

COMPATIBILITY OF EXTENSIONS OF A COMBINATORIAL GEOMETRY

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ABSTRACT

Two extensions of a geometry are compatible with each other if they have a common extension. If the given extensions are elementary, their compatibility can be intrinsically described in terms of their corresponding linear subclasses. Certain adjointness relation between an extension of a geometry and the geometry itself is also discussed.

Any extension of a geometry G by a geometry F determines and is determined by a unique quotient bundle on G indexed by F . As a study of the compatibility among given quotients of a geometry, we look at the possibility of completing to F -bundles a family of quotients indexed by a set I of flats of F . If the indexing geometry F is free and if the set I is a Boolean subalgebra or a sublattice of F , for any family $Q(I)$ of quotients of a geometry G , there is a canonical construction which determines its completability and at the same time produces the extremal completion if it is a partial bundle.

Geometries studied in this dissertation are furnished with the weak order. Almost invariably, the Higgs' lift construction, in a somewhat generalized sense, constitutes a convenient and indispensable means in various of the extremal constructions.

THESIS SUPERVISOR: Henry H. Crapo.

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0. INTRODUCTION

A new trend in the studies of combinatorial geometries began a decade ago with Crapo's famous work on single-element extensions of geometries [5]. Higgs' work on factorizations of strong maps [11], which appeared in 1968, made significant contributions to the theory of extensions and provided a setting in which rapid progress could be made by others. Thereafter, a number of papers have appeared, building on the work of Crapo and Higgs, which described geometric constructions or proved that various classes of geometries can be obtained by the judicious use of these constructions. Among the most recent contributions, which focus upon the representation of quotients by extensions of geometries, are Brylawski's Modular constructions [2], Dowling and Kelly's elementary strong map factorizations [9], Kennedy's canonical major constructions [12] and the relative position in extensions studied by the author jointly with Crapo [4].

The purpose of our present work is to study the compatibility of extensions of a geometry. Basically, a family of extensions G_i of a geometry is said to be compatible if the G_i 's have a common extension. The original problem of compatibility can be traced back to Crapo's counter-example for the existence of a pushout in the category of geometries and strong maps [6]. In a quick response, Brylawski gave a partial solution with a certain modularity assumption [2].

In this dissertation, we start with the problem of the compatibility of a pair of elementary extensions of a geometry. By making use of the correspondence between elementary extensions and linear subclasses of a geometry, we obtained an intrinsic description of the compatibility. For any two linear subclasses, their compatibility can be determined by observing a sequence of sets of flats, recursively defined.

There is in fact a basic philosophy of gain and loss which motivated our present study of compatibilities: by adding more points to a geometry, we have a richer description in terms of relative positions for the purpose of further extensions, but certain extensions of the original geometry may have been destroyed. If we restrict ourselves to elementary extensions, this situation can be appropriately described by a Galois connection between the lattice of elementary extensions of a geometry G and the lattice of elementary extensions of any given extension of G . This adjointness relation does not hold for more general extensions (vide chapter III), so we have to invent some other techniques for our compatibility studies.

Any extension of a geometry G by another geometry F can be characterized by a family $\{Q(a)\}_{a \in F}$ of quotients of G indexed by F , satisfying the following two properties:

- 1) $Q(0) = G$; if b covers a in F , then $Q(b)$ is an elementary quotient of $Q(a)$
- 2) for any $a, b \in F$ such that a, b cover $a \wedge b$, if $x \in Q(a \wedge b)$ and

$$r_{Q(a)}(x) = r_{Q(b)}(x) = r_{Q(a \wedge b)}(x) + 1, \text{ then}$$

$$r_{Q(a \vee b)}(x) = r_{Q(a \wedge b)}(x) + 2.$$

(In chapter IV, the terminology of modular filters will be used instead of the rank functions in 2)). Any such family of quotients is called an F-bundle on G and determines a unique extension of G by F. The concept of a quotient bundle gives a new direction for compatibility studies—simultaneous representations of quotients of a geometry.

For a single quotient Q of a geometry G, its representability is affirmative, as proved by Higgs [11]. Any such representation is an extension R of G together with a flat z of R such that

$$\text{Quo}(G \rightarrow R \rightarrow R/z) = Q.$$

For two or more quotients Q_i of a geometry G, their representability is still an unsettled problem. There are two possible directions we can pursue: firstly, we can ask for a simultaneous representation in the most general sense—an extension R of G together with flats z_i of R such that

$$\text{Quo}(G \rightarrow R \rightarrow R/z_i) = Q_i;$$

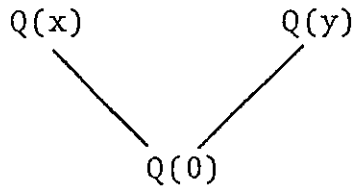
secondly, if we assume that the flats z_i are (R-closures of) preassigned flats of a geometry F, we come to the problem of completing the quotients Q_i to an F-bundle on G.

To illustrate the distinct nature between these two types of problems, let us consider for example the quotients

$$Q_1 = \begin{array}{c} \text{ab} \quad \text{cd} \quad \text{e} \quad \text{f} \\ \text{-----} \end{array}$$

$$\text{and } Q_2 = \begin{array}{c} \text{ab} \quad \text{cd} \quad \text{ef} \\ \text{-----} \end{array}$$

of the plane geometry G of six points a, b, c, d, e, f in general position. While Q_1 and Q_2 , as we shall show, are simultaneously representable, however, if F is the line geometry of two points x, y and we let $Q(0) = G$, $Q(x) = Q_1$ and $Q(y) = Q_2$, then the following diagram of quotients



cannot be completed to any F -bundle on G (vide chapter IV).

With regard to the first direction, we prove that any pair of elementary quotients of a geometry is representable. However, the second problem is our next goal. We call a family $Q(I)$ of quotients of a geometry G indexed by a set I of flats of a geometry F a partial bundle if $Q(I)$ can be completed to an F -bundle on G . Our main concern then is to explore the possible completions of a partial bundle.

Several partial orders can be defined for geometries (on the same set of points) [4,10]. To facilitate our work in the next stage we consider in particular the weak order, which is the geometric analogue of specializations of classical alge-

braic varieties. A geometry G is weaker than a geometry H if and only if flats of G are in positions more general than flats of H (vide section 1.6).

In chapter V, completions of a partial bundle are discussed with the consideration of the weak order. For an arbitrary partial bundle $Q(I)$, its completions need not form a Λ -semilattice. Nor does a least completion of $Q(I)$ necessarily exist. In stages we develop a sufficient condition on the indexing set I so as to guarantee that the completions of a partial bundle $Q(I)$ are Λ -closed (and thus form a Λ -semilattice).

Ever since it was invented by Higgs, the lift construction has demonstrated its indispensability and handiness as a construction technique for quotients in most extremal problems of combinatorial geometries. We give a thorough treatment (chapters IV,VI) of various lift sequences of quotients as preliminaries for our completion constructions. In chapter VI, we consider in detail a special class of partial bundles $Q(I)$ where I is a Boolean subalgebra of a free geometry. For any such partial bundle, its least completion always exists, which can be canonically constructed by means of appropriate lift sequences.

In chapter VII we proceed to a more general class of partial bundles $Q(I)$ where I is a sublattice of a free geometry. While a treatment as detailed as the previous one is not allowed, we still prove that any such partial bundle has a least completion, which can be constructed with a special lifting scheme.

The importance of the above construction is two-fold. Firstly, it links the concept of lifting of quotients to certain principal extensions and thus exhibits the geometric nature of the lift construction. Secondly, it serves as a test for completability of an arbitrary family $Q(I)$ of quotients where I is a sublattice of a free geometry.

The author would like to take this opportunity to thank Professor Henry Crapo for his kind guidance and generous help during the various stages in the preparation of this dissertation and during many other occasions.

I. COMBINATORIAL GEOMETRIES: BASIC CONCEPTS

1.1 Combinatorial Geometries

A geometry $G(X)$ is a set X together with a closure operator $\bar{}$ satisfying the Steinitz-MacLane exchange property:

$$\forall p, q \in X, \forall A \subseteq X, p \notin \bar{A} \text{ and } p \in \overline{A \cup q} \implies q \in \overline{A \cup p},$$

and the finite basis property:

$$\forall A \subseteq X, \exists \text{finite } A_f \subseteq A \text{ such that } \overline{A_f} = \bar{A}.$$

With no essential loss of generality, we often assume that the empty set and one-element sets are closed. Closed sets are generically called flats. The complete lattice G of flats is a geometric lattice, characterized as a semimodular atomistic lattice without infinite chains. In such a lattice, each element x is the supremum of atoms and has a well-defined rank $r(x)$, equal to the length of any maximal chain from the 0-element to x , satisfying the semimodular inequality:

$$r(x \wedge y) + r(x \vee y) \leq r(x) + r(y).$$

The height of G is called the rank of the geometry $G(X)$, denoted by $r(G)$. Flats of ranks 1, 2, 3 and $r(G)-1$, $r(G)-2$, $r(G)-3$ are specifically called points, lines, planes and copoints, colines, coplanes respectively. Two flats x, y form a modular pair if $r(x \wedge y) + r(x \vee y) = r(x) + r(y)$, a locally modular pair if x, y cover $x \wedge y$. For any subset $A \subseteq X$, the rank $r(A)$ of A is defined to be the rank of \bar{A} in G , and if A is finite, the nullity $n(A)$ of A is the non-negative integer $|A| - r(A)$. A subset $A \subseteq X$ is said to be independent if $n(A) = 0$, and dependent otherwise. A maximal independent set is called a basis and a

minimal dependent set is called a circuit. A geometry $G(X)$ is uniquely determined by its rank (and nullity) functions on subsets of X , its independent sets, its bases, its circuits, ..., and most importantly, by its associated geometric lattice (with the assumption that \emptyset and one-element subsets are closed). If there is no confusion caused, we prefer to simply write G for the geometry $G(X)$.

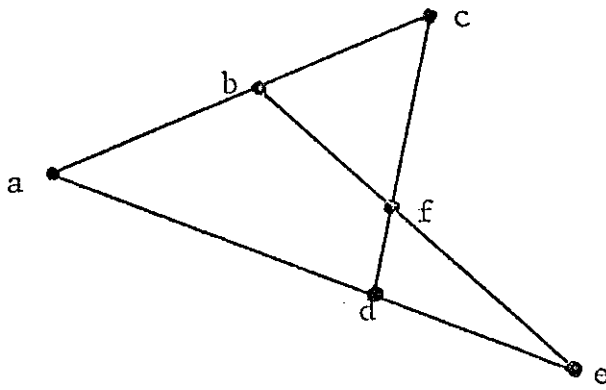
If G is a geometry of rank $r \geq 1$, then the flats of rank not equal to $r-1$ form a geometry, called the truncation of G . Recursively, the k^{th} truncation of G is also defined.

The free geometry $B(X)$ on a set X is the geometry whose flats are all subsets of X . Its geometric lattice is a Boolean algebra. If k is a positive integer not greater than $|X|$, then the rank- k geometry on X in general position is the $(|X| - k)^{\text{th}}$ truncation of $B(X)$, whose flats are subsets of X of cardinality less than k together with X .

The direct sum $G \oplus H$ of geometries $G(X)$ and $H(Y)$ is the geometry on $X \cup Y$ whose flats are all of $x \cup y$ with $x \in G$ and $y \in H$.

Two geometries are said to be equivalent or isomorphic if their associated geometric lattices are isomorphic.

Whenever possible, we shall take the liberty to picture geometries in the real affine space. In such a diagram, only non-trivial lines and planes together with the points are drawn. For example, in the following rank three geometry on six points:



af, bd and ce are lines but are not drawn in the diagram.

If necessary, we will adapt the notation with subscripts or the like to specify a particular geometry. For example, instead of $r(x)$ for the rank of a flat x of G , we may choose to say $r_G(x)$ for the G -rank of a G -flat x .

1.2 Minors

Given a geometry $G(X)$ and a subset $S \subseteq X$, the subgeometry of $G(X)$ on S is the geometry $H(S)$ on the set S with the following induced closure operator:

$$A \longrightarrow \overline{A}^G \cap S, \quad \forall A \subseteq S.$$

A subset $A \subseteq X$ is H -closed if and only if $A = A' \cap S$ for some G -flat A' . The associated geometric lattice H of the subgeometry $H(S)$ is isomorphic to the sub- \mathcal{V} -semilattice of G generated by the atoms of G contained in A .

If $H(S)$ is a subgeometry of $G(X)$, then $G(X)$ is called an extension of $H(S)$.

Next, given a geometry $G(X)$ and a subset $C \subseteq X$, the contraction of $G(X)$ by C is the geometry on the set $X \setminus C$ with the following closure operator:

$$A \longrightarrow \overline{A \cup C}^G \setminus C, \quad \forall A \subseteq X \setminus C,$$

which is equivalent to the geometry on the set X with the following closure operator:

$$A \longrightarrow \overline{A \cup C}^G, \quad \forall A \subseteq X.$$

The associated geometric lattice, denoted G/C , of the contraction is isomorphic to the interval $[\overline{C}^G, 1_G]$ in the associated geometric lattice G of $G(X)$. For any subset $A \subseteq X$,

$$r_{G/C}(A) = r_G(A) - r_G(C).$$

A minor of a geometry $G(X)$ is defined to be a contraction of a subgeometry of $G(X)$, which is always a subgeometry of a contraction of $G(X)$.

1.3 Elementary extensions

A modular filter of a geometry G is a set M of G -flats satisfying the following two properties:

1) if $x \in M$ and $x \leq y$, then $y \in M$,

and 2) if $x, y \in M$ and x, y form a modular pair, then $x \wedge y \in M$.

With 1) given, 2) is equivalent to the following:

2') if $x, y \in M$ and x, y form a locally modular pair, then $x \wedge y \in M$.

Given a modular filter M of $G(X)$, we can always put a new element e "within" M and thus obtain an elementary extension $H(X \cup e)$ of $G(X)$ by the element e . There are three distinct types of H -flats:

1) all G -flats A not in M ,

2) $A \cup e$, where $A \in M$,

and 3) $A \cup e$, where A is a G -flat not in M and covered by no flats in M .

We write $G \xrightarrow{M} H$ to indicate that H is the elementary extension of G determined by the modular filter M . The rank of H is related to that of G as follows:

$$r(H) = \begin{cases} r(G) + 1 & \text{if } M = \emptyset \\ r(G) & \text{otherwise.} \end{cases}$$

If $H(X \cup e)$ is an elementary extension of $G(X)$, then the set $\{x \in G \mid \bar{x}^R \geq \bar{e}^R\}$ is a modular filter of G , denoted by $\text{Mod}(G \rightarrow H)$. There is a precise correspondence between elementary extensions and modular filters of G . The elementary

extensions of G , ordered by containment of their respective modular filters in G , form a lattice.

Given a modular filter M of G , the set L of copoints in M satisfies the following property:

for any copoints x, y, z in G , if $x, y \in L$ and x, y, z cover $x \wedge y \wedge z$, then $z \in L$.

Any such set of copoints is called a linear subclass. Any linear subclass L of G determines a non-empty modular filter

$\{x \in G \mid x \leq y \text{ and } y \text{ is a copoint of } G \text{ implies } y \in L\}$, denoted by $MF(L)$. This establishes a correspondence between linear subclasses and non-empty modular filters of G .

Any set A of flats of G generates a modular filter of G , namely, the intersection of all modular filters of G containing A . Similarly, any set A of copoints of G generates a linear subclass of G , namely, the intersection of all linear subclasses of G containing A .

1.4 Strong maps and quotients

A strong map from a geometry G to a geometry H is a function from G to H which takes points of G to points of H or to the zero flat of H , and preserves all suprema. Equivalently, $f : G \rightarrow H$ is a strong map if and only if the inverse image of any H -closed set of points is G -closed. Composites of strong maps are also strong maps.

The following are examples of strong maps:

1) Injection (or embedding) of a subgeometry into a geometry.

2) Contraction by a flat z in a geometry G , ie. the map $x \rightarrow x \vee z$ from G onto the interval $[z, 1]$ in G .

3) The canonical surjection from a free geometry $B(X)$ onto any given geometry $G(X)$: $A \rightarrow \bar{A}^G$.

If G and Q are geometries on the same set X such that any Q -flat is a G -flat, then Q is a quotient of G . Equivalently, Q is a quotient of G if and only if the identity map on X extends to a strong map from G to Q .

Given any strong map $f: G \rightarrow H$, the f -nullity of a G -flat x is the nonnegative integer $r_G(x) - r_H(f(x))$, denoted by $n_f(x)$ or $n_{G \rightarrow H}(x)$. A G -flat x is said to be f -independent if $n_f(x) = 0$, and f -closed if x is a maximal G -flat with given image in H . The set of all f -closed G -flats is a quotient of G , called the quotient of the strong map f and denoted by $\text{Quo}(G \rightarrow H)$. The quotient $\text{Quo}(G \rightarrow H)$ is isomorphic to the image $f(G)$ of G .

So, if a strong map $f:G \rightarrow Q$ is onto, then without loss of generality, we can assume that G and Q are geometries on the same set and hence that Q is a quotient of G . The degree of the quotient Q of G , denoted by $\text{deg}(G \rightarrow Q)$, is defined to be the nonnegative integer $r(G) - r(Q)$. The set

$$\{x \in G \mid n_{G \rightarrow Q}(x) = \text{deg}(G \rightarrow Q)\},$$

denoted by $M(G \rightarrow Q)$, is a modular filter of G ; in fact it is the largest modular filter of G contained in Q . More generally, if k is an integer not less than $\text{deg}(G \rightarrow Q)$, we define

$$M_k(G \rightarrow Q) = \{x \in G \mid n_{G \rightarrow Q}(x) = k\}.$$

If Q is a quotient of G and if $\text{deg}(G \rightarrow Q) = 0$ or 1 , then the quotient Q and the strong map $G \rightarrow Q$ are said to be elementary. It is clear that $M_1(G \rightarrow Q)$ is non-empty if and only if the elementary quotient Q is proper (ie, $Q \neq G$). If M is a modular filter of G , then the set

$$M \cup \{x \in G \mid x \text{ is covered by no flats in } M\}$$

is an elementary quotient of G . This establishes a one-to-one correspondence between elementary quotients and proper (ie. $\neq G$) modular filters of G .

1.5 Orthogonality

For any geometry $G(X)$, there is a uniquely defined dual (or orthogonal) geometry $G^*(X)$ given by the following closure operator:

$$\overline{A}^{G^*} = A \cup \{p \in X \mid p \notin \overline{X \setminus (A \cup p)}^G\} \quad \forall A \subseteq X.$$

Duality of geometries is involutory, ie. $(G^*)^* = G$. The rank and nullity functions in G and G^* are related as follows:

$$r_G(A) + n_{G^*}(X \setminus A) = r(G), \quad \forall A \subseteq X.$$

If Q is a quotient of G , then G^* is a quotient of Q^* and

$$\deg(G \rightarrow Q) = \deg(Q^* \rightarrow G^*).$$

1.6 Weak order

Denote by $\mathcal{G}[X]$ the set of all geometries on X . The weak (map) order on $\mathcal{G}[X]$ is the partial order \leq defined as follows:

$G \leq H$ if and only if any of the following equivalent statements holds:

- 1) every independent set of H is independent in G
- 2) $r_G(A) \geq r_H(A) \quad \forall A \subseteq X$
- 3) $r_G(x) \geq r_H(x) \quad \forall G\text{-flat } x$
- 4) $n_G(A) \leq n_H(A) \quad \forall A \subseteq X$

etc.

We say that G is weaker than H if $G \leq H$. The weak order is not a lattice order, as demonstrated in the following examples:

EXAMPLE 1.1 Let G_1, G_2, H_1, H_2 be geometries on the set $\{a, b, c, d\}$ as shown in figure 1.1. In the weak order, G_1, G_2 cover H_1, H_2 , so neither $G_1 \wedge G_2$ nor $H_1 \vee H_2$ exists.

EXAMPLE 1.2 Let G_1, G_2, H_1, H_2 be rank four geometries on the set $\{a, b, c, d, e, f\}$ as shown in figure 1.2. Both H_1 and H_2 are maximal geometries weaker than G_1 and G_2 , so $G_1 \wedge G_2$ does not exist.

If Q is a quotient of G , then $G \leq Q$. Given elementary quotients P, Q of G , $P \leq Q$ if and only if $M_1(G, P) \subseteq M_1(G, Q)$. The elementary quotients of G form a semilattice[†] under the weak order, which is isomorphic to the semilattice of proper modular filters of G .

PROPOSITION 1.3 Let $G(X)$ and $H(X)$ be geometries of the same rank. Then $G \leq H$ if and only if $G^* \leq H^*$.

Proof: Let r be the rank of G . Then

$$\begin{aligned} G \leq H &\iff r_G(A) \geq r_H(A) \quad \forall A \subseteq X \\ &\iff r - n_{G^*}(X \setminus A) \geq r - n_{H^*}(X \setminus A) \quad \forall A \subseteq X \\ &\iff n_{G^*}(A) \leq n_{H^*}(A) \quad \forall A \subseteq X \\ &\iff G^* \leq H^*. \quad \square \end{aligned}$$

The above proposition is definitely false without the assumption that G and H have the same rank.

[†]Throughout this paper we use the term semilattice for what should strictly be called a \wedge -semilattice.

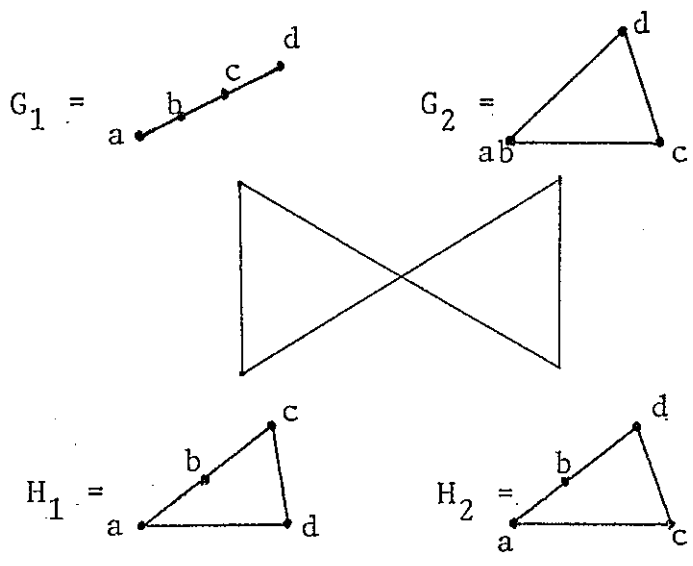


figure 1.1

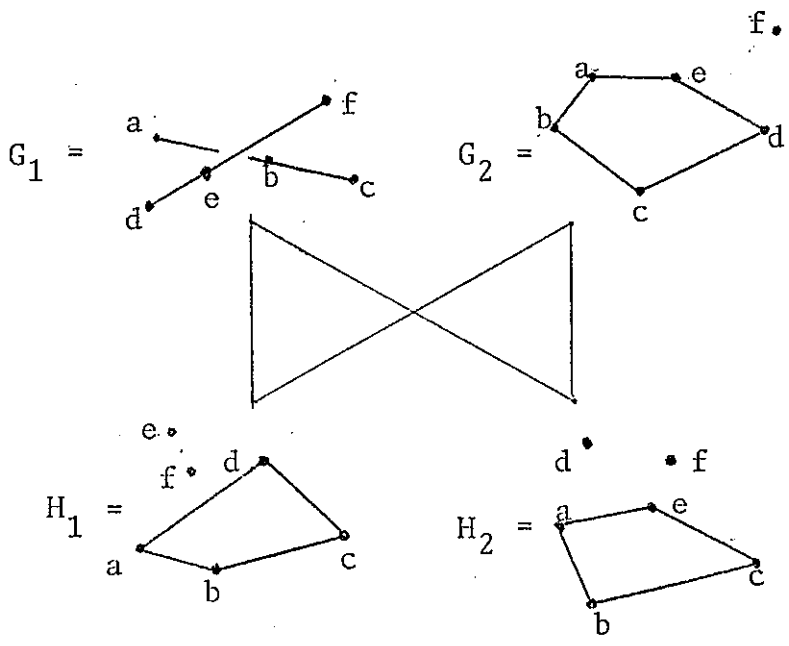
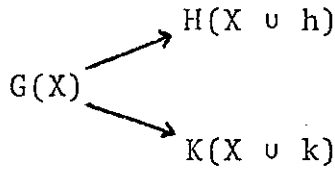


figure 1.2

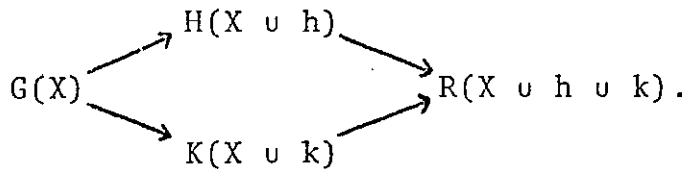
II. COMPATIBILITY OF TWO ELEMENTARY EXTENSIONS OF A GEOMETRY

2.1 Definitions and examples

A pair of elementary extensions



of a geometry $G(X)$ [†] is compatible if there exists a geometry $R(X \cup h \cup k)$ together with the following (commutative) embedding diagram:



Otherwise, the pair is said to be incompatible.

Incompatible pairs exist. For example, let G be the geometry of points a, b, c, d, e, f in general position on a plane and let H and K be elementary extensions of G as shown in figure 2.1.

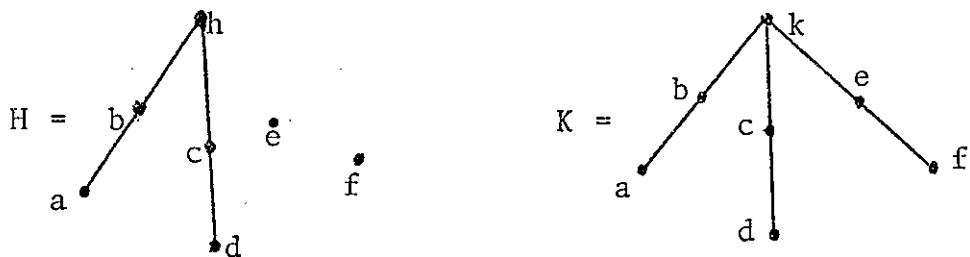


figure 2.1

There is no geometry on $\{a, b, c, d, e, f, h, k\}$ which yields H and K when restricted to $\{a, b, c, d, e, f, h\}$ and $\{a, b, c, d, e, f, k\}$ respect-

[†]We assume that $h \neq k$.

ively. So the pair of extensions H and K of G is incompatible.

