Essays in Corporate Prediction Markets

by

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I hereby declare that I am the sole author of this thesis. This is a true copy o the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Personal subjective opinions are one of the most important assets in management. Prediction markets are mechanisms that can be deployed to elicit and aggregate a group of people's opinions regarding the outcome of future events at any point in time. Prediction markets are exchange-traded markets where security values are tied to the outcome of future events. Prediction markets are systematically designed in a way that that their market prices capture the crowd's consensus about the probability of a future event. Corporations harness internal prediction markets for managerial decision making and business forecasting. Prediction markets are traditionally designed for large and diverse populations, two properties that are not often displayed in corporate settings. Therefore special considerations must be given to prediction markets used in corporations.

Our first contribution in this thesis is in addressing the issue of diversity, in the sense of risk preferences, in corporate prediction markets. We study prediction markets in the presence of risk averse or risk seeking agents that have unknown risk preferences. We show that such agents' behavior is not desirable for the purpose of information aggregation. We then characterize the agents' behavior with respect to prediction market parameters and offer a systematic method to market organizers that fine tunes market parameters so at to best mitigate the impact of a pool agents' risk-preferences.

Our Second contribution in this thesis is in recommending prediction market mechanisms in different settings. There are many prediction market mechanisms with various advantages and weaknesses. The choice of a market mechanism can heavily affect the market accuracy and in turn, the success of a managerial decision, or a forecast based on prediction markets' prices. Using trade data from two real-world prediction markets, we study the two main types of prediction markets mechanism and provide the much-needed insight as to what market mechanism to choose in various situations.

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To my love, *Niloufar*

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Managers and practitioners can utilize employees' opinion and probabilistic estimates to make better decisions and generate more accurate forecasts, than when employees' perspectives are not considered. From the chance of meeting a project's deadline (Ortner 1998) to the quality of a new software program (Cowgill and Zitzewitz 2015), employees' collected wisdom can improve any managerial decision. The need for opinion aggregation and the inability of conventional mechanisms has led corporations to harness internal prediction markets to collect the wisdom of the crowd (Surowiecki 2005, Arrow et al. 2008).

Prediction markets are mechanisms that involve the purchase and sale of securities with outcome-contingent payoffs. Prediction markets are designed to aggregate traders' opinions, also called beliefs, while simple inference can be made regarding the crowd's consensus by simply monitoring the market prices (Wolfers and Zitzewitz 2004).

The use of prediction markets in decision makings can lead to significant gains/saving in corporations (Arrow et al. 2008, Tziralis and Tatsiopoulos 2012), which according to one estimate (Christiansen 2009) can exceed millions of dollars. Prediction markets are expected to evolve and grow, centering on better predictions and new application areas (Shrier et al. 2016). Practitioners have envisioned prediction markets as tools that remake collective decision making, and thus reducing the number of poor decisions (Hanson 2017).

Corporations use prediction markets internally in decision making and forecasting applications, we refer to these prediction markets as corporate prediction markets (Arrow et al. 2008, Cowgill and Zitzewitz 2015). Prediction markets are often designed for large and diverse groups (Surowiecki 2005, Wolfers and Zitzewitz 2004). The idea behind large groups is to ensure a liquid market and the diversity ensures that the market participant would be correcting each others biases. Large populations and diversity are not often observed in general corporate settings, thus special considerations must be given to the design of corporate prediction markets. In this introduction, we outline two problems regarding the design of corporate prediction markets.

Problem 1 THE IMPACT OF RISK-AVERSE AND RISK-SEEKING TRADERS IN PREDICTION MARKETS.

In recent times some media outlets are stating that prediction markets are failing (Ledbetter 2008, D.R. 2016). Besides the natural argument that prediction markets provide probability estimates for the event and not a deterministic estimate, popular media provides arguments such as cognitive biases, lack of liquidity, low stakes, and slow to react to events as reasons behind prediction market failure. These may all be viable reasons, but one reason not mentioned is risk bias that induces rational individuals to change their trading behavior in prediction markets due to their risk preferences. Due to the small number of participants, risk bias may be an issue in corporate prediction markets (Cowgill and Zitzewitz 2015). We analyze the behavior of risk-averse and risk-seeking agents in prediction markets. We show that such agentsâĂŹ may not reveal their exact opinion and their trading bahvior often deviates from the ideal scenario that a central market organizer expects. We then analytically characterize agents' trading behavior as a function of prediction market parameters and offer solutions to market organizers to be able to tune the market parameters to restrain agents' behavior.

Problem 2 Subsidized Versus Unsubsidized Prediction Markets.

Many market organizers employ traditional mechanisms to handle buying and selling shares. Such mechanisms are often peer-to-peer where agents trade with each other. One of the main advantages of peer-to-peer market mechanisms is that they are costless. Such markets can successfully elicit and aggregate agents' opinion so long as the market consists of a large group of traders (Hanson 2003, Milgrom and Stokey 1982), however, when the number of trades is low, these markets face various issues concerning belief aggregation. Subsidized prediction markets are designed to circumvent the negative issues of peer-to-peer markets in situations where the trade frequencies might be low (Hanson 2003, Dimitrov et al. 2015). Subsidized prediction markets provide subsidy in the shape of a central market maker where agents trade with, as opposed to agents trading with each other. Subsidized prediction markets, however, are not cost-free and require an endowment to operate (Hanson 2003). We use an empirical framework to compare various prediction market mechanisms, using real-world market price data. We provide recommendations to market organizers as to what mechanism to choose in different situations.

1.1 Contributions

The contributions of this dissertation are:

Problem 1 Considering binary outcome space:

We prove that there exists no incentive compatible deterministic *Market Scoring Rule* (MSR) prediction market, a form of subsidized prediction markets we consider in this thesis, for agents with unknown, yet bounded, risk preferences. We introduce the notion of an agent's deviation in a prediction market. We present the necessary and sufficient conditions of the structural properties of two MSRs such that one MSR yields lower deviation relative to another MSR for all agents in a population. We present the relationship between deviation and liquidity for cost-function market makers, defined in Section 2.3, using two different measures of liquidity, namely inverse liquidity and market depth (Chen and Pennock 2007, Abernethy et al. 2014). We show that for each MSR in a family of MSRs, higher (lower) deviation implies higher (lower) liquidity. Using our derived relationship between deviation and liquidity, we present an optimization problem that market orginizers may use to determine the desired liquidity when running a subsidized prediction market.

Considering any finite space:

We expand the literature of belief elicitation mechanisms to introduce the concept of dominance, Definition 1, to draw comparisons between two arbitrary MSRs. We show that the classical intuition behind agents' belief and their report (Kadane and Winkler 1988, Murphy and Winkler 1970) cannot be generalized to any finite outcome space beyond binary spaces. We introduce the concept of flatness in prediction markets. We expand the literature of MSRs and risk-aversion by showing a relationship between flatness and dominance in any finite outcome space. We further present a relationship between flatness and liquidity for cost-function market makers. We show that our derived relationship between flatness and liquidity can be used by market organizers to determine a suitable amount of liquidity when running a subsidized prediction market.

Problem 2 We provide the much-needed insight for prediction market operators when determining if they should run a subsidized or unsubsidized prediction market. The current literature on prediction markets suggests using subsidized prediction markets when the number of trades is small (Chen and Pennock 2010, Hanson 2003). Our empirical study supports this idea. We show that when the number of trades is less than 310, then a subsidized market aggregates information faster than a prediction market that is not subsidized with 95% confidence. Moreover, we show that when both subsidized prediction markets and unsubsidized prediction markets aggregate traders' belief simultaneously, subsidized prediction markets require a lower number of trades compared to unsubsidized prediction markets. Our results can help a growing number of managers and practitioners, that are using prediction markets, choose the suitable mechanism based on their needs and expected trade volume.

1.2 Outline of the Dissertation

In Chapter 2, we describe the main concepts and terminologies needed in this dissertation. In Chapter 2 we introduce prediction markets and give a detailed summary of the current related works. Furthermore, we define the concept of dominance in MSR prediction markets which extend the current literature and enables a direct comparison between two arbitrary MSRs. Chapter 3 details the impact of risk preferences in binary prediction markets. In Chapter 4 we discuss the limitation of the binary results derived in Chapter 3 and set a new analytical framework to generalize the study of risk-aversion in prediction markets to any finite outcome space. Chapter 5 details our empirical study that enables comparison between prediction market mechanisms from an information aggregation point of view. In Chapter 6, we conclude by presenting a summary of out contributions and we leave the reader with a list of our future work.

Chapter 2

Background and Related Work

Large groups of people are collectively smarter than individual experts when it comes to decision making, innovating and predicting (Surowiecki 2005). The superiority of the collective intelligence over individual decision makers is not a new observation. In the famous essay, *The Use of Knowledge in Our Society* (Hayek 1945), the Nobel laureate Fredrik Hayek explains how individual decision makers are unable to efficiently allocate resources because regardless of their expertise, they would never have all the information required to make better decisions. Hayek (1945) argues that pricing methods, (e.g. stock markets), which reflect the collective knowledge of individuals are a much better approach for performing resource-allocation decisions. Hayek (1945) refers to mechanisms which signal the value of a resource through numerical indices, e.g. stock market prices, as "mechanisms for communicating information". These information-communication mechanisms are platforms that collect and aggregate individuals' information. Other than resource allocation, prediction is another task in

Figure 2.1: A prediction market running on the outcome of 2017 German Chancellor election (source: predictit.com).

which large groups of people outperform individuals. But what mechanism can express individuals' information in terms of numerical indices? The answer is *Prediction Markets*.

Prediction markets are similar to financial markets in that they have assets with values contingent on the outcome of future events (Wolfers and Zitzewitz 2004). Figure 2.1 is an example of an online prediction market about the outcome of the 2017 German Chancellor election. The market displayed in Figure 2.1 is selling multiple securities with their values corresponding to the outcome of the future event, the outcome of the German Chancellor election. For instance, the security "MERKEL.CHANCELLOR.GERMANY.2017" is paying $1 for every share, if candidate Angela Merkel wins the 2017 election and nothing otherwise. As the Merkel security, at the time of taking the snapshot of the market, traded at \$0.88, assuming traders are risk-neutral, we interpret the likelihood of Merkel winning the election as 0.88. Practitioners extend this idea further, by saying that prediction market prices represent the aggregated opinions of all traders in the market up to that point in time Wolfers and Zitzewitz (2004), Surowiecki (2005).

Firms also use prediction markets as a management tool to elicit employees' beliefs on the outcome of a future event by incentivizing employees to provide probability estimates. Schlack (2015), Filios (2016) argue that prediction markets are a form of predictive analytics for scenarios with no historical data available. Prediction markets have successfully been used in project management (Ortner 1998), new product sales and development (Plott and Chen 2002) and disease spread forecasting (Polgreen et al. 2007), just to name a few examples. Prediction markets are also used privately within corporations. Prediction markets appear in corporations like General Electric, Google, Hewlett-Packard, IBM, Intel, Microsoft, Siemens, and Yahoo! (Arrow et al. 2008). There is a broad range of literature on prediction markets' accuracy, information elicitation and information aggregation. Tziralis and Tatsiopoulos (2012) provide a literature review on prediction markets' history and success, as well as challenges that prediction markets face. Prediction markets' performance and information aggregation capabilities are widely studied in application areas such as politics (Mellers et al. 2015, Atanasov et al. 2016, Chen et al. 2008, 2004), economics (Ostrovsky 2012), finance (Bossaerts et al. 2002, Palfrey and Wang 2012), health (Polgreen et al. 2007), and corporations (Cowgill and Zitzewitz 2015, Csaszar and Eggers 2013, Arrow et al. 2008, Healy et al. 2010, Berg et al. 2009). Prediction markets advances are based on the scientific foundation of the efficient market hypothesis (Wolfers and Zitzewitz 2004, Fama 1970). As Wolfers and Zitzewitz (2004) explain: "In a truly efficient prediction market, the market price will be the best predictor of the event and no combination of available polls or other information can be used to improve on the market-generated forecasts." Prediction markets are known to provide good, if not better, forecasts related to other means of prediction (Arrow et al. 2008, Wolfers and Zitzewitz 2004).

There are two general types of prediction markets: subsidized and unsubsidized. In an *unsubsidized prediction market*, USPM hereafter, traders exchange contracts with each other. USPMs require no cost to run. Moreover, market administrators can also make a profit by charging participant a small percentage of their winnings, something observed in practice (Luckner et al. 2011). USPMs are shown to produce accurate forecasts when the number of trades is high. However, when trading volume is small, USPM's prices are often highly volatile and non-indicative of market consensus. *Subsidized prediction markets*, SPMs hereafter, are alternatives to USPMs that perform well regardless of the volume of trades. SPMs, however, are not costless and require a market maker to facilitate trade within the market.

Prediction markets use different mechanisms to handle trades and satisfy buy/sell orders. The two most common mechanisms to run a prediction market are *Continuous Double Auctions*, CDAs hereafter, and *Market Scoring Rules*, MSRs hereafter, in which the former is an USPM mechanism and the latter is a SPM mechanism. Due to their similarities to financial markets, CDAs are arguably the most natural choice to run a prediction market. CDAs match buyers and sellers for each security. A buyer (seller) purchases selling (buying) orders where the order price is referred to the bid (ask) price. CDAs execute all feasible orders by matching buyers to sellers with bid prices higher than ask price, with all other orders placed in an order book. The current price of an asset in a CDA prediction market is determined by the price of the most recently executed trade. The difference between the highest bid and the lowest ask price is also referred to as the bid-ask spread (Pennock 2004). CDAs are shown to be efficient and reliable in settings in which there are a large amount of trades and traders' population is diverse in their beliefs and risk preferences (Wolfers and Zitzewitz 2004, Hanson 2003, Luckner et al. 2011). However, when there is a low number of trades, the CDA market prices are often prone to two main issues. First, as the number of trades diminishes, CDA market prices do not represent the current consensus and become highly volatile (Bossaerts et al. 2002). Second, when the bid-ask spread is high, some traders may not reveal their information because they cannot profit from it (Milgrom and Stokey 1982). The former issue is being referred to as *thin market*, and the latter is classified as a *no-trade* situation. Both of these problems are very common in small corporate settings where the number of trades may not be large, and the loss of an individual's information may be costly (Cowgill and Zitzewitz 2015, Horn and Ivens 2015). Hanson (2003) introduces MSRs as an alternative platform that enables information aggregation and elicitation even in thin markets. MSRs are derived from proper scoring rules. A proper scoring rule is a mechanism that provides incentives to promote honest probabilistic forecasts. Proper scoring rules are capable of eliciting a single agents belief. MSRs can elicit a group of agents' belief, and aggregate those beliefs into a single estimate that is the MSR prediction market's current price. MSRs were later generalized into a broader class of automated market makers called cost-function market makers (Chen and Pennock 2007, Chen and Vaughan 2010). More suitable than CDAs, MSR prediction markets accurately capture the market information, even in thin markets (Hanson 2003). The key is using subsidy and creating an automated market maker for traders to trade with, as opposed to traders trading with each other. MSRs have zero bid-ask spreads; and are not susceptible to thin markets, or no-trade issues. The current understandings of subsidy in prediction market is: CDAs are reliable when a large number of trades is expected, and MSRs are utilized when there are concerns over issues such as thin markets and no trade situations. Given the fact that CDAs are costless, and potentially profitable, and MSRs are costly and need an initial endowment to run, we would like to address the following question. What is the number of trades needed for a MSR to outperform a CDA? In Chapter 5, we compare MSRs and CDAs in an empirical setting to answer this question.

Due to their capability of dealing with smaller populations and circumventing no-trade situation, MSRs are suited for corporate settings where the number of trades may not be large, and the loss of an individual's information may be costly (Cowgill and Zitzewitz 2015, Horn and Ivens 2015). MSRs however, are not perfect. In fact the result of Hanson (2003) that indicates a MSR can elicit an agent's belief is based on the assumption that all agents are risk neutral. The underlying assumptions of MSRs are similar to proper scoring rules that assume agents are risk neutral and myopic. Therefore, we analyze the impact of risk preference on agent's behavior in MSR prediction markets in Chapter 3, and Chapter 4.

The rest of this chapter details the theoretical concepts of this dissertation. We start with introducing scoring rules in Section 2.1 and explain how MSRs are derived from scoring rules. A summary of the literature on risk in MSRs is presented in Section 2.2. In Section 2.3, we present the related work on cost-function market makers along with most recent findings involving MSR prediction markets and liquidity.

2.1 Scoring Rules and Market Scoring Rules

Brier (1950) introduced the *quadratic proper scoring rule* as a reward mechanism designed to induce truthful reporting from weather forecasters. Proper scoring rules were later generalized to include a larger class of functions applied to subjective probability elicitation (Good 1952, Winkler 1969, Savage 1971, McCarthy 1956). Formally, a *scoring rule* is a function $S(\cdot) : \Delta_{N-1} \times \Omega \mapsto \mathbb{R}$, where $\Omega = \{1, 2, \cdots, N\}$ is the discrete outcome space of a future event represented by a random variable ω, and Δ_{N-1} is the $N-1$-simplex. Consider a prediction market designed to elicit a set of myopic, expected-utility-maximizing agents' beliefs on $\omega \in \Omega$. An incentive compatible scoring rule is called a *proper* scoring rule. In other words, a scoring rule $S(\cdot)$ is proper for a risk neutral agent, when the following is satisfied:

$$\forall \mathbf{q} \in \Delta_{N-1} : E_{\mathbf{p}}[S(\mathbf{p}, \omega)] \geq E_{\mathbf{p}}[S(\mathbf{q}, \omega)], \tag{2.1}$$

where $\mathbf{p}$ is the individual forecaster's belief on ω, and

$$E_{\mathbf{p}}[S(\mathbf{q}, \omega)] \triangleq \sum_{\omega \in \Omega} \mathbf{q}_{\omega} S(\mathbf{q}, \omega), \tag{2.2}$$

is the expected score of reporting a feasible report of $\mathbf{q}$. A scoring rule is called *strictly* *proper* when it satisfies (2.1) strictly, whenever $\mathbf{q} \neq \mathbf{p}$.

For example, the following scoring rule, called the Quadratic scoring rule (Brier 1950), is strictly proper:

$$S(\mathbf{q}, \omega) = 2\mathbf{q}_\omega - \sum_{\omega \in \Omega} \mathbf{q}_\omega^2.$$

Another example of a strictly proper scoring rule is Spherical scoring rule (Gneiting and Raftery 2007, Savage 1971, McCarthy 1956)

$$S(\mathbf{q}, \omega) = \frac{\mathbf{q}_\omega}{\sqrt{\sum_{i \in \Omega} \mathbf{q}_i^2}}$$

Another popular class of strictly proper scoring rule is logarithmic scoring rule (Savage 1971, McCarthy 1956), defined as follows:

$$S(\mathbf{q}, \omega) = \ln(\mathbf{q}_\omega).$$

We can verify that if the score function $S(\cdot)$ is strictly proper, then the function $b\, S(\cdot)$ is also strictly proper for any given positive scalar b. In general if $S(\cdot)$ is strictly proper, then any affine transformation of $S(\cdot)$ is also strictly proper. However, the shifting factor of an affine transformation has no effect on the properties that we are interested in this thesis. We say two scoring functions are from the same *family*, if one is a positive scalar of the other. Moreover, a strictly proper scoring rule $S(\mathbf{q}, \omega)$ is strictly increasing in $\mathbf{q}_i$ for $\omega = i$, and non-increasing in $\mathbf{q}_i$ for $\omega \neq i$ (Gneiting and Raftery 2007, Schervish 1989).

MSRs are derived from proper scoring rules and are used to elicit the belief of each individual in a group and *aggregate* the group's beliefs into a single estimate (Hanson

2003). A MSR takes two sequential reports, $\mathbf{q}^{(t)}$ and $\mathbf{q}^{(t-1)}$, and the observed outcome to determine the score of each agent's report. Similar to Hanson (2003), we define the MSR functions as follows: an agent who reports $\mathbf{q}^{(t)}$, at step $t \geq 1$, will receive:

$$\text{MSR}(\mathbf{q}^t, \mathbf{q}^{t-1}, \omega) = S(\mathbf{q}^t, \omega) - S(\mathbf{q}^{t-1}, \omega), \tag{2.3}$$

where $S(\cdot)$ is a proper scoring rule and $\mathbf{q}^{(t-1)}$ is the previously submitted report. The initial report, $\mathbf{q}^{(0)}$, is made by the market maker and is referred to as the market's *initial estimate*. Similar to proper scoring rules, in a MSR, a risk-neutral, myopic, and expected utility-maximizing individual reports truthfully. That is:

$$\mathbf{p} \in \arg \max_{\mathbf{q}^{(t)} \in \Delta_{N-1}} E_{\mathbf{p}} \left[\text{MSR}\left(\mathbf{q}^{(t)}, \mathbf{q}^{(t-1)}, \omega\right)\right] \tag{2.4}$$

where $\mathbf{p}$ is the agent's belief on ω. A MSR is *strictly proper* if it is proper and $\mathbf{p}$ in (2.4) is unique. Since MSRs are derived from scoring rules, if $X(\cdot)$, the underlying scoring rule of a MSR, $\mathcal{X}$, is strictly proper then we also get:

$$\forall i, j \in \Omega, \forall \mathbf{r}^{(0)} \in \Delta_{\Omega-1} : \begin{cases} \frac{\partial}{\partial \mathbf{q}_i} \mathcal{X}\left(\mathbf{q}, \mathbf{r}^{(0)}, j\right) > 0 & : j = i \\ \\ \frac{\partial}{\partial \mathbf{q}_i} \mathcal{X}\left(\mathbf{q}, \mathbf{r}^{(0)}, j\right) \leq 0 & : j \neq i \end{cases}. \tag{2.5}$$

We say two MSRs are from the same *family*, if their underlying scoring functions are from the same family. To compare two MSRs from the same family to find out which one provide higher score (lower score), we can easily compare their scale factors. However, when two MSRs do not belong to the same family we cannot directly compare their MSR score functions. We provide a new definition that enables a direct comparison between two arbitrary MSRs.

Definition 1. *(Dominance) Let $\mathcal{X}$, and $\mathcal{Y}$ be two MSRs. We say $\mathcal{Y}$ dominates $\mathcal{X}$ when the following holds*

$$\forall i,j \in \Omega, \forall \mathbf{r}^{(0)} \in \Delta_{\Omega-1} \begin{cases} \mathcal{Y}\left(\mathbf{q},\mathbf{r}^{(0)},i\right) \leq \mathcal{X}\left(\mathbf{q},\mathbf{r}^{(0)},i\right) \leq \mathcal{X}\left(\mathbf{q},\mathbf{r}^{(0)},j\right) \leq \mathcal{Y}\left(\mathbf{q},\mathbf{r}^{(0)},j\right), \mathbf{q}_i \leq \mathbf{r}^{(0)}{}_i \\ \mathcal{Y}\left(\mathbf{q},\mathbf{r}^{(0)},j\right) < \mathcal{X}\left(\mathbf{q},\mathbf{r}^{(0)},j\right) < \mathcal{X}\left(\mathbf{q},\mathbf{r}^{(0)},i\right) < \mathcal{Y}\left(\mathbf{q},\mathbf{r}^{(0)},i\right), \mathbf{q}_i > \mathbf{r}^{(0)}{}_i \end{cases} . \quad (2.6)$$

Intuitively, dominance enables us to compare two given MSRs, not necessarily from the same family, to deduce which one provides a higher amount of reward or punishment for any given market estimate. The notion of dominance is adapted from the definition of dominance in functional analysis. Following the notion of dominance in functional analysis (Rudin 1991, Chapter 3), (2.6) implies $\left|\mathcal{X}\left(\mathbf{q},\mathbf{r}^{(0)},j\right)\right| \leq \left|\mathcal{Y}\left(\mathbf{q},\mathbf{r}^{(0)},j\right)\right|$ for all $\mathbf{q},\mathbf{r}^{(0)} \in \Delta_{N-1}$ and for all $j \in \Omega$. Note that if $\mathcal{X}$ and $\mathcal{Y}$ are from the same family in which $\mathcal{X} = b\mathcal{Y}$, where $b > 1$, $\mathcal{X}$ and $\mathcal{Y}$ satisfy (2.6).

2.2 Risk Aversion in Scoring Rules and Market Scoring Rules

Kadane and Winkler (1988) and Murphy and Winkler (1970) show that when a myopic agent that is not risk neutral, is asked about her belief on the outcome of an event and is rewarded using a proper scoring rule, she may not report truthfully. Such an agent may hedge her expected losses by under or over reporting her belief on ω. As shown by Kadane and Winkler (1988) and also by Lambert (2011), when agent's preferences are known, the scoring rule can be corrected to retain its incentive compatibility. Following a similar line of thought, Offerman et al. (2009) propose a new mechanism that includes a two-stage process. In the first stage, individual agents are prescreened, and their risk preferences are elicited. In the second stage, each agent is scored with a tailor made proper scoring rule. In contrast, we do not prescreen agents with unknown risk preferences, in particular, in Chapter 3, and Chapter 4 we propose an online mechanism, similar to the work of Dimitrov et al. (2015).

2.3 Market Scoring Rules and Cost Function Market Makers

In addition to having agents report probability estimates, agents may also buy and sell shares of securities with values contingent on a future event. For example if agents are trading on a binary event, E, they trade one or two securities, one "Yes" security and one "No" security, similar to the concept of binary derivative or binary options in financial markets. With the "Yes" security having a value of $1 if E occurs, and $0 otherwise, the "No" security is similarly defined. The process of buying and selling is considered to be more intuitive to prediction market participants relative to reporting probability estimates (Chen and Pennock 2007). Chen and Vaughan (2010), later extended by Abernethy et al. (2013), show that for every given MSR market maker, there exists a cost-function prediction market, such that the two markets are equivalent, same prices, reward, etc., and vice versa. In other words, every MSR prediction market can be transformed into a buying/selling share market via a cost-function and for every cost-function prediction market, there exists a MSR such that an agent's transactions can be interpreted in terms of reporting probability estimates.

The new market maker, called a cost function market maker, operates via a cost function $C : \mathbb{R}^N \mapsto \mathbb{R}$ that determines the cost of transactions. In particular, let $\mathbf{s}_0 = (s_{01}, s_{02}, \cdots, s_{0N})$ be the current bundle of outstanding shares of all securities for all mutually exclusive outcomes on a traded event. When an agent enters the market and changes the outstanding shares to $\mathbf{s} = (s_1, s_2, \cdots, s_N)$, the agent is required to pay $C(\mathbf{s}) - C(\mathbf{s}_0)$. The (instantaneous) price for each security ω, given an outstanding bundle of shares, is also defined by the following partial derivative:

$$\mathrm{Pr}_\omega(\mathbf{s}) = \frac{\partial}{\partial s_\omega} C(\mathbf{s}), \omega \in \Omega.$$

As defined by Abernethy et al. (2013), a cost function is valid when it satisfies five properties. Though all five properties must hold for there to be a cost-function for a given MSR, in this thesis we only require the following three properties, plus an additional property, to hold for our results:

(path independent) $\forall \mathbf{s}, \mathbf{s}', \mathbf{s}'' \in \mathbb{R}^N : \mathbf{s} = \mathbf{s}' + \mathbf{s}'' \implies C(\mathbf{s}) = C(\mathbf{s}') + C(\mathbf{s}'')$,

(price existence) the function $C(\cdot)$ is continuous and differentiable everywhere on $\mathbb{R}^N$,

(no arbitrage) $\forall \omega \in \Omega, \mathbf{s} \in \mathbb{R}^N : \mathrm{Pr}_\omega(\mathbf{s}) \geq 0, \sum_{\omega \in \Omega} \mathrm{Pr}_\omega(\mathbf{s}) = 1$.

