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Superconformal ruled surfaces in $\mathbb{E}^4$

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Abstract. In the present study we consider ruled surfaces imbedded in a Euclidean space of four dimensions. We also give some special examples of ruled surfaces in $\mathbb{E}^4$. Further, the curvature properties of these surface are investigated with respect to variation of normal vectors and curvature ellipse. Finally, we give a necessary and sufficient condition for ruled surfaces in $\mathbb{E}^4$ to become superconformal. We also show that superconformal ruled surfaces in $\mathbb{E}^4$ are Chen surfaces.
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1. Introduction

Differential geometry of ruled surfaces has been studied in classical geometry using various approaches (see [9] and [13]). They have also been studied in kinematics by many investigators based primarily on line geometry (see [2], [21] and [23]). For a $CAGD$ type representation of ruled surfaces based on line geometry see [17]. Developable surfaces are special ruled surfaces [12].

The study of ruled hypersurfaces in higher dimensions have also been studied by many authors (see, e.g. [1]). Although ruled hypersurfaces have singularities, in general there have been very few studies of ruled hypersurfaces with singularities [11]. The 2-ruled hypersurfaces in $\mathbb{E}^4$ is a one-parameter family of planes in $\mathbb{E}^4$, which is a generation of ruled surfaces in $\mathbb{E}^3$ (see [20]).

In 1936 Plass studied ruled surfaces imbedded in a Euclidean space of four dimensions. Curvature properties of the surface are investigated with respect to the variation of normal vectors and a curvature conic along a generator of the surface [18]. A theory of ruled surface in $\mathbb{E}^4$ was developed by T. Otsuiki and K. Shiohama in [16].

In [4] B.Y. Chen defined the allied vector field $a(v)$ of a normal vector field v. In particular, the allied mean curvature vector field is orthogonal to H. Further, B.Y. Chen defined the $\mathcal{A}$-surface to be the surfaces for which $a(H)$ vanishes identically.

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Such surfaces are also called Chen surfaces [7]. The class of Chen surfaces contains all minimal and pseudo-umbilical surfaces, and also all surfaces for which dim $N_1 \leq 1$, in particular all hypersurfaces. These Chen surfaces are said to be trivial $\mathcal{A}$-surfaces [8]. In [19], B. Rouxel considered ruled Chen surfaces in $\mathbb{E}^n$. For more details, see also [6] and [10].

General aspects of the ellipse of curvature for surfaces in $\mathbb{E}^4$ were studied by Wong [22]. This is the subset of the normal space defined as $\{h(X,X) : X \in TpM, \|X\| = 1\}$, where h is the second fundamental form of the immersion. A surface in $\mathbb{E}^4$ is called superconformal if at any point the ellipse of curvature is a circle. The condition of superconformality shows up in several interesting geometric situations [3].

This paper is organized as follows: Section 2 explains some geometric properties of surfaces in $\mathbb{E}^4$. Further, this section provides some basic properties of surfaces in $\mathbb{E}^4$ and the structure of their curvatures. Section 3 discusses ruled surfaces in $\mathbb{E}^4$. Some examples are presented in this section. In Section 4 we investigate the curvature ellipse of ruled surfaces in $\mathbb{E}^4$. Additionally, we give a necessary and sufficient condition of ruled surfaces in $\mathbb{E}^4$ to become superconformal. Finally, in Section 5 we consider Chen ruled surfaces in $\mathbb{E}^4$. We also show that every superconformal ruled surface in $\mathbb{E}^4$ is a Chen surface.

2. Basic concepts

Let M be a smooth surface in $\mathbb{E}^4$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, \tag{1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $g = EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular.

For each $p \in M$ consider the decomposition $T_p\mathbb{E}^4 = T_pM \oplus T_p^\perp M$ where $T_p^\perp M$ is the orthogonal component of T_pM in $\mathbb{E}^4$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Given any local vector fields X_1, X_2 tangent to M, the induced Riemannian connection on M is defined by

$$\nabla_{X_1} X_2 = (\widetilde{\nabla}_{X_1} X_2)^T, \tag{2}$$

where T denotes the tangent component.