If two elementary extensions H and K of G are compatible with each other, there may be more than one common elementary extensions of both H and K. As an example, consider the pair of elementary extensions shown in figure 2.2

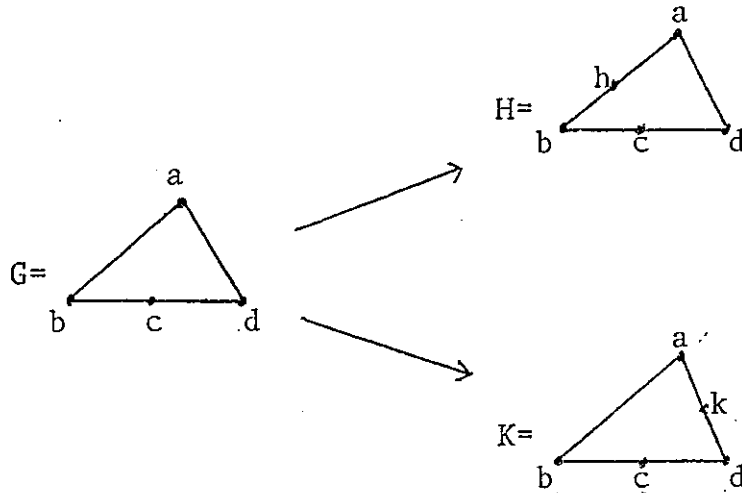


figure 2.2

There are two common elementary extensions of both H and K, as shown in figure 2.3:



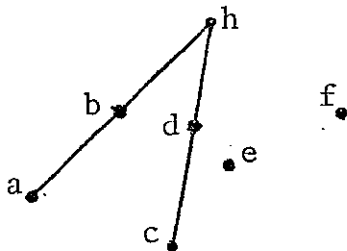
figure 2.3

where in R_2 chk is a three-point line.

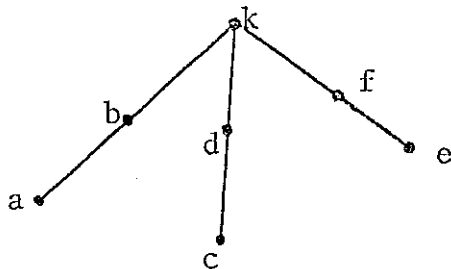
We'll see later (chapter III) that all such possible common elementary extensions of both H and K form a semilattice (in the weak order).

2.2 Compatibility of two modular filters.

Given two modular filters M and N of a geometry G, we enquire if it is possible to put a point on precisely the flats of G in M and at the same time another point on precisely the flats of G in N. Consider for example the geometry G of six points a,b,c,d,e,f in general position on a plane and modular filters $M = \{ab, cd, 1_G\}$ and $N = \{ab, cd, ef, 1_G\}$. Putting a point h on the lines ab and cd, we have the extension:



if in which a point k is put on the lines ab, cd and ef, then h is forced to lie on ef, a contradiction. If in the geometry G, a point k is put on the lines ab, cd and ef, we have the extension



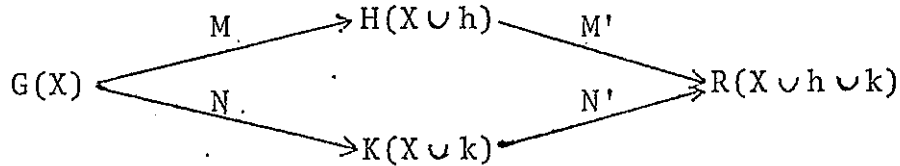
in which it is impossible to put a point h on only the lines ab and cd .

Two modular filters of a geometry G are said to be compatible if their corresponding elementary extensions of G are compatible. We will show in proposition 2.3 that compatibility is indeed equivalent to the above geometric version of the problem.

LEMMA 2.1 If $G(X)$ is a subgeometry of $H(Y)$, then for any $A \subseteq X$ and $x \in X$,

$$x \in \overline{A}^H \text{ if and only if } x \in \overline{A}^G. \quad \square$$

PROPOSITION 2.2 Given elementary extensions as shown in the following diagram:



where M, N, M' and N' are the corresponding modular filters, we have

$$M = \{x \in G \mid \overline{x}^K \in N'\}$$

$$\text{and } N = \{x \in G \mid \overline{x}^H \in M'\}.$$

Proof: If $x \in M$, then $h \in \overline{x}^H \subseteq \overline{x}^R = \overline{\overline{x}^K}^R$, so $\overline{x}^K \in N'$. Conversely, if $x \in G$ and $\overline{x}^K \in N'$, then from $\overline{\overline{x}^K}^R = \overline{x}^R$ we obtain $h \in \overline{x}^H$ by lemma 2.1, so that $x \in M$. So $M = \{x \in G \mid \overline{x}^K \in N'\}$. Similarly, $N = \{x \in G \mid \overline{x}^H \in M'\}$. \square

PROPOSITION 2.3 Given modular filters M and N of $G(X)$, they are compatible with each other if and only if there exists a two-element extension

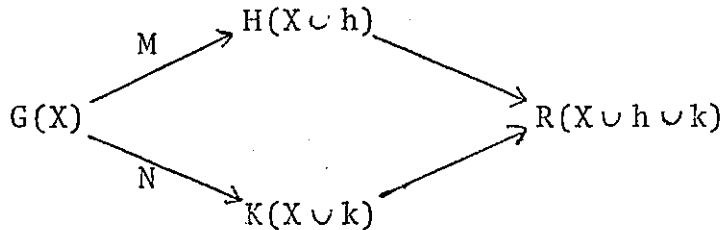
$$G(X) \longrightarrow R(X \cup h \cup k)$$

such that for any $x \in G$,

$$h \in \overline{x}^R \iff x \in M$$

$$\text{and } k \in \overline{x}^R \iff x \in N.$$

Proof: If modular filters M, N are compatible with each other, then there exist extensions

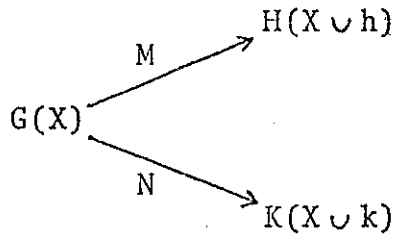


where $M = \text{Mod}(G \rightarrow H)$ and $N = \text{Mod}(G \rightarrow K)$. Let $x \in G(X)$. Then $x \in M \iff h \in \overline{x}^H \iff h \in \overline{x}^R$ and similarly $x \in N \iff h \in \overline{x}^R$.

Conversely, assume the condition of the proposition holds. Let $H(X \cup h)$ and $K(X \cup k)$ be the subgeometries of $R(X \cup h \cup k)$ on $X \cup h$ and $X \cup k$ respectively. If $x \in G$, then $x \in M \iff h \in \overline{x}^R \iff h \in \overline{x}^H \iff x \in \text{Mod}(G \rightarrow H)$, so $M = \text{Mod}(G \rightarrow H)$. Similarly $N = \text{Mod}(G \rightarrow K)$ and hence M, N are compatible. \square

NOTATION: Given a geometry $G(X)$, if M is a set of subsets of X , then denote by \overline{M}^G the set $\{ \overline{x}^G \mid x \in M \}$.

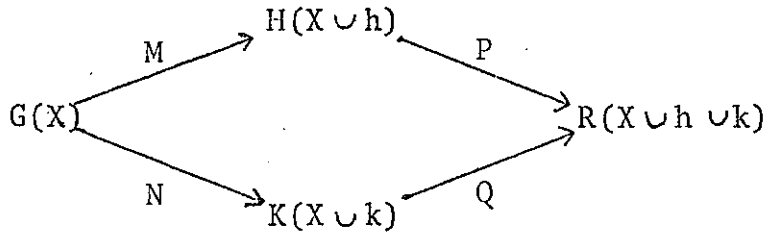
PROPOSITION 2.4 Given elementary extensions:



if N' is the modular filter of H generated by \overline{N}^H and if M' is the modular filter of K generated by \overline{M}^K , then the following are equivalent:

- 1) M, N are compatible
- 2) $\{x \in G \mid \overline{x}^H \in N'\} = N$
- 3) $\{x \in G \mid \overline{x}^K \in M'\} = M$.

Proof: If M, N are compatible, then there exist elementary extensions as in the following diagram:



By proposition 2.2, $\{x \in G \mid \overline{x}^H \in P\} = N$. so $\overline{N}^H \subseteq P$ and therefore $N' \subseteq P$. Thus $\{x \in G \mid \overline{x}^H \in N'\} \subseteq N$. On the other hand, $N = \{x \in G \mid \overline{x}^H \in \overline{N}^H\} \subseteq \{x \in G \mid \overline{x}^H \in N'\}$, so $N = \{x \in G \mid \overline{x}^H \in N'\}$.

Assume now that 2) holds. Let $R(X \cup h \cup k)$ be the elementary extension of $H(X \cup h)$ corresponding to the modular filter N' , and let $F(X \cup k)$ be the subgeometry of $R(X \cup h \cup k)$ on $X \cup k$. Then $x \in \text{Mod}(G \rightarrow F) \iff k \in \overline{x}^F \iff k \in \overline{x}^R \iff \overline{x}^H \in N' \iff x \in N$. So M, N are compatible.

Hence we have established that 1) \iff 2). By symmetry, we also have 1) \iff 3). \square

2.3 Principal modular filters

Given a geometry G , any flat E of G together with all flats of G above it form a modular filter of G , called a principal modular filter and denoted by $\{E\}$ or $\{E\}_G$. $E \longrightarrow \{E\}_G$ is an (order-) embedding from the inverted lattice of G into its lattice of modular filters. This embedding is \wedge -preserving but not \vee -preserving [6].

PROPOSITION 2.5 A principal modular filter of G is compatible with any modular filter of G .

Proof: Let E be any flat of G and let M be any modular filter of G . Let H be the elementary extension of G corresponding to M . $\{\overline{E^H}\}_H$ is the modular filter of H generated by $\overline{\{E\}_G^H}$. For any $x \in G$, $\overline{x^H} \in \{\overline{E^H}\}_H \iff \overline{x^H} \geq \overline{E^H} \iff x \geq E \iff x \in \{E\}_G$, so by proposition 2.4, M is compatible with $\{E\}_G$. \square

2.4 Linear subclass generating sequences

It is clear that the empty modular filter is compatible with every other modular filter. In the rest of this chapter, we will exclude the empty modular filter in our discussion of compatibilities. There then is an exact correspondence between the linear subclasses and the modular filters of a geometry. For a set A of copoints of a geometry G , there is a sequential construction for the linear subclass of G generated by A , a construction which will prove useful in later sections of this chapter.

Given a geometry G , three distinct copoints a, b and c of G are said to form a generating triple if $a \wedge b \wedge c$ is a coline, ie, a, b and c cover $a \wedge b \wedge c$. For any set A of copoints of G , we define a sequence

$$A = A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$$

of sets of copoints of G by the following:

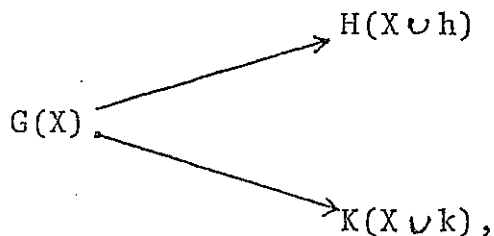
$$x \in A_i \iff x \in A_{i-1} \text{ or } \exists a, b \in A_{i-1} \text{ such that } a, b, x \text{ form a generating triple, } \forall i \geq 1.$$

This sequence is called the generating sequence of A in G . The linear subclass of G generated by A is easily constructed via this sequence, as stated in the following:

PROPOSITION 2.6 If A is a set of copoints in a geometry G , and if $\{A_i\}_{i=0}^{\infty}$ is its generating sequence, then $\bigcup_{i=0}^{\infty} A_i$ is the least linear subclass of G containing A . \square

As a corollary of propositions 2.4 and 2.6, we now have the following:

PROPOSITION 2.7 Given elementary extensions:



let M, N be the linear subclasses of G corresponding to H, K respectively. If M' is the linear subclass of K generated by \overline{M}^K with generating sequence $\{M_i\}_{i=0}$ and if N' is the linear subclass of H generated by \overline{N}^H with generating sequence $\{N_i\}_{i=0}$,

then the following are equivalent:

- 1) M, N are compatible
- 2) $\{x \in G \mid \bar{x}^H \in N'\} = N$
- 3) $\{x \in G \mid \bar{x}^K \in M'\} = M$
- 4) $\{x \in G \mid \bar{x}^H \in N_i\} = N \quad \forall i = 0, 1, 2, \dots$
- 5) $\{x \in G \mid \bar{x}^K \in M_i\} = M \quad \forall i = 0, 1, 2, \dots \quad \square$

2.5 Compatibility theorems

For any linear subclasses $M \supseteq N$ of $G(X)$, we define a sequence

$$\emptyset = K_0(M, N) \subseteq K_1(M, N) \subseteq K_2(M, N) \subseteq \dots$$

of sets of colines of G covered by no copoints in M by the following:

$$x \in K_i(M, N) \setminus K_{i-1}(M, N)$$

\iff x is a coline of G covered by no copoints in M

and $\exists a, b \in N \cup K_{i-1}(M, N)$ such that x covers $a \wedge b$, $i \geq 1$.

We denote the union $\bigcup_{i=0}^{\infty} K_i(M, N)$ by $K(M, N)$.

LEMMA 2.8 Given linear subclasses $M \supseteq N$ of $G(X)$, if $H(X \cup h)$ is the elementary extension of $G(X)$ corresponding to M and if $\{N_i\}_{i=0}$ is the generating sequence of \bar{N}^H in H , then for any i ,

$$\{x \cup h \mid x \in N \cup K_i(M, N)\} \subseteq N_i.$$

Proof: By induction on i . Since $K_0(M, N) = \emptyset$ and $N_0 = \bar{N}^H = \{x \cup h \mid x \in N\}$, so induction starts when $x = 0$. Suppose $\{x \cup h \mid x \in N \cup K_{i-1}(M, N)\} \subseteq N_{i-1}$ and let $a \in K_i(M, N)$. There exist $x, y \in N \cup K_{i-1}(M, N)$ such that a covers $x \wedge y$. But then $a \cup h, x \cup h, y \cup h$ form a generating triple in H . By induction hypothesis, $x \cup h, y \cup h \in N_{i-1}$, so $a \cup h \in N_i$. Thus $\{x \cup h \mid x \in N \cup K_i(M, N)\} \subseteq N_i. \quad \square$

THEOREM 2.9 Two linear subclasses $M \supseteq N$ are incompatible if and only if the following condition holds:

$\exists a, b \in N \cup K(M, N)$ and $c \in M \setminus N$ such that $a \wedge b$ is a coplane not contained in $MF(M)$ and $c > a \wedge b$.

Proof: Let $H(X \cup h)$ be the elementary extension of $G(X)$ corresponding to M and let $\{N_i\}_{i=0}$ be the generating sequence of \overline{N}^H in H .

Assume the condition of the theorem holds. Then $a, b \in N \cup K_i(M, N)$ for some i and so by lemma 2.8, $a \cup h, b \cup h \in N_i$. Since $a \wedge b$ is a coplane of G not contained in $MF(M)$, $(a \wedge b) \cup h$ is a coline in H . $c \cup h$ is a copoint in H and $(a \cup h) \wedge (b \cup h) = (a \wedge b) \cup h < c \cup h$, so $c \cup h \in N_i$. But then by proposition 2.7, M, N are incompatible.

Conversely, assume the condition of the theorem does not hold. We want to show that $N_i \subseteq \{x \cup h \mid x \in N \cup K_i(M, N)\}$, $\forall i$.

By induction on i . When $i = 0$, $K_i = \emptyset$ and $N_i = \overline{N}^H = \{x \cup h \mid x \in N\}$, so inclusion holds. Suppose $N_{i-1} \subseteq \{x \cup h \mid x \in N \cup K_{i-1}(M, N)\}$ and let $c \in N_i \setminus N_{i-1}$. There exist $a, b \in N_{i-1}$ such that a, b, c form a generating triple in H . Consider now the G -flats $a \setminus h, b \setminus h$ and $c \setminus h$. By induction hypothesis, $a \setminus h, b \setminus h \in N \cup K_{i-1}(M, N)$. $(a \setminus h) \wedge (b \setminus h) = (a \wedge b) \setminus h$ must be a coplane of G not contained in $MF(M)$, for otherwise both $a \setminus h$ and $b \setminus h$ are in N and so $c \setminus h$ would also be in N . Now, $c \setminus h$ can only be a coline of G covered by no copoint in M for otherwise if $c \setminus h \in M \setminus N$, then the condition of the theorem would

hold. Thus $c \setminus h \in K_i(M, N)$.

Since $N_i \subseteq \{x \cup h \mid x \in N \cup K_i(M, N)\}$ implies $\{x \in G \mid \bar{x}^H \in N_i\} \subseteq N$, so by proposition 2.7, M, N are compatible. \square

LEMMA 2.10 If M, N, P are linear subclasses of G such that $M \supseteq N \supseteq P$, then for any i ,

$$1) K_i(M, P) \subseteq K_i(N, P)$$

$$\text{and } 2) K_i(M, P) \subseteq K_i(M, N).$$

Proof: 1) By induction on i . When $i = 0$, $K_i(M, P) = \emptyset = K_i(N, P)$. Suppose inclusion holds for $i-1$. Let $x \in K_i(M, P) \setminus K_{i-1}(M, P)$. There exist $a, b \in P \cup K_{i-1}(M, P)$ such that x covers $a \wedge b$. Since x is covered by no copoint in N and since by induction hypothesis $a, b \in P \cup K_{i-1}(N, P)$, so $x \in K_i(N, P)$.

2) Again by induction on i . When $i = 0$, $K_i(M, P) = \emptyset = K_i(M, N)$. Suppose inclusion holds for $i-1$. Let $x \in K_i(M, P) \setminus K_{i-1}(M, P)$. There exist $a, b \in P \cup K_{i-1}(M, P)$ such that x covers $a \wedge b$. By induction hypothesis, $a, b \in N \cup K_{i-1}(M, N)$, so $x \in K_i(M, N)$. \square

THEOREM 2.11 Let M, N, P be linear subclasses of G such that $M \supseteq N \supseteq P$. If M, N are compatible and N, P are compatible, then M, P are compatible.

Proof: Suppose that M, P are not compatible. There exist $a, b \in P \cup K_i(M, P)$ for some i and $c \in M \setminus P$ such that $a \wedge b$ is a coplane not contained in $MF(M)$ and $c > a \wedge b$. If $c \in N$, then N, P are incompatible because $a, b \in P \cup K_i(N, P)$ and $a \wedge b \notin MF(N)$. If $c \notin N$, then M, N are incompatible because $a, b \in N \cup K_i(M, N)$ [Lemma 2.9]. \square

THEOREM 2.12 Two linear subclasses M, N of $G(X)$ are compatible with each other if and only if both M and N are compatible with $M \cap N$.

Proof: If M, N are compatible, then there exists a two-element extension $R(X \cup h \cup k)$ of $G(X)$ such that for any copoint $x \in G$,

$$x \in M \iff h \in \overline{x}^R$$

$$\text{and } x \in N \iff k \in \overline{x}^R.$$

Let $T(X \cup h \cup k \cup t)$ be the elementary extension of $R(X \cup h \cup k)$ corresponding to the linear subclass of R consisting of all copoints of R above the flat $\overline{h \cup k}^R$. The subgeometry $H(X \cup h \cup t)$ of $T(X \cup h \cup k \cup t)$ on $X \cup h \cup t$ exhibits the compatibility between M and $M \cap N$, because for any copoint $x \in G$,

$$x \in M \iff h \in \overline{x}^R \iff h \in \overline{x}^T \iff h \in \overline{x}^H,$$

$$\text{and } x \in M \cap N \iff h, k \in \overline{x}^R \iff \overline{h \cup k}^R \leq \overline{x}^R \iff t \in \overline{x}^H.$$

Similarly, N is compatible with $M \cap N$.

Conversely, assume both linear subclasses M and N are compatible with $M \cap N$. Let $H(X \cup h)$ be the elementary extension of $G(X)$ corresponding to M . Let D be the set of colines of G in $MF(N)$ covered by no copoint in M and let $\{N_i\}_{i=0}^{\infty}$ be the generating sequence of \overline{N}^H in H . For each i , let

$$L_i = \{x \setminus h \mid x \in N_i \setminus \overline{N}^H\}.$$

To complete the proof, we show by induction on i that

1) L_i contains no copoint of G ,

together with

2) $L_i \subseteq D \cup K_i(M, M \cap N)$,

and 3) $L_i \subseteq D \cup K_i(N, M \cap N)$.

Since $N_0 = \overline{N}^H$, so $L_0 = \emptyset$ and induction starts when $i = 0$. Suppose 1),2),3) hold for $i-1$, and let $c \in N_i \setminus N_{i-1}$. There exist $a, b \in N_{i-1}$ such that a, b, c form a generating triple in H .

Suppose $h \notin a$ say. $a \wedge b$ must then be a coline of G . If $h \notin b$, then either $a, b \setminus h, c \setminus h$ form a generating triple in G or $c \setminus h = a \wedge b$, ie, either $c \in \overline{N}^H$, a contradiction, or $c \setminus h \in D$, showing that 1),2),3) hold for i . If $h \in b$, then $a \wedge b = b \setminus h \in D \cup K_i(N, M \cap N)$. If $a \wedge b \in D$, then $c \in \overline{N}^H$, a contradiction, and if $a \wedge b \in K_{i-1}(N, M \cap N)$ then $a \wedge b$ is covered by $a \in N$, again a contradiction.

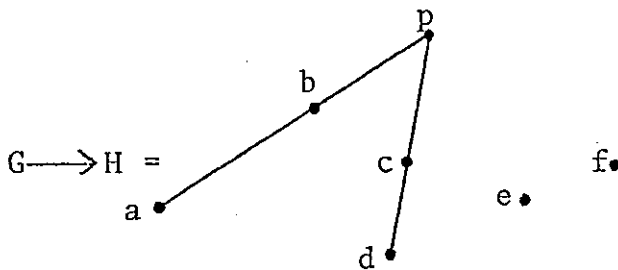
So we can now assume that $h \in a, b$. $(a \setminus h) \wedge (b \setminus h)$ cannot be a coline of G for otherwise $a \setminus h, b \setminus h \in M \cap N$ and so $c \setminus h \in M \cap N$, a contradiction. Also, $(a \setminus h) \wedge (b \setminus h) \notin MF(N)$ for otherwise $c \in \overline{N}^H$, a contradiction.

Furthermore, $a \setminus h, b \setminus h \notin D$. Suppose $a \setminus h \in D$ say. Then $a \setminus h, b \setminus h$ form a modular pair in G . But $b \setminus h \notin N$, so $b \setminus h \in D \cup K_{i-1}(N, M \cap N)$. If $b \setminus h \in D$, then $(a \setminus h) \wedge (b \setminus h) \in MF(N)$, a contradiction; and if $b \setminus h \in K_{i-1}(N, M \cap N)$, then $b \setminus h$ is covered the the copoint $(a \setminus h) \vee (b \setminus h)$ in N , also a contradiction.

So we have shown that both $a \setminus h$ and $b \setminus h$ are contained in both $(M \cap N) \cup K_{i-1}(M, M \cap N)$ and $(M \cap N) \cup K_{i-1}(N, M \cap N)$ such that $(a \setminus h) \wedge (b \setminus h)$ is a coplane of G contained in neither $MF(M)$ nor $MF(N)$. If $c \setminus h$ is a copoint of G , then $c \setminus h \in M \setminus N$ and so by theorem 2.9, M, N are incompatible. Thus $c \setminus h$ is a coline of G covered by no copoint in M . Thus $c \setminus h \in K_i(M, M \cap N)$. Since $N, M \cap N$ are compatible, so $c \setminus h$ is covered by no copoint in $N \setminus M$ and therefore $c \setminus h \in K_i(N, M \cap N)$. Hence 1),2),3) hold for i . \square

III. ADJOINTNESS OF EXTENSIONS

In this chapter, we will discuss, for a given extension $G \rightarrow H$, the adjointness relation between extensions of G and extensions of H . To illustrate the idea, let us consider the effect which adding an extra point p to G has on the existence of further extensions of G . First of all, it is clear that the elements or objects defining relative positions for further extensions are increased. On the other hand, the possibility of extensions originally definable in G may be cut down. For example, let G be the plane geometry of six points a, b, c, d, e, f in general position and suppose a point p is added to G on the lines ab and cd , as shown in the following diagram:



In the extension H , there are more definable positions for putting points, e.g., we can put a point on the lines ac and ep , or on the lines ac and fp ; but it is impossible to put a point on the lines ab , cd and ef which do in G define a relative position for a point extension.

LEMMA 3.1 Given extensions as shown in the following diagram:

$$\begin{array}{ccc}
K(X \cup e) & \longrightarrow & R(Y \cup e) \\
\uparrow M & & \uparrow N \\
G(X) & \xrightarrow{i} & H(Y)
\end{array}$$

if M, N are modular filters corresponding to the elementary extensions $G \rightarrow K$ and $H \rightarrow R$ respectively, then $M = i^{-1}(N)$.

Proof: Let $x \in G$. Then

$$\begin{aligned}
x \in M &\iff e \in \overline{x}^K \\
&\iff e \in \overline{x}^R \\
&\iff i(x) \in N.
\end{aligned}$$

So $M = i^{-1}(N)$. \square

COROLLARY 3.2 Given any extension $G \xrightarrow{i} H$, if M is a modular filter of H , then $i^{-1}(M)$ is a modular filter of G . \square

In what follows, we will assume some familiarity with Galois connections between two ordered sets.

Given an extension $G \xrightarrow{i} H$, let $E(G)$ and $E(H)$ denote the lattices of modular filters of G and H respectively. Two functions

$$E(G) \begin{array}{c} \xrightarrow{\sigma} \\ \xleftarrow{\tau} \end{array} E(H)$$

are defined as follows:

$$\sigma(M) = \text{the modular filter of } H \text{ generated by } i(M) \quad \forall M \in E(G)$$

$$\text{and } \tau(N) = i^{-1}(N) \quad \forall N \in E(H).$$

It is immediate that both σ and τ are order-preserving; moreover, for any $M \in E(G), N \in E(H)$,

$$\begin{aligned}
M \subseteq \tau(N) &\iff M \subseteq i^{-1}(N) \\
&\iff i(M) \subseteq N \\
&\iff \sigma(M) \subseteq N.
\end{aligned}$$

Thus σ, τ form a Galois connection between the lattices $E(G)$ and $E(H)$. As a consequence, $M \rightarrow \tau\sigma(M)$ is a closure operator on $E(G)$. $M = \tau\sigma(M)$ if and only if $M = \tau(N)$ for some $N \in E(H)$; such a modular filter is said to be Galois-closed with respect to the extension $G \rightarrow H$. The Galois-closed modular filters form a sublattice of $E(G)$. For any modular filter $M \in E(G)$, the set $\{N \in E(H) \mid \tau(N) = M\}$ is a sub-semilattice of $E(H)$.

The following theorem links this Galois connection treatment to the compatibility of extensions.

THEOREM 3.3 Given extensions as shown in the following diagram:

$$\begin{array}{ccc} & K(X \cup e) & \\ M \uparrow & & \\ G(X) & \xrightarrow{i} & H(Y) \end{array}$$

where $G \rightarrow K$ is elementary with modular filter M , the extensions H, K are compatible with each other if and only if M is Galois-closed with respect to the extension $G \rightarrow H$.

Proof: If the extensions H, K are compatible with each other, we have the following extension diagram:

$$\begin{array}{ccc} K(X \cup e) & \longrightarrow & R(Y \cup e) \\ \uparrow & & \uparrow \\ G(X) & \xrightarrow{i} & H(Y) \end{array}$$

Let $N = \text{Mod}(H \rightarrow R)$. By proposition 3.1, $M = i^{-1}(N)$, i.e., $M = \tau(N)$. Thus M is Galois-closed.