As defined by Abernethy et al. (2013) a valid cost function is defined as a function $C(\cdot)$ that satisfies the three above properties as well as the following two properties.

$$\text{(information incorporation)} \qquad \forall \mathbf{s}, \mathbf{s}', \in \mathbb{R}^N : C(\mathbf{s} + 2\mathbf{s}') - C(\mathbf{s} + \mathbf{s}') \geq C(\mathbf{s} + \mathbf{s}') - C(\mathbf{s})$$
$$\text{(expressiveness)} \qquad \forall \mathbf{p} \in \Delta_{|\Omega|-1}, \epsilon > 0, \mathbf{x}^{\mathbf{p}} := \mathbb{E}_{\mathbf{p}}[\mu(\omega)] : \exists \epsilon > 0, \mathbf{q} \in \Delta_{|\Omega|-1}, ||\nabla C(\mathbf{q}) - \mathbf{x}^{\mathbf{p}}|| < \epsilon.,$$

in which $\mu(\omega)$ is the vector of payoffs for each security for outcome ω. Note that we do not consider the two properties of information incorporation and expressiveness in this dissertation. Moreover, understanding the two conditions of information incorporation and expressiveness requires tools and technicalities which we do not cover in this dissertation. Please see Abernethy et al. (2013) for a complete discussion on the five properties required for the validity of a cost-function.

The additional property that we require to hold is for the cost-function, $C(\mathbf{s})$, to be twice differentiable. Abernethy et al. (2013) show an equivalence relationship between valid cost functions, and Hanson's MSRs. In other words, there exist a cost function market maker for a large class of MSRs and vice versa. In particular for a MSR $\mathcal{X}$, we can find $C^{\mathcal{X}}(\cdot)$, the corresponding cost function, by solving the following problem:

$$C^{\mathcal{X}}(\mathbf{s}) = \max_{\mathbf{q} \in \Delta_{N-1}} \mathbf{s}^T \mathbf{q} - \sum_{\omega \in \Omega} q_i X(\mathbf{q}, \omega). \tag{2.7}$$

Accordingly the price function is defined as:

$$\Pr^{\mathcal{X}}_{\omega}(\mathbf{s}) = \frac{\partial}{\partial s_{\omega}} C^{\mathcal{X}}(\mathbf{s}) = \arg \max_{\mathbf{q} \in \Delta_{N-1}} \mathbf{s}^T \mathbf{q} - \sum_{\omega \in \Omega} q_i X(\mathbf{q}, \omega). \tag{2.8}$$

For instance, the Logarithmic MSR for a binary outcome event:

$$\text{LMSR}(\mathbf{q}, \mathbf{r}^{(0)}, \omega) = \begin{cases} b\log(\mathbf{q}_1) - b\log(\mathbf{r}^{(0)}{}_1) & ,\omega = 1 \\ b\log(\mathbf{q}_0) - b\log(\mathbf{r}^{(0)}{}_0) & ,\omega = 0 \end{cases}$$

in which b is a positive scalar, has the following *logarithmic cost function* (Chen and Vaughan 2010):

$$C_{\text{LMSR}}(\underbrace{\mathbf{s}}_{(s_1,s_0)}) = b\log\left(\exp\left(\frac{s_1}{b}\right) + \exp\left(\frac{s_0}{b}\right)\right). \tag{2.9}$$

Despite the equivalence relation between MSRs and cost function market makers, some issues are unique to the latter and may not be directly applied to the former. For instance, the ability to handle limit orders, an order to buy or sell a bundle of securities at a given price, is one issue that is directly applied to cost function market makers (Heidari et al. 2015) but has no well-defined equivalence in MSRs. Another issue that is unique to cost-function market makers is determining the appropriate amount of liquidity to provide by the market maker.

2.4 Liquidity Measures in Cost-function Prediction Markets

Regardless of the type of prediction market used, MSR or cost-function, a market maker must determine how to facilitate trade within the market, especially if the market is thin. As such, the market market must provide some liquidity in a market.

We now discuss liquidity, and various measures proposed in the literature. Liquidity of a market is defined as the market's price responsiveness to trade. The idea of liquidity is that: in a more liquid market, larger trade volume is required to change the market's price compared to the trade volume that is required in a less liquid market. Prediction market literature uses the following three approaches to measure cost-functions' market liquidity:

- *Inverse liquidity* (Frongillo et al. 2012, Brahma et al. 2012, Abernethy et al. 2011, Wah et al. 2016, Abernethy et al. 2014, Slamka et al. 2013, Othman et al. 2013) Given a base valid cost-function market maker $C(\cdot)$, as defined in Section 2.1, one can verify that the cost-function:

$$C_b(\mathbf{s}) \triangleq b\, C\left(\frac{1}{b}\mathbf{s}\right), \tag{2.10}$$

  is a valid cost-function market maker. The parameter b determines the liquidity of the market. The higher the parameter b, the lower prices change given a fixed volume of trade. This definition is adopted to compare different cost-function market makers from the same family. For instance, when two Logarithmic cost-functions, see (2.9), with different b parameters, $C_{\mathrm{LMSR}_{b_1}}$ and $C_{\mathrm{LMSR}_{b_2}}$ in which $b_2 > b_1$, are compared; $C_{\mathrm{LMSR}_{b_2}}$ is considered more liquid than $C_{\mathrm{LMSR}_{b_1}}$. Given a base cost-function market maker, a family of cost-function market makers can be generated using different values of b, and liquidity of each market maker can be easily measured by simply comparing the market makers' b parameters. By definition, inverse liquidity is unable to compare two cost-function market makers from different families.

- *Market depth* (Chen and Pennock 2007, Li and Vaughan 2013) Given the cost-function $C^{\mathcal{X}}(\mathbf{s})$, a security's market depth, also known as instantaneous liquidity, is defined as follows:

$$\rho^{\mathcal{X}}_{\omega}(\mathbf{s}) = \frac{1}{\frac{\partial}{\partial s_{\omega}} Pr^{\mathcal{X}}_{\omega}(\mathbf{s})} \tag{2.11}$$

in which $Pr^{\mathcal{X}}_{\omega}(\mathbf{s})$ is the security ω's price function, defined in (2.8). Comparing two cost-function market makers' liquidity using market depth is not as straightforward as using inverse liquidity. To compare two cost-function market makers using market depth, one should examine the rate of change in each market's prices when both markets have equal prices. Definition 2 describes this comparison more precisely.

Definition 2. *Let $C^{\mathcal{X}}$ and $C^{\mathcal{Y}}$ define the cost-function market makers of the two MSRs $\mathcal{X}$ and $\mathcal{Y}$, respectively. We say $C^{\mathcal{Y}}$ has more market depth relative $C^{\mathcal{X}}$ when:*

$$\forall \omega \in \Omega, \mathbf{p} \in \Delta_{N-1} : \rho^{C^{\mathcal{X}}}_{\omega}(\mathbf{s}) \leq \rho^{C^{\mathcal{Y}}}_{\omega}(\mathbf{s}') \tag{2.12}$$

in which $\mathbf{s}, \mathbf{s}' \in \mathbb{R}^N$ such that $Pr^{\mathcal{Y}}_{\omega}(\mathbf{s}) = Pr^{\mathcal{X}}_{\omega}(\mathbf{s}') = \mathbf{p}$.

Definition 2 says: the change in price for all possible prices in one cost-function market maker, $C^{\mathcal{Y}}$, must be less than the change in price for the same prices in another market, $C^{\mathcal{X}}$, for a given security. We note that we compare the depth of each market to one another for a fixed price, instead of number of outstanding shares. We use price because for a fixed number of outstanding shares, there may be different price estimates in two different markets, meaning that the projected likelihood of the security is not the same in each market. Figure 2.2 illustrates a comparison of the market depth between three well-known cost function market makers.

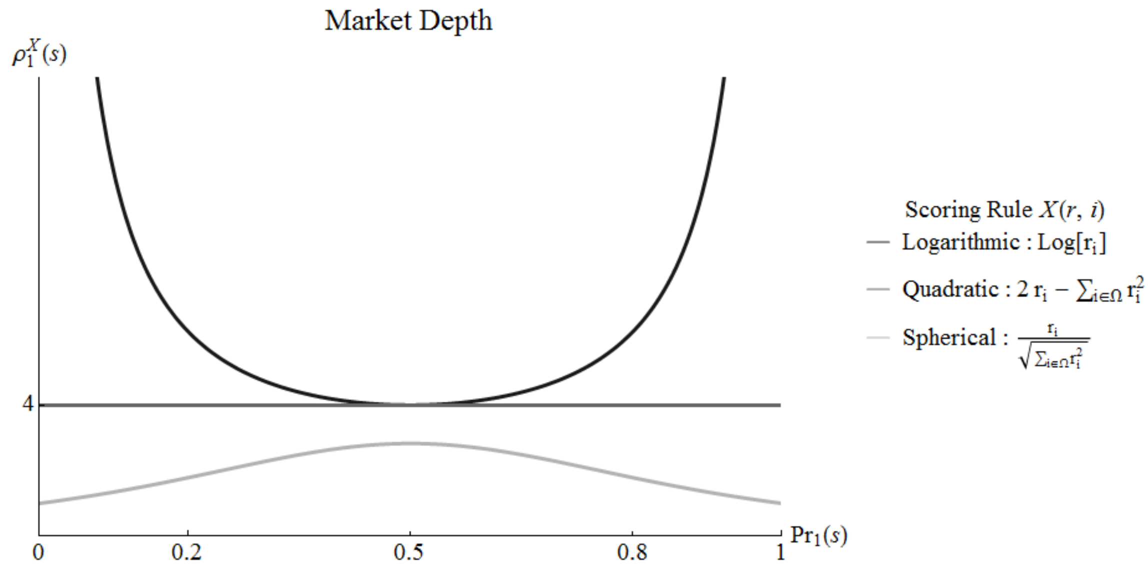


Figure 2.2: Market depth comparison across three cost functions corresponding to the three well-known scoring rules with $b = 1$.

- *Bid-ask spread* (Abernethy et al. 2013) In theory, cost-function market makers generate zero bid-ask spread, as traders can purchase infinitely small bundles. In practice, however, market makers determine a minimum trading unit, say $\mathbf{r}$, and thus the market bid-ask spread can be defined as follows. For a given bundle of shares, $\mathbf{s}$, the bid-ask spread is defined as

$$(C(\mathbf{s}+\mathbf{r}) - C(\mathbf{s})) - (C(\mathbf{s}-\mathbf{r}) - C(\mathbf{s})) \,, \tag{2.13}$$

  which is the difference between the current cost of buying the $\mathbf{r}$ bundle of shares and the cost at which $\mathbf{r}$ could be sold (Abernethy et al. 2013).

In our study, we only use the first two measures of liquidity, inverse liquidity, and market depth. The third measure, bid-ask spread, does not directly apply to our study. This is because we assume the agents' beliefs and agents' reports are probability distributions in Δ_{N-1}. The concept of a minimum bundle of shares restricts the agents' report space, something we do not consider in our study, though something worth exploring in the future.

Chapter 3

Deviation and Liquidity in Binary Outcome Space

Recently, a number of popular media outlets state that prediction markets are failing (Ledbetter 2008, D.R. 2016). Besides the natural argument that prediction markets provide probability estimates on the event, and not a deterministic estimate, popular media provides arguments such as cognitive biases, thin markets, low stakes, and slow to react to events as reasons behind prediction market failure. These may all be viable reasons, but one reason not mentioned is risk bias that induces rational individuals to change their behavior (i.e., reported probability estimates) in prediction markets due to their risk preferences. Due to the small number of participants, risk bias may be an issue in MSR prediction markets, in particular, in corporate settings. The impact of risk bias is studied by Dimitrov et al. (2015), in which the authors show that the reward given in a subsidized prediction market must decrease exponentially. However, the results of Dimitrov et al. (2015) are contingent on agents' having unbounded risk preferences, such as an agent who prefers receiving \$0.01 for certain to a lottery that pays \$1,000,000 with probability 0.99 and \$0 otherwise. Such agents are not observed in practice (Cox and Harrison 2008), and there is often a bound on the maximum risk-averse and risk-seeking preferences of a population. In this chapter, we study the impact of risk bias on agents' behavior in MSRs, when a population's risk preferences are unknown yet bounded. Sethi and Vaughan (2016) show that the price of a subsidized prediction market with risk-averse traders who are budget-limited is a good approximation of the agents' aggregated belief. The result of Sethi and Vaughan (2016) suggests that characteristics of a subsidized prediction market, namely liquidity, can be adjusted such that the market organizer can make better inferences about agents' beliefs. Abernethy et al. (2014) analytically draw the connection between a risk-averse agent, with an exponential utility function, and exponential belief distribution, and a subsidized prediction market's prices. In their work, Abernethy et al. (2014) show how a market's liquidity can be adjusted to alter the market's belief elicitation. Abernethy et al. (2014) use inverse liquidity to measure a market's liquidity. By definition, inverse liquidity can only compare the provided liquidity of different market makers from the same family as defined in Section 2.1. The work of Sethi and Vaughan (2016) and Abernethy et al. (2014) attempt to bridge the gap in the interpretation of prediction markets' prices in terms of agents' beliefs, and the characteristics of subsidized prediction markets. In this chapter, we extend this stream of literature in five ways.

Our five extensions of the current literature are as follows. First, we do not make any assumption on the agents' belief distribution, for example the work of Abernethy et al. (2014) assumed agents' beliefs are exponentially distributed. Second, we consider both risk-averse and risk-seeking individuals in our study, not only risk-averse as is currently assumed. Third, we do not make any assumption about the type of risk-aversion or risk-seeking utility of agents other than the fact that: an agent cannot be risk-averse and risk-seeking at the same time, and the risk-aversion/risk-seeking preference of agents are bounded. Our setting is an extension of those considered in the literature that in the most general case, only considers risk-averse agents (Sethi and Vaughan 2016). Fourth, we use two measures of liquidity, inverse liquidity and market depth, to show that our results are consistent regardless of how the liquidity is measured, this allows our analysis to be carried across different prediction markets from varying market maker families, while current literature only considers inverse liquidiy (Abernethy et al. 2014). Last, we do not consider bounded-budget agents, and therefore the behavior of an agent considered in our study is the same behavior exhibited by the same agent in the presence of learning, as considered by Abernethy et al. (2014), Sethi and Vaughan (2016).

In this chapter, we first show that there is no deterministic prediction market mechanism that can truthfully elicit the beliefs of agents with unknown, yet bounded, risk preferences. This result follows from the works of Lambert (2011), Schlag and van der Weele (2013) that show the same result for proper scoring rules. Fortunately, a group of papers shows that if probabilistic payment mechanisms are used instead of deterministic mechanisms (all market scoring rule and cost function subsidized prediction markets are deterministic mechanisms), then incentive compatibility may be restored in scoring rules (Allen 1987, Karni 2009). Unfortunately, in practice, individuals inherently dislike probabilistic payments (Wakker et al. 1997), and we are not aware of any probabilistic prediction market mechanisms in use today. It is results such as Wakker et al. (1997) that motivate us to only consider deterministic subsidized prediction market mechanisms in this study. As we cannot eliminate risk-bias using deterministic mechanisms, our only alternative is to minimize risk bias in subsidized prediction markets. In order to reduce risk bias in subsidized prediction markets, the market liquidity must also be reduced. In fact, we find the analytical relationship that exists between risk-bias and market liquidity for the subsidized prediction markets we consider.

With the analytical relationship between market liquidity and maximum agent deviation, we address a practical problem that has plagued subsidized prediction markets for years: "how much liquidity should be provided in a subsidized market?" This question has until now been answered with what is described by some prediction market researchers as the "art" of prediction markets (Pennock 2010). Our results enable practitioners to move out of the realm of "art" and into the realm of science, by carefully trading off market liquidity with the maximum agent deviation. With a bound on a population's risk preferences (Babcock et al. 1993) and our analytical results, we solve a series of mathematical programs for the Logarithmic MSR (LMSR), Quadratic MSR (QMSR), and Spherical MSR (SMSR) subsidized prediction markets, the type of markets used most frequently in practice, to determine the trade-off made in setting the market liquidity and the maximum agent deviation. It is this trade-off that a market organizer, the individual interested in running a market, must make when designing the market. We note that our results generalize to more than just the LMSR, QMSR, and SMSR prediction markets, and may be applied to any strictly proper MSR prediction market.

Managers, practitioners, and prediction market designers will find this work valuable in that we propose a systematic method to set up prediction market mechanisms that elicit probability estimates closer to agents' beliefs than those used in practice today, and gain a new understanding of the relationship between agents' deviation and prediction markets' liquidity. More practically, our results on the LMSR may be used for practitioners to set their market depth to their liking, knowing the maximum deviation that they may expect from a population of agents.

3.1 Model Set-up

Consider a binary event E, with two possible outcomes in $\{0, 1\}$. Let ω represent the corresponding random variable where $\omega \in \{0, 1\}$. In this thesis, we only analyze the behavior of myopic agents one at a time and therefore we can focus on two consecutive reports at a time. Thus we use the similar, but simpler notation of $\mathcal{X}(r, r^0, \omega)$ instead of $\mathcal{X}(r^{(t)}, r^{(t-1)}, \omega)$, where r is the agents report and r^0 is the market's current estimate, at the time of making report[1]. An agent with personal belief of p on ω, is asked to submit her probability estimates denoted by r. When the outcome of the event is observed, the prediction market rewards according to a market scoring rule function:

$$\mathcal{X}\left(r, r^0, \omega\right) = X\left(r, \omega\right) - X\left(r^0, \omega\right), \tag{3.1}$$

[1]Note that since the outcome space is binary, and we use coherent probabilities, we may use the variable r, and r^0 to represent the tuples like $\mathbf{r} = (r, 1-r)$, and $\mathbf{r^0} = (r^0, 1-r^0)$.

where the strictly proper score function $X(\cdot)$ has the following property:

$$X : \Delta_1 \times \{0,1\} \mapsto \mathbb{R} \text{ is smooth on } \Delta_1 \text{ for any given } \omega. \tag{3.2}$$

In addition to being myopic and expected utility maximizers, we further assume agents may be risk-averse or risk-seeking. As for risk preference, we assume agents have a concave or convex, utility function. We also refer to maximum risk aversion or risk seeking as to the utility of the most risk-averse or risk-seeking agent. We also assume such utility function always exist. We assume that the market organizer has a bound on the Arrow-Pratt measure of the agents risk preferences, but does not know any individuals' risk preferences. An agent with a utility function of the form $u \in \mathcal{U}$ is rational, and thus maximizes her expected utility given her belief of p on ω. The set $\mathcal{U} \subset \mathbb{R}^{\mathbb{R}}$ is the set of all utility functions that are monotonically increasing, twice differentiable, and are either convex, representing risk-seeking agents, or concave, representing risk-averse agents. Such an agent's expected utility from reporting any feasible report of q is as follows:

$$E_p\left[u\left(\mathcal{X}(q,r^0,\omega)\right)\right] = \; p\; u\left(X\left(q,1\right) - X\left(r^0,1\right)\right) + (1-p)u\left(X\left(q,0\right) - X\left(r^0,0\right)\right) \tag{3.3}$$

In any strictly proper MSR, a risk neutral agent will report truthfully (i.e., the utility maximizing report of a risk-neutral agent is the same as her belief). Given a strictly proper scoring rule $X(\cdot)$, and its corresponding MSR $\mathcal{X}$, we formally define the expected utility maximizing report of $r^u_{\mathcal{X}}\left(p, r^0\right)$ as:

$$r^u_{\mathcal{X}}\left(p, r^0\right) \triangleq \arg\max_{q\in[0,1]} E_p\left[u\left(\mathcal{X}\left(q, r^0, \omega\right)\right)\right]. \tag{3.4}$$

We note that as we assume all agents are myopic and rational, all reports made by agents will be those that satisfy (3.4) and we thus refer to all reports adhering to (3.4) simply as reports. As we show in Section 3.2, for an arbitrary agent with belief p and a non-linear utility function, the value of $r^{u}_{\mathcal{X}}(\cdot)$ almost always differs from p (i.e., an agent might under or over report her belief). As a consequence, in MSRs, beliefs are under and over reported. In this chapter, we compare these under/over reporting in different MSRs and we find necessary and sufficient conditions for a MSR to yield lower under/over reporting relative to another MSR. Accordingly, we refine our definition of *deviation* to be the element-wise difference between an agent's report and belief.

3.2 Deviation

In this section we show that in presence of agents with unknown and non-linear utilities, risk averse or risk seeking, no Hanson's MSRs with deterministic rewards can be incentive compatible. This result is similar to the work of Lambert (2011) and Schlag and van der Weele (2013), in which they prove that no proper scoring rule with deterministic rewards can be incentive compatible. We also note that for the impossibility result to hold, the assumption of bounded risk-preferences is not necessary. Our impossibility result is similar to the result of Lambert (2011). Lambert (2011) shows that when agents' risk preferences are not known, no scoring rule can elicit agents' beliefs. In Lemma 2, we show that when an agent has unknown utility, concave or convex, no Hanson's MSR can elicit her belief, unless her belief is identical to the market's current estimate. An implication of this finding is that in a MSR, the market's current estimate acts as a reference point. That is, if a risk-averse/risk-seeking agent with belief of p on ω, participates in a MSR with a current estimate of r^0; the only information that can be extracted from her report is either $p = r^0$, $p < r^0$, or $p > r^0$. Being able to extract this information is a new finding and was not explicitly discussed by Lambert (2011). We note that with an affine transformation of the scoring rules used by Lambert (2011), one may be able to recreate this feature. As the main focus of this section we analyze an agent's optimal report in MSRs, and we find that two or more MSRs may be compared to each other, to conclude which one provides a better estimate on agents subjective probability estimates. In other words, which market induces a lower deviation for the same agent for any given market estimate and any given belief. In Proposition 1, we provide the necessary and sufficient conditions for a MSR to yield a lower deviation compared to another MSR.

We next present a definition of deviation which we use throughout this dissertation. Intuitively, deviation may be captured by simply getting the difference of the agent's report and her belief.

Definition 3. *(Deviation) For a given market estimate, r^0, in a MSR $\mathcal{X}$, the following is defined as an agent's deviation function of $\mathcal{X}$ for any belief and current market estimate:*

$$\mathcal{F}^u_{\mathcal{X}}\left(p, r^0\right) = \left| r^u_{\mathcal{X}}\left(p, r^0\right) - p \right| \tag{3.5}$$

Before deriving the main result of this section, Proposition 1, we need to show two preliminary results in Lemma 1, and Lemma 2. Lemma 1 shows two properties that follow from the definition of strictly proper scoring rules and market scoring rules. Lemma 1 shows that the optimal report for the expected score of a risk-neutral agent is her belief. Moreover, for a given outcome, say $\omega = 1$, the score of a report is monotonically increasing in q.

Lemma 1. *Let $S(\cdot)$ be a strictly proper scoring rule.*

i)

$$\forall r \in (0,1): q\frac{\partial}{\partial q}S(q,1) + (1-q)\frac{\partial}{\partial q}S(q,0)\bigg|_{q=r} = 0. \tag{3.6}$$

ii)

$$\forall r \in (0,1): \frac{\partial}{\partial q}S(q,1)\,|_{q=r} > 0 \textit{ and } \frac{\partial}{\partial q}S(q,0)\,|_{q=r} < 0. \tag{3.7}$$

Proof. i) The claim in this case follows from the standard properties of strictly proper scoring rules. A proof has been presented for completeness. Since $S(\cdot,\omega)$ is a strictly proper scoring rule, for given $r \in (0,1)$ we have:

$$\{r\} = \underset{q\in[0,1]}{\arg\max} E_r\left[S(q,\omega)\right].$$

As $S(\cdot)$ is also smooth, r is an interior maximizer of $E_r\left[S(\cdot,\omega)\right]$ and therefore is a stationary point of $E_r\left[S(\cdot,\omega)\right]$. That is:

$$\nabla\Big(E_r\left[S(q,\omega)\right]\Big) = 0 \quad \Longrightarrow \frac{\partial}{\partial q}E_r\left[S(q,\omega)\right] = 0.$$

The definition of $E_r\left[S(q,\omega)\right]$ (2.2) for $\mathbf{q} = (q, 1-q) \in \Delta_1$ gives:

$$q\frac{\partial}{\partial q}S(q,1) + (1-q)\frac{\partial}{\partial q}S(q,0)\bigg|_{q=r} = 0,$$

that proves the claim in part (i).

ii) We prove the first part of the claim, $\frac{\partial}{\partial q}S\left(q,1\right)|_{q=r} > 0$, by contradiction; the second part can be prooven similarly. To the contrary assume $\frac{\partial}{\partial q}S\left(q,1\right)|_{q=y} \leq 0$ for some $y \in (0,1)$. Since $S(q,\omega)$ is smooth with respect to q, $\frac{\partial}{\partial q}S\left(q,1\right)|_{q=y} \leq 0$, implies that $S\left(\cdot,1\right)$ has a secant with a negative slope, that is:

$$\exists a, b \in (0,1) : 0 < a < y < b < 1, S(b,1) \leq S(a,1). \tag{3.8}$$

Since $S(\cdot)$ is a strictly proper scoring rule, we have $\arg\max\limits_{q\in[0,1]} E_p\left[S(q,\omega)\right]\Big|_{p=1} = \arg\max_{q\in[0,1]} S(q,1) = 1$, that is:

$$\forall q \in [0,1] : S(q,1) \leq S(1,1) \tag{3.9}$$

(3.8) and (3.9) imply there exists $c \in (0,1) : a < c < b$ such that:

$$S(a,1) = S(c,1). \tag{3.10}$$

Using strict properness of $S(\cdot)$ again we get:

$$aS(a,1) + (1-a)S(a,0) \quad > aS(c,1) + (1-a)S(c,0)\ , \tag{3.11}$$

$$cS(c,1) + (1-c)S(c,0) \quad > cS(a,1) + (1-c)S(a,0)\ . \tag{3.12}$$

By (3.10) we get:

$$
\begin{aligned}
(3.11) \quad &\implies \quad aS(a,1) + (1-a)S(a,0) > aS(a,1) + (1-a)S(c,0) \quad \implies S(a,0) > S(c,0), \\
(3.12) \quad &\implies \quad cS(a,1) + (1-c)S(c,0) > cS(a,1) + (1-c)S(c,0) \quad \implies S(c,0) > S(a,0),
\end{aligned}
$$

which is a contradiction. □

Lemma 2 provides a useful relationship between the agent's report and her belief that includes the agent's marginal utility in each outcome. To prove Lemma 2, we solve for the first order conditions of (3.4) and use Lemma 1 to obtain the relationship between an agent's report, her belief, and the market's current estimate.

