Planar graphs without 3-cycles and with 4-cycles far apart are 3-choosable

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2016

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

A graph G is said to be L-colourable if for a given list assignment $L = \{L(v)|v \in V(G)\}$ there is a proper colouring c of G such that $c(v) \in L(v)$ for all v in $V(G)$. If G is L-colourable for all L with $|L(v)| \geq k$ for all v in $V(G)$, then G is said to be k-choosable.

This paper focuses on two different ways to prove list colouring results on planar graphs. The first method will be discharging, which will be used to fuse multiple results into one theorem. The second method will be restricting the lists of vertices on the boundary and applying induction, which will show that planar graphs without 3-cycles and 4-cycles distance 8 apart are 3-choosable.

Acknowledgements

I would like to thank:

- my advisor Dr. Bruce R. Richter for suggesting the topic for my thesis, suggesting edits to my thesis, and for all around putting up with my crazy train of thought;
- my friends for keeping me on track when needed and for supplying distractions when I needed a break;
- the Department of Combinatorics and Optimization for giving me the opportunity to pursue my interests; and
- my readers for taking the time to read my thesis.

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Chapter 1: Introduction

H. Grötzsch published one of the most famous theorems in graph colouring in 1959. He stated that every planar graph without 3-cycles is 3-colourable and today this is known as Grötzsch's Theorem. His proof, although correct, was also complex. Therefore, short and concise proofs of this theorem were well sought after. In 1995, Thomassen published [11], which contained a similar result for list colouring planar graphs. Thomassen's theorem stated that every planar graph with girth 5 is 3-choosable. He then went on to use this result to publish [12], which contains a new proof of Grötzsch's Theorem. To date, Thomassen's proof of Grötzsch's Theorem is one of the shortest and most concise proofs we know.

The results in this paper focus on families of planar graphs that are 3-choosable that were not covered by Thomassen's theorem. After Thomassen released his list colouring result for planar graph, there has been a vast influx of results pertaining to planar graphs without certain cycles. Theorem 1 shows some recent results in this area.

Theorem 1. Every planar graph without x-cycles is 3-choosable.

3	4	5	6	7	8	9	Authors	Year
x	x						[11] Thomassen	1995
	x	x	x	x	x	x	[2] Borodin	1996
x			x	x		x	[20] Zhang and Xu	2004
x		x			x	x	[16] Zhang	2005
x		x	x				[8] Lam, Shiu and Song	2005
x					x	x	[21] Zhang, Xu and Sun	2006
x					x	x	[22] Zhu, Miao and Wang	2007
x				x	x		[5] Dvořák, Lidický and Škrekovski	2009
x			x	x			[6] Dvořák, Lidický and Škrekovski	2010
	x	x	x	x	x		[7] Dvořák and Postle	2015
	x	*	*	*	*	x	[17]-[15] Varies	Varies

Note that the last theorem has stars for cycles of length 5 to 8. This is because it has been proven that every planar graph without 4- and 9-cycles, and without any two cycles of length between 5 and 8 is 3-choosable.

Along with removing cycles from planar graphs, we ask what happens when cycles of certain length are further and further apart. Theorem 2 is an example of some of the results in this area that have appeared after Thomassen's theorem.

Theorem 2. Every planar graph without x-cycles and 3-cycles distance $d^\triangledown$ apart is 3-choosable.

4	5	6	7	8	9	$d^\triangledown$	Authors	Year
x	x	x	x	x	x	0	[2] Borodin	1996
x	x	x				3	[10] Montassier, Raspaud, Wang	2006
x	x					4	[10] Montassier, Raspaud, Wang	2006
	x	x	x			3	[19] Zhang and Sun	2008
	x	x		x		2	[19] Zhang and Sun	2008
x	x	x	x	x		0	[7] Dvořák and Postle	2015
x	x	x	x			2	**Theorem 3**	2016

In Chapter 2 we will see how a new results along with results from Borodin, Montassier, Raspaud, and Wang can be combined and proven using discharging. But results in this area do not solely depend on discharging. As an example, in Chapter 3, we will show a proof of Theorem 4 by restricting the lists of the vertices on the boundary of planar graphs and applying induction.

Theorem 4. Let G be a planar graph without 3-cycles such that any two 4-cycles are distance at least 8 apart. Then G is 3-choosable.

Theorem 4 extends Thommasen's Theorem about girth 5 planar graphs. The goal of this theorem was to improve upon a theorem by Dvořák [4], that states planar graphs with 3- and 4-cycles distance 26 apart are 3-choosable. This thesis can be viewed as a proof of concept and we intend to continue simplifying and extending the arguments to include 3-cycles.

Chapter 2: Discharging

The discharging method was developed in the 1970s to attempt to prove the 4-colour theorem. Ultimately, it would be one of the key components to proving one of the biggest theorems in graph colouring and become an important tool in graph theory. As an example of discharging we show that the results in [2] and [10] can be combined using one discharging proof.

Theorem 3. Let G be a planar graph without 4- and 5-cycles. Then G is 3-choosable if it satisfies one of the following conditions:

1. G has no 6- to 9-cycles, (Borodin,1996)
2. G has no 6- to 7-cycles and $d^{\triangledown}(G) \geq 2$,
3. G has no 6-cycles and $d^{\triangledown}(G) \geq 3$, (Montassier-Raspaud-Wang,2006)
4. $d^{\triangledown}(G) \geq 4$. (Montassier-Raspaud-Wang,2006)

Proof. Define $\mathcal{G}$ to be the set of plane graphs with no 4- and 5-cycles. Let $G \in \mathcal{G}$ be a minimal counter example to the theorem and let L be a 3-list assignment on the vertices of G such that G is not L-colourable.

Claim 1. $\delta(G) \geq 3$.

Proof. Suppose there was a vertex v in G such that $deg_G(v) \leq 2$. By minimality of G, there is an L-colouring of $G - v$. Since $deg_G(v) \leq 2$ and $|L(v)| = 3$ we can colour v with a colour from $L(v)\backslash\{c(x)|x \in N_G(v)\}$. This yeields an L-colouring of G, and a contradiction. $\square$

For the sake of the reader, we will note that an i-vertex is a vertex of degree i and an i^+-vertex is a vertex of degree at least i.

Claim 2. For a postive even integer k, G does not contain any chordless k-cycles comprised of all 3-vertices.

Proof. Suppose there was an chordless even cycle $C = (x_1, ..., x_k, x_1)$ such that every vertex in V had degree three. By minimality of G, there is an L-colouring of $G - V(C)$. Place the cycle C back into the graph. For every vertex x_i in $V(C)$ let c_i be the colour of the neighbour of x_i in $G - V(C)$. Define $L'(x_i) = L(x_i)\backslash\{c_i\}$. We have that for all i, $|L(x_i)| \geq 2$. Since even cycles are 2-choosable, there is an L'-colouring of C, yielding an L-colouring of G, and a contradiction. $\square$

Now we use discharging to show that if G has one of the four properties, then G does not exist. Define ℓ_f to be the length of face f. Assign a charge to each face and vertex denoted $ch(x)$ for $x \in V(G) \cup F(G)$. We define the charges to be:

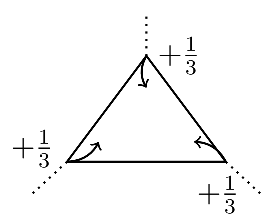
- $ch(v) = deg_G(v) - 4$, if $v \in V(G)$; and
- $ch(f) = \ell_f - 4$, if $f \in F(G)$.

We note that by Euler's formula $|V(G)| - |E(G)| + |F(G)| = 2$ gives us

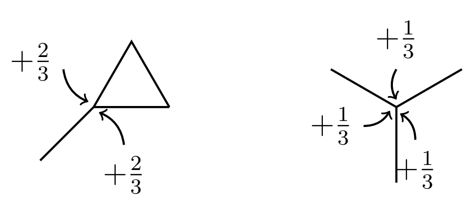
$$\begin{aligned}\sum_{v\in V(G)} ch(v) + \sum_{f\in F(G)} ch(f) &= \sum_{v\in V(G)} (deg_G(v) - 4) + \sum_{f\in F(G)} (\ell_f - 4) \\ &= -4|V(G)| + 4|E(G)| - 4|F(G)| = -8.\end{aligned}$$

Now we define discharging rules on the charged graph in a way that preserves the total charge of the graph.

R1. Each vertex incident to a 3-face gives $+\frac{1}{3}$ to the 3-face.



R2. For $k \geq 6$, each k-face gives $+\frac{2}{3}$ to each incident 3-vertex also incident with a 3-face and $+\frac{1}{3}$ to each 3-vertex not also incident with a 3-face.



R3. For $k \geq 6$, each k-face gives $+\frac{1}{3}$ to each incident 4-vertex that is also either incident to two 3-faces or incident with a 3-face that does not share an edge with the k-face.

It is clear that **R1** makes every 3-face non-negative. Suppose we have a k-face f_k for $k \geq 12$. From the three discharging rules we see that

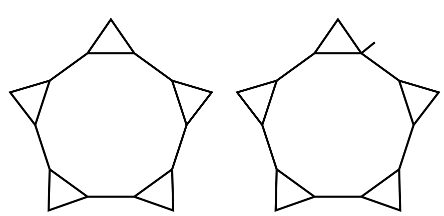
$$ch(f_k) \geq k - 4 - \frac{2k}{3} = \frac{k}{3} - 4 \geq \frac{12}{3} - 4 = 0.$$

If every vertex in a face of length k has degree three, then there are at most $\lfloor \frac{k}{2} \rfloor$ edges that are incident with two 3-vertices and a 3-cycle. This implies that there are at most $2\lfloor \frac{k}{2} \rfloor$ 3-vertices incident with a 3-face in this face of length k. This implies that an 11-face f_{11} has charge

$$ch(f_{11}) \geq 11 - 4 - \frac{2 * 10}{3} - \frac{1}{3} = 0.$$

If a 10-face has negative charge, then it must have nine 3-vertices that are each incident to a 3-cycle. Let f_{10} be such a face. Let v be a vertex of maximum degree in f_{10}. If v has degree three, then f_{10} is a chordless even cycle with every vertex having degree three, a contradiction with Claim 2. Therefore, v has degree at least four. If each of nine 3-vertices is incident to a 3-cycle, then f_{10} must share an edge with exactly five 3-cycles. Since there is no 4-cycle, this implies that v is incident to one 3-cycle and this 3-cycle shares an edge with f_{10}. We count to find that

$$ch(f_{10}) \geq 10 - 4 - \frac{2 * 9}{3} = 0.$$



By **R1**, if v_k is a vertex of degree $k \geq 5$, then

$$ch(v_k) \geq k - 4 - \frac{1}{3}\left\lfloor \frac{k}{2} \right\rfloor \geq 5 - 4 - \frac{1}{3}\left\lfloor \frac{5}{2} \right\rfloor = \frac{1}{3}.$$

Let $v \in V(G)$. It is clear that if $deg_G(v) = 3$, then v is incident with at most one 3-face, as otherwise G has a 4-cycle. Thus, **R2** implies that $ch(v) = 0$. Similarly, if $deg_G(v) = 4$, then **R3** implies that $ch(v) = 0$.

We have that the only possible negative faces are 6- to 9-faces. Since the sum of the charges in G is negative, G must have 6- to 9-cycles.

Using the same argument on 7- to 9- faces we get the following claim.

Claim 3. For $7 \leq k \leq 9$, a k-face f_k has negative charge only if there are at least $k-5$ 3-faces that share an edge with f_k.

Proof. Suppose f_k is a k-face such that $7 \leq k \leq 9$ and f_k has negative charge. Let t be the number 3-vertices in f_k that are incident with a 3-face. It follows that

$$0 > ch(f_k) \geq k - 4 - t\frac{2}{3} - (k-t)\frac{1}{3},$$

or

$$\frac{t}{2} > k - 6.$$

The number of edges in f_k that are in a 3-face is at least $\left\lceil \frac{t}{2} \right\rceil$. The result follows by plugging in values for k. □

Claim 4. G does not have any of the following properties:

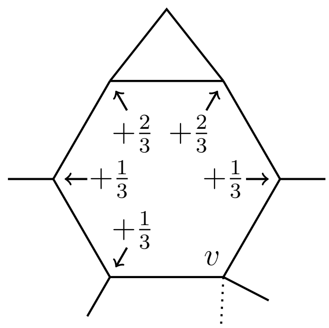
- G has no 6- to 7-cycles and $d^{\triangledown}(G) \geq 2$,
- G has no 6-cycles and $d^{\triangledown}(G) \geq 3$, (Montassier-Raspaud-Wang,2006).

Proof. It follows from Claim 3. □

Therefore, it must be the case that $d^{\triangledown}(G) \geq 4$.

Definition 5. A face f is *bad* if it shares an edge with a 3-face and both vertices incident with this edge are degree three, and a face is *good* otherwise. Similarly, we say a vertex v is *bad* if it is incident with a 3-face and *good* otherwise.

Under our current discharging rules the only negative faces in G are bad 6-faces. Since 6-faces are chordless in G and $d^{\triangledown}(G) \geq 4$, we have that all 6-faces have a good vertex v of degree at least 4 as shown.



Under this case we add two new rules to take care of the bad 6-faces.

- **R4.** Good 4^{+}-vertices give $+\frac{1}{3}$ to each incident bad 6-face.
- **R5.** 7^{+}-faces and good 6-faces give $+\frac{1}{3}$ to each incident good 4- and 5-vertex.

Under these new rules we count the charge of f_k a face of length k. Since 3-faces are distance at least four apart we have that there are at most $\left\lfloor \frac{k}{5} \right\rfloor$ 3-faces that share an edge with a k face. This results in

$$ch(f_k) \geq k-4-2*\frac{2}{3}\left\lfloor \frac{k}{5} \right\rfloor-\frac{1}{3}\left(k-2\left\lfloor \frac{k}{5} \right\rfloor\right) \geq 7-4-2*\frac{2}{3}\left\lfloor \frac{7}{5} \right\rfloor-\frac{1}{3}\left(7-2\left\lfloor \frac{7}{5} \right\rfloor\right) = 0.$$

Originally bad 6-faces had charge at least $6-4-3\frac{1}{3}-2\frac{2}{3} = -\frac{1}{3}$. It is clear that **R4** shows that bad 6-faces now have non-negative charge.

Good 6-faces have charge at least $6-4-6\frac{1}{3} = 6-4-4\frac{1}{3}-\frac{2}{3} = 0$, therefore all faces have non-negative charge.

By **R1** and **R4**, we have that k-vertices have charge at least $k-4-k\frac{1}{3}$. Therefore, for $k \geq 6$, k-vertices have non-negative charge. The counting for 3-vertices, bad 4-vertices, and bad 5-vertices stays unaffected by **R4** and **R5**, therefore they have non-negative charge.

The only thing left to check is that the charge on good 4-vertices and good

5-vertices is non-negative. By **R4** and **R5**, we want to show that good 4-vertices are incident with at most 2 bad 6-faces and good 5-vertices are not incident with 5 bad 6-faces.

The following claim is helpful in doing this.

Claim 6. Suppose a vertex v is incident with distinct 6-faces C_1 and C_2, and T is a 3-face having an edge in each of C_1 and C_2. Then, for some $i \in \{1, 2\}$, an end of an edge in $T \cap C_i$ is within distance 1 of v.

Proof. Suppose v is incident with two 6-faces and a 3-cycle T has an edge in each of the 6-faces. Let one 6-face be bounded by the cycle C_1 and the other bounded by the cycle C_2. Let $x_i y \in E(C_i)$ be in the 3-cycle. By way of contradiction, we may assume that both x_i and y_i are distance ≥ 2 to v in C_i.

Since only one vertex is distance greater than 2 to v in C_i, we may assume x_i is distance 2 from v in C_i. Let P_i be the shortest x_i, v path in C_i. The path obtained by extending P_2 with the edge $x_1 x_2$ and the path P_1 are distinct v, x_1-paths since the edge $x_2 x_1$ is in one and not the other. The two paths together from a closed walk of length 5 that does not contain every edge of T. Since it was formed by two distinct v, x_1-paths, there is a cycle C of length at most 5 in this closed walk that is not T. Since 3-cycles are distance 4 apart, C is not a 3-cycle. Since G does not contain 4- or 5-cycle, C can not exist, therefore our assumption is false and one of x_i or y_i is distance at most 1 to v, as desired. □

Suppose that G has a negatively charged good 4-vertex v. Then v is incident with at least 3 bad 6-faces and let each of these faces be bounded by a cycle C_i.

We begin by proving that none of the 3-faces having an edge in common with any of the C_i contains v. By way of contradiction, suppose that T contains v. Then T is the boundary of the fourth face incident with v. In particular, $T \cap C_1$ and $T \cap C_3$ have edges vx_1 and vx_2, respectively.

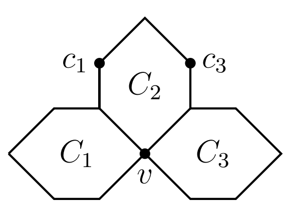
Suppose $x_1 x_2$ is in $E(C_2)$. Since $x_1 \neq v$ and there are no parallel edges incident with v, x_i has distance 2 or 3 from v in C_2. Let P be a shortest $x_1 v$-path in C_2. Adding the edge vx_i to P yields a cycle of length 3 or 4. The latter does not exist and the former is a 3-cycle through v that does not use the edge x_1x_2. Both conclusions are contradictions. Therefore, T does not have a vertex in C_2 other than v.

There is another 3-face T' that has an edge in C_2. One end of this edge is within distance at most 2 of v in C_2. But then T and T' are only distance at most 2 apart, a contradiction. Thus, none of the 3-faces with edges in a C_i contains v.

Now suppose that some 3-face T has edges in all of C_1, C_2 and C_3. Since v is not incident with any of these edges, Claim 6, implies two of the C_i have an edge of T having an end adjacent in C_i to v. The short paths from these ends to v in the C_i together with an edge of T yields a 3-, 4-, or 5-cycle in G containing v, a contradiction.

