists and if it is continuous on $(a-\delta, b+\delta) \subset (0,\infty), \delta > 0$, then for sufficiently large n,*

$$\big\|(V_n^{(r)} f)(x) - f^{(r)}(x)\big\| \leqslant C_1 n^{-1}\big(\|f^{(r)}\| + \|f^{(r+1)}\|\big) + C_2 n^{-1/2}\omega\big(f^{(r+1)}, n^{-1/2}\big) + O(n^{-s/2}) \text{ for any } s > 0,$$

where C_1 and C_2 are both independent of f and n, and $\|.\|$ is sup-norm on $[a,b]$.

Proof. By finite Taylor's expansion of f we write

$$f(t) = \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!}(t-x)^i + \frac{\big\{f^{(r+1)}(\xi) - f^{(r+1)}(x)\big\}}{(r+1)!}(t-x)^{r+1}\chi(t) + h(t,x)\big(1-\chi(t)\big),$$

where ξ lies between t and x and $\chi(t)$ is the characteristic function of $(a-\delta, b+\delta)$.

For $t \in (a-\delta, b+\delta)$ and $x \in [a,b]$ we have

$$f(t) = \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!}(t-x)^r + \frac{\big\{f^{(r+1)}(\xi) - f^{(r+1)}(x)\big\}}{(r+1)!}(t-x)^{r+1}.$$

For $t \in [0,\infty) \setminus (a-\delta, b+\delta)$ and $x \in [a,b]$ we define

$$h(t,x) = f(t) - \sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!}(t-x)^i.$$

Now

$$\begin{aligned}(V_n^{(r)}f)(x) - f^{(r)}(x) &= (n-c)\beta(n,r)\Bigg[\sum_{i=0}^{r+1} \frac{f^{(i)}(x)}{i!} \sum_{k=0}^{\infty} p_{n+cr,k}(x) \\ &\quad \times \int_I p_{n-cr,k-r}(t) \frac{d^r}{dx^r}(t-x)^i\,dt\Bigg] - f^{(r)}(x) \\ &\quad + (n-c)\sum_{k=0}^{\infty} p_{n,k}^{(r)}(x) \int_I p_{n,k}(t) \Bigg[\frac{\big\{f^{(r+1)}(\xi) - f^{(r+1)}(x)\big\}}{(r+1)!} \\ &\quad \times (t-x)^{r+1}\chi(t) + h(t,x)\big(1-\chi(t)\big)\Bigg]\,dt \\ &= I_1 + I_2 + I_3, \text{ say.}\end{aligned}$$

Using [5, Lemma 2.1], we obtain

$$I_1 = \left(\frac{(n-c)\beta(n,r)}{[n-c(r+1)]}\mu_{r,n,0}(x) - 1\right) f^{(r)}(x) + \frac{(n-c)\beta(n,r)}{[n-c(r+1)]}\mu_{r,n,1}(x) f^{(r+1)}(x),$$

in view of $\dfrac{d^r}{dx^r}(t-x)^i = 0$ for $i < r$.

Next, using Lorentz type lemma, we get

$$\begin{aligned} I_2 &\leqslant (n-c) \sum^{\otimes} \sum_{\substack{2i+j \leqslant r \\ i,j \geqslant 0}} n^i |k-nx|^j \frac{|q_{i,j,r}(x)|}{x^r(1+cx)^r} p_{n,k}(x) \\ &\quad \times \int_I p_{n,k}(t) \frac{\big|f^{(r+1)}(\xi) - f^{(r+1)}(x)\big|}{(r+1)!}(t-x)^{r+1}\chi(t)\,dt \\ &\leqslant (n-c) \sum_{\substack{2i+j \leqslant r \\ i,j \geqslant 0}} n^i \sum^{\otimes} p_{n,k}(x)|k-nx|^j \\ &\quad \times \int_I p_{n,k}(t)\Big(1 + \frac{|t-x|}{\delta}\Big)\omega\big(f^{(r+1)},\delta\big)|t-x|^{r+1}\,dt, \text{ for all } \delta > 0,\end{aligned}$$

i.e

$$= C(n-c)\omega\big(f^{(r+1)},\delta\big)\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \sum^{\otimes} p_{n,k}(x)|k-nx|^j$$

$$\times \int_I p_{n,k}(t)\Big(|t-x|^{r+1}+\frac{|t-x|^{r+2}}{\delta}\Big)\,dt.$$