Lemma 2. *Let* $r^u_{\mathcal{X}} \triangleq r^u_{\mathcal{X}}\left(p, r^0\right)$*, be the expected utility-maximizing report, defined in* (3.4)*, of an agent in the Hanson MSR* $\mathcal{X}$*, in which* $\mathbf{p} = (p, 1-p)$ *is the agent's belief on* ω *and* $\mathbf{r^0} = (r^0, 1-r^0)$ *is the market's current estimate.*

i)

$$
\frac{1-p}{p} = \frac{1-r^u_{\mathcal{X}}}{r^u_{\mathcal{X}}} \frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)}. \tag{3.13}
$$

ii)

$$
\text{(risk averse agent) if } u \text{ is concave} \quad \Longrightarrow \begin{cases} p < & r^u_{\mathcal{X}} & < r^0 & : p < r^0 \\ p = & r^u_{\mathcal{X}} & = r^0 & : p = r^0 \\ r^0 < & r^u_{\mathcal{X}} & < p & : p > r^0 \end{cases} .
$$

$$
\text{(risk seeking agent) if } u \text{ is covex} \quad \Longrightarrow \begin{cases} p < & r^u_{\mathcal{X}} & < r^0 & : p > r^0 \\ p = & r^u_{\mathcal{X}} & = r^0 & : p = r^0 \\ r^0 < & r^u_{\mathcal{X}} & < p & : p < r^0 \end{cases} . \tag{3.14}
$$

Proof. i) Since X, the underlying score function of $\mathcal{X}$ is smooth, $\mathcal{X}\left(\cdot, r^0, \omega\right)$ is also smooth, thus $r^u_{\mathcal{X}}$ is an interior point and it satisfies the first order condition. That is:

$$
\nabla\left(E_p\left[u\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, \omega\right)\right)\right]\right) = 0.
$$

By chain rule we get:

$$
p\tfrac{\partial}{\partial q}\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right) u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right) + (1-p)\tfrac{\partial}{\partial q}\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right) u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right) = 0.
$$

By definition of a MSR (3.1) we further have:

$$
p\tfrac{\partial}{\partial q}X\left(r^u_{\mathcal{X}}, 1\right) u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right) + (1-p)\tfrac{\partial}{\partial q}X\left(r^u_{\mathcal{X}}, 0\right) u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right) = 0.
$$

Rearrangement gives:

$$
\frac{1-p}{p} = \frac{\frac{\partial}{\partial q}X\left(r^u_{\mathcal{X}}, 1\right)}{\frac{\partial}{\partial q}X\left(r^u_{\mathcal{X}}, 0\right)} \frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)}. \tag{3.15}
$$

By a rearrangement of Lemma 1-i, (3.15) can be reduced to the followings:

$$\frac{1-p}{p} = \frac{1-r^u_{\mathcal{X}}}{r^u_{\mathcal{X}}} \frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)}. \tag{3.16}$$

This proves the claim in part (i).

ii) By (3.13), to compare $r^u_{\mathcal{X}}$ to p, we need to determine how $\frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)}$ compares to 1. Without loss of generality, let $r^u_{\mathcal{X}} > r^0$ and u be a concave function. By Lemma 1-ii we get:

$$\begin{cases} \mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right) = X\left(r^u_{\mathcal{X}}, 1\right) - X\left(r^0, 1\right) > 0 \\ \\ \mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right) = X\left(r^u_{\mathcal{X}}, 0\right) - X\left(r^0, 0\right) < 0 \end{cases}. \tag{3.17}$$

Since u is concave and non-decreasing, the function u' is decreasing and therefore (3.17) implies:

$$\frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)} < 1. \tag{3.18}$$

Therefore by (3.18) and (3.13) we get:

$$r^u_{\mathcal{X}} < p.$$

Similarly we can show the claim holds in the cases in which $r^u_{\mathcal{X}} = r^0$, $r^u_{\mathcal{X}} < r^0$, or u is convex. □

Proposition 1 below, shows that to compare the deviation in two MSRs, it is necessary and sufficient to compare the projection of the two MSRs' functions under the first derivative of an agent's utility. The optimal report of an agent, required for deviation function (3.5), is calculated by solving the first order equation of the agent's expected utility maximizing problem (3.4). By Lemma 2, we may directly relate an agent's belief to her optimal report. Intuitively, Lemma 2 states that an agent's report is a non-linear scale of her belief, where the non-linear scale is a function of the agent's utility and the market's current estimate. Proposition 1 states that for a given agent, given the definition of deviation, in order to compare two MSRs using deviation, one only needs to compare the non-linear scale factor, derived in Lemma 2. The comparison between an agent's reports, in different MSRs, to her belief, provides the necessary and sufficient conditions for a MSR to yield a lower deviation relative to another MSR.

Proposition 1. *Let $\mathcal{X}(\cdot)$ and $\mathcal{Y}(\cdot)$ be two market scoring rules. Also let $u \in \mathcal{U}$ be a concave (convex) function. The following holds for all $r^0 \in [0,1]$:*

$$\forall p \in [0,1] : \mathcal{F}^u_{\mathcal{X}}\left(p, r^0\right) \leq \mathcal{F}^u_{\mathcal{Y}}\left(p, r^0\right) \iff \begin{cases} \frac{u'\left(\mathcal{Y}\left(q,r^0,1\right)\right)}{u'\left(\mathcal{Y}\left(q,r^0,0\right)\right)} \leq (\geq) \frac{u'\left(\mathcal{X}\left(q,r^0,1\right)\right)}{u'\left(\mathcal{X}\left(q,r^0,0\right)\right)}, & \forall q \in \left[0, r^0\right] \\ \\ \frac{u'\left(\mathcal{Y}\left(q,r^0,1\right)\right)}{u'\left(\mathcal{Y}\left(q,r^0,0\right)\right)} \geq (\leq) \frac{u'\left(\mathcal{X}\left(q,r^0,1\right)\right)}{u'\left(\mathcal{X}\left(q,r^0,0\right)\right)}, & \forall q \in \left[r^0, 1\right] \end{cases} \tag{3.19}$$

Proof. Let $r^u_{\mathcal{X}} \triangleq r^u_{\mathcal{X}}\left(p, r^0\right)$ and $r^u_{\mathcal{Y}} \triangleq r^u_{\mathcal{Y}}\left(p, r^0\right)$, be the expected utility-maximizing report, defined in (3.4), of an agent in the Hanson MSR $\mathcal{X}$ and $\mathcal{Y}$, respectively. By the definition of deviation, Definition 3, we have:

$$\mathcal{F}^u_{\mathcal{X}}\left(p, r^0\right) \leq \mathcal{F}^u_{\mathcal{Y}}\left(p, r^0\right) \iff \left|r^u_{\mathcal{X}} - p\right| \leq \left|r^u_{\mathcal{Y}} - p\right|.$$

Without loss of generality let $p < r^0$ and u be a concave utility function. By Lemma 2-ii, we have $p < r^u_{\mathcal{X}}, r^u_{\mathcal{Y}} < r^0$ and thus we get:

$$\mathcal{F}^u_{\mathcal{X}}\left(p, r^0\right) \leq \mathcal{F}^u_{\mathcal{Y}}\left(p, r^0\right) \iff r^u_{\mathcal{X}} - p \leq r^u_{\mathcal{Y}} - p \iff r^u_{\mathcal{X}} \leq r^u_{\mathcal{Y}}.$$

By Lemma 2-i we have:

$$\tfrac{1-r^u_{\mathcal{X}}}{r^u_{\mathcal{X}}} \tfrac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)} = \tfrac{1-p}{p} = \tfrac{1-r^u_{\mathcal{Y}}}{r^u_{\mathcal{Y}}} \tfrac{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 0\right)\right)} \cdot \tag{3.20}$$

By simplifying (3.20) we get:

$$\frac{1-r^u_{\mathcal{X}}}{r^u_{\mathcal{X}}} \frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)} = \frac{1-r^u_{\mathcal{Y}}}{r^u_{\mathcal{Y}}} \frac{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 0\right)\right)}. \tag{3.21}$$

By (3.21) and simple comparison we get:

$$r^u_{\mathcal{X}} \leq r^u_{\mathcal{Y}} \iff \tfrac{1-r^u_{\mathcal{X}}}{r^u_{\mathcal{X}}} \geq \tfrac{1-r^u_{\mathcal{Y}}}{r^u_{\mathcal{Y}}} \iff \frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)} \leq \frac{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 0\right)\right)}. \tag{3.22}$$

Consider $q \in \left[0, r^0\right]$ and assume:

$$\frac{u'\left(\mathcal{X}\left(q, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(q, r^0, 0\right)\right)} \leq \frac{u'\left(\mathcal{Y}\left(q, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(q, r^0, 0\right)\right)}. \tag{3.23}$$

To prove (3.19), in the case where $q \in \left[0, r^0\right]$, we need to show (3.22) $\iff$ (3.23).

- (3.22) $\implies$ (3.23): By (3.22), $r^u_{\mathcal{X}} \leq r^u_{\mathcal{Y}}$. By Lemma 2-ii, and the fact that u' is decreasing, we get:

$$\begin{cases} \mathcal{Y}\left(r^u_{\mathcal{X}}, r^0, 1\right) \leq \mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 1\right) \\ \\ \mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 0\right) \leq \mathcal{Y}\left(r^u_{\mathcal{X}}, r^0, 0\right) \end{cases} \implies \frac{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(r^u_{\mathcal{Y}}, r^0, 0\right)\right)} \leq \frac{u'\left(\mathcal{Y}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)}. \tag{3.24}$$

By (3.22), the right hand side of (3.24) implies:

$$\frac{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)} \leq \frac{u'\left(\mathcal{Y}\left(r^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(r^u_{\mathcal{X}}, r^0, 0\right)\right)}. \tag{3.25}$$

Before we can proceed we need to show that the image of the univariate function $r^u_{\mathcal{X}}\left(\cdot, r^0\right)$ is $[0, 1]$. By definition, we have:

$$Im\left(r^u_{\mathcal{X}}\left(p, r^0\right)\right) \subseteq [0, 1]. \tag{3.26}$$

Also

$$r^u_{\mathcal{X}}\left(p, r^0\right)\Big|_{p=1} = \arg\max_{q\in[0,1]} E_1\left[u\left(\mathcal{X}\left(q, r^0, \omega\right)\right)\right] = 1$$

and

$$r^u_{\mathcal{X}}\left(p, r^0\right)\Big|_{p=0} = \arg\max_{q\in[0,1]} E_0\left[u\left(\mathcal{X}\left(q, r^0, \omega\right)\right)\right] = 0$$

$$\{0, 1\} \in Im\left(r^u_{\mathcal{X}}\left(\cdot, r^0\right)\right). \tag{3.27}$$

The function $r^u_{\mathcal{X}r^0}(p)$ is continuous and since the continuous image of a compact set is compact, (3.26) and (3.27) gives:

$$[0,1] \subseteq Im\left(r^u_{\mathcal{X}}\left(\cdot, r^0\right)\right),$$

which proves that the image of the univariate function $r^u_{\mathcal{X}}\left(\cdot, r^0\right)$ is $[0,1]$. Therefore (3.25) can be rewritten as:

$$\frac{u'\left(\mathcal{X}\left(q, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(q, r^0, 0\right)\right)} \leq \frac{u'\left(\mathcal{Y}\left(q, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(q, r^0, 0\right)\right)},$$

which proves (3.22) $\implies$ (3.23).

- (3.23) $\implies$ (3.22): Assume to the contrary that there exists $p^{(0)} \in \left[0, r^0\right]$ and let $a^u_{\mathcal{X}} \triangleq r^u_{\mathcal{X}}\left(p^{(0)}, r^0\right)$ and $b^u_{\mathcal{X}} \triangleq r^u_{\mathcal{Y}}\left(p^{(0)}, r^0\right)$, such that:

$$\frac{u'\left(\mathcal{Y}\left(b^u_{\mathcal{Y}}, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(b^u_{\mathcal{Y}}, r^0, 0\right)\right)} < \frac{u'\left(\mathcal{X}\left(a^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(a^u_{\mathcal{X}}, r^0, 0\right)\right)}. \tag{3.28}$$

By (3.21), we get:

$$b^u_{\mathcal{X}} < a^u_{\mathcal{X}}. \tag{3.29}$$

By (3.23), (3.29), and the fact that u' is decreasing, we get:

$$\begin{cases} \mathcal{X}\left(a^u_{\mathcal{X}}, r^0, 1\right) \leq \mathcal{X}\left(b^u_{\mathcal{Y}}, r^0, 1\right) \\ \mathcal{X}\left(b^u_{\mathcal{Y}}, r^0, 0\right) \leq \mathcal{X}\left(a^u_{\mathcal{X}}, r^0, 0\right) \end{cases} \implies \frac{u'\left(\mathcal{X}\left(a^u_{\mathcal{X}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(a^u_{\mathcal{X}}, r^0, 0\right)\right)} \leq \frac{u'\left(\mathcal{X}\left(b^u_{\mathcal{Y}}, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(b^u_{\mathcal{Y}}, r^0, 0\right)\right)}. \tag{3.30}$$

By (3.28), the right-hand side of (3.30) gives:

$$\frac{u'\left(\mathcal{Y}\left(b_{\mathcal{Y}}^{u}, r^{0}, 1\right)\right)}{u'\left(\mathcal{Y}\left(b_{\mathcal{Y}}^{u}, r^{0}, 0\right)\right)} < \frac{u'\left(\mathcal{X}\left(b_{\mathcal{Y}}^{u}, r^{0}, 1\right)\right)}{u'\left(\mathcal{X}\left(b_{\mathcal{Y}}^{u}, r^{0}, 0\right)\right)}. \tag{3.31}$$

Since u is concave, $p^{(0)} \in \left[0, r^{0}\right]$ and $b_{\mathcal{X}}^{u} \triangleq r_{\mathcal{Y}}^{u}\left(p^{(0)}, r^{0}\right)$, we conclude that $b_{\mathcal{X}}^{u} \in \left[0, r^{0}\right]$. Thus (3.31) is a contradiction with (3.23). □

Corollary 1 is a special case of Proposition 1, as not all MSR pairs satisfy (3.32). Equation 3.32 states that MSR $\mathcal{Y}$ dominates MSR $\mathcal{X}$ in a binary outcome space, see Definition 1. However, if (3.32) is satisfied, then we no longer need condition (3.19) that is a function of an agent's utility. With (3.32) we are able to only compare the underlying MSR function, as MSRs and the utility functions we consider are monotone. Corollary 1 states that for a family of MSRs, including LMSRs, the deviation can be compared between the two MSRs without considering agents' utility.

Corollary 1. *Let $\mathcal{X}(\cdot)$ and $\mathcal{Y}(\cdot)$ be two market scoring rules that satisfy:*

$$\forall r^{0} \in [0,1] : \begin{cases} \mathcal{Y}\left(q, r^{0}, 1\right) \leq \mathcal{X}\left(q, r^{0}, 1\right) \leq \mathcal{X}\left(q, r^{0}, 0\right) \leq \mathcal{Y}\left(q, r^{0}, 0\right), \forall q \in \left[0, r^{0}\right] \\ \mathcal{Y}\left(q, r^{0}, 0\right) \leq \mathcal{X}\left(q, r^{0}, 0\right) \leq \mathcal{X}\left(q, r^{0}, 1\right) \leq \mathcal{Y}\left(q, r^{0}, 1\right), \forall q \in \left[r^{0}, 1\right] \end{cases}. \tag{3.32}$$

Therefore the following holds:

$$\forall u \in \mathcal{U}, \forall p \in [0,1] : \mathcal{F}_{\mathcal{X}}^{u}\left(p, r^{0}\right) \leq \mathcal{F}_{\mathcal{Y}}^{u}\left(p, r^{0}\right).$$

Proof. Let (3.32) hold, $u(\cdot)$ be concave and without loss of generality let $q \in \left[r^0, 1\right]$. By Proposition 1, to prove the claim we need to prove the following:

$$\frac{u'\left(\mathcal{X}\left(q, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(q, r^0, 0\right)\right)} \leq \frac{u'\left(\mathcal{Y}\left(q, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(q, r^0, 0\right)\right)}. \tag{3.33}$$

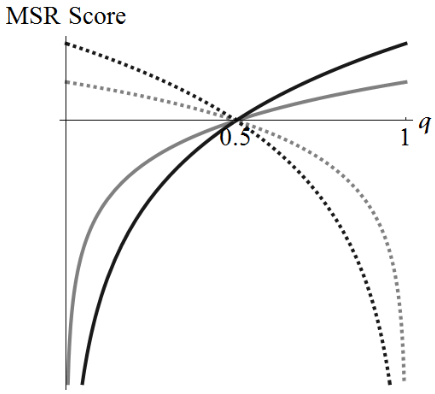
Since $u(\cdot)$ is concave and monotonically increasing, u', the first derivative of the function $u(\cdot)$, is monotonically decreasing. (3.32) and Lemma 1-ii gives:

$$\begin{cases} \mathcal{Y}\left(q, r^0, 1\right) \leq \mathcal{X}\left(q, r^0, 1\right) \\ \\ \mathcal{X}\left(q, r^0, 0\right) \leq \mathcal{Y}\left(q, r^0, 0\right) \end{cases} \implies \frac{u'\left(\mathcal{X}\left(q, r^0, 1\right)\right)}{u'\left(\mathcal{X}\left(q, r^0, 0\right)\right)} \leq \frac{u'\left(\mathcal{Y}\left(q, r^0, 1\right)\right)}{u'\left(\mathcal{Y}\left(q, r^0, 0\right)\right)}, \tag{3.34}$$

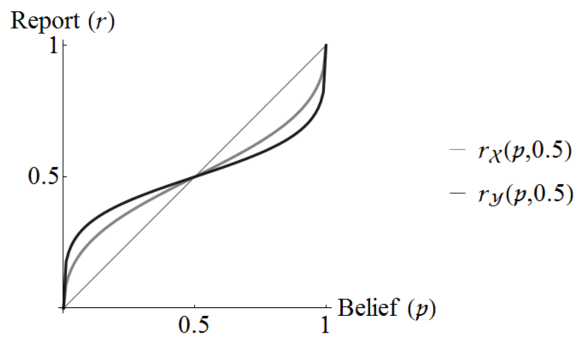
which completes the proof in this case.

When $u(\cdot)$ is convex, by Proposition 1 we need to prove the reverse of (3.33). However, since $u(\cdot)$ is convex and monotonically increasing, $u'(\cdot)$ is monotonically increasing, and thus (3.32) is enough to show the reverse of (3.33). □

In addition to LMSRs, two other families of MSRs satisfy (3.32), Quadratic MSRs, and Spherical MSRs (Hanson 2003, Gneiting and Raftery 2007), with different b parameters. Thus, when comparing the deviation between two MSRs from the same family, two LMSRs for example, it is sufficient to compare their b parameters, as the MSR with the larger b will have a larger deviation relative to the other MSR. For example, let $r^0 = 0.5$, and consider a risk-averse agent with utility $u(x) = 1 - e^{-x}$, and the belief of $p = 0.75$ on $\omega = 1$. This agent's report in $\mathcal{X}$, a LMSR with $b = 1$, found by solving[2] the optimization problem (3.4) is ~ 0.634. The same agent's report in $\mathcal{Y}$, a LMSR with $b = 2$, however, is ~ 0.590 which is further from her belief of 0.75, relative to 0.634, which shows $\mathcal{F}^u_{\mathcal{X}}(0.75, 0.5) < \mathcal{F}^u_{\mathcal{Y}}(0.75, 0.5)$. Figure 3.1 is an extended illustration of this numerical example for any given belief $p \in [0, 1]$.



(a) MSR comparison. The solid black line is $\mathcal{Y}(q, 0.5, 1) = 2\ln(q) - 2\ln(0.5)$, the solid grey line is $\mathcal{X}(q, 0.5, 1) = \ln(q) - \ln(0.5)$, the dashed black line is $\mathcal{Y}(q, 0.5, 0) = 2\ln(1-q) - 2\ln(1-0.5)$, and the dashed grey line is $\mathcal{X}(q, 0.5, 0) = \ln(1-q) - \ln(1-0.5)$.



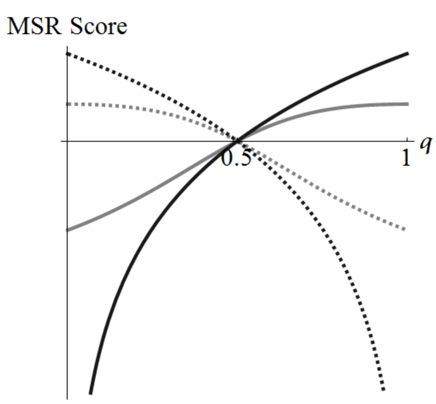
(b) Report comparison for the two MSRs $\mathcal{X}$ and $\mathcal{Y}$ in Figure 3.1 (a), showing $\mathcal{F}^u_{\mathcal{X}}(p, 0.5) \leq \mathcal{F}^u_{\mathcal{Y}}(p, 0.5)$ for any given $p \in [0, 1]$.

Figure 3.1: An extended numerical example of Corollary 1 for two MSRs from the same family that satisfy (3.32), using $u(x) = 1 - e^{-x}$, and $\mathbf{r}^{(0)} = 0.5$.

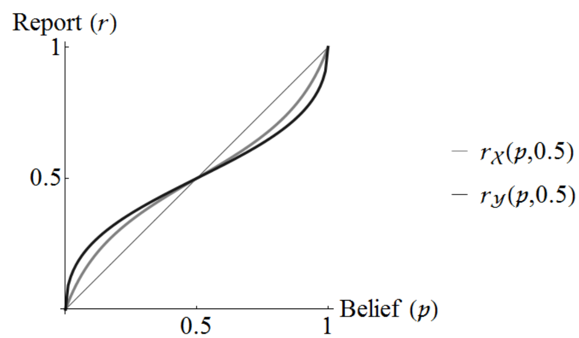
Similar to the case in which we compare different MSRs from the same family, when we compare the deviation between two MSRs from different families, a LMSR and a Spherical MSR for instance, it is sufficient to compare their MSR functions as both MSRs satisfy (3.32). In particular, we can verify that if $\mathcal{Y}$ is a LMSR with $b = 1$, and $\mathcal{X}$ is Spherical MSR with $b = 1$, then $\mathcal{X}$ and $\mathcal{Y}$ satisfy (3.32), thus $\mathcal{Y}$ has more deviation compare to $\mathcal{X}$. For example, let $r^0 = 0.5$, and consider a risk-averse agent

[2] We use the function ArgMax in Mathematica (Wolfram Research, Inc.) to numerically solve the optimization problem.

with utility $u(x) = 1 - e^{-x}$, and the belief of $p = 0.9$ on $\omega = 1$. This agent's report in $\mathcal{Y}$, a LMSR with $b = 1$, found by solving the optimization problem (3.4) is ~ 0.750. The same agent's report in $\mathcal{X}$, a Spherical MSR with $b = 1$, however, is ~ 0.810 which is closer to her belief of 0.9, relative to 0.750, which shows $\mathcal{F}^u_{\mathcal{X}}(0.9, 0.5) < \mathcal{F}^u_{\mathcal{Y}}(0.9, 0.5)$. Figure 3.2 is an extended illustration of this numerical example for any given belief $p \in [0, 1]$. Please note that Figures 3.1 and 3.2 are not a complete illustration of Corollary 1 as the two variables of utility and market's current estimate are fixed. However, if we allow utility, market's current estimate, and belief to all vary, then Corollary 1 still holds, but we are not able to visually present this result, and the numerical example simply highlights one situation.



(a) MSR comparison. The solid black line is $\mathcal{Y}(q, 0.5, 1) = \ln(q) - \ln(0.5)$, the solid grey line is $\mathcal{X}(q, 0.5, 1) = \frac{q}{\sqrt{q^2+(1-q)^2}} - \frac{0.5}{\sqrt{0.5^2+(1-0.5)^2}}$, the dashed black line is $\mathcal{Y}(q, 0.5, 0) = \ln(1-q) - \ln(0.5)$, and the dashed grey line is $\mathcal{X}(q, 0.5, 0) = \frac{1-q}{\sqrt{q^2+(1-q)^2}} - \frac{0.5}{\sqrt{0.5^2+(1-0.5)^2}}$.



(b) Report comparison for the two MSRs $\mathcal{X}$ and $\mathcal{Y}$ in Figure 3.2 (a), showing $\mathcal{F}^u_{\mathcal{X}}(p, 0.5) \leq \mathcal{F}^u_{\mathcal{Y}}(p, 0.5)$ for any given $p \in [0, 1]$.

Figure 3.2: An extended numerical example of Corollary 1 for two MSRs from different families that satisfy (3.32), using $u(x) = 1 - e^{-x}$, and $\mathbf{r}^{(0)} = 0.5$.

3.3 Deviation and Liquidity

In this section, we present the analytical results that show the amount of liquidity provided by a valid cost-function prediction market, as defined in Section 2.3, is closely related to the deviation of reports in its corresponding MSR prediction market. We use two measures of liquidity: inverse liquidity, and market depth, to illustrate the relation between deviation and liquidity. In Section 3.3.1, we show that when inverse liquidity is used to measure the amount of provided liquidity, a higher deviation implies a higher liquidity. In Section 3.3.2, we show that when market depth is being used to measure liquidity, the same results holds and a higher deviation implies a higher liquidity. In Section 3.3.3, we discuss the other, perhaps more practical, implication of our results that can help a market organizer determine the amount of market liquidity required for a desired level of belief elicitation. Given a bound on a prediction market populations' risk preferences, a market organizer can optimize the market's liquidity constrained by the maximum allowed deviation. The established optimization problem introduces an analytical criterion to determine the amount of market liquidity for a MSR cost-function market maker while maintaining bounded subsidy.

3.3.1 Inverse Liquidity and Deviation

Recall that for a valid cost-function, say $C^{\mathcal{X}}$, another valid cost-function $C_b^{\mathcal{X}}(\mathbf{s}) \triangleq b\, C^{\mathcal{X}}\left(\frac{1}{b}\mathbf{s}\right)$ can be generated for any positive scalar b. It is also straightforward to show that the corresponding MSR for the cost-function $C_b^{\mathcal{X}}$ is $b\, \mathcal{X}(\cdot)$, in which $\mathcal{X}(\cdot)$ is the underlying MSR of $C^{\mathcal{X}}$. Thus, given a cost-function, a family of cost-functions can be generated using different b parameters.