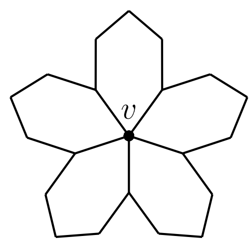
Let {i,j,k}={1,2,3} and suppose that some 3-face T has edges in C_i and C_j , but not C_k. There is another 3-face T' with an edge in C_k. One end of this edge is within distance at most 2 of v in C_2. By Claim 5, T is at most distance 1 to v. Therefore, T and T' are within distance at most 3, a contradiction.

Thus there are three different 3-faces T_1, T_2, T_3 such that , for $i = 1, 2, 3$, $T_i \cap C_i$ is an edge. Each of these edges must be at distance 2 from v in each C_i as otherwise to of the T_1, T_2, T_3 are at distance at most 3 in G, a contradiction. However, the end of $T_2 \cap C_2$ that is distance at most two from v is necessarily at distance at most 3 in G from the end of either $T_1 \cap C_1$ or $T_3 \cap C_3$ that is distance from v in its C_i. This is the final contradiction that shows such a v does not exist. That is, good 4-vertices have non-negative charge.



It remains to show that no good 5-vertex v can be incident with five bad 6-faces but this is actually an easier case. If a 3-cycle shares an edge with two bad 6-faces, then the 3-cycle is distance at most 1 from v. If this is the case, then every edge in the 6-cycles are distance at most 3 from this 3-cycle. This implies that not all of the 6-cycles can be bad, a contradiction.

Therefore, no 3-cycle is incident with more than one bad 6-cycle. It follows that we are looking for a set of five edges that are all distance 4 apart in these five 6-cycles such that no edge has v as its end. Now if we let C be the cycle formed by these five 6-faces that does not include v, C has length 20. It follows that for this set of edges to exist, C must have length at least 25, a contradiction. Therefore, such a v does not exist and good 5-vertices have non-negative charge.



By discharging we have a contradiction with the total charge of the graph being negative, as desired. $\square$

Chapter 3: Boundary Colouring and Induction

Discharging is not the only tool to be used in graph colouring. Another common tool is to work around the boundary of a planar graph and apply induction. This particular work involves restricting the lists of some boundary vertices and applying induction. This technique originated from Thomassen [12] where he originally proves Grötzsch's theorem. Thomassen used a list colouring approach along with restricting a path of vertices on the boundary to be 1-lists and an independent set of vertices on the boundary to be 2-lists.

This technique was then modified in the unpublished work of C. Nuñes da Silva, R.B. Richter, and D.H. Younger to allow for an adjacency between 2-list vertices. This modified technique was then used by N. Asghar in [1] to present another proof of Grötzsch's theorem using list colouring. We will use this technique to show that every planar graph without 3-cycles has 4-cycles distance 8 apart is 3-choosable. This result may seem random but the motivation is to improve a result of Dvořák's result that planar graphs with cycle of length at most 4 being distance at least 26 from each other are 3-choosable.

Definition 7. Let G be a graph embedded in the plane, let C be a cycle and P a path in G. Then:

(a) $Int(C)$ is the subgraph of G contained in the closed disc bounded by C, $int(C) = Int(C) - V(C)$, $Ext(C) = G - int(C)$, and $ext(C) = Ext(C) - V(C)$;

(b) the *outer face*, also known as the *infinite face*, is the unique face of G that is unbounded;

(c) the *boundary* $Bnd(G)$ is the subgraph consisting of the vertices and edges incident with the outer face on the boundary of G and its vertices and edges of $Bnd(G)$ are the *boundary vertices and edges*;

(d) $int(G) = G - V(Bnd(G))$;

(e) a cycle or path of length i is an i-cycle or i-path; and

(f) *P separates the boundary* of G if $G - P$ has two disjoint connected components G_1 and G_2 such that $V(G_1) \cap V(Bnd(G)) \neq \varnothing$ and $V(G_2) \cap V(Bnd(G)) \neq \varnothing$.

Definition 8. Let G be a graph, L a list-assignment of G, G_1 and G_2 subgraphs of G, and c_1 and c_2 be colourings of G_1 and G_2, respectively. Then:

(a) a vertex v with $|L(v)| = 1$ has a *proper 1-list* if for every vertex u adjacent to v with $|L(u)| = 1$ we have $L(v) \cap L(u) = \varnothing$; and

(b) If for all $v \in V(G_1) \cap V(G_2)$ we have $c_1(v) = c_2(v)$, then *the union of the colourings c_1 and c_2* on $G_1 \cup G_2$ is

$$c(v) = \begin{cases} c_1(v), & \text{if } v \in V(G_1) \\ c_2(v), & \text{if } v \in V(G_2). \end{cases}$$

We are trying to prove that planar graphs without 3-cycles and 4-cycles distance at least 8 apart are 3-choosable and so it makes sense to define this set of graphs.

Definition 9. The set $\mathcal{G}$ is the set of planar graphs G such that:

- G has no 3-cycles; and
- each pair of 4-cycles in G is distance at least 8 apart.

From now on we will just say that the 4-cycles are far apart to reference that they are distance at least 8 apart.

Theorem 4. Suppose $G \in \mathcal{G}$. If we have an L-list assignment of the vertices of G such that:

4.1 there is a set T of boundary vertices such that (i) each vertex in T has a list size at least 2, (ii) there are no S-T adjacencies, and (iii) $|E(G[T])| \leq 1$;

4.2 there is a set S of consecutive boundary vertices such that $|S| \leq 6$, and each S-vertex has a proper 1-list;

4.3 internal vertices and boundary vertices that are not in $S \cup T$ have 3-lists;

4.4 If $|E(G[T])| = 1$, then $|S| \leq 4$;

4.5 no edge in $E(G[T])$ is in a 4-cycle;

4.6 $G[N_G(S) \cup T]$ contains no odd cycles; and

4.7 no 3-list vertex is adjacent to three S-vertices.

Then G is L-colourable.

Theorem 4 is proven by first showing that there are no short separating paths. We then colour and delete some of the boundary vertices (S vertices if there is no T-adjacency, or vertices around the T-adjacency if it exists) and define a new S set and T set. We then show that there is no adjacency between an old S-vertex and a new T-vertex and vice versa since there are no separating paths across the boundary. We then show through case analysis that the remaining properties hold on the new S and T sets and apply induction to find an L-colouring of G.

Let P be a path in a graph G and v a vertex adjacent to at least three vertices of P. If $|V(P)| \leq 6$, then $G[V(P) \cup \{v\}]$ contains either a 3-cycle or a 4-cycle. If $|V(P)| \leq 5$, then it contains a 3-cycle or distinct 4-cycles. As we think about Theorem 4 (4.7), if S has at most five vertices, then no vertex will be adjacent to three vertices of S. For $|S| = 6$, if there is another 4-cycle nearby, then no vertex will be adjacent to three vertices of S.

Similarly, we see that no vertex is adjacent to three vertices in a cycle of length at most 7.

As for Item 4.6, it seems to be a cumbersome property, but Lemma 10 will give us a much more concrete property to deal with if the property does not hold true.

Lemma 10. Let $H \in \mathcal{G}$ satisfy (4.1)-(4.5). If $H[N_H(S) \cup T]$ contains an odd cycle C, then:

- C is a 5-cycle;
- there is a vertex of S in a 4-cycle in $H[S \cup C]$;
- every 3-list in C is adjacent to an S-vertex;
- every vertex of C not in T-vertex has a 3-list; and

- either;
  - $C \cap T$ is precisely the T-adjacency and $|S| = 4$; or
  - $C \cap T$ is a single vertex and $|S| = 6$.

Proof. Let $H \in \mathcal{G}$ such that H satisfies (4.1)-(4.5). Suppose $H[N_H(S) \cup T]$ contains an odd cycle C.

First note that no vertex in $H[N_H(S) \cup T]$ is an S-vertex, therefore C is comprised of T-vertices and 3-list vertices. Consecutively label the S-vertices in $Bnd(H)$, $s_1, ..., s_n$. Let $C = (c_1, ..., c_m)$.

Case 10.1 H does not have a T-adjacency.

Since H does not have a T-adjacency, C has at least three vertices in $N_H(S)$. Let s_i be the first S-vertex that is adjacent to a C vertex c_l and let s_j be the last S-vertex that is adjacent to a C-vertex c_h. Let P and R be the two c_l, c_h-paths in C. Without loss of generality, suppose every vertex in $V(P)\backslash\{c_l, c_h\}$ is in the interior of the cycle formed by the paths $Bnd(H)[S]$, $s_i c_l$, R, $c_h s_j$.

Therefore, for every vertex in $v \in V(P)$, $v \notin T$ and v must be a neighbour of an S-vertex. Since S-vertices are on the boundary of H, there are no vertices in $V(R)\backslash\{c_l, c_h\}$ that are adjacent to an S-vertex. Since there is no T-adjacency, this tells us that $V(R)\backslash\{c_l, c_h\}$ is empty or is a set containing one T-vertex.

Since 3-cycle do not exist and C is an odd cycle we have $|V(P)| = |V(C)| - |V(R)| + 2 \geq 5 - 3 + 2 = 4$. Because $|S| \leq 6$, $H[S \cup P]$ has a 4-cycle. Since 4-cycles are far apart, $H[S \cup P]$ can only have one 4-cycle, so $|S| > 5$. Therefore, $|S| = 6$, $|V(P)| = 4$, R has precisely one T-vertex, and $m = 5$, as desired.

Case 10.2 H has a T-adjacency.

Since H does have a T-adjacency, C has at least two vertices in $N_H(S)$. Let s_i be the first S-vertex that is adjacent to a C vertex c_l and let s_j be the last S-vertex that is adjacent to a C-vertex c_h. Let P and R be the two c_l, c_h-paths in C. Without loss of generality, suppose every vertex in $V(P)\backslash\{c_l, c_h\}$ is in the interior of the cycle formed by the paths $Bnd(H)[S]$, s_ic_l, R, c_hs_j.

Therefore, for every vertex in $v \in V(P)$, $v \notin T$ and v must be a neighbour of an S-vertex. Since S-vertices are on the boundary of H, there are no vertices in $V(R)\backslash\{c_l, c_h\}$ that are adjacent to an S-vertex. Since there is a T-adjacency, this tells us that $V(R)\backslash\{c_l, c_h\}$ is empty, is a set containing one T-vertex or a set containing two T-vertices.

Since 3-cycle do not exist and C is an odd cycle we have $|V(P)| = |V(C)| - |V(R)| + 2 \geq 5 - 4 + 2 = 3$. Because $|S| \leq 4$, $H[S \cup P]$ has a 4-cycle. Since 4-cycles are far apart, $H[S \cup P]$ can only have one 4-cycle, so $|S| > 3$. Therefore, $|S| = 4$, $|V(P)| = 3$, R has precisely the T-adjacency, and $m = 5$, as desired. □

Proof of Theorem 4. Let $G \in \mathcal{G}$ be a minimum counterexample and let L be a list-assignment as described such that G is not L-colourable. The hypothesis being satisfied for G implies the hypothesis is satisfied for each connected component of G; therefore, G is connected by minimality of G.

Claim 11. *G has no vertex v such that $|L(v)| > deg(v)$.*

Proof. Suppose G has a vertex v as described. Define G' to be $G - v$ and L' to be L restricted to G'. By the minimality of G , G' is L'-colourable. Let this L'-colouring be a partial L-colouring of G with v not coloured. Since $|L(v)| \geq deg(v) + 1$, we can properly colour v and find an L-colouring for G, a contradiction. □

Claim 12. *G has no k-cycle C such that $k \leq 7$, $int(C) \neq \varnothing$, and $ext(C) \neq \varnothing$.*

Proof. Suppose first that C has legnth at most 6. By minimality of G. By minimality of G we can L-colour $Ext(C)$ with L restricted to the vertices of $Ext(C)$. This colouring induces a new list assignment on $Int(C)$ where all vertices of C have proper 1-lists. Clearly no vertex in $int(C)$ is adjacent to three vertices in $V(C)$ since that would form a pair of cycles of length four that are adjacent. Therefore, $Int(C)$ satisfies the hypothesis and by minimality of G we can extend the L-colouring of $Ext(C)$ to G itself, a contradiction.

Now suppose C has length 7. By minimality of G, we can L-colour $Ext(C)$ with L restricted to the vertices of $Ext(C)$. This colouring induces a new list assignment on $Int(C)$ where all vertices of C have proper 1-lists. Let $v \in V(C)$ such that v is not in a 4-cycle. Since v is not in a 4-cycle, no vertex in $N_G(v) \cap V(int(C))$ is adjacent to a vertex of $V(C)$ other than v, else G contains a separating k-cycle for $k \leq 6$, contradicting Claim 12.

Define $S' = V(C) \backslash \{v\}$ and $T' = N_{int(C)}(v)$. No vertex in $int(C)$ is adjacent to three vertices of V(C) since 4-cycles are far apart. Since 4-cycles are far apart, no neighbour of v is in a 5-cycle with two vertices that are adjacent to the ends of $C - v$ (this would form two different 4-cycles with v). Delete v, and adjust the lists of the vertices of T'. By minimality of G, we can find an L-colouring of $Int(C) - v$. Placing v back in the graph we find an L-colouring of $Int(C)$. The union of the colourings of $Int(C)$ and $Ext(C)$ form an L-colouring of G, and a contradiction. $\square$

A principle point in the induction is dealing with short paths that allow for a partition of G into two smaller subgraphs. Our next definition is a first step to finding the paths that allow for an inductive argument.

Definition 13. Let $\mathcal{P}$ be the set of triples (G_1, G_2, P) with the properties:

- P is a path;
- $G_1 \cap G_2 = P$;
- $(V(G_2) \backslash V(P)) \cap V(Bnd(G)) \neq \varnothing$;
- $(V(G_1) \backslash V(P)) \cap V(Bnd(G)) \neq \varnothing$;
- If $S \neq \varnothing$, then $V(G_1) \cap S \neq \varnothing$;
- the vertices of $V(G_1) \cap S$ are consecutive on the boundary of G_1; and
- $G_1 \cup G_2 = G$;

We will call these properties the *separating path properties.*

Claim 14. *G has no 0-path that separates the boundary and no 1-path with one end in T that separates the boundary.*

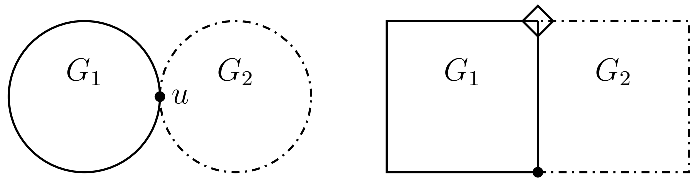


Figure 1: 0-path separating the boundary and 1-path separating the boundary with one end in T.

Proof. Let $\bar{\mathcal{P}} \subseteq \mathcal{P}$ be the set of triples (G_1, G_2, P) where P is a 0-path or a 1-path with one end in T.

Choose the triple $(G_1, G_2, P) \in \bar{\mathcal{P}}$ that maximizes $|V(G_1)\backslash V(P)|$. Note that $G[V(G_2)\backslash V(P)]$ is connected, else $|V(G_1)\backslash V(P)|$ is not maximum.

Suppose a vertex v in the boundary of $Bnd(G_2)$ is adjacent to three or more vertices of $Bnd(G_2)$. Let v_1, v_2, v_3 be three vertices in $Bnd(G_2)$ adjacent to v.

By our choice of G_1 and G_2, v is not a cut vertex, therefore $Bnd(G_2) - v$ is connected. There is a path P in $Bnd(G_2)$ joining, say, v_1 and v_2 that does not go through v_3. In $Bnd(G_2)$, there is a v_3P-path Q such that $Q \cap P$ is a single vertex w. The 2-connected subgraph of $Bnd(G_2)$ consisting of v, its three incident edges, P , and Q has three faces, all incident with two of the three edges incident with v. One of these faces is the outer face, and is not incident with one of these three edges. Therefore, v has degree at most 2 in $Bnd(G_2)$

Suppose v has degree 1 in $Bnd(G_2)$ and is adjacent to u in $Bnd(G_2)$. It is clear that$deg_{G_2}(v) = 1$. By Claim 11, $v \in S$. If $u \in S$, then by minimality of G we colour $G - v$. Since $c(u) \notin L(v)$, we place v back in the graph and colour it to find an L-colouring of G and a contradiction.

Therefore, $u \notin S$. Let $S' = (N_G(u) \cap T) \cap \{u\}$ and let $T' = T\backslash T'$ and define $L'(u) = L(u)\backslash L(v)$. There is an L-colouring of $G - v$. Colouring v and placing it back in the graphs yields an L-colouring of G, a contradiction. Therefore v does not exist.

Every vertex in the boundary of G_2 is incident with exactly two boundary edges, therefore the boundary of G_2 is a cycle.

Case 14.1 $V(P) \subseteq S$.

By the hypothesis of this claim, it must be the case that $V(P) = \{u\}$ and $u \in S$. Under this case we colour all of G_1, then we colour all of G_2 by minimality of G. The union of these colourings results in an L-colouring of G, a contradiction.

Case 14.2 $V(P) \nsubseteq S$.

As $V(P) \nsubseteq S$, there exists $b \in V(P)$ such that $b \notin S$. Since $|V(P)| \leq 2$, $S \subseteq V(G_1)$. Thus P is either a 0-path that separates the boundary and has no vertex in S or a 1-path with one end in T that separates the boundary.

Since G is connected, we know that both G_1 and G_2 are connected. We define H to be the component of $G[V(G_1) \cup T]$ that contains G_1. We will colour H by minimality of G, but first we must show that $H \neq G$. It is enough to show that G_2 has a vertex with list of size three not in P.

