QUOTIENT COMPLEXITY OF STAR-FREE LANGUAGES

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The quotient complexity, also known as state complexity, of a regular language is the number of distinct left quotients of the language. The quotient complexity of an operation is the maximal quotient complexity of the language resulting from the operation, as a function of the quotient complexities of the operands. The class of star-free languages is the smallest class containing the finite languages and closed under boolean operations and concatenation. We prove that the tight bounds on the quotient complexities of union, intersection, difference, symmetric difference, concatenation and star for star-free languages are the same as those for regular languages, with some small exceptions, whereas $2^n - 1$ is a lower bound for reversal.

1. Introduction

The class of regular languages can be defined as the smallest class containing the finite languages and closed under union, concatenation and star. Since regular languages are also closed under complementation, one can redefine them as the smallest class containing the finite languages and closed under boolean operations, concatenation and star. In this new formulation, a natural question is that of the *generalized star height* of a regular language, which is the minimum number of nested stars required to define the language when boolean operations are allowed. It is not clear who first considered the problem of generalized star height, but McNaughton and Papert reported in their 1971 monograph [13] that this problem had been open "for many years". There exist regular languages of star height 0 and 1, but it is not even known whether there exists a language of star height 2. See `http://liafa.jussieu.fr/~jep/Problemes/starheight.html`.

We consider regular languages of star height 0, which are also called *star-free.* In 1965, Schützenberger proved [16] that a language is star-free if and only if its syntactic monoid is *group-free*, that is, has only trivial subgroups. An equivalent condition is that the minimal deterministic automaton of a star-free language is

permutation-free, in the sense that no input word can perform a permutation of any non-empty subset S of the set of states, unless that permutation is the identity mapping. Another point of view is that these automata are *counter-free*, since they cannot count modulo any integer greater than 1. They can, however, *count to a threshold*, that is, $0, 1, \ldots, n-2$, and ($n-1$ *or more*). Such automata are called *aperiodic,* and this is the term that we use.

The *state complexity* of a regular language [18] is the number of states in the minimal deterministic finite automaton accepting that language. We prefer the equivalent concept of *quotient complexity* [1], which is the number of distinct left quotients of the language, because quotient complexity is a language-theoretic concept. The *quotient complexity of an operation* in a subclass of regular languages is the maximal quotient complexity of the language resulting from the operation, as a function of the quotient complexities of the operands, when they range over all the languages in the subclass. The complexities of basic operations in the class of regular languages were studied by Maslov [12] and Yu, Zhuang and Salomaa [19].

The complexities of operations were also considered in several subclasses of regular languages: unary [15, 19], finite [6, 18], ideal [3], closed [5], prefix-free [10], suffix-free [9], bifix-, factor-, and subword-free [4], and convex [2]. The complexity of operations can be significantly lower in a subclass of regular languages than in the general case. We prove that this is *not* the case for star-free languages, which meet the bounds for regular languages, with small exceptions. These results parallel those of Holzer, Kutrib and Meckel [11] who proved in 2011 that, in most cases, exactly the same tight state complexity bounds are reached by operations on aperiodic nondeterministic finite automata (NFA's) as on general NFA's.

In Section 2 we define our terminology and notation. Boolean operations, concatenation, star, and reversal are studied in Sections 3–6, respectively. Unary languages are treated in Section 7, and Section 8 concludes the paper.

2. Terminology and Notation

For general background on regular languages we refer the reader to [17]. If Σ is a finite non-empty alphabet, then Σ^* is the set of all words over this alphabet, with ε as the empty word. For $w \in \Sigma^*$, $a \in \Sigma$, let $|w|$ be the length of w, and $|w|_a$, the number of a's in w. A language is any subset of Σ^*.

The following are set operations on languages: *complement* ($\overline{L} = \Sigma^* \setminus L$), *union* ($K \cup L$), *intersection* ($K \cap L$), *difference* ($K \setminus L$), and *symmetric difference* ($K \oplus L$). We also use *product*, also called *(con)catenation* ($KL = \{w \in \Sigma^* \mid w = uv, u \in K, v \in L\}$) and *star* ($K^* = \bigcup_{i \geqslant 0} K^i$). The *reverse* w^R *of a word* $w \in \Sigma^*$ is defined by: $\varepsilon^R = \varepsilon$, and $(wa)^R = aw^R$. The *reverse of a language* L is $L^R = \{w^R \mid w \in L\}$.

Regular languages are the smallest class of languages containing the finite languages and closed under boolean operations, product and star. *Star-free* languages are the languages one can construct from finite languages using only boolean operations and concatenation. Some examples of star-free languages are the empty set $\emptyset$,

$\Sigma^* = \overline{\emptyset}$, $\{b\}^* = \overline{\Sigma^*\{a\}\Sigma^*} = \overline{\overline{\emptyset}\{a\}\overline{\emptyset}}$ over $\Sigma = \{a, b\}$, and $\{a\}\{a\}^* = \overline{\overline{\varepsilon}}$ over $\Sigma = \{a\}$. We do not write such expressions for star-free languages, but denote them using the usual notation of regular expressions.

The *(left) quotient* of a language L by a word w is defined as $L_w = \{x \in \Sigma^* \mid wx \in L\}$. The number of distinct quotients of a language is called its *quotient complexity* and is denoted by $\kappa(L)$. A quotient L_w is *final* if $\varepsilon \in L_w$; otherwise it is *non-final*.

A *deterministic finite automaton (DFA)* is a quintuple $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$, where Q is a finite set of *states*, Σ is a finite *alphabet*, $\delta : Q \times \Sigma \to Q$ is the *transition function*, q_0 is the *initial state*, and $F \subseteq Q$ is the set of *final states*. As usual, the transition function is extended to $Q \times \Sigma^*$. A DFA $\mathcal{D}$ accepts $w \in \Sigma^*$ if $\delta(q_0, w) \in F$, and the *language* accepted by $\mathcal{D}$ is $L(\mathcal{D})$. The *language of a state* q of $\mathcal{D}$ is the language L_q accepted by the automaton $(Q, \Sigma, \delta, q, F)$. If the language of a state is empty, that state is *empty*.

The *quotient DFA* of a regular language L is $\mathcal{D} = (Q, \Sigma, \delta, q_0, F)$, where $Q = \{L_w \mid w \in \Sigma^*\}$, $\delta(L_w, a) = L_{wa}$, $q_0 = L_\varepsilon = L$, and $F = \{L_w \mid L_w \text{ is final}\}$. Since this is the minimal DFA accepting L, the quotient complexity of L is equal to the state complexity of L, and we call it simply *complexity*.