Proposition 2 shows that when two cost-functions from the same family are compared using inverse liquidity; higher liquidity is equivalent to higher deviation. To see this, recall that, by definition, higher b parameter equals to higher liquidity. Moreover, a higher b parameter also increases the value of the underlying MSR function which by Corollary 1, implies more deviation.

Proposition 2. *Let $C^{\mathcal{X}}$ and $C^{\mathcal{Y}}$ be the cost function market makers of the two MSRs $\mathcal{X}$ and $\mathcal{Y}$ respectively, in which $C^{\mathcal{Y}}(\mathbf{s}) = b\, C^{\mathcal{X}}(\mathbf{s}/b)$ for some $b > 1$. The market maker $C^{\mathcal{Y}}$ has more inverse liquidity if and only if $\mathcal{Y}$ has more deviation compared to $\mathcal{X}$.*

Given the definition of inverse liquidity, the implication of Proposition 2 is simple: a higher b parameter presents a tension between two desirable properties of less deviation and higher liquidity. Moreover, when inverse liquidity is used to measure the market's liquidity, deviation and liquidity are equivalent. However, by definition of inverse liquidity, (2.10), we are unable to compare two MSRs from different families, leading us to consider market depth when comparing MSRs from different families.

3.3.2 Market Depth and Deviation

When we use market depth to measure liquidity, we can compare cost-functions from different families. For instance, we can conclude that the Logarithmic cost-function market maker with b parameter of 1, has more market depth relative to a Quadratic cost-function market maker (Chen and Pennock 2007) with b parameter of 1. Proposition 3 shows that a higher deviation implies a higher market depth. To prove Proposition 3, we need to find the relation between a share quantity and the price function. Fortunately (2.7) presents the unconstrained optimization problem we need to solve to find the relation. In order to solve the unconstrained optimization problem we consider the first order conditions of the objective function with respect to q and derive the relation between the price and the share quantity, referred to as the *price-share* relation. The price-share relationship allows us to implicitly calculate the derivative of the price w.r.t. the quantity of outstanding shares for a given MSR. It follows from the definition of market depth, Definition 2, that computing the price-share relation allows us to compare the market depths of the cost-functions of two MSRs. In particular, we show that for a cost-function $C^{\mathcal{X}}$, the depth of the market is equal to $\frac{\partial}{\partial q} X(q,1) - X(q,0)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})}$. Using the previous equation, we relate the depth of a cost-function market maker to its underlying proper scoring rule. Using Corollary 1, and the simple definition of the first derivative, we can conclude that more deviation in a MSR implies more market depth in its corresponding cost-function market maker.

Proposition 3. *Let $C^{\mathcal{X}}$ and $C^{\mathcal{Y}}$ be the cost-function market makers of the two MSRs $\mathcal{X}$, and $\mathcal{Y}$ respectively. $C^{\mathcal{X}}$ has more market depth relative to $C^{\mathcal{Y}}$, only if $\mathcal{X}$ has more deviation compared to $\mathcal{Y}$.*

Proof. By Equivalence relation (2.7) we have:

$$C^{\mathcal{X}}(\mathbf{s}) = \max_{q \in [0,1]} s_1 q + s_0(1-q) - \left(qX(q,1) + (1-q)X(q,0)\right). \tag{3.35}$$

By definition of the price function (2.8), (3.35), and the first order condition of (3.35);

we have:

$$\frac{\partial}{\partial q}\left(s_1 q + s_0(1-q) - (qX(q,1) + (1-q)X(q,0))\right)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})} = 0. \tag{3.36}$$

Expanding (3.36) gives:

$$s_1 - s_0 - \left(X\left(q,1\right) - X\left(q,0\right)\right) + q\frac{\partial}{\partial q}X(q,1) + (1-q)\frac{\partial}{\partial q}X(q,0)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})} = 0. \tag{3.37}$$

By Lemma 1-i, $q\frac{\partial}{\partial q}X(q,1) + (1-q)\frac{\partial}{\partial q}X(q,0)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})} = 0$, thus (3.37) reduces to:

$$s_1 - s_0 - \left(X\left(Pr_1^{\mathcal{X}}(\mathbf{s},1)\right) - X\left(Pr_1^{\mathcal{X}}(\mathbf{s},0)\right)\right) = 0. \tag{3.38}$$

From (3.38), we can implicitly derive the rate of change in $Pr_1^{\mathcal{X}}$ w.r.t. s_1. By implicit differentiation we get:

$$\frac{\partial}{\partial s_1}Pr_1^{\mathcal{X}}(\mathbf{s}) = \frac{1}{\frac{\partial}{\partial q}\left(X\left(q,1\right) - X\left(q,0\right)\right)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})}}. \tag{3.39}$$

Thus by definition of market depth, (2.11), we get:

$$\rho_1^{\mathcal{X}}\left(\mathbf{s}\right) = \frac{\partial}{\partial q}\left(X\left(q,1\right) - X\left(q,0\right)\right)\Big|_{q=Pr_1^{\mathcal{X}}(\mathbf{s})} \tag{3.40}$$

Carrying our the procedure above, after replacing $C^{\mathcal{Y}}$ for $C^{\mathcal{X}}$ we find:

$$\rho_1^{\mathcal{Y}}\left(\mathbf{s}\right) = \frac{\partial}{\partial q}\left(Y\left(q,1\right) - Y\left(q,0\right)\right)\Big|_{q=Pr_1^{\mathcal{Y}}(\mathbf{s})}. \tag{3.41}$$

Thus by Definition 2, $C^{\mathcal{Y}}$ has more market depth relative to $C^{\mathcal{X}}$, only if[3]:

$$\forall a \in [0,1] : \left.\frac{\partial}{\partial q}\left(X\left(q,1\right) - X\left(q,0\right)\right)\right|_{q=a} \leq \left.\frac{\partial}{\partial q}\left(Y\left(q,1\right) - Y\left(q,0\right)\right)\right|_{q=a}. \tag{3.42}$$

Therefore, to show that $C^{\mathcal{Y}}$ has more market depth relative to $C^{\mathcal{X}}$ it is enough to show that (3.42) is satisfied.

By assumption, $\mathcal{Y}$ has more deviation compared to $\mathcal{X}$. Corollary 1, for a given $a \in [0,1]$ gives:

$$\begin{cases} \mathcal{Y}\left(q,a,1\right) \leq \mathcal{X}\left(q,a,1\right) \leq \mathcal{X}\left(q,a,0\right) \leq \mathcal{Y}\left(q,a,0\right), & q \leq a \\ \mathcal{Y}\left(q,a,0\right) \leq \mathcal{X}\left(q,a,0\right) \leq \mathcal{X}\left(q,a,1\right) \leq \mathcal{Y}\left(q,a,1\right), & q > a \end{cases} \tag{3.43}$$

Equation (3.43) gives:

$$\begin{cases} \mathcal{X}\left(q,a,0\right) - \mathcal{X}\left(q,a,1\right) \leq \mathcal{Y}\left(q,a,0\right) - \mathcal{Y}\left(q,a,1\right) & : q \leq a \\ \mathcal{X}\left(q,a,1\right) - \mathcal{X}\left(q,a,0\right) \leq \mathcal{Y}\left(q,a,1\right) - \mathcal{Y}\left(q,a,0\right) & : q > a \end{cases}. \tag{3.44}$$

Thus by definition of a MSR (3.1), (3.44) can be expanded to:

$$\begin{cases} 0 \leq X\left(q,0\right) - X\left(a,0\right) - \left(X\left(q,1\right) - X\left(a,1\right)\right) \leq Y\left(q,0\right) - Y\left(a,0\right) - \left(Y\left(q,1\right) - Y\left(a,1\right)\right) & : q \leq a \\ 0 \leq X\left(q,1\right) - X\left(a,1\right) - \left(X\left(q,0\right) - X\left(a,0\right)\right) \leq Y\left(q,1\right) - Y\left(a,1\right) - \left(Y\left(q,0\right) - Y\left(a,0\right)\right) & : q > a \end{cases}. \tag{3.45}$$

[3] Note that since $|\Omega| = 2$, we only require to show (3.42) to conclude that $C^{\mathcal{Y}}$ has more market depth relative to $C^{\mathcal{X}}$.

Rearranging (3.45) gives:

$$\begin{cases} \frac{Y(q,1)-Y(q,0)-\left(Y(a,1)-Y(a,0)\right)}{q-a} \leq \frac{X(q,1)-X(q,0)-\left(X(a,1)-X(a,0)\right)}{q-a} & q \leq a \\ \\ \frac{X(q,1)-X(q,0)-\left(X(a,1)-X(a,0)\right)}{q-a} \leq \frac{Y(q,1)-Y(q,0)-\left(Y(a,1)-Y(a,0)\right)}{q-a} & q > a \end{cases} . \tag{3.46}$$

By (3.46) and the limit definition of the first derivative we get:

$$\frac{\partial}{\partial q}\left(X\left(q,1\right)-X\left(q,0\right)\right)\bigg|_{q=a} \leq \frac{\partial}{\partial q}\left(Y\left(q,1\right)-Y\left(q,0\right)\right)\bigg|_{q=a} . \tag{3.47}$$

Note that the above equation holds since both score functions $X(\cdot)$ and $Y(\cdot)$ are differentiable and continuous. Equation 3.47 shows that (3.42) is satisfied and the proof is complete. □

Similar to Proposition 2, Proposition 3 also shows that higher deviation implies higher market depth. As both market depth and inverse liquidity are the two measures of liquidity used in the literature, the two propositions, Proposition 2 and Proposition 3, collectively state the same result: higher deviation implies higher liquidity. In the next section, we discuss how this result can be used to determine an optimal amount of liquidity given a desired level of belief elicitation.

3.3.3 Optimizing Market Depth and Report Deviation

As discussed in Section 2.1, setting the market depth of a market maker is often described as "art." In this section, we apply the results of Proposition 2 to show how a market organizer can optimize inverse liquidity to bound a maximum value of deviation. Our result is similar to the work of Abernethy et al. (2014), where authors show that lower inverse liquidity increases the difference between agents' belief and the market price, what we call deviation. The results of Abernethy et al. (2014), Sethi and Vaughan (2016) suggest that a market organizer can change the market's inverse liquidity parameter to achieve a desirable value of belief elicitation. Unfortunately, neither of these two papers formalize the relationship that exists between inverse liquidity and deviation. As mentioned earlier, we fill in this gap by introducing an optimization problem that can determine a minimum amount of inverse liquidity for a desired level of deviation.

We can verify that for two binary LMSRs with parameter b_1 and b_2 in which $b_2 > b_1$, the following holds:

$$\forall r^0 \in [0,1] : \begin{cases} \text{LMSR}_{b_2}\left(q, r^0, 0\right) \leq \text{LMSR}_{b_1}\left(q, r^0, 0\right) \leq \text{LMSR}_{b_1}\left(q, r^0, 1\right) \leq \text{LMSR}_{b_2}\left(q, r^0, 1\right), \; q \geq r^0 \\ \\ \text{LMSR}_{b_2}\left(q, r^0, 1\right) < \text{LMSR}_{b_1}\left(q, r^0, 1\right) < \text{LMSR}_{b_1}\left(q, r^0, 0\right) < \text{LMSR}_{b_2}\left(q, r^0, 0\right), \; q < r^0 \end{cases} . \tag{3.48}$$

Thus by Corollary 1, the $LMSR_{b_2}$ has more deviation compared to $LMSR_{b_1}$, By Proposition 2, $C^{\text{LMSR}_{b_2}}$ has more inverse liquidity compared to $C^{\text{LMSR}_{b_1}}$. Now assume that $\bar{u}(\cdot)$ is the utility function of the most risk-averse/risk-seeking agent, as defined in Section 2.2. We define the function $DevMax(b) \triangleq \max_{r^0, p \in [0,1]} \mathcal{F}^{\bar{u}}_{\text{LMSR}_b}(p, r^0)$ to be the maximum possible deviation. The following optimization problem can determine the maximum amount of market depth while guaranteeing the maximum market maker losses to $\bar{B}$, and a maximum deviation of $\bar{D}$.

$$
\begin{aligned}
\max \quad & b \\
\text{subject to:} \quad & \\
& DevMax(b) \leq \bar{D} \quad \text{(deviation constraint)} \\
& b\ln(2) \leq \bar{B} \\
& b \geq 0
\end{aligned} \tag{CP}
$$

The budget constraint ensures that the liquidity parameter b does not exceed the maximum subsidy requires to run a market (Chen and Pennock 2007). Using the same analysis, we can set similar optimization problems to (CP), for other MSRs such as QMSRs, and SMSRs. Figure 1 shows the value of parameter b as a function of maximum deviation, $\bar{D}$, accepted by the market maker for three different MSRs. In the example of Figure 3.3, the value of b is determined by program (CP) for agents with risk preferences determined by Babcock et al. (1993)[4].

However, the results of Proposition 2, similar to the results of Abernethy et al. (2014), Sethi and Vaughan (2016), only allow us to compare MSRs that belong to the same family (for example, two LMSRs or two Quadratic MSRs, etc.). The fact that Figure 3.3 was derived from the results of Proposition 2, it follows that the lines in Figure 3.3 cannot be used to compare different MSR families to one another. Fortunately, the results of Proposition 3 allows us to compare MSRs from different families to one another. We note that for a comparison of two MSRs to be valid

[4] Babcock et al. (1993) models individuals' risk attitudes using the constant absolute risk-aversion (CARA) utility function, that is $u(x) = 1 - e^{-\alpha x}$, in which α determines the amount of absolute risk aversion for an individual's risk preference. Babcock et al. (1993) show that for a gamble sizes of less than or equal to \$100, the value of the parameter α ranges from 0.0002 to 0.046204. In the example of Figure 3.3, we use the value of $\bar{\alpha} = 0.046204$ to to determine the maximum deviation of $\bar{D}$

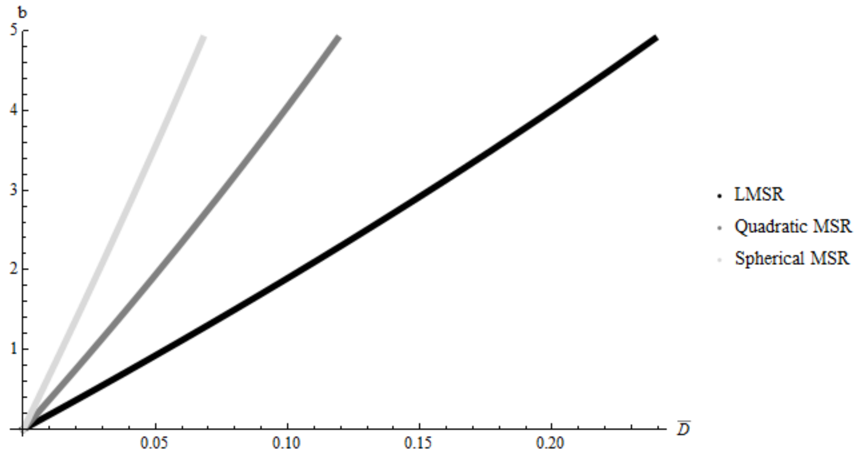


Figure 3.3: Approximations of parameter b as a function of $\bar{D}$ for different MSRs.

our definition of one market having more market depth than another, we assume the property, described in Definition 2, holds for all possible market prices. It is not always possible to guarantee that one market will have a larger market depth than another market for all market prices. We would like to point out that the results of Proposition 3 may be extended to a locally defined variation of market depth. However, we choose not to use this definition as the comparison of MSRs with a locally defined variation of market depth will require the market maker to have prior knowledge on the distribution of the market prices. If a market maker, has knowledge on the distribution of market prices, then it is no longer clear why the market should exist, as a distribution on market prices is equivalent to a distribution on the outcome of the traded event.

To illustrate an application of Proposition 3, consider a set of MSRs $\mathfrak{X}$, not necessarily from the same families of MSRs, and the set of their corresponding cost-functions $\mathfrak{C}^{\mathfrak{X}}$. Given a maximum value of deviation, say $\bar{D}$, we may choose the MSR $\mathcal{X}^\star \in \mathfrak{X}$, in which: $\mathcal{X}^\star$ has a maximum deviation of $\bar{D}$, and $C^{\mathcal{X}^\star}(\cdot)$ has more market

(a) Logarithmic and Quadratic Cost-functions.
(b) Logarithmic and Spherical Cost-functions.
(c) Spherical and Quadratic Cost-functions.

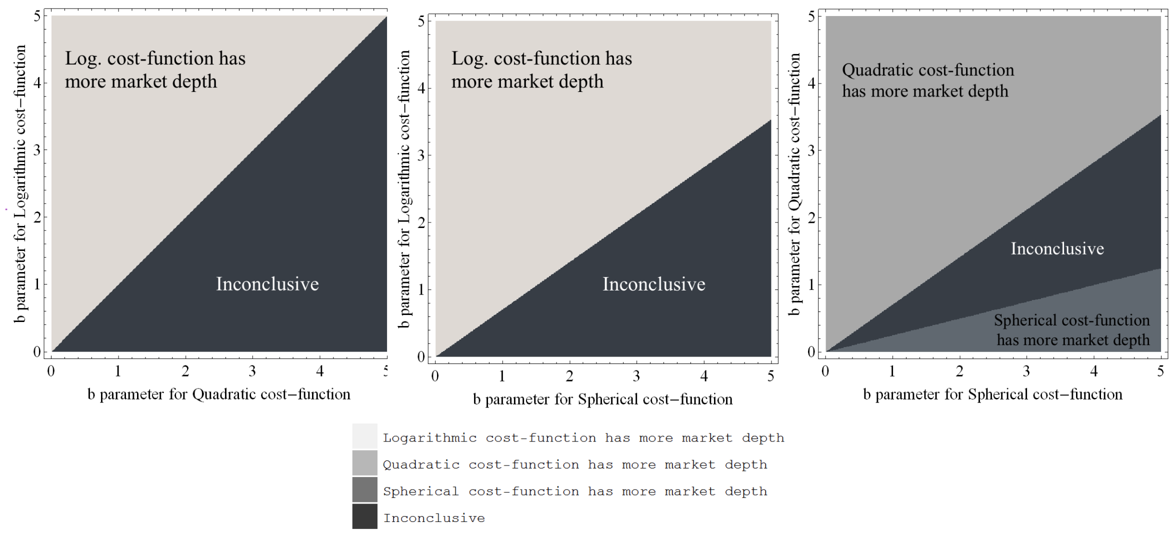


Figure 3.4: Depth comparison between Logarithmic, Quadratic, and Spherical cost-function market makers. Inconclusive means that the conditions in the definition of Market Depth, Definition 2, are violated.

depth relative to all other cost-functions in $\mathfrak{C}^{\mathfrak{X}}$, if comparable. Figure 3.4 illustrates the head-to-head comparison of the three popular variants of MSRs.

Using the depth comparison illustrated in Figure 3.4, we can numerically show that when $\mathfrak{X}$ includes three families of MSRs, Logarithmic, Quadratic, and Spherical MSRs (Chen and Pennock 2007), for any given $\bar{D}$, $\mathcal{X}^\star$ is a LMSR. This is because a logarithmic cost-function either: has more depth relative to any quadratic or spherical cost-function market makers, or is not comparable to a quadratic or spherical cost-function market maker. Moreover, the fact that $\mathcal{X}^\star$ is a LMSR is independent of the maximum risk preference. Using a similar numerical analysis, and assuming any population has the same maximum risk preference found by Babcock et al. (1993), we can show that when $\mathfrak{X}$ includes only two families of Quadratic, and Spherical

MSRs (Chen and Pennock 2007), $\mathcal{X}^\star$ is a Quadratic MSR for any given $\bar{D}$. We note that, unlike the case with three families of MSRs, Logarithmic, Quadratic, and Spherical MSRs, when $\mathfrak{X}$ includes only two families of Quadratic, and Spherical MSRs, the utility function of the most risk-averse/risk seeking agent determines if $\mathcal{X}^\star$ is a Quadratic or a Spherical MSR. Moreover, the class of MSRs where $\mathcal{X}^\star$ belongs to, may also depend on $\bar{D}$.

3.4 Conclusion

In this chapter we study the effect of agent risk-bias on belief reporting in MSR prediction markets and characterized the close relation with cost-functions' market liquidity and risk-bias in MSRs. Our analytical results suggest that myopic, utility maximizing agents do not report their exact beliefs, unless their beliefs are identical to the current market estimate. We introduced the concept of deviation to measure the difference between an agent's reported belief and personal belief. Our first finding is that we can compare the value of deviation across all MSRs by comparing the reward functions provided to agents. The higher the reward functions, the higher the deviation. We also showed that for all MSRs, decreasing the deviation of a MSR implies decreasing the liquidity of the MSR's corresponding cost-function market maker. We used this relation to introduce an analytical approach to determine the amount of liquidity to use to ensure a desirable belief elicitation. Market organizers who are interested in eliciting and aggregating traders' belief on future outcomes can utilize our findings by optimizing a prediction market's liquidity to ensure a desired level of belief elicitation.

Chapter 4

Flatness and Liquidity in Finite Outcome Space

Recall that in Chapter 3, considering a binary outcome space, we find an analytical relationship between deviation and market liquidity for the subsidized prediction markets we consider. When a non-binary finite outcome space is considered, the relationship between deviation and market liquidity may not hold.

For the deviation result found in Section 3.2 to hold, Lemma 2-ii is crucial. However, Lemma 2-ii, cannot be generalized for any non-binary outcome space. The main intuition provided by Lemma 2-ii is that a risk-averse agent's report is always bounded by the market's current estimate and her belief. This is not the case for a non-binary finite outcome space. For example, consider $\Omega = \{1, 2, 3\}$, $\mathbf{p} = (0.3, 0.5, 0.2)$, and $\mathbf{r}^{(0)} = (0.4, 0.1, 0.5)$. Also, let the agent's utility be $u(x) = 1 - e^{-2x}$, where she is participating in the MSR $\mathcal{X}$ in which $X(\mathbf{q}, i) = 2 \log \mathbf{q}_i$. Using a numerical optimization problem we have:

$$\mathbf{r}^{\mathcal{X}} = (0.405238, 0.148058, 0.446704)\,, \tag{4.1}$$

which indicates a violation of a generalized version of Lemma 2-ii for $\omega = 1$. Recently, in the work of Peysakhovich and Plagborg-Møller (2012), a similar finding has been noted. As explained by Peysakhovich and Plagborg-Møller (2012), when strictly proper scoring rules are considered, the classic results such as the work by Winkler and Murphy (1970) and Kadane and Winkler (1988) on binary state spaces do not generalize to cases where $N > 2$.

Fortunately, the deviation result found in Section 3.2, presents a different intuition that can be generalized to any finite outcome space. The deviation result in binary space, Corollary 1, also shows that given two MSRs, when a MSR function, say $\mathcal{Y}$, dominates another MSR function say $\mathcal{X}$, as defined in (2.6); a risk-averse agent's report is closer to the market's current estimate in $\mathcal{Y}$ compared to $\mathcal{X}$. In this chapter, we show that this results can be generalized to any finite outcome space. In particular, we show that for any given finite outcome space, and a given market estimate, when a MSR function dominates another MSR function, a risk-averse agent reports closer to the market's current estimate in the dominant MSR compared to the dominated MSR. We use the concept of flatness, as defined in Definition 4, to measure the closeness of an agent's report to the markets current estimate. Furthermore, we show that the liquidity result provided in Section 3.2 can also be generalized to any finite outcome space. In combination, the generalized results can help market organizers to set the liquidity parameters of a prediction market, in any finite outcome space, to control the maximum allowed flatness in the market. The rest of this chapter is organized as follows. In Section 4.2 we utilize an optimization approach to study the behavior of a risk-averse agent in a MSR prediction market. In Section 4.2, we show that for any finite outcome space, the report of a risk-averse agent is closer to the market's current estimate in a dominant MSR market relative to a dominated MSR market. In Section 4.3, we prove that for any finite outcome space, the cost-function prediction market corresponding to a dominant MSR provides higher liquidity compared to the cost-function prediction market corresponding to a dominated MSR.

4.1 Model Set-up

Consider a finite outcome space $\Omega = \{1, 2, \cdots, N\}$. Let ω represent the corresponding random variable where $\omega \in \Omega$. Similar to our analysis in Chapter 3, we only analyze the behavior of myopic agents one at a time, and therefore we use a similar notation of $\mathcal{X}(\mathbf{r}, \mathbf{r}^{(0)}, \omega)$ instead of $\mathcal{X}(\mathbf{r}^{(t)}, \mathbf{r}^{(t-1)}, \omega)$, where $\mathbf{r}$ is the agents report and $\mathbf{r}^{(0)}$ is the market's current estimate, at the time of making report. An agent with a personal belief of $\mathbf{p}$ on ω, is asked to submit her probability estimates denoted by $\mathbf{r}$. Once the outcome is realised, the prediction market rewards the agent according to a MSR function:

$$\mathcal{X}\left(\mathbf{r}, \mathbf{r}^{(0)}, \omega\right) = X\left(\mathbf{r}, \omega\right) - X\left(\mathbf{r}^{(0)}, \omega\right), \tag{4.2}$$

where the strictly proper score function $X(\cdot)$ has the following property:

$$X : \Delta_{N-1} \times \Omega \mapsto \mathbb{R} \text{ is smooth on } \Delta_{N-1} \text{ for any given } \omega, \tag{4.3}$$

We assume agents are myopic, expected utility maximizers, and risk-averse. Note that we do not address risk-seeking agents in this chapter. We assume agents have a monotonically increasing, twice differentiable, and a concave utility. We refer to the set of all such utilities as $\mathcal{U}^c \subset \mathbb{R}^{\mathbb{R}}$. Given a market's current estimate, an agent with a utility function of the form $u \in \mathcal{U}^c$ maximizes her expected utility on ω. Such an agent's expected utility from reporting any feasible report of $\mathbf{q}$ is as follows:

$$E_{\mathbf{p}}\left[u\left(\mathcal{X}(\mathbf{q}, \mathbf{r}^{(0)}, \omega)\right)\right] = \sum_{i\in\Omega} \mathbf{p}_i u\left(\mathcal{X}\left(\mathbf{q}, \mathbf{r}^{(0)}, i\right)\right) . \tag{4.4}$$

A risk neutral agent will report truthfully in any strictly proper MSR. Given a strictly proper scoring rule $X(\cdot)$, and its corresponding MSR $\mathcal{X}$, we formally define the expected utility maximizing report of $\mathbf{r}_u^{\mathcal{X}}\left(\mathbf{p}, \mathbf{r}^{(0)}\right)$ as:

$$\mathbf{r}_u^{\mathcal{X}}\left(\mathbf{p}, \mathbf{r}^{(0)}\right) \triangleq \arg\max_{\mathbf{q}\in[0,1]} E_{\mathbf{p}}\left[u\left(\mathcal{X}\left(\mathbf{q}, \mathbf{r}^{(0)}, \omega\right)\right)\right]. \tag{4.5}$$

We assume all agents are myopic and rational, hence all reports made by agents will be those that satisfy (4.5) and we thus refer to all reports adhering to (4.5) simply as reports.

