Strong Morita Equivalence and Imprimitivity Theorems

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Pure Mathematics

Waterloo, Ontario, Canada, 2016

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

The purpose of this thesis is to give an exposition of two topics, mostly following the books [6] and [9]. First, we wish to investigate crossed product C^*-algebras in its most general form. Crossed product C^*-algebras are C^*-algebras which encode information about the action of a locally compact Hausdorff group G as automorphisms on a C^*-algebra A. One of the prettiest example of such a dynamical system that I have observed in the wild arises in the gauge-invariant uniqueness theorem [5], which assigns to every C^*-algebra $C^*(E)$ associated with a graph E a *gauge action* of the unit circle $\mathbf{T}$ to automorphisms on $C^*(E)$. Group C^*-algebras also arise as a crossed product of a dynamical system. I found crossed products in its most general form very abstract and much of its constructions motivated by phenomena in a simpler case. Because of this, much of the initial portion of this exposition is dedicated to the action of a discrete group on a unital C^*-algebra, where most of the examples are given.

I must admit that I find calculations of crossed products when one has an indiscrete group G acting on our C^*-algebra daunting except under very simple cases. This leads to our second topic, on imprimitivity theorems of crossed product C^*-algebras. Imprimitivity theorems are machines that output (strong) Morita equivalences between crossed products. Morita equivalence is an invariant on C^*-algebras which preserve properties like the ideal structure and the associated K-groups. For example, no two commutative C^*-algebras are Morita equivalent, but $C(X) \otimes M_n$ is Morita equivalent to $C(X)$ whenever n is a positive integer and X is a compact Hausdorff space. Notice that Morita equivalence can be used to prove that a given C^*-algebra is simple.

All this leads to our concluding application: Takai duality. The set-up is as follows: we have an action α of an abelian group G on a C^*-algebra A. On the associated crossed product $A \rtimes_\alpha G$, there is a dual action $\widehat{\alpha}$ from the Pontryagin dual $\widehat{G}$. Takai duality states that the iterated crossed product $(A \rtimes_\alpha G) \rtimes \widehat{G}$ is isomorphic to $A \otimes \mathcal{K}(L^2(G))$ in a canonical way. This theorem is used to show for example that all graph C^*-algebras are nuclear or to establish theorems on the K-theory on crossed product C^*-algebras.

Acknowledgements

I would firstly like to thank my readers Nico Spronk and Matt Kennedy for taking time out of their busy lives to go through this thesis. I would also like to thank my advisor Ken Davidson for all the help he provided in the past year. I could not have asked for a beter advisor.

Table of Contents

List of Notations

1. $0 \notin \mathbf{N}$. We set $\mathbf{N}_0 := \mathbf{N} \cup \{0\}$.
2. $A \Subset B$: A is a compact subset of B.
3. $A \leq_{\text{closed}} B$: A is a closed subgroup of B.
4. $A \subset_p B$: A is a subset of B with property p. For example, $A \subset_{\text{closed}} B$ means that A is a closed subset of B.
5. $G \curvearrowright_{\text{free}} B$: G acts freely on X.
6. $x \leftarrow y$: replace the variable y by x. For example,
$$\int_{\mathbf{R}} \sin(2x)dx = \int_{\mathbf{R}} \sin(x)\frac{dx}{2}$$
where we make the change of variables $x/2 \leftarrow x$.
7. $A \odot B$: the algebraic tensor product of A and B. That is,
$$A \odot B := \operatorname{span}\{a \otimes b : a \in A, b \in B\} .$$
8. Rep_A : the class of non-degenerate representations of a C^*-algebra A.
9. $f_i \to_{i.l} f$: the net $(f_i)_{i\in I}$ inductive limit converges to the function f.
10. The notation $X\text{-}\bullet \uparrow_B^A$ is non-standard. [6] denotes this operation by $X\text{-}\mathrm{Ind}_B^A(\bullet)$ instead.

Chapter 1

Dynamical systems

Note: All topological groups and topological spaces will be assumed to be locally compact and Hausdorff unless stated otherwise.

1.1 Dynamical systems

An old result of groups is Cayley's theorem.

Theorem 1.1 (Cayley)**.** *If G is a group and H is a closed subgroup of G, then there is a group morphism*

$$G \xrightarrow{\lambda} \mathrm{Homeo}(G/H)$$

from G into the group of homeomorphisms of the topological space G/H given by left translation with kernel $H_G := \bigcap_{g\in G} gHg^{-1}$.

Cayley's theorem comes up time and time again in the study of groups because it tells us that much of our understanding of abstract groups can reduced to studying the *dynamics* of G as it acts on various spaces.[1] Before I define what I mean by dynamics, let us topologize the space of homeomorphisms on a topological space X.

[1] For instance, if we have a finite group G and a subgroup H with index $|G : H| = p$, where p is the smallest prime dividing G, then $|G : H_G| = p$, which tells us $H = H_G$ and is hence normal using Cayley's theorem.

Definition 1.2. Let X be a locally compact Hausdorff space. The space $\operatorname{Homeo} X$ is endowed with the topology specified by the following convergence condition: a net $(f_i)_{i\in I}$ converges to a point f in $\operatorname{Homeo} X$ if whenever $(x_i)_{i\in I}$ is a net in X which converges to a point $x \in X$, the nets $(f_i(x_i))_{i\in I}$ and $(f_i^{-1}(x_i))_{i\in I}$ converge to the points $f(x)$ and $f^{-1}(x)$ respectively.

Here is an easy exercise: check that λ in Cayley's theorem is continuous.

Definition 1.3. A dynamical system (or a transformation group) is a triple (X, G, σ) where G is a topological group, X is a topological space, and $\sigma : G \to \operatorname{Homeo} X$ is a continuous group morphism. The map σ is called a group action. It is standard to denote $s \cdot x := \sigma_s x$ for any $s \in G$ and $x \in X$. We denote by $G \curvearrowright_\sigma X$ for the fact that σ is a group action on X.

A fundamental result on commutative C^*-algebras is the theorem of Gelfand and Naimark. It tells us that there is a duality between commutative C^*-algebras and locally compact topological spaces given by the cofunctor $C_0 : X \mapsto C_0(X)$. Studying commutative C^*-algebras is equivalent to studying topological spaces! If we have a transformation group (X, G, σ), then how will the group action arise in $C_0(X)$?

Lemma 1.4. *There is a homeomorphism*

$$\Phi : \operatorname{Aut} C_0(X) \to \operatorname{Homeo} X$$

(where $\operatorname{Homeo} X$ *has the topology given by 1.2) given as follows: let us assume by Gelfand-Naimark that* $X = \widehat{C_0(X)}$*. Given* $\alpha \in \operatorname{Aut} C_0(X)$*, let* $\Phi(\alpha) : \