A *transformation* of a set $S = \{1, \ldots, n\}$ is a mapping

$$t = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ i_1 & i_2 & \cdots & i_{n-1} & i_n \end{pmatrix},$$

where $i_k \in S$ for $1 \leqslant k \leqslant n$. Each word in Σ^* performs a transformation of the set Q of states of a DFA $\mathcal{D}$. A DFA is *aperiodic* if no word performs a permutation, other than the identity permutation, of a non-empty subset of Q. Since testing if a DFA is aperiodic is PSPACE-complete [7], we use a subclass of aperiodic automata. Without loss of generality, we assume that $Q = \{1, \ldots, n\}$. A transformation is *non-decreasing* if $j < k$ implies $i_j \leqslant i_k$. Such a transformation cannot have a non-trivial permutation, and the composition of non-decreasing transformations is non-decreasing. Hence a DFA with non-decreasing input transformations is aperiodic.

A *nondeterministic finite automaton (NFA)* is defined as a quintuple $\mathcal{N} = (Q, \Sigma, \eta, I, F)$, where Q, Σ, and F are as in a DFA, $\eta : Q \times \Sigma \to 2^Q$ is the *transition function* and $I \subseteq Q$ is the *set of initial states*. If η also allows ε, that is, $\eta : Q \times (\Sigma \cup \{\varepsilon\}) \to 2^Q$, we call $\mathcal{N}$ an *ε-NFA*.

3. Boolean Operations

We now consider the quotient complexity of union, intersection, symmetric difference, and difference in the class of star-free languages. The upper bound for these four operations in the class of regular languages is mn [1, 12, 19].

Theorem 1. *Given $m, n \geqslant 1$, for each of the operations union, intersection, symmetric difference, and difference, there exist binary star-free languages K and L with quotient complexities m and n, respectively, that meet the bound mn.*
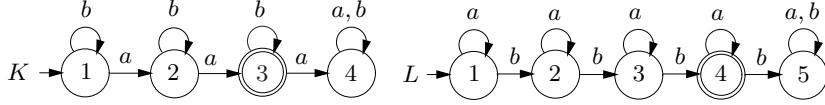
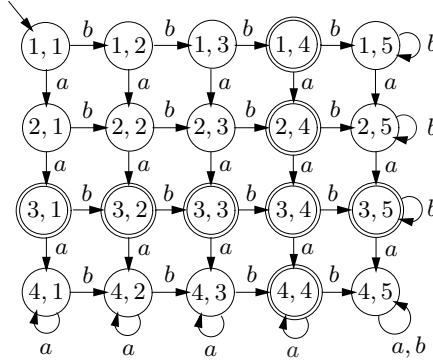Fig. 1. Witnesses K and L for union with $m = 4$ and $n = 5$.

Fig. 2. Quotient automaton of $K \cup L$.

Proof. Let $\Sigma = \{a, b\}$. We examine union first. For $m = 1$, let $K = \emptyset$ and let L be any binary star-free language with $\kappa(L) = n$. Then $\kappa(K \cup L) = \kappa(L) = n = mn$. Similarly, if $n = 1$, let $L = \emptyset$ and let K be any binary star-free language with $\kappa(K) = m$. Then $\kappa(K \cup L) = mn$.

For $m, n \geqslant 2$, let $K = (b^*a)^{m-2}b^* = \{w \in \Sigma^* \mid |w|_a = m - 2\}$, and $L = (a^*b)^{n-2}a^* = \{w \in \Sigma^* \mid |w|_b = n - 2\}$; then $\kappa(K) = m$ and $\kappa(L) = n$, and both K and L are star-free since their quotient DFA's are non-decreasing. The quotient DFA's of K and L are in Fig. 1 for $m = 4$ and $n = 5$, and their direct product for $K \cup L$, in Fig. 2.

Let $M = K \cup L$, and consider the quotients of M by the mn words $a^i b^j$, $i = 0, \ldots, m - 1$, and $j = 0, \ldots, n - 1$; these quotients $M_{a^i b^j}$ correspond to states $(i + 1, j + 1)$ in the direct-product automaton for M. We begin with the non-final quotients of M. First, $M_{a^{m-1}b^{n-1}} = \emptyset$, and all the other quotients are non-empty. Next, if $i < m - 2$ and $j < n - 2$ (rows 1 to $m - 2$, columns 1 to $n - 2$), then the pair (a^{m-2-i}, b^{n-2-j}) of non-empty words belongs to $M_{a^i b^j}$ and to no other non-final quotient. If $i < m - 2$, then $M_{a^i b^{n-1}}$ (rows 1 to $m - 2$, column n) contains a^{m-2-i}, but has no words from b^*. If $j < n - 2$, then $M_{a^{m-1}b^j}$ (row m, columns 1 to $n - 2$) contains b^{n-2-j}, but has no words from a^*. So all non-final quotients are distinct.

Now turn to the final quotients. For $i, k \leqslant m - 2$, quotient $M_{a^i b^{n-2}}$ (rows 1 to $m - 1$, column $n - 1$) contains ba^{m-2-i}, and this word is not contained in any

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other quotient $M_{a^k b^{n-2}}$ with $k \neq i$, and $M_{a^{m-1} b^{n-2}}$ has no words from ba^*. Thus all the quotients in column $n-1$ are distinct. For $j, \ell \leqslant n-2$, $M_{a^{m-2} b^j}$ (row $m-1$, columns 1 to $n-1$) contains ab^{n-2-j}, and this word is not contained in any other quotient $M_{a^{m-2} b^\ell}$ with $\ell \neq j$, and $M_{a^{m-2} b^{n-1}}$ has no words from ab^*. Thus all the quotients in row $m-1$ are distinct. Excluding $M_{a^{m-2} b^{n-2}}$, each quotient in column $n-1$ contains a but not b, each quotient in row $m-1$ contains b but not a, and $M_{a^{m-2} b^{n-2}}$ contains both a and b. Hence all final quotients are distinct, and our claim holds for union.

For difference, we can use $\overline{K}$ and L, where K and L meet the bound mn for union, because $\kappa(\overline{K} \setminus L) = \kappa(\overline{K} \cap \overline{L}) = \kappa(\overline{\overline{K} \cap \overline{L}}) = \kappa(K \cup L)$.