Let $\chi(M)$ and $\chi^\perp(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map $h : \chi(M) \times \chi(M) \to \chi^\perp(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \leq i,j \leq 2. \tag{3}$$

This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal frame field $\{N_1, N_2\}$ of M, recall the shape operator $A : \chi^\perp(M) \times \chi(M) \to \chi(M)$;

$$A_{N_i} X = -(\widetilde{\nabla}_{X_i} N_i)^T, \quad X_i \in \chi(M). \tag{4}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k,\ 1 \le i, j, k \le 2. \tag{5}$$

Equation (3) is called a Gaussian formula, where

$$\nabla_{X_i} X_j = \sum_{k=1}^{2} \Gamma_{ij}^k X_k, \quad 1 \le i, j \le 2 \tag{6}$$

and

$$h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k, \quad 1 \le i, j \le 2 \tag{7}$$

where Γ_{ij}^k are the Christoffel symbols and c_{ij}^k are the coefficients of the second fundamental form.

Further, the Gaussian and mean curvature of a regular patch are given by

$$K = \frac{1}{g}(\langle h(X_1, X_1), h(X_2, X_2) \rangle - \|h(X_1, X_2)\|^2) \tag{8}$$

and

$$\|H\| = \frac{1}{4g^2} \langle h(X_1, X_1) + h(X_2, X_2), h(X_1, X_1) + h(X_2, X_2) \rangle \tag{9}$$

respectively, where h is the second fundamental form of M and

$$g = \|X_1\|^2 \|X_2\|^2 - \langle X_1, X_2 \rangle^2 .$$

Recall that a surface in $\mathbb{E}^n$ is said to be minimal if its mean curvature vanishes identically [4].

3. Ruled surfaces in $\mathbb{E}^4$

A ruled surface M in a Euclidean space of four dimension $\mathbb{E}^4$ may be considered as generated by a vector moving along a curve. If the curve C is represented by

$$\alpha(u) = (f_1(u), f_2(u), f_3(u), f_4(u)), \tag{10}$$

and the moving vector by

$$\beta(u) = (g_1(u), g_2(u), g_3(u), g_4(u)), \tag{11}$$

where the functions of the parameter u sufficiently regular to permit differentiation as may be required, of any point p on the surface, with the coordinates X_i, will be given by

$$M : X(u, v) = \alpha(u) + v\beta(u), \tag{12}$$

where if $\beta(u)$ is a unit vector (i.e. $\langle \beta, \beta \rangle = 1$), v is the distance of p from the curve C in the positive direction of $\beta(u)$. Curve C is called directrix of the surface and vector $\beta(u)$ is the ruling of generators [18]. If all the vectors $\beta(u)$ are moved to the same point, they form a cone which cuts a unit hypersphere on the origin in a curve. This cone is called a director-cone of the surface. From now on we assume that $\alpha(u)$ is a unit speed curve and $\langle \alpha'(u), \beta(u) \rangle = 0$.

We prove the following result.

Proposition 1. *Let M be a ruled surface in $\mathbb{E}^4$ given with parametrization (12). Then the Gaussian curvature of M at point p is*

$$K = -\frac{1}{g}\{\langle X_{uv}, X_{uv} \rangle - \frac{1}{E} \langle X_{uv}, X_u \rangle^2\}. \tag{13}$$

Proof. The tangent space to M at an arbitrary point $P = X(u, v)$ of M is spanned by

$$X_u = \alpha'(u) + v\beta'(u), \ X_v = \beta(u). \tag{14}$$

Further, the coefficient of the first fundamental form becomes

$$\begin{aligned} E &= \langle X_u, X_u \rangle = 1 + 2v \langle \alpha'(u), \beta'(u) \rangle + v^2 \langle \beta'(u), \beta'(u) \rangle, \\ F &= \langle X_u, X_v \rangle = 0, \\ G &= \langle X_v, X_v \rangle = 1. \end{aligned} \tag{15}$$