We know that the boundary of G_2 is a cycle C of length at least four. If P is a 0-path, then C has three consecutive vertices that are not in $P \cup S$, and therefore the definition of T implies one of them must have a list of size three.

If P is a 1-path, then P has a T-vertex u. Observe that $G[V(C - P) \cup \{u\}]$ is a path of length two with one vertex in T. Since none of these vertices is in S and there is at most one T-adjacency in G, we have that one of the vertices in $V(C - P) \subseteq V(G_2 - P)$ has a list of size three.

We now colour H by minimality of G, thereby inducing a proper 1-list on the set of vertices $S' = V(H \cap G_2)$ on the boundary of G_2. By maximality of $|V(G_1 - P)|$, G_2 has no 1-path with one end in T that separates the boundary. Therefore, the vertices of S' must be consecutive on the boundary of G_2. Define $T' = (V(G_2) \cap T) \backslash S'$. No S'-vertex is adjacent to a T'-vertex by definition of H and S'.

To help determine $|S'|$, notice that $S' \subseteq S$ or

$$S' \leq \begin{cases} |P| + 3 \leq 5, & \text{if } G_2 - S' \text{ does not contain a } T\text{-adjacency} \\ 3, & \text{if } G_2 - S' \text{ contains a } T\text{-adjacency} \ . \end{cases}$$

It is clear that no vertices with list size three in G_2 are adjacent to three S'-vertices since G_2 does not contain any 3-cycles, and 4-cycles are far apart. Since $|S'|$ is one less than its bound of 6 or 4 depending on the situation, this implies that $G[N_G(S') \cup T']$ contains no odd cycles. Therefore, with S' and T', we get that G_2 satisfies the hypothesis and by minimality of G, there is an L-colouring of G_2. The colouring of H and G_2 agree on their intersection and therefore the union of the two colorings results in an L-colouring of G, a contradiction. □

Claim 15. G is 2-connected.

Proof. By Claim 14, we need only suppose G has a cut-vertex v that does not separate the boundary. Let G_1 be the component of $G - v$ graph that contains the boundary of G, and let $G_2 = G - V(G_1)$. By minimality of G, we can extend L to all of $G_1 + v$. This L-colouring of G_1 defines a new L'-list assignment on G_2 where v is an S-vertex and every other vertex in G_2 has a list of size three. By minimality of G, we have an L'-colouring of G_2. These two colourings result in an L-colouring of G, a contradiction. □

Note by Claim 15, If $(G_1, G_2, P) \in \mathcal{P}$, then G_1 and G_2 are 2-connected; in particular the boundaries are cycles.

Observation 16. [Separating Path] For every path P that separates the boundary of G, there is a triple $(G_1, G_2, P) \in \mathcal{P}$.

Proof. Let u and v be the ends of P. Since the boundary of G is a cycle we have that there are two internally disjoint (u, v)-paths H_1 and H_2 that form the boundary of G. Let C_1 be the cycle $H_1 \cup P$ and C_2 be the cycle $H_2 \cup P$. Since the S-vertices are consecutive on the boundary of G, we have that if $S \neq \varnothing$, then a non-empty subset of the S-vertices are consecutive on H_1 or H_2. Without loss of generality, suppose if $S \neq \varnothing$, then a non-empty subset of the S-vertices are consecutive on H_1. Let $G_1 = Int(C_1)$ and $G_2 = Int(C_2)$. It is clear that (G_1, G_2, P) have the desired properties. □

Definition 17. Given a triple $(G_1, G_2, P) \in \mathcal{P}$, we define H_P to be the component of $G[V(G_1) \cup S \cup T]$ containing G_1 and S_P to be the set containing the vertices of the component of $H_P \cap Bnd(G_2)$ that contains P.

Note that we can find an upper bound on $|S_P|$ by considering the length of P along with the length of the extension of P in the boundary of G_2 using $S \cup T$-vertices.

Definition 18. A triple $(G_1, G_2, P) \in \mathcal{P}$ is *good* if it satisfies:

- $|S_P| \leq 6$, if $G_2 - H_P$ does not contain a T-adjacency;
- $|S_P| \leq 4$, if $G_2 - H_P$ does contain a T-adjacency;
- there exists $v \in V(G_2) \backslash V(H_P)$ such that $|L(v)| = 3$;
- No vertex with list size three in $V(G_2) \backslash V(H_P)$ is adjacent to three vertices in S_P; and
- $G_2[N_{G_2}(S_P) \cup (T \cap V(G_2))]$ contains no odd cycles.

We must consider the number of vertices in S_P because S_P becomes our new "S-set" in G_2 after we do induction on G_1. Because of this, if (G_1, G_2, P) is a good triple, then $(G[V(G_1) \cup S_P], G_2, S_P)$ is a good triple as well.

Claim 19. G has no good triples (G_1, G_2, P).

Proof. By way of contradiction, let $(G_1, G_2, P) \in \mathcal{P}$ be a good triple as described with the additional property that $|V(G_2)|$ is minimal. We have two cases, either the boundary of $G_2 - H_P$ does or does not contain a vertex v with $|L(v)| = 3$.

Case 19.1 The boundary of $G_2 - H_P$ does not contain a vertex v with $|L(v)| = 3$.

This means that the boundary of G_2 is bounded by the vertices of S_P. Therefore, G_2 is bounded by a cycle C of length at most six. Since there is a vertex v with $|L(v)| = 3$ in $V(G_2)$ that is not in the boundary of G_2, it must be in $int(C)$. Therefore, $int(C) \neq \emptyset$. Since P separates the boundary, $V(Bnd(G_1)) \backslash V(P) \neq \emptyset$. Therefore $ext(C) \neq \emptyset$. By Claim 12, we have a contradiction.

Case 19.2 The boundary of $G_2 - H_P$ contains a vertex v with $|L(v)| = 3$.

Since $G_2 - H_P$ contains a vertex v with $|L(v)| = 3$, we have that $H_P \neq G$. We show that $V(H_P) \cap V(G_2) = S_P$.

If not, then there is a vertex $a \in S_P$ adjacent to x (a $S \cup T$-vertex) through an edge that is not on the boundary of G_2. The vertices of S are consecutive on the boundary of G and there is an S-vertex in G_1, therefore if $x \in S$, then $x \in S_P$. Since $x \notin S_P$, x is a T-vertex. By Claim 14, $a \in V(P) \cap V(int(G))$.

Let u be the closest endpoint of S_P to x in $G[V(S_P) \cup \{x\}]$, and y be the endpoint of P that is in the shortest (u, x)-path in $G[V(S_P) \cup \{x\}]$. Let P' be the shortest (x, y)-path in $G[V(S_P) \cup \{x\}]$. Let R be the unique (x, y)-path on the boundary of G_2 that does not contain an edge of P. Since $x \notin S_P$, $x \in T$, R must contain a vertex $v \neq u$ with $|L(v)| = 3$ (in particular u is not adjacent to an $S \cup T$-vertex on R).

Take $G_2' = Int(P' \cup R)$ (the union of two internally disjoint (x, y)-paths). Let $G_1' = G[V(ext(P' \cup R)) \cup V(P')]$. We have a triple (G_1', G_2', P') such that:

- **(1)** $G_1 \subset G_1'$;
- **(2)** $G_2' \subset G_2$;
- **(3)** $G_1' \cap G_2' = P'$;
- **(4)** $G_1' \cup G_2' = G$; and
- **(5)** G_2' has a vertex v with $|L(v)| = 3$ in the boundary of G not in P'.

(1), **(2)**, **(3)**, and **(4)** are clear by definition of G_1' and G_2'. **(5)** is true since R has a vertex with list size three.

Let $H_{P'}$ and $S_{P'}$ be as defined in Definition 17 on the triple (G_1', G_2', P').

We would like to find out the number of vertices in $S_{P'}$, which is the component of $H_{P'} \cap Bnd(G_2')$ that contains P'. We count the vertices of $S_{P'}$ to find that

$$|S_{P'}| \leq \left\lceil \frac{|V(S_P)|}{2} \right\rceil + 1 + t \leq \begin{cases} 3, & \text{if } G_2' - H_{P'} \text{ contains a } T\text{-adjacency}, \\ 4, & \text{otherwise,} \end{cases}$$

where t is the number of T-vertices adjacent to the T-vertex x.

Note that $|V(G_2')| < |V(G_2)|$, and $|S_{P'}|$ does not meet its upper bounds, therefore, by Lemma 10, $G_2'[N_{G_2'}(S_{P'}) \cup ((T \cap V(G_2'))\backslash S_{P'})]$ contains no odd cycles. Since $|S_{P'}|$ does not meet its upper bound we also have that no 3-list vertex is adjacent to three $S_{P'}$ in G_2'.

Therefore, (G_1', G_2', P') is a good triple, a contradiction with the minimality of $|V(G_2)|$. Therefore, $G_2 \cap H_P$ is a path. In other words, $H_P = S_P$.

We now colour G_1 by minimality of G, which induces a proper 1-list on the set of vertices S_P; these are consecutive on the boundary of G_2. Define $T' = T \cap V(G_2)\backslash S_P$. No S_P-vertex is adjacent to a T'-vertex since $S_P = H_P$. $G_2[N_{G_2}(S_P) \cup (T \cap V(G_2))]$ contains no odd cycles by hypothesis of good triple.

Since no vertex in $V(G_2)\backslash S_P$ is adjacent to three vertices of S_P, we have that S_P and T' satisfy the hypothesis of the main theorem on G_2. Therefore, we colour G_2 by minimality of G and take the union of the colorings of G_1 and G_2 to form an L-colouring of G, a contradiction. □

Claim 20. G has no 1-path containing an $S \cup T$-vertex that separates the boundary.

Proof. First we may assume by Claim 14 that every separating 1-path has no T-vertex, and therefore the paths we consider contain an S-vertex. Suppose G has 1-paths that separate the boundary. Choose a triple $(G_1, G_2, P) \in \mathcal{P}_1$ such that $|V(S) \cap V(G_1)| \geq |V(S) \cap V(G_2)|$ and there does not exist a 1-path with one end in S that separates the boundary of G_2. We now have two cases. Either $|E(T)| = 0$ or $|E(T)| = 1$.

Case 20.1 $|E(T)| = 0$.

Let H_P and S_P be as in Definition 17 on the triple (G_1, G_2, P), and H_P' and S_P' be as defined on the triple (G_2, G_1, P). We have two subcases. Either $V(G_2)\backslash V(P)$ has a vertex v with $|L(v)| = 3$ or it does not.

Subcase 20.1.1 $V(G_2)\backslash V(P)$ has a vertex v with $|L(v)| = 3$.

By Claim 12, $V(Bnd(G_2))\backslash V(P)$ contains a vertex u with $|L(u)| = 3$, since $v \in Int(Bnd(G_2))$ and $V(G_1)\backslash V(G_2)$ is non-empty.

Since G does not contain a T-adjacency, we have $|S_P| \leq 4$. Therefore, no vertex is adjacent to three vertices of S_P. Similarly, since $|S_P| \leq 4$, we have that $G_2[N_{G_2}(S_P) \cup (T \cap V(G_2))]$ contains no odd cycles. Therefore, (G_1, G_2, P) is a good triple, and we have a contradiction with Claim 19.

Subcase 20.1.2 $V(G_2)\backslash V(P)$ does not have a vertex v with $|L(v)| = 3$.

Note that $|V(G_2)\backslash V(P)| \geq 2$ since G_2 is bounded by a cycle of length at least four. Since $|E(T)| = 0$, If there is a T-vertex in $V(G_2 - P)$, then there is also a vertex with list size three in $V(G_2 - P)$. Therefore, $V(G_2 - P) \subset S$. Since $|V(S) \cap V(G_1)| \geq |V(S) \cap V(G_2)|$, we have that G_2 is a 4-cycle by Claim 12.

If $V(P) \subseteq S$, then we colour G_1, by minimality of G which results in an L-colouring G, and a contradiction.

Therefore, there must be a vertex $u \in P$ such that $|L(u)| = 3$ and $V(P) = \{u, v\}$ where $v \in S$. Since no vertex with list size three is adjacent to three S-vertices and u is adjacent to two S vertices in G_2, it must be the case that u is adjacent to a T-vertex or another vertex with list size three on the boundary of G_1.

In either case, since a T-vertex in $G_1 - V(P)$ implies a vertex with list size three in $G_1 - V(P)$, we have that $G_1 - V(P)$ contains a vertex with list size three. We also note that, since $|V(S) \cap V(G_1)| \geq |V(S) \cap V(G_2)|$ and $V(G_2 - P) \subset S$, we have that G_2 is a 4-cycle.

Since $G_1 - V(P)$ has a vertex with list size three, $H'_P \neq G$. Since G has no T-adjacency, G_1 has no T-adjacency. By Lemma 10, since G_2 is a 4-cycle, we have that no vertex with list size three in G_1 is adjacent to three vertices in S'_P. Similarly, since G_2 is a 4-cycle $G_1[N_{G_1}(S'_P) \cup (T \cap V(G_1))]$ contains no odd cycles. It is clear that $|S'_P| \leq |S| - 2 + 2 \leq 6$, therefore (G_2, G_1, P)

is a good triple, contradicting Claim 19.

Case 20.2 $|E(G[T])| = 1$.

Either $G_2 - V(H_P)$ has a vertex with list size three or it does not.

Subcase 20.2.1 $G_2 - V(H_P)$ has a vertex v with list size three.

If $V(Bnd(G_2)) = S_P$, then $Bnd(G_2)$ is a separating cycle with v on its interior, a contradiction with Claim 12. Therefore, $V(Bnd(G_2)) \neq S_P$ and $V(Bnd(G_2)) \backslash V(S_P)$ contains a 3-list vertex u.

If $G_2 - V(H_P)$ does not contain a T-adjacency, then $|S_P| \leq 5$. By Lemma 10, since $|S_P| \leq 5$, no vertex with list size three in G_2 is adjacent to three S_P-vertices and $G_2[N_{G_2}(S_P) \cup ((T \cap V(G_2)) \backslash S_P)]$ contains no odd cycles. Therefore, (G_1, G_2, P) would be a good triple, contradicting Claim 19.

Therefore, $G_2 - V(H_P)$ does contain a T-adjacency. In this case, $|S_P| \leq 4$. By Claim 19, we must have that $G_2[N_{G_2}(S_P) \cup ((T \cap V(G_2)) \backslash S_P)]$ contains an odd cycle. By Lemma 10, a vertex in S_P is in a 4-cycle in G_2.

Now we consider the triple (G_2, G_1, P) and let H'_P and S'_P be as defined in Definition 17 on this triple. Since $|V(G_1) \cap S| \geq |V(G_2) \cap S|$ and $|S_P| = 4$, we have that $|S'_P| \leq 5$.

If there does not exist a vertex with list size three in $G_1 - H'_P$, then $|S'_P| = 4$ and G_1 is a 4-cycle. This is not possible since P has a vertex in a 4-cycle in G_2, therefore the same vertex is not close to a 4-cycle in G_1. Therefore, $G_1 - H'_P$ has a vertex with list size three. By Lemma 10, since $|S'_P| = 5$, we have that no vertex with list size three in G_1 is adjacent to three S'_P-vertices and $G_1[N_{G_1}(S'_P) \cup (T \cap V(G_1))]$ contains no odd cycles. Therefore, (G_2, G_1, P) is a good triple, a contradiction with Claim 19.

Subcase 20.2.2 $G_2 - V(H_P)$ does not contain a vertex with list size three.

Since $V(G_2) - V(P)$ does not contain a vertex with list size three, G_2 is bounded by a cycle of length four. From this, we have that $Bnd(G_2) - V(P)$ is a path of length two. Since $G_2 - V(H_P)$ does not contain a vertex with list size three, we have that if one of the vertices on the path $Bnd(G_2) - V(P)$ is an S-vertex, then both of them are S-vertices. Therefore, either the vertices in $Bnd(G_2) - H_P$ are both in S or both in T.

Since the T-adjacency is not in a 4-cycle, it must be the case that $G_2 - V(P)$ has two S-vertices. Since S-vertices are consecutive on the boundary of G, we have that $G_1 - V(P)$ contains at least two S-vertices and P contains at least one S vertex. We have that $|S| \geq 5$, a contradiction with $|E(G[T])| = 1$ and $|S| \leq 4$. $\square$

Claim 21. *G has no separating 2-paths with both ends in S.*

Proof. Let P be a separating 2-path in G with ends in S, let $s, u \in S$ be the ends of P, and let $v \in V(P) \backslash \{s, u\}$. By the Separating Paths Observation 16, a triple (G_1, G_2, P) exists with the separating graph properties.

Case 21.1 $|V(Bnd(G)) \backslash S| \leq 1$.

By Claim 20, no vertex on the boundary is adjacent to another on the boundary through a non-boundary edge. Since there are no 3-cycles, 4-cycles are far apart and $|V(Bnd(G))| \leq 7$, we also have that there are no internal vertices that are adjacent to three boundary vertices. Since v is adjacent to two vertices on the boundary (s and u), it can not be adjacent to a third.

Let S_P be as defined in Definition 17 on the triple (G_1, G_2, P). Either $Bnd(G_1)$ is a 4-cycle and $|S_P| \leq 6$, or $|S_P| \leq 5$. By Lemma 10, since $Bnd(G_1)$ is a 4-cycle or $|S_P| \leq 5$, no vertex is adjacent to three vertices of S_P and $G_2[N_{G_2}(S_P) \cup (T \cap V(G_2))]$ does not contain an odd cycle.

There is no T-adjacency in G_2 since $T = \varnothing$. Note $|S_P \leq 6$, therefore (G_1, G_2, P) is a good triple, contradicting Claim 19.

Case 21.2 $|V(Bnd(G)) \backslash S| \geq 2$.