Conversely, if M is Galois-closed, then $M = i^{-1}(N)$ for some modular filter N of H . Let $R(Y \cup e)$ be the elementary extension of H corresponding to the modular filter N . By proposition 3.1 again, $K(X \cup e)$ is the subgeometry of $R(Y \cup e)$ on $X \cup e$. Hence $H(Y)$ and $K(X \cup e)$ are compatible with each other. \square

If in the above discussion, the extension $G \rightarrow H$ is elementary with modular filter N , then for any modular filter M of G , $\tau\sigma(M)$ is called the N -closure of M , and denoted by \bar{M}^N . The following two theorems are immediate from the previous discussions.

THEOREM 3.4 Given a modular filter N of a geometry G ,

- 1) for any modular filter M of G , there is a least modular filter \bar{M}^N of G such that \bar{M}^N is compatible with N ;
- 2) $M \rightarrow \bar{M}^N$ is a closure operator on the lattice of modular filters of G ;
- 3) the modular filters of G which are compatible with N form a closure system in $E(G)$. \square

THEOREM 3.5 Given two elementary extensions

$$\begin{array}{ccc} & K & \\ & \uparrow & \\ & G & \longrightarrow H, \end{array}$$

the common elementary extensions of both H and K form a semi-lattice (in the weak order). \square

Any partially ordered set is also a category if we take its elements to be the objects of the category, its ordered pairs to be the morphisms of the category. For any extension $G(X) \rightarrow H(Y)$ and a set Z , we consider two categories: 1), the category $E_Z(G)$ of extensions of G by Z , in the weak order, 2), the category $E_Z(H)$ of extensions of H by Z , in the weak order.

The retraction functor Ret

$$E_Z(H) \xrightarrow{\text{Ret}} E_Z(G)$$

is defined as follows:

for any $R(Y \cup Z) \in E_Z(H)$, $\text{Ret}(R)$ = the subgeometry of R on $X \cup Z$ (which is an extension of G by Z).

If Z is a one-element set, the functor Ret has an adjoint, as we discussed above. In general, the adjoint of Ret fails to exist.

EXAMPLE 3.6 Let G be the rank-four geometry on nine points $1, 2, 3, \dots, 9$ in general position. Let H, K, K' be extensions of G and let R, R' be extensions of H as shown in figure 3.1. If the functor $E_Z(H) \xrightarrow{\text{Ret}} E_Z(G)$ were to have an adjoint, say σ , for $Z = \{a, b\}$, then the extension $R'' = \sigma(K)$ of H would have to satisfy $R'' = \sigma(K) \leq R, R'$ because $K \leq \text{Ret}(R), \text{Ret}(R')$, and $\text{Ret}(R'') = \text{Ret}(\sigma(K)) \geq K$. But there is no object R'' in $E_Z(H)$ satisfying $R'' \leq R, R'' \leq R'$ and $\text{Ret}(R'') \geq K$.

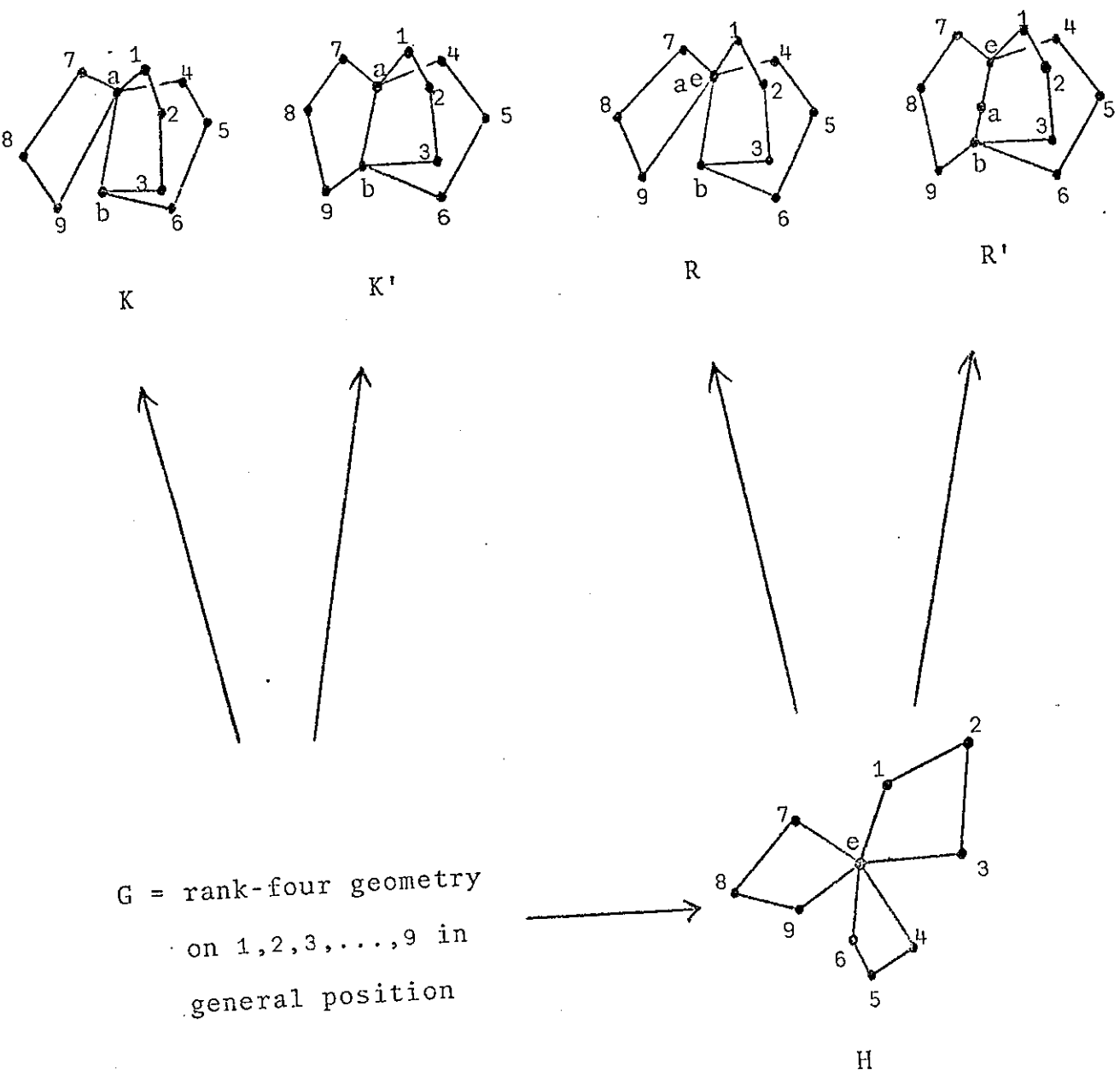


figure 3.1

IV. REPRESENTATIONS OF QUOTIENTS

4.1 Quotient bundles

The quotient bundle was introduced and studied extensively in [4] by the author jointly with H. Crapo; it provides a complete description of extensions (up to isomorphisms) of any given geometry. A concise exposition of the theory is given in this section; the omitted proofs could be found in [4].

We shall consider geometries G and F together with G -quotients $Q(a)$ indexed by flats $a \in F$. For any F -flats $a \leq b$, we shall require that $Q(b)$ is a quotient of $Q(a)$, and also, we shall have the inequality

$$(*) \quad \deg(Q(a) \rightarrow Q(b)) \leq r(b) - r(a);$$

a modular filter $M(a,b)$ of $Q(a)$ is defined by the following:

$$\begin{aligned} M(a,b) &= M_{r(b)-r(a)}(Q(a) \rightarrow Q(b)) \\ &= \left\{ x \in Q(a) \mid n_{Q(a) \rightarrow Q(b)}(x) = r(b) - r(a) \right\}. \end{aligned}$$

Note that if equality holds in (*), then $M(a,b)$ is just the modular filter $M(Q(a) \rightarrow Q(b))$; otherwise $M(a,b) = \emptyset$.

THEOREM 4.1 If R is an extension of a geometry G by a geometry F , the family $\{ Q(a) \mid a \in F \}$ of quotients defined by

$$Q(a) = \text{Quo}(G \rightarrow R \rightarrow R/a)$$

satisfies the following two properties:

- 1) $Q(0) = G$; and if a flat b covers a flat a in F , then $Q(b)$ is an elementary quotient of $Q(a)$.
- 2) For any locally modular pair of flats a, b in F ,

$$M(a \wedge b, a) \cap M(a \wedge b, b) = M(a \wedge b, a \vee b). \quad \square$$

We use these properties to define a quotient bundle. Specifically, we define an F-indexed bundle of quotients of a geometry G to be a family $\{Q(a) \mid a \in F\}$ of quotients of G, satisfying conditions 1) and 2) of theorem 4.1. Abbreviations such as F-bundles on G are also appropriate.

THEOREM 4.2 Any quotient bundle on a geometry G, indexed by the flats of a geometry F, is the bundle of a unique extension of G by F. \square

The following theorem shows how the rank-function as well as the flats of the corresponding extension are determined by a quotient bundle.

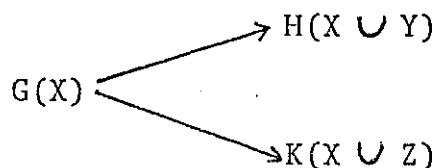
THEOREM 4.3 Any extension $R(X \cup Y)$ of a geometry $G(X)$ by a geometry $F(Y)$ is a quotient of the direct sum $G \oplus F$. Assume that $\{Q(a) \mid a \in F\}$ is the quotient bundle for an extension R of G by F. Then for any $(x,a) \in G \oplus F$,

- 1) $r_R(x,a) = r_{Q(a)}(x) + r_F(a)$
- 2) $\overline{(x,a)}^R = (y,b)$, where y is the least flat in $Q(a)$ above x, and b is the greatest among those flats $c \in F$ with the property that $y \in M(a,c)$. \square

An F-bundle $\{Q(a) \mid a \in F\}$ on G is said to be strict if and only if any one of the following equivalent statements holds,

- 1) $\deg(G \rightarrow Q(a)) = r(a)$ for all $a \in F$
- 2) for any $a, b \in F$, if $a \leq b$, then $\deg(Q(a) \rightarrow Q(b)) = r(b) - r(a)$
- 3) $M(a,b) \neq \emptyset$ for all $a \leq b \in F$
- 4) the extension of G determined by the F-bundle has the same rank as G.

In concluding this section, we look at T. Brylawski's pushout theorem [2]: given any pair of extensions



such that X is a modular flat of H , the pushout of H and K relative to G (in the category of geometries and strong maps) exists. If we let $R(X \cup Y \cup Z)$ be the pushout and let $F(Y)$ be the subgeometry of H on Y , then R is an extension of K by F . The F -bundle $\{Q(a)\}_{a \in F}$ on K corresponding to this extension is given by the following:

$$Q(a) = K/\frac{H}{a} \cap X.$$

4.2 Lift and drop sequences

In this section, we introduce the concept of quotient sequences as well as the lift and drop constructions as studied in [4] and develop the mathematical background for the representation theory of quotients later on.

For any F-bundle $\{ Q(a) \mid a \in F \}$ on a geometry G, any chain of F-flats

$$C: a_0 \leq a_1 \leq a_2 \leq \dots \leq a_k$$

give a sequence (=chain) of quotients of G:

$$Q(C): Q(a_0) \rightarrow Q(a_1) \rightarrow Q(a_2) \rightarrow \dots \rightarrow Q(a_k).$$

If the F-bundle is strict, then any maximal chain will give a sequence of proper elementary quotients.

The lift construction of quotients was first invented by D. Higgs in order to prove Edmonds' strong map factorization theorem: every strong map can be factored as an injection followed by a surjection. If Q is a quotient of a geometry G, the lift of Q toward G, written

$$L = L(G \rightarrow Q),$$

consists of precisely those flats $x \in G$ such that

$$x \in Q \text{ or } n_{G \rightarrow Q}(x) = 0$$

that is, the Q-flats together with all (G \rightarrow Q)-independent flats of G. The lift L is a quotient of G, so we have a sequence of quotients:

$$G \rightarrow L \rightarrow Q,$$

with $r(L) = r(Q) + 1$ unless $G = Q$. Since L is a quotient of

G, we may iterate the lift. Thus the kth lift $L^k(G \rightarrow Q)$ of Q toward G can be defined by

$$L^k(G \rightarrow Q) = L(G \rightarrow L^{k-1}(G \rightarrow Q))$$

along with $L^0(Q \rightarrow Q) = Q$.

Most proofs in this section are omitted, and the reader is referred to [4].

PROPOSITION 4.4 For any quotient sequence

$$G \rightarrow H \rightarrow L \rightarrow Q,$$

if L is the lift of Q toward G, then L is also the lift of Q toward H. \square

If Q is a quotient of G and $\text{deg}(G \rightarrow Q) = k$ say, then by lifting Q toward G k-1 times, we obtained a sequence of proper elementary quotients:

$$G = Q_0 \rightarrow Q_1 \rightarrow Q_2 \rightarrow \dots \rightarrow Q_k = Q,$$

called the strict lift sequence from G to Q, where

$$Q_{i-1} = L(G \rightarrow Q_i) \quad \text{for any } 1 \leq i \leq k.$$

The lift sequence will be used very frequently in this paper; it provides a standard construction for quotient bundles in certain interesting cases.

PROPOSITION 4.5 In any quotient sequence

$$G \rightarrow P \rightarrow Q,$$

$$M(G \rightarrow Q) = M(G \rightarrow P) \cap M(P \rightarrow Q). \quad \square$$

THEOREM 4.6 A sequence

$$Q_0 \rightarrow Q_1 \rightarrow Q_2 \rightarrow \dots \rightarrow Q_k$$

of proper elementary quotients is a strict lift sequence if and only if any of the following equivalent statements holds,

- 1) $M(Q_{i-1} \twoheadrightarrow Q_i) \subseteq M(Q_i \twoheadrightarrow Q_{i+1})$ for any $1 \leq i < k$,
- 2) $M(Q_{i-1} \twoheadrightarrow Q_i) = M(Q_{i-1} \twoheadrightarrow Q_{i+1})$ for any $1 \leq i < k$,
- 3) $M(Q_{i-1} \twoheadrightarrow Q_i) = M(Q_{i-1} \twoheadrightarrow Q_k)$ for any $1 \leq i < k$.

THEOREM 4.7 Any interval of a strict lift sequence is also a strict lift sequence \square

Every strict lift sequence

$$G=Q_0 \twoheadrightarrow Q_1 \twoheadrightarrow Q_2 \twoheadrightarrow \dots \twoheadrightarrow Q_k=Q$$

is completely determined by its terminal members G and Q .

As we are now going to show, the same sequence can be produced by another construction which proceeds from left to right, beginning from G . For any quotient Q of a geometry G , the drop of G toward the quotient Q , denoted $D(G \twoheadrightarrow Q)$ is defined to be the unique elementary quotient of G with modular filter $M(G \twoheadrightarrow Q)$, that is,

$$D(G \twoheadrightarrow Q) = \left\{ x \in G \mid n_{G \twoheadrightarrow Q}(x) \neq \deg(G \twoheadrightarrow Q) - 1 \right\}.$$

For any positive integer k , the k^{th} drop of G toward Q , denoted $D^k(G \twoheadrightarrow Q)$, is defined by

$$D^k(G \twoheadrightarrow Q) = D(D^{k-1}(G \twoheadrightarrow Q) \twoheadrightarrow Q)$$

along with $D^0(G \twoheadrightarrow Q) = G$. The following is an analogue of proposition 4.4:

PROPOSITION 4.8 For any quotient sequence

$$G \twoheadrightarrow D \twoheadrightarrow H \twoheadrightarrow Q$$

if D is the drop of G toward Q , D is also the drop of G toward H . \square

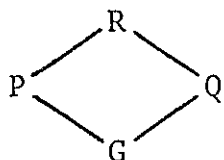
A strict drop sequence from G to a quotient Q is a sequence of proper elementary quotients

$$G=Q_0 \rightarrow Q_1 \rightarrow \dots \rightarrow Q_k=Q$$

such that $Q_i = D^i(G \rightarrow D)$ for all $1 \leq i \leq k$.

THEOREM 4.9 Every strict lift sequence is a strict drop sequence, and vice versa. \square

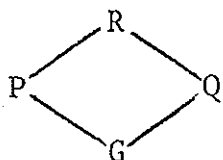
We call an order-preserving function from a partially ordered set I to $\mathcal{L}_f(X)$ (with the strong map order) a quotient diagram, which can be indicated by labelling the elements of I with their images in $\mathcal{L}_f(X)$. For example, the following diagram of elementary quotients, indexed by the free geometry on two points.



represents a quotient bundle if and only if

$$M_1(G \rightarrow P) \cap M_1(G \rightarrow Q) = M_2(G \rightarrow R).$$

PROPOSITION 4.10 Given quotients as shown in the following diagram

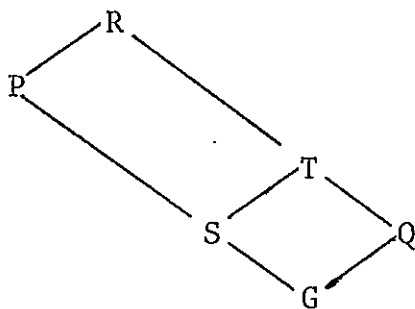


with $\deg(G \rightarrow Q) = \deg(P \rightarrow R) = 1$, $D(Q \rightarrow R)$ is a (proper elementary) quotient of $D(G \rightarrow P)$ if the following condition is satisfied:

$$M(G \rightarrow P) \cap M(G \rightarrow Q) = M(G \rightarrow R).$$

Proof: Let $x \in D(Q \rightarrow R)$. If $x \in M(Q \rightarrow R)$, then $x \in R \subseteq D(G \rightarrow P)$. So we assume $x \notin M(Q \rightarrow R)$. Let $n = \deg(G \rightarrow P)$. If $x \notin M(G \rightarrow Q)$, then $n_{G \rightarrow P}(x) \leq n_{G \rightarrow R}(x) \leq n-2$ and so $x \in D(G \rightarrow P)$. If otherwise $x \in M(G \rightarrow Q)$, then $n_{G \rightarrow H}(x) \leq n-1$. Suppose $x \notin D(G \rightarrow P)$. x is then covered by some $x' \in M(G \rightarrow P)$. But then $x' \in M(G \rightarrow Q)$ and so $x' \in M(G \rightarrow R)$, ie, $n_{G \rightarrow R}(x) = n-1$, and therefore $n_{G \rightarrow R}(x) \geq n$, a contradiction. Thus $D(Q \rightarrow R) \subseteq D(G \rightarrow P)$ and so $D(Q \rightarrow R)$ is a quotient of $D(G \rightarrow P)$. \square

REMARK 4.11 In the above proposition, although $T = D(Q \rightarrow R)$ is a quotient of $S = D(G \rightarrow P)$, as shown in the following diagram



the intersection $M(S \rightarrow P) \cap M(S \rightarrow T)$ may not be equal to $M(S \rightarrow R)$ and so $D(T \rightarrow R)$ may not be a quotient of $D(S \rightarrow P)$, ie, $D^2(Q \rightarrow R)$ may not be a quotient of $D^2(G \rightarrow P)$. Also, $L(Q \rightarrow R)$ may not be a quotient of $L(G \rightarrow P)$. For example, consider the quotient diagram in figure 4.1.

figure 4.1

Since $M(S \rightarrow P) = [ab]$, $M(S \rightarrow T) = [a]$, $M(S \rightarrow R) = \{abcd\}$, so $M(S \rightarrow P) \cap M(S \rightarrow T) = [ab] \neq M(S \rightarrow R)$. Thus $L(Q \rightarrow R) = D^2(Q \rightarrow R)$ is not a quotient of $L(G \rightarrow P) = D^2(G \rightarrow P)$.

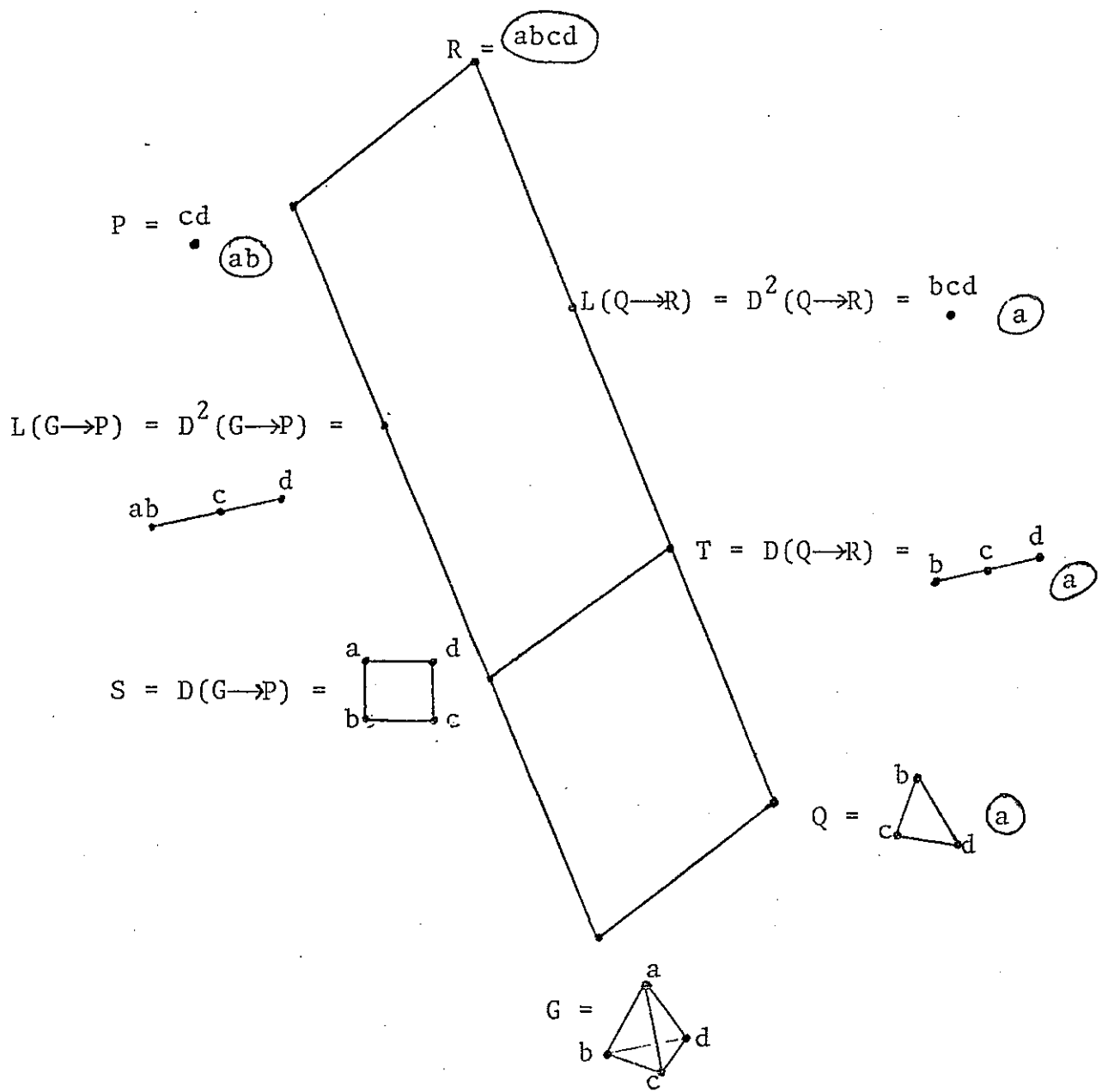


figure 4.1

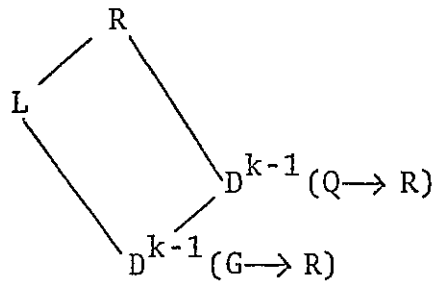
As a particular case of proposition 4.10, we have the following:

PROPOSITION 4.12 Given a quotient sequence

$$G \rightarrow Q \rightarrow R$$

with $\deg(G \rightarrow Q) = 1$, $D^k(Q \rightarrow R)$ is a proper elementary quotient of $D^k(G \rightarrow R)$ for any $0 \leq k \leq \deg(Q \rightarrow R)$.

Proof: By induction on k . Since $D^0(Q \rightarrow R) = Q$, $D^0(G \rightarrow R) = G$, so $D^0(Q \rightarrow R)$ is a proper elementary quotient of $D^0(G \rightarrow R)$. Assume that $D^{k-1}(Q \rightarrow R)$ is a proper elementary quotient of $D^{k-1}(G \rightarrow R)$. Consider the following quotient diagram



where $L = L(G \rightarrow R)$. Since

$$\begin{aligned}
 & M(D^{k-1}(G \rightarrow R) \rightarrow L) \cap M(D^{k-1}(G \rightarrow R) \rightarrow D^{k-1}(Q \rightarrow R)) \\
 & \subseteq M(D^{k-1}(G \rightarrow R) \rightarrow L) \\
 & \subseteq M(D^{k-1}(G \rightarrow R) \rightarrow R),
 \end{aligned}$$

so by proposition 4.10, $D(D^{k-1}(Q \rightarrow R) \rightarrow R)$ is a proper elementary quotient of $D(D^{k-1}(G \rightarrow R) \rightarrow L)$, that is, $D^k(Q \rightarrow R)$ is a proper elementary quotient of $D^k(G \rightarrow R)$. \square

4.3 Representations of a single quotient of a geometry

A representation of a quotient Q of a geometry $G(X)$ is an extension $R(X \cup Y)$ of G together with a flat z of R , called the representing flat, such that

$$\text{Quo}(G \rightarrow R \rightarrow R/z) = Q.$$

Without loss of generality, we can assume that $z = \bar{y}^R$.

As a result of theorem 4.3, we know that in such a representation, the flat z is in a position relative to all flats $x \in G$ prescribed by the following equation:

$$r(x \vee z) = r_Q(x) + r(z).$$

In particular, the inequality

$$(*) \text{deg}(G \rightarrow Q) \leq r(z)$$

is always satisfied. If equality holds in (*), then the representation is said to be strict. D. Higgs first proved that any quotient of a geometry has a strict representation [1].

Let Q be a quotient of a geometry G and let F be any geometry of rank $r \geq \text{deg}(G \rightarrow Q)$. The lift F -bundle of Q toward G is the F -bundle $\{Q(a)\}_{a \in F}$ on G defined by

$$Q(a) = L^{r-r(a)}(G \rightarrow Q) \quad \forall a \in F.$$

The extension R determined by this bundle is a representation of the quotient Q . If $r = \text{deg}(G \rightarrow Q)$, then R is the Higgs representation [4].

4.4 Simultaneous representations of quotients of a geometry

A representation of a family $\{Q_i\}_{i \in I}$ of quotients of a geometry G is an extension R of G together with a family $\{z_i\}_{i \in I}$ of flats of R , called the representing flats of the representation, such that for any $i \in I$,

$$\text{Quo}(G \rightarrow R \rightarrow R/z_i) = Q_i.$$

In such a representation, the following equation holds for each $i \in I$:

$$r(x \vee z_i) = r_{Q_i}(x) + r(z_i) \quad \forall x \in G.$$

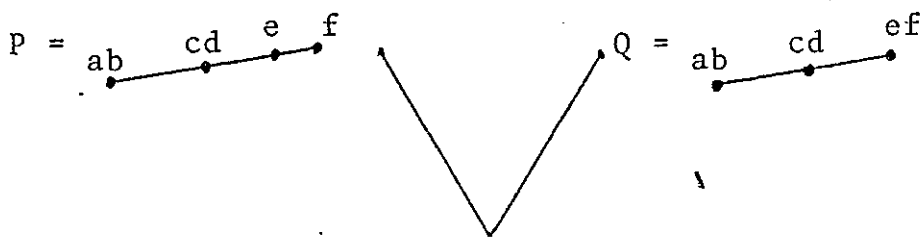
Also, for each $i \in I$, the inequality

$$(*) \text{deg}(G \rightarrow Q_i) \leq r(z_i)$$

is satisfied. If equality holds in (*) for all $i \in I$, then the representation is said to be strict.