4.2 Risk-Averse Agent's behavior in Finite Space

Following the new intuition of comparing agents' report to the market's current estimate, as oppose to the agent's belief, we propose an alternate way to compare two market scoring rules, we call this comparison flatness.

Definition 4. *(Flatness) Let $\mathcal{X}$ and $\mathcal{Y}$ be two MSRs. We say $\mathcal{Y}$ is flatter than $\mathcal{X}$ if for every agent, the following holds:*

$$\forall \mathbf{r}^{(0)}, \mathbf{p} \in \Delta_{N-1} : \left| \mathbf{r}_u^{\mathcal{Y}} \left(\mathbf{p}, \mathbf{r}^{(0)} \right) - \mathbf{r}^{(0)} \right|_1 < \left| \mathbf{r}_u^{\mathcal{X}} \left(\mathbf{p}, \mathbf{r}^{(0)} \right) - \mathbf{r}^{(0)} \right|_1, \tag{4.6}$$

in which $| \cdot |_1$ is the L_1 norm.

Flatness and deviation are similar concepts. Both flatness and deviation indicate a similar behavior in risk-averse agents' reporting. A more deviated market causes every risk-averse agent's report to be less indicative of their beliefs. Similarly, in a flatter market, a risk-averse agent's report is closer to the market's current estimate and consequently less indicative of her belief.

The main result of this section, Proposition 4, is fundamentally based on the findings in Lemma 3, and the use of the *total variance distance*. Lemma 3 is a generalization of Lemma 2-i that finds a relationship between a risk-averse agent's report, her belief, and the market's current estimate. Lemma 3 is similar to result of (Peysakhovich and Plagborg-Møller 2012, Lemma 1), in which they show a similar result considering strictly proper scoring rules. Our results in Lemma 3 is different from the result of Peysakhovich and Plagborg-Møller (2012), as we consider MSRs, and not scoring rules. Moreover, we require no further assumption on the underlying scoring rule function other than smoothness, while Peysakhovich and Plagborg-Møller (2012), require the underlying strictly proper score functions to be neutral, bounded above, extensively continuous and semi quasi-convex.

In Lemma 3, we show that an agent's report is a weighted distribution of the agent's belief and her marginal utility in each outcome. To prove Lemma 3, we solve for the first order condition of the problem in (4.5), to find the agent's report, which we show is unique and feasible. We show that for a risk-averse agent, the two distributions of: the agent's report, $\mathbf{r}_u^{\mathcal{X}}\left(\mathbf{p}, \mathbf{r}^{(0)}\right)$, and a weighted vector of the agent's belief and her marginal utility in each outcome, $\mathbf{p}^u \triangleq \left\{\mathbf{p}_i u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, 1\right)\right)\right\}_{i\in\Omega}$; are two vectors in the null space of the MSR's Jacobian matrix. By showing that the MSR's Jacobian matrix has a null space of dimension one, we show that the two distributions of, $\mathbf{r}_u^{\mathcal{X}}\left(\mathbf{p}, \mathbf{r}^{(0)}\right)$, and $\mathbf{p}^u$ are scalar factors of each other.

Lemma 3. *Let $\mathbf{r}^{\mathcal{X}} \triangleq \mathbf{r}_u^{\mathcal{X}}\left(\mathbf{p}, \mathbf{r}^{(0)}\right)$, be the expected utility-maximizing report, defined in (4.5), of an agent in the MSR $\mathcal{X}$, in which $\mathbf{p}$ is the agent's belief on ω and $\mathbf{r}^{(\mathbf{0})}$ is the market's current estimate. The following equations hold.*

$$\mathbf{r}_i^{\mathcal{X}} = \frac{\mathbf{p}_i u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right)\right)}{\sum_{j=1}^{N} \mathbf{p}_j u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)\right)} : \forall i \in \Omega. \tag{4.7}$$

Proof. Without loss of generality, let $\mathbf{p}$ be an interior point of Δ_{N-1}. Note that if $\mathbf{p}$ is in closure of Δ_{N-1}, thus there exist an outcome, say $j \in \Omega$, where $\mathbf{p}_j = 0$, and the claim can proven by induction on N, where Lemma 2-i is the base, and the remaining of this proof is the induction step. Since X, the underlying score function of $\mathcal{X}$ is smooth, $\mathcal{X}\left(\cdot, \mathbf{r}^{(0)}, \omega\right)$ is also smooth, thus $\mathbf{r}^{\mathcal{X}}$ is an interior point and it satisfies the first order conditions. That is:

$$\nabla\left(E_{\mathbf{p}}\left[u\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, \omega\right)\right)\right]\right) = 0.$$

By chain rule we get:

$$\forall j \in \Omega : \sum_{i=1}^{N} \mathbf{p}_i u' \left(\mathcal{X} \left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i \right) \right) \frac{\partial}{\partial \mathbf{q}_j} \mathcal{X} \left(\mathbf{q}, \mathbf{r}^{(0)}, i \right) \bigg|_{\mathbf{q}=\mathbf{r}_u^{\mathcal{X}}} = 0\cdot \tag{4.8}$$

By definition of MSR, (2.3), (4.8) can be simplified as

$$\forall j \in \Omega : \sum_{i=1}^{N} \mathbf{p}_i u' \left(\mathcal{X} \left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i \right) \right) \frac{\partial}{\partial \mathbf{q}_j} X \left(\mathbf{q}, i \right) \bigg|_{\mathbf{q}=\mathbf{r}_u^{\mathcal{X}}} = 0\cdot \tag{4.9}$$

Consider the matrix $J^{\mathcal{X}}(\mathbf{q})$ defined as

$$J^{\mathcal{X}}(\mathbf{q}) = \left\{ \frac{\partial}{\partial \mathbf{q}_j} X(\mathbf{q}, i) \right\}_{(i,j)\in\Omega^2},$$

thus (4.8) can be rewritten as

$$J^{\mathcal{X}}\left(\mathbf{r}^{\mathcal{X}}\right) \times \mathbf{p}^u = \mathbf{0} : \mathbf{p}^u = \left\{ \mathbf{p}_i u' \left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, 1\right)\right) \right\}_{i\in\Omega} \cdot \tag{4.10}$$

By definition of proper scoring rules, and the use of function $J^{\mathcal{X}}(\cdot)$ we also know that

$$J^{\mathcal{X}}\left(\mathbf{r}^{\mathcal{X}}\right)\mathbf{r}^{\mathcal{X}} = 0\cdot \tag{4.11}$$

By (4.9) and (4.11) we get

$$\text{nullity}\left(J^{\mathcal{X}}\left(\mathbf{r}^{\mathcal{X}}\right)\right) \geq 1.$$

We show that the nullity of the matrix $J^{\mathcal{X}}(\mathbf{q})$ is exactly one for all $\mathbf{q} \notin \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_N\}$.

We prove the nullity claim by contradiction. Assume to the contrary that

$$\text{nullity}\left(J^{\mathcal{X}}\left(\mathbf{q}\right)\right) > 1. \tag{4.12}$$

By row-rank theorem we get

$$\text{rank}\left(J^{\mathcal{X}}\left(\mathbf{q}\right)\right) \leq N - 2\cdot \tag{4.13}$$

Let $J_i^{\mathcal{X}}$ be the i^{th} row of the matrix $J^{\mathcal{X}}\left(\mathbf{q}\right)$. By (4.13), we know that a set of $N-1$ rows of the matrix $J^{\mathcal{X}}(\mathbf{q})$ are linearly dependent, thus we get

$$\exists \alpha \in \mathbb{R}^{N-1} - \{\mathbf{0}\} : \sum_{i=1}^{N-1} \alpha_i J_i^{\mathcal{X}} = \mathbf{0}\cdot \tag{4.14}$$

Let k be the index of the greatest entry of α, that is:

$$k = \arg\max_{i \in \Omega} \alpha_i. \tag{4.15}$$

Note that since $\alpha \neq \mathbf{0}$, thus $\alpha_k \neq 0$. By (4.14) we get:

$$\sum_{i=1}^{N-1} \frac{\alpha_i}{\alpha_k} J_i^{\mathcal{X}} = \mathbf{0} \tag{4.16}$$

By definition of strictly proper scoring rule, and the no arbitrage property, we get:

$$\forall i \in \Omega : \sum_{j=1}^{N} \frac{\partial}{\partial \mathbf{q}_j} X\left(\mathbf{q}, i\right)\Bigg|_{\mathbf{q}=\mathbf{r}^{\mathcal{X}}} = 0 \implies \sum_{i=1}^{N} J_i^{\mathcal{X}} = \mathbf{0}\cdot \tag{4.17}$$

(4.16) and the right-hand-side of (4.17) gives:

$$J_N^{\mathcal{X}} + \sum_{i=1}^{N-1} \left(1 - \frac{\alpha_i}{\alpha_k}\right) J_i^{\mathcal{X}} = \mathbf{0}. \tag{4.18}$$

By (4.18), for index k we get:

$$\left(1 - \tfrac{\alpha_1}{\alpha_k}\right) {J_1^{\mathcal{X}}}_k + \left(1 - \tfrac{\alpha_2}{\alpha_k}\right) {J_2^{\mathcal{X}}}_k + \cdots + \left(1 - \tfrac{\alpha_{k-1}}{\alpha_k}\right) J^{\mathcal{X}}_{k-1_k} + \left(1 - \tfrac{\alpha_{k+1}}{\alpha_k}\right) J^{\mathcal{X}}_{k+1_k} + \cdots + \left(1 - \tfrac{\alpha_1}{\alpha_{N-1}}\right) J^{\mathcal{X}}_{N-1_k} + {J_N^{\mathcal{X}}}_k = 0. \tag{4.19}$$

By the fact the change in score $X(\cdot, j)$ is non-increasing while $\mathbf{q}_k$ is increasing for all $j \neq k$, (2.5), $\forall i \in \{1, 2, \cdots, k-1, k+1, \cdots, N\}$, we get

$$\frac{\partial}{\partial \mathbf{q}_k} X(\mathbf{q}, i) = {J_i^{\mathcal{X}}}_k \leq 0. \tag{4.20}$$

By (4.20) and (4.19) we get:

$$\exists l \in \{1, 2, \cdots, k-1, k+1, \cdots, N-1\} \text{ in which } \left(1 - \frac{\alpha_l}{\alpha_k}\right) < 0, \tag{4.21}$$

otherwise, the left-hand-side of (4.19) is negative, which is not possible. Thus we have $1 < \frac{\alpha_l}{\alpha_k}$ or equivalently $\alpha_k < \alpha_l$, which is a contradiction with (4.15). Thus nullity $\left(J^{\mathcal{X}}\left(\mathbf{r}_u^{\mathcal{X}}\right)\right) = 1$, hence by (4.10) and (4.11) we get:

$$\exists a \in \mathbb{R} : \mathbf{r}_u^{\mathcal{X}} = a\mathbf{p}^u.$$

Since $\sum_{i \in \Omega} \mathbf{r}^{\mathcal{X}} = 1$, we get $a = \sum_{i \in \Omega} \mathbf{p}_i^u$, and the above equation can equivalently be written as follows:

$$\mathbf{r}_i^{\mathcal{X}} = \frac{\mathbf{p}_i u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right)\right)}{\sum_{j=1}^{N} \mathbf{p}_j u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)\right)} : \forall i \in \Omega.$$

Note that by definition $\mathbf{r}^{\mathcal{X}}$ is unique and feasible, and thus the proof is complete.

□

To prove the main result of this section, Proposition 4, and a preliminary result, Lemma 4, we use the concept of total variance distance. The total variance distance is a distance measure for probability distributions. Following Levin et al. (2009), the total variation distance between two probability distributions $\mathbf{x}, \mathbf{y} \in \Delta_{N-1}$ is defined as

$$|\mathbf{x} - \mathbf{y}|_{TV} = \arg\max_{A \subset \Omega} |\mathbf{x}(A) - \mathbf{y}(A)|, \tag{4.22}$$

in which $\mathbf{x}(A) = \sum_{i \in A} \mathbf{x}_i$. As defined, the total variance distance is determined over all possible subsets of Ω, which results in computational difficulty in calculating the distance between two probability distributions. Fortunately, there is a useful characterization of the total variation distance, that relates the total variance distance to a simple L_1 norm.

$$|\mathbf{x} - \mathbf{y}|_{TV} = \frac{1}{2} \sum_{i \in \Omega} |\mathbf{x}_i - \mathbf{y}_i|. \tag{4.23}$$

For a proof see Proposition 4.2 in (Levin et al. 2009). Moreover, the proof of Proposition 4.2 in (Levin et al. 2009) also shows the following.

$$|\mathbf{x} - \mathbf{y}|_{TV} = \sum_{j : \mathbf{x}_j \geq \mathbf{y}_j} \mathbf{x}_j - \mathbf{y}_j. \tag{4.24}$$

Also note that since the total variance distance is symmetric, that is, $|\mathbf{x}-\mathbf{y}|_{TV} = |\mathbf{y}-\mathbf{x}|_{TV}$, by 4.24 we also have:

$$|\mathbf{x}-\mathbf{y}|_{TV} = \sum_{j:\mathbf{x}_j \geq \mathbf{y}_j} \mathbf{x}_j - \mathbf{y}_j = \sum_{k:\mathbf{y}_k \geq \mathbf{x}_k} \mathbf{y}_k - \mathbf{x}_k. \tag{4.25}$$

To make our notations cleaner, we also use the following definition.

Definition 5. *Let $\mathbf{r}^{\mathcal{X}} \triangleq \mathbf{r}_u^{\mathcal{X}}\left(\mathbf{p}, \mathbf{r}^{(0)}\right)$, be the expected utility-maximizing report, defined in (4.5). The following two index sets $I_{\mathcal{X}}^{+}$, and $I_{\mathcal{X}}^{-}$ are called positive and negative index sets respectively.*

$$\begin{aligned} I_{\mathcal{X}}^{+} &= \{i : \mathbf{r}_i^{\mathcal{X}} > \mathbf{r}^{(0)}{}_i\} \\ I_{\mathcal{X}}^{-} &= \{i : \mathbf{r}_i^{\mathcal{X}} \leq \mathbf{r}^{(0)}{}_i\} \end{aligned} . \tag{4.26}$$

Lemma 4 shows that for given MSRs $\mathcal{Y}$ and $\mathcal{X}$, where $\mathcal{Y}$ dominates $\mathcal{X}$, the difference of the agent's report and the current market estimate is smaller in $\mathcal{Y}$ relative to $\mathcal{X}$, for those states in which the agent's report is greater than the market's current estimate. To prove Lemma 4, we consider the two possible cases when we compare $\mathbf{r}_i^{\mathcal{X}}$ and $\mathbf{r}_i^{\mathcal{Y}}$ for a given $i \in I_{\mathcal{Y}}^{+}$. We then use Lemma 3, to show that the claim's inequality holds in all cases.

Lemma 4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two MSRs that satisfies (2.6). Moreover let $\mathbf{r}^{\mathcal{X}} \triangleq \mathbf{r}^{\mathcal{X}}\left(\mathbf{p}, \mathbf{r}^{(0)}\right)$, and $\mathbf{r}^{\mathcal{Y}} \triangleq \mathbf{r}_{\mathcal{Y}}^{u}\left(\mathbf{p}, \mathbf{r}^{(0)}\right)$ be the expected utility-maximizing reports, defined in (4.5). The following holds:*

$$\sum_{i \in I_{\mathcal{Y}}^{+}} \mathbf{r}_i^{\mathcal{Y}} - \mathbf{r}^{(0)}{}_i < \sum_{j \in I_{\mathcal{X}}^{+}} \mathbf{r}_j^{\mathcal{X}} - \mathbf{r}^{(0)}{}_j. \tag{4.27}$$

Proof. We consider the following two possible cases and show that the claim in (4.27) holds in each case.

- *Case 1:* For all $i \in I^+_{\mathcal{Y}}$, $\mathbf{r}^{\mathcal{Y}}_i \leq \mathbf{r}^{\mathcal{X}}_i$.

  By the assumption in this case, and Definition 5, we get $I^+_{\mathcal{Y}} \subset I^+_{\mathcal{X}}$. Moreover, we have

$$\forall i \in I^+_{\mathcal{Y}} : \mathbf{r}^{(0)}{}_i < \mathbf{r}^{\mathcal{Y}}_i \leq \mathbf{r}^{\mathcal{X}}_i.$$

  Thus we get

$$\sum_{i \in I^+_{\mathcal{Y}}} \mathbf{r}^{\mathcal{Y}}_i - \mathbf{r}^{(0)}{}_i < \sum_{i \in I^+_{\mathcal{Y}}} \mathbf{r}^{\mathcal{X}}_i - \mathbf{r}^{(0)}{}_i.$$

  Note that the strict inequality holds as $\mathbf{r}^{\mathcal{X}} \neq \mathbf{r}^{\mathcal{Y}}$, hence there exist $k, l \in \Omega$, such that $\mathbf{r}^{\mathcal{X}}_k < \mathbf{r}^{\mathcal{Y}}_k$ and $\mathbf{r}^{\mathcal{Y}}_k > \mathbf{r}^{\mathcal{X}}_k$. Since $I^+_{\mathcal{Y}} \subset I^+_{\mathcal{X}}$, we get

$$\sum_{i \in I^+_{\mathcal{Y}}} \mathbf{r}^{\mathcal{Y}}_i - \mathbf{r}^{(0)}{}_i < \sum_{j \in I^+_{\mathcal{X}}} \mathbf{r}^{\mathcal{X}}_j - \mathbf{r}^{(0)}{}_j,$$

  which proves the claim in this case.

- *Case 2:* There exist $i \in I^+_{\mathcal{Y}}$ in which $\mathbf{r}^{\mathcal{X}}_i < \mathbf{r}^{\mathcal{Y}}_i$.

  By (2.5), and the fact that $\mathbf{r}^{\mathcal{X}}_i < \mathbf{r}^{\mathcal{Y}}_i$ we get

$$\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right) < \mathcal{X}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right). \tag{4.28}$$

  When we compare $\mathbf{r}^{(0)}{}_i$ to $\mathbf{r}^{\mathcal{X}}_i$ we have the following two possibilities (note that $\mathbf{r}^{(0)}{}_i$ cannot exceed $\mathbf{r}^{\mathcal{Y}}_i$, as $i \in I^+_{\mathcal{Y}}$).

(I) $\mathbf{r}^{(0)}{}_i \leq \mathbf{r}^{\mathcal{X}}_i < \mathbf{r}^{\mathcal{Y}}_i$. In this instance, by (2.6) we have

$$\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right) < \mathcal{Y}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right). \tag{4.29}$$

Moreover, by (4.29) and (2.5) we have

$$\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right) < \mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right). \tag{4.30}$$

(II) $\mathbf{r}^{\mathcal{X}}_i \leq \mathbf{r}^{(0)}{}_i < \mathbf{r}^{\mathcal{Y}}_i$. In this instance, by (2.5), and definition of MSRs we know that $\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right)$ is not positive and $\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right)$ is positive. Thus we have

$$\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right) < \mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right). \tag{4.31}$$

Given that for both possibilities of (I) and (II), we get

$$\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right) < \mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right); \tag{4.32}$$

the fact that $u(\cdot)$ is concave and non-decreasing, and thus $u'(\cdot)$ is non-increasing, we get

$$u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right)\right) \leq u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right)\right). \tag{4.33}$$

We now show that:

$$\forall j \in I^-_{\mathcal{Y}} : \mathbf{r}^{\mathcal{X}}_j < \mathbf{r}^{\mathcal{Y}}_j \leq \mathbf{r}^{(0)}{}_j. \tag{4.34}$$

To show (4.34), for a given $j \in I^-_{\mathcal{Y}}$, we compare $\mathbf{r}^{\mathcal{X}}_j$ to $\mathbf{r}^{\mathcal{Y}}_j$ in the following cases and we show that the only possibility is the one in (4.34).

($\mathcal{I}$) $\mathbf{r}_j^{\mathcal{Y}} \leq {\mathbf{r}^{(0)}}_j < \mathbf{r}_j^{\mathcal{X}}$.

In this instance, by (2.5), and definition of MSRs we know that $\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)$ is positive and $\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j\right)$ is non-positive. Moreover, since $u(\cdot)$ is concave and non-decreasing, $u'(\cdot)$ is non-increasing and we have

$$u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)\right) \leq u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j\right)\right). \tag{4.35}$$

On the other hand, by Lemma 3 we have

$$\frac{\mathbf{r}_i^{\mathcal{X}}}{\mathbf{r}_j^{\mathcal{X}}} \frac{u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)\right)}{u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right)\right)} = \frac{\mathbf{p}_i}{\mathbf{p}_j} = \frac{\mathbf{r}_i^{\mathcal{Y}}}{\mathbf{r}_j^{\mathcal{Y}}} \frac{u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j\right)\right)}{u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right)\right)}. \tag{4.36}$$

By (4.36), and the fact that $\mathbf{r}_i^{\mathcal{X}} < \mathbf{r}_i^{\mathcal{Y}}$, and $\mathbf{r}_j^{\mathcal{Y}} < \mathbf{r}_j^{\mathcal{X}}$, we get

$$\frac{u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)\right)}{u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right)\right)} > \frac{u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j\right)\right)}{u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right)\right)}. \tag{4.37}$$

Since (4.33) holds, (4.37) implies

$$u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)\right) > u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j\right)\right), \tag{4.38}$$

which is a contradiction with (4.35). Thus this case is not possible.

($\mathcal{II}$) $\mathbf{r}_j^{\mathcal{Y}} < \mathbf{r}_j^{\mathcal{X}} \leq {\mathbf{r}^{(0)}}_j$.

In this instance, by (2.5), and (2.6) we get

$$\mathcal{Y}(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j) \leq \mathcal{X}(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j) < \mathcal{X}(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j). \tag{4.39}$$

Again, since $u(\cdot)$ is concave and non-decreasing, $u'(\cdot)$ is non-increasing, thus (4.39) implies

$$u'\left(\mathcal{X}(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j)\right) \leq u'\left(\mathcal{Y}(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j)\right). \tag{4.40}$$

On the other hand, by Lemma 3 we have

$$\frac{\mathbf{r}_i^{\mathcal{X}}}{\mathbf{r}_j^{\mathcal{X}}} \frac{u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)\right)}{u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right)\right)} = \frac{\mathbf{p}_i}{\mathbf{p}_j} = \frac{\mathbf{r}_i^{\mathcal{Y}}}{\mathbf{r}_j^{\mathcal{Y}}} \frac{u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j\right)\right)}{u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right)\right)}. \tag{4.41}$$

By (4.41), and the fact that $\mathbf{r}_i^{\mathcal{X}} < \mathbf{r}_i^{\mathcal{Y}}$, and $\mathbf{r}_j^{\mathcal{Y}} < \mathbf{r}_j^{\mathcal{X}}$, we get

$$\frac{u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)\right)}{u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, i\right)\right)} > \frac{u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j\right)\right)}{u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, i\right)\right)}. \tag{4.42}$$

Since (4.33) holds, (4.42) implies

$$u'\left(\mathcal{X}\left(\mathbf{r}^{\mathcal{X}}, \mathbf{r}^{(0)}, j\right)\right) > u'\left(\mathcal{Y}\left(\mathbf{r}^{\mathcal{Y}}, \mathbf{r}^{(0)}, j\right)\right), \tag{4.43}$$

which is a contradiction with (4.40). Thus this case is also not possible.

By considering the two cases ($\mathcal{I}$), and ($\mathcal{II}$), we conclude that (4.34), holds.

(4.34) also shows that $I_{\mathcal{Y}}^{-} \subset I_{\mathcal{X}}^{-}$. Hence, we have

$$\sum_{k \in I_{\mathcal{Y}}^{-}} \mathbf{r}^{(0)}{}_{k} - \mathbf{r}_{k}^{\mathcal{Y}} < \sum_{i \in I_{\mathcal{Y}}^{-}} \mathbf{r}^{(0)}{}_{k} - \mathbf{r}_{k}^{\mathcal{X}}.$$

Note that the strict inequality holds as $\mathbf{r}^{\mathcal{X}} \neq \mathbf{r}^{\mathcal{Y}}$, hence there exist $s, t \in \Omega$, such that $\mathbf{r}_{s}^{\mathcal{X}} < \mathbf{r}_{t}^{\mathcal{Y}}$ and $\mathbf{r}_{s}^{\mathcal{Y}} > \mathbf{r}_{t}^{\mathcal{X}}$. Since $I_{\mathcal{Y}}^{-} \subset I_{\mathcal{X}}^{-}$, we get

$$\sum_{k \in I_{\mathcal{Y}}^{-}} \mathbf{r}^{(0)}{}_{k} - \mathbf{r}_{k}^{\mathcal{Y}} < \sum_{l \in I_{\mathcal{X}}^{-}} \mathbf{r}^{(0)}{}_{l} - \mathbf{r}_{l}^{\mathcal{X}}. \tag{4.44}$$

By (4.25) we know that

$$\sum_{k \in I_{\mathcal{Y}}^{-}} \mathbf{r}^{(0)}{}_{k} - \mathbf{r}_{k}^{\mathcal{Y}} = \sum_{\mathfrak{k} \in I_{\mathcal{Y}}^{+}} \mathbf{r}_{\mathfrak{k}}^{\mathcal{Y}} - \mathbf{r}^{(0)}{}_{\mathfrak{k}} \text{ and } \sum_{l \in I_{\mathcal{X}}^{-}} \mathbf{r}^{(0)}{}_{l} - \mathbf{r}_{l}^{\mathcal{X}} = \sum_{\mathfrak{l} \in I_{\mathcal{X}}^{+}} \mathbf{r}_{\mathfrak{l}}^{\mathcal{X}} - \mathbf{r}^{(0)}{}_{\mathfrak{l}}. \tag{4.45}$$

Combining (4.44), and (4.45), we get

$$\sum_{k \in I_{\mathcal{Y}}^{+}} \mathbf{r}^{(0)}{}_{k} - \mathbf{r}_{k}^{\mathcal{Y}} < \sum_{l \in I_{\mathcal{X}}^{+}} \mathbf{r}^{(0)}{}_{l} - \mathbf{r}_{l}^{\mathcal{X}}, \tag{4.46}$$

which proves the claim in Case 2 as well.

□

Lemma 4 compares the difference of an agent's report and the market's current estimate in those outcomes where the agent receives a positive score. Lemma 4 then shows that when $\mathcal{Y}$ dominates $\mathcal{X}$, an agent's report is closer to the market's current estimate in $\mathcal{Y}$ than the same agent's report in $\mathcal{X}$, across all those outcomes where the agent receives a positive score. Using Lemma 4 and the characterization of the total variance distance (4.24), we can prove Proposition 4 which states that the agent's report is closer to the market's current estimate in a dominant MSR compared to a dominated MSR.