Let H_P and S_P be as in Definition 17. Since G_1 is not bounded by a 3-cycle we have that if there is no T-adjacency, then

$$|S_P| \leq \begin{cases} 6, & \text{if } Bnd(G_1) \text{ is a 4-cycle} \\ 5, & \text{otherwise,} \end{cases}$$

and if there is a T-adjacency, then

$$|S_P| \leq \begin{cases} 4, & \text{if } Bnd(G_1) \text{ is a 4-cycle} \\ 3, & \text{otherwise.} \end{cases}$$

Since $|V(Bnd(G))\backslash S| \neq 0$ we have that there exists a vertex v with $|L(v)| = 3$ on the boundary of G. This vertex is not on the boundary of G_1 since the ends of P are in S and the S-vertices are consecutive on the boundary of G_1. Therefore, $v \in V(G_2)\backslash V(G_1)$ and hence v is in $G_2 - V(H_P)$. If $|S_P| = 6$, then the boundary of G_1 is a 4-cycle. Similarly, if $|S_P| = 4$ and there is a T-adjacency, then the boundary of G_1 is a 4-cycle.

By Lemma 10, if $Bnd(G_1)$ is a 4-cycle, then no vertex with list size three in G_2 is adjacent to three S_P-vertices and $G_2[N_{G_2}(S_P) \cup (T \cap V(G_2))]$ does not contain an odd cycle. Therefore, either $|S_P| \leq 5$ and there is no T-adjacency, or $|S_P| \leq 3$ and there is a T- adjacency.

In both cases, by Lemma 10, no vertex with list size three in G_2 is adjacent to three S_P-vertices and $G_2[N_{G_2}(S_P) \cup (T \cap V(G_2))]$ does not contain an odd cycle.Therefore, (G_1, G_2, P) is a good triple, contradicting Claim 19. $\square$

Claim 22. *G has no separating 2-,3-paths with one end in T and the other in $S \cup T$.*

Proof. By way of contradiction, suppose G has a separating 2- or 3-path P with one end in T and the other in $S \cup T$. Let (G_1, G_2, P) be as defined in the Separating Path Observation 16 and without loss of generality, let $|V(G_1) \cap S| \geq |V(G_2) \cap S|$. Define H_P and S_P as in Definition 17. We have two cases, either both ends of P are T-vertices, or one end is not a T-vertex.

Case 22.1 Both ends of P are T-vertices.

In this case,

$$|S_P| \leq \begin{cases} 4, & \text{if } G_2 - H_P \text{ has a T-adjacency} \\ 5, & \text{if } G_2 - H_P \text{ does not have a T-adjacency.} \end{cases}$$

Since both ends of P are T-vertices, we have that $G_2 - H_P$ has a vertex with list size three. By Claim 19, (G_1, G_2, P) is not a good triple, therefore it must be the case that $G_2 - H_P$ has a T-adjacency, $|S_P| = 4$, and $N_G(S_P) \cup ((T \cap V(G_2))\backslash S_P)$ contains an odd cycle. By Lemma 10 there is a cycle C such that

- C is a 5-cycle;
- every 3-list vertex of C is adjacent to an S-vertex;
- C contains the T-adjacency; and
- all the vertices in C have 3-lists other than the vertices in the T-adjacency.

Let $C = (z_1, ..., z_5)$ such that z_1z_5 is the T-adjacency. Consecutively label the vertices of S_P $x_1, ..., x_4$ such that $x_1z_2, x_4z_4 \in E(G_2)$. Since both ends of S_P are in T, we can assume without loss of generality that z_3 is adjacent to x_2.

Note that if x_2 is adjacent to an $S \cup T$-vertex v other than x_1, then the path $P' = (x_1, x_2, v)$ would be a separating path. This separating path would induce a triple (G_1', G_2', P') such that G_2 is contained in G_1'. Define $S_{P'}$ and $H_{P'}$ as in Definition 17 on the triple (G_1', G_2', P').

Since $S \subset V(G_1)$, we have that $|S_{P'}| \leq 6$. Since the T-adjacency is in G_2 we have that there is no T-adjacency in G_2'. Since the ends of $S_{P'}$ are in $S \cup T$, there is a 3-list vertex in $G_2' - V(H_P')$. By Lemma 10, since a vertex of S_P is in a 4-cycle in G_2, there are no odd cycles in $G_2'[N_{G_2'}(S_{P'}) \cup ((T \cap V(G_2'))\backslash S_{P'})]$. Similarly, there is no 3-list vertex adjacent to three $S_{P'}$-vertices in G_2'.

Therefore, the triple (G_1', G_2', P') would be a good triple, contradicting Claim 19. All this to say x_2 is not adjacent to any $S \cup T$-vertices other than x_1.

By Claim 12, the 4- and 5-cycles formed by the vertices of C and the vertices of $N_{Bnd(G_2)}(S_P) \cup S_P$ have empty interior. Let $c \in L(z_3)\backslash L(z_1)$. Redefine the list of x_2 to be $L'(x_2) = L(x_2)\backslash\{c\}$.

On G_1 we define $T' = (T \cup \{x_2\} \cap V(G_1))$ and by minimality of G, we find and L-colouring of G_1. This induces a colouring on the vertices of P in G_2. We delete those vertices and adjust the lists of the vertices of C accordingly. Since $c(x_2) \neq c$, $c \in L(z_3)$ and $c \notin L(z_1)$, we have that the remaining

5-cycle is L-colourable (All remaining vertices have 2-list that are not all the same). Giving us an L-colouring of G, and a contradiction.

Case 22.2 One end of P is not a T-vertex.

This implies that one end of P is an S-vertex. We consider two separate cases, whether $G_2 - H_P$ does or does not contain a T-adjacency.

Subcase 22.2.1 $G_2 - H_P$ does not contain a T-adjacency.

If $G_2 - H_P$ does not contain a T-adjacency, then

$$|S_P| \leq \begin{cases} \left\lceil \frac{|S|}{2} \right\rceil + (|V(P)| - 1) + 1, & G \text{ has a T-adjacency} \\ \\ \left\lceil \frac{|S|}{2} \right\rceil + (|V(P)| - 1), & G \text{ does not have a T-adjacency.} \end{cases}$$

In either case $|S_P| \leq 6$.

If $|S_P| \leq 5$, then we can use Lemma 10 to show that (G_1, G_2, P) is a good triple, contradicting Claim 19. Therefore, $|V(P)| = 4$ and $|S_P| = 6$. By Claim 19, it must be the case that either there is a vertex with list size three in G_2 that is adjacent to three S_P-vertices or $G_2[N_{G_2}(S_P) \cup (T \cap V(G_2))]$ contains a 5-cycle containing one T-vertex.

Consecutively label the vertices of the path $G[S_P] \cap Bnd(G_2)$ starting at the T-vertex $x_1, x_2, ..., x_6$.

Subcase 22.2.1.1 There is a vertex with list size three in G_2 that is adjacent to three S_P-vertices and that $|S_P| = 6$.

Let v be such a vertex. Then v must be adjacent to x_1 and x_6. Since 4-cycles are far apart and v is adjacent to three S-vertices, it is either v is adjacent to x_3 or x_4.

If $Bnd(G_2) \backslash (S_P \cup \{v\}) \neq \varnothing$, then the path $P' = (x_1, v, x_6)$ is a separating 2-path with one end in S and the other end in T, which have shown already in this lemma do not exist.

It is clear that v forms a 4-cycle and a 5-cycle with the vertices of S_P, and by Claim 12, the interior of these cycles are empty (in particular, the only vertex in $V(G_2)\backslash V(G_1)$ is v). We break this into two cases whether v is adjacent to x_3 or x_4.

Subcase 22.2.1.1.1 v is adjacent to x_3.

It is either x_2 has a 3-list or $x_2 \in T$.

Subcase 22.2.1.1.1.1 x_2 has a 3-list.

Since $|S_P| = 6$, it must be the case that $x_4, x_5, x_6 \in S$. It must be the case that $L(x_6) \subset L(v)$ else we would delete x_6 and do induction on $G - x_6$. Colour x_3 with $c \in L(x_3)\backslash(L(v)\backslash L(x_6))$.

Let $S' = (S \cup \{x_3\})\backslash\{x_5, x_6\}$ and $T' = T\backslash\{v\}$. There is no T'-adjacency in G_1. If there is a T-adjacency in G, then $|S'| \leq 4$. If there is no T-adjacency in G, then $|S'| = 5$. Since we have proven the length two version of this theorem already, there are no S', T'-adjacencies in G_1.

By Lemma 10, since (v, x_1, x_2, x_3, v) is a 4-cycle in G_2, we know that there are no odd cycles in $G_1[N_{G_1}(S') \cup T']$. Since $|S'| \leq 5$, we know that there is not a 3-list vertex in G_1 adjacent to three S'-vertices. Therefore, we can colour G_1 by induction such that $c(x_3) = c$. Since $c \in L(x_3)\backslash(L(v)\backslash L(x_6))$ we can colour v and find an L-colouring of G, and a contradiction.

Subcase 22.2.1.1.1.2 $x_2 \in T$.

Define H'_P and S'_P as in Definition 17 on the triple (G_2, G_1, P). Since $x_5, x_6 \in S \cap V(G_2)\backslash V(G_1)$ and $|V(G_1) \cap S| \geq |V(G_2) \cap S|$, we have that $|S| \geq 5$ and there is no T-adjacency in G. This tells us that $|S'_P| \leq 6$.

Clearly there is a 3-list vertex in $V(G_1)\backslash V(H'_P)$ since the one end of S'_P is in S and the other is in T. By Lemma 10 and by the existence of the 4-cycle (v, x_1, x_2, x_3, v) in G_2, no vertex with list size three in $V(G_1)\backslash V(H'_P)$ is adjacent to three vertices in S'_P and $G_1[N_{G_1}(S'_P) \cup (T \cap V(G_1))]$ contains no odd cycles.

Therefore, (G_2, G_1, P) is a good triple contradicting Claim 19.

Subcase 22.2.1.1.2 v is adjacent to x_4.

It is either x_4 is an S-vertex or $|L(x_4)| = 3$.

Subcase 22.2.1.1.2.1 x_4 is an S-vertex.

Colour and delete x_5 and x_6, and colour v and x_1. Let $T' = T \backslash \{x_1\}$ and $S' = (S \cup \{v, x_1\}) \backslash \{x_5, x_6\}$. There are no T-adjacencies in $G - \{x_5, x_6\}$, no S', T'-adjacencies and $|S'| = 6$. By Lemma 10, since (x_4, x_5, x_6, v, x_4) was a 4-cycle, no three list vertex in $G - \{x_5, x_6\}$ is adjacent to three S'-vertices. By minimality of G, we can find and L-colouring of $G - \{x_5, x_6\}$. Placing the vertices x_5 and x_6 back into the graph we find an L-colouring of G, and a contradiction.

Subcase 22.2.1.1.2.2 $|L(x_4)| = 3$.

Since $|S_P| = 6$, we have that $x_2 \in T$. Let $G'_1 = G_2$ and $G'_2 = G_1$. Consider the triple (G'_1, G'_2, P) and define H'_P and S'_P as in 17. We know that no vertex with list size three in G'_2 is adjacent to three vertices in S'_P since (v, x_4, x_5, x_6, v) is a 4-cycle in G'_1. We also know that $|S'_P| \leq (|S| - 1) + 3 = 6$. Since one end of S'_P is in T and the other is in S we know that there is a vertex with list size three in G'_2. Therefore, (G'_1, G'_2, P) is a good triple, a contradiction with 19.

Subcase 22.2.1.2 $G_2[N_{G_2}(S_P) \cup (T \cap V(G_2))]$ contains a 5-cycle containing one T-vertex.

Let this 5-cycle be $C = (z_1, .., z_5, z_1)$ such that z_5 is the T-vertex. Now it must be the case that z_1 and z_4 are adjacent to x_1 and x_6 respectively. Since we have already proved this lemma for paths of length 2, it must be the case that z_1 and z_4 are boundary vertices. By Claim 12, the faces of $G[V(C) \cup S_P]$ have empty interior in G.

We will now break this into two cases depending on whether $x_2 \in T$ or $x_2 \notin T$.

Subcase 22.2.1.2.1 $x_2 \in T$.

Since 4-cycles are far apart, the T-adjacency is not in a 4-cycle, and four vertices of C are adjacent to vertices of S_P, we have z_2 is adjacent to x_3.

Let $c \in L(z_2)\backslash(z_5)$. Reduce the list of x_3 to be $L'(x_3) = L(x_3)\backslash\{c\}$. Let $S' = (S \cap V(G_1))$ and $T' = (T \cap V(G_1)) \cup \{x_3\}$. Since we have already proven this lemma for 2-paths and $x_2 \in T$, x_3 is not adjacent to any $S \cup T$-vertices other than x_2. Therefore, there are no S', T'-adjacencies and the only T'-adjacency is the edge x_2x_3.

We have that $|S'| = |S| - 1 = 3$ and that G_1 has exactly one T'-adjacency (x_2x_3). Therefore, there is no need to check if anything bad happens around the S'-vertices. By minimality of G, we have an L-colouring of G_1 such that $c(x_3) \notin (L(z_3)\backslash L(z_5))$. We colour S_P and look at the graph $C = G - (V(G_1) \cup S_P)$ after adjusting the lists of the vertices of C.

Every vertex in C now has list size two after the deletion. But we now have that c is in the list of z_3 and is not in the list of z_5. This implies we can find an L-colouring of C, and therefore, an L-colouring of G, a contradiction.

Subcase 22.2.1.2.2 $x_2 \notin T$.

Since 4-cycles are far apart, it is either the case that z_2 is adjacent to x_2 or x_3.

Subcase 22.2.1.2.2.1 z_2 is adjacent to x_2.

Let $c \in L(z_2)\backslash L(z_5)$. Reduce the list of x_2 to be $L'(x_2) = L(x_2)\backslash\{c\}$, define $T' = (T \cap V(G_1)) \cup \{x_2\}$ and $S' = S\backslash\{x_5, x_6\}$. Since x_2 is in the 4-cycle $(z_1, x_1, x_2, z_2, x_1)$ in G_2, we have that the T'-adjacency x_1x_2 is not in a 4-cycle in G_1. Since we have proven this theorem for length two paths, x_2 is not adjacent to any $S \cup T$-vertices other than x_1.

By Lemma 10, since x_2 is in a 4-cycle in G_2, there are no odd cycle in $G_1[N_{G_1}(S') \cup T']$. We also have that $|S'| = |S| - 2 \leq 4$, therefore, we do induction on G_1. This induces a colouring on the vertices of $G - V(C)$. Adjust the lists of the vertices of C accordingly. Every vertex in C now has list size at least two after the deletion. But we now have that c is in the list of z_2 and is not in the list of z_5. This implies we can find an L-colouring of C, and therefore, an L-colouring of G, a contradiction.

Subcase 22.2.1.2.2.2 z_2 is adjacent to x_3.

If x_3 is adjacent to exactly one S-vertex, then we apply the same argument as in Subcase 22.2.1.2.1.1 with z_2 and find an L-colouring of G, and a contradiction. Therefore, it must be the case that x_3 is adjacent to two S-vertices.

Since 4-cycles are far apart, and a vertex of S_P is in a 4-cycle in G_2, it must be the case that x_3 is adjacent to the end of the path $Bnd(G)[S]$ in G_1. Call this end y. Notice that x_1,x_2, and x_3 are all not adjacent to T-vertices and x_1 and x_2 are not adjacent to S-vertices since we have proven the 2-path version of this lemma.

Since x_1 is an end of $Bnd(G_2)[S_P]$ we have that x_1 is adjacent to z_1. Let $c \in L(z_1)\backslash L(z_5)$. Reduce the list of x_1 to be $L'(x_1) = L(x_1)\backslash\{c\}$, then reduce the lists of x_2 and x_3 to be proper 1-lists (this is possible since x_2 is not adjacent to any S-vertices).

By minimality of G, we can L-colour $G[(V(G_1)\backslash S) \cup \{y\}]$. We then colour the S-vertices and the rest of S_P. This induces 2-lists on every vertices of C, which are the only remaining uncoloured vertices. Since $c \notin L(z_1)\backslash L(z_5)$ we have that C is L-colourable, resulting in an L-colouring of G, and a contradiction.

Subcase 22.2.2 $G_2 - H_P$ does contain a T-adjacency.

Let S'_P and H'_P be as in Definition 17 on the triple (G_2, G_1, P). We first note that $|S_P| + |S'_P| \leq 2|V(P)| + 3 \leq 11$. It is important to note that $|V(P)| = 4$ and at this point we have already proven the theorem when $|V(P)| = 3$, this will be useful for the end of the proof. By Claim 19, neither (G_1, G_2, P) nor (G_2, G_1, P) is a good triple. Therefore, this leaves us with two cases, either $|S_P| = 4$ and $|S'_P| = 7$, or $|S_P| = 5$ and $|S'_P| = 6$.

Subcase 22.2.2.1 $|S_P| = 4$ and $|S'_P| = 7$.

Consecutively label the vertices of the path $G[S_P] \cap Bnd(G_2)$ starting at the T-vertex $x_1, x_2, ..., x_4$.

By Claim 19 it must be the case that there is a 5-cycle C in G_2 comprised of the T-adjacency and three 3-list vertices such that each 3-list vertex is adjacent to exactly one vertex of S_P. Let $C = (z_1, z_2, ..., z_5, z_1)$ such that $z_1, z_5 \in T$ and $z_2x_1, z_4x_4 \in E(G_2)$.

Since this claim has been proven for paths of length 2, we have that the vertices of C other than z_3 are boundary vertices. By Claim 12, the 4- and 5-cycles in the graph $G[S_P \cup V(C)]$ have empty interior. Therefore, $V(G_2) = S_P \cup V(C)$.

Let y be the vertex adjacent to z_3 on S_P. Note that since y is adjacent to either x_1 or x_4 it is not adjacent to any other $S \cup T$-vertices by Claim 21 and by the length 2 version of this lemma.