If P, Q are elementary quotients of a geometry G , they are strictly representable if and only if their corresponding elementary extensions are compatible. It is possible that two elementary quotients of a geometry are representable even though they are not strictly representable.

EXAMPLE 4.13 Let P, Q be elementary quotients of G as shown in figure 4.2. The elementary extensions corresponding to P, Q are incompatible (page 2.1). Consider the geometry in figure 4.3, where the non-trivial planes are $abcdef, abgh, abij, cdgh, cdij, efij$ and $ghij$. R is an extension of G and $\text{Quo}(G \rightarrow R \rightarrow R/gh) = P, \text{Quo}(G \rightarrow R \rightarrow R/ij) = Q$. So the pair of extensions P, Q are representable.



G = rank-three geometry on $\{a,b,c,d,e,f\}$
 in general position

figure 4.2

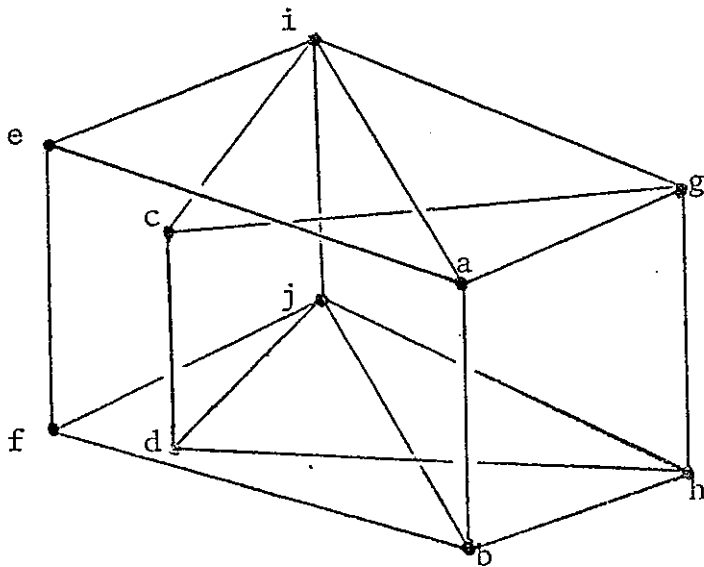


figure 4.3

While the general representation problem has still been unsettled, there is a positive result for representing elementary quotients. In the proof of the following theorem, we construct a suitable quotient bundle to guarantee the existence of a representation.

THEOREM 4.14 Any pair of elementary quotients P, Q of G is representable.

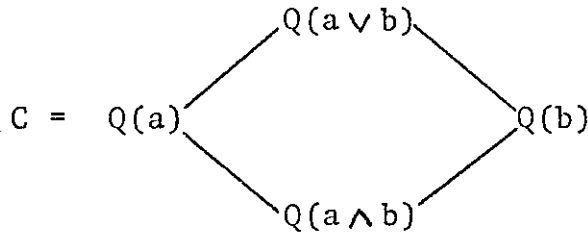
Proof: Without loss of generality, we assume that neither P nor Q is equal to G . There is some quotient R of both P and Q . Let $n = \deg(G \rightarrow R)$ and let F be the truncation of the free geometry on points $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n$. Denote the flats $a_1 a_2 \dots a_n$ and $b_1 b_2 \dots b_n$ by A and B respectively. For each $a \in F$, define

$$Q(a) = \begin{cases} L^{2n-1-r(a)} (P \rightarrow R) & \text{if } a \geq A \\ L^{2n-1-r(a)} (Q \rightarrow R) & \text{if } a \geq B \\ L^{2n-1-r(a)} (G \rightarrow R) & \text{otherwise.} \end{cases}$$

It is clear that $Q(A) = P$ and $Q(B) = Q$. We claim that $\{Q(a)\}_{a \in F}$ is an F -bundle on G .

Let $a, b \in F$ be such that b covers a . It is obvious that if $a \geq A$ or $a \geq B$, or if $B \not\geq A, B$, $Q(b)$ is an elementary quotient of $Q(a)$. If otherwise $a \not\geq A, B$ and $b \geq A$ say, then $Q(a) = L^{2n-1-r(a)} (G \rightarrow R)$ and $Q(b) = L^{2n-1-r(b)} (P \rightarrow R)$ and so by proposition 4.12 $Q(b)$ is an elementary quotient of $Q(a)$.

Next, let a, b be a locally modular pair in F . We want to show that in the quotient diagram:



$$(*) \quad M(a \wedge b, a) \cap M(a \wedge b, b) = M(a \wedge b, a \vee b).$$

It is not possible that $a \geq A$ and $b \geq B$. If this happens, $a \vee b = 1_F$ and so a, b are copoints of F . But then $r(a \wedge b) = |a \wedge b| \leq 2n-2$ and thus $a \wedge b$ cannot be a coplane of F , a contradiction. Similarly, it is not possible that $a \geq B$ and $b \geq A$. If $a \wedge b \geq A$ (or B), then the quotient diagram C lies in a lift bundle from R toward P (Q respectively) and so $(*)$ is satisfied. Finally, consider the case when $a \wedge b, a, b \not\geq A, B$. If $a \vee b \not\geq A, B$, the quotient diagram C lies in the lift bundle from R toward G and so $(*)$ is satisfied. If otherwise $a \vee b \geq A$ (or B), then $M(a \wedge b, a) \cap M(a \wedge b, b) = M(a \wedge b, a) = M(a \wedge b, 1_F) \subseteq M(a \wedge b, a \vee b)$ and so $(*)$ is satisfied.

In the extension R of G determined by the F -bundle $\{Q(a)\}_{a \in F}$,

$$\text{Quo}(G \rightarrow R \rightarrow R/A) = Q(A) = P$$

$$\text{and } \text{Quo}(G \rightarrow R \rightarrow R/B) = Q(B) = Q.$$

Thus the pair of elementary quotients P, Q is representable. \square

4.5 Partial Bundles

This section introduces a new manner of representing quotients of a geometry, and lays the groundwork for the remaining chapters. This idea is most clearly expressed in the language of "relative position", as follows. If a quotient Q of a geometry G is represented by a flat z in an extension of G , we say the quotient Q determines "the position of z relative to the geometry G ". It frequently happens that we have two geometries G and F , and wish to find a common extension of G and F in which certain flats z_i of F have certain positions relative to G , given by quotients Q_i of G . Such an extension of G by F exists if and only if the correspondence

$$z_i \rightarrow Q_i$$

can be completed to an F -indexed bundle of quotients of G .

It is impossible that there are flats $z_i \leq z_j$ in F for which Q_j is not a quotient of Q_i , so we may as well assume that

$$z_i \leq z_j \text{ implies } Q_j \text{ is a quotient of } Q_i.$$

Let us recall that a diagram of quotients (indexed by an ordered set I) is an order-preserving assignment of quotients of G to the elements of I . We are thus led naturally to the problem: "given a geometry F and a diagram of quotients of G indexed by a subset I of the set of flats of F , when can the diagram be extended to an F -indexed bundle of quotients of G ?"

Think, for example, what happens if for each flat $z \in I$, the quotient Q_z has degree equal to the rank of z . Then any completion to an F -bundle will be a strict representation of

these quotients, as discussed in the preceding section. The present formulation of the problem will also guarantee that the representing flats have given positions relative to one another, a property not guaranteed in an arbitrary strict representation. For instance, three elementary quotients may be strictly represented by three independent points, or by three colinear points.

Given geometries G and F , a family of quotients of G

$$\{Q(a)\}_{a \in I}$$

indexed by a set I of F -flats is called a partial bundle [4] on G if it can be completed to an F -bundle on G . Here we give a non-trivial example of an I -indexed family of quotients which is not a partial bundle.

EXAMPLE 4.15 Let F be the free geometry on three points 1,2,3 and let $I = \{0,12,23,123\}$. Consider the quotients shown in figure 4.4.

figure 4.4

If $Q(I)$ is completable to an F -bundle $\{Q(a)\}_{a \in F}$ on G , then the quotient $Q(2)$ would be equal to one of the three quotients in figure 4.5

figure 4.5

But for each $i = 1,2,3$,

$$M(P_i \rightarrow Q(12)) \cap M(P_i \rightarrow Q(23)) \neq M(P_i \rightarrow Q(123)),$$

a contradiction. Thus $Q(I)$ is not a partial bundle.

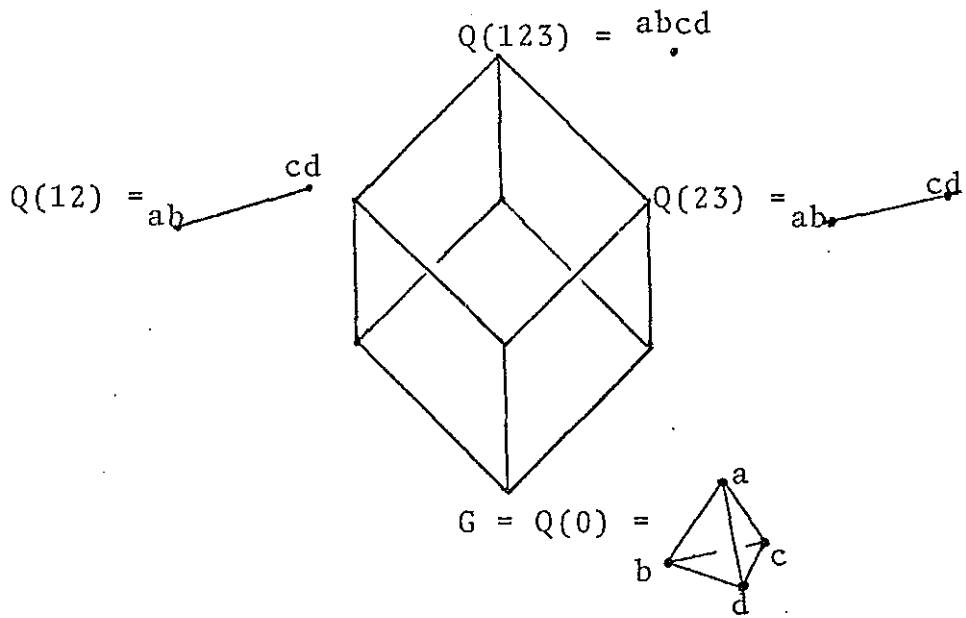


figure 4.4

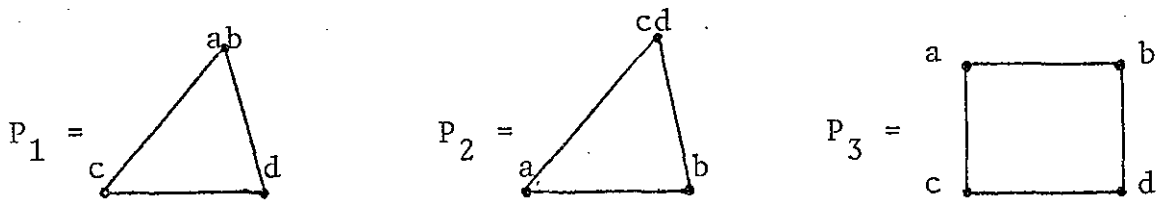


figure 4.5

V. COMPLETIONS OF PARTIAL BUNDLES

5.1 Weak Order on Partial Bundles

Given any set I of flats of a geometry F , a family of geometries on X $\{Q(a)\}_{a \in I}$ will simply be denoted by $Q(I)$. Further, by an F -bundle on X , or just an F -bundle, we understand an F -bundle on $G(X)$, where $G(X)$ is some geometry on X . Recall that we denote by $\mathcal{G}(X)$ the set of all geometries on X together with the weak order. Each F -bundle is naturally an element of \mathcal{G}^F , the F -fold product of $\mathcal{G}(X)$ together with the component-wise order. Thus any possible completion of $Q(I)$ to an F -bundle is an element of \mathcal{G}^F .

The weak order on bundles is consistent with that on the corresponding extensions, as we now show in the following:

PROPOSITION 5.1 Given geometries $G(X)$ and $F(Y)$, let $Q_i(F)$ be F -bundles on G and $R_i(X \cup Y)$ be the corresponding extensions, $i = 1, 2$. Then

$$R_1 \leq R_2 \iff Q_1(a) \leq Q_2(a) \text{ for all } a \in F.$$

Proof: The rank equation in theorem 4.3 says that for each i

$$r_{R_i}(xua) = r_{Q_i(a)}(x) + r_F(a) \quad \forall x \in G, a \in F.$$

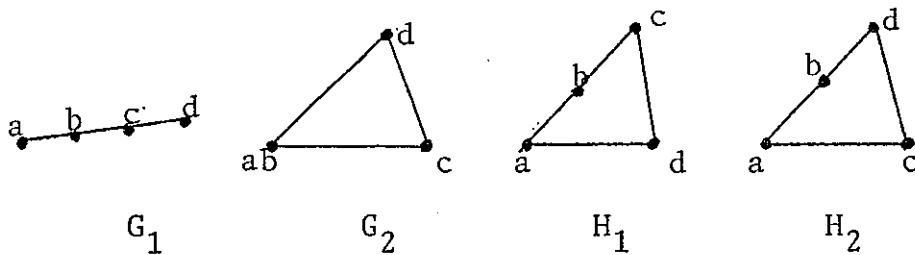
$$\text{Thus } R_1 \leq R_2 \iff r_{R_1}(xua) \leq r_{R_2}(xua) \quad \forall x \in G, a \in F$$

$$\iff r_{Q_1(a)}(x) \leq r_{Q_2(a)}(x) \quad \forall x \in G, a \in F$$

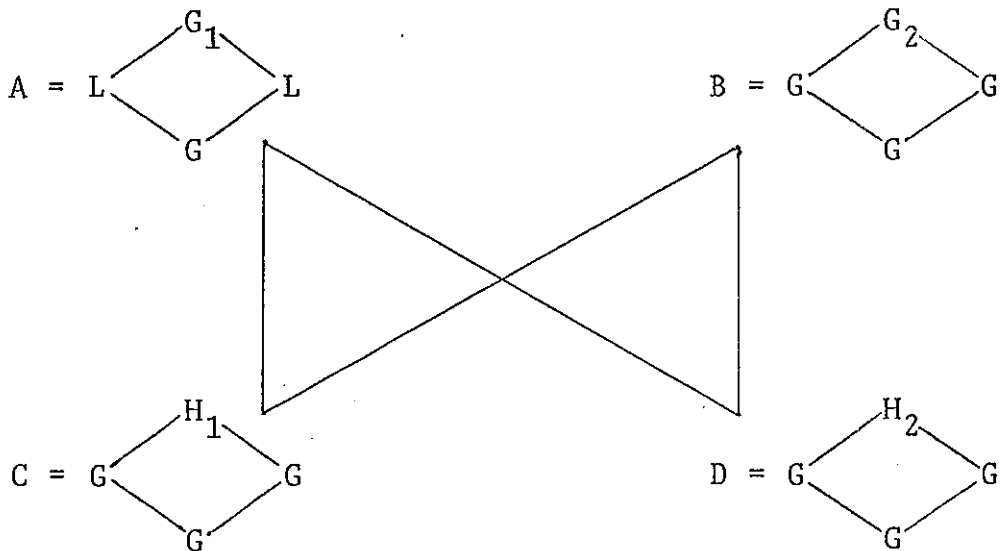
$$\iff Q_1(a) \leq Q_2(a) \quad \forall a \in F. \quad \square$$

Given a set I of F -flats and a partial bundle $Q(I)$ on X , the completions of $Q(I)$ to F -bundles need not form a semilattice under the induced weak order in $\mathcal{L}_f(X)^F$.

EXAMPLE 5.2 Let F be the geometry of a two-point line and let $I = \{0\}$ consist of only the zero-flat of F . Let $Q(0) = G$ be the free geometry on four points a, b, c, d . Consider the geometries G_1, G_2, H_1, H_2 in example 1.1:



Let $L = L(G - G_1)$. The following F -bundles are completions of $Q(I)$:



Both C and D are maximal completions weaker than A and B ; so the meet of A, B does not exist.

5.2 Duality of strict 2-bundles

If F is the geometry of a two-point line, then an F -bundle is called a 2-bundle. A strict 2-bundle corresponds to an extension by a two-point line without increasing the rank of the geometry.

In order to prove the duality proposition, we need the following:

LEMMA 5.3 Let Q be a quotient of a geometry $G(X)$. For any subset A of X ,

$$n_{G \rightarrow Q}(A) + n_{Q^* \rightarrow G^*}(X \setminus A) = \deg(G \rightarrow Q).$$

Proof: $n_{G \rightarrow Q}(A) + n_{Q^* \rightarrow G^*}(X \setminus A)$
 $= r_G(A) - r_Q(A) + n_{G^*}(X \setminus A) - n_{Q^*}(X \setminus A)$
 $= r_G(A) + n_{G^*}(X \setminus A) - (r_Q(A) + n_{Q^*}(X \setminus A))$
 $= r(G) - r(Q)$
 $= \deg(G \rightarrow Q). \quad \square$

PROPOSITION 5.4 Given any strict 2-bundle

$$B = \begin{array}{ccc} & R & \\ & \swarrow \quad \searrow & \\ P & & Q \\ & \nwarrow \quad \nearrow & \\ & G & \end{array}$$

on X , its inverted dual

$$B' = \begin{array}{ccc} & G^* & \\ & \swarrow \quad \searrow & \\ P^* & & Q^* \\ & \nwarrow \quad \nearrow & \\ & R^* & \end{array}$$

is also a strict 2-bundle.

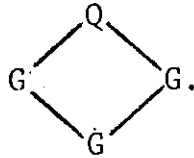
Proof: Let $A \subseteq X$. Then

$$A \in M(R^* \rightarrow P^*) \cap M(R^* \rightarrow Q^*)$$

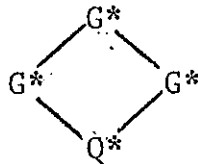
$$\begin{aligned}
\implies & r_{P^*}(A) = r_{R^*}(A) - 1 = r_{Q^*}(A) \\
\implies & r_P(X \setminus A) = r_R(X \setminus A) = r_Q(X \setminus A) \\
\implies & r_R(X \setminus A) = r_G(X \setminus A) \\
\implies & A = \bar{A}^{-R^*} \in M(R^* \rightarrow G^*).
\end{aligned}$$

Thus B' satisfies the two bundle properties. \square

If in the above proposition the 2-bundle B is not strict, then its inverted dual B' may not be a 2-bundle. For example, let Q be a proper elementary quotient of G and we have a 2-bundle



But its inverted dual



is not a 2-bundle because $M(Q^* \rightarrow G^*) \neq \emptyset$ while the modular filter $\{x \in Q^* \mid n_{Q^* \rightarrow G^*}(x) = 2\}$ is empty.

5.3 Completions to 2-bundles

As we mentioned earlier in 1.4, for any geometry G , the semilattice of its elementary quotients is isomorphic to the semilattice of its proper modular filters. The modular filter corresponding to an elementary quotient Q of G is $M_1(G \rightarrow Q) = \{x \in G \mid n_{G \rightarrow Q}(x) = 1\}$; $M_1(G \rightarrow Q) = \emptyset$ if and only if $Q = G$. We are now going to show that the elementary quotients of a geometry $G(X)$ are actually \wedge -closed in $\mathcal{L}_f(X)$.

PROPOSITION 5.5 Given a non-empty set $\{Q_i\}_{i \in I}$ of elementary quotients of a geometry $G(X)$, $\bigwedge_{i \in I} Q_i$ exists and is equal to the elementary quotient of G with modular filter $\bigcap_{i \in I} M_1(G \rightarrow Q_i)$.

Proof: If there is some $j \in I$ such that $Q_j = G$, then $Q_j \leq Q_i$ for any $i \in I$, so $\bigwedge_{i \in I} Q_i = G$ which is the elementary quotient of G corresponding to the modular filter $\emptyset = \bigcap_{i \in I} M_1(G \rightarrow Q_i)$. So we assume that $Q_i \neq G$ for all $i \in I$. The modular filter $M = \bigcap_{i \in I} M_1(G \rightarrow Q_i)$ is non-empty and we let Q be the elementary quotient of G with modular filter M . We want to show that $Q = \bigwedge_{i \in I} Q_i$.

Suppose $T(X)$ is a geometry such that $T \leq Q_i$ for all $i \in I$. To complete the proof, we need to show that for any flat $x \in G$, $r_T(x) \geq r_Q(x)$. Since $r_T(x) \geq r_{Q_i}(x)$ for all $i \in I$, so if $x \in M$, then $r_T(x) \geq r_G(x) - 1 = r_Q(x)$, and if otherwise $x \notin M_j$ for some $j \in I$, then $r_T(x) \geq r_{Q_j}(x) = r_G(x) = r_Q(x)$. \square

Next, instead of elementary quotients of a geometry, we look at geometries which have a common elementary quotient.

PROPOSITION 5.6 If $\{Q_i\}_{i \in I}$ is a non-empty set of geometries (on the same set X) with a common elementary quotient $G(X)$, then $\bigwedge_{i \in I} Q_i$ exists and G is an elementary quotient of it.

Proof: Let $I' = \{i \in I \mid Q_i \neq G\}$. If $I' = \emptyset$, then $\bigwedge_{i \in I} Q_i = G$. So we now assume $I' \neq \emptyset$ and whence $\bigwedge_{i \in I} Q_i = \bigwedge_{i \in I'} Q_i$.

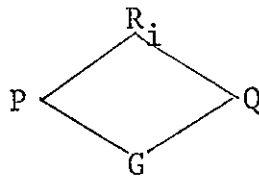
For each $i \in I'$, Q_i^* is a proper elementary quotient of G^* ,

so by the preceding proposition $\bigwedge_{i \in I'} Q_i^*$ exists and equals Q say. We claim that $Q^* = \bigwedge_{i \in I'} Q_i$.

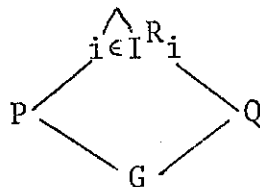
For each $i \in I'$, since $Q \leq Q_i^*$ and both Q and Q_i^* have the same rank, by proposition 1.3 $Q^* \leq Q_i$. Suppose $T(X)$ is a geometry such that $T \leq Q_i$ for all $i \in I'$. To complete the proof, we need to show that for any subset A of X , $n_T(x) \leq n_{Q^*}(x)$. Let $A' = X \setminus A$ and let r be the rank of Q . If $\overline{A}^{G^*} \in \bigcap_{i \in I'} M_1(G^* \rightarrow Q_i^*)$, then $n_T(A) \leq n_{Q_i}(A) = r - r_{Q_i^*}(A')$ for all $i \in I'$. But $r_{Q_i^*}(A) = r_{G^*}(A') - 1 = r_Q(A')$ for all $i \in I'$, so $n_T(A) \leq r - r_Q(A') = n_{Q^*}(A)$. If otherwise $\overline{A}^{G^*} \notin M_1(G^* \rightarrow Q_j^*)$ for some $j \in I'$, then $n_T(A) \leq n_{Q_j}(A) = r - r_{Q_j^*}(A') = r - r_{G^*}(A') = r - r_Q(A') = n_{Q^*}(A)$. \square

The following two propositions will be used in proving the completion theorem in the next section.

PROPOSITION 5.7 Let P, Q be elementary quotients of G and let $\{R_i\}_{i \in I}$ be a non-empty set of elementary quotients of P, Q such that for each $i \in I$,



is a 2-bundle. Then $\bigwedge_{i \in I} R_i$ exists and

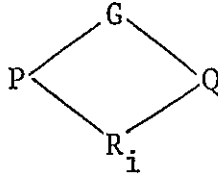


is also a 2-bundle.

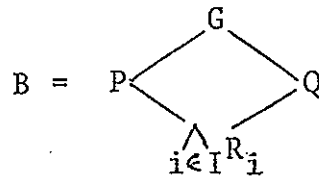
Proof: By proposition 5.5, $\bigwedge_{i \in I} R_i$ exists and is an elementary quotient of both P and Q. For each $i \in I$, $M_1(G \rightarrow P) \cap M_1(G \rightarrow Q) = M_1(G \rightarrow P) \cap M_1(P \rightarrow R_i)$. Let $R = \bigwedge_{i \in I} R_i$; then

$$\begin{aligned} M_1(G \rightarrow P) \cap M_1(G \rightarrow Q) &= M_1(G \rightarrow P) \cap \left(\bigwedge_{i \in I} M_1(P \rightarrow R_i) \right) \\ &= M_1(G \rightarrow P) \cap M_1(P \rightarrow R) \\ &= M_2(G \rightarrow R). \quad \square \end{aligned}$$

PROPOSITION 5.8 Given an elementary quotient G of geometries P and Q, If $\{R_i\}_{i \in I}$ is a non-empty set of geometries such that for each $i \in I$,



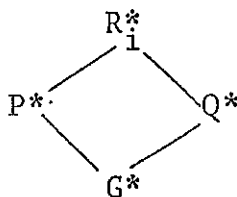
is a 2-bundle, then $\bigwedge_{i \in I} R_i$ exists and



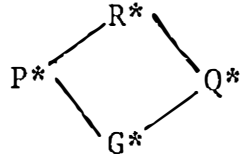
is also a 2-bundle.

Proof: By proposition 5.6, $\bigwedge_{i \in I} R_i$ exists. Let $R = \bigwedge_{i \in I} R_i$.

Both P, Q are elementary quotients of R. If $R = P$ say, then $M_1(R \rightarrow P) = \emptyset$ and so B is a 2-bundle. So we now assume that R is equal to neither P nor Q. The set $I' = \{i \in I \mid R_i \neq P\} = \{i \in I \mid R_i \neq Q\}$ is non-empty and G is equal to neither P nor Q. By proposition 5.4, for each $i \in I'$,



is a 2-bundle. Since $\bigwedge_{i \in I} R_i^* = (\bigwedge_{i \in I} R_i)^* = (\bigwedge_{i \in I} R_i)^* = R^*$, so



is a 2-bundle. By proposition 5.4 again, B is a 2-bundle. \square

5.4 A completion theorem

Given a set I of F-flats and a partial bundle Q(I) on X, the completions of Q(I) to F-bundles may not form a semilattice. The purpose of this section is to give a sufficient condition on I under which the completions are \wedge -closed in $\mathcal{G}(X)$ and thus form a semilattice.

THEOREM 5.10 Let F be a geometry and let I be a set of F-flats such that

$$x < y \in F \setminus I \implies r(y) - r(x) > 2.$$

If Q(I) is a partial bundle on X, then the completions of Q(I) to F-bundles form a sub- \wedge -semilattice of $\mathcal{G}(X)^F$.