For intersection, it was shown in [3] that the languages $K = (b^* a)^{m-1} \Sigma^* = \{w \in \Sigma^* \mid |w|_a \geqslant m-1\}$ and $L = (a^* b)^{n-1} \Sigma^* = \{w \in \Sigma^* \mid |w|_b \geqslant n-1\}$ meet the bound mn. Since both languages are star-free, our claim holds for intersection. These languages also meet the bound mn for symmetric difference [3]. $\square$

4. Product

The tight bound for product of regular languages [12, 19] is $(m-1)2^n + 2^{n-1}$. We show that this bound can be met by star-free languages, with some exceptions.

In subset constructions, we use the notation $S \xrightarrow{w} T$ to mean that subset S under input word w moves to subset T.

Theorem 2. *Given* $m \geqslant 1$, $n \geqslant 3$, *there exist quaternary star-free languages* K *and* L *with quotient complexities* m *and* n, *respectively, such that* $\kappa(KL) = (m-1)2^n + 2^{n-1}$.

**Proof.** The languages that meet the bound are defined by their quotient DFA's:

The quotient DFA for K is $\mathcal{D}_K = (Q_K, \Sigma, \delta_K, q_0, F_K)$, where $Q_K = \{q_1, q_2, \ldots, q_m\}$, $\Sigma = \{a, b, c, d\}$, $q_0 = q_1$, $F_K = \{q_m\}$, and

$$\delta_K(q_i, a) = q_{i+1} \text{ for } i = 1, \ldots, m-1, \quad \delta_K(q_m, a) = q_m,$$
$$\delta_K(q_i, b) = q_{i-1} \text{ for } i = 2, \ldots, m, \quad \delta_K(q_1, b) = q_1,$$
$$\delta_K(q_i, c) = q_i \text{ for } i = 1, \ldots, m,$$
$$\delta_K(q_i, d) = q_m \text{ for } i = 1, \ldots, m.$$

The quotient DFA for L is $\mathcal{D}_L = (Q_L, \Sigma, \delta_L, p_0, F_L)$, where $Q_L = \{1, 2, \ldots, n\}$, $\Sigma = \{a, b, c, d\}$, $p_0 = 1$, $F_L = \{n-1\}$, and

$$\delta_L(i, c) = i+1 \text{ for } i = 1, \ldots, n-1, \quad \delta_L(n, c) = n,$$
$$\delta_L(i, d) = i-1 \text{ for } i = 2, \ldots, n, \quad \delta_L(1, d) = 1,$$
$$\delta_L(i, a) = i+1 \text{ for } i = 2, \ldots, n-1, \quad \delta_L(1, a) = 1, \quad \delta_L(n, a) = n,$$
$$\delta_L(i, b) = i \text{ for } i = 1, \ldots, n.$$

The automaton $\mathcal{D}_K$ for $m = 4$ is shown in Fig. 3, where the transition labeled ε should be ignored for now. The automaton $\mathcal{D}_L$ for $n = 5$ is also shown in Fig. 3.
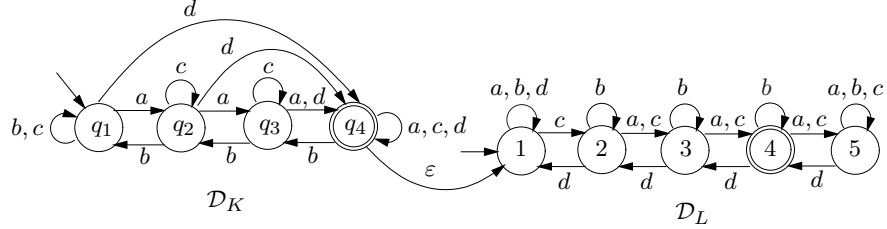
Fig. 3. ε-NFA $\mathcal{N}$ of KL.

Now we construct an ε-NFA which accepts the language KL, as illustrated in Fig. 3. The transition labeled ε is introduced and q_4 is made a non-final state. Here the only initial state is q_1, the set of final states is $\{4\}$, and the transitions are as shown in the figure.

Next, we need to find the minimal DFA $\mathcal{D}_{KL}$ accepting the language KL. This is done by carrying out a subset construction on the ε-NFA $\mathcal{N}$; it will be shown that $(m-1)2^n + 2^{n-1}$ subsets of states of $\mathcal{N}$ are reachable from the set $\{q_1\}$ of initial states of $\mathcal{N}$, and that these subsets are pairwise distinguishable. This will imply that $\mathcal{D}_{KL}$ has at least $(m-1)2^n + 2^{n-1}$ distinct states. Since it is known that $\mathcal{D}_{KL}$ cannot have more than that many states [12, 19], the quotient complexity of L is precisely $(m-1)2^n + 2^{n-1}$.

Our first task is to show the reachability of the required number of states. The following notation will be used. For $1 \leqslant s_k \leqslant n-1$, $S = \{s_1, \ldots, s_k\}$, $s_1 < s_2 < \cdots < s_k$, with $s_1, s_2, \ldots, s_k \in Q_L$, and $0 \leqslant x \leqslant n - s_k$, denote $(s_1 + x, \ldots, s_k + x)$ by S_{+x}. Similarly, for $2 \leqslant s_1 \leqslant n$, and $0 \leqslant x \leqslant s_1 - 1$, denote $(s_1 - x, \ldots, s_k - x)$ by S_{-x}.

We first show by induction on the size of S that all $(m-1)2^{n-1}$ subsets of the form $\{q_i\} \cup S$, where $q_i \in Q_K$, $q_i \neq q_m$, and $S \subseteq Q_L \setminus \{1\}$, are reachable.

When $S = \emptyset$, the set $\{q_i\}$ is reached by a^{i-1}, for $i = 1, \ldots, m-1$. Now suppose we want to reach $\{q_i\} \cup T$, where $i \neq m$, $T = \{s_0, s_1, \ldots, s_k\}$, $k \geqslant 0$, and $1 < s_0 < s_1 < \cdots < s_k$. Let $S = \{s_1, \ldots, s_k\}$; by the induction assumption, $\{q_i\} \cup S$ is reachable. Then $\{q_i\} \cup S \xrightarrow{d^{s_0-1}} \{q_m, 1\} \cup S_{-(s_0-1)} \xrightarrow{b^{m-i}} \{q_i, 1\} \cup S_{-(s_0-1)} \xrightarrow{c^{s_0-1}} \{q_i\} \cup \{s_0\} \cup S = \{q_i\} \cup T$. Thus $\{q_i\} \cup T$ is also reachable.

Next, we prove that the 2^{n-1} subsets of the form $\{q_m, 1\} \cup S$, where S is any subset of $Q_L \setminus \{1\}$, are reachable. If $m = 1$, then $\{q_1, 1\}$ is the initial subset. Let S and T be as above. Then $\{q_1, 1\} \cup S \xrightarrow{d^{s_0-1}} \{q_1, 1\} \cup S_{-(s_0-1)} \xrightarrow{c} \{q_1, 1\} \cup \{2\} \cup S_{-(s_0-2)} \xrightarrow{a^{s_0-2}} \{q_1, 1\} \cup \{s_0\} \cup S = \{q_1, 1\} \cup T$.