Let $c \in L(z_3) \backslash L(z_1)$. Redefine the list of y to be

$$L'(y) = \begin{cases} L(y) \backslash \{c\}, & \text{if } y = x_2 \\ \\ \{c(y)\}, & c(y) \in L(y) \backslash (\{c\} \cup L(x_4)) \text{ , if } y = x_3. \end{cases}$$

On G_1 we define

$$T' = \begin{cases} (T \cap V(G_1)) \cup \{y\}, & \text{if } y = x_2 \\ \\ T \cap V(G_1), & \text{if } y = x_3 \end{cases}$$

and

$$S' = \begin{cases} S, & \text{if } y = x_2 \\ \\ S \cup \{y\}, & \text{if } y = x_3. \end{cases}$$

By minimality of G, we find and L-colouring of G_1. This induces a colouring on the vertices of P in G_2. We delete those vertices and adjust the lists of the vertices of the 5-cycle that contains the T-adjacency and the two ends of the path $Bnd(G_2)[S_P \cup N_{Bnd(G_2)}(S_P)]$ accordingly. Since $c(y) \neq c$, $c \in L(v)$ and $c \notin L(x)$, we have that the remaining 5-cycle is L-colourable (All remaining vertices have 2-list that are not all the same). Giving us an L-colouring of G, and a contradiction.

Subcase 22.2.2.2 $|S_P| = 5$ and $|S'_P| = 6$.

Consecutively label the vertices of the path $G[S'_P] \cap Bnd(G_1)$ starting at the T-vertex $x_1, x_2, ..., x_6$.

Since (G_2, G_1, P) is not a good triple it must be the case that there is a vertex $v \in V(G_1)$ such that $|L(v)| = 3$, and v is adjacent to three S'_P-vertices or $G_1[N_{G_1}(S'_P) \cup (T \cap V(G_1))]$ contains a 5-cycle containing one T-vertex.

Subcase 22.2.2.2.1 There is a vertex $v \in V(G_1)$ such that $|L(v)| = 3$, and v is adjacent to three S'_P-vertices.

Since v is adjacent to three S'_P-vertices, it is adjacent to x_1 and x_6. v is not adjacent to x_4 by Claim 20, therefore v must be adjacent to x_3.

We have already proven that there are no separating $S \cup T, T$-paths of length two, therefore the path (x_1, v, x_6) is not a separating path. This implies that $G_1 = G[\{x_1, x_2, ..., x_6, v\}]$. Let u be the neighbour of x_1 in $Bnd(G_2)$ that is not x_2 and let u_1 be the neighbour of u on the boundary of G_2 that is not x_1. Either u_1 has list size three, or it does not.

Subcase 22.2.2.2.1.1 $|L(u_1)| = 3$.

Colour and delete x_1 and adjust the lists of the neighbours of x_1 accordingly. Call this new list assignment L'. We also have a new set $T' = (T \cup N_G(x_1)) \backslash \{x_1\}$ of T'-vertices on $G - \{x_1\}$. Colour and delete x_5,x_6 and v. Then colour x_3 and x_2.

Define $S' = (S \backslash \{x_5, x_6\}) \cup \{x_2, x_3\}$, let $H = G_2 - x_1$ and restrict T' to H. Note $N_G(x_1)$ is an independence set. Since no separating $S \cup T, T$-paths of length two exist, we also have that no vertex in $N_G(x_1)$ is adjacent to an $S \cup T$-vertex in $G_2 - x_1$. Therefore, the only T'-adjacency in $G_2 - x_1$ is the T-adjacency.

By Lemma 10, since (v, x_1, x_2, x_3, v) is a 4-cycle in G_1, there are no odd cycles in $H[N_H(S') \cup T']$. Since $|S'| = 4$, we can use the minimality of G to find an L'-colouring of H. Placing the coloured verticesx_1, v, x_5, x_6 back in the graph, we find an L-colouring of G, and a contradiction.

Subcase 22.2.2.2.1.2 $|L(u_1)| \neq 3$.

By the existence of the T-adjacency in G_2 and $x_4 \in S$, u_1 must be a T-vertex. Colour and delete u and x_1 as to not disturb the list of u_1 and adjust the lists of the neighbours of u and x_1 accordingly. Call this new list assignment L'. We also have a new set $T' = (T \cup N_G(\{x_1, u\})) \backslash \{x_1, u\}$ of T'-vertices on $G - \{x_1\}$. Colour and delete x_5,x_6 and v. Then colour x_3 and x_2.

Define $S' = (S \backslash \{x_5, x_6\}) \cup \{x_2, x_3\}$ and restrict T' to $G_2 - x_1$. Note $N_G(\{u, x_1\})$ is an independence set by the existence of the cycle (x_1, x_2, x_3, v, x_1) in G. Since no separating $S \cup T, T$-paths of length two exist, we also have that no vertex in $N_G(x_1)$ is adjacent to an $S \cup T$-vertex in $G_2 - x_1$.By the existence of the cycle (x_1, x_2, x_3, v, x_1), if a vertex in $N_{G_2}(u) \cap int(G_2)$ were adjacent to an $S \cup T$-vertex, then there would be a separating path of length two that would induce a good triple in G, a contradiction with Claim 19.

Therefore, the only T'-adjacency in $G_2 - x_1$ is the T-adjacency. Since $|S'| = 4$, we can use the minimality of G to fine an L'-colouring of $G - x_1$. Placing the coloured vertex x_1 back in the graph, we find an L-colouring of G, and a contradiction.

Subcase 22.2.2.2.2 $G_1[N_{G_1}(S'_P) \cup (T \cap V(G_1))]$ contains a 5-cycle containing one T-vertex.

Let this 5-cycle be $C = (z_1, .., z_5, z_1)$ such that z_5 is the T-vertex. Now it must be the case that z_1 and z_4 are adjacent to the ends of $Bnd(G_1)[S'_P]$, without loss of generality, suppose that z_1 is adjacent to x_1 and z_4 is adjacent to x_6. Since we have already proved this lemma for paths of length 2, it must be the case that z_1 and z_4 are boundary vertices. By Claim 12, the faces of $G[V(C) \cup S'_P]$ have empty interior in G.

Since 4-cycles are far apart, it is either the case that z_2 is adjacent to w_2 or w_3.

Subcase 22.2.2.2.2.2.1 z_2 is adjacent to x_2.

This case is the same as Subcase 22.2.1.2.2.1, only on G_1 and S'_P.

Subcase 22.2.2.2.2.2.2 z_2 is adjacent to x_3.

Since $|S| = 4$ and one of the vertices of C is in a 4-cycle, we have that x_3 is adjacent to exactly one S-vertex. We apply the same argument as in Subcase 22.2.1.2.1.1. with z_2 to find an L-colouring of G, and a contradiciton. □

Claim 23. G has a T-adjacency.

Proof. Suppose G does not have a T-adjacency. We will show without loss of generality that $|S| \geq 3$.

Suppose $|S| < 3$. Since G does not contain any 3-cycles, we know that $|V(Bnd(G))| \geq 4$. Since $Bnd(G)$ can not have all T-vertices, we have that $Bnd(G)$ has a vertex v such that $|L(v)| = 3$. Let P be a path in $Bnd(G)$ such that:

- $S \subset V(P)$;
- $v \notin V(P)$; and
- $3 \leq |V(P)| \leq 4$.

Let H be the component in $G[V(P) \cup T]$ that contains P and let $S' = V(H)$. Let $T' = T \backslash S'$. By Claim 20, G has no 1-paths that separate the boundary with one end in $S \cup T$. Therefore, the vertices of S' are consecutive on the boundary. Reduce the lists of the vertices of S' so that they have proper 1-lists (we can do this since paths with one end coloured and every other vertex having a list of size at least two are L-colourable). Call this new list assignment L'. An L'-colouring is clearly an L-colouring, therefore we can assume $|S| \geq 3$. Furthermore $G[S] \cap Bnd(G)$ is either a path or a cycle.

Case 23.1 $|V(Bnd(G)) \backslash S| \leq 1$.

By Claims 20 and 21, if $Bnd(G)[S]$ is not a cycle, then both ends of $Bnd(G)[S]$

are not in the same 4-cycle. Since 4-cycles are far apart, it must be the case that one of these ends is not in a 4-cycle. If $Bnd(G)[S]$ is not a cycle. then let $s \in S$ be an end of $Bnd(G)[S]$ that is not in a 4-cycle. If $Bnd(G)[S]$ induces a cycle, then s can be any S-vertex not in a 4-cycle with 3-list vertices.

Colour and delete s. Let $c(s)$ be the only colour in $L(s)$. For every neighbor x of s, redefine the list of x to be $L(x)\backslash\{c(s)\}$, and let $T' = N_G(s) \cap int(C)$ and $S' = V(Bnd(G))\backslash\{s\}$ (If needed, reduce the list of S'-vertices to be proper 1-lists). Note that $T = \varnothing$, so $T \subset T'$. There are no T'-adjacencies since there are no 3-cycles in the graph. There are no S, T'-adjacencies by Claim 21 and therefore, no S', T'-adjacencies since s is not in a 4-cycle. Reduce the lists of the vertices of S' to be proper 1-lists (this only happens when x was adjacent to a list three vertex on the boundary).

Now S' and T' satisfy the hypothesis and by minimality of G we colour $G - s$. This induces a partial colouring on G where no neighbor of s is coloured $c(s)$, therefore we colour s with $c(s)$ and we have an L-colouring of G, a contradiction.

Case 23.2 $|V(Bnd(G))\backslash S| \geq 2$.

Let u_1 and v_1 be the first and last S-vertices, and let u_2 and v_2 be their S-neighbours via a boundary edge, respectively. By Claims 20 and 21, u_1 and v_1 can not be in the same 4-cycle. Since 4-cycles are far apart we have that one pair of neighbours is not in a 4-cycle together. Without loss of generality, suppose u_1 and u_2 are not in a 4-cycle together.

Let x_1 be the neighbour of u_1 in $Bnd(G)$ not in S and let x_2 be the neighbour of x_1 in $Bnd(G)$ that is not in S. There are two cases, either x_2 is a 3-list vertex or $x_2 \in T$.

Case 23.2.1 x_2 is a 3-list vertex.

Colour and delete u_1 and let $H = G - u_1$. Adjust the lists of the vertices in $N_G(u_1)\backslash\{u_2\}$ to be 2-lists (either by deleting $c(u_1)$ from their lists, or deleting a colour from their lists). Define $T' = T \cup N_G(u_1)\backslash\{u_2\}$ and $S' = S\backslash\{u_1\}$.

By Claim 21, there are no S, T'-adjacencies, and therefore there are no S', T'-

adjacencies. $N_G(u_1)$ is an independence set since there are no 3-cycles in G. By Claim 22, there are no adjacencies between vertices in$N_G(u_1)$ and vertices in T. Therefore, there are no T'-adjacencies.

By Lemma 10, since $|S'| = |S|-1 \leq 5$, there are no odd cycles in $(H)[N_H(S') \cup T']$. Similarly, there is no 3-list vertex in H adjacent to three S' vertices. Therefore, we can apply induction to H, resulting in an L-colouring of H. Placing the coloured vertex u_1 back in H, we get an L-colouring of G, and a contradiction.

Case 23.2.1 $x_2 \in T$.

Colour and delete u_1 and u_2, and let $H = G - \{u_1, u_2\}$. Adjust the lists of the vertices in $N_G(\{u_1, u_2\})$ to be 2-lists (either by deleting $c(u_1)$ or $c(u_2)$ from their lists, or deleting a colour from their lists). Define $T' = T \cup N_G(\{u_1, u_2\})$ and $S' = S \backslash \{u_1, u_2\}$.

By Claim 21, there are no S, T'-adjacencies, and therefore there are no S', T'-adjacencies. $N_G(\{u_1, u_2\})$ is an independence set since there are no 3-cycles, and u_1 and u_2 are not in a 4-cycle. By Claim 22, there are no adjacencies between vertices in $N_G(\{u_1, u_2\})$ and vertices in T other than the edge x_1x_2. Therefore, is exactly one T'-adjacency.

If the T'-adjacency is in a 4-cycle, then by Claim 20 this 4-cycle does not contain v_1 and v_2. Therefore, we would have coloured and deleted the vertices v_1 and v_2 over u_1 and u_2. We may assume without loss of generality that the T-adjacency is not in a 4-cycle.

The last thing we must check is for odd cycles in $H[N_H(S') \cup T']$. Suppose by way of contradiction that a cycle C in Lemma 10 exists in H. Since x_1 was originally adjacent to u_1 in G and $x_2 \in T$, C would be an odd cycle in $G[N_G(S) \cup T]$, a contradiction with G being a counter example.

Therefore, we can apply induction to H, resulting in an L-colouring of H. Placing the coloured vertices u_1, u_2 back in H, we get an L-colouring of G, and a contradiction. □

Claim 24. *G has no T-adjacency.*

By way of contradiction, suppose G has a T-adjacency. We will first show that $|Bnd(G)| > 6$.

Suppose $|Bnd(G)| \leq 6$. Since G has a exactly one T-adjacency and no S,T-adjacencies, there exists a vertex $v \in V(Bnd(G))$ such that $|L(v)| = 3$. Observe by the existence of v, $Bnd(G)$ is L-colourable. Therefore, we can define L' to be the new list assignment with the lists of vertices in $V(Bnd(G)$ to be proper 1-lists and the remaining vertices keeping their original lists from L. Letting $S' = V(Bnd(G))$, we have that $|S'| \leq 6$, and that G under the list assignment L' and with the set S' satisfies the hypothesis. By Claim 23, we have that G is L'-colourable, and therefore L-colourable, a contradiction.

Let $u, v \in T$ be adjacent and let $(u_2, u_1, u, v, v_1, v_2)$ be the subpath of $Bnd(G)$ of length five with the T-adjacency in the middle. Clearly we have that $|L(u_1)| = 3 = |L(v_1)|$. The rest of this proof will follow by considering cases of u_2 and v_2.

Case 24.1 $u_2 \in S$ and $|L(v_2)| = 3$.

Colour and delete v. Adjust the lists of the vertices in $N_G(v)$, reduce the lists of u_1 and u to be proper 1-lists and call the resulting graph and list assignment G' and L' respectively. If there is an L'-colouring of G', then there is an L-colouring of G by placing the coloured vertex v in G'. Define $S' = S \cup \{u_1, u\}$ and $T' = (T \cup N_G(v)) \backslash \{u, v\}$.

First let us check for any T'-adjacencies. Note that $T' \backslash T \subseteq N_G(v)$ and there are no 3-cycles in G; therefore $T' \backslash T$ is an independence set. As $G[T \backslash \{v\}]$ is an independent set, so; by Claim 22, there are no edges between $T' \backslash T$ and $T \backslash \{v\}$. Therefore T' is an independent set.

Now let us check that there are no S', T'-adjacencies. We know there are no S', T-adjacencies by Claim 20. There are also no adjacencies between S-vertices and vertices in $N_G(v) \cap V(int(G))$ by Claim 22. Since the T-adjacency is not in a 4-cycle, u_1 is not adjacent to anything in $N_G(v) \cap V(int(G))$. There are no 3-cycles, therefore u is not adjacent to anything in $N_G(v) \cap V(int(G))$. Hence there are no S', T'-adjacencies.

If G' satisfies the hypothesis on L' with the sets S' and T', then we colour

G', find an L-colouring for G, and end with a contradiction. Therefore, G' does not satisfy the hypothesis under the list assignment L' and the sets S' and T'. This imples that there exists an odd cycle in $G'[N_{G'}(S') \cup T']$. This means that there is a 5-cycle $C = (z_1 z_2 ... z_5 z_1)$ such that $z_5 \in T$ and for each vertex that is not z_5 in the cycle, it must have list size three and be adjacent to an S'-vertex. Since this is the case, $|S'| = 6$, z_1 is adjacent to w_1 and z_4 is adjacent to w_2 where w_1 and w_2 are the different ends of $Bnd(G)[S']$.

Let $P = (w_2 z_4 z_3 z_2 z_1 w_1)$ be a path of length four. Clearly P is separating by the existence of v_1. This path P induces a triple (G_1, G_2, P). Define H_P and S_P as in Definition 17 on the triple (G_1, G_2, P). Note that both z_1 and z_4 are each at most distance one from a 4-cycle in G_1. Therefore, there is no vertex in $G_2 - V(H_P)$ with list size three adjacent to three vertices of S_P in G_2 and $G_2[N_{G_2}(S_P) \cup (T \cap V(G_2 - V(H_P)))]$ does not contain an odd cycle. Thus (G_1, G_2, P) is a good triple, contradicting Claim 19.

Case 24.2 $u_2 \in S$ and $v_2 \in T$.

There are two cases: u_1 and v_1 either have a common neighbour in $V(int(G))$ or they do not.

Subcase 24.2.1 u_1 and v_1 have a common neighbour $x \in V(int(G))$.

We will break this up into two cases, depending on whether x is adjacent to an S-vertex or not.

Subcase 24.2.1.1 x is adjacent to a vertex $z \in S$.

This implies that both z and u_1 are incident with an interior face F. We colour and delete all S-vertices in F that have degree two, u_1, u, and v. Reduce the lists of x, v_1, and v_2 to be proper 1-lists and call this new graph G' with new list assignment L'. Let $S' = \{v \in V(G')|L'(v) = 1\}$ and $T' = T \backslash \{u, v, v_2\}$.

Note that either $|S'| \leq 5$ or x is in a 4-cycle in G that is not in G'. Either way there is no need to check if there is a vertex in G' adjacent to three S'-vertices or for odd cycles in $G'[N_{G'}(S') \cup T']$. If x is adjacent to a T-vertex, then there is a separating 2-path in G with one end in S and the other in

T, a contradiction with Claim 22. Therefore, there are no S', T'-adjacencies and G' is L'-colourable, a contradiction.