Proof: Let $\{Q_j(F) \mid j \in J\}$ be any non-empty set of completions of Q(I) to F-bundles. For each $a \in I$, $Q_j(a) = Q(a)$ for all $j \in J$, so $\bigwedge_{j \in J} Q_j(a) = Q(a)$. For any flat $a \in F \setminus I$, there exists a flat $b \in I$ such that either a covers b or b covers a. If a covers b, then $Q_j(a)$ is an elementary quotient of $Q(b)$ for all $j \in J$, and so by proposition 5.5, $\bigwedge_{j \in J} Q_j(a)$ exists. Similarly, if b covers a, then $Q(b)$ is an elementary quotient of $Q_j(a)$ for all $j \in J$, and so by proposition 5.6, $\bigwedge_{j \in J} Q_j(a)$ exists. Define $Q(a)$ to be $\bigwedge_{j \in J} Q_j(a)$ for each $a \in F$. Claim that $\{Q(a) \mid a \in F\}$ is indeed an F-bundle on X.

First of all, by propositions 5.5 and 5.6, for any flats $a, b \in F$, if b covers a , then $Q(b)$ is an elementary quotient of $Q(a)$. So the first property of a quotient bundle is satisfied.

Next, let a, b be a locally modular pair in F . We want to show that

$$(*) \quad M(a \wedge b, a) \cap M(a \wedge b, b) = M(a \wedge b, a \vee b).$$

If all of $a, b, a \wedge b, a \vee b$ are in I , then it is clear that $(*)$ is satisfied. Otherwise, we have the following three cases to consider:

1) $a, b, a \wedge b \in I$; proposition 5.7 shows that $(*)$ holds.

2) $a, b, a \vee b \in I$; proposition 5.8 shows that $(*)$ holds.

$$\begin{aligned} 3) \quad a \wedge b, a \vee b \in I; \quad & M(a \wedge b, a) \cap M(a \wedge b, b) \\ &= \left(\bigcap_{j \in J} M(a \wedge b, a) \right) \cap \left(\bigcap_{j \in J} M(a \wedge b, b) \right) \\ &= \bigcap_{j \in J} (M_1(Q(a \wedge b), Q_j(a) \cap M_1(Q(a \wedge b), Q_j(b)))) \\ &= \bigcap_{j \in J} M(a \wedge b, a \vee b) \\ &= M(a \wedge b, a \vee b). \end{aligned}$$

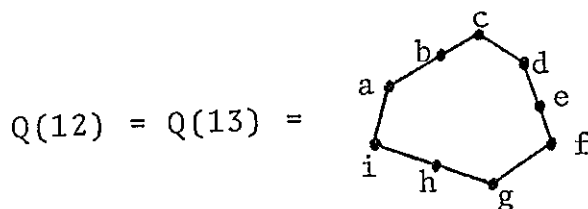
Thus $\{Q(a) \mid a \in F\}$ is a completion of $Q(I)$ to an F -bundle on X and this completes the proof. \square

5.5 A counter-example for the existence of a least completion of a partial bundle

We have shown earlier in this chapter that the completions of a partial bundle need not form a semilattice. As a matter of fact, a partial bundle may not even have a least completion.

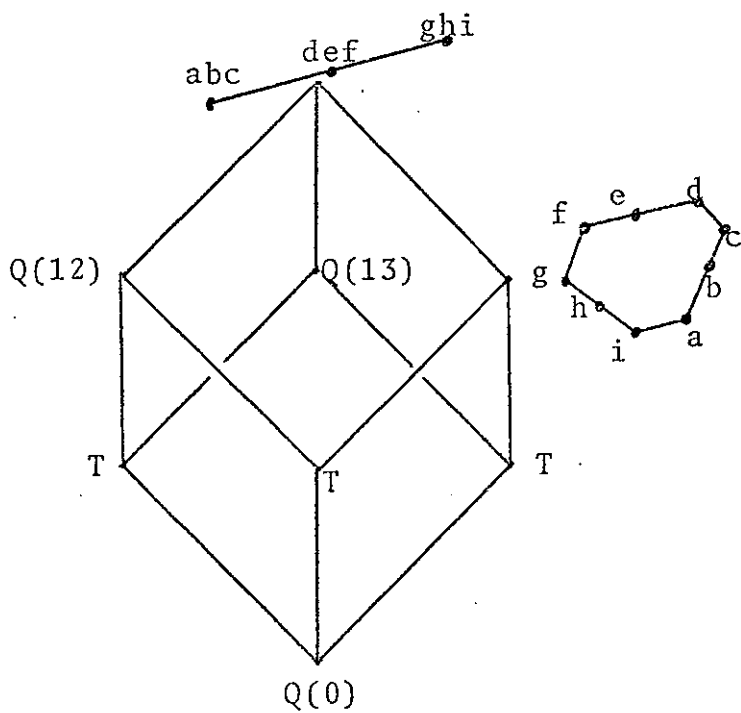
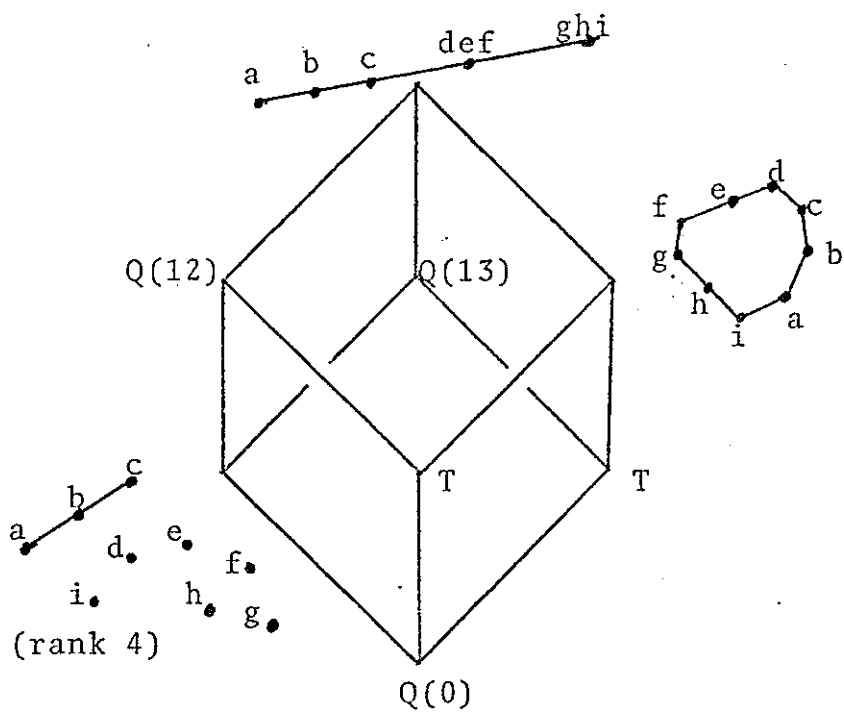
EXAMPLE 5.11 Let F be the plane geometry of three points 1,2,3 and let $I = \{0, 12, 13\}$. Let

$Q(0)$ = the rank-five geometry of nine points a, b, c, \dots, i in general position



The two completions of $Q(I)$ shown in figure 5.1 are minimal. Thus $Q(I)$ has no least completion.

We shall see in the next two chapters that if I is a sublattice of a free geometry F , then any partial bundle $Q(I)$ has a least completion, a canonical construction for which will be given.



T = truncation of G.

figure 5.1

VI. EXTREMAL COMPLETIONS OF PARTIAL BUNDLES I

The purpose of this chapter is to develop a canonical construction scheme for completions of partial bundles. The partial bundles we consider will be indexed by a Boolean subalgebra of a free geometry. For this class of partial bundles, there is always a least completion achieved by successively lifting quotients along certain paths which eventually fill up the indexing geometry F .

In the first three sections of this chapter, we will give the materials necessary for the development of the construction scheme.

6.1 Commutative and totally compatible extensions

Let M_1, M_2, \dots, M_n be modular filters of a geometry $G(X)$. For each $i = 1, 2, \dots, n$, let $H_i(X \cup e_i)$ be the elementary extension of $G(X)$ corresponding to M_i . The elementary extensions $G \rightarrow H_i$ are said to be totally compatible if there exists an extension $G \rightarrow H(X \cup e_1 \cup e_2 \cup \dots \cup e_n)$ such that for each i , H is an extension of H_i , or equivalently,

$$x \in M_i \iff e_i \in \overline{x}^H, \quad \forall x \in G.$$

Consider a geometry $G(X)$ and a list of modular filters M_1, M_2, \dots, M_n (with repetition allowed) of G . Let $E = \{e_1, e_2, \dots, e_n\}$ be an n -element set. For any permutation π on $\{1, 2, \dots, n\}$, there is a sequence

$$\begin{aligned} G(X) = G_0(X) \rightarrow G_1(X \cup e_{\pi(1)}) \rightarrow G_2(X \cup e_{\pi(1)} \cup e_{\pi(2)}) \rightarrow \\ \dots \rightarrow G_n(X \cup E) \end{aligned}$$

of elementary extensions, called a minimal extension sequence, defined by the prescription:

$$\text{Mod}(G_{i-1} \rightarrow G_i) = \text{the modular filter of } G_{i-1} \\ \text{generated by } \overline{M}_i^{G_{i-1}} \quad \forall 1 \leq i \leq n.$$

The modular filters M_1, M_2, \dots, M_n are said to be commutative if the extension $G_n(X \cup E)$ does not depend on the choice of the permutation π .

PROPOSITION 6.1 If modular filters M_1, M_2, \dots, M_n of $G(X)$ commute, then their corresponding elementary extensions $H_1(X \cup e_1), H_2(X \cup e_2), \dots, H_n(X \cup e_n)$ are totally compatible.

Proof: Let $H(X \cup e_1 \cup e_2 \cup \dots \cup e_n)$ be the extension of $G(X)$ determined by the commutative modular filters M_1, M_2, \dots, M_n .

For any $i = 1, 2, \dots, n$, there exists a permutation π on $\{1, 2, \dots, n\}$ such that $\pi(i) = 1$ and so the minimal extension sequence defined with respect to π has a subsequence

$$G(X) \rightarrow H_i(X \cup e_i) \rightarrow H(X \cup e_1 \cup e_2 \cup \dots \cup e_n).$$

Thus the elementary extensions $H_i(X \cup e_i)$, $i = 1, 2, \dots, n$, are totally compatible. \square

PROPOSITION 6.2 Two modular filters M, N of $G(X)$ commute if and only if their corresponding elementary extensions $H(X \cup e_1)$ and $K(X \cup e_2)$ are compatible (= totally compatible).

Proof: If M, N are commutative, by proposition 6.1, H, K are compatible. Conversely, if H, K are compatible, there exists a least elementary extension R of H, K as shown in the following diagram:

$$\begin{array}{ccccc}
 & & & \rightarrow & H(X \cup e_1) \\
 & & & \nearrow & \\
 G(X) & & & & \\
 & & & \searrow & \\
 & & & \rightarrow & K(X \cup e_2) \\
 & & & \searrow & \\
 & & & \rightarrow & R(X \cup e_1 \cup e_2).
 \end{array}$$

By proposition 2.4, $\text{Mod}(H \rightarrow R) =$ the modular filter of H generated by \bar{N}^H and $\text{Mod}(K \rightarrow R) =$ the modular filter of K generated by \bar{M}^K . Thus M, N are commutative. \square

The converse of proposition 6.1 is not true. There are totally compatible modular filters which are not commutative.

EXAMPLE 6.3 Let G be the rank-four geometry on eight points a, b, \dots, h in general position. Let $M_1 = M_2 = \{abc, def, 1_G\}$ and $M_3 = \{abc, def, agh, bgh, \dots, fgh, 1_G\}$. The modular filters M_1, M_2, M_3 are totally compatible, as exhibited in the extension shown in figure 6.1.

figure 6.1

But for different orders M_1, M_2, M_3 and M_1, M_3, M_2 of the modular filters, the minimal extension sequences give different extensions, as shown in figure 6.2.

figyre 6.2

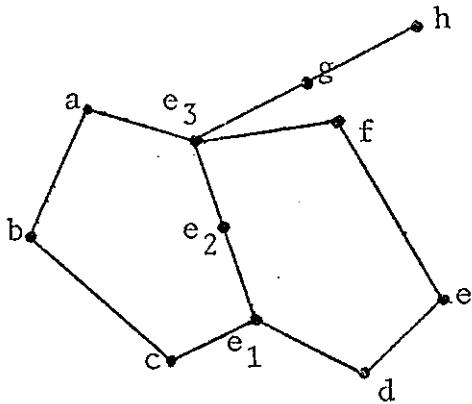


figure 6.1

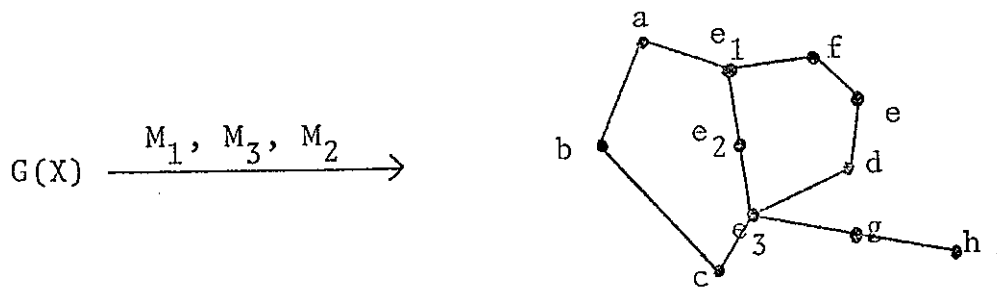
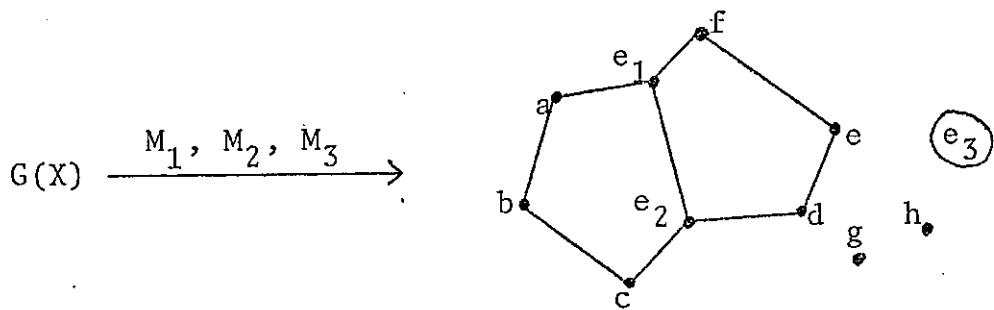


figure 6.2

6.2 Principal extensions

Given a geometry $G(X)$ together with a flat E in G , by putting an extra point e in general position on E , we mean the elementary extension

$$G(X) \longrightarrow H(X \cup e)$$

corresponding to the principal modular filter $[E]$ of G .

Crapo and Roulet [8] first observed that for any list of flats $\{E_i\}$ of a geometry G , an extension of G can be constructed by putting, for each i , one extra point e_i in general position on the flat E_i .

A theorem of Brown [1] on exchange closures showed, as noted by Crapo and Roulet [8], that any two such elementary extensions of a geometry commute:

THEOREM 6.4 Given two flats E, F of a geometry $G(X)$, there exist elementary extensions as shown in the following diagram:

$$\begin{array}{ccccc} & & & \rightarrow G_1(X \cup e_1) & & \\ & \nearrow & & & \searrow & \\ G(X) & & & & & \rightarrow H(X \cup e_1 \cup e_2) \\ & \searrow & & & \nearrow & \\ & & & \rightarrow G_2(X \cup e_2) & & \end{array}$$

where $\text{Mod}(G \rightarrow G_1) = [E]_G$

$\text{Mod}(G \rightarrow G_2) = [F]_G$

$\text{Mod}(G_1 \rightarrow H) = [\bar{F}^{G_1}]_{G_1}$

and $\text{Mod}(G_2 \rightarrow H) = [\bar{E}^{G_2}]_{G_2}$. \square

In order to prove the next theorem on commutativity, we recall a fundamental result on permutations of integers:

LEMMA 6.5 Let π and σ be permutations on $\{1, 2, \dots, n\}$. There exists a sequence

$$\pi = \pi_0, \pi_1, \pi_2, \dots, \pi_k = \sigma$$

of k permutations on $\{1, 2, \dots, k\}$ for some integer k , such that for any $0 < j \leq k$

$$(*) \pi_{j-1}(i) = \pi_j(i) \text{ for all } i=1, 2, \dots, n \text{ except some two consecutive integers.} \quad \square$$

THEOREM 6.6 For any flats E_1, E_2, \dots, E_n of a geometry $G(X)$, the principal modular filters $[E_1], [E_2], \dots, [E_n]$ are commutative.

Proof: Let $\{e_1, e_2, \dots, e_n\} = E$ be an n -element set and let π, σ be any two permutations on $\{1, 2, \dots, n\}$. There exists a sequence of permutations

$$\pi = \pi_0, \pi_1, \dots, \pi_k = \sigma$$

satisfying condition $(*)$ in lemma 6.5. By theorem 6.4, for any $0 < i \leq k$, the minimal extension sequences of M_1, M_2, \dots, M_n defined with respect to the permutations π_{i-1} and π_i give the same extension $R(X \cup E)$ of $G(X)$. Thus the modular filters $[E_1], [E_2], \dots, [E_n]$ are commutative. \square

Given flats E_1, E_2, \dots, E_n of a geometry G , the extension of G determined by the commutative modular filters $[E_1], [E_2], \dots, [E_n]$ is called a principal extension, which is often described as the extension of G obtained by adding, for each $i = 1, 2, \dots, n$, a point e_i in general position on the flat E_i .

LEMMA 6.7 Given a principal extension $G(X) \rightarrow R(X \cup E)$ in which among others a point $a \in E$ is put in general position on a flat C of G , for any subset $A \subseteq E$ not containing a ,

$$a \in \bar{A}^R \text{ if and only if } \bar{C}^R \leq \bar{A}^R.$$

Proof: Consider the extension sequence

$$G(X) \rightarrow H(X \cup A) \rightarrow K(X \cup A \cup a) \rightarrow R(X \cup E)$$

where H and K are subgeometries of R on $X \cup A$ and $X \cup A \cup a$ respectively. By theorem 6.6, $\text{Mod}(H \rightarrow K) = [\bar{C}^H]_H$. But then $a \in \bar{A}^R \iff a \in \bar{A}^K \iff \bar{C}^H \leq \bar{A}^H \iff \bar{C}^R \leq \bar{A}^R$. \square

PROPOSITION 6.8 Let C be a flat of rank $r > 0$ in a geometry $G(X)$. If R is the extension obtained by putting points a_1, a_2, \dots, a_r in general position on C , then a_1, a_2, \dots, a_r are independent in R .

Proof: Consider the minimal extension sequence

$$G = G_0(X) \rightarrow G_1(X \cup a_1) \rightarrow G_2(X \cup a_1 \cup a_2) \rightarrow \dots \rightarrow R(X \cup a_1 \cup a_2 \cup \dots \cup a_r)$$

from G to R , where for each $1 \leq i \leq r$, $\text{Mod}(G_{i-1} \rightarrow G_i) = [\bar{C}^{G_{i-1}}]_{G_{i-1}}$.

Suppose there is some $i \leq r$ such that a_i depends on $\{a_1, a_2, \dots, a_{i-1}\}$. Then by lemma 6.7, $\overline{\{a_1, a_2, \dots, a_{i-1}\}}^{G_{i-1}} \geq \bar{C}^{G_{i-1}}$ and so $i-1 \geq r$, a contradiction. Thus a_1, a_2, \dots, a_r are independent. \square

PROPOSITION 6.9 Let C_1, C_2, \dots, C_n be flats of ranks r_1, r_2, \dots, r_n respectively and be skew to each other (i.e., $r(C_1 \vee C_2 \vee \dots \vee C_n) = r_1 + r_2 + \dots + r_n$) in a geometry $G(X)$. If for each $i=1, 2, \dots, n$, points $a_1^i, a_2^i, \dots, a_{r_i}^i$ are put in general position on C_i , then the set C of added points is independent in the extension R thus defined.

Proof: Let $a \in C$ and $A \subseteq C$ be such that $a \notin A$. We want to show that a does not depend on A in the extension R . Suppose not. Let a be put on the flat C_k say. Then $\overline{C_k} \leq \overline{A}$. For each $i = 1, 2, \dots, n$, let $C_i^! = \{a_1^i, a_2^i, \dots, a_{r_i}^i\}$. By proposition 6.8; $\overline{C_i^!} = \overline{C_i}$. Let $B = (\bigcup_{i \neq k} C_i^!) \cup (A \cap C_k^!)$. Then $\overline{B} \geq \bigvee_{i=1, 2, \dots, n} \overline{C_i}$. But this is impossible because $|B| < |C|$. \square

6.3 Extremal sequences of elementary quotients

As we have pointed out in 4.2, given a quotient bundle $Q(F)$, any maximal chain

$$c_0 < c_1 < \dots < c_n$$

in F gives a sequence

$$Q(c_0) \rightarrow Q(c_1) \rightarrow \dots \rightarrow Q(c_n)$$

of elementary quotients. The purpose of this section is to study these sequences with the consideration of the weak order.

If $S: P_0 \rightarrow P_1 \rightarrow \dots \rightarrow P_n$

and $T: Q_0 \rightarrow Q_1 \rightarrow \dots \rightarrow Q_n$

are sequences of elementary quotients of equal length, define

$$S \leq T$$

if and only if

$$P_i \leq Q_i \quad \forall i = 0, 1, \dots, n.$$

Sequences of proper elementary quotients with the weak order were studied in [4]. The treatment there is now generalized in our present discussion.

To start with, let Q be a quotient of a geometry G and look at sequences of elementary quotients from G to Q of length n for some integer $n \geq \deg(G \rightarrow Q)$. There is no surprise that such sequences need not form a semilattice although there is a least such.

EXAMPLE 6.10 Let Q be a quotient of a geometry G as shown in figure 6.3. Consider sequences S_1, S_2, S_3 and S_4 as shown in figure 6.4. In the ordered set of sequences of elementary quotients from G to Q of length 3, S_1, S_2 cover S_3, S_4 . Thus

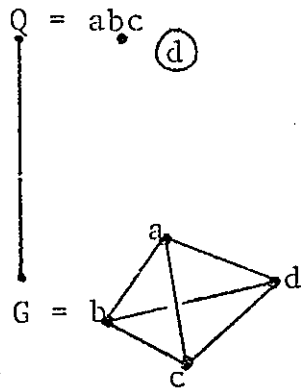


figure 6.3

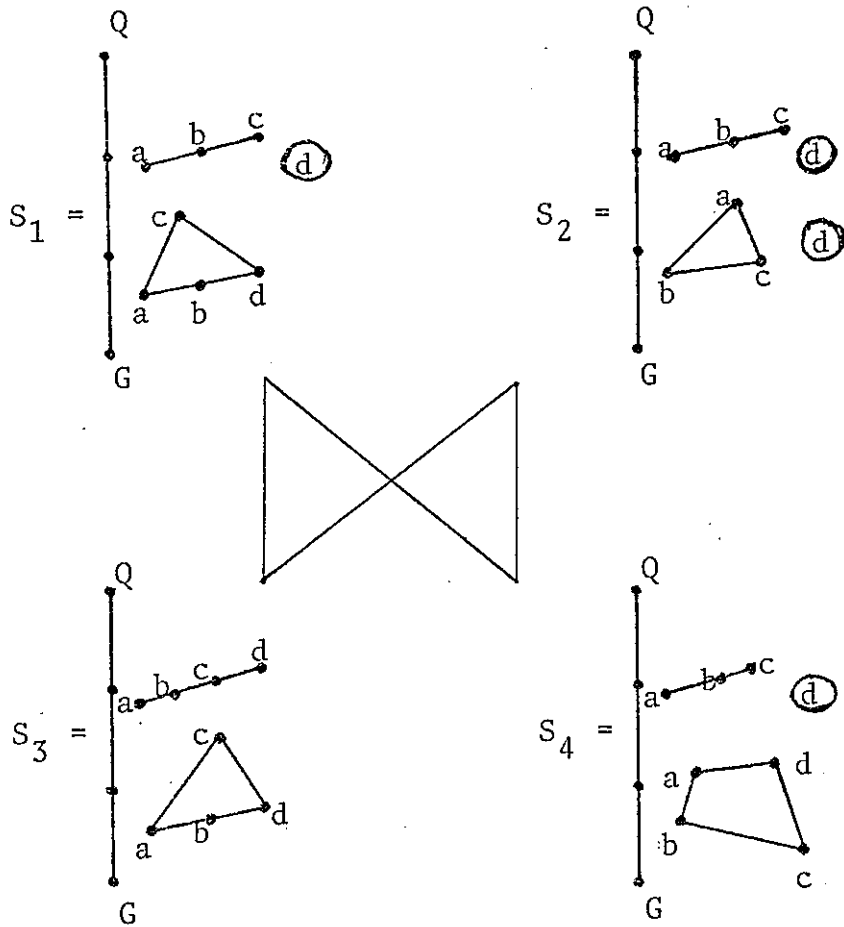


figure 6.4

$S_1 \wedge S_2$ does not exist.

figure 6.3

figure 6.4

Given a lift bundle $Q(F)$ of Q toward G , any maximal chain

$$C: 0_F = c_0 < c_1 < \dots < c_n = 1_F$$

in F gives a sequence

$$Q(C): G = Q(c_0) \rightarrow Q(c_1) \rightarrow \dots \rightarrow Q(c_n) = Q$$

of elementary quotients. $Q(C)$ may not be the strict lift sequence from G to Q unless the bundle $Q(F)$ is strict, i.e., $r(F) := \text{deg}(G \rightarrow Q)$.

The lift sequence from G to Q of length n is defined to be the sequence

$$G = Q_0 \rightarrow Q_1 \rightarrow \dots \rightarrow Q_n = Q$$

of elementary quotients such that

$$Q_{i-1} = L(G \rightarrow Q_i) \quad \forall i = 1, 2, \dots, n.$$

It is clear that if $k = n - \text{deg}(G \rightarrow Q)$, then

$$Q_i = G \quad \forall i \leq k$$

and the subsequence

$$Q_k \rightarrow Q_{k+1} \rightarrow \dots \rightarrow Q_n$$

is the strict lift sequence from G to the quotient Q .

The following is a slight modification of proposition 4.5:

PROPOSITION 6.11 Given any sequence of quotients

$$G \rightarrow P \rightarrow Q$$

and integers m, n such that $m \geq \text{deg}(G \rightarrow P)$ and $n \geq \text{deg}(P \rightarrow Q)$,

$$M_n(G \rightarrow P) \cap M_n(P \rightarrow Q) = M_{m+n}(G \rightarrow Q).$$

Proof: If $m = \deg(G \rightarrow P)$ and $n = \deg(P \rightarrow Q)$, then $m + n = \deg(G \rightarrow Q)$ and so

$$\begin{aligned} M_m(G \rightarrow P) \cap M_n(P \rightarrow Q) &= M(G \rightarrow P) \cap M(P \rightarrow Q) \\ &= M(G \rightarrow Q) \\ &= M_{m+n}(G \rightarrow Q). \end{aligned}$$

If otherwise $m > \deg(G \rightarrow P)$ say, then $m+n > \deg(G \rightarrow Q)$ and so

$$M_m(G \rightarrow P) \cap M_n(P \rightarrow Q) = \emptyset = M_{m+n}(G \rightarrow Q). \quad \square$$

As a corollary of proposition 6.11 and theorem 4.6, we have the following characterization of lift sequences:

THEOREM 6.12 For any sequence

$$S: G = G_0 \rightarrow G_1 \rightarrow \dots \rightarrow G_n = Q$$

of elementary quotients, the followings are equivalent:

- 1) S is the lift sequence from G to Q of length n
- 2) $M_1(G_{i-1} \rightarrow G_i) \subseteq M_1(G_i \rightarrow G_{i+1}) \quad \forall i = 1, 2, \dots, n-1$
- 3) $M_1(G_{i-1} \rightarrow G_i) = M_{n-i-1}(G_{i-1} \rightarrow G_n) \quad \forall i = 1, 2, \dots, n-1. \quad \square$

Finally, we have the extremal property of lift sequences:

THEOREM 6.13 Given a lift sequence

$$S: G = L_0 \rightarrow L_1 \rightarrow \dots \rightarrow L_n = Q$$

from G to Q , if

$$T: H = Q_0 \rightarrow Q_1 \rightarrow \dots \rightarrow Q_n = P$$

is any sequence of elementary quotients such that $G \leq H$ and $Q \leq P$, then $S \leq T$.