If $m \geqslant 2$, there are two cases. If $2 \notin S$, then start with $\{q_1\} \cup S$, which has already been shown to be reachable. We then have $\{q_1\} \cup S \xrightarrow{d} \{q_m, 1\} \cup S_{-1} \xrightarrow{a} \{q_m, 1\} \cup S$. If $2 \in S$, then start with $\{q_1\} \cup S \setminus \{2\}$. Now $\{q_1\} \cup S \setminus \{2\} \xrightarrow{d} \{q_m, 1\} \cup (S \setminus \{2\})_{-1} \xrightarrow{c}$

$\{q_m, 1\} \cup \{2\} \cup (S \setminus \{2\}) = \{q_m, 1\} \cup S$.

Finally, we show that the $(m-1)2^{n-1}$ subsets of the form $\{q_i, 1\} \cup S$, where $i < m$, and $S \subseteq Q_L \setminus \{1\}$ are reachable. We have $\{q_m, 1\} \cup S \xrightarrow{b^{m-i}} \{q_i, 1\} \cup S$.

In summary, $(m-1)2^n + 2^{n-1}$ different subsets are reachable. We now prove that all these subsets are pairwise distinguishable.

For $1 \leqslant k \leqslant n-1$, state k of Q_L accepts the word $w_k = c^{n-1-k}$, and state n accepts the word $w_n = d$; moreover, each of these words w_h is accepted by only that one state h of Q_L, and none of these words is accepted by state q_i, if $i \neq m$. Hence, if h is in $S \setminus T$ or in $T \setminus S$, then S and T are distinguished by w_h.

First, let $1 \leqslant i \leqslant j < m$, and consider $\{q_i\} \cup S$ and $\{q_j\} \cup T$, where $S, T \subseteq Q_L$, and S and T differ by state h. Then $\{q_i\} \cup S$ and $\{q_j\} \cup T$ are distinguished by w_h. Next, let $1 \leqslant i < j < m$ and take $\{q_i\} \cup S$ and $\{q_j\} \cup S$, where $S \subseteq Q_L$. First apply c; then we reach $\{q_i\} \cup R$ and $\{q_j\} \cup R$, where $1 \notin R$. Then $\{q_j\} \cup R$ accepts $a^{m-j}c^{n-2}$, whereas $\{q_i\} \cup R$ rejects this word.

Second, suppose $S, T \subseteq Q_L \setminus \{1\}$ and S and T differ by state h; then $\{q_m, 1\} \cup S$ and $\{q_m, 1\} \cup T$ are distinguished by w_h.

Third, consider $\{q_i\} \cup S$, where $S \subseteq Q_L$ and $\{q_m, 1\} \cup T$, where $T \subseteq Q_L \setminus \{1\}$ and $i < m$. Then c^{n-1} is accepted by $\{q_m, 1\} \cup T$ but not by $\{q_i\} \cup S$.

Since all reachable sets are pairwise distinguishable, the bound is met. $\qquad\square$

Corollary 3. *Given $n \geqslant 1$, there exists a ternary star-free language L with quotient complexity n such that $\kappa(\Sigma^* L) = 2^{n-1}$.*

Proof. If $K = \Sigma^*$, the DFA $\mathcal{D}_K$ has one state, which is both initial and final. Now b is not needed in the proofs of reachability and distinguishability. $\qquad\square$

A *right (left) ideal* [3] is a language L satisfying $L = L\Sigma^*$ ($L = \Sigma^* L$). If $M = K\Sigma^*$ ($M = \Sigma^* K$), then M is the right (left) ideal generated by K. Corollary 3 shows that the bound 2^{n-1} on the quotient complexity of the left ideal generated by a regular language can also be met by a star-free language.

If $n = 1$ in Theorem 2, then either $KL = \emptyset$ and $\kappa(KL) = 1$, or $KL = K\Sigma^*$ is the right ideal generated by K. In the second case, it is known [19] that m is a tight bound for $\kappa(K\Sigma^*)$, and that the language $a^{m-1}a^*$ is a witness [3]. Since that witness is star-free, the general bound holds also for star-free languages.

The case $m \geqslant 2$ and $n = 2$ remains. For $m = n = 2$, the best bound for product of regular languages is 6, whereas it is 4 for star-free languages. This was verified with the *GAP* package *Automata* [8] by enumerating all products of 2-state aperiodic automata.

There are only three types of inputs possible for a 2-state aperiodic DFA: the input that takes both states to state 1, the input that takes both states to state 2, and the identity input. If 1 is the final state, then subsets $\{1\}$ and $\{1, 2\}$ are not distinguishable. Therefore a non-final quotient of $\mathcal{D}_K$ can appear with only three subsets of quotients of $\mathcal{D}_L$ in the DFA of KL instead of $2^2 = 4$, and a final quotient,

only with one subset instead of two. The complexity is maximized when there is only one final quotient of K. Hence $\kappa(KL) \leqslant (m-1)3+1 = 3m-2$. If 2 is the final state, then $\{2\}$ and $\{1,2\}$ are not distinguishable. Hence $\kappa(KL) \leqslant (m-1)3+2 = 3m-1$ in this case.

Theorem 4. *Given $m \geqslant 2$, there exist ternary star-free languages K and L with quotient complexities m and 2, respectively, such that $\kappa(KL) = 3m - 2$.*

Proof. Let $\mathcal{D}_K(a, b, c)$ be the DFA in the proof of Theorem 2 restricted to input alphabet $\{a, b, c\}$. Let $\mathcal{D}_L = (\{1, 2\}, \{a, b, c\}, \delta, 1, \{1\})$, where

$$\delta_L(i, a) = i \text{ for } i = 1, 2,$$
$$\delta_L(i, b) = 1 \text{ for } i = 1, 2,$$
$$\delta_L(i, c) = 2 \text{ for } i = 1, 2.$$

For $i \neq m$, subset $\{q_i\}$ is reached by a^{i-1}, $\{q_i\} \cup \{1\}$, by $a^{m-1}b^{m-i}$, and $\{q_i\} \cup \{2\}$, by $a^{m-1}b^{m-i}c$. Finally, $\{q_m\} \cup \{1\}$ is reached by a^{m-1}. This gives $3m - 2$ subsets.