Subcase 24.2.1.2 x is not adjacent to an S-vertex.

Colour and delete v_1, v, and u such that $c(v_1) \notin L(v_2)$. Adjust the lists of the vertices in $N_G(\{u, v_1\})$ to be 2-lists.

If u_1 can not be reduced to a proper 1-list, then it is adjacent to an S-vertex through an edge that is not on the boundary, a contradiction with Claim 20.

If x can not be reduced to a proper 1-list, then it is adjacent to an S-vertex, contradicting the hypothesis of this case.

Reduce the lists of u_1 and x to be proper 1-lists and call the resulting graph and list assignment G' and L' respectively. Note that by Claim 12, u, v have degree two in G since the cycle (u_1, u, v, v_1, s, u_1) bounds a face of length five, therefore the only neighbour of u that remains in G' is u_1.

Define $S' = S \cup \{u_1, x\}$ and $T' = (T \cup N_G(v_1)) \backslash \{u, v, x\}$.

Clearly there are no L'-colourings of G', else we could place the coloured vertices u, v, v_1 back in G' to find a L-colouring of G, and a contradiction.

Let us check for any T'-adjacencies. Recall $(T' \backslash T) \subseteq N_G(v_1)$. Since there are no 3-cycles in G, we have that $T' \backslash T$ is an independent set. By defiinition $T \backslash \{u, v, v_1\}$ is an independent set. By Claim 22, the only possible edge between $T' \backslash T$ and $T \backslash \{u, v\}$ is between a T-vertex that is not in a T-adjacency and a vertex in $N_G(v_1) \cap V(int(G))$. This adjacency is not possible because it would form a good separating 2-path, which by Claim 19 does not exist. Therefore T' is an independent set.

By Claim 20, we know there are no $S' \backslash \{x\}, T$-adjacencies. Suppose there is a vertex $w \in S$ that is adjacent to a vertex $y \in N_G(v_1) \cap V(int(G))$. Since the path v_1, y, w does not induce a good triple it must be the case that $(w, y, v_1, v, u, u_1, u_2, w)$ is a 6-cycle. By Claim 12, we have that $deg_G(x) = 2$, a contradiction with Claim 11. Therefore, no vertex in $N_G(v_1)$ is adjacent to an S-vertex.

If u_1 is adjacent to anything in $(N_G(v_1)\backslash\{x\})$, then u_1 is in a 5-cycle that has x in its interior and u_2 in its exterior, contradicting Claim 12.

If x is adjacent to a T-vertex, then it forms a good separating 2-path, contradicting Claim 19. Since there are no 3-cycles in the graph, x is also not adjacent to a vertex in $T'\backslash T \subseteq N_G(v_1)$.

We need to check two things, first that there are no vertices that are adjacent to three vertices in S', and second that $H[N_H(S') \cup T']$ does not contain an odd cycle.

Suppose there is a vertex z adjacent to three vertices in $s_1, s_i, x \in S'$ such that s_1 and x are the ends of the path $Bnd(G')[S']$. Similar to Subcase 24.2.1.1, colour and delete every vertex in $(S' \cup \{u_1, u, v)\}\backslash\{s_1, x\}$ and call this new graph G''. Then define $S'' = \{s_1, z, x, v_1, v_2\}$ and $T'' = T\backslash\{u, v, v_2\}$ and reduce the list of the vertices of S'' to be proper 1-lists and call this list assignments L''.

By minimality of G, G'' is L''-colourable. Placing the coloured vertices of $V(G)\backslash V(G'')$ back into the graph G'' we get an L-colouring of G and a contradiction. Therefore, there is no vertex adjacent to three vertices of S'.

Suppose there is an odd cycle in $G'[N_{G'}(S') \cup T']$. Let C be a cycle as described in Lemma 10 such that $C = (z_1z_2...z_5z_1)$ and $z_5 \in T'$. Since this is the case, $|S'| = 6$, z_1 is adjacent to w_1 and z_4 is adjacent to w_2, where w_1 and w_2 are the different ends of $Bnd(G)[S']$.

Without loss of generality, let $w_1 = u_2$. If $z_5 \in T$, then the path $(z_5z_1u_2)$ is a separating 2-path,a contradiction with Claim 22. Therefore, $z_5 \in V(int(G))$ which tells us z_5 is adjacent to v_1 since $z_5 \in T'\backslash T$. Since there is a 4-cycle comprised of only $S' \cup V(C)$-vertices, we have that $x \neq z_5$, else (x, u_1, u_2, z_1, x) would be another 4-cycle too close to the previous.

We have (u_1, u, v, v_1, x) is a 5-cycle with empty interior by Claim 12, and $(v_1, z_5, z_1, u_2, u_1, x, v_1)$ is a 6-cycle with empty interior for the same reason. This implies that $deg_G(x) = 2$, a contradiction with Claim 11. Therefore there is no odd cycle in $H[N_{G'}(S') \cup T']$.

By minimality of G, we get an L'-colouring of G'; placing the coloured vertices of $V(G)\backslash V(G')$ back yields an L-colouring of G and a contradiction.

Subcase 24.2.2 u_1 and v_1 do not have a common neighbour in $V(int(G))$.

We have either both v and v_1 are in a 4-cycle or they are not.

Subcase 24.2.2.1 One of v and v_1 is not in a 4-cycle.

Colour and delete v_1 and v so as not to disturb the list of v_2 and call this new graph H. Following this reduce the lists of $N_G(\{v, v_1\})$. Assign u a 1-list that is not the colour of v, and u_1 a 1-list that is not the colour of u_2 or v. Define $S' = S \cup \{u, u_1\}$ and $T' = (T \cup N_G(\{v, v_1\}))\backslash\{u, v\}$.

There are no adjacencies between a vertex in $N_G(v_1) \cap V(int(G))$ and an $S \cup T$-vertex, else there is a separating path P of length two form v_1 to an S-vertex that would induce a triples (G_1, G_2, P), (G_2, G_1, P) such that one of the triples is good, a contradiction with Claim 19. There are also no adjacencies between u or u_1, and a vertex in $N_G(v_1) \cap V(int(G))$ since the T-adjacency is not in a 4-cycle and u_1 and v_1 do not have a common neighbour in $V(G)$.

There are no adjacencies between a vertex in $N_G(v) \cap V(int(G))$ and an $S \cup T$-vertex by Claim 22 and no adjacencies between a vertex in $N_G(v) \cap V(int(G))$ and a vertex in $S'\backslash S$ since the T-adjacency is not in a 3- or 4-cycle.

By Claim 20 no vertex in S' is adjacent to a T-vertex. Therefore, we have that there are no S', T'-adjacencies. Since one of v and v_1 is not in a 4-cycle, we have that there are no T'-adjacencies in $G[N_G(\{v, v_1\})]$. Since the T-adjacency no longer exists in H, we have that there are no T'-adjacencies in H.

By Claim 22, since $u \in T$ and u is an end of the path $Bnd(H)[S']$), there is no vertex with list size three in H that is adjacent to three S-vertices. Therefore, we need only check that there is not a 5-cycle in H that has one vertex in T', and the rest having list size and each being adjacent to one

S'-vertex.

Suppose by way of contradiction that there exists such a cycle $C = (z_1 z_2 ... z_5 z_1)$ such that $z_5 \in T'$, z_1 is adjacent to u_1 and z_4 is adjacent to a vertex w that is the end of both paths $Bnd(G)[S]$ and $Bnd(H)[S']$. If z_5 is adjacent to v, then the path (v, z_5, z_4, w) is a separating 3-path with one end in S and the other in T, a contradiction with Claim 22.

If z_5 is adjacent to v_1, then the path (v_1, z_5, z_4, w) will induce a good triple since one side of this path contains the S-vertices and the T-adjacency, and the other side contains a list size three vertex (i.e. v_1).

If $z_5 \in T$, then the path (z_5, z_1, u_1) is a separating 2-path since $z_5 \neq u$ ($z_5 \in V(H)$, $u \notin V(H)$).

Therefore, $z_5 \notin T'$, a contradiction with $z_5 \in T'$. This gives us that there are no odd cycles in $(N_H(S') \cup T')$ with every list three vertex adjacent to an S'-vertex. By minimality of G, we L-colour H, and place the coloured vertices v and v_1 back into H to find an L-colouring of G, and a contradiction.

Subcase 24.2.2.2 Both of v and v_1 are in 4-cycles.

Since 4-cycles are far apart it must be the case that v and v_1 are in the same 4-cycle. Let this 4-cycle be $C = (v, v_1, x_1, x)$. Since the T-adjacency is not in a 4-cycle and $deg_G(v_1) = 3$, by Claim 11 we know that neither x nor x_1 is either of u or v_2.

Now we have that either there is a vertex in $N_G(u_1)$ that is adjacent to a vertex in $N_G(v)$ or not.

Subcase 24.2.2.2.1 There does not exist a vertex in $N_G(u_1)$ that is adjacent to a vertex in $N_G(v)$.

Colour and delete u_1, u and v so as not to disturb the list of u_2 and call this new graph H. Following this reduce the lists of the vertices of $N_G(\{u_1, u, v\})$ in H. Define $T' = (T \cup N_G(\{u_1, u, v, \})) \backslash \{u, v\}$.

Since $N_G(\{u_1, v\})$ is an independent set, and by the existence of C we have that $N_G(\{u_1, v, v_1\})$ is an independent set. We also have that $T\backslash\{u, v\}$ is an independent set. Now we check for $(T'\backslash T, S \cup T\backslash T')$-adjacencies.

By Claim 22, no vertex in $N_G(\{u, v\})\backslash\{u_1\}$ is adjacent to an $S\cup T$-vertex. By the existence of C, if a vertex in $N_G(u_1)$ is adjacent to a $S\cup T$-vertex, then it would form a separating path that would induce a good triple, contradicting Claim 19. Therefore, there are no T'-adjacencies and no (S, T')-adjacencies.

By Lemma 10, since C is in G, and not in H, we know the remaining properties to the hypothesis hold. Therefore, with the sets S and T' and by minimality of G, we find an L-colouring of H. Placing the coloured vertices u, v, and v_1 back in the graph, we find an L-colouring of G, and a contradiction.

Subcase 24.2.2.2.2.2 There does exist a vertex in $N_G(u_1)$ that is adjacent to a vertex in $N_G(v)$.

Let these two vertices be $w_1 \in N_G(u_1)$ and $w \in N_G(v)$. There are two cases to consider, whether $w = x$ or $w \neq x$.

Subcase 24.2.2.2.2.2.1 $w \neq x$.

By Claim 22, no vertex in $N_G(w)$ is adjacent to an S-vertex.

Colour and delete u_1, u, v, v_1, w and w_1 so as not to disturb the list of v_2 and call this new graph H. Following this reduce the lists of the vertices of $N_G(\{u_1, u, v, v_1, w, w_1\})$ in H. Define $T' = (T \cup N_G(\{u_1, u, v, v_1, w, w_1\}))\backslash\{u, v\}$.

By the existence of C, and Claims 12, 11 we have that $E(G[N_G(\{w, w_1, u_1, u, v, v_1\})]) = \{xx_1\}$. We already have that $T\backslash\{u, v\}$ is an independence set and that there are no (S, T)-adjacencies. Now we check for $(T'\backslash T, S \cup (T\backslash T'))$-adjacencies.

Since u is in the cycle (u_1, u, v, w, w_1, u_1), Claim 12 implies $deg_G(u)$, and that all of its neighbours were deleted. By Claim 22, no vertex in $N_G(v, w)\backslash\{u_2\}$ is adjacent to an $S \cup T$-vertex.

If a vertex in $N_G(\{u_1, v_1\})\backslash\{u, v\}$ is adjacent to an $S \cup T$-vertex, then it would form a separating path. By Lemma 10, since C is close to this separating path, the path would induce a good triple, contradiction Claim 19.

Liekwise, no vertex in $N_G(w_1)\backslash\{v_1, w\}$ is adjacent to an $S \cup T$-vertex.

Therefore, there is one T'-adjacency and no (S, T')-adjacencies.

Since C is in G, and not in H, we know that the T'-adjacency is not in a 4-cycle and that it is not in an odd cycle with the remaining vertices adjacent to S-vertices. Therefore, by minimality of G, an L-colouring of H. Placing the coloured vertices u_1, u, v, v_1, w, w_1 back in the graph, yields an L-colouring of G, and a contradiction.

Subcase 24.2.2.2.2.2 $w = x$.

By Claim 22, no vertex in $N_G(x_1)$ is adjacent to an S-vertex.

Colour and delete u, v, v_1, x and x_1. Call this new graph H. Following this reduce the lists of the vertices of $N_G(\{v_1, x, x_1\})$ in H and then reduce the lists of u_1 and w_1 to be proper 1-lists. Define $T' = (T \cup N_G(\{v_1, x, x_1\}))\backslash\{u, v, w_1\}$ and $S' = S \cup \{u_1, w_1\}$.

By the existence of C, Claims 11 and 12 imply that $N_G(\{x, x_1, v_1\})\backslash\{u, v\}$ is an independent set. We also have that $T\backslash\{u, v\}$ is an independent set. Now we check for $(T'\backslash T, S' \cup (T\backslash T'))$-adjacencies.

By Claim 22, no vertex in $N_G(x)\backslash\{u_2\}$ is adjacent to an $S \cup T$-vertex. If a vertex in $N_G(\{v_1\})\backslash\{u, v\}$ is adjacent to an $S \cup T$-vertex, then it along with its $S \cup T$ neighbour and v_1 would form a separating path. By Lemma 10, since C is close to this path, the path would induce a good triple, contradicting Claim 19.

By Claim 19 and by the hypothesis of this case, no vertex in $N_G(x_1)\backslash\{v_1, w\}$ is adjacent to an $S \cup T$-vertex. Therefore, there are no T'-adjacencies and no (S, T')-adjacencies.

By the existence of C, and Claims 12, 11 we have that w_1 and u_1 are not adjacent to any $T'\backslash T$-vertices. By Claim 20, u_1 is not adjacent to any T-vertices. If w_1 is adjacent to a T-vertex, then the 2-path from u_1 to this T-vertex would induce a good triple, contradicting Claim 19. Therefore, there are no S', T'-adjacencies.

By Lemma 10, since C is in G, and not in H, we know the remaining hypothes hold. Therefore, by minimality of G, there is an L-colouring of H. Placing the coloured vertices u, v, v_1, w, w_1 back in the graph, produces an L-colouring of G, and a contradiction.

Case 24.3 $u_2, v_2 \in S$.

We will break this up into cases, depending on whether $N_G(u_1) \cap N_G(v_1) = \varnothing$ or not.

Subcase 24.3.1 $z \in N_G(u_1) \cap N_G(v_1)$.

Colour and delete u, v and reduce the lists of u_1 and v_1 to be proper 1-lists. Let $S' = S \cup \{u_1, v_1\}$, $T' = \varnothing$ and let this this new list assignment be L' on $G' = G - \{u, v\}$. Clearly z is not adjacent to an S vertex, else the 5-cycle (z, u_1, u, v, v_1, z) is in G and has every three list adjacent to an S-vertex, which is not possible by hypothesis.

There are also no odd cycles in $N_{G'}(S') \cup T'$ because $T' = \varnothing$. If there is a list size three vertex adjacent to three S'-vertices, then is must be adjacent to both u_1 and v_1. This vertex would then form a 5-cycle in G with z on its interior and u_2 on its exterior, a contradiction with Claim 12.

Therefore, by minimality of G, G' satisfies the hypothesis and is L'-colourable. Placing the coloured vertices back into G' we get that G is L-colourable, a contradiction.

Subcase 24.3.2 $N_G(u_1) \cap N_G(v_1) = \varnothing$

Since the T-adjacency is not in a 4-cycle, we may assume without loss of generality that v is not in a 4-cycle, else we look at u. Colour and delete v and v_1. Adjust the lists of $N_G(\{v, v_1\})$ accordingly and reduce the lists of u_1 and u to be proper 1-lists. Let $S' = S \cup \{u_1, u\}$ and $T' = N_G(\{v, v_1\}) \backslash \{u, v_2\}$ and let this new list assignment be L' on $G' = G - \{v, v_1\}$.

Since v is not in a 4-cycle, we have that there are no T'-adjacencies. Since the T-adjacency is not in a 4-cycle and $N_G(u_1) \cap N_G(v_1) = \varnothing$, we have that there are no $(S' \backslash S, T')$-adjacencies. If there a was an S, T'-adjacency, then we would have a good separating 2-path, a contradiction with Claim 19.

Suppose there is a vertex with list size three adjacent to three S'-vertices in G' and let this vertex be z. z would be adjacent to both u and v_2 and therefore (u, v, v_1, v_2, s, u) would be a 5-cycle with empty interior by Claim 12. This would imply that $deg_G(v_1) = 2$, and by Claim 11, this is a contradiction. Therefore, thre is no vertex adjacent to three S'-vertices.

Suppose there is a 5-cycle $C = (z_1, z_2, ..., z_5, z_1)$ in G', such that $z_5 \in T'$, z_1 is adjacent to u and z_4 is adjacent to v_2 and the graph $G[V(C) \cup S']$ has a 4-cycle C_4. Since $z_5 \in T'$, it is either $z_5 v \in E(G)$ or $z_5 v_1 \in E(G)$. If $z_5 v \in E(G)$, then the 4-cycle (v, z_5, z_1, u, v) is too close to C_4. Therefore, $z_5 v_1 \in E(G)$. We have that the 4-cycle $(v_1, z_5, z_4, v_2, v_1)$ is too close to C_4, therefore C does not exists.

G' satisfies the hypothesis and by minimality of G, we have that G' is L'-colourable. Placing the coloured vertices back into G', we get that G is L-colourable, and a contradiction.

Case 24.4 $u_2, v_2 \notin S \cup T$.