Proposition 4. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two MSRs that satisfies (2.6). $\mathcal{Y}$ is flatter than $\mathcal{X}$.*

Proof. For a given u, $\mathbf{p}$, and $\mathbf{r}^{(0)}$, let $\mathbf{r}^{\mathcal{X}} \triangleq \mathbf{r}^{\mathcal{X}}\left(\mathbf{p}, \mathbf{r}^{(0)}\right)$, and $\mathbf{r}^{\mathcal{Y}} \triangleq \mathbf{r}^{u}_{\mathcal{Y}}\left(\mathbf{p}, \mathbf{r}^{(0)}\right)$ be the expected utility-maximizing report, defined in (4.5). By Lemma 4, we have

$$\sum_{i \in I^{+}_{\mathcal{Y}}} \mathbf{r}^{\mathcal{Y}}_{i} - \mathbf{r}^{(0)}{}_{i} < \sum_{j \in I^{+}_{\mathcal{X}}} \mathbf{r}^{\mathcal{X}}_{j} - \mathbf{r}^{(0)}{}_{j}. \tag{4.47}$$

By (4.24), and (4.47) we get

$$\sum_{i \in \Omega} \left| \mathbf{r}^{\mathcal{Y}}_{i} - \mathbf{r}^{(0)}{}_{i} \right|_{TV} < \sum_{i \in \Omega} \left| \mathbf{r}^{\mathcal{X}}_{i} - \mathbf{r}^{(0)}{}_{j} \right|_{TV}. \tag{4.48}$$

The above equation and (4.23) imply

$$\sum_{i \in \Omega} \left| \mathbf{r}^{\mathcal{Y}}_{i} - \mathbf{r}^{(0)}{}_{i} \right| < \sum_{i \in \Omega} \left| \mathbf{r}^{\mathcal{X}}_{i} - \mathbf{r}^{(0)}{}_{j} \right|, \tag{4.49}$$

which shows that $\mathcal{Y}$ is flatter than $\mathcal{X}$.

□

4.3 Liquidity in General Cost Function Markets

In this section we show that similar to the liquidity-deviation relationship in binary outcome space, the amount of liquidity provided by a valid cost-function prediction market is closely related to the concept of flatness defined in Definition 4.

In particular, Proposition 5 shows that when $\mathcal{Y}$ is flatter than $\mathcal{X}$, then $\mathcal{C}^{\mathcal{Y}}$ provides more liquidity compared to $\mathcal{C}^{\mathcal{X}}$. The proof of Proposition 5 is similar to the proof of Proposition 3. To prove Proposition 5, we first find a relation between share quantities and the price function by solving the first order conditions of the optimization problem in (2.7). The unique and feasible solution of the first order conditions presents the familiar *price-share* relation, similar to Section 3.3.2, in any finite outcome space. The price-share relationship allows us to implicitly calculate the derivative of the price w.r.t. the quantity of outstanding shares. We show that for a cost-function $C^{\mathcal{X}}$, the depth of the market for security i is equal to

$$\frac{\partial}{\partial \mathbf{q}_i} \left(X\left(\mathbf{q}, i\right) - X\left(\mathbf{q}, N\right) \right) \Big|_{\mathbf{q}=Pr^{\mathcal{X}}(\mathbf{s})}$$

Using the above equation, we relate the depth of a cost-function market maker to its underlying proper scoring rule. Using Proposition 5, and the simple definition of the first derivative, we can conclude the cost-function prediction market corresponding to the flatter MSR, provides more market depth than the cost function market maker that corresponds to the MSR that is not flatter.

Proposition 5. *Let $C^{\mathcal{X}}$ and $C^{\mathcal{Y}}$ be the cost function market makers of the two MSRs $\mathcal{X}$ and $\mathcal{Y}$ respectively. $C^{\mathcal{Y}}$ has more market depth relative to $C^{\mathcal{X}}$, only if $\mathcal{Y}$ and $\mathcal{X}$ are two MSRs that satisfy* (2.6).

Proof. We start with a change of a variable from $\mathbf{q} \in \Delta_{N-1}$, to $\mathbf{q} \in \mathbb{R}_{+}^{N-1}$, and we define $\mathbf{q}_N = \left(1 - \sum_{j\in\Omega-\{N\}} \mathbf{q}_j\right)$. Note that this change of variable preserves the optimal point of the optimization problem in (2.7). By Equivalence relation (2.7) we have

$$C^{\mathcal{X}}(\mathbf{s}) = \max_{\mathbf{q}\in\mathbb{R}_+^{N-1}} \mathbf{s}^T\mathbf{q} - \sum_{\omega\in\Omega} q_i X(\mathbf{q},\omega). \tag{4.50}$$

By definition of the price function (2.8), and the first order condition of (4.50); we have

$$\forall k \in \Omega - \{N\} : \tfrac{\partial}{\partial \mathbf{q}_k} \left(\textstyle\sum_{i\in\Omega-\{N\}} s_i \mathbf{q}_i + s_N \left(1 - \sum_{j\in\Omega-\{N\}} \mathbf{q}_j\right) - \left(\sum_{i\in\Omega-\{N\}} \mathbf{q}_i X\left(\mathbf{q}, i\right) + \left(1 - \sum_{j\in\Omega-\{N\}} \mathbf{q}_j\right) X(\mathbf{q}, N)\right)\right)\Big|_{\mathbf{q}=Pr^{\mathcal{X}}(\mathbf{s})} = 0. \tag{4.51}$$

Expanding (4.51) gives

$$\forall k \in \Omega - \{N\} : s_k - s_N - \left(X\left(\mathbf{q}, k\right) - X\left(\mathbf{q}, N\right)\right) - \left(\textstyle\sum_{i\in\Omega-\{N\}} \mathbf{q}_i \tfrac{\partial}{\partial \mathbf{q}_k} X\left(\mathbf{q}, i\right) + \left(1 - \sum_{j\in\Omega-\{N\}} \mathbf{q}_j\right) \tfrac{\partial}{\partial \mathbf{q}_k} X(\mathbf{q}, N)\right)\Big|_{\mathbf{q}=Pr^{\mathcal{X}}(\mathbf{s})} = 0. \tag{4.52}$$

Since $X(\cdot)$ is a strictly proper scoring rule, for given $\mathbf{r} \in \mathbb{R}_+^N$, in which $\mathbf{r}_N = \left(1 - \sum_{j\in\Omega-\{N\}} \mathbf{r}_j\right)$ we have:

$$\{\mathbf{r}\} = \arg\max_{\mathbf{q}\in\mathbb{R}_+^N} E_{\mathbf{r}}\left[X(\mathbf{q},\omega)\right].$$

As $X(\cdot)$ is also smooth, $\mathbf{r}$ is a stationary point of $E_{\mathbf{r}}\left[X(\mathbf{q},\omega)\right]$. That is:

$$\nabla \left(E_{\mathbf{r}}\left[X(\mathbf{q},\omega)\right]\right)\Big|_{\mathbf{q}=\mathbf{r}} = 0 \quad \Longrightarrow \forall i \in \Omega - \{N\}\ \frac{\partial}{\partial \mathbf{q}_i} E_{\mathbf{r}}\left[X(\mathbf{q},\omega)\right]\Big|_{\mathbf{q}=\mathbf{r}} = 0. \tag{4.53}$$

The right-hand-side of the above equation can be written as follows:

$$\forall i \in \Omega - \{N\} : \sum_{j\in\Omega-\{N\}} \mathbf{q}_j \frac{\partial}{\partial \mathbf{q}_i} X\left(\mathbf{q}, j\right) + \left(\left(1 - \sum_{k\in\Omega-\{N\}} \mathbf{q}_j\right) \frac{\partial}{\partial \mathbf{q}_i} X(\mathbf{q}, N)\right)\Bigg|_{\mathbf{q}=\mathbf{r}} = 0.$$

Using the above equation for $\mathbf{r}^{\mathcal{X}}$ in place of $\mathbf{r}$, (4.52) reduces to

$$s_k - s_N - \left(X\left(Pr^{\mathcal{X}}(\mathbf{s}), k\right) - X\left(Pr^{\mathcal{X}}(\mathbf{s}), N\right)\right) = 0. \tag{4.54}$$

From (4.54), we can implicitly derive the rate of change in $Pr_k^{\mathcal{X}}$ w.r.t. s_k. By implicit differentiation we get

$$\frac{\partial}{\partial s_k} Pr_k^{\mathcal{X}}(\mathbf{s}) = \frac{1}{\frac{\partial}{\partial \mathbf{q}_k}\left(X\left(\mathbf{q}, k\right) - X\left(\mathbf{q}, N\right)\right)\Big|_{\mathbf{q}=Pr^{\mathcal{X}}(\mathbf{s})}}. \tag{4.55}$$

Thus by definition of market depth, (2.11), we get

$$\rho_k^{\mathcal{X}}\left(\mathbf{s}\right) = \frac{\partial}{\partial \mathbf{q}_k}\left(X\left(\mathbf{q}, k\right) - X\left(\mathbf{q}, N\right)\right)\Bigg|_{\mathbf{q}=Pr^{\mathcal{X}}(\mathbf{s})} \tag{4.56}$$

Carrying our the procedure above, after replacing $C^{\mathcal{Y}}$ for $C^{\mathcal{X}}$ we find

$$\rho_k^{\mathcal{Y}}\left(\mathbf{s}\right) = \frac{\partial}{\partial \mathbf{q}_k}\left(Y\left(\mathbf{q}, k\right) - Y\left(\mathbf{q}, N\right)\right)\Bigg|_{\mathbf{q}=Pr^{\mathcal{Y}}(\mathbf{s})}. \tag{4.57}$$

Thus by Definition 2, $C^{\mathcal{Y}}$ has more market depth relative to $C^{\mathcal{X}}$, only if

$$\forall \mathbf{a} \in \Delta_{N-1} : \tfrac{\partial}{\partial \mathbf{q}_k}\left(X\left(\mathbf{q}, k\right) - X\left(\mathbf{q}, N\right)\right)\Big|_{\mathbf{q}=\mathbf{a}} \leq \tfrac{\partial}{\partial \mathbf{q}_k}\left(Y\left(\mathbf{q}, k\right) - Y\left(\mathbf{q}, N\right)\right)\Big|_{\mathbf{q}=\mathbf{a}}. \tag{4.58}$$

Therefore, to show that $C^{\mathcal{Y}}$ has more market depth relative to $C^{\mathcal{X}}$ it is enough to show that (4.58) is satisfied.

By assumption, $\mathcal{X}$ and $\mathcal{Y}$, satisfy the followings

$$\begin{cases} \mathcal{Y}(\mathbf{q},\mathbf{a},k) \leq \mathcal{X}(\mathbf{q},\mathbf{a},k) \leq \mathcal{X}(\mathbf{q},\mathbf{a},N) \leq \mathcal{Y}(\mathbf{q},\mathbf{a},N), & \mathbf{q}_k \leq \mathbf{a}_k \\ \mathcal{Y}(\mathbf{q},\mathbf{a},N) < \mathcal{X}(\mathbf{q},\mathbf{a},N) < \mathcal{X}(\mathbf{q},\mathbf{a},k) < \mathcal{Y}(\mathbf{q},\mathbf{a},k), & \mathbf{q}_k > \mathbf{a}_k \end{cases}. \tag{4.59}$$

Equation (4.59) gives

$$\begin{cases} \mathcal{X}(\mathbf{q},\mathbf{a},N) - \mathcal{X}(\mathbf{q},\mathbf{a},k) \leq \mathcal{Y}(\mathbf{q},\mathbf{a},N) - \mathcal{Y}(\mathbf{q},\mathbf{a},k) & : \mathbf{q}_k \leq \mathbf{a}_k \\ \mathcal{X}(\mathbf{q},\mathbf{a},k) - \mathcal{X}(\mathbf{q},\mathbf{a},N) < \mathcal{Y}(\mathbf{q},\mathbf{a},k) - \mathcal{Y}(\mathbf{q},\mathbf{a},N) & : \mathbf{q}_k > \mathbf{a}_k \end{cases}. \tag{4.60}$$

Thus by definition of a MSR (4.2), (4.60) can be expanded to

$$\begin{cases} 0 \leq X(\mathbf{q},N) - X(\mathbf{a},N) - (X(\mathbf{q},k) - X(\mathbf{a},k)) \leq Y(\mathbf{q},N) - Y(\mathbf{a},N) - (Y(\mathbf{q},k) - Y(\mathbf{a},k)) & : \mathbf{q}_k \leq \mathbf{a}_k \\ 0 \leq X(\mathbf{q},k) - X(\mathbf{a},k) - (X(\mathbf{q},N) - X(\mathbf{a},N)) \leq Y(\mathbf{q},k) - Y(\mathbf{a},1) - (Y(\mathbf{q},N) - Y(\mathbf{a},N)) & : \mathbf{q}_k > \mathbf{a}_k \end{cases}. \tag{4.61}$$

Rearranging (4.61) gives

$$\begin{cases} \frac{Y(\mathbf{q},k) - Y(\mathbf{q},N) - \big(Y(\mathbf{a},k) - Y(\mathbf{a},N)\big)}{\mathbf{q}_k - \mathbf{a}_k} \leq \frac{X(\mathbf{q},k) - X(\mathbf{q},N) - \big(X(\mathbf{a},k) - X(\mathbf{a},N)\big)}{\mathbf{q}_k - \mathbf{a}_k} & : \mathbf{q}_k \leq \mathbf{a}_k \\ \\ \frac{X(\mathbf{q},k) - X(\mathbf{q},N) - \big(X(\mathbf{a},k) - X(\mathbf{a},N)\big)}{\mathbf{q}_k - \mathbf{a}_k} \leq \frac{Y(\mathbf{q},k) - Y(\mathbf{q},N) - \big(Y(\mathbf{a},k) - Y(\mathbf{a},N)\big)}{\mathbf{q}_k - \mathbf{a}_k} & : \mathbf{q}_k > \mathbf{a}_k \end{cases}. \tag{4.62}$$

By (4.62) and the limit definition of the first derivative we get

$$\frac{\partial}{\partial \mathbf{q}_k} \left(X(\mathbf{q},k) - X(\mathbf{q},N)\right)\Big|_{\mathbf{q}=\mathbf{a}} \leq \frac{\partial}{\partial \mathbf{q}_k} \left(Y(\mathbf{q},k) - Y(\mathbf{q},N)\right)\Big|_{\mathbf{q}=\mathbf{a}}. \tag{4.63}$$

Note that the above equation holds since both score functions $X(\cdot)$ and $Y(\cdot)$ are differentiable on the interior of Δ_{N-1} and continuous on Δ_{N-1}. Equation 4.63 shows that (4.58) is satisfied and thus the proof is complete. □

When Proposition 5, and Proposition 4 are combined, we find the coveted relationship between flatness, as defined in Definition 4, and market liquidity, Definition 2. Proposition 4 show that the agent's report is closer to the market's current estimate in the MSR $\mathcal{Y}$ compared to a MSR $\mathcal{X}$, when $\mathcal{Y}$ dominates $\mathcal{X}$. This is a negative situation that a market maker interested in eliciting agent's belief would like to avoid. In other words, as flatness increases, agents' report can get arbitrary close to the market's current estimate. As such, a market organizer may not be able to interpret any reports. The result points toward selecting $\mathcal{X}$ as oppose to $\mathcal{Y}$. On the other hand, Proposition 5 shows that when $\mathcal{Y}$ dominates $\mathcal{X}$, a market maker can inject a higher amount of liquidity by choosing the cost-function $\mathcal{C}^{\mathcal{Y}}$ instead of $\mathcal{C}^{\mathcal{X}}$. A market maker prefers a more liquid market to a less liquid market. The result of Proposition 5, and Proposition 4 present a tension between two desired properties of higher liquidity and lower flatness. Similar to the result of Section 3.3.2, given a set of MSRs $\mathfrak{X}$, that may or may not belong to the same family, and the set of their corresponding cost-functions $\mathfrak{C}^{\mathfrak{X}}$, we can choose a MSR $\mathcal{X}^{\star}$ with a maximum flatness of $\bar{F}$, in which $\mathcal{C}^{\mathcal{X}^{\star}}$ has more market depth relative to all other cost-functions in $\mathfrak{C}^{\mathfrak{X}}$.

4.4 Conclusion

In this chapter, we characterize the behavior of risk-averse agents in prediction markets using MSRs for non-binary state spaces. We show that similar to MSRs with a binary state space, myopic, utility maximizing agents do not report their exact beliefs, unless their beliefs are identical to the current market estimate. We introduced the concepts of flatness to measure the difference between an agent's reported belief and the market's current estimate. Our primary finding is that we can compare the flatness of MSRs by comparing their corresponding reward functions provided to agents. We showed that a MSR with a higher reward and a higher penalty is flatter than a MSR with lower reward and a lower penalty. We also show that flatter MSRs provide higher liquidity as well. We show that a market organizer can take advantage of the relationship between flatness and liquidity to find a better market that provides the maximum amount of liquidity while maintaining a desirable flatness.

Chapter 5

Subsidized Versus Unsubsidized Prediction Markets

The prediction market literature states that in thin markets MSRs aggregate the information quicker than CDAs. Researchers studied the accuracy of CDAs and MSRs extensively (Hanson 2003, Jian and Sami 2012, Wolfers and Zitzewitz 2004, Hanson 2012, Chen et al. 2010, Milgrom and Stokey 1982, Pennock 2004). However, to the best of our knowledge, there has been few empirical studies comparing the information flow across CDAs and MSRs aside from arbitrage opportunities (Kildal et al. 2012). Given the fact that CDAs, USPMs in general, are costless, and potentially profitable, and MSRs, SPMs in general, are costly and need an initial endowment to run, we would like to address the following question. What is the number of trades needed for a MSR to aggregate information quicker than a CDA?

Currently, we are only aware of one study that considers the differences across the two types of markets, but that study considers arbitrage opportunities that exist for only one market with multiple outcomes (Kildal et al. 2012). In their paper the authors consider how much profit can be made by an agent that buys in a CDA market and sells in a MSR market and vise versa. They show that arbitrage exists on average on an hour over hour time scale. However, it is not clear if the arbitrage opportunity exists because the CDA market price trails the MSR market price or vise versa, something we address in our study.

Perhaps the closest work to our study is the research by Jian and Sami (2012). In their work, Jian and Sami (2012) run experiments to investigate the effect of the prediction market mechanism on information aggregation by utilizing an experiment on human subjects. However, their choice of prediction market mechanisms is between two SPMs. The current understandings of subsidy in prediction market is: CDAs are reliable when a large number of trades is expected, and MSRs are utilized when there are concerns over issues such as thin markets and no trade situations. See Figure 5.1 for an illustration of SPM and USPM comparison.

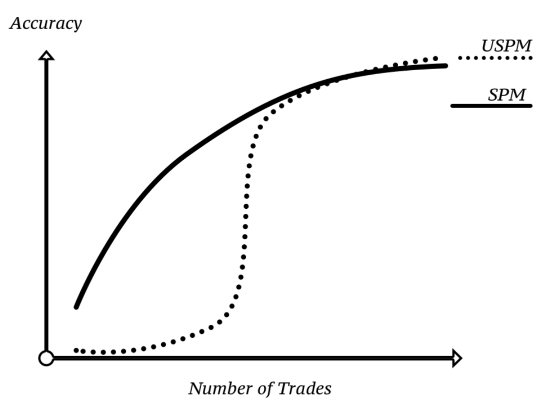


Figure 5.1: Comparison between SPMs and USPMs (Adapted from Hanson (2003)).

We use prediction market data gathered from a series of CDA and MSR markets to characterize the switching point before which a MSR market price leads a CDA market price, and in turn you may use the MSR market price to predict the CDA market price, and after which the CDA market may be used to predict the MSR market. We utilize time-series analysis to study the information flow between market prices and determine the leading and the following markets. In this chapter, we say market X aggregates information faster, or quicker, or with more speed than market Y, when market X's price leads market Y's price, that is market X's price causes market Y's price as explained in Section 5.3.

Some of the contributions of this chapter are as follows. This chapter provides much-needed insight for prediction market operators when determining if they should run a subsidized or unsubsidized prediction market. The current literature on prediction markets suggests using MSRs, SPMs in general, when the number of trades is small. Our empirical study supports this idea. We show that when the number of trades is below a certain threshold, a CDA market, may not aggregate the information quicker than a SPM market. Moreover, we show that when CDA and MSR markets aggregate market informations simultaneously, MSRs require a lower number of trades compared to CDA markets. Our study can help a growing number of managers and practitioners, that are using prediction markets, choose the appropriate mechanism based on their needs and expected trade volume.

The remainder of this chapter is organized as follows. Section 5.1 details our hypotheses. In Section 5.2 we briefly describe the data that we use in our study. We present our time series model in Section 5.3. We then discuss the causality results and its implications in Section 5.4.

5.1 Hypotheses

In order to extend the literature further, our first hypothesis is:

HYPOTHESIS 1. There exists a threshold in the number of trades, say κ, such that a CDA market aggregates information faster after κ trades than a MSR market after the same number of trades.

Hypothesis 1 only considers the cases in which a CDA market aggregates information quicker than a MSR, or vice versa. However, when both markets capture the market consensus with the same speed, it is interesting to investigate which market requires a lower number of trades to do so. This is in particular interesting in application areas in which the market organizer cannot expect a large number of trades. The current prediction market literature states that when the number of trades is high both MSR and CDA aggregate information quickly. However, to our knowledge, there has been no studies on the values of such large number of trades and if it differs across different market mechanisms. On the other hand, there has been much evidence that indicates that MSR markets encourage *rational* trades and there are fewer irrational trades in MSRs compared to CDAs due to the presence of a market maker (Hanson 2003). As such to extend the literature further our second hypothesis is:

HYPOTHESIS 2. When MSR and CDA markets aggregate information with the same speed, a MSR market requires fewer number of trades, compared to its corresponding CDA market, to do so.

We use the prices of two prediction markets: Intrade, a CDA market, and iPredict, a MSR market. Intrade and iPredict run prediction markets on a series of political, economic, financial, and scientific events. Intrade and iPredict ran some markets, we consider 129 as some markets had fewer than 10 trades or no trades at all, on the same events. For two pairs of markets on the same event, we use time-series analysis on the pair to determine if information is discovered faster in one relative to the other. We use Granger-causality analysis to compare the information aggregation within each pair of markets. The idea that prediction markets, or financial market, can aggregate traders' information is based on the *Efficient Market Hypothesis* (Malkiel and Fama 1970). In particular, semi-strong efficient market hypothesis (Malkiel and Fama 1970) states that stock market prices quickly adjust to new information available to the public, in such a way that trading can earn no excess return on that information. It is often suggested that for the semi-strong efficient market hypothesis to stand, the alteration of stock prices to the new information must happen very quickly (Malkiel and Fama 1970). The application of semi-strong efficient market hypothesis in prediction markets states that any prediction market price can quickly reflect all of the information currently available to market participants, and newly available information causes the market price to change immediately (Wolfers and Zitzewitz 2004, Luckner et al. 2011, Wolfers and Zitzewitz 2006, Bell 2008). For example, without loss of generality if market A's price, say an Intrade security, changes to newly available information quicker than market B's price, say an iPredict security; market A's price can then be used to predict market B's price. With this analytical approach, for each of 129 securities, that are shared between iPredict and Intrade, we test if iPredict causes Intrade, Intrade causes iPredict, they both cause each other, or there is no causal relationship.

5.2 Data

We use the market prices of the two markets: Intrade (Intrade 2017) and iPredict (iPredict 2017). Intrade was an USPM that ran a wide variety of prediction markets on sport, political, financial, economic and social events from 2007 to 2013. Intrade is considered a successful prediction market; provided with approximately $13.5 million in venture capital (Intrade 2017). Intrade prediction markets' raw data is publicly available for researchers (Data 2017). iPredict, on the other hand, is a SPM founded jointly by the New Zealand Institute for the Study of Competition and Regulation and Victoria University of Wellington. The iPredict data was acquired through personal communication with the site operators. During the period of January, 2010 to March, 2013, iPredict and Intrade ran multiple markets on the same events. We find the intersecting markets and normalize their corresponding prices. The normalization consists of converting transaction times to a common time-zone and converting market prices to the same scale, iPredict used values between $0.00 to $1.00, and Intrade used values between 0 to 100 cents. After the normalization, a combined time series is created for each intersecting market. The combined time series is an irregular, with different time increments, binary variable observation. We then create a more detailed series by incrementing the combined irregular series on 5 minutes intervals. Finally, we consider the most recent observations with up to 8760 observations (one month). We consider the most recent market prices as it can be shown that squared errors of prediction market prices relative to the true outcome decrease with time (Arrow et al.

2008). For a given pair of intersecting markets, we refer to the Intrade and iPredict regular series as INT and IPR respectively. An example of a combined time series for the security: *"Mitt Romney to win the 2012 South Carolina Primary."* is illustrated in Figure 5.2.

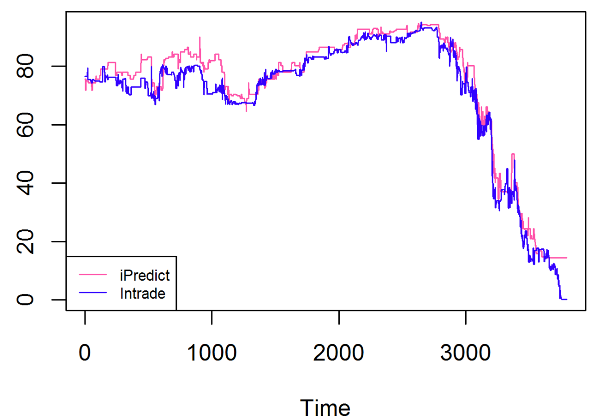


Figure 5.2: The combined normalized time series for the security of: *Mitt Romney to win the 2012 South Carolina Primary.*

5.3 Methodology

We use multiple bivariate techniques to investigate the relationship between INT and IPR pairs. In order to conduct our analysis, we follow the outline below propsed by Lütkepohl (2005).

1. Test each of the time-series INT and IPR to determine their order of integration using an Augmented Dicky Fuller (ADF) test.

2. Set up a VAR model in the levels of the data. To determine the appropriate maximum lag length for the variables in the VAR, say K, we use two information criteria (AIC and BIC (Lütkepohl 2005)) for selecting the best parameterization. In the case where different information criteria suggest different values of K, a likelihood ratio test is utilized to choose the appropriate lag value.

3. Test to ensure the validity of the VAR model. In particular, we utilize a Breusch-Godfrey test (Breusch 1978, Lütkepohl 2005, Section 4.4.4) to ensure that there is no serial correlation in the residuals. In the presence of any serial correlation, the value of K is increased until all autocorrelations are removed.

4. In the case where INT and IPR have the same order of integration, at Step 1, we utilize a Johansen test, Section 5.3.2, to see if the two series are cointegrated.

5. Test for Granger causality via a Wald test. We use the approach introduced by Toda and Yamamoto (1995) to avoid adjusting the Wald test in cases where at least one of the series is not stationary, Step 1.