For $i \neq m$, $\{q_i\}$ accepts no words from b^*, $\{q_i\} \cup \{1\}$ accepts ε, and $\{q_i\} \cup \{2\}$ accepts b but not ε. Hence subsets $\{q_i\} \cup S$ and $\{q_i\} \cup T$ with $i \neq m$, $S, T \in \{\emptyset, \{1\}, \{2\}\}$, and $S \neq T$, are distinguishable. Next, $\{q_i\} \cup S$ and $\{q_j\} \cup S$ with $i < j < m$ are distinguished by ca^{m-j}. Also, $\{q_i\}$ and $\{q_i\} \cup \{2\}$ are distinguished from $\{q_m\} \cup \{1\}$ by ε, and $\{q_i\} \cup \{1\}$ from $\{q_m\} \cup \{1\}$ by c. Therefore all $3m - 2$ subsets are distinguishable. □

We do not know whether the bound $3m - 1$ can be reached. However, we have verified with *GAP* that it cannot be reached if $m = 2$.

5. Star

The following DFA plays a key part in finding bounds on the quotient complexities of stars of star-free languages. Let $n \geqslant 3$, and $\mathcal{D}_n = \mathcal{D}_n(a, b, c, d) = (Q, \{a, b, c, d\}, \delta, 1, \{n-1\})$, where $Q = \{1, 2, \ldots, n\}$ and

$$\delta(i, a) = i + 1 \text{ for } i = 1, \ldots, n-1, \quad \delta(n, a) = n,$$
$$\delta(i, b) = i - 1 \text{ for } i = 2, \ldots, n, \quad \delta(1, b) = 1,$$
$$\delta(i, c) = i - 1 \text{ for } i = 2, \ldots, n-1, \quad \delta(1, c) = 1, \quad \delta(n, c) = n,$$
$$\delta(i, d) = n \text{ for } i = 1, \ldots, n.$$

Since all the inputs perform non-decreasing transformations, $\mathcal{D}_n$ is aperiodic.

In Fig. 4, if we ignore state 0 and its outgoing transitions, and also the ε transition, and use state 1 as the initial state, then the figure shows the minimal automaton $\mathcal{D}_7(a, b, c, d)$ of a star-free language L. After state 0, transitions from state 0, and the ε transition are added, the figure depicts the ε-NFA of L^*.
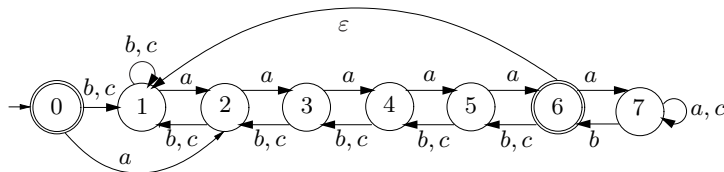
Fig. 4. ε-NFA $\mathcal{N}$ of L^*, $\kappa(L) = 7$. Transitions under d (not shown) are all to state 7.

We first study $\mathcal{D}_n(a, b)$, the restriction of $\mathcal{D}_n(a, b, c, d)$ to the alphabet $\{a, b\}$. We now show that $\mathcal{D}_n(a, b)$ is minimal. For $1 \leqslant i \leqslant n - 2$, only state i accepts a^{n-1-i}, and only state n accepts b. Hence all the non-final states are distinguishable. Since there is only one final state, there is nothing further to prove.

DFA $\mathcal{D}_n(a, b)$ with final state changed to n has been studied in detail by McNaughton and Papert [13] (p. 27), who showed that the language accepted by $\mathcal{D}_n(a, b)$ is not locally testable, and derived a star-free expression for it. For our version with final state $n - 1$, the language accepted by $\mathcal{D}_3$ can be described by the regular expression: $(b \cup a(aa^*b)^*b)^*a(aa^*b)^*$.

Lemma 5. *If $n \geqslant 3$, and L is the star-free language accepted by $\mathcal{D}_n(a, b)$, then $\kappa(L^*) = 2^{n-1} + 2^{n-3} - 1$.*

Proof. Consider the subsets of $\{0\} \cup Q$ in the subset construction of the DFA for L^*. Since 0 can only appear in $\{0\}$, the remaining reachable subsets are subsets of Q. The empty subset cannot be reached because there is a transition from each state under every letter. Since state $n - 1$ cannot occur without state 1, we eliminate 2^{n-2} subsets. Because state $n - 1$ always appears with state 1, and state n can only be reached from state $n - 1$ by a, the subset $\{n\}$ first appears with state 2, and afterwards, always with a state from $\{1, \ldots, n - 1\}$; hence $\{n\}$ cannot be reached. Also, 1 and n cannot appear together without $n - 1$, because n cannot be reached by b, and 1 cannot be reached by a without including $n - 1$. This eliminates another 2^{n-3} subsets. So $1 + 2^{n-2} + 1 + 2^{n-3}$ subsets are unreachable, and $\kappa(L^*) \leqslant 2^n + 1 - (2^{n-2} + 2^{n-3} + 2) = 2^{n-1} + 2^{n-3} - 1$.

Now turn to the reachable subsets, and note that subsets $\{0\}$ and $\{1\}$ are reached by ε and b, respectively.

First, let $\mathbb{P} = \{S \subseteq \{2, \ldots, n - 2\} \mid S \neq \emptyset\}$. All singleton sets $\{i\} \in \mathbb{P}$ are reached by a^{i-1} from $\{1\}$. Now let $S = \{s_1, \ldots, s_k\}$, $T = \{s_0, s_1, \ldots, s_k\}$, where $0 < k$, $1 < s_0 < s_1 < \cdots < s_k < n - 1$, and $h = n - 1 - s_k$; then $S \xrightarrow{a^h} \{1\} \cup S_{+h} \xrightarrow{b^h} \{1\} \cup S \xrightarrow{b^{s_0-1}} \{1\} \cup S_{-(s_0-1)} \xrightarrow{a^{s_0-1}} \{s_0\} \cup S$. Thus any $T \in \mathbb{P}$ can be reached from a smaller $S \in \mathbb{P}$, and so all subsets in $\mathbb{P}$ are reachable.

Second, let $\mathbb{Q} = \{\{1\} \cup S \mid S \in \mathbb{P}\}$; then $S \xrightarrow{a^h b^h} \{1\} \cup S$, as above, and all subsets in $\mathbb{Q}$ are reachable.