In this case we colour and delete u and v. For every vertex $x \in N_G(u)$, redefine the list of x to be $L'(x) = L(x) \backslash c(u)$. Similarly, redefine the lists of the neighbours of v. Define $T' = (T \cup N_G(\{u, v\})) \backslash \{u, v\}$. By Claim 22, no vertex in $N_G(\{u, v\}) \backslash \{u_1, v_1\}$ is adjacent to an $S \cup T'$-vertex. By Claim 20, u_1 and v_1 are not adjacent to any $S \cup T'$-vertices either.

Since the T-adjacency is not in a 4-cycle, we have that there is no T'-adjacency. Since $|S| \leq 4$ and there is no T'-adjacency, we have that there is no odd cycle in $G[N_G(S) \cup T']$

Therefore $G - \{u, v\}$ satisfies the hypothesis, and by minimality of G, $G - \{u, v\}$ has an L'-colouring. Placing the coloured vertices u, v back in the graph we find an L-colouring of G, and a contradiction.

Case 24.5 $u_2 \in T$ and $v_2 \notin S \cup T$.

Colour and delete u_1, u, v so as not to disturb the list of u_2 and let $G' = G - \{u_1, u, v\}$. Adjust the list of each vertex in $N_G(\{u_1, u, v\})$ accordingly and call this list assignment L'. Note that since the list of u_2 was not disturb by the colouring of these vertices, $|L'(u_2)| = 2$. Define $T' = T \cup (N_G(\{u_1, u, v, s\}) \cap V(G'))$.

Since the T-adjacency is not in a 4-cycle we have that $N_G(u_1) \cap N_G(v) \cap Int(G) = \varnothing$. This implies that there are no vertices outside of the S-vertices that have an L' list size of one in G'.

By Claim 22, we know that no neighbours of u or v are adjacent to a $S \cup T$-vertex other than u and v. By Claim 19, we know that no neighbour of u_1 is adjacent to an $S \cup T$-vertex.

Now the question becomes, where might a new T'-adjacency xy appear, and can it be in a 4-cycle in G'. Since the T-adjacency can not be in a 4-cycle, there are two cases for x and y that are independent of each other by Claim 12. Either

- $x \in N_G(u_1) \cap int(G)$ and $y \in N_G(u) \cap int(G)$,or
- $x \in N_G(u_1) \cap int(G)$ and $y \in N_G(v)$.

Subcase 24.5.1 Suppose $x \in N_G(u_1) \cap int(G)$ and $y \in N_G(u) \cap int(G)$.

This immediately implies that the cycle $C = (u_1, x, y, u, u_1)$ is a 4-cycle. It follows that the T'-adjacency in G' is not in a 4-cycle since 4-cycles are far apart. Similarly there are no odd cycles in $N_{G'}(S) \cup T'$ since the T'-adjacencies are close to C in G, and C is not in G'. Therefore, under this constraint G' has an L'-colouring. Placing the deleted vertices back into the graph, we find an L-colouring of G, and a contradiction.

Subcase 24.5.2 $x \in N_G(u_1) \cap int(G)$ and $y \in N_G(v)$.

First let us show that the edge xy is not in a 5-cycle in G' with the property that the three vertices in the 5-cycle that are not x or y are each adjacent to an S-vertex. Suppose this is the case and let z be the vertex adjacent to y in the 5-cycle that is not x and let s be the S-vertex adjacent to z. The path (v, y, z, s) is a separating 3-path, a contradiction with Claim 22.

Therefore, we may assume that xy is in a 4-cycle C in G', else we can find an L'-colouring of G' and a contradiction. By Claim 22, no vertex of this 4-cycle is in $S \cup T$.

Extend the labelling of the vertices, so u_3 is the neighbour of u_2 on the boundary that is not u_1 and u_4 is the neighbour of u_3 on the boundary that is not u_2. Colour and delete u_1 and u_2 as to not disturb the list of u and call the resulting graph H. For every vertex in $N_G(\{u_1, u_2\})$ adjust their lists accordingly and call this list assignment L''. Note that since the list of u was not disturbed by the colouring of these vertices, $|L''(u)| = 2$. Define $T'' = T \cup (N_G(\{u_1, u_2\})$.

If a neighbour of u_1 that is not u is adjacent to an $S \cup T$-vertex other than v, then this neighbour along with the $S \cup T$-vertex and u_1 would form a separating path. By Lemma 10 and the existence of the cycle C, this path would induce a good triple, contradicting Claim 19. If a neighbour of u_1 other than u is adjacent to v, then this neighbour along with u_1, u and v would form a 4-cycle that is close to C, a contradiction with 4-cycles far apart.

By Claim 22, at most one neighbour of u_2 is adjacent to an $S \cup T$-vertex, specifically u_3.

Since 4-cycles are far apart, either $N_G(\{u_1, u_2\})$ is an independent set, or $y \in N_G(u_2)$. If $y \in N_G(u_2)$, then (y, u_2, u_1, u, v, y) is a 5-cycle with x on the interior and v_2 on the exterior, a contradiction by Claim 12.

Therefore, $N_G(\{u_1, u_2\})$ is an independence set. Since $x \in N_G(u_1) \cap int(G)$ and $y \in N_G(v)$, Claim 12 shows $deg_H(u) = 1$. So we have that there are no S, T''-adjacencies in $H - u$ and there is at most one T''-adjacency in $H - u$.

If there is an L''-colouring of $H - u$, then we can we place the coloured vertices back into the graph so that we have an L-colouring of $G - \{u\}$. Since $c(u_1) \notin L(u)$, we can let $c(u) \in L(u) \backslash \{c(v)\}$ to get an L-colouring of G, and a contradiction. Therefore, we need only find an L''-colouring of $H - u$. Either u_3 is adjacent to an S-vertex or it is not.

Subcase 24.5.2.1 u_3 is adjacent to an S-vertex.

In $H - u$, if u_3 is adjacent to an S-vertex, then reduce its list to be a proper 1-list, let $S' = S \cup \{u_3\}$ and remove u_3 from T''. Note that $|S'| \leq 5$, therefore $H - u$ satisfies the hypothesis with S' and T'' and we use the minimality of G to find an L''-colouring of $H - u$, and a contradiction.

Subcase 24.5.2.2 u_3 is not adjacent to an S-vertex.

Suppose $N_{H-u}(S) \cup T''$ contains an odd cycle. By Lemma 10, $(H-u)[V(C') \cup S]$ contains a 4-cycle. Note that this 4-cycle is different from C because C contains no $S \cup T$ vertices. These two cycles contradict the fact that 4-cycles are far apart. Therefore, $N_{H-u}(S) \cup T''$ does not contain an odd cycle.

Similarly, if there is a T''-adjacency, then the T''-adjacency would be u_2u_3 and it is it not in a 4-cycle because it would be too close to C.

The minimality of G shows there is an L''-colouring of $H - u$, and a contradiction.

Case 24.6 $u_2, v_2 \in T$.

Without loss of generality, we may assume that $N_G(u_1)$ does not have a vertex adjacent to a vertex in $N_G(v)$, else by planarity the vertices v_1 and u do not have adjacent neighbours.

Define $T' = (T \backslash \{u, v\}) \cup N_G(\{u_1, u, v\})$. We would like to colour and delete u_1, u, v such that the colour of u_1 is not in $L(u_2)$ and proceed as in previous cases by reducing the lists of their neighbours, but unwelcome things can happen. In particular, there could be an odd cycle in $G[N_G(S) \cup T']$, $N_G(u_1) \cap N_G(v_1) \cap Int(G) \neq \varnothing$, there could be a vertex in $N_G(u_1)$ adjacent to a vertex in $N_G(u)$, or v_1 and v_2 could be in a 4-cycle.

Claim 24.6.1 There is no odd cycle in $G[N_G(S) \cup T']$.

Suppose by way of contradiction that there is an odd cycle in $G[N_G(S) \cup T']$.

This implies that there is a cycle C such that the edge v_1v_2 is in C, and every other vertex of C has list size three and is adjacent to an S-vertex. Let z be the neighbour of v_1 in $V(C)\backslash\{uv2\}$. Since 4-cycles are far apart, we have that z is adjacent to a vertex s that is the end of $Bnd(G)[S]$. The path (v_1, z, s) is a separating 2-path, that induces triples (G_1, G_2, P) and (G_2, G_1, P), in which one of them is a good triple, a contradiction with Claim 19.

Claim 24.6.2 $N_G(u_1) \cap N_G(v_1) \cap Int(G) = \varnothing$.

Suppose to the contrary that $z \in N_G(u_1) \cap N_G(v_1) \cap Int(G)$.

As before let u_3 be the neighbour of u_2 on the boundary that is not u_1 and let u_4 be the neighbour of u_3 on the boundary that is not u_2. Define v_3 and v_4 in a similar manner on the other side. The $Bnd(G)$ is really two internally disjoint u_1, v_1-paths.

Let R be the path of the two that is not the path (u_1, u, v, v_1).By Claim 12, if the cycle formed by R and z has length less than 7, then its interior must be empty. But the interior of the cycle (z, u_1, u, v, v_1, z) is empty. This would imply that $deg_G(z) = 2$, a contradiction with Claim 11. Therefore, we know that $|V(Bnd(G))| \geq 9$. This says that if $u_i = v_j$, then $i = j = 4$.

First we show that no vertex in the set $\{u_1, u_2, u_3\}$ is adjacent to a vertex in $\{v_1, v_2, v_3\}$.

Suppose this is not the case, so some u_i and v_j are adjacent for $i, j \in \{1, 2, 3\}$. If $i \neq 3$ and $j \neq 3$, then this would give us a separating cycle $(u_i, ..., u, v, ..., v_j)$ of length at most 7, a contradiction with Claim 12. Therefore, it must be the case that $i = j = 3$. Apply the same argument that shows $|V(Bnd(G))| \geq 9$ to show u_3 is not adjacent to u_3, except define $R = (u_1, u_2, u_3, v_3, v_2, v_1)$.

Without loss of generality we may assume that no pair of vertices in $\{u_1, u_2, u_3\}$ is in a 4-cycle, else we look at v's instead of u's. Colour and delete u_1 and u_2 as to not disturb u. Delete u and v and call the resulting graph H. Adjust the lists of neighbours of u_1 and u_2 accordingly and call this list assignment L'.

Either $u_4 \in S$ or $u_4 \notin S$.

Case 24.6.2.1 $u_4 \in S$.

Define $T' = (T \cup N_G(\{u_1, u_2\}))\backslash\{v, u, u_2, u_3\}$ and $S' = S \cup \{u_3\}$. If a vertex y in $N_G(u_1)\backslash\{u_1, u\}$ is adjacent to an $S \cup T$-vertex, then the $S \cup T$-vertex along with y, u_1 and u_2 would a separating path, contradicting Claim 22. By Claim 22, no vertex in $N_G(u_2)\backslash\{u_1, u_3\}$ is adjacent to an $S \cup T$-vertex.

Since u_1 and u_2 are not in a 4-cycle together, we have that $N_G(\{u_1, u_2\})$ is an independent set. For the same reason, we have that u_3 is not adjacent to any internal neighbours of u_1. By Claim 20, we have that u_3 is not adjacent to any T'-vertices. Reduce the list of u_3 to be a proper 1-list.

From our arguments, there are no S', T'-adjacencies, and there are no T'-adjacencies. As $|S'| \leq 5$, there are no odd cycles in $N_H(S') \cup T'$, and no list size three vertex is adjacent to three S'-vertices. With sets S' and T', H satisfies the hypothesis of the theorem, and by minimality of G, there is an L'-colouring of H. Place the coloured vertices u_1, u_2 back in the graph along with the non-coloured vertices u, v. Since u_1 does not disturb the list of u we can colour v, u without conflict, resulting in an L-colouring of G, a contradiction.

Case 24.6.2.2 $u_4 \notin S$.

Define $T'' = (T \cup N_G(\{u_1, u_2\}))\backslash\{v, u, u_2\}$ and $S' = S$. Apply the same arguments as in Case 24.6.2.1 to show that there are no S', T''-adjacencies and that the only possible T''-adjacency is u_3u_4. By Lemma 10, the only thing we need worry about is if u_3u_4 is a T'-adjacency in a 5-cycle C, with the other vertices in C having list size three and each being adjacent to an S'-vertex.

Let the vertices of $C = (z_1, ..., z_5)$ be such that $u_3 = z_4$ and $u_4 = z_5$. Let s be the S-vertex adjacent to z_3. Then the path $P = (z_4, z_3, s)$ induces a good triple (G_1, G_2, P), a contradiction with Claim 19.

Therefore, there are no odd cycles in $N_H(S') \cup T'$, and we have an L-colouring of H. Place the coloured vertices u_1 and u_2 back in the graph along with u and v. Since u_1 does not disturb the list of u we can colour v, u without conflict, resulting in an L-colouring of G, a contradiction.

Claim 24.6.3 There is no vertex in $N_G(u_1)$ that is adjacent to a vertex in $N_G(u)$.

Suppose there exists a vertex $x \in N_G(u_1)$ that is adjacent to a vertex $y \in N_G(u)$.

Since the T-adjacency is not in a 4-cycle, $y \neq v$. By Claim 11, $deg_G(u_1) \geq 3$. If $x = u_2$, then (x, u_1, u, y, x) would be a 4-cycle with non-empty interior and non-empty exterior, contradicting Claim 12. As such, $x \in N_G(u_1) \cap int(G)$ and $y \in N_G(u) \cap int(G)$. Let C be the 4-cycle (u_1, u, y, x, u_1). Either no vertex in $N_G(u)$ is adjacent to a vertex in $N_G(v_1)$ or some vertex in $N_G(u)$ is adjacent to a vertex in $N_G(v_1)$.

Case 24.6.3.1 No vertex in $N_G(u)$ is adjacent to a vertex in $N_G(v_1)$.

Colour and delete u, v, v_1 so as not to disturb the list of v_2. Adjust the lists of the neighbours of u, v, v_1 accordingly and call this new graph H. Let $T'' = T \cup N_G(\{u, v, v_1\})$.

Clearly no neighbour of u or v is adjacent to an $S \cup T$-vertex by Claim 22. If a neighbour of v_1 is adjacent to an $S \cup T$-vertex, then there is a separating path of length two across the boundary of G with both sides of the path containing a 3-list vertex on the boundary. Using Claim 19, we get that no neighbour of v_1 is adjacent to an $S \cup T$-vertex.

Since C is a 4-cycle, we have that $N_G(\{v, v_1\})$ is an independent set. Similarly, since the T-adjacency is not in a 4-cycle, we have $N_G(\{u, v\})$ is an independent set. Since no vertex in $N_G(u)$ is adjacent to a vertex in $N_G(v_1)$, we have that $N_G(\{u, v_1\}) \cap int(G)$ is an independent set. Thus, $N_G(\{u, v, v_1\}) \cap int(G)$ is an independent set.

Therefore the only T''-adjacency is between v_1 and v_2. Since C is a 4-cycle, we have that the T''-adjacency is not in a 4-cycle. Similarly, by the existence of C, we have that $H[N_H(S) \cup T']'$ has no odd cycles. Taking the sets S and T'' along with the list assignment L', we get that $G - \{u, v, v_1\}$ is L'-colourable by minimality of G. Placing the coloured vertices u, v, v_1 back in the graph, results in an L-colouring of G, and a contradiction.

Case 24.6.3.2 There does exist a vertex in $N_G(u)$ that is adjacent to a vertex in $N_G(v_1)$.

As before, let u_3 be the neighbour of u_2 on the boundary that is not u_1 and let u_4 be the neighbour of u_3 on the boundary that is not u_2. Similarly, extend the labelling of the vertices for v up to v_4.

It is clear that by Claim 12, since u, v, v_1 are in a 5-face on the interior of the graph, $deg_G(v) = 2$. It also follows from Claim 12 that neither u_1 nor u is adjacent to v_1, v_2, v_3 or v_4.

Colour and delete v_1 and v_2 so as not to disturb the list of v. Following this delete v and let $H = G - \{v, v_1, v_2\}$. Define $T' = (T \cup N_G(v_1, v_2)) \backslash \{v, v_3\}$ and $S' = S \cup \{v_3\}$ and call this new list-assignment L'. Adjust the list of the vertices in $N_G(v_1, v_2)$ accordingly, and reduce the list of v_3 to be a proper 1-list.

Clearly no neighbour of v_2 is adjacent to an $S \cup T$-vertex by Claim 22. If a neighbour of v_1 is adjacent to an $S \cup T$-vertex, then there is a separating path of length two across the boundary of G with both sides of the path containing a 3-list vertex on the boundary. Using Claim 19, we get that no neighbour of v_1 is adjacent to an $S \cup T$-vertex.

We can break this up into two similar subcases. Either $v_4 \in S$ or $v_4 \notin S$.

Subcase 24.6.3.2.1 $v_4 \in S$.

C is close to v_1 and v_2, therefore, $N_G(v_1, v_2)$ is an independence set and there is no T'-adjacency. Since $|S'| \leq 5$, $N_H(S') \cup T'$ has no odd cycles.

By minimality of G, we have that H is L'-colourable, by placing the coloured vertices v_1 and v_2 back in the graph, we get an L-colouring of $G - \{v\}$. Apply this colouring to G and let $c(v) \in L(v) \backslash \{c(u)\}$. This is an L-colouring of G, and a contradiction.

Subcase 24.6.3.2.2 $v_4 \notin S$.

C is close to v_1 and v_2, therefore, $N_G(v_1, v_2)$ is an independence set and at most one new T'-adjacencies occurs at the edge v_3v_4. Since C is close to the T'-adjacency, we have that $N_H(S') \cup T'$ has no odd cycles.

Since C is close to the edge v_3v_4, the T'-adjacency is not is a 4-cycle. Since $|S'| = |S| \leq 4$, we may use the minimality of G to get that H is L'-colourable, by placing the coloured vertices v_1 and v_2 back in the graph, we get an L-colouring of $G - \{v\}$. Apply this colouring to G and let $c(v) \in L(v) \backslash \{c(u)\}$. This is an L-colouring of G, and a contradiction.