The Christoffel symbols Γ_{ij}^k of the ruled surface M are given by

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2E}\partial_u(E) = \frac{1}{E} \langle X_{uu}, X_u \rangle, \\ \Gamma_{11}^2 &= -\frac{1}{2G}\partial_v(E) = -\frac{1}{G} \langle X_{uv}, X_u \rangle, \\ \Gamma_{12}^1 &= \frac{1}{2E}\partial_v(E) = \frac{1}{E} \langle X_{uv}, X_u \rangle, \\ \Gamma_{12}^2 &= \Gamma_{22}^1 = \Gamma_{22}^2 = 0, \end{aligned} \tag{16}$$

which are symmetric with respect to the covariant indices.

Hence, taking into account (3), the Gauss equation implies the following equations for the second fundamental form;

$$\begin{aligned} \widetilde{\nabla}_{X_u} X_u &= X_{uu} = \nabla_{X_u} X_u + h(X_u, X_u), \\ \widetilde{\nabla}_{X_u} X_v &= X_{uv} = \nabla_{X_u} X_v + h(X_u, X_v), \\ \widetilde{\nabla}_{X_v} X_v &= X_{vv} = 0, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \nabla_{X_u} X_u &= \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v, \\ \nabla_{X_u} X_v &= \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v. \end{aligned} \tag{18}$$

Taking in mind (16), (17) and (18) we get

$$\begin{aligned} h(X_u, X_u) &= X_{uu} - \frac{1}{E} \langle X_{uu}, X_u \rangle X_u + \frac{1}{G} \langle X_{uv}, X_u \rangle X_v, \\ h(X_u, X_v) &= X_{uv} - \frac{1}{E} \langle X_{uv}, X_u \rangle X_u, \\ h(X_v, X_v) &= 0. \end{aligned} \tag{19}$$

With the help of (8), the Gaussian curvature of a real ruled surface in a four dimensional Euclidean space becomes

$$K = -\frac{\langle h(X_u, X_v), h(X_u, X_v)\rangle}{g}, \tag{20}$$

where $g = EG - F^2$.

Taking into account (19) with (20) we obtain (13). This completes the proof of the theorem. □

Consequently, substituting (19) into (13) we get the following result.

Corollary 1. *Let M be a ruled surface in $\mathbb{E}^4$ given with parametrization (12). Then the Gaussian curvature of M at point p is*

$$K = -\frac{1}{g}\left(\frac{\langle \beta'(u), \beta'(u)\rangle - \langle \alpha'(u), \beta'(u)\rangle^2}{E}\right), \tag{21}$$

where $g = EG - F^2$ and E is defined in Eq.(15).

Remark 1. *The ruled surface in $\mathbb{E}^4$ for which $K = 0$ is a developable surface. However, in $\mathbb{E}^4$ all surfaces for which $K = 0$ are not necessarily ruled developable surfaces, see [18].*

Let $\|H\|$ be the mean curvature of the ruled surface M in $\mathbb{E}^4$. Since $h(X_v, X_v) = 0$, from equality (9)

$$\|H\| = \frac{\langle h(X_u, X_u), h(X_u, X_u)\rangle}{4g^2}. \tag{22}$$

Consequently, taking into account (19) with (22) we get

Proposition 2. *Let M be a ruled surface in $\mathbb{E}^4$ given with parametrization (12). Then the mean curvature of M at point p is*

$$\begin{aligned} 4\|H\| = &\frac{1}{g^2}\{\langle X_{uu}, X_{uu}\rangle - \frac{1}{E}\langle X_{uu}, X_u\rangle^2 \\ &+\frac{1}{G}\langle X_{uv}, X_u\rangle [2\langle X_{uu}, X_v\rangle + \langle X_{uv}, X_u\rangle] \\ &-\frac{2}{EG}\langle X_{uu}, X_u\rangle \langle X_{uv}, X_u\rangle \langle X_u, X_v\rangle\}. \end{aligned} \tag{23}$$

For a vanishing mean curvature of M, we have the following result of ([18], p.17).

Corollary 2 (see [18])**.** *The only minimal ruled surfaces in $\mathbb{E}^4$ are those of $\mathbb{E}^3$, namely the right helicoid.*

In the remaining part of this section we give some examples.