Proof: Let $k = n - \deg(G \rightarrow Q)$. For any $i \leq k$,

$$L_i = G \leq H \leq Q_i.$$

To complete the proof, we want to show that for any $k < i \leq n$,

if $L_{i-1} \leq Q_{i-1}$, then $L_i \leq Q_i$. Let $x \in L_i$. If $x \notin M(L_{i-1} \rightarrow L_i)$

then

$$r_{L_i}(x) = r_{L_{i-1}}(x) \geq r_{Q_{i-1}}(x) \geq r_{Q_i}(x).$$

If $x \in M(L_{i-1} \rightarrow L_i)$, then by theorem 4.6 $x \in M(L_{i-1} \rightarrow Q)$ and so

$$r_{L_i}(x) = r_Q(x) + n - 1 \geq r_P(x) + n - 1 \geq r_{Q_i}(x).$$

Thus $L_i \leq Q_i$ for all $0 \leq i \leq n$ and so $S \leq T$. \square

6.4 The weakest completion of a partial bundle

For the rest of this chapter, $F(E)$ will be the free geometry on a (finite) set E and I will be a set of flats of F such that:

$$1) \quad a, b \in I \implies a \wedge b (= a \cap b), a \vee b (= a \cup b) \in I$$

$$\text{and } 2) \quad a \in I \implies E \setminus a \in I,$$

that is, I is a Boolean subalgebra of F .

As subsets of E , the atoms E_1, E_2, \dots, E_n of I partition the set E , where n equals the order of I . For each $a \in F$, let

$$E(a) = \{E_i \mid \emptyset \neq a \wedge E_i \neq E_i\}$$

$$\text{and } l(a) = |E(a)|.$$

For any flat $a \in F$ and any atom E_i of I , if $E_i \notin E(a)$, then

$$l(a \vee E_i) = l(a), \text{ and if } E_i \in E(a), \text{ then } l(a \setminus E_i) = l(a \vee E_i) = l(a) - 1.$$

For any $0 \leq i \leq n$, let

$$I_i = \{a \in F \mid l(a) \leq i\}.$$

Then we have a sequence

$$I = I_0 \subseteq I_1 \subseteq \dots \subseteq I_n = F$$

of subsets from I to F .

Let $Q(I) = \{Q(a)\}_{a \in I}$ be an arbitrary partial bundle of quotients of a geometry $G(X)$ indexed by I . The purpose of this section is to show that the weakest completion of $Q(I)$ always exists.

Let $Q(F)$ be any completion of $Q(I)$ to an F -bundle, and let $R(X \cup E)$ be the extension of $G(X)$ determined by $Q(F)$. For notational convenience, we produce an extra copy of E :

$$E' = \{a' \mid a \in E\}.$$

Consider the flats $\overline{E}_1^R, \overline{E}_2^R, \dots, \overline{E}_n^R$ of R . For each $i = 1, 2, \dots, n$, we put in general position on \overline{E}_i^R the points p' , with $p \in E_i$. This gives an extension

$$R(X \cup E) \longrightarrow H(X \cup E \cup E').$$

By proposition 6.8, the set E' is independent in H , and so the subgeometry $F'(E')$ of H on E' is free and isomorphic to $F(E)$. For each flat $a \in F$, let

$$a' = \{p' \mid p \in a\}$$

be the corresponding flat in F' . The subgeometry $R'(X \cup E')$ of H on $X \cup E'$ is an extension of both $G(X)$ and $F'(E')$ and so determines an F' -bundle $Q'(F')$ on $G(X)$. If we let $Q'(a) = Q'(a')$ for all $a \in F$, then $Q'(F) = \{Q'(a)\}_{a \in F}$ is an F -bundle on $G(X)$.

LEMMA 6.14 For each $a \in I$, $Q'(a) = Q(a)$, ie., $Q'(F)$ is a completion of $Q(I)$.

Proof: If $a \in I$, then $\overline{a}^R = \overline{a'}^{R'}$ and so

$$\begin{aligned} Q'(a) &= \text{Quo}(G \rightarrow R' \rightarrow R'/a') \\ &= \text{Quo}(G \rightarrow H \rightarrow H/a') \\ &= \text{Quo}(G \rightarrow H \rightarrow H/a) \\ &= \text{Quo}(G \rightarrow R \rightarrow R/a) \\ &= Q(a). \quad \square \end{aligned}$$

If $a \leq b \in F$, we denote the modular filter

$$M_{r(b)-r(a)}(Q'(a') \rightarrow Q'(b'))$$

by $M(a, b)$.

LEMMA 6.15 Given an F -flat a and an atom E_i of I disjoint from a , if

$$a = a_0 < a_1 < \dots < a_k = a \vee E_i$$

is a maximal chain in F from a to $a \vee E_i$ ($k = |E_i|$), then for any $0 < j < k$,

$$M(a_{j-1}, a_j) = M(a_{j-1}, a \vee E_i),$$

that is, the sequence of elementary quotients

$$Q'(a) = Q'(a_0) \longrightarrow Q'(a_1) \longrightarrow \dots \longrightarrow Q'(a_k) = Q'(a \vee E_i)$$

is a lift sequence from $Q'(a)$ to $Q'(a \vee E_i)$.

Proof: Let $R_{j-1}(X \cup E \cup a_{j-1}')$ and $R_j(X \cup E \cup a_j')$ be subgeometries of $H(X \cup E \cup E')$ on $X \cup E \cup a_{j-1}'$ and $X \cup E \cup a_j'$ respectively.

Since $a_j \setminus a_{j-1} \in E_i$, by theorem 6.6, the modular filter of

R_{j-1} corresponding to the elementary extension $R_{j-1} \rightarrow R_j$ is

$[\overline{E_i}^{R_{j-1}}]_{R_{j-1}}$. Let $x \in M(a_{j-1}, a_j)$. Then

$$\overline{E_i}^{R_{j-1}} \leq \overline{x \cup a_{j-1}}^{R_{j-1}}$$

and so

$$\overline{E_i}^H = \overline{E_i}^H \leq \overline{x \cup a_{j-1}}^H.$$

Thus

$$\overline{x \cup a' \cup E_i}^H = \overline{x \cup a_{j-1}}^H.$$

It follows that

$$\overline{x \cup a' \cup E_i}^{R_{j-1}} = \overline{x \cup a_{j-1}}^{R_{j-1}}$$

and so $x \in M(a_{j-1}, a \cup E_i)$. Hence $M(a_{j-1}, a_j) = M(a_{j-1}, a \vee E_i)$. \square

THEOREM 6.16 The F -bundle $Q'(F)$ defined above is the weakest possible completion of the partial bundle $Q(I)$.

Proof: Let $P(F)$ be any completion of $Q(I)$. We are going to show by induction on $l(a)$ that $Q'(a) \leq P(a)$ for all $a \in F$.

If $l(a) = 0$, then $Q'(a) = Q(a) = P(a)$. Suppose $Q'(a) \leq P(a)$ for all $a \in F$ such that $l(a) < i$. Let $b \in F$ be such that $l(b) = i$. Then there exist $a \in F$ and atom E_i of I such that

$$a < b < a \vee E_i$$

$$\text{and } l(a) = l(a \vee E_i) = i-1.$$

By induction hypothesis, $Q'(a) \leq P(a)$ and $Q'(a \vee E_i) \leq P(a \vee E_i)$.

By lemma 6.15 and theorem 6.13, $Q'(b) \leq P(b)$. \square

Combining lemma 6.15 and theorem 6.16, we have the following construction theorem:

THEOREM 6.17 Given a partial bundle $Q(I)$, its least completion $Q'(F)$ is constructable via the sequence

$$Q(I) = Q'(I_0), Q'(I_1), \dots, Q'(I_n) = Q'(F),$$

where for any $i > 0$, $Q'(I_i)$ is definable from $Q'(I_{i-1})$ as follows:

$$\forall a \in I_i, \forall E_j \in E(a),$$

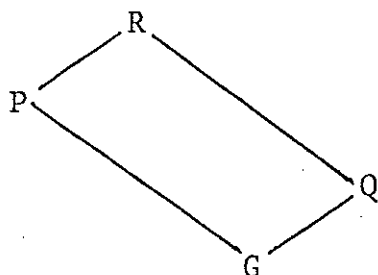
$$Q'(a) = L^{E_j \setminus a} (Q'(a \setminus E_j) \rightarrow Q'(a \vee E_j)). \quad \square$$

6.5 Some preliminary completion constructions

The purpose of this section is to prove some results related to lift sequences, which will be needed in the next section for completion testings.

We start with a slight modification of proposition 4.10:

PROPOSITION 6.18 Suppose in the following diagram of quotients



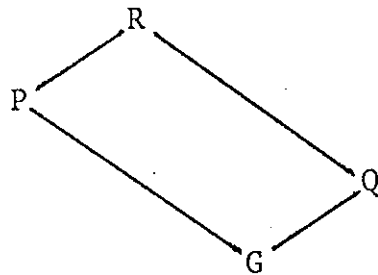
$G \rightarrow Q$ and $P \rightarrow R$ are elementary. For any integer k not less than $\deg(G \rightarrow P)$ or $\deg(Q \rightarrow R)$, $L^{k-1}(Q \rightarrow R)$ is an elementary quotient of $L^{k-1}(G \rightarrow P)$ if the following condition is satisfied:

$$M_1(G \rightarrow Q) \cap M_k(G \rightarrow P) = M_{k+1}(G \rightarrow R).$$

Proof: Consider first the case when $k > \deg(G \rightarrow P)$. If $G = Q$, then $L^{k-1}(Q \rightarrow R) = L^{k-1}(G \rightarrow R)$ is an elementary quotient of $G = L^{k-1}(G \rightarrow P)$. If $G \neq Q$, then $\deg(Q \rightarrow R) = \deg(G \rightarrow R) - 1 \leq \deg(G \rightarrow P) < k$, so $L^{k-1}(Q \rightarrow R) = Q$ is an elementary quotient of $L^{k-1}(G \rightarrow P)$.

Next, assume that $k = \deg(G \rightarrow P)$. If $G \neq Q$ then condition of the proposition implies that $\deg(P \rightarrow R) = 1$ and $\deg(Q \rightarrow R) = k$, and so by proposition 4.10 $L^{k-1}(Q \rightarrow R) = D(Q \rightarrow R)$ is an elementary quotient of $D(G \rightarrow P) = L^{k-1}(G \rightarrow P)$. If $G = Q$, it is impossible that $P \neq R$ for otherwise $\deg(Q \rightarrow R) = \deg(G \rightarrow R) = k+1$, a contradiction. If $G = Q$ and $P = R$, the proposition is trivially true. \square

Given quotients P, Q, R of a geometry G as shown in the following diagram



$Q \rightarrow R$ is said to be elementary to $G \rightarrow P$ if the following two conditions are satisfied:

- 1) $L^i(Q \rightarrow R)$ is an elementary quotient of $L^i(G \rightarrow P)$ for any $i = 0, 1, 2, \dots$
- and 2) $M_1(L^i(G \rightarrow P) \rightarrow L^i(Q \rightarrow R)) \cap M_i(L^i(G \rightarrow P) \rightarrow P) = M_{i+1}(L^i(G \rightarrow P) \rightarrow R)$ for any $i = 0, 1, 2, \dots$

Note that $Q \rightarrow R$ is elementary to $G \rightarrow P$ if and only if for any lift sequences of equal length

$$Q = Q_0 \rightarrow Q_1 \rightarrow \dots \rightarrow Q_n = R$$

$$\text{and } G = G_0 \rightarrow G_1 \rightarrow \dots \rightarrow G_n = P,$$

Q_i is an elementary quotient of G_i and $M_1(G_i \rightarrow Q_i) \cap M_i(G_i \rightarrow P) = M_{i+1}(G_i \rightarrow R)$ for any $i = 0, 1, 2, \dots, n$. As pointed out in remark 4.11, in order that $Q \rightarrow R$ is elementary to $G \rightarrow P$, it is not sufficient to assume $M_1(G \rightarrow Q) \cap M_n(G \rightarrow P) = M_{n+1}(G \rightarrow P)$.

PROPOSITION 6.19 Let lift sequences

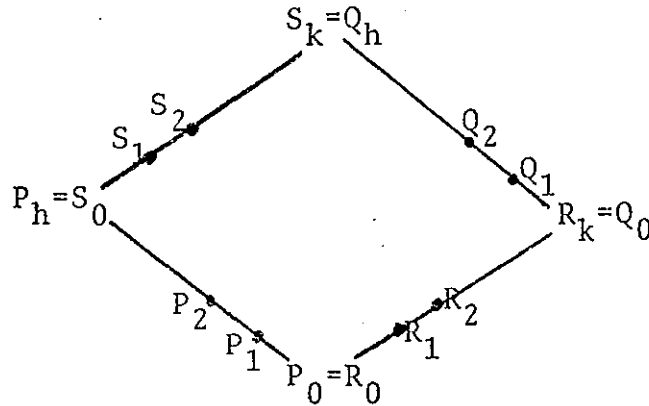
$$P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow \dots \rightarrow P_h$$

$$Q_0 \rightarrow Q_1 \rightarrow Q_2 \rightarrow \dots \rightarrow Q_h$$

$$R_0 \rightarrow R_1 \rightarrow R_2 \rightarrow \dots \rightarrow R_k$$

$$\text{and } S_0 \rightarrow S_1 \rightarrow S_2 \rightarrow \dots \rightarrow S_k$$

be given as shown in the following diagram:



and suppose that they have the following two properties:

- 1) Q_i is a quotient of P_i , $\forall i=1,2,\dots,h$;
 S_j is a quotient of R_j , $\forall j=1,2,\dots,k$;
- 2) $P_i \rightarrow Q_i$ is elementary to $P_{i-1} \rightarrow Q_{i-1}$, $\forall i=1,2,\dots,h$,
 $R_j \rightarrow S_j$ is elementary to $R_{j-1} \rightarrow S_{j-1}$, $\forall j=1,2,\dots,k$.

Then for any $0 \leq i \leq h$ and $0 \leq j \leq k$, we have

$$L^{k-j}(P_i \rightarrow Q_i) = L^{h-i}(R_j \rightarrow S_j).$$

Proof: For any $0 < i \leq h$ and $0 \leq j \leq k$, $L^{k-j}(P_i \rightarrow Q_i)$ is an elementary quotient of $L^{k-j}(P_{i-1} \rightarrow Q_{i-1})$. For each $0 \leq j \leq k$, by comparing the sequence of elementary quotients

$$\begin{aligned} R_j = L^{k-j}(P_0 \rightarrow Q_0) &\rightarrow L^{k-j}(P_1 \rightarrow Q_1) \rightarrow \dots \\ &\rightarrow L^{k-j}(P_n \rightarrow Q_n) = S_j \end{aligned}$$

to the lift sequence of length h from R_j to S_j , we have

$$L^{h-i}(R_j \rightarrow S_j) \leq L^{k-j}(P_i \rightarrow Q_i) \quad \forall 0 \leq i \leq h.$$

Similarly, we also have $L^{k-j}(P_i \rightarrow Q_i) \leq L^{h-i}(R_j \rightarrow S_j)$ for any $0 \leq i \leq h$ and $0 \leq j \leq k$. \square

Note that in the above proposition, condition 2) could be replaced by the following weaker one:

2) $L^j(P_i \rightarrow Q_i)$ is an elementary quotient of $L^j(P_{i-1} \rightarrow Q_{i-1})$ and

$L^i(R_j \rightarrow S_j)$ is an elementary quotient of $L^i(R_{j-1} \rightarrow S_{j-1})$

for any $0 < i \leq h$ and $0 < j \leq k$.

Condition 2) is used in the proposition mainly because of convenience, as we can see in the next section.

6.6 Completabilities and Examples

Given a family of quotients $Q(I)$ of G , theorem 6.17 suggests a standard test for its completability to an F -bundle; such a test we will obtain in this section.

LEMMA 6.20 For any $0 < k < n$, if a family of quotients $Q(I_k)$ of G satisfies the following properties:

0) for any $a \in I_{k-1}$ and for any $E_i \in E(a)$ together with a maximal chain

$$a = a_0 < a_1 < \dots < a_h = a \vee E_i \quad (h = |E_i|)$$

from a to $a \vee E_i$ in F , the quotients $Q(a_0), Q(a_1), \dots, Q(a_h)$ form a lift sequence from $Q(a)$ to $Q(a \vee E_i)$

1) for any $a \in I_k$ and for any $E_i \in E(a)$, $Q(a \vee E_i)$ is a quotient of $Q(a)$

2) for any $a, b \in I_k$ such that b covers a and for any $E_i \in E(a) \cup E(b)$, $Q(b) \rightarrow Q(b \vee E_i)$ is elementary to $Q(a) \rightarrow Q(a \vee E_i)$,

then for any $a \in I_{k+1} \setminus I_k$ and for any distinct $E_i, E_j \in E(a)$,

$$L^{|E_i \setminus a|}(Q(a \setminus E_i) \rightarrow Q(a \vee E_i)) = L^{|E_j \setminus a|}(Q(a \setminus E_j) \rightarrow Q(a \vee E_j)).$$

Proof: Let $a \in I_{k+1} \setminus I_k$ and let $E_i \neq E_j \in E(a)$. Consider the F -flats in I_{k+1} as shown in figure 6.5

figure 6.5

Let

$$a \setminus (E_i \vee E_j) = p_0 < p_1 < \dots < p_{|a \wedge E_j|} = a \setminus E_i < \dots < p_{|E_j|} = a \vee E_j \setminus E_i$$

$$\text{and } a \setminus (E_i \vee E_j) = q_0 < q_1 < \dots < q_{|a \wedge E_i|} = a \setminus E_j < \dots < q_{|E_i|} = a \vee E_i \setminus E_j$$

be maximal chains in F . The lift sequences

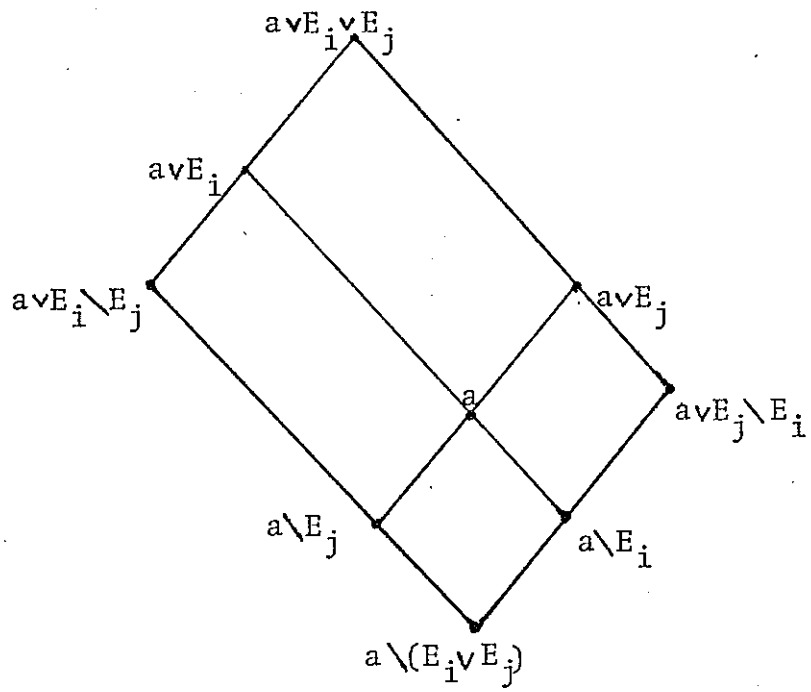


figure 6.5

$$Q(p_0) \rightarrow Q(p_1) \rightarrow \dots \rightarrow Q(p_{|E_j|})$$

$$Q(p_0 \vee E_i) \rightarrow Q(p_1 \vee E_i) \rightarrow \dots \rightarrow Q(p_{|E_j|} \vee E_i)$$

$$Q(q_0) \rightarrow Q(q_1) \rightarrow \dots \rightarrow Q(q_{|E_i|})$$

$$\text{and } Q(q_0 \vee E_j) \rightarrow Q(q_1 \vee E_j) \rightarrow \dots \rightarrow Q(q_{|E_i|} \vee E_j)$$

satisfy the conditions of proposition 6.19. Thus

$$L^{|E_i \setminus a|} (Q(a \setminus E_i) \rightarrow Q(a \vee E_i)) = L^{|E_j \setminus a|} (Q(a \setminus E_j) \rightarrow Q(a \vee E_j)). \quad \square$$

Next, given a family of quotients $Q(I)$ of G , for various $1 \leq k \leq n$, we define its k -completabilities recursively as follows: $Q(I)$ is 1 -completable if

for any $a, b \in I$ such that $b > a$, $Q(b)$ is a quotient of $Q(a)$.

$Q(I)$ is $(k+1)$ -completable, $k < n$, if

$Q(I)$ is k -completable and the family of quotients

$Q(I_k)$ satisfies conditions 1) and 2) of lemma 6.20.

If $Q(I)$ is $(k+1)$ -completable, then by lemma 6.20, a family of quotients $Q(I_{k+1})$ of G is uniquely determined, viz.,

for any $a \in I_{k+1} \setminus I_k$, let $E_i \in E(a)$, then

$$Q(a) = L^{|E_i \setminus a|} (Q(a \setminus E_i) \rightarrow Q(a \vee E_i)).$$

(Note that if $Q(I)$ is k -completable, condition 0) of lemma 6.20 is automatically satisfied by $Q(I_k)$.)

THEOREM 6.21 A family of quotients $Q(I)$ of G is completable to an F -bundle if and only if it is n -completable.

Proof: If $Q(I)$ is completable to an F -bundle, let $Q(F)$ be the least completion. For each $k = 1, 2, \dots, n$, by lemma 6.15

and theorem 6.17, the subfamily $Q(I_k)$ satisfies the conditions of lemma 6.20. Thus $Q(I)$ is k -completable for $k = 1, 2, \dots, n$.

Conversely, if $Q(I)$ is n -completable, there is a family of quotients $Q(F)$ defined accordingly. We want to show that $Q(F)$ satisfies the two properties of a quotient bundle.

Let $a, b \in F$ be such that b covers a . If $b \setminus a \in E_i$ say, then

$$Q(a) = L^{|E_i \setminus a|} (Q(a \setminus E_i) \rightarrow Q(a \vee E_i))$$

$$\text{and } Q(b) = L^{|E_i \setminus b|} (Q(b \setminus E_i) \rightarrow Q(b \vee E_i)).$$

Thus $Q(b)$ is an elementary quotient of $Q(a)$.

If a, b is a locally modular pair of flats in F , let $a \setminus b \in E_i$ and $b \setminus a \in E_j$. For any flats $c, d \in F$, if $c \leq d$, we denote by $M(c, d)$ the modular filter $M_{|d \setminus c|} (Q(c) \rightarrow Q(d))$. If $E_i = E_j$, then all $M(a \wedge b, a)$, $M(a \wedge b, b)$ and $M(a \wedge b, a \vee b)$ are equal to $M(a \wedge b, a \vee E_i)$. If $E_i \neq E_j$, then

$$\begin{aligned} M(a \wedge b, a) \cap M(a \wedge b, b) &= M(a \wedge b, a) \cap M(a \wedge b, b \vee E_j) \\ &= M(a \wedge b, a \vee E_j) \\ &\subseteq M(a \wedge b, a \vee b). \end{aligned}$$

Thus $Q(F)$ is a completion of $Q(I)$. □

To conclude this section, we give some examples:

EXAMPLE 6.22 Let F be the free geometry of four points $1, 2, 3, 4$, and let $I = \{0, 12, 34, 1234\}$. Consider the family of quotients $Q(I)$ as shown in figure 6.6

figure 6.6

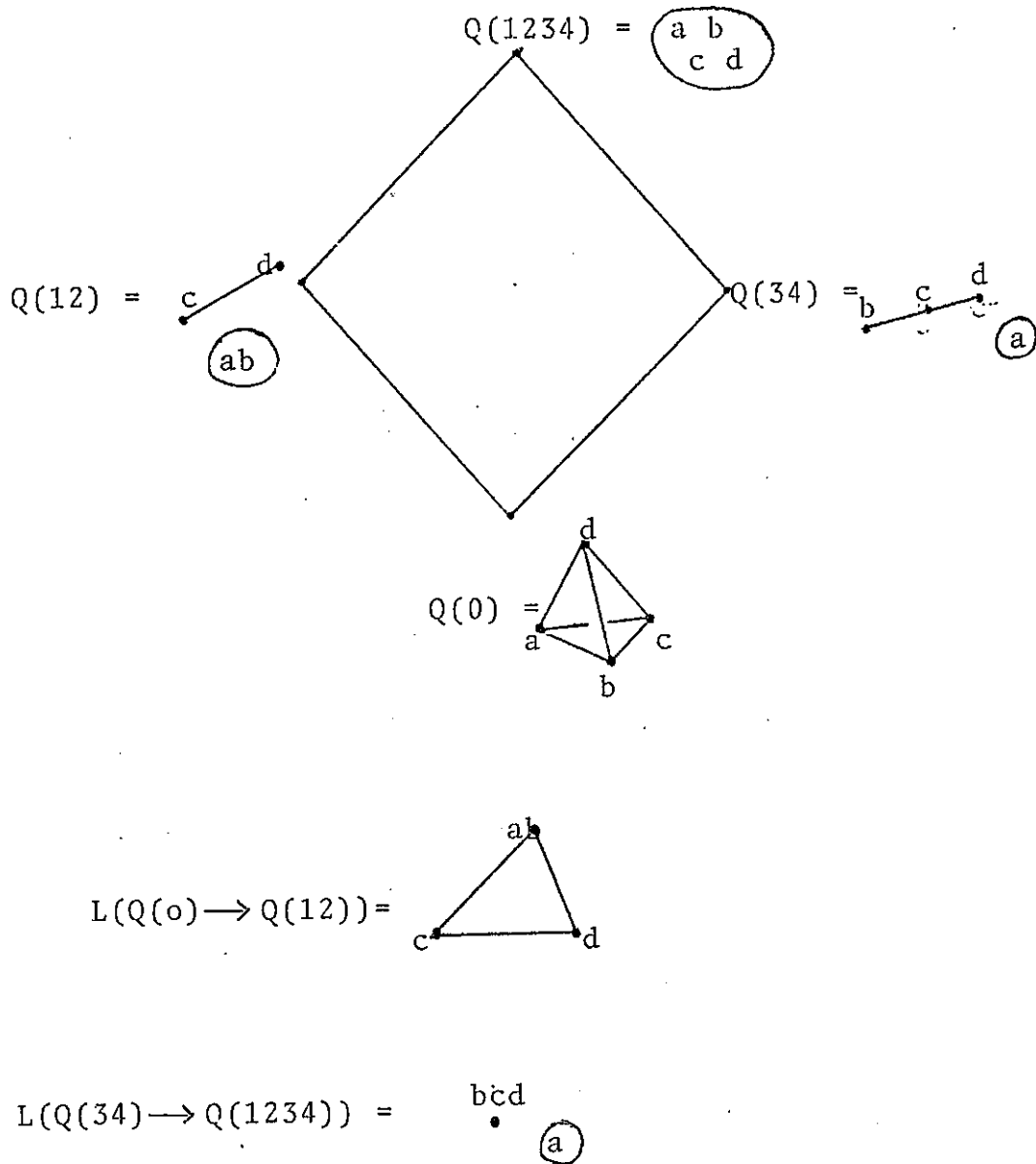


figure 6.6

$L(Q(34) \rightarrow Q(1234))$ is not a quotient of $L(Q(0) \rightarrow Q(12))$,
 so $Q(I)$ is not completable.