Claim 24.6.4 There is no vertex in $N_G(v_1)$ that is adjacent to a vertex in $N_G(v_2)$.

Suppose by way of contradiction there there exists a vertex $x \in N_G(v_1)$ that is adjacent to a vertex $y \in N_G(v_2)$. First we know by Claims 12 and 11 that $x \neq v$. Let C be the 4-cycle (v_1, v_2, y, x, v_1).

There are two cases, either there is a vertex in $N_G(u)$ adjacent to a vertex of $N_G(v_1)$ or not.

Case 24.6.4.1 No vertex in $N_G(u)$ is adjacent to a vertex of $N_G(v_1)$.

Colour and delete v_1, v, u so as not to disturb the list of v_2. Adjust the lists of the neighbours of u, v, v_1 and call this new graph H and list assignment L'. Let $T' = (T \cup \{u_1\}) \backslash \{u, v\}$ and $S' = S$. By hypothesis, and the existence of C we know that $N_G(u, v, v_1)$ is an independent set.

By Claim 12 and $v_2 \in T$, no vertex in $N_G(v_1)$ is adjacent to an $S \cup T$-vertex. By Claim 22, no vertex in $N_G(\{u, v\})$ is adjacent to an $S \cup T$-vertex other than the vertex u_1. From Claim 20 we know the only T'-adjacency is the edge u_1u_2.

By the existence of C and Lemma 10, the T'-adjacency is not in a 4-cycle, and there is no odd cycle in $G[N_G(S) \cup T']$.

As a consequence of the minimality of G, there is an L'-colouring of H. Placing the coloured vertices u, v, v_1 back into H, yields an L-colouring of G, and a contradiction.

Case 24.6.4.2 $z \in N_G(u)$ such that z is adjacent to a vertex $z_1 \in N_G(v_1)$.

Now we either have $z_1 \neq x$ or $z_1 = x$.

Subcase 24.6.4.2.1 $z_1 \neq x$.

Colour and delete v_1, v, u, z_1, and z so as not to disturb the list of v_2 and call the resulting graph H. Adjust the lists of the vertices of $N_G(\{v_1, v, u, z_1, z\})$ accordingly, define $T' = (T \cup N_G(v_1, v, u, z_1, z)) \backslash \{u, v\}$ and $S' = S$.

By Claims 12 and 11, and by the existence of C, we have that $N_G(\{v_1, v, u, z_1, z\})$ is an independent set. By Claims 20 and 22, none of $\{v_1, v, u, z_1, z\}$ is adjacent to an $S \cup T$-vertex therefore, every T'-vertex has a 2-list and every S'-vertex has a 1-list.

We claim that no vertex in $N_G(\{v_1, v, u, z_1, z\})$ is adjacent to an $S' \cup T'$-vertex except u_1 adjacent to u_2.

Claim 22, no vertex in $N_G(\{u, z\})$ is adjacent to an $S \cup T$-vertex, except for u_1. No neighbour of v_1 is adjacent to an $S \cup T$-vertex, else it would form a path that would induce a good triple, contradicting Claim 19. Suppose w is an $S \cup T$-vertex adjacent to $w_1 \in N_G(z_1)$. One of the paths (v_1, z_1, w_1, w) or (u, z, z_1, w_1, w) would induce a good triple, contradicting Claim 19, therefore, no such $S \cup T$-vertex exists.

Clearly Claim 20 and the existence of C imply that u_1 is not adjacent to any T'-vertices other than u_2. By the existence of C, we have that there is no odd cycle in $H[N_H(S) \cup T']$. Therefore, H satisfies the hypothesis with S and T', and by minimality of G, there is an L-colouring of H. Placing the coloured vertices back into the graph, yields an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2 $z_1 = x$.

We prove a claim that will break this down into easier cases.

Claim 24.6.4.2.2 One of the following four sets is non-empty:

1. $[L(v_1) \cap L(y)] \backslash L(v)$;
2. $L(v_1) \backslash [L(v) \cup L(x)]$;
3. $[L(x) \cap L(v_2)] \backslash L(y)$;
4. $L(v_1) \backslash [L(v) \cup L(v_2)]$

Proof.
Suppose all of these sets are empty. We have $|L(v_1) \backslash L(v)| \geq 1$, therefore, let $c \in L(v_1) \backslash L(v)$,

By 4), $c \in L(v_2)$,

By 2), $c \in L(x)$,

By 3), $c \in L(y)$,

By 1), $c \in L(v)$,

and we have a contradiction with $c \notin L(v)$. $\square$

Let v_3 be the neighbour of v_2 on the boundary that is not v_1 and let v_4 be the neighbour of v_3 on the boundary that is not v_2. From here we consider two cases, whether $y = v_3$ or not.

Subcase 24.6.4.2.2.1 $y \neq v_3$.

From Claim 24.6.4.2.2, we have four cases.

Subcase 24.6.4.2.2.1.1 There exists $c \in [L(v_1) \cap L(y)] \backslash L(v)$.

Colour and delete v_1, v_2 and y so that v_1 and y get coloured with c. Following this delete v, adjust the lists of the vertices in $N_G(\{v_1, v_2, y\})$ accordingly, and call the resulting graph H. Let $T' = (T \cup N_G(v_1, v_2, y\})) \backslash \{v\}$. By Claims 12 and 11, and the existence of C, we have that $N_G(\{v_1, v_2, y\})$ is an independence set.

If x is adjacent to any $S \cup T$-vertices, then would form a path that induces a good triple, and a contradiction with Claim 19. By Claim 22, $N_G(\{v_2, y\})$ contains no vertex adjacent to an $S \cup T$-vertex other than v_3. If $v_4 \in S$, then reduce the list of v_3 to be a proper 1-list, remove it from T' and place it in S. By the existence of C, there are no odd cycles in $H[N_H(S) \cup T']$ in either case. Similarly, the existence of C implies v_3v_4 is not in a 4-cycle. Since $|S| \leq 5$, we also have that there is no 3-list vertex adjacent to three S-vertices.

Therefore, there is an L-colouring of H by minimality of G. Place the coloured and uncoloured vertices back into H. Since $c(v_1) \notin L(v)$, we colour v, and this yields an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2.1.2 There exists $c \in L(v_1) \backslash [L(v) \cup L(x)]$.

Colour and delete v_1 and v_2 so that v_1 gets coloured with c. Following this delete v and adjust the lists of the vertices accordingly, and call the resulting graph H. Let $T' = (T \cup N_G(\{v_1, v_2\})) \backslash \{v\}$.

By the analysis of Subcase 24.6.4.2.2.1.1, there are no S, T'-adjacencies, the only possible T'-adjacency is v_3v_4, v_3v_4 is not in a 4-cycle, there are no odd cycles in $H[N_H(S) \cup T']$, and there is no 3-list vertex adjacent to three S-vertices. Therefore, there is an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2.1.3 There exists $c \in [L(x) \cap L(v_2)] \backslash L(y)$.

There are two cases here depending on whether $v_4 \in S \cup T$ or not.

Subcase 24.6.4.2.2.1.3.1 $v_4 \in S \cup T$.

Colour and delete $v_3, v_2, v_1, v, u, z, x$ so as to not disturb the list of v_4 and so that the colour of x is c. Following this adjust the lists of the vertices accordingly, and call the resulting graph H. Let $T' = (T \cup N_G(\{v_3, v_2, v_1, v, u, z, x\})) \backslash \{v\}$. By Claims 12 and 11, and the existence of C, we have that $N_G(\{v_3, v_2, v_1, v, u, z, x\})$ is an independent set.

Claim 22 implies that no vertex in $N_G(v_2)$ is adjacent an $S \cup T$-vertex. We need not worry about the neighbours of v and v_1 since they were all deleted. The paths (v_2, y, x, z) and (u, z) show that no neighbour of z is adjacent to an $S \cup T$-vertex, else it would form a path that would induce a good triple, contradicting Claim 19. Similarly, by paths (u, z, x) and (v_2, y, x), no neighbour of x is adjacent to an $(S \cup T)$-vertex. By Claim 20, other than v_4, no vertex in $N_H(v_3)$ is an $(S \cup T)$-vertex. By Claim 19, no vertex other than u_1 in $N_H(\{u, v_2\})$ is adjacent an $S \cup T$-vertex.

Now we have that the only T'-adjacency is the edge u_1u_2 and it is clearly not in a 4-cycle by the existence of C in G. Lemma 10 tells us that the existence of C in G implies that $H[N_H(S) \cup T']$ does not contain an odd cycle. Therefore, by minimality of G, there is an L-colouring of H. Place the coloured vertices back into H to get an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2.1.3.2 $v_4 \notin S \cup T$.

Colour and delete v_2, v_1, v, u, z, x so that the colour of x is c. Following this adjust the list of the vertices in $N_G(\{v_2, v_1, v, u, z, x\})$ accordingly, and call the resulting graph H. Let $T' = (T \cup N_G(\{v_2, v_1, v, u, z, x\}) \backslash \{v\}$. The rest of the analysis of this case is contained in the analysis Subcase 24.6.4.2.2.1.3.1, and so we can find an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2.1.4 There exists $c \in L(v_1) \backslash [L(v) \cup L(v_2)]$.

Colour v_1 with c and delete it. Reduce the list of x and define $T' = T \cup \{x\}$. By the existence of the paths (u, z, x) and (v_2, y, x), x is not adjacent to an $S \cup T$-vertex, else there would be a path that induces a good triple, contradicting Claim 19. Since G satisfies the hypothesis with sets S and T, we have $G - v_1$ satisfies the hypothesis with sets S and T'. By minimality of G, there is an L-colouring $G - v_1$. Placing the coloured vertex v_1 back in the graph, yields an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2.2 $y = v_3$.

By Claim 22, other than u_1 and v no vertex in $N_G(\{u, z\})$ is adjacent to an $(S \cup T)$-vertex. By paths (u, z, x) and (y, x), no vertex in $N_G(x)$ is an $S \cup T$-vertex. By a similar argument, no vertex other than possibly v_4 and v_2 in $N_G(v_3)$ is an $(S \cup T)$-vertex and no vertex in $N_G(v_3)$ is adjacent to an $(S \cup T)$-vertex.

From Claim 24.6.4.2.2, we have four cases.

Subcase 24.6.4.2.2.2.1 There exists $c \in [L(v_1) \cap L(y)] \backslash L(v)$.

There are three options for the size of the lists of v_4

Subcase 24.6.4.2.2.2.1.1 $v_4 \in S$.

Colour and delete v_1 and v_2 such that v_1 is coloured with c, delete v and call the resulting graph H. Reduce the lists of v_3 and x to be proper 1-lists, and define $T' = T \backslash \{v, v_2\}$ and $S' = S \cup \{v_3, x\}$. Clearly $T' \subset T$ is an independent set, since the T-adjacency was deleted. We already know that there are no (S', T)-adjacencies, therefore there are no (S', T')-adjacencies. By the existence of C in G, we know that there is no vertex with list size three adjacent to three S'-vertices, and that there are no odd cyccles in $H[N_H(S') \cup T')]$.

There is an L-colouring H, by minimality of G, and place the coloured vertices along with v back in the graph. By our choice of $c(v_1)$, it is possible to colour v to find an L-colouring of G, a contradiction.

Subcase 24.6.4.2.2.2.1.2 $v_4 \in T$.

By Claim 12, we know that $v_4 \neq u_2$, and since the T-adjacency is already defined, we know that v_4 is not adjacent to u_2. Extend the labelling two steps further to v_5 and v_6. Colour and delete v_1, v_2, v_3 and v_4 so that v_1 and v_3 are coloured with c. Following this delete v, reduce the lists of the vertices of $N_G(\{v_1, v_2, v_3, v_4\})$ accordingly, and call the resulting graph H. Define $T' = (T \cup N_G(\{v_1, v_2, v_3, v_4\})) \backslash \{v, v_2, v_4\}$. By the existence of C, $N_G(\{v_1, v_2, v_3, v_4\})$ is an independent set.

By Claim 22, other than v_5, no vertex in $N_G(v_4)$ is adjacent to an $(S \cup T)$-vertex. If $v_6 \in S$, then we reduce the list of v_5 to be a proper 1-list and define $S' = S \cup \{v_5\}$ and remove v_5 form T'. In any case, we have that by the existence of C in G, there is no list size three vertex adjacent to three S or S' vertices, that $H[N_H(S) \cup T']$ and $H[N_H(S') \cup T')]$ both do not contain an odd cycle, and that the T'-adjacency v_5v_6 if it exists is not in a 4-cycle.

Therefore, H satisfies the hypothesis, and by minimality of G, there is an L-colouring of H. Placing the coloured vertices back in the graph yields an L-colouring of $G - v$. By our choice of $c(v_1)$, we can colour v, resulting in an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2.2.1.3 $v_4 \notin S \cup T$.

Colour and delete v_1, v_2, v_3 so that v_1 and v_3 get coloured with c. Following this, delete v, adjust the lists of $N_G(\{v_1, v_2, v_3\})$ accordingly, and call the resulting graph H. Define $T' = (T \cup N_G(v_4))\backslash\{v, v_2\}$. The analysis of this case is contained in the analysis of Subcase 24.6.4.2.2.2.1.2, therefore, there is an L-colouring of H by minimality of G.

Placing the coloured vertices back in the graph yields an L-colouring of $G-v$. By our choice of $c(v_1)$, we colour v, resulting in an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2.2.2 There exists $c \in L(v_1)\backslash[L(v) \cup L(x)]$.

Colour and delete v_1 and v_2 such that v_1 is coloured with c. Following this delete v, reduce the lists of vertices in $N_G(\{v_1, v_2\})$ and call the resulting graph H. Define $S' = S$ and $T' = (T \cup N_G(\{v_1, v_2\})\backslash\{v_2\}$. By Claim 22, no vertex in $N_G(\{v_1, v_2\}$ is adjacent to an $S \cup T$ vertex and therefore, there are no S', T'-adjacencies. Since $c \notin L(x)$, the only possible T'-adjacency is v_3v_4.

If v_4 is not an S-vertex, then we can L-colour H and we are done by placing the vertices back in the graph and colouring v.

Therefore, v_4 is an S-vertex. Remove v_3 from T' and add it to S'. We reduce the list of v_3 to be a proper 1-list. Yet again, we find that we can L-colour H, which results in an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2.2.3 There exists $c \in [L(x) \cap L(v_2)]\backslash L(y)$.

Colour x and v_2 such that they are both coloured with c and delete them. Following this, delete v_1 and v, reduce the list of $N_G(x)$ other than v_1, and call the resulting graph H. Define $T' = (T \cup N_G(x))\backslash\{v_2, v_1, v\}$. We have already checked for $(S \cup T, T')$-adjacencies. By the existence of C, the only T'-adjacency is the edge uz. By the existence of C, the T'-adjacency is not in a 4-cycle and $H[N_H(S) \cup T']$ contains no odd cycle.

Therefore, there is an L-colouring of H. By placing the vertices back in the graph, and by our choice of $c(v_2) = c(x_2) = c$ we can colour v and v_1, resulting in an L-colouring of G, and a contradiction.

Subcase 24.6.4.2.2.2.4 There exists $c \in L(v_1)\backslash[L(v) \cup L(v_2)]$.

Apply the same argument as in Subcase 24.6.4.2.2.1.4. There is an L-colouring of G, and a contradiction.

And so ends the cases of Claim 24.6.4.

Now that these claims are finished we may resume the last case of the main theorem, in particular we are in the case where $u_2, v_2 \in T$.

We colour and delete u_1, u, v so as to not disturb the list of u_2. Adjust the lists of the neighbours of u_1, u, v accordingly, and call this new graph H. Define $T' = T \cup N_G(\{u_1, u, v\})\backslash\{u_1, u, v\}$. Since the T-adjacency is not in a 4-cycle, Claim 24.6.3 shows $N_G(u_1, u, v)$ is an independent set.

By Claims 22 and 19, other than v_1 no vertex in $N_G(\{u_1, u, v\})$ is adjacent to an $S \cup T$-vertex. Therefore, since there is no (S, T)-adjacencies, there is also no (S, T')-adjacencies.

By Claim 24.6.2, no neighbour of u_1 is adjacent to v_1. Since the T-adjacency is not in a 4-cycle, no neighbour of u is adjacent to v_1. Therefore, the only T'-adjacency is the edge v_1v_2. By Claim 24.6.1, there is no odd cycle in $G[N_G(S) \cup T']$. By Claim 24.6.4, the T'-adjacency is not in a 4-cycle.

Therefore, H satisfies the hypothesis, and by minimality of G, there is an L-colouring of H. Placing the coloured vertices back in the graph yields an L-colouring of G, and our final contradiction to the proof. □

Claim 24 and Claim 23 are contradictory, therefore, G does not exist and Theorem 4 holds true. □

Corollary 25. Every planar graph without 3-cycles and with 4-cycles distance 8 apart is 3-choosable.

Proof. Let G be a planar graph without 3-cycles and with 4-cycles distance 8 apart and L a list assignment of the $V(G)$ such that $|L(v)| = 3$ for all

$v \in V(G)$. Note that $G \in \mathcal{G}$. Let $S = T = \varnothing$. Every internal vertex in G has a 3-list and 4.2-4.7 are vacuously true. Therefore, G satisfies the hypothesis of Theorem 4 and G has an L-colouring. $\square$

The goal of the thesis was to improve Dvořák's theorem. Dealing with the case of 4-cycles has provided strong evidence that this is possible. In the immediate future we will pursue the case of 3-cycle. This thesis also gives the reader a complicated proof to a generalization of Thomassen's theorem. By removing the tedious case work of the 4-cycles, this proof gives way to a tidy proof of Thomassen's theorem.

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