Third, let $\mathbb{R} = \{\{1, n - 1\} \cup S \mid S = \emptyset \text{ or } S \in \mathbb{P}\}$. If $S = \emptyset$, then $\{1, n - 1\}$

is reachable from $\{1\}$ by a^{n-2}. Now suppose $S \in \mathbb{P}$ is not empty. If $i \in S$, then $\{i\} \overset{a^{n-1-i}}{\to} \{1, n-1\} \overset{a^{i-1}}{\to} \{i, n\}$. So $S \overset{a^{n-2}}{\to} \{n\} \cup S$. Now, if $s_k = n - 2$, then $\{n\} \cup S \overset{a}{\to} \{1, n-1, n\} \cup S_{+1} \overset{b}{\to} \{1, n-1\} \cup S$. If $s_k < n - 2$, then $\{n\} \cup S \overset{a}{\to} \{n\} \cup S_{+1} \overset{b}{\to} \{1, n-1\} \cup S$. In either case, $S \overset{a^{n-1}b}{\to} \{1, n-1\} \cup S$, and all 2^{n-3} subsets in $\mathbb{R}$ are reachable.

Fourth, let $\mathbb{S} = \{\{n\} \cup T \mid T \in \mathbb{P} \cup \mathbb{R}\}$. We have shown that $S \overset{a^{n-2}}{\to} \{n\} \cup S$, if $S \in \mathbb{P}$. Since also $\{1, n-1\} \overset{a^{n-2}}{\to} \{1, n-1, n\}$, we have $\{1, n-1\} \cup S \overset{a^{n-2}}{\to} \{1, n-1, n\} \cup S$. Hence all $2^{n-2} - 1$ subsets $\{n\} \cup T$ in $\mathbb{S}$ are reachable.

Altogether, $2^{n-1} + 2^{n-3} - 1$ subsets are reachable. It remains to be shown that all the reachable subsets are pairwise distinguishable. State 0 does not accept ab, while $n - 1$ accepts it. Each state i with $1 \leqslant i \leqslant n - 2$ accepts a^{n-1-i} and each of these words is accepted by only that one state, and n accepts b. So any two subsets S and $T \neq S$ considered above are distinguishable. $\qquad \Box$

Theorem 6. *For $n \geqslant 2$, there exists a quaternary star-free language L with $\kappa(L) = n$ such that $\kappa(L^*) = 2^{n-1} + 2^{n-2}$. For $n = 1$, the tight upper bound is 2.*

Proof. For $n = 1$, there are only two languages, $\emptyset$ and Σ^*, and both are star-free. We have $\kappa(\emptyset^*) = 2$, and $\kappa((\Sigma^*)^*) = 1$. For $n = 2$, there are two star-free unary languages, ε and aa^*, and the bound cannot be met if $|\Sigma| = 1$. If $\Sigma = \{a, b\}$, then $b^*a\Sigma^*$ meets the bound 3. For $n = 3$, we analyzed all 3-state aperiodic automata using GAP. The bound 6 is met by $\mathcal{D}_3(a, b, c, d)$ defined above, and bounds 5 and 4 are met by $\mathcal{D}_3(a, b, c)$ and $\mathcal{D}_3(a, b)$, respectively. These bounds cannot be improved.

We now turn to the general case. We will show that the following sets of states are reachable in the nondeterministic automaton $\mathcal{N}$ (see Fig. 4) from the initial state 0: the set $\{0\}$, all subsets of Q containing $\{1, n-1\}$, and all non-empty subsets of $Q \setminus \{n-1\}$. By Lemma 5, we can reach all these subsets by words in $\{a, b\}^*$, except $\{n\}$ and the subsets of $Q \setminus \{n-1\}$ containing $\{1, n\}$.

We have $\{1, n-1\} \overset{a}{\to} \{2, n\} \overset{c}{\to} \{1, n\}$; hence $\{1, n\}$ is reachable. Now consider $\{n\} \cup S$, where $S = \{s_1, s_2, \ldots, s_k\} \in \mathbb{P}$. Let $h = n - 1 - s_k$; then using a^h we move to $\{1, n\} \cup S_{+h}$, and by c^h we reach $T = \{1, n\} \cup S$. Since $\{n\} \cup S$ is reachable by Lemma 5, T is also reachable. Thus we can reach all the subsets of $Q \setminus \{n-1\}$ containing $\{1, n\}$ by words in $\{a, b, c\}^*$. The only set missing now is $\{n\}$, and it is reached by d.

In Lemma 5, we have already shown that any two subsets $S, T \subseteq Q$ such that $T \neq S$ are distinguishable by words in $\{a, b\}^*$. $\qquad \Box$

Table 1 summarizes our results for the quotient complexity of L^* in case L is star-free. For unary languages, see Section 7. The figures in boldface type are known to be tight upper bounds. For $n = 4$, we analyzed all 4-state automata with non-decreasing input transformations. Automata $\mathcal{D}_4(a, b, c, d)$, $\mathcal{D}_4(a, b, c)$, and $\mathcal{D}_4(a, b)$ meet the bounds 12, 11, and 9, respectively. The bounds 11 and 9 cannot

be improved in the class of automata with non-decreasing input transformations. For the rest, the bounds for $|\Sigma| = 3$ and $|\Sigma| = 2$ are met by $\mathcal{D}_n(a, b, c)$, and $\mathcal{D}_n(a, b)$, respectively.

Table 1. Quotient complexities for stars of star-free languages.

n	1	2	3	4	5	6	7	8	$\cdots$	
$|\Sigma| = 1$	**2**	**2**	**3**	**4**	**5**	**7**	**13**	**21**	$\cdots$	$\mathbf{n^2 - 7n + 13}$
$|\Sigma| = 2$	$-$	**3**	**4**	9	19	39	79	159	$\cdots$	$2^{n-1} + 2^{n-3} - 1$
$|\Sigma| = 3$	$-$	$-$	**5**	11	23	47	95	191	$\cdots$	$2^{n-1} + 2^{n-2} - 1$
$|\Sigma| = 4$	$-$	$-$	**6**	**12**	**24**	**48**	**96**	**192**	$\cdots$	$\mathbf{2^{n-1} + 2^{n-2}}$

6. Reversal

For regular languages, it was shown by Mirkin [14] in 1966 that the tight bound for reversal is 2^n. One easily verifies that this bound 4 cannot be met for $n = 2$; there are only three inputs possible, the permutation of the two states being forbidden. For $n = 3$, we have enumerated all the DFA's with non-decreasing transformations, and none of them meets the bound 8 for reversal. In general, we conjecture that the bound 2^n cannot be met by a star-free language. However, the bound $2^n - 1$ can be met by such a language over an alphabet of $n - 1$ letters, as we now show.