3.1. Examples

Let C be a smooth closed regular curve in $\mathbb{E}^4$ given by the arclength parameter with positive curvatures κ_1, κ_2, not signed curvatures κ_3 and the Frenet frame $\{t, n_1, n_2, n_3\}$. The Frenet equation of C are given as follows:

$$\begin{aligned} t' &= \kappa_1 n_1 \\ n_1' &= -\kappa_1 t + \kappa_2 n_2 \\ n_2' &= -\kappa_2 n_1 + \kappa_3 n_3 \\ n_3' &= -\kappa_3 n_2 \end{aligned} \tag{24}$$

Let C be a curve as above and consider the ruled surfaces

$$M_i : X(u,v) = \alpha(u) + v n_i, \ i = 1,2,3. \tag{25}$$

Then, by using of (16) it is easy to calculate the Gaussian curvatures of these surfaces (see Table 1).

Surface	K
M_1	$-\frac{\kappa_2^2}{((1-\kappa_1 v)^2+\kappa_2^2 v^2)^2}$
M_2	$-\frac{\kappa_2^2+\kappa_3^2}{(1+\kappa_2^2 v^2+\kappa_3^2 v^2)^2}$
M_3	$-\frac{\kappa_3^2}{(1+\kappa_3^2 v^2)^2}$

Table 1. *Gaussian curvatures of ruled surfaces*

Hence, the following results are obtained:

i) For a planar directrix curve C the surfaces M_1 and M_2 are flat,

ii) For a space directrix curve C the surface M_3 is flat.

4. Ellipse of curvature of ruled surfaces in $\mathbb{E}^4$

Let M be a smooth surface in $\mathbb{E}^4$ given with the surface patch $X(u,v) : (u,v) \in \mathbb{E}^2$. Let γ_θ be the normal section of M in the direction of θ. Given an orthonormal basis $\{Y_1, Y_2\}$ of the tangent space $T_p(M)$ at $p \in M$ denote by $\gamma_\theta' = X = \cos\theta Y_1 + \sin\theta Y_2$ the unit vector of the normal section. A subset of the normal space defined as

$$\{h(X,X) : X \in TpM, \|X\| = 1\}$$

is called the ellipse of curvature of M and denoted by $E(p)$, where h is the second fundamental form of the surface patch $X(u,v)$. To see that this is an ellipse, we just have to look at the following formula:

$$X = \cos\theta Y_1 + \sin\theta Y_2$$

the unit vector that

$$h(X,X) = \overrightarrow{H} + cos2\theta \overrightarrow{B} + \sin 2\theta \overrightarrow{C}, \tag{26}$$

where $\overrightarrow{H} = \frac{1}{2}(h(Y_1, Y_1) + h(Y_2, Y_2))$ is the mean curvature vector of M at p and

$$\overrightarrow{B} = \frac{1}{2}(h(Y_1, Y_1) - h(Y_2, Y_2)), \overrightarrow{C} = h(Y_1, Y_2) \tag{27}$$

are the normal vectors. This shows that when X goes once around the unit tangent circle, the vector $h(X, X)$ goes twice around an ellipse centered at $\overrightarrow{H}$, the ellipse of curvature $E(p)$ of $X(u, v)$ at p. Clearly $E(p)$ can degenerate into a line segment or a point. It follows from (26) that $E(p)$ is a circle if and only if for some (and hence for any) orthonormal basis of $T_p(M)$ it holds that

$$\langle h(Y_1, Y_2), h(Y_1, Y_1) - h(Y_2, Y_2)\rangle = 0 \tag{28}$$

and

$$\|h(Y_1, Y_1) - h(Y_2, Y_2)\| = 2\,\|h(Y_1, Y_2)\|\,. \tag{29}$$

General aspects of the ellipse of curvature for surfaces in $\mathbb{E}^4$ were studied by Wong [22]. For more details see also [14], [15], and [19]. We have the following functions associated with the coefficients of the second fundamental form [15]:

$$\Delta(p) = \frac{1}{4g}\det\begin{bmatrix} c_{11}^1 & 2c_{12}^1 & c_{22}^1 & 0 \\ c_{11}^2 & 2c_{12}^2 & c_{22}^2 & 0 \\ 0 & c_{11}^1 & 2c_{12}^1 & c_{22}^1 \\ 0 & c_{11}^2 & 2c_{12}^2 & c_{22}^2 \end{bmatrix}(p) \tag{30}$$

and the matrix

$$\alpha(p) = \begin{bmatrix} c_{11}^1 & c_{12}^1 & c_{22}^1 \\ c_{11}^2 & c_{12}^2 & c_{22}^2 \end{bmatrix}(p). \tag{31}$$

By identifying p with the origin of $N_p(M)$ it can be shown that:

a) $\Delta(p) < 0 \Rightarrow p$ lies outside of the curvature ellipse (such a point is said to be a hyperbolic point of M),

b) $\Delta(p) > 0 \Rightarrow p$ lies inside the curvature ellipse (elliptic point),

c) $\Delta(p) = 0 \Rightarrow p$ lies on the curvature ellipse (parabolic point).

A more detailed study of this case allows us to distinguish among the following possibilities:

d) $\Delta(p) = 0$, $K(p) > 0 \Rightarrow p$ is an inflection point of imaginary type,

e) $\Delta(p) = 0$, $K(p) < 0$ and $\begin{cases} \text{rank}\,\alpha(p) = 2 \Rightarrow \text{ellipse is non-degenerate} \\ \text{rank}\,\alpha(p) = 1 \Rightarrow p \text{ is an inflection point} \\ \text{of real type,} \end{cases}$

f) $\Delta(p) = 0$, $K(p) = 0 \Rightarrow p$ is an inflection point of flat type.

Consequently we have the following result.

Proposition 3. *Let M be a ruled surface in $\mathbb{E}^4$ given with parametrization (12). Then the origin p of N_pM is non-degenerate and lies on the ellipse of curvature $E(p)$ of M.*

Proof. Since $h(X_v, X_v) = 0$, then using (5) we get $\Delta(p) = 0$, which means that the point p lies on the ellipse of curvature (parabolic point) of M. Further, $K(p) < 0$ and rank $\alpha(p) = 2$. So the ellipse of curvature $E(p)$ is non-degenerate. □

Definition 1. *The surface M is called superconformal if its curvature ellipse is a circle, i.e. $<\overrightarrow{B}, \overrightarrow{C}> = 0$ and $\left\|\overrightarrow{B}\right\| = 2\left\|\overrightarrow{C}\right\|$ holds (see [5]).*

We prove the following result.

Theorem 1. *Let M be a ruled surface in $\mathbb{E}^4$ given with parametrization (12). Then M is superconformal if and only if the equalities*

$$\langle h(X_u, X_u), h(X_u, X_v)\rangle = 0 \quad and \quad \left\|\frac{1}{\sqrt{EG}}h(X_u, X_v)\right\| = \left\|\frac{1}{E}h(X_u, X_u)\right\| \tag{32}$$

hold, where $h(X_u, X_u)$ and $h(X_u, X_v)$ are given in (19).

Proof. It is convenient to use the orthonormal frame

$$Y_1 = \frac{X_1 + X_2}{\sqrt{2}},\ Y_2 = \frac{X_1 - X_2}{\sqrt{2}}, \tag{33}$$

where

$$X_1 = \frac{X_u}{\|X_u\|}, X_2 = \frac{X_v}{\|X_v\|}. \tag{34}$$

So, we get

$$\begin{aligned} h(Y_1, Y_1) &= \frac{1}{2E}h(X_u, X_u) + \frac{1}{\sqrt{EG}}h(X_u, X_v), \\ h(Y_1, Y_2) &= \frac{1}{2E}h(X_u, X_u), \\ h(Y_2, Y_2) &= \frac{1}{2E}h(X_u, X_u) - \frac{1}{\sqrt{EG}}h(X_u, X_v). \end{aligned} \tag{35}$$