To be able to draw a reliable conclusion from our statistical tests, following the steps above are crucial. Lütkepohl (2005) provides a complete technical explanation regarding the importance of each step. In summary, before we are able to compare two models in Step 5, we are required to fit the best model to our data, Step 2; and diagnose the model to ensure proper statistical inference, Step 1, 3, and 4. Following the above 5 steps, in Section 5.3.1, we begin our analysis by testing the order of integration for each series in a pair of INT and IPR time series. Section 5.3.2, details the Johansen test that is utilized to find any long-run relationship in terms of co-integration. In Section 5.3.3, we investigate any short-run relationship between each pair of time-series by performing a Granger-causality test.

5.3.1 Order of Integration and the general Vector Auto Regression model

VAR is one of the widely used econometric techniques to model a data set for inference and forecasting (Sims 1980, Enders 2004, Kennedy 2003). VAR models are commonly used in application areas such as finance, policy making, health care, information systems, etc (Adomavicius et al. 2012, Enders 2004, Kennedy 2003). Simply put, a VAR model is a multi-variable linear model where each variable is regressed on its own and others' past values. The main advantages of VAR models compared to most other traditional models is that they treat all variables as a priori endogenous. This makes them ideal for our study as the relationship between SPMs' and USPMs' prices are endogenous. As discussed in Chapter 2, prediction markets are theorized to react to the change of traders' available information, hence the endogenous relationship between two market prices that run on the outcome of the same event.

A VAR model is often defined for a stationary time series and special care must be taken when inference is taken from VAR models of time series that are not stationary. A stationary time series is a stochastic process whose joint probability distribution does not change when shifted in time. A series that is not stationary is called non-stationary. The term *order of integration*, is defined as the minimum number of differences required to obtain a covariance stationary series and is shown via I(Order of Integration). A stationary series must be $I(0)$, however, the converse is not necessarily true. Determining the order of integration is crucial in multiple steps of our analysis of causality and cointegration. Given a pair of INT and IPR time series, we detect the order of integration for each series via an Augmented Dicky Fuller (ADF) test (Fuller 2009). A summary of the results to determine the order of integration is shown in Table 5.7. See Table 5.1, 5.2 and 5.3 for a mapping of market ID's to their security descriptions. Further, see Table 5.4, 5.5, and 5.6 for a summary of markets' statistics.

Regardless of the order of integration, we also estimate a two-variable VAR model for each INT and IPR series as follows.

$$\begin{aligned} INT_t &= u_{1,t} + \textstyle\sum_{i=1}^{K} \theta_{i,1} INT_{t-i} \\ IPR_t &= u_{2,t} + \textstyle\sum_{i=1}^{K} \theta_{i,2} IPR_{t-i} \end{aligned} \tag{5.1}$$

in which $u_{1,t}$, and $u_{2,t}$ are white noise. White noise is a processes with a mean zero and no correlation between its values at different times (Lütkepohl 2005, Section 2.1). Given a sample of INT and IPR time-series, a VAR(K) can be efficiently estimated by Ordinary Least-Squares (OLS) applied separately to each of the equations in (5.1). Of course, a critical part of this model specification is to determine the number K. The common approach to choosing the proper value of K is to use a combination of information criteria (Lütkepohl 2005), that finds the best fit between the series and its estimated VAR series. We use *Schwartz Criterion* and *Akaike's Information Criterion* (Lütkepohl 2005) to determine the lag length, K. In the case that the two criteria are do not agree on the same number we use a likelihood ratio (LR) test to determine the optimal lag. To ensure proper model selection, we also test the VAR(K) model to make sure no serial correlation in the residuals is present. This is because a VAR(K) model with serially correlated residuals can produce inefficient forecasts, that is, OLS

Market ID	Security Description
1073	Republican Party candidate to win 2012 Presidential Election
21	John McCain to win 2008 US Presidential Election
22	Barack Obama to win 2008 US Presidential Election
2595	Newt Gingrich to win the 2012 Iowa Caucus
2596	Ron Paul to win the 2012 Iowa Caucus
2603	Mitt Romney to win the 2012 New Hampshire Primary
2604	John Kerry to win the New Hampshire Primary
2606	Ron Paul to win the 2012 New Hampshire Primary
2607	Jon Huntsman to win the 2012 New Hampshire Primary
2623	Rick Santorum to win the 2012 South Carolina Primary
2634	Rick Santorum to win the 2012 Florida Primary
2636	Mitt Romney to win the 2012 Nevada Caucus
2637	Ron Paul to win the 2012 Nevada Caucus
2705	Nicolas Sarkozy to be elected President of France in 2012
2706	Francois Hollande to be elected President of France in 2012
2710	Mitt Romney to win the 2012 Maine Caucus
2711	Newt Gingrich to win the 2012 Maine Caucus
2712	Rick Santorum to win the 2012 Maine Caucus
2713	Ron Paul to win the 2012 Maine Caucus
2721	Mitt Romney to win the 2012 Colorado Caucus
2722	Newt Gingrich to win the 2012 Colorado Caucus
2723	Rick Santorum to win the 2012 Colorado Caucus
2724	Ron Paul to win the 2012 Colorado Caucus
2726	Mitt Romney to win the 2012 Minnesota Caucus
2727	Newt Gingrich to win the 2012 Minnesota Caucus
2728	Rick Santorum to win the 2012 Minnesota Caucus
2729	Ron Paul to win the 2012 Minnesota Caucus
2731	Mitt Romney to win the 2012 Arizona Primary
2732	Newt Gingrich to win the 2012 Arizona Primary
2733	Rick Santorum to win the 2012 Arizona Primary
2734	Ron Paul to win the 2012 Arizona Primary
2736	Mitt Romney to win the 2012 Michigan Primary
2737	Newt Gingrich to win the 2012 Michigan Primary
2738	Rick Santorum to win the 2012 Michigan Primary
2742	Mitt Romney to win the 2012 Washington Caucus
2745	Ron Paul to win the 2012 Washington Caucus
2748	Newt Gingrich to win the 2012 Alaska Caucus
2749	Rick Santorum to win the 2012 Alaska Caucus
2750	Ron Paul to win the 2012 Alaska Caucus
2752	Mitt Romney to win the 2012 Idaho Caucus
2754	Rick Santorum to win the 2012 Idaho Caucus
2757	Mitt Romney to win the 2012 North Dakota Caucus
2759	Rick Santorum to win the 2012 North Dakota Caucus
2760	Ron Paul to win the 2012 North Dakota Caucus
2762	Mitt Romney to win the 2012 Georgia Primary

Table 5.1: Market ID's and their corresponding securities.

Market ID	Security Description
2763	Newt Gingrich to win the 2012 Georgia Primary
2764	Rick Santorum to win the 2012 Georgia Primary
2863	Mitt Romney to win the 2012 Ohio Primary
2864	Newt Gingrich to win the 2012 Ohio Primary
2865	Rick Santorum to win the 2012 Ohio Primary
2868	Mitt Romney to win the 2012 Oklahoma Primary
2869	Newt Gingrich to win the 2012 Oklahoma Primary
2870	Rick Santorum to win the 2012 Oklahoma Primary
2874	Mitt Romney to win the 2012 Tennessee Primary
2875	Newt Gingrich to win the 2012 Tennessee Primary
2876	Rick Santorum to win the 2012 Tennessee Primary
2882	Mitt Romney to win the 2012 Vermont Primary
2885	Ron Paul to win the 2012 Vermont Primary
2887	Mitt Romney to win the 2012 Virginia Primary
2889	Rick Santorum to win the 2012 Virginia Primary
2890	Ron Paul to win the 2012 Virginia Primary
2973	Mitt Romney to win the 2012 Kansas Caucus
2976	Rick Santorum to win the 2012 Kansas Caucus
2994	Mitt Romney to win the 2012 Alabama Primary
2995	Newt Gingrich to win the 2012 Alabama Primary
2997	Rick Santorum to win the 2012 Alabama Primary
3004	Mitt Romney to win the 2012 Hawaii Caucus
3007	Rick Santorum to win the 2012 Hawaii Caucus
3014	Mitt Romney to win the 2012 Mississippi Primary
3015	Newt Gingrich to win the 2012 Mississippi Primary
3017	Rick Santorum to win the 2012 Mississippi Primary
3024	Mitt Romney to win the 2012 Illinois Primary
3027	Rick Santorum to win the 2012 Illinois Primary
3029	Mitt Romney to win the 2012 Missouri Caucus
3032	Rick Santorum to win the 2012 Missouri Caucus
3034	Mitt Romney to win the 2012 Louisiana Primary
3035	Newt Gingrich to win the 2012 Louisiana Primary
3037	Rick Santorum to win the 2012 Louisiana Primary
3217	Rick Santorum to win the 2012 Maryland Primary
3220	Mitt Romney to win the 2012 Wisconsin Primary
3222	Rick Santorum to win the 2012 Wisconsin Primary
3242	Rick Santorum to win the Pennsylvania Primary
3287	Rick Santorum to win the 2012 Texas Primary
3713	Democratic nominee to win North Carolina
3714	Republican nominee to win North Carolina
3719	Republican nominee to win Florida
3720	Democratic nominee to win Florida
3722	Republican nominee to win Ohio
3723	Democratic nominee to win Ohio
3725	Republican nominee to win Virginia

Table 5.2: Market ID's and their corresponding securities (Continued).

Market ID	Security Description
3726	Democratic nominee to win Virginia
3728	Republican nominee to win Arizona
3729	Democratic nominee to win Arizona
3731	Democratic nominee to win Colorado
3732	Republican nominee to win Colorado
3734	Democratic nominee to win Iowa
3735	Republican nominee to win Iowa
3739	Democratic nominee to win Minnesota
3740	Republican nominee to win Minnesota
3743	Democratic nominee to win New Hampshire
3746	Democratic nominee to win Nevada
3747	Republican nominee to win Nevada
3749	Democratic nominee to win Wisconsin
3750	Republican nominee to win Wisconsin
3753	Democratic nominee to win Michigan
3754	Republican nominee to win Michigan
3823	Republican nominee to win Montana
3832	Democratic nominee to win Oregon
3833	Republican nominee to win Oregon
3844	Democratic nominee to win Pennsylvania
4834	Republican candidate to win
4851	Republican candidate to win
4852	Democratic candidate to win
5573	Cardinal Peter Turkson (Ghana) to succeed Benedict XVI
5574	Cardinal Francis Arinze (Nigeria) to succeed Benedict XVI
5575	Cardinal Marc Ouellet (Canada) to succeed Benedict XVI
5576	Archbishop Angelo Scola (Italy) to succeed Benedict XVI
5578	Cardinal Gianfranco Ravasi (Italy) to succeed Benedict XVI
814	Neither Party to control the Senate after 2010 Congressional Elections
815	The Democrats to control the Senate after 2010 Congressional Elections
816	The Republicans to control the Senate after 2010 Congressional Elections
818	The Democrats to control the House of Representatives after 2010 Congressional Elections
819	The Republicans to control the House of Representatives after 2010 Congressional Elections

Table 5.3: Market ID's and their corresponding securities (Continued).

Market ID	IPR Mean	INT Mean	IPR SD	INT SD	IPR Min	INT Min	IPR Max	INT Max	Period (min)
21	13.003	17.48	5.317	5.983	0.01	0.9	30.15	36	8641
22	86.226	82.862	5.582	6.035	69	64.6	99.41	99.3	8641
814	28.082	29.37	7.269	7.295	0.1	0.8	59.97	50	8641
815	56.058	56.345	9.746	9.904	35	35.1	99.9	99.6	8641
816	16.5	18.029	4.657	5.189	0.01	0.7	29.59	29.4	8641
818	12.57	15.716	6.362	7.036	0.01	0.8	24.35	30.8	8641
819	87.447	85.169	6.47	6.843	70	70	99.99	99.5	8641
1073	28.218	36.766	4.349	4.403	0.5	0.2	44.99	46.8	8641
2594	42.777	38.279	9.78	10.862	9.39	22	99.98	99.8	4488
2595	6.148	6.817	3.996	4.559	0.01	0.1	11.92	15	4447
2596	44.041	42.516	12.907	11.57	0.03	0.1	70.41	59.8	4465
2599	6.929	9.448	8.569	9.004	0.01	0.2	88.76	75	4480
2603	96.737	96.514	1.903	1.647	88.76	91.9	99.99	99.9	1833
2604	1.257	0.665	0.597	0.651	0.5	0.1	3.05	2.4	1692
2606	1.423	1.515	0.839	0.406	0.01	0.1	5.38	2.4	1764
2607	1.755	1.199	0.684	0.48	0.41	0.1	5.38	2.9	1760
2622	73.233	70.01	21.923	22.638	14.44	0.1	94.62	95	3795
2623	4.028	4.145	4.283	4.112	0.05	0.1	15.89	20.9	3747
2624	21.654	25.209	23.318	24.484	3.74	4.5	84.98	99.9	3795
2632	81.035	81.097	18.254	19.314	35.43	37.6	100	99.9	5554
2633	18.065	18.293	18.358	20.033	0.02	0.1	63.03	62.6	5529
2634	1.02	0.768	0.688	0.709	0.11	0.1	3.84	2.7	5228
2636	89.976	89.545	8.495	8.792	66.08	68	99.98	99.9	6753
2637	3.451	3.899	1.627	2.172	0.02	0.1	36.97	8.7	6754
2705	17.951	18.308	8.724	7.101	0.02	0.1	40.92	35	8641
2706	81.647	81.961	9.228	7.761	31	60	98.27	99.9	8641
2710	80.498	79.629	9.038	10.872	19.78	22	99.97	99.9	5217
2711	4.349	2.807	5.977	4.797	0.21	0.1	26.89	38	4901
2712	2.013	1.78	1.393	1.79	0.5	0.1	9.99	12	4936
2713	14.495	18.598	9.52	10.57	0.01	0.1	71.78	80	5407
2721	85.724	87.851	10.967	9.701	2	0.1	99.99	99.4	3814
2722	11.821	8.522	6.82	7.572	0.2	0.1	32.45	27	3776
2723	2.911	2.987	7.784	7.448	1	0.6	99.99	99.9	3770
2724	1.951	5.31	0.994	4.504	0.01	0.1	13.39	12	3747
2726	67.798	69.946	16.92	18.18	2.48	0.1	81.26	85	3799
2727	17.198	14.105	8.848	8.97	1.3	0.1	43.37	50	3794
2728	12.187	15.772	22.907	23.246	1.8	1	99.99	99.9	3817
2729	3.681	4.205	3.521	1.942	0.01	0.1	35.43	9	3794
2731	85.663	86.282	6.559	8.909	70.41	52.5	99.95	99.9	8641
2732	4.354	3.56	5.186	5.272	0.01	0.1	15.89	24	8641
2733	10.402	11.643	8.281	9.75	0.01	0.1	29.59	40	8641
2734	1.13	1.675	0.882	2.405	0.01	0.1	2.49	19	8641
2736	71.275	71.14	15.874	16.714	35.43	32.5	99.99	99.9	8641

Table 5.4: Statistic summary of Intrade and iPredict markets.

Market ID	IPR Mean	INT Mean	IPR SD	INT SD	IPR Min	INT Min	IPR Max	INT Max	Period (min)
2737	4.708	4.796	4.093	7.918	1.38	0.1	15.02	22	8641
2738	25.414	28.41	19.009	18.371	0.08	0.1	66.08	68.7	8641
2742	59.705	56.351	19.929	18.723	25.6	20.1	99.99	99.9	8641
2744	30.785	35.847	18.365	19.94	0.01	0.1	61.4	72.9	8641
2745	7.299	11.303	3.081	5.204	0.01	0.1	13.39	24.8	8641
2747	57.459	54.897	15.373	16.216	40.13	30	99.5	99.9	5892
2748	6.927	5.409	2.971	4.747	0.11	0.1	11.24	17.9	5384
2749	20.147	29.732	10.351	15.314	0.02	0.1	35.43	49.5	5646
2750	15.843	18.318	3.535	6.233	0.08	0.1	19.78	40	5801
2752	84.962	86.848	8.11	8.305	73.11	75	97.52	99.9	5606
2754	9.511	8.571	6.862	7.332	0.73	0.1	24.35	20	5808
2757	47.236	43.344	12.57	14.265	2	0.2	85.81	74.9	5533
2759	35.469	45.882	9.881	17.075	8.68	12	99	99.8	5494
2760	15.062	22.4	3.801	8.353	2.48	0.1	25.6	40	5535
2762	14.459	11.784	7.218	6.842	0.1	0.1	31	35.3	5820
2763	69.549	74.951	16.804	14.17	45.43	50	99.5	99.9	5809
2764	15.606	17.972	8.273	13.385	2.36	0.1	28.22	50	5736
2863	53.159	55.817	18.71	17.615	12.64	19	100	99.9	6088
2864	1.519	2.054	1.702	1.589	0.01	0.1	20.86	6.7	6103
2865	45.259	42.804	18.097	17.227	0.02	0.1	78.02	77	6103
2868	14.453	15.655	6.963	8.997	0.02	0.1	21.98	29.8	3858
2869	11.429	11.85	4.616	5.81	1.23	0.1	17.75	25	3874
2870	67.11	75.473	14.447	10.155	29.59	60	99.5	99.9	5796
2874	21.906	24.874	7.151	7.092	0.02	0.1	46.67	46.5	5735
2875	19.534	18.08	8.389	12.07	0.03	0.1	32.45	60	5736
2876	57.263	61.56	11.703	11.353	31	35.1	99	99.9	5792
2882	91.293	92.909	3.948	4.901	87.36	86	98.26	99.9	3171
2885	4.717	7.831	1.637	3.477	1	0.1	6.1	15	5469
2887	92.752	92.702	4.335	3.927	51.67	82.2	99.5	99.9	5560
2889	1.075	0.961	1.603	1.826	0.2	0.1	16.8	12.8	5460
2890	6.07	6.878	3.158	3.413	1.06	0.1	11.92	11.8	5484
2973	3.055	3.671	2.113	1.336	1.03	0.1	12	8	778
2976	96.618	96.223	2.044	1.766	87.36	92	99.25	99.9	832
2994	22.657	21.563	8.661	7.46	0.01	0.1	45	55.6	1776
2995	41.835	38.581	19.043	17.138	0.2	0.1	91.68	67.4	1776
2997	35.25	40.399	20.129	19.769	9.39	12.5	99.5	99.9	1759
3004	96.185	94.08	3.12	3.79	86.61	76.1	99.95	99.9	1583
3007	3.49	3.312	3.052	2.955	1	0.1	9.98	19.7	1584
3014	32.305	32.074	17.819	14.688	0.01	0.1	91.68	89.5	1696
3015	31.749	38.937	19.924	17.137	0.01	0.1	65	69.1	1719
3017	36.284	33.467	29.212	26.56	1.74	3	99	99.9	1747
3024	87.891	85.018	6.729	7.529	71.78	60	99.99	99.9	3687
3027	11.964	15.235	6.874	7.539	0.01	0.1	29.59	34.9	3680
3029	14.55	17.978	6.737	3.627	5.05	4	40.13	25	1494
3032	81.906	79.617	5.835	5.581	60	60.1	94.95	95.6	1566
3034	12.184	15.551	7.33	10.464	1.5	0.4	89.41	50	3902

Table 5.5: Statistic summary of Intrade and iPredict markets (Continued).

Market ID	IPR Mean	INT Mean	IPR SD	INT SD	IPR Min	INT Min	IPR Max	INT Max	Period (min)
3035	9.053	15.142	10.872	17.571	1	0.1	33.92	46	4760
3037	78.606	75.197	15.644	16.967	24.35	30.1	99.99	99.9	4769
3217	3.351	3.813	2.414	2.873	1.74	0.2	15.02	10.9	5438
3220	76.415	76.845	23.394	21.566	20.86	33.4	99.94	99.9	5524
3222	23.191	24.859	23.297	23.471	1	0.1	80.22	70.5	5516
3240	84.56	85.283	15.046	15.198	36.97	38.2	99.99	99.9	8641
3242	46.533	45.138	27.033	25.413	0	0.3	84.98	84.7	7427
3287	23.144	20.766	12.339	14.117	5.05	0.2	56.63	62	7417
3713	24.869	22.664	4.03	3.988	1.74	0.1	33.92	39.4	8641
3714	75.444	77.604	4.375	4.18	67	68.9	99	99.9	8641
3719	55.465	57.956	23.845	24.981	0.5	0.1	90	84.9	8641
3720	43.673	41.922	22.427	24.384	18.74	15.3	99	99.9	8641
3722	32.606	37.528	6.404	6.15	1.74	0.1	61.46	58	8641
3723	67.806	63.018	6.083	6.228	26.89	42	90	99.9	8641
3725	48.221	51.156	8.152	7.768	1.74	0.1	64.57	80	8641
3726	51.599	49.102	8.539	7.705	36.97	20	98.26	99.9	8641
3728	90.949	91.321	1.491	3.286	88.08	81	99	98.8	8641
3729	9.159	9.228	1.561	3.045	1.74	0.1	12	18.7	8641
3731	54.374	50.416	7.283	7.24	46.67	40	98.26	99.9	8641
3732	45.935	50.155	5.927	5.487	1.74	0.1	51.67	60	8641
3734	67.605	63.246	5.58	6.575	61.46	40	98.26	99.5	8641
3735	32.678	37.013	4.892	5.184	2.74	0.1	38.54	59.8	8641
3739	92.459	88.781	0.801	4.204	90.02	80.4	99	99.9	8641
3740	7.476	12.687	0.883	3.703	1.74	0.1	9.39	24.5	8641
3743	67.906	62.28	7.377	7.51	54.98	44	99.99	99.9	8641
3746	74.423	75.443	6.174	6.569	67.55	55	99	99	8641
3747	25.371	25.349	6.273	7.389	1.74	0.1	32.45	47.9	8641
3749	73.409	70.206	5.752	5.856	56.63	55.3	98.5	99.8	8641
3750	27.035	30.222	5.219	4.864	1.5	0.5	40.13	49	8641
3753	88.358	84.857	2.84	3.842	83.2	55	99	99.7	8641
3754	11.604	15.424	2.879	4.069	1.74	0.1	16.8	44.9	8641
3823	93.153	95.984	1.223	2.95	88.76	90	96.26	99	8641
3832	93.183	92.67	1.533	4.237	91.16	75	96.26	99	8641
3833	6.483	9.912	1.489	4.326	1.74	0.1	8.32	15	8641
3844	84.942	81.218	3.19	4.245	40.13	70.1	98.5	99.8	8641
4834	30.212	27.31	7.533	7.827	9.35	0.1	40.13	49.9	8641
4851	35.863	36.374	7.318	4.345	12.64	0.3	45.02	49.6	8641
4852	64.253	64.497	8.222	5.081	54.98	50.5	94.95	99.7	8641
5573	18.652	22.71	4.611	5.446	7.36	6.1	30	34.9	7322
5574	2.884	2.857	1.803	1.818	1	0.6	5.73	5.5	5584
5575	16.468	11.918	10.608	2.556	5.85	6.1	90	20	6914
5576	32.909	23.116	4.138	3.024	27.5	15	48	27.5	5797
5578	6.088	9.739	3.202	7.133	1	3	12.03	25	6209

Table 5.6: Statistic summary of Intrade and iPredict markets (Continued).

Market ID	I(IPR)	I(INT)	Market ID	I(IPR)	I(INT)	Market ID	I(IPR)	I(INT)	Market ID	I(IPR)	I(INT)
1073	1**	1**	3731	1**	1**	2713	1	0**	3024	1**	1**
21	0**	1**	3732	1**	1**	2721	1**	1**	3027	1**	1**
22	0**	1**	3734	1**	0**	2722	1**	1	3029	1**	1**
2622	1**	1**	3735	1**	0**	2723	1**	1**	3032	1**	1*
2623	1**	0**	3739	1**	0**	2724	0**	1**	3034	0**	1**
2624	1**	1**	3740	1**	0**	2726	1**	1**	3035	1**	1**
2636	1**	1**	3743	1**	1**	2727	0**	0**	3037	1*	1**
2637	0**	0**	3746	1**	0**	2728	1**	1**	3217	0**	0**
2742	1**	1**	3747	1**	0**	2729	0**	0**	5573	1	1**
2744	1**	1**	3749	1**	0**	2731	1**	1**	5574	1**	1**
2745	1**	0**	3750	1**	0**	2732	1**	0**	5575	0**	1**
2863	0**	0**	3753	1**	0**	2733	1**	1**	5576	0**	1
2864	0**	0**	3754	1**	0**	2734	1**	0**	5578	1**	1**
2865	0**	0**	3823	1	1*	2736	1**	1**	818	1**	1**
2868	1**	0**	3832	1	1	2737	1**	1**	819	1	0**
2869	1**	1**	3833	1**	1*	2738	1**	1**			
2870	1**	0**	3844	1**	0**	2747	1	0**			
2874	1**	1*	4834	1**	0**	2748	1**	1**			
2875	1**	1**	4851	1**	0**	2749	1	0**			
2876	1**	1**	4852	1**	1	2750	1	1*			
2882	1**	1**	814	1**	1**	2752	1**	1**			
2885	1**	0**	815	1**	1**	2754	1**	1			
2887	0**	0**	816	1**	1**	2755	1**	0**			
2889	0**	0**	2594	0**	0**	2757	1**	1**			
2890	0**	1	2595	1**	1**	2759	1**	1**			
3220	1**	1**	2596	1	1**	2760	0**	1			
3222	1**	1**	2599	0**	0**	2762	1**	0**			
3240	1	0**	2603	0**	0**	2763	1**	1*			
3242	1**	0**	2604	0**	1**	2764	1**	0**			
3287	1**	1	2605	1*	0**	2973	0**	1			
3713	1**	0**	2606	0**	0**	2976	0**	1			
3714	1**	0**	2607	1	0**	2994	1**	1			
3719	1**	1**	2632	1**	1**	2995	1**	1**			
3720	1**	1**	2633	1**	1**	2997	1**	1**			
3722	1**	0**	2634	0**	1	3004	1	0**			
3723	1**	0**	2705	1*	0**	3006	0**	1**			
3725	1**	1**	2706	0**	0**	3007	1**	0**			
3726	1**	1**	2710	0**	0**	3014	1**	1**			
3728	1**	0**	2711	1	0**	3015	1**	1**			
3729	1**	0**	2712	0**	0**	3017	1**	1**			

Table 5.7: Markets order of integration determined by an ADF test at 90% level of confidence (**, * indicate significance at 1%, and 5% probability levels respectively)

will under/overestimate the sampling variances (McGuigan et al. 2013, Appendix 4A). We use a Breusch-Godfrey test (Breusch 1978, Lütkepohl 2005, Section 4.4.4) with the null hypothesis that residual errors are not serially correlated. Following the strategy proposed by (McGuigan et al. 2013, Appendix 4A), the lag value K is increased until the serial correlation in residuals is removed. The final result is a proper two-variable VAR model.