Theorem 7. *For each $n \geqslant 1$, there exists a star-free language L with quotient complexity n such that $\kappa(L^R) = 2^n - 1$. For $n = 1$, the bound is met if $|\Sigma| \geqslant 1$, for $n = 2$, if $|\Sigma| \geqslant 2$, and for $n \geqslant 3$, if $|\Sigma| \geqslant n - 1$.*

Proof. For $n = 1$ and $\Sigma = \{a\}$, a^* is a witness. For $n = 2$ and $\Sigma = \{a, b\}$, Σ^*a is a witness. We have verified using *GAP* that all star-free languages L with $n = 2$ satisfy $\kappa(L^R) \leqslant 3$; hence this bound cannot be increased.

Now let $n \geqslant 3$, and let $\mathcal{D}_n = (Q, \Sigma, \delta, 1, E)$, where $Q = \{1, 2, \ldots, n\}$, $\Sigma = \{a, b, c_3, \ldots, c_{n-1}\}$, $E = \{i \in Q \mid i \text{ is even}\}$, and

$$\delta(i, a) = i + 1 \text{ for } i = 1, \ldots, n - 1, \quad \delta(n, a) = n,$$
$$\delta(i, b) = i - 1 \text{ for } i = 2, \ldots, n, \quad \delta(1, b) = 1,$$
$$\delta(i, c_j) = i \text{ for } i \neq j, \quad \delta(j, c_j) = j - 1 \quad \text{for } j = 3, \ldots, n - 1.$$

Since all the inputs perform non-decreasing transformations, $\mathcal{D}_n$ is aperiodic. Figure 5 shows the NFA $\mathcal{N}$ which is the reverse of DFA $\mathcal{D}_7$.

Assume initially that n is odd. Let $S = \{s_1, \ldots, s_k\}$ be a subset of Q, and let $1 \leqslant s_1 < \cdots < s_k \leqslant n$. Then NFA $\mathcal{N}$ has the following properties:
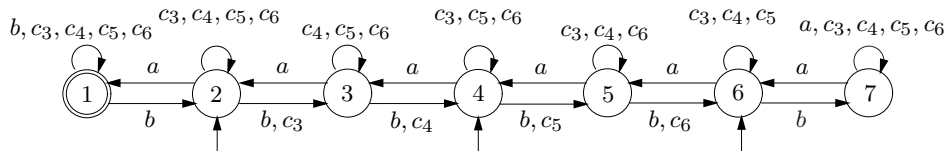
Fig. 5. NFA $\mathcal{N}$ of L^R, n odd.

P1 If $3 \leqslant j \leqslant n-1$, $j \in S$ and $j-1 \notin S$, then input c_j deletes state j from S without changing any of the other states.

P2 If $3 \leqslant j \leqslant n-1$, $j \notin S$, and $j-1 \in S$, then input c_j adds state j to S without changing any of the other states.

We now examine the sets of reachable states in $\mathcal{N}$. The set O of all the odd states cannot be reached. For suppose that it is reached from some set S. If it is reached by a, then S must be a subset of $E \cup \{n\}$. However, the successor under a of such a set S also contains $n-1$ if it contains n. If we use b, then S must be a subset of $E \cup \{1\}$. But then the successor of S also contains 2 if it contains 1. If we use c_i with i odd, then S must be a subset of $O \setminus \{i\}$, and S must also have $i-1$. But then the successor of S also contains $i-1$, which is even, if it contains i. If we use c_i with i even, then we also get i.

If $n = 3$, there are no c_i inputs. Set $\{2\}$ is initial, $\{1\}$ can be reached by a and $\{3\}$ by b. We can get $\emptyset$ by aa, $\{1,2\}$ and $\{2,3\}$ by ab and ba, respectively, and $\{1,2,3\}$ by abb. Set $\{1,3\}$ is unreachable. So assume $n \geqslant 5$.

Let $M = S \setminus \{1, n\}$ be the set of *middle states*. First consider subsets S of M. If $2 \in S$, start with $E = \{2, 4, \ldots, n-1\}$. By using inputs c_i, delete $n-1$ or not, add $n-2$ or not, etc., until we reach 2, which cannot be removed by any c_i. If $2 \notin S$, then $S_{-(s_1-2)}$ has 2, is a subset of M, and so is reachable; then S is reached by b^{s_1-2} from $S_{-(s_1-2)}$.

Second, consider subsets S of Q containing 1 but not n. If $2 \in S$, start with E and apply ab to reach $\{1\} \cup E$. Each state in E, except 2, is without a predecessor in $\{1\} \cup E$. Hence, by using inputs c_i, we can construct any such S. If $2 \notin S$, start with E and apply a to reach $O \setminus \{n\}$, where O of all the odd states. By using inputs c_i, we can construct any such set S.

Third, examine subsets S of Q containing n but not 1. If $2 \in S$, start with E and apply b to reach $E_{+1} = \{3, 5, \ldots, n\} = O \setminus \{1\}$, and then apply a to get $E \cup \{n\}$. Construct any such set S using inputs c_i. If $2 \notin S$, then S is a subset of $\{3, \ldots, n\}$ containing n. Since the set S_{-1} is a subset of M, it is reachable; then S is reached by b from S_{-1}.

Finally, consider subsets S containing both 1 and n. Apply $baab$ to E to reach $\{1, n\} \cup E$. From this set we can reach any set containing $\{1, 2, n\}$.

Now assume that $2 \notin S$. We now show that $\{i\} \cup O$ is reachable for every even $i > 2$ in Q. Apply baa to E to reach $\{n-1\} \cup O$. If $i = n-1$, we are done; otherwise,

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delete $n-2$ and $n-1$ by c_{n-2} and c_{n-1} in that order. Then insert $n-3$ and $n-2$ by c_{n-3} and c_{n-2} in that order. If $i = n-3$, we are done; otherwise, continue in this fashion. If we reach $\{3, 4, 5\}$, then $i = 4$, and the process stops.

If $n = 5$, then we can reach $\{1, 3, 4, 5\}$. From $\{1, 3, 4, 5\}$ we can get $\{1, 5\}$, $\{1, 4, 5\}$, and $\{1, 3, 4, 5\}$. We are missing only $\{1, 3, 5\}$, which is unreachable.

If $n \geqslant 7$, from $\{n-1\} \cup O$ we can reach by c_i inputs all the subsets containing $\{1, n\}$ but not $\{2\}$, except those subsets containing $n-2$ without $n-1$. From now on, we are interested only in the missing subsets, which are with $\{1, n\}$, without 2, and have $n-2$ without $n-1$. Then take $\{n-3\} \cup O$. From here we can reach all subsets containing $\{1, n-2, n\}$ without $\{2, n-1\}$, except those containing $n-4$ without $n-3$. If $n = 7$, then $n-4 = 3$, and we are missing only $\{1, 3, 5, 7\}$, which is unreachable.