By induction it can be easily shown that for $p = 0, 1, 2, ...$

$$\sum^{\otimes} p_{n,k}(x)|k-nx|^j \times \int_I p_{n,k}(t)|t-x|^p\,dt = \frac{1}{\sqrt{n-c}}O\big(n^{(j-p)/2}\big).$$

Hence, choosing $\delta = n^{-1/2}$ we have

$$|I_2| \leqslant Cn^{-1/2}\omega\big(f^{(r+1)}, n^{-1/2}\big).$$

Now, from the definition of $h(t,x)$, we have $h(t,x) = O\big(\phi_\alpha(t)\big) \Rightarrow h(t,x) = O(t-x)^s$, for any $s \in N$ with $s \geqslant \alpha$.

$$|I_3| \leqslant M' \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i |k-nx|^j p_{n,k}(x)$$

$$\times \int_{|t-x|\geqslant\delta} p_{n,k}(t)|h(t,x)|\,dt.$$

Applying Cauchy's inequality [5, Lemma 2.1] we obtain

$$|I_3| \leqslant M'(n-c)\sum^{\otimes}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i p_{n,k}(x)|k-nx|^j$$

$$\times \int_{|t-x|\geqslant\delta} p_{n,k}(t)M''|t-x|^s\,dt$$

$$\leqslant C\,n^{(1+r-s)/2}\,\omega\big(f^{(r+1)},\delta\big).$$

Choosing $s > r+1$, we get the limit $I_3 \to 0$ as $n \to \infty$.

Combining the estimates of I_1, I_2 and I_3, we get the required result. □

4. Simultaneous approximation

Theorem 5. *Let $f \in \mathcal{H}$ and let it be bounded on every finite subinterval of $[0,\infty)$ admitting a derivative of order $2k+r+2$ at a point $x \in (0,\infty)$. Let $f(t) = O(\phi_\alpha(t))$ as $t \to \infty$ for some $\alpha > 0$. Then*

$$\lim_{n\to\infty} n^{k+1}\Big[V_n^{(r)}(f,k,x) - f^{(r)}(x)\Big] = \sum_{j=r}^{2k+2+r} \frac{f^{(j)}(x)}{j!}Q(j,k,r,c,x) \tag{1}$$

and

$$\lim_{n\to\infty} n^{k+1}\left[V_n^{(r)}(f,k+1,x)-f^{(r)}(x)\right]=0, \tag{2}$$

where $Q(j,k,r,c,x)$ are certain polynomials in x. Further, if $f^{(2k+2+r)}$ exists and if it is continuous on $(a-\eta,b+\eta)\subset(0,\infty),\eta>0$, then (1) and (2) hold uniformly on $[a,b]$.

Proof. The proof is similar to [1, Theorem 2]. □

In our next result we obtain an estimate of the degree of approximation.

Theorem 6. *Let $f\in\mathcal{H}$ be bounded on every finite subinterval of $[0,\infty)$ and $f(t)=\ O(\phi_\alpha(t))$ as $t\to\infty$ for some $\alpha>0$. Further, let $1\leqslant p\leqslant 2k+2$ and $r\in N$. If $f^{(p+r)}$ exists and if it is continuous on $(a-\delta,b+\delta)\subset(0,\infty),\delta>0$, then for sufficiently large n,*

$$\|V_n^{(r)}(f,k,.)-f^{(r)}(.)\|\leqslant\max\left\{C_1n^{-p/2}\omega\big(f^{(p+r)},n^{-1/2}\big),C_2n^{-(k+1)}\right\},$$

where $C_1=C_1(k,p,c,r)$, $C_2=C_2(k,p,r,c,f)$ and $\omega\big(f^{(p+r)},.\big)$ denotes the modulus of continuity of $f^{(p+r)}$ on $(a-\delta;b+\delta)$.

Proof. The proof is similar to [1, Theorem 3] and hence it is omitted. □

Theorem 7. *Let $f\in\mathcal{H}$ be bounded on $[0,\infty)$ and $f(t)=O(\phi_\alpha(t))$ as $t\to\infty$ for some $\alpha>0$. If $f^{(r)}$ exists and if it is continuous on $(a-\eta,b+\eta),\eta>0$, then for sufficiently large n,*

$$\begin{aligned}\big\|V_n^{(r)}(f,k,.)-f^{(r)}(.)\big\|_{C(I)}&\leqslant C\,n^{-(k+1)}\\ &\times\left\{\|f\|_{C_\alpha}+\omega_{2k+2}\big(f^{(r)};n^{-1/2};(a-\eta,b+\eta)\big)\right\},\end{aligned}$$

where C is independent of f and n.

Proof. The proof follows along the lines [3, Theorem 3] □

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