EXAMPLE 6.23 Let F be the free geometry of four points 1,2,3,4,
 and let $I = \{0,12,34,1234\}$. Consider the family of quotients
 $Q(I)$ as shown in figure 6.7.

figure 6.7

Let $H = L(Q(0) \rightarrow Q(34))$ and $K = L(Q(12) \rightarrow Q(1234))$. Since

$$\begin{aligned} M(H \rightarrow K) \cap M(H \rightarrow Q(34)) &= \{ab,abcd\} \cap \{ab,cd,abcd\} \\ &= \{ab,abcd\} \\ &\neq M(H \rightarrow Q(1234)), \end{aligned}$$

so $Q(I)$ is not completable.

EXAMPLE 6.24 Let F be the free geometry of four points 1,2,3,4,
 and let $I = \{0,12,34,1234\}$. Let $Q(I)$ be a family of quotients
 of a geometry G as shown in figure 6.8

figure 6.8

$Q(I)$ is 1-completable; the family of quotients $Q(I_1)$ is shown
 in figure 6.9

figure 6.9

$Q(I)$ is also 2-completable and the least completion $Q(F)$ of
 $Q(I)$ is shown in figure 6.10.

figure 6.10

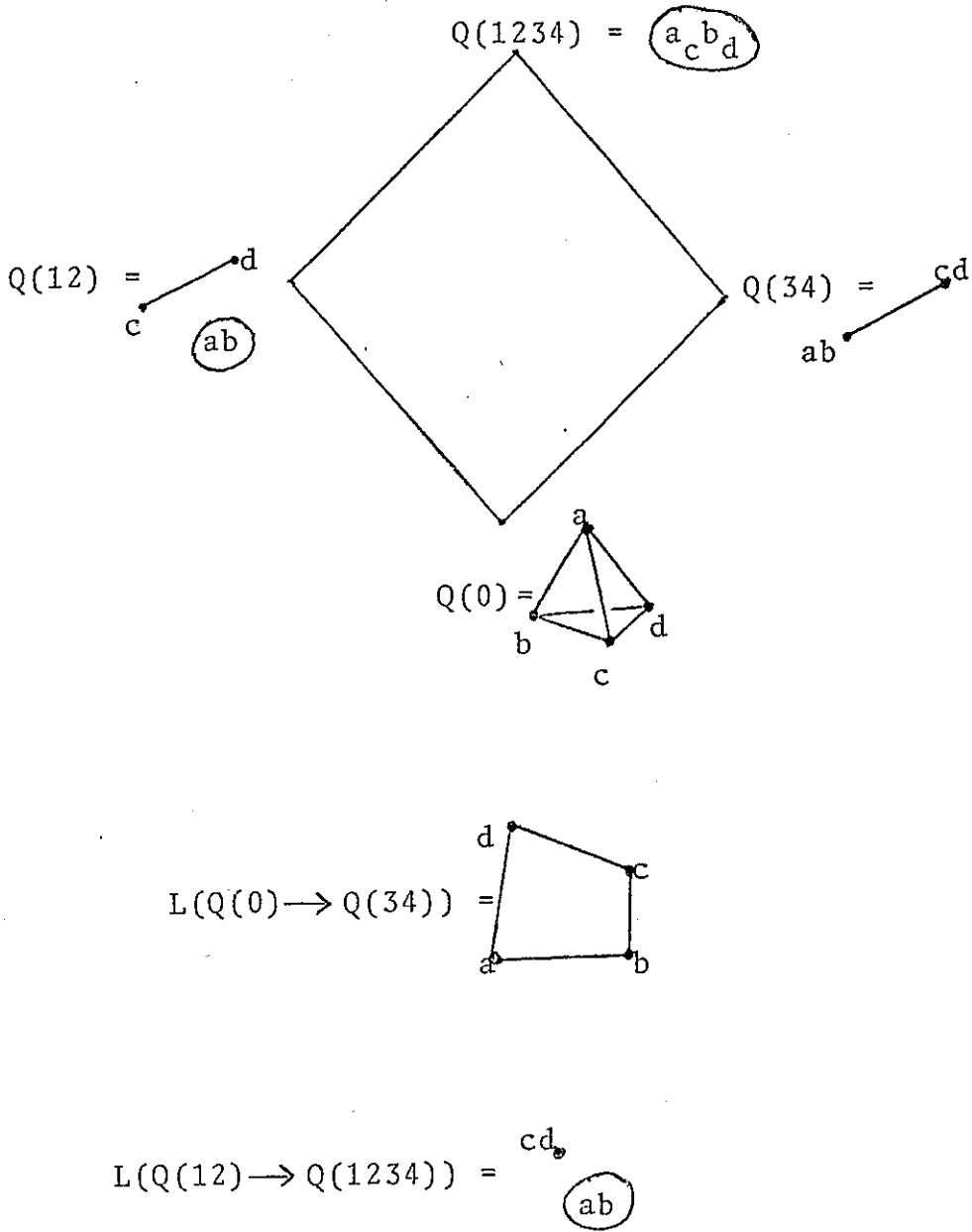


figure 6.7

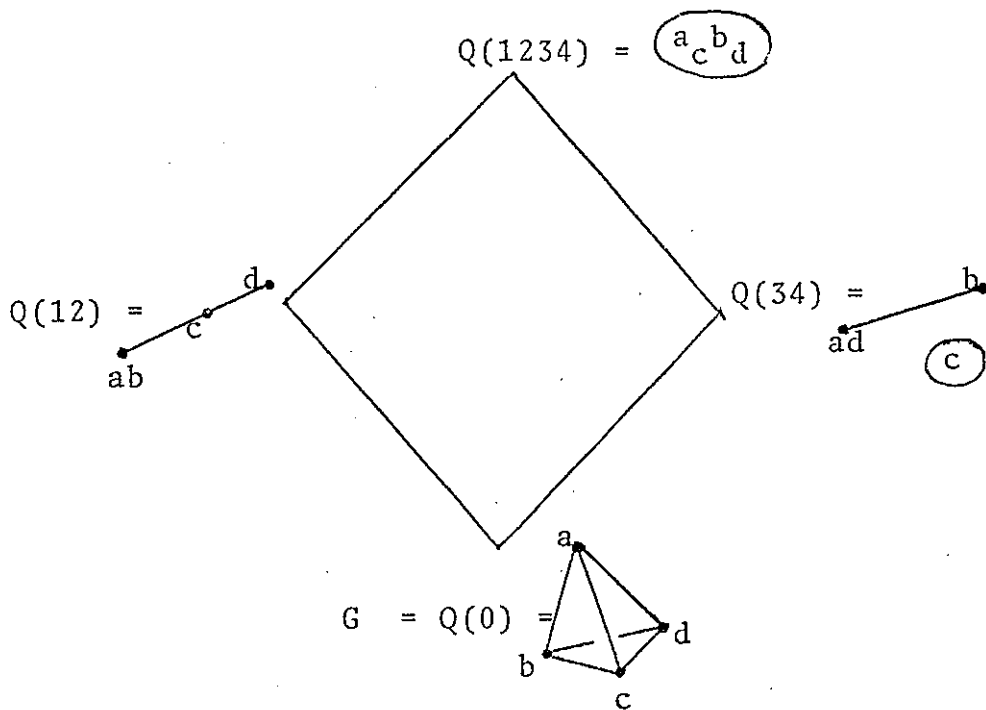


figure 6.8

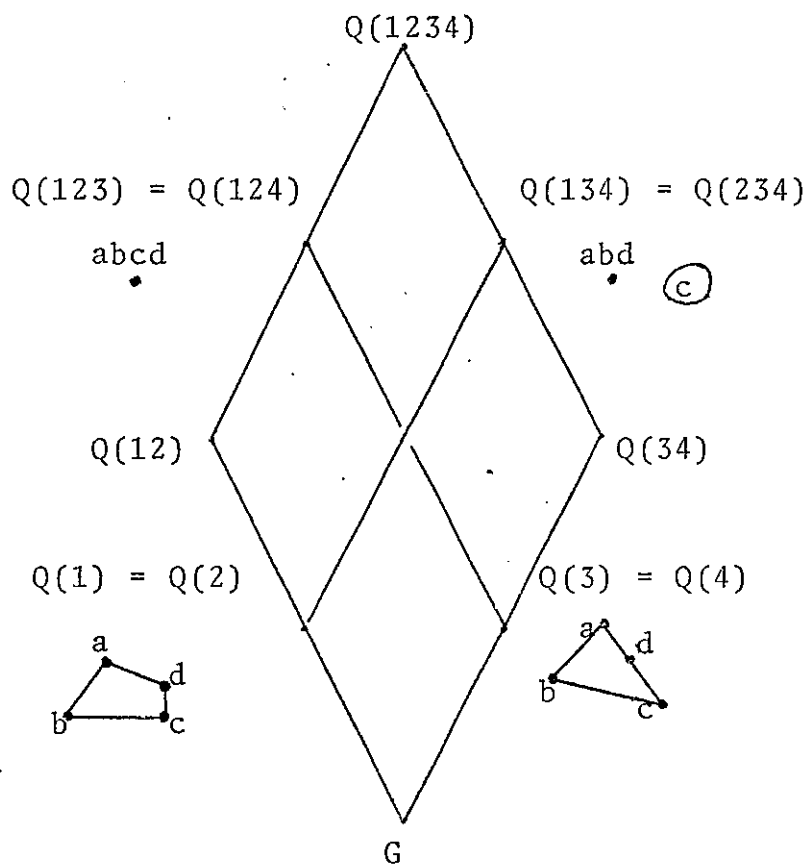
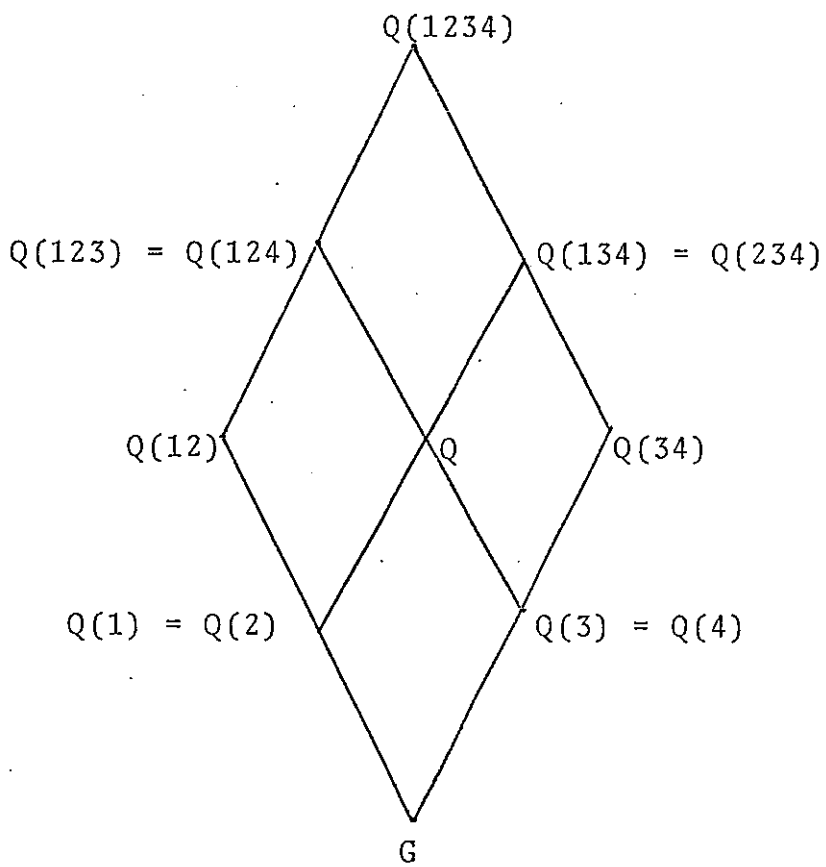


figure 6.9



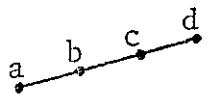
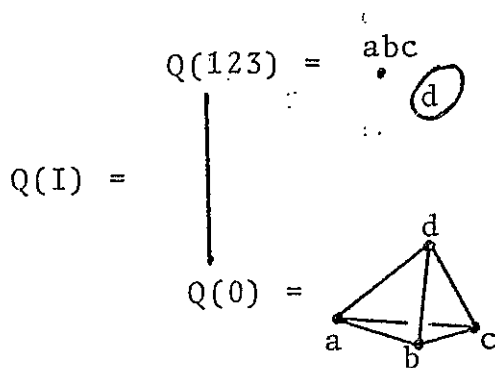
$Q = Q(13) = Q(14) = Q(23) = Q(24) =$


figure 6.10

Given a Boolean subalgebra I of a free geometry, the completions of a partial bundle $Q(I)$ need not form a semi-lattice.

EXAMPLE 6.25 Let F be the plane geometry of three points 1,2,3 and let $I=\{0, 123\}$. Consider the following partial bundle:



The two completions of $Q(I)$ shown in figure 6.11 have no infimum.

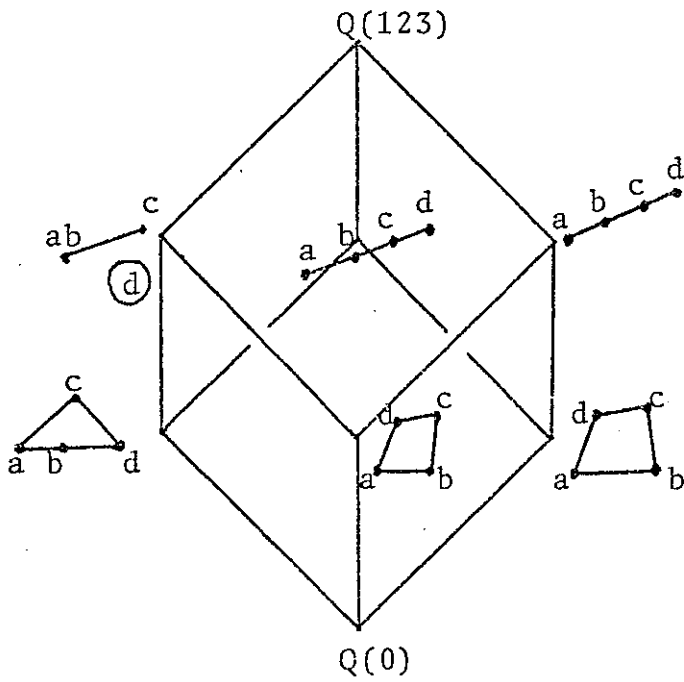
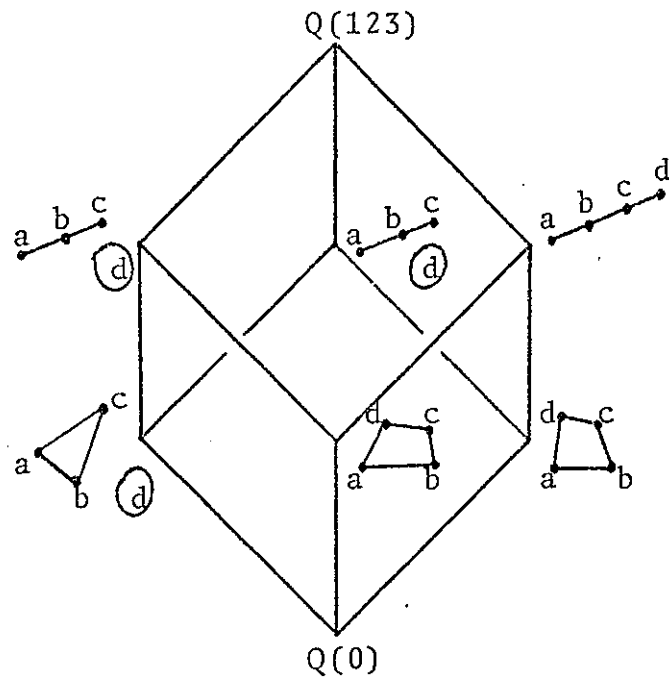


figure 6.11

VII. EXTREMAL COMPLETIONS OF PARTIAL BUNDLES II

7.1 Some lattice-theoretic preliminaries

Throughout this chapter, we let $F(E)$ be the free geometry on a (finite) set E and let I be a complete sublattice of F , i.e., a sublattice of F containing 0 and 1.

For any flat $a \in F$, let

$$I(a) = \bigwedge \{x \in I \mid x \geq a\};$$

and for any point $a \in E$, let

$$\underline{a} = \{b \in E \mid I(a) = I(b)\}.$$

Thus $\underline{E} = \{\underline{a} \mid a \in E\}$ is a partition of the set E of points.

As an illustration, consider the free geometry F on six points a, b, c, d, e, f , and a sublattice I of F as shown in figure 7.1.

figure 7.1

The set $\{a, b, c, d, e, f\}$ is partitioned into $\{\{a\}, \{b, c\}, \{d\}, \{e, f\}\}$.

PROPOSITION 7.1 For any subset $A \subseteq E$, $A \in \underline{E}$ if and only if $A = y \setminus x$ for some covering pair x, y in I (such that y is \vee -irreducible).

Proof: If y covers x in I , we let $a \in y \setminus x$ and claim that $y \setminus x = \underline{a}$. Since $I(a) \leq y$, $I(a) \not\leq x$ and y covers x , so $I(a) \vee x = y$ and therefore $y \setminus x \leq I(a)$. Thus $y \setminus x \leq \underline{a}$. For any $b \in \underline{a}$, $b \leq I(b) = I(a) \leq y$, and since $I(b) = I(a) \not\leq x$, so $b \not\leq x$ and therefore $b \in y \setminus x$. Hence $y \setminus x = \underline{a}$.

On the other hand, if $A = \underline{a}$ for some $a \in E$, let $y = I(a)$. y must be \vee -irreducible for if otherwise $x_1 \neq x_2$ are covered by y in I , then either $a \leq x_1$ or $a \leq x_2$, contradicting the fact that $I(a) = y$. Thus y covers a unique element x in I .

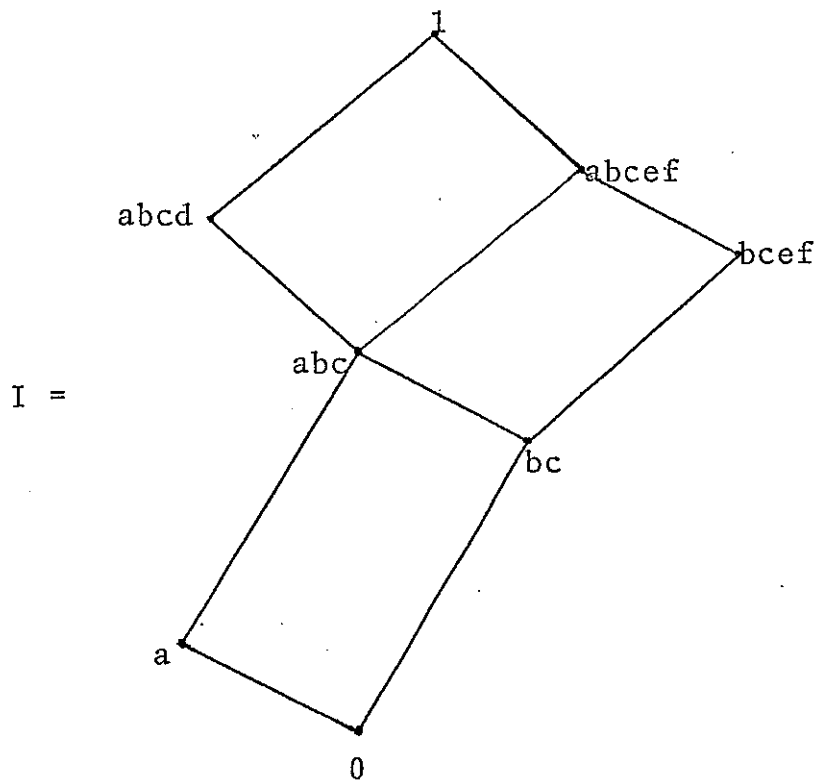


figure 7.1

Since $a \notin x$, so $a \in y \setminus x$ and hence $\underline{a} = y \setminus x$. \square

Define a sequence

$$0 = u_0 < u_1 < \dots < u_n = 1$$

in I recursively by the following prescription:

$$u_i = \bigvee \{x \in I \mid x \text{ covers } u_{i-1} \text{ in } I\}, \quad 1 \leq i \leq n.$$

Also, for each $i=1,2,\dots,n$, define

$$P_i = \{x \setminus u_{i-1} \mid x \text{ covers } u_{i-1} \text{ in } I\}$$

$$\text{and } A_i = [0, u_i \setminus u_{i-1}]_F (= \{x \in F \mid 0 \leq x \leq u_i \setminus u_{i-1}\}).$$

It is immediate that for any $i=1,2,\dots,n$,

$$P_i = \{y \setminus x \mid y \text{ covers } x \text{ in } [u_{i-1}, u_i]_I\}$$

$$\text{and } u_i = u_{i-1} \vee (\bigvee P_i) = \bigvee_{k \leq i} (\bigvee P_k).$$

PROPOSITION 7.2 \underline{E} is partitioned by P_1, P_2, \dots, P_n .

Proof: It is clear that by proposition 7.1, $P_i \subseteq \underline{E}$ for any $1 \leq i \leq n$. If $x \in P_i$, then $x \leq u_i$ and $x \notin u_{i-1}$. It follows that the P_i 's are pairwise disjoint.

If $\underline{a} \in \underline{E}$, then by proposition 7.1, $\underline{a} = I(a) \setminus b$ for some $b \in I$. Let i be the least integer such that $I(a) \leq u_i$. But then

$$\underline{a} = (I(a) \vee u_{i-1}) \setminus (b \vee u_{i-1})$$

and so $\underline{a} \in P_i$. \square

A sequence of sublattices from I to F :

$$I = I_0 \subset I_1 \subset \dots \subset I_n = F$$

is defined recursively by the following:

$$I_i = I_{i-1} \vee A_i \quad (= \{s \vee a \mid s \in I_{i-1}, a \in A_i\})..$$

7.2 An extremal completion construction

Let $Q(I)$ be a partial bundle of quotients of a geometry G , where the indexing set I is a complete sublattice of a free geometry. The purpose of this section is to show that the least completion of $Q(I)$ always exists and to give a direct construction.

Let $Q(F)$ be any completion of $Q(I)$ and let $R(X \cup E)$ be the corresponding extension. As in the previous chapter, we produce an extra copy of E :

$$E' = \{a' \mid a \in E\}.$$

Putting each point $a' \in E'$ in general position on the flat $\overline{I(a)}^R$ in $R(X \cup E)$, we have a further extension

$$R(X \cup E) \rightarrow H(X \cup E \cup E').$$

We are going to show that E' is an independent set in H . If x is a subset of E , we denote by x' the corresponding subset $\{a' \mid a \in x\}$ of E' .

LEMMA 7.3 Let $\underline{a} \in P_i$, $1 < i < n$. If $\overline{u_{i-1}'}^H = \overline{u_{i-1}}^H$, then $u_{i-1}' \cup \underline{a}'$ is an independent set in $H(X \cup E \cup E')$.

Proof: We want to show that for any $b \in \underline{a}$ and $B \leq \underline{a}$ such that $b \notin B$, $b' \notin \overline{u_{i-1}'} \cup B'^H$. Suppose otherwise. By lemma 6.7, $\overline{u_{i-1}'} \cup B'^H \geq \overline{I(b)}^H$ and so

$$\begin{aligned} \overline{u_{i-1}'} \cup B'^H &\geq \overline{I(b)}^H \vee \overline{u_{i-1}'}^H \\ &= \overline{I(a)}^H \vee \overline{u_{i-1}}^H \\ &= \overline{I(a) \vee u_{i-1}}^H \\ &= \overline{\underline{a} \vee u_{i-1}}^H \\ &= \overline{\underline{a} \vee u_{i-1}}^H. \end{aligned}$$

$$\begin{aligned}
\text{But then } r_H(u_{i-1}') + r_H(B') &\geq r_H(u_{i-1}' \cup B') \\
&\geq r_H(\underline{a} \vee u_{i-1}') \\
&= |\underline{a}| + |u_{i-1}'|
\end{aligned}$$

and so $r_H(B') \geq |\underline{a}|$ which is a contradiction. \square

PROPOSITION 7.4 The set E' is independent in $H(X \cup E \cup E')$.

Proof: We show by induction on k that $u_k' (= \bigcup_{i \leq k} (\bigcup P_i'))$ is independent in H for any $k = 1, 2, \dots, n$.

For any $\underline{a} \in P_1$, $I(\underline{a}) = \underline{a}$, so by proposition 6.9, the set $\bigcup P_1'$ is independent. Suppose u_{k-1}' is independent and let $\underline{a} \in P_k$ and $B \subseteq P_k$ be such that $\underline{a} \notin B$. To complete the proof, we need to show that $\underline{a}' \notin \overline{u_{k-1}' \cup B}^H$. Suppose otherwise. Let $C = (\bigcup \{\underline{b} \in P_k \mid \underline{b} \neq \underline{a}\}) \cup B$. By lemma 6.7,

$$\overline{u_{k-1}' \cup C}^H \geq \overline{u_{k-1}' \cup B}^H \geq \overline{I(\underline{a})}^H,$$

and by lemma 7.3,

$$\overline{u_{k-1}' \cup \underline{b}}^H = \overline{u_{k-1}' \cup \underline{b}}^H, \text{ for any } \underline{b} \in P_k.$$

Thus

$$\begin{aligned}
\overline{u_{k-1}' \cup C}^H &\geq \overline{(\bigcup \{u_{k-1}' \cup \underline{b}' \mid \underline{b} \in P_k, \underline{b} \neq \underline{a}\}) \cup (u_{k-1}' \cup I(\underline{a}))}^H \\
&= \overline{(\bigcup \{u_{k-1}' \cup \underline{b} \mid \underline{b} \in P_k, \underline{b} \neq \underline{a}\}) \cup (u_{k-1}' \cup \underline{a})}^H \\
&= \overline{\bigcup \{u_{k-1}' \cup \underline{b} \mid \underline{b} \in P_k\}}^H \\
&= \overline{u_k}^H.
\end{aligned}$$

But then $|\overline{u_{k-1}' \cup C}^H| \geq r_H(u_{k-1}' \cup C) \geq |r_H(u_k)| = |u_k|$, a contradiction. \square

Since E' is independent in $H(X \cup E \cup E')$, the subgeometry $F'(E')$ of $H(X \cup E \cup E')$ on E' is free and isomorphic to $F(E)$. The subgeometry $R'(X \cup E')$ of $H(X \cup E \cup E')$ on $X \cup E'$ is an extension of both $G(X)$ and $F'(E')$ and so determines an F' -bundle $Q'(F')$ on $G(X)$. If we let $Q'(a) = Q'(a')$ for all $a \in F$, then $Q'(F)$ is an F -bundle on $G(X)$.