Continuing in this fashion, we can reach all the subsets containing $\{1, n\}$ but not 2, except O. Together with the case where $2 \in S$, we have all the states containing $\{1, n\}$, except O.
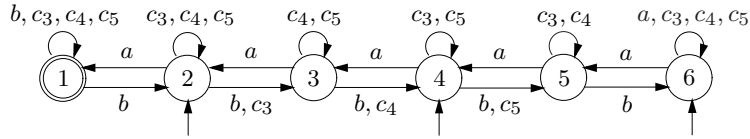

Fig. 6. NFA $\mathcal{N}$ of L^R, n even.

The case where n is even is similar. The NFA $\mathcal{N}$ is shown in Fig. 6 for $n = 6$. By an argument similar to that for n odd, O cannot be reached.

Any subset of $M = Q \setminus \{1, n\}$ can be reached as follows. If $2 \in S$, apply b to E to get $O \setminus \{1\}$, and then a to get to $E \setminus \{n\}$. Now any subset of M containing 2 can be reached by inputs c_i. If $2 \notin S$, then any subset of $M \setminus \{2\}$ can be reached from $O \setminus \{1\}$ by inputs c_i.

Second, consider subsets S of Q containing 1 but not n. If $2 \in S$, start with E and apply ba to reach $E \setminus \{n\}$. Then apply ab to get $E \setminus \{n\} \cup \{1\}$. Now any subset of $\{1\} \cup M$ containing $\{1, 2\}$ can be reached by inputs c_i. If $2 \notin S$, start with E and apply baa to reach $O \setminus \{n-1\}$. By using inputs c_i, we can construct any subset S of $\{1\} \cup M$ containing 1 and not 2, except the subsets that have $\{n-3, n-1\}$ without $n-2$. In case $n = 4$, we can reach $\{1, 2\}$, $\{1, 2, 3\}$, and $\{1\}$, but not $\{1, 3\}$. From now on, we are interested only in the missing subsets. As in the odd case, we can get subsets containing $\{n-3, n-1\}$ without $n-2$ by deleting $n-3$ and $n-2$, adding $n-4$, and re-inserting $n-3$. Now we are unable to reach states having $\{n-5, n-3\}$ without $n-4$. We verify that $\{i\} \cup O$ is reachable for every even i with $4 \leqslant i \leqslant n-2$, and continue as in the odd case. We can keep moving this problem to the left, until we reach $\{3, 4, 5\}$. Then state 4 cannot be removed because O is

not reachable.

Third, examine subsets S of Q containing n but not 1. If $2 \in S$, all such subsets are reachable by inputs c_i from E. If $2 \notin S$, then S is a subset of $\{3, \dots, n\}$ containing n. Since S_{-1} is a subset of M, it is reachable; then S is reached by b from S_{-1}.

Finally, consider subsets S containing both 1 and n. If $2 \in S$, apply ab to reach $\{1\} \cup E$. From here we can reach any set containing $\{1, 2, n\}$ by inputs c_i. If $2 \notin S$, we reach $O \cup \{n\}$ from E by a. From here we can reach any set containing $\{1, n\}$ but not 2 by inputs c_i.

We still need to verify that all the reachable subsets are pairwise distinguishable. State i, and only state i, accepts a^{i-1}. Hence, if $S, T \subseteq Q$ and S and T differ by state i, then they are distinguishable by a^{i-1}. □

7. Unary Languages

The case of unary languages is special. For regular unary languages, the tight bounds for each boolean operation $K \circ L$, product KL, star L^*, and reversal L^R are mn, mn, $n^2 - 2n + 2$, and n, respectively [19]. With the exception of the bound for reversal, these bounds cannot be met by star-free unary languages.

Theorem 8. *Given $m, n \geqslant 1$, let K and L be unary star-free languages with quotient complexities m and n, respectively.*

1. For each boolean operation $\circ$, $\kappa(K \circ L) \leqslant \max(m, n)$ and the bound is tight.

2. For product, $\kappa(KL) \leqslant m + n - 1$, and the bound is tight.

3. For the star, the tight bound is

$$\kappa(L^*) \leqslant \begin{cases} 2, & \text{if } n = 1; \\ n, & \text{if } 2 \leqslant n \leqslant 5; \\ n^2 - 7n + 13, & \text{otherwise} . \end{cases}$$

4. For reversal, $\kappa(L^R) = n$.

Proof. If a unary star-free language L is finite and $\kappa(L) = n$, its longest word has length $n - 2$; if it is infinite, the longest word not in L has length $n - 2$.

1. One verifies that $\kappa(K \circ L) \leqslant \max(m, n)$. The witness languages are $K = a^{m-2}$ and $L = a^{n-2}$ for union and symmetric difference, $K' = a^{m-1}a^*$ and $L' = a^{n-1}a^*$ for intersection, and K' and $\overline{L'}$ for difference, since $K' \setminus \overline{L'} = K' \cap L'$.

2. One verifies that $\kappa(KL) \leqslant m + n - 1$, and $K = a^{m-1}a^*$ and $L = a^{n-1}a^*$ are witnesses.

3. If L is infinite, then $L \supseteq a^{n-1}a^*$, and $L^* \supseteq a^{n-1}a^*$; hence $\kappa(L^*) \leqslant n$. For $n = 1, 2, 3, 4, 5$, the bounds actually met in the infinite case are $1, 1, 3, 4, 5$, respectively. If L is finite, it must contain a^{n-2}, and if it has a, then $\kappa(L^*) = 1$. The tight bounds for finite unary star-free languages are $2, 2, 1, 2, 3$, respectively. Hence the tight bounds for all unary star-free languages for the first five values of n are $2, 2, 3, 4, 5$, and the witnesses are $\emptyset, \varepsilon, a^2a^*, a^3a^*$, and a^4a^*, respectively.

It was shown in [6] that for a finite unary language L, $\kappa(L^*) \leqslant n^2 - 7n + 13$ for $n \geqslant 5$. For $n \geqslant 6$, this bound applies here, and a witness is $a^{n-3} \cup a^{n-2}$.

4. For unary languages, we have $L^R = L$; hence $\kappa(L^R) = \kappa(L)$.   □

8. Conclusions

We have shown that all the commonly used regular operations in the class of star-free languages meet the quotient complexity bounds of arbitrary regular languages. The only exceptions are in the product for $n = 2$, reversal, and operations on unary languages.

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