5.3.2 Long-Run Relationship, Co-integration

When two series have the same order of integration, it is of great interest to investigate whether the two series are co-integrated. Two series, say X and Y, are co-integrated if there exist a stationary non-trivial linear combination between them, that is there exist β in which the series u_t, defined as

$$u_t = Y_t - \beta X_t, \tag{5.2}$$

is a stationary series. Such relation is often referred to as “long-run equilibria”, or “long-run relationship” since it can be shown that co-integrated variables do not diverge from one another in the long-run. Moreover, according to Granger (1988), there must exist a causal relation between two co-integrated variables, one way or the other. The converse however is not necessary true. Hence it is critical to test whether a pair of INT, and IPR series are co-integrated before we test for the presence of any causality. If a given pair of non-stationary INT, and IPR series have the same order of integration, we use a co-integration test to investigate any long-run relationship between them. We use the Johansen test (Johansen 1988, Lütkepohl 2005, Section

8.2) to test whether a given pair of series are co-integrated. Johansen test uses the VAR(K) model (5.1), rewritten as follows.

$$\Delta\left(INT_t, IPR_t\right)^T = \mathbf{u}_t + \mathcal{B}\left(INT_{t-1}, IPR_{t-1}\right)^T + \sum_{i=1}^{K-1} \mathcal{C}_i \Delta\left(INT_{t-i}, IPR_{t-i}\right)^T, \quad (5.3)$$

in which $A = \begin{pmatrix} \theta_{i,1} & 0 \\ 0 & \theta_{i,2} \end{pmatrix}$, $\mathcal{B} = \sum_{i=1}^{K} A_i - I$, and $\mathcal{C}_i = -\sum_{j=i+1}^{K} A_j$. It can be shown that rank($\mathcal{B}$) is the number of co-integrated factors. For a technical explanation of the Johansen test see (Lütkepohl 2005, Section 8.2). For instance if the two series are not co-integrated, rank($\mathcal{B}$) $= 0$. Let λ_1 and λ_2 be the eigenvalues of the matrix $\mathcal{B}$ in which $\lambda_1 > \lambda_2$; If $\lambda_1 = 0$, then rank($\mathcal{B}$) $= 0$ and there are no co-integrating vectors, otherwise the two series are co-integrated. Johansen's *eigenvalue* test utilizes a likelihood ratio test to examine whether the largest eigenvalue of $\mathcal{B}$ is zero, the null hypothesis, versus the alternative that the largest eigenvalue is non-zero. Table 5.8 includes the result of Johansen's maximum eigenvalue test on INT and IPR pairs that have the same order of integration.

5.3.3 Short-Run Relationship, Granger Causality Test

We use Granger-causality to investigate any short-run interdependencies between INT, and IPR series. Granger-causality is a popular approach to determine any short-term predictive relationship between series. According to Granger (1969), when past observations of series X and previous observations of the series Y can help us better predict future values of Y, as oppose to using only past values of Y, we imply that there exists a predictive relationship from X to Y. When we find statistically

Makret ID	Cointegrated?	Makret ID	Cointegrated?	Makret ID	Cointegrated?
1073	Yes**	3739	Yes*	2731	Yes**
21	Yes**	3740	Yes*	2732	Yes**
22	Yes**	3743	Yes**	2733	Yes**
2623	No**	3746	Yes**	2734	Yes*
2636	Yes**	3747	Yes**	2736	No**
2637	Yes**	3749	Yes**	2737	No**
2742	Yes**	3750	Yes**	2738	No**
2745	Yes*	3753	Yes**	2748	No**
2863	Yes**	3754	Yes**	2749	No**
2864	Yes**	3823	Yes	2750	Yes*
2865	Yes**	3832	No**	2752	No**
2868	Yes*	3833	No**	2754	Yes**
2869	No**	3844	Yes**	2757	Yes
2870	Yes**	4834	Yes**	2759	No**
2874	Yes**	4851	Yes**	2760	Yes
2875	Yes	4852	Yes**	2762	Yes**
2876	Yes*	814	Yes**	2763	Yes**
2882	No**	815	No**	2764	No**
2885	No**	816	Yes*	2973	No**
2887	Yes**	2595	Yes**	2976	Yes*
2889	Yes**	2596	Yes	2994	No**
2890	Yes**	2603	Yes**	2995	Yes**
3220	Yes**	2604	Yes**	2997	No**
3222	Yes**	2606	Yes**	3004	Yes**
3242	Yes**	2607	Yes*	3007	Yes*
3287	Yes**	2634	Yes*	3014	Yes*
3713	Yes**	2705	Yes**	3015	Yes*
3714	No**	2706	Yes**	3017	Yes
3719	Yes**	2710	Yes**	3024	Yes**
3720	Yes**	2711	Yes*	3027	Yes*
3722	Yes**	2712	Yes**	3029	No**
3723	Yes**	2713	Yes**	3032	Yes*
3725	Yes**	2721	No**	3034	Yes**
3726	Yes**	2722	Yes	3035	Yes
3728	Yes**	2723	Yes*	3037	Yes**
3729	No**	2724	No**	3217	Yes**
3731	Yes**	2726	Yes**	5573	Yes
3732	Yes**	2727	Yes**	5574	No**
3734	Yes**	2728	Yes**	5575	Yes**
3735	Yes**	2729	Yes**	5576	Yes*
819	Yes*	818	No**	5578	No**

Table 5.8: Johansen's maximum eigenvalue test results at 90% level of confidence (**, * indicate significance at 1%, and 5% probability levels respectively)

significant evidence of such interdependency we say "X *Granger causes* Y". Formally, we use the following VAR(K) model of two series X and Y,

$$Y_t = \mathbf{u}_t + \sum_{i=1}^{K} \alpha_i Y_{t-i} + \sum_{i=1}^{K} \beta_i X_{t-i}. \tag{5.4}$$

If no lagged values of X are retained in the above regression, that is $\beta_i = 0$ for $i \in \{1, 2, , \cdots , , K\}$, we imply that X does not Granger-causes Y. Similarly we can set another VAR(K) model as follows

$$X_t = \epsilon_t + \sum_{i=1}^{K} \gamma_i X_{t-i} + \sum_{i=1}^{K} \delta_i Y_{t-i}, \tag{5.5}$$

and if $\delta_i = 0$ for $i \in \{1, 2, , \cdots , , K\}$, we imply that Y does not Granger-causes X. Granger-causality test utilizes a Wald test (Granger 1969) with the null hypothesis that there does not exist any statistically significant β_i (δ_i), i.e., X does not Granger-causes Y (Y does not Granger-causes X). Thus, if we have enough evidence to reject the null hypothesis, we may conclude that there exists a short-run causal predictive relationship from X to Y, or in short, X Granger causes Y, or vice versa. Granger-causality is ideally suited for our analysis of information flow in Intrade and iPredict markets. According to the prediction market literature, market prices react to the change of information. Hence, if a market captures the newly available information quicker than another market, then the latter market's prices must be statistically significant when predicting the former market's prices.

When both time-series, X and Y, are stationary, the result of a Wald test on the models (5.4) (or (5.5)) enables us to draw proper inferences regarding any causal relationship. However, when either of time series are non-stationary the inference cannot be drawn because the VAR model does not have the usual asymptotic distribution (Toda and Yamamoto 1995). Many studies are dedicated to this issue and a proper inference is made possible via different techniques, many of such requires transformation of the original series, or changing the VAR model (Lütkepohl 2005). A common characteristic of these solution is their reliance on unit-root test which have "notoriously low-power" (Giles et al. 2002). Toda and Yamamoto (1995) offer a technique that bypasses the need for such unit-root tests and introduce a *modified* Wald (MWALD) test on the following two augmented VAR models.

$$\begin{aligned} Y_t = & \quad \mathbf{u}_t + \sum_{i=1}^{K} \alpha_i Y_{t-i} + \sum_{i=1}^{K} \beta_i X_{t-i} + \sum_{j=K+1}^{K+m_{\max}} \beta_j X_{t-j} + \sum_{j=K+1}^{K+m_{\max}} \alpha_j Y_{t-j} \\ X_t = & \quad \epsilon_t + \sum_{i=1}^{K} \gamma_i X_{t-i} + \sum_{i=1}^{K} \delta_i Y_{t-i} + \sum_{j=K+1}^{K+m_{\max}} \gamma_j Y_{t-j} + \sum_{j=K+1}^{K+m_{\max}} \delta_j X_{t-j} \end{aligned} \tag{5.6}$$

in which $m_{\max}$ is the maximum order of integration of the two series. To test for causality we carry the same approach to test the null hypothesis of $\beta_1 = \beta_2 = \cdots = \beta_K = 0$ ($\delta_1 = \delta_2 = \cdots = \delta_K = 0$) to test if X Granger-causes Y (X Granger-causes X). According to Toda and Yamamoto (1995), the MWALD test on the above augmented VAR models, ensures the proper asymptotic distribution and enables us to draw valid inferences. The results of the proposed causality procedure to INT and IPR series are shown in Table 5.9.

Market ID	G-Causality Results	Market ID	G-Causality Results	Market ID	G-Causality Results
1073	Feedback**	2757	Feedback**	3287	Intrade
21	Inconclusive**	2759	Feedback**	3713	Feedback**
22	Inconclusive**	2760	Feedback**	3714	Feedback**
2595	iPredict*	2762	Intrade*	3719	Feedback**
2596	Feedback**	2763	Feedback	3720	Feedback**
2603	Inconclusive**	2764	Inconclusive**	3722	Intrade**
2604	Feedback*	2863	Feedback**	3723	Feedback*
2606	Inconclusive**	2864	Intrade**	3725	Feedback**
2607	iPredict**	2865	Feedback**	3726	Feedback**
2623	Feedback**	2868	Intrade**	3728	Feedback**
2634	Inconclusive**	2869	Feedback**	3729	Feedback**
2636	Inconclusive**	2870	Inconclusive**	3731	Feedback**
2637	Intrade*	2874	Feedback**	3732	Feedback**
2705	Intrade**	2875	Feedback**	3734	Feedback**
2706	Inconclusive**	2876	Feedback**	3735	Feedback**
2710	Feedback**	2882	Feedback**	3739	Feedback**
2711	Feedback**	2885	Inconclusive**	3740	Feedback**
2712	Feedback**	2887	Inconclusive**	3743	Feedback**
2713	Feedback**	2889	Inconclusive**	3746	Feedback**
2721	Feedback**	2890	Inconclusive**	3747	Inconclusive**
2722	Intrade**	2973	Intrade**	3749	Feedback**
2723	Feedback**	2976	Inconclusive**	3750	Feedback**
2724	Inconclusive**	2994	Feedback**	3753	Feedback*
2726	Feedback**	2995	Feedback**	3754	Intrade**
2727	Feedback**	2997	Feedback**	3823	Feedback**
2728	Feedback**	3004	Feedback**	3832	Inconclusive**
2729	Inconclusive**	3007	Feedback**	3833	iPredict**
2731	Feedback	3014	Feedback**	3844	Inconclusive**
2732	Intrade**	3015	Feedback**	4834	Feedback**
2733	Feedback**	3017	Feedback**	4851	Feedback**
2734	Inconclusive**	3024	Feedback**	4852	Feedback**
2736	Feedback**	3027	Feedback**	5573	Inconclusive**
2737	Inconclusive**	3029	Inconclusive**	5574	Intrade**
2738	Feedback	3032	Inconclusive**	5575	Intrade**
2742	Feedback**	3034	Feedback**	5576	iPredict*
2745	Feedback**	3035	Intrade**	5578	Inconclusive**
2748	Inconclusive**	3037	Feedback**	814	Feedback**
2749	Feedback**	3217	Feedback**	815	Feedback**
2750	Feedback**	3220	Intrade**	816	Feedback**
2752	Feedback**	3222	Feedback**	818	Intrade**
2754	Feedback**	3242	Intrade**	819	Intrade

Table 5.9: Granger Causality results at 90% level of confidence (**, * indicate significance at 1%, and 5% probability levels respectively)

5.4 Results

We divide the causality results into three groups of INT $\Longrightarrow$ IPR, IPR $\Longrightarrow$ INT, and INT $\Longleftrightarrow$ IPR in which the direction of causality is presented using the implication sign. We discard the 26 cases in which there is no evidence of any short-run or long-run relation between the two market. These market pairs, with no evidence of a short-run relationship between them, illustrate a fourth option in which INT does not capture information quicker IPR, and IPR does not capture information quicker that INT, and INT and IPR do not capture information simultaneously. We do not consider this fourth option in our study.

Let us recall that when INT $\Longrightarrow$ IPR (IPR $\Longrightarrow$ INT), we imply that there is significant evidence that INT (IPR) captures the market information quicker than IPR (INT). However, when the causality is significant in both direction, i.e., the case in which INT $\Longleftrightarrow$ IPR, the interpretation is slightly different. Most economists associate bi-directional causality with feedback systems, and the inference from feedback systems are often complex. In our framework, however, bi-directional causality between IPR and INT series indicates that both markets capture the market information simultaneously. In other words, since both market prices are influenced by the change in information available to the market participants, the feedback system between IPR and INT indicates that there is significant evidence that market prices simultaneously react to the change of information. Hence we may imply that both markets capture the traders' information with the same speed.

Our main objective of this study is to investigate the empirical relation between information flow and the number of market trades. To illustrate the connection

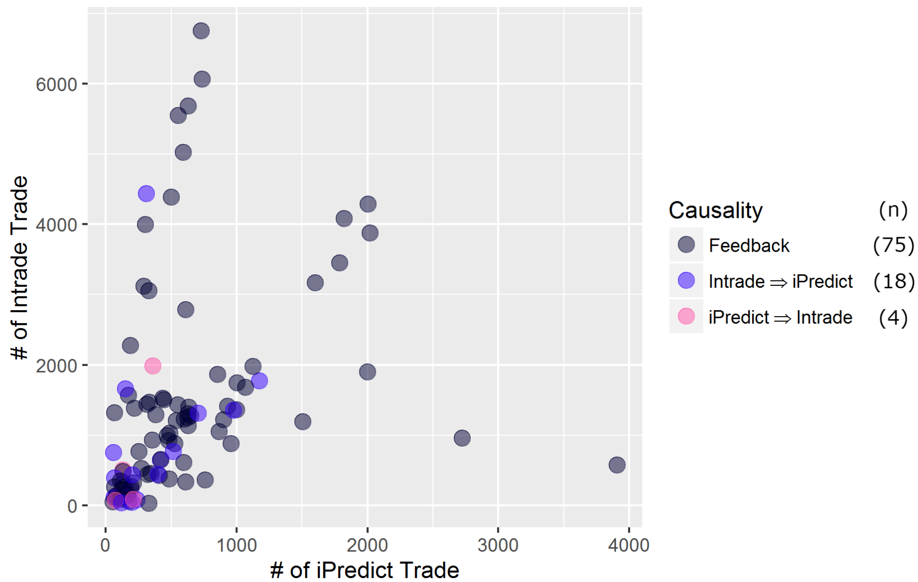


Figure 5.3: IPR and INT trades across market pairs labelled by the causality results derived in Section 5.3.3.

between causality results and the number of trades we use a simple graph of market pairs using their corresponding iPredict Trades, and Intrade Trades. We further label these data points using their casual status derived in Section 5.3.3. Figure 5.3 illustrates the simple plot of labelled pairs. For example consider the market pair 2622, which represents the security *Mitt Romney to win the 2012 South Carolina Primary*. This market pair includes an INT market with 4670 trades, and an IPR with 1742 trades. According to our analysis in Section 5.3.3, for this pair, there exist a feedback mechanism between IPR and INT market prices. Thus the data point representing this pair is coordinated at (1742, 4670) with the label *"Feedback"*. Figure 5.3 illustrates the two main intuitions behind the causality results.

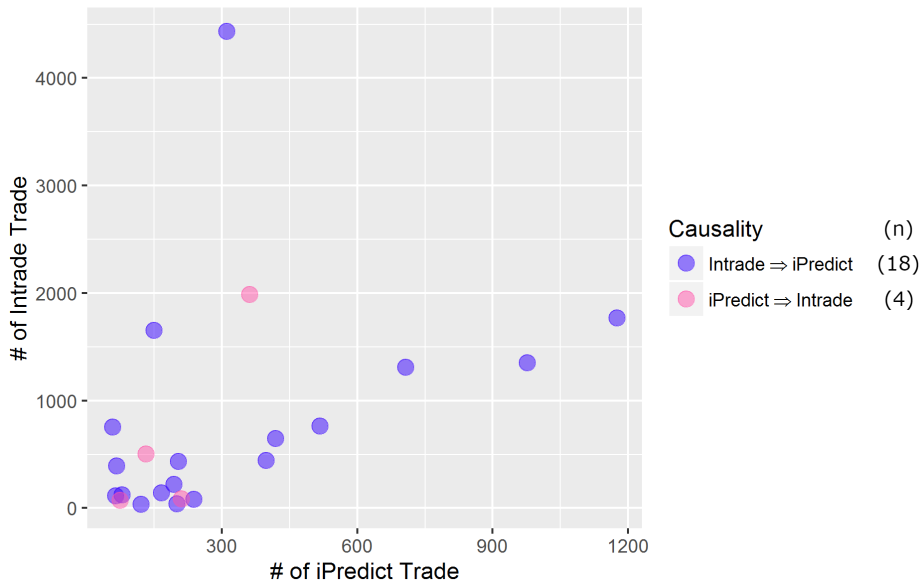


Figure 5.4: Market pairs with unidirectional causality results.

5.4.1 MSRs information aggregation in thin markets.

To study the effect of MSRs in thin markets, we need to investigate the cases in which there exist a unidirectional causality, i.e., the cases labelled as INT $\implies$ IPR, or IPR $\implies$ INT. Figure 5.4 isolates the cases with unidirectional causality from the rest.

To identify what part of the "iPredict's trade - Intrade's trade" space corresponds to the cases in which INT $\implies$ IPR, or vice versa, we are required to identify what group, INT $\implies$ IPRor IPR $\implies$ INT, does the causality status of a new, unknown, market pair belongs to. Fortunately, the answer to this problem can be found by applying a simple classification algorithm using the labelled data. In particular, we use *Quadratic Discriminant Analysis* with Leave-One-Out cross-validation (Kohavi et al. 1995) to find the best classifier that describes the data presented in Figure 5.4.

Figure 5.5: A probabilistic binary classification of unidirectional causality results.

Due to uneven number of observation points, we use the area under the curve as our metric to choose the best model amongst linear and non-linear logistic regression, linear and quadratic discriminant analysis, and recursive partitioning and regression trees (Kuhn 2008).

For a given number of trades the estimated probability of Intrade Granger causing iPredict can be determined using the posterior probability given by a quadratic discriminant analysis model. As illustrated in Figure 5.5, the maximum number of trades that results in a CDA performing better than a MSR is never more than 288, 310, and 352 at 90%, 95%, and 99% level respectively. Although this result is consistent with prediction market literature (Hanson 2003) that argue MSRs should be used in thin markets, we do not have sufficient data points to make such conclusion, hence the first hypothesis which states:

> There exists a threshold in the number of trades, say κ, such that a CDA market aggregates information faster after κ trades than a MSR market after the same number of trades.

is inconclusive. This inconclusive result could be down to two possible reasons. First, the thin market in which a MSR could outperform a CDA market may have a number of trades lower than the minimum number of trades in our data. Second, it might simply be the case that with more market data, the significance of a classification results, similar to the results shown in Figure 5.5, may increase.

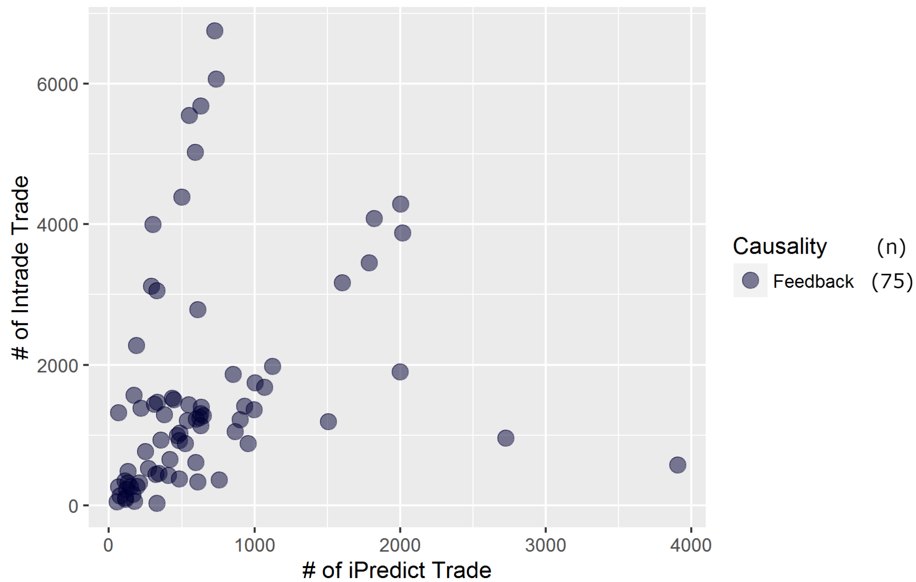


Figure 5.6: Market pairs with a causal feedback.

5.4.2 Feedback Analysis

As illustrated in Figure 5.3, in the majority of market pairs, there is a causal feedback between CDAs' and MSRs' market prices. As discussed in Section 5.3, a causal feedback between INT and IPR series can be interpreted as the case in which both markets capture the available information with the same speed. The results from Section 5.3.3 also offers us the opportunity to monitor the markets in which INT $\Longleftrightarrow$ IPR. Similar to Figure 5.5, Figure 5.6 illustrates the simple plot of feedback pairs using their corresponding market trades. To compare Intrade and iPredict markets using their trade quantities, we find the distribution of the number of Intrade trades relative to iPredict trades shown in Figure 5.7. As shown in Figure 5.7, in 80% of cases, in a pair of markets in which INT $\Longleftrightarrow$ IPR, IPR has a lower number of trades. Moreover, in more than 46% of those cases, INT has at least twice as many

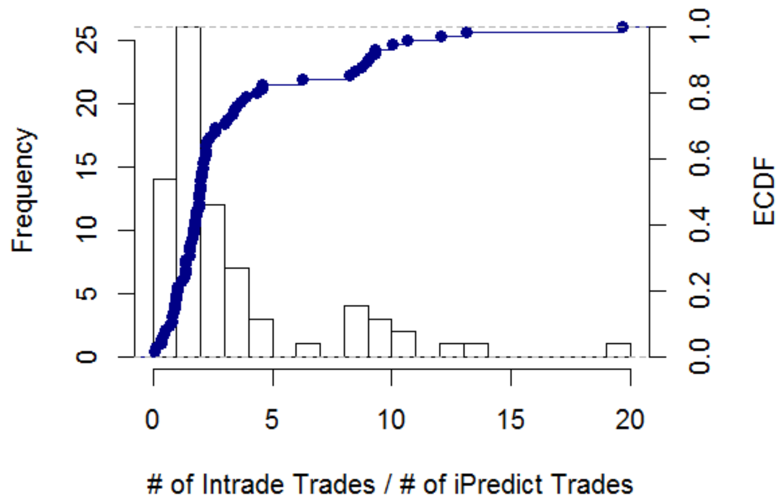


Figure 5.7: Distribution of the number of Intrade trades relative to iPredict trades for feedback market pairs.

trades as the corresponding IPR market. We further utilize a Wilcoxon signed-rank test (Wilcoxon 1945, R Core Team 2013) to compare the two distribution of IPR trades, and INT trades. The Wilcoxon signed-rank test, examines the hypothesis that the distribution of IPR trades and INT trades are identical, versus the alternative that INT trades are higher than IPR trades on average. The result of the Wilcoxon signed-rank test indicates that there are significant evidence, at less than the 0.001 probability level, which indicates that INT trades are higher than IPR trades on average. This observation indicates that when both CDAs and MSRs aggregate the information with the same speed, MSRs often require a lower number of trades, which confirms the second hypothesis.

When MSR and CDA markets aggregate information with the same speed, a MSR market requires fewer number of trades, compared to its corresponding CDA market, to do so.

This result is in particular interesting in corporate settings in which achieving a high number of trades may not be feasible.

5.5 Conclusion and Future Work

We conducted a time-series analysis to analyze the information aggregation performance of subsidized and unsubsidized prediction markets. We find that, affirmative to the current theory on prediction market literature, when subsidized and unsubsidized prediction markets aggregate information simultaneously, subsidized prediction markets require a lower number of trades. Our findings suggest that subsidized prediction markets must be used when the trade frequencies are a concern to market organizers interested in better information aggregation performance. We further examine a threshold in the number of trades in which a market organizer can drop a subsidized prediction market in favor of a subsidized prediction market without concerns regarding the information aggregation performance. Although our findings indicate a small threshold, in the number of 310 trades, the unidirectional subset of our dataset did not contain sufficient observations to indicate a significance conclusion.

Chapter 6

Concluding Remarks

In this thesis, we address issues in corporate prediction markets using an optimization and statistical perspective. In particular, we covered the two issues of risk-preferences in subsidized prediction markets and the selection criteria for using subsidy in prediction markets. In particular, we showed that when designing a prediction market, ignoring the risk-attitude of market participants may have negative consequences. Moreover, we provide a systematic approach for market organizers to tune market parameters to better control such consequences. Furthermore, we showed that our findings generalize to any finite outcome space and are not limited to binary cases.

Concerning the subsidy in prediction markets, we utilized a statistical approach to compare the information aggregation capabilities of different prediction market mechanisms in different situations. Using an empirical framework, we showed that unsubsidized prediction markets are ideal in situations when there is low trade frequency and unsubsidized prediction markets can be used for markets with high number of trades. Furthermore, we showed that when both subsidized and unsubsidized prediction markets aggregate traders' information simultaneously, subsidized prediction markets require a lower number of trades compared to unsubsidized prediction markets.

Prediction market organizers can utilize our findings when designing prediction markets for corporations. We illustrate a systematic approach that analyzes different prediction market mechanisms and recommends the best mechanism to ensure proper elicitation and fast information aggregation. Moreover, corporate prediction market organizers can use our results concerning subsidy to save on the operational costs of running internal prediction markets by using unsubsidized prediction markets whenever appropriate.

It is worth mentioning that Chapter 3, and Chapter 4 present sufficient conditions for deviation-liquidity, and flatness-liquidity results and in the future we would like to find necessary conditions. We also would like to extend our results concerning risk attitudes in prediction markets, with other risky choice theories such as prospect theory. In addition, the presented work in Chapter 3 and Chapter 4 only shows how to choose the most liquid market maker amongst three popular MSR families. As a future direction of study, we would like to optimize over the set of all possible MSRs to find the one that has the optimal liquidity across all MSRs. We further plan to test both deviation-liquidity and flatness-liquidity results in laboratory settings. Using our empirical study in Chapter 5, we would also like to investigate the information aggregation performance of prediction market mechanisms amongst subsidized prediction markets.

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