PROPOSITION 7.5 $Q'(F)$ is a completion of $Q(I)$.

Proof: For each $a \in I$, $\overline{a'}^H \leq \overline{a}^H$ and $r_H(a') = [a']$, so $\overline{a'}^H = \overline{a}^H$.

$$\begin{aligned} \text{Thus } Q'(a) &= \text{Quo}(G \rightarrow R' \rightarrow R'/a') \\ &= \text{Quo}(G \rightarrow H \rightarrow H/a') \\ &= \text{Quo}(G \rightarrow H \rightarrow H/a) \\ &= \text{Quo}(G \rightarrow R \rightarrow R/a) \\ &= Q(a). \quad \square \end{aligned}$$

For any $a \leq b \in F$, we let $M'(a,b)$ denote the modular filter $M_{|b \setminus a|}(Q'(a) \rightarrow Q'(b))$.

PROPOSITION 7.6 Let $s \in I_{i-1}$, and let b cover a in A_i .

If $s \vee a \neq s \vee b$, then

$$M'(s \vee a, s \vee b) = M'(s \vee a, s \vee a \vee I(b \setminus a)).$$

Proof: Let $x \in M'(s \vee a, s \vee b)$. Then $\overline{x \cup s' \vee a'}^H \geq \overline{I(b \setminus a)}^H = \overline{I(b' \setminus a')^H}$ and so $\overline{x \cup s' \cup a'}^H \geq \overline{s' \cup a' \cup I(b' \setminus a')^H}$. Thus

$$x \in M'(s \vee a, s \vee a \vee I(b \vee a)). \quad \square$$

THEOREM 7.7 The F -bundle $Q'(F)$ defined above is the weakest completion of the partial bundle $Q(I)$.

Proof: Let $P(F)$ be any completion of $Q(I)$. We show by induction on i that $Q'(I_i) \leq P(I_i)$ for all $0 \leq i \leq n$.

If $i = 0$, $Q'(I) = Q(I) = P(I)$. Suppose $Q'(I_{i-1}) \leq P(I_{i-1})$.

For each $0 \leq j \leq r(u_i \setminus u_{i-1})$, we let

$$A_i^j = \{x \in A_i \mid r(x) \leq j\},$$

$$\text{and } J_j = I_{i-1} \vee A_i^j.$$

We want to show by induction on j that $Q'(J_j) \leq P(J_j)$ for all $0 \leq j \leq r(u_i \setminus u_{i-1})$.

When $j = 0$, $Q'(J_0) = Q'(I_{i-1}) = P(J_0)$. Suppose $Q'(J_{j-1}) \leq P(J_{j-1})$ and let $s \in I_{i-1}$ and $b \in A_i^j$. Let a be any element in A_i^j covered by b . Then $s \vee a$, $I(b \vee a) \vee s \vee a \in J_{j-1}$, so by induction hypothesis,

$$Q'(s \vee a) \leq P(s \vee a)$$

$$\text{and } Q'(I(b \setminus a) \vee s \vee a) \leq P(I(b \setminus a) \vee s \vee a).$$

If $s \vee a = s \vee b$, then $Q'(s \vee b) \leq P(s \vee b)$. Otherwise, by proposition 7.6 and theorem 6.13, $Q'(s \vee b) \leq P(s \vee b)$ \square

For every $i = 1, 2, \dots, n$ and every $j = 0, 1, \dots, r(u_i \setminus u_{i-1})$, we define

$$I_i^j = I_{i-1} \vee \{x \in A_i \mid r(x) \leq j\}.$$

Thus we have a super-sequence

$$\begin{aligned} I = I_0 \leq \dots \leq I_1 \leq \dots \leq I_{i-1} = I_i^0 \leq I_i^1 \leq \dots \leq I_i^{r(u_i \setminus u_{i-1})} = I_i = I_{i+1}^0 \\ \leq \dots \leq I_n = F \end{aligned}$$

of the sequence

$$I = I_0 \leq I_1 \leq \dots \leq I_n = F$$

we defined earlier.

If a family $Q(I)$ of quotients of a geometry G is completable

to an F-bundle, its least completion $Q(F)$ gives subfamilies $Q(I_i^j)$ in such a way that if $i \neq 0$, $Q(I_i^j)$ (we can assume that $j \neq 0$) is definable from the preceding $Q(I_i^{j-1})$ as follows:

for any $s \in I_{i-1}$ and $b \in A_i$ such that $r(b) = j$,

$$Q(s \vee b) = \begin{cases} Q(s \vee a), & \text{if } s \vee b = s \vee a \text{ for some } a \text{ covered} \\ & \text{by } b \text{ in } A_i \\ \text{the elementary quotient of } Q(s \vee a) \text{ correspond-} \\ \text{ing to the modular filter } M(s \vee a, I(b \setminus a) \vee s \vee a), \\ \text{where } a \text{ is any element in } A_i \text{ covered by } b, \\ \text{if otherwise.} \end{cases}$$

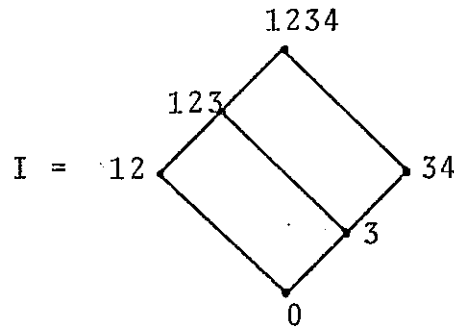
The above sequential construction provides a standard procedure to determine the completability of a family $Q(I)$ of quotients of a geometry G . Having constructed $Q(I_i^{j-1})$, for any $s \in I_{i-1}$ and for any $b \in A_i$ of rank j such that $s \vee a \neq s \vee b$, the quotient $Q(s \vee b)$ should not depend on the choice of the element in A_i covered by b . It is necessarily true that the following two properties are satisfied by all the $Q(I_i^j)$'s:

- 1) if $b \geq a$ in I_i^j , then $Q(b)$ is a quotient of $Q(a)$,
- and 2) if $a, b, a \wedge b, a \vee b \in I_i^j$ and a, b cover $a \wedge b$, then

$$M(a \wedge b, a) \cap M(a \vee b, b) = M(a \wedge b, a \vee b).$$

The family $Q(I)$ is completable to an F-bundle if and only if each $Q(I_i^j)$ is constructable from $Q(I_i^{j-1})$ in the above manner, and if so, the construction gives the least completion.

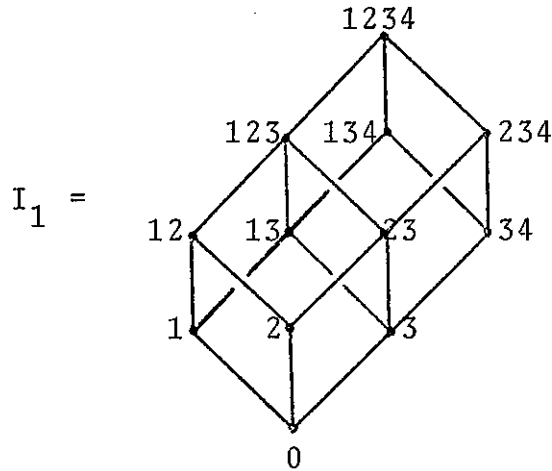
EXAMPLE 7.8 Let I be the sublattice of the free geometry F of four points 1,2,3,4, as shown below:



In accordance with the notations developed in this chapter, the sequence of sublattices from I to F is:

$$I=I_0 \subset I_1 \subset I_2=F$$

where



We want to discuss the completability of a family of quotients $Q(I)$, which is shown in figure 7.2

figure 7.2

The first step of the completion construction gives the family of quotients $Q(I_1)$ as shown in figure 7.3.

figure 7.3

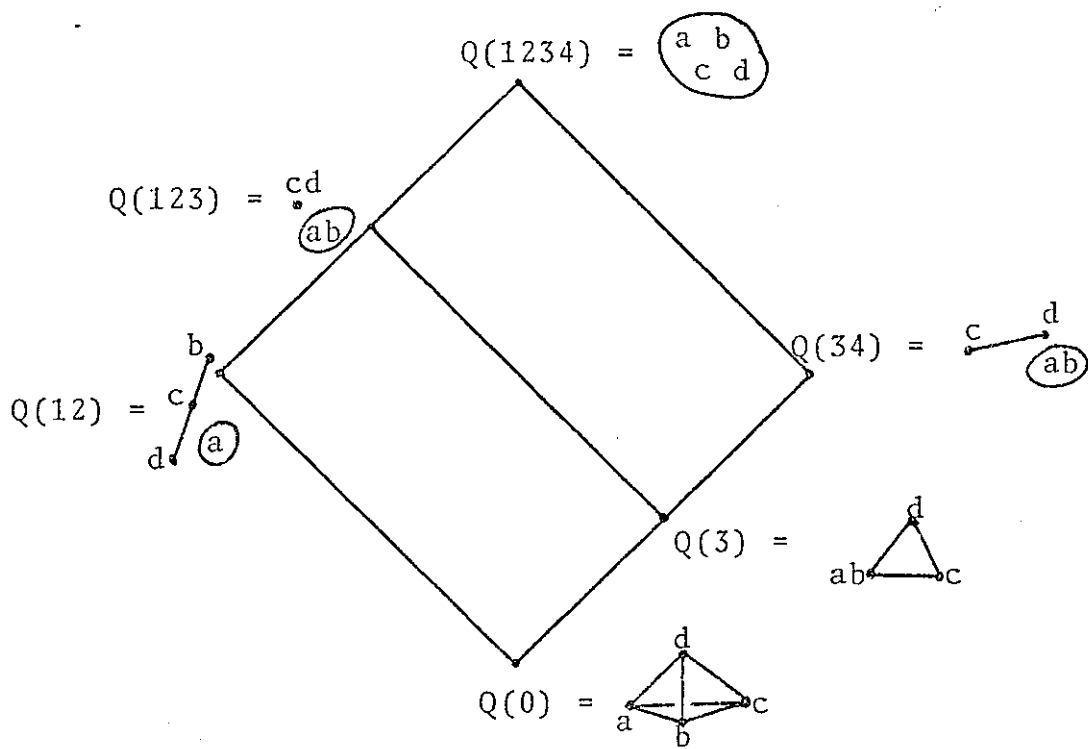
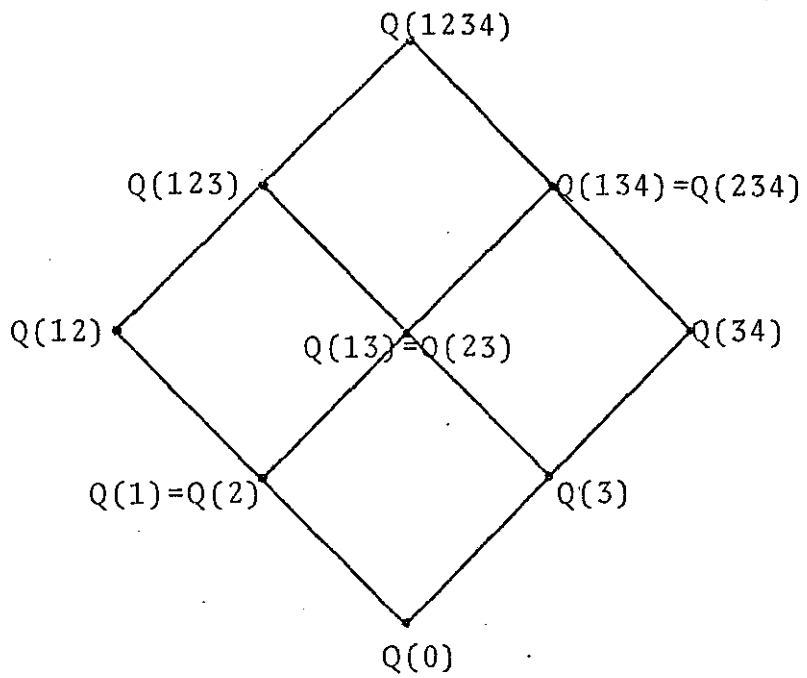


figure 7.2



$$Q(1) = Q(2) = D(Q(0) \rightarrow Q(12)) = \begin{array}{c} d \\ \diagup \quad \diagdown \\ c \quad b \\ \diagdown \quad \diagup \\ a \end{array}$$

$$Q(13) = Q(23) = D(Q(3) \rightarrow Q(123)) = \begin{array}{c} d \\ \diagup \quad \diagdown \\ c \quad b \end{array}$$

$$Q(134) = Q(234) = D(Q(34) \rightarrow Q(1234)) = \begin{array}{c} d \\ \diagup \quad \diagdown \\ c \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} \quad \textcircled{ab}$$

figure 7.3

If the least completion $Q(F)$ of $Q(I)$ exists, then

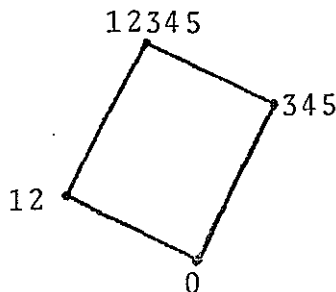
$$Q(4) = D(Q(0) \rightarrow Q(34)) = \begin{array}{c} d \\ \diagup \quad \diagdown \\ ab \quad c \end{array}$$

$$\text{and } Q(124) = D(Q(12) \rightarrow Q(1234)) = \begin{array}{c} bcd \\ \bullet \\ a \end{array}$$

and so $Q(124)$ is not a quotient of $Q(4)$, a contradiction. Thus $Q(I)$ is not a partial bundle.

Finally, we want to remark that if I is a Boolean subalgebra of a free geometry F , then for any partial bundle $Q(I)$, the construction for the least completion of $Q(I)$ given in this chapter is not necessarily the same as that in the previous chapter.

EXAMPLE 7.9 Let F be the free geometry of five points 1,2,3,4,5, and let I be the sublattice of F shown below:



Let $Q(I)$ be a partial bundle as shown in figure 7.4.

figure 7.4

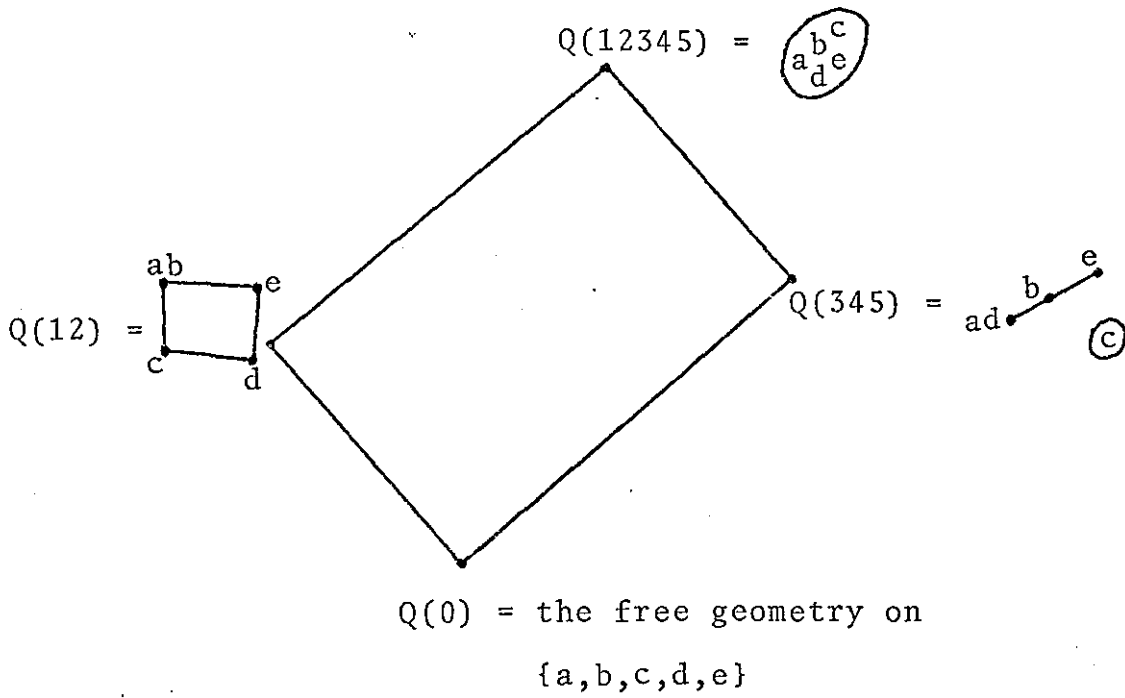
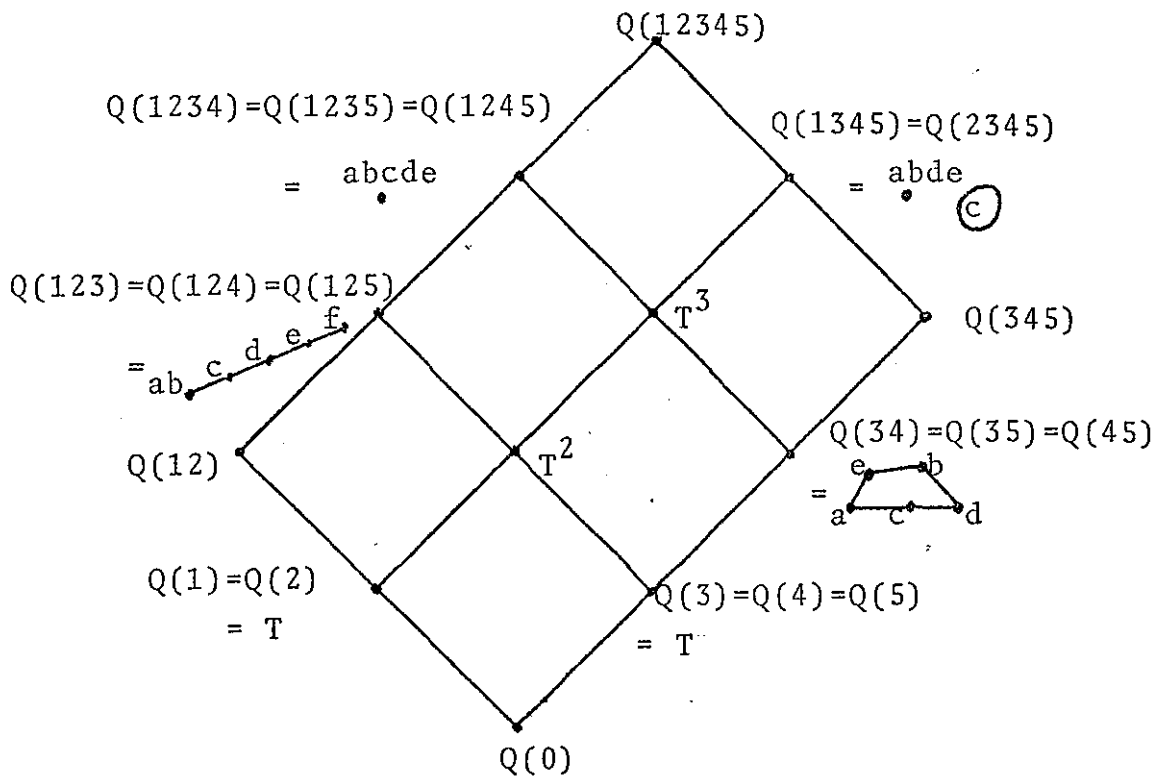


figure 7.4



T = truncation of $Q(0)$
 $Q(13)=Q(14)=Q(15)=Q(23)=Q(24)=Q(25)$
 $= T^2$ = second truncation of $Q(0)$
 $Q(134)=Q(135)=Q(145)=Q(234)=Q(235)=Q(245)$
 $= T^3$ = third truncation of $Q(0)$.

figure 7.5

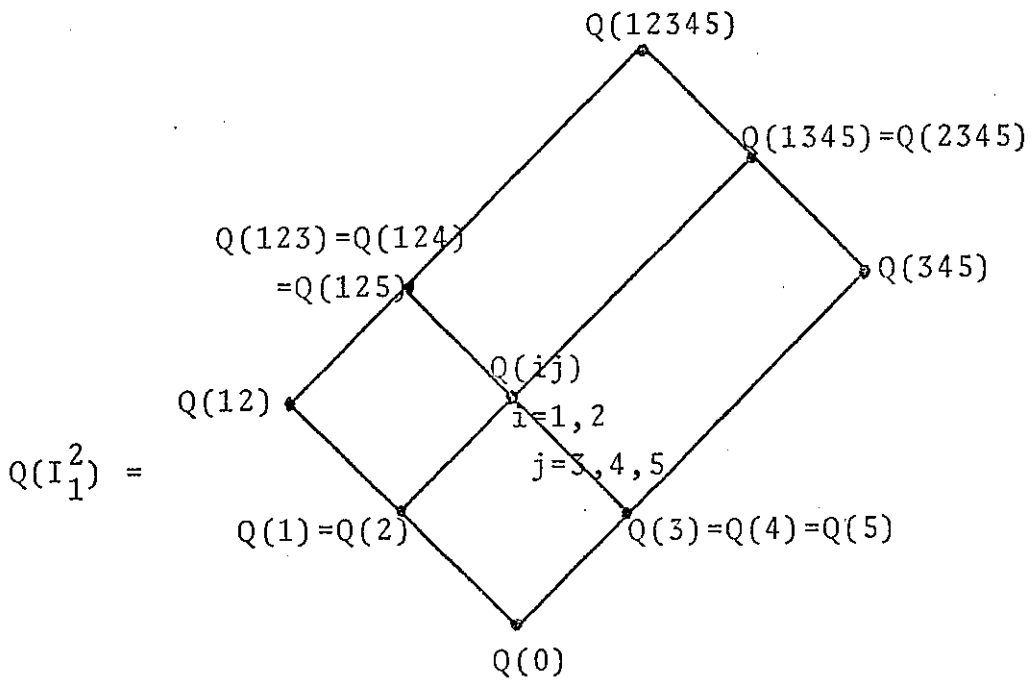
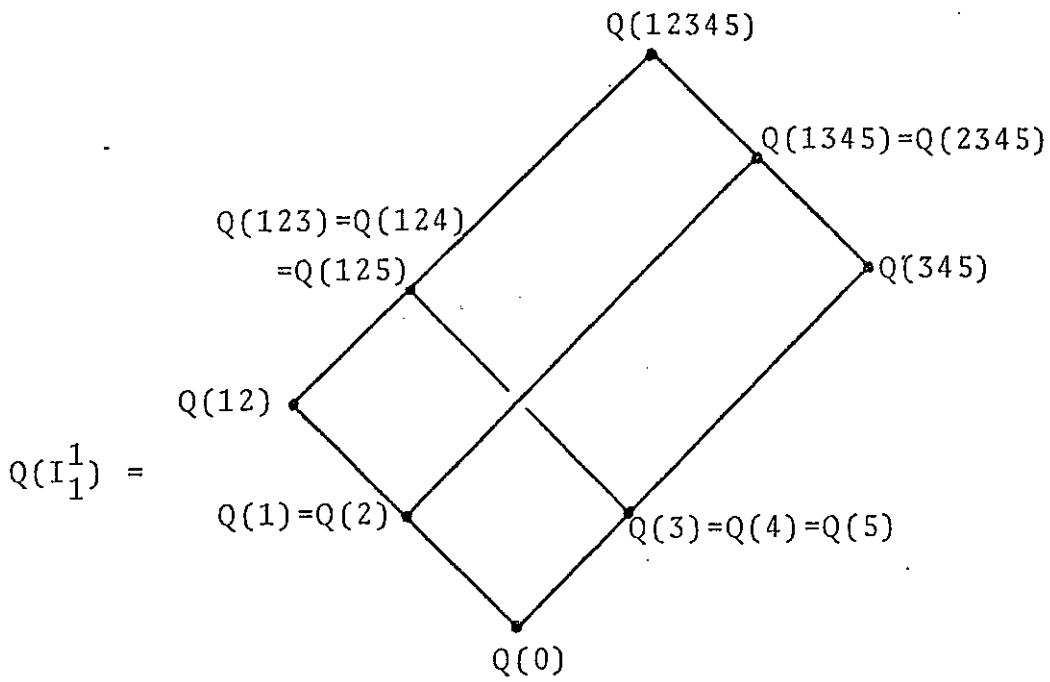


figure 7.6

The least completion $Q(F)$ of $Q(I)$, as shown in figure 7.5

figure 7.5

is constructed via the sequence

$$Q(I) = Q(I_1^0) \rightarrow Q(I_1^1) \rightarrow Q(I_1^2) \rightarrow Q(I_1^3) = Q(F)$$

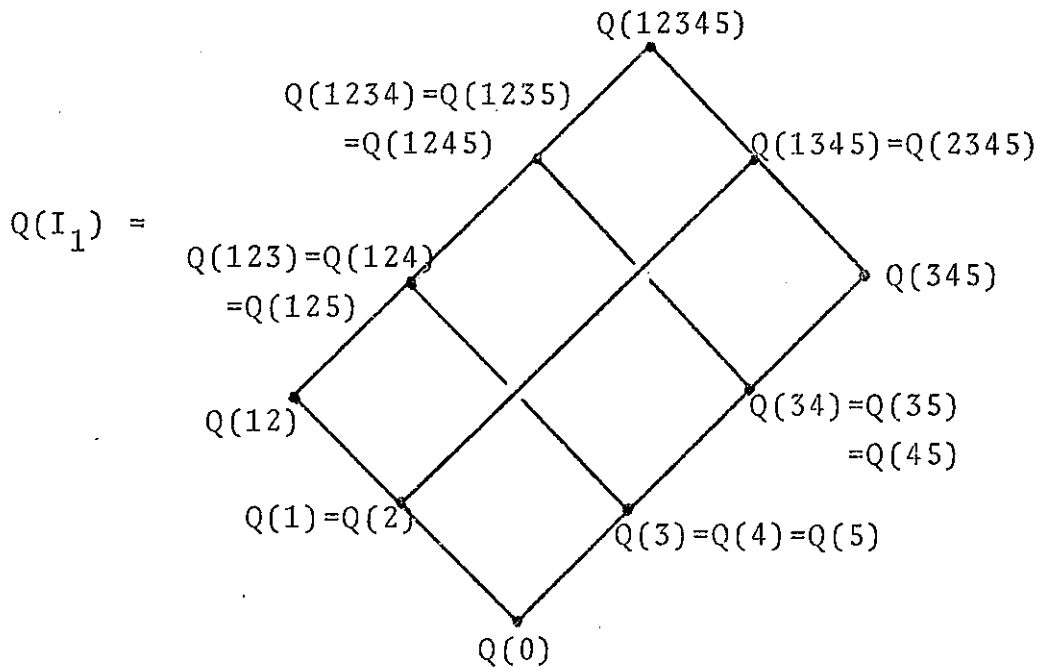
where $Q(I_1^1)$ and $Q(I_1^2)$ are shown in figure 7.6.

But, with the construction given in the last chapter,

$Q(F)$ is obtained via the following sequence:

$$Q(I) = Q(I_0) \rightarrow Q(I_1) \rightarrow Q(I_2) = Q(F)$$

where



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