Therefore, normal vectors $\overrightarrow{B}$ and $\overrightarrow{C}$ become

$$\overrightarrow{C} = h(Y_1, Y_2) = \frac{1}{2E}h(X_u, X_u), \tag{36}$$

and

$$\begin{aligned} \overrightarrow{B} &= \frac{1}{2}(h(Y_1, Y_1) - h(Y_2, Y_2)) \\ &= \frac{1}{\sqrt{EG}}h(X_u, X_v). \end{aligned} \tag{37}$$

Suppose M is superconformal; then by Definition 1 $\langle\overrightarrow{B}, \overrightarrow{C}\rangle = 0$ and $\left\|\overrightarrow{B}\right\| = 2\left\|\overrightarrow{C}\right\|$ holds. Thus by using equalities (36)-(37) we get the result.

Conversely, if (32) holds, then by using equalities (36) and (37) we get $\langle\overrightarrow{B}, \overrightarrow{C}\rangle = 0$ and $\left\|\overrightarrow{B}\right\| = 2\left\|\overrightarrow{C}\right\|$, which means that M is superconformal. □

5. Ruled Chen surfaces in $\mathbb{E}^4$

Let M be an n-dimensional smooth submanifold of m-dimensional Riemannian manifold N and ζ a normal vector field of M. Let ξ_x be $m-n$ mutually orthogonal unit normal vector fields of M such that $\zeta = \|\zeta\|\,\xi_1$. In [4] B.Y. Chen defined the allied vector field $a(\zeta)$ of a normal vector field ζ by the formula

$$a(v) = \frac{\|\zeta\|}{n}\sum_{x=2}^{m-n}\{tr(A_1A_x)\}\,\xi_x,$$

where $A_x = A_{\xi_x}$ is the shape operator. In particular, the allied mean curvature vector field $a(\overrightarrow{H})$ of the mean curvature vector $\overrightarrow{H}$ is a well-defined normal vector field orthogonal to $\overrightarrow{H}$. If the allied mean vector $a(\overrightarrow{H})$ vanishes identically, then the submanifold M is called $\mathcal{A}$-submanifold of N. Furthermore, $\mathcal{A}$-submanifolds are also called Chen submanifolds [7].

For the case M is a smooth surface of $\mathbb{E}^4$ the allied vector $a(\overrightarrow{H})$ becomes

$$a(\overrightarrow{H}) = \frac{\left\|\overrightarrow{H}\right\|}{2}\{tr(A_{N_1}A_{N_2})\}\,N_2 \tag{38}$$

where $\{N_1, N_2\}$ is an orthonormal basis of $N(M)$.

In particular, the following result of B. Rouxel determines Chen surfaces among the ruled surfaces in Euclidean spaces.

Theorem 2 (see [19]). *A ruled surface in $\mathbb{E}^n$ $(n > 3)$ is a Chen surface if and only if it is one of the following surfaces:*

i) a developable ruled surface,

ii) a ruled surface generated by the n-th vector of the Frenet frame of a curve in $\mathbb{E}^n$ with constant (n-1)-st curvature,

iii) a "helicoid" with a constant distribution parameter.

We prove the following result.

Proposition 4. *Let M be a ruled surface given by parametrization (12). If M is non-minimal superconformal, then it is a Chen surface.*

Proof. Suppose M is a superconformal ruled surface in $\mathbb{E}^4$. Then by Theorem 1 normal vectors $\overrightarrow{B} = \frac{1}{\sqrt{EG}}h(X_u, X_v)$ and $\overrightarrow{C} = \frac{1}{2E}h(X_u, X_u)$ are orthogonal to each other. So, we can choose an orthonormal normal frame field $\{N_1, N_2\}$ of M with

$$N_1 = \frac{h(X_u, X_u)}{\|h(X_u, X_u)\|} \text{ and } N_2 = \frac{h(X_u, X_v)}{\|h(X_u, X_v)\|}. \tag{39}$$

Hence, by using (5), (38) with (39) we conclude that $tr(A_{N_1}A_{N_2}) = 0$. So M is a Chen surface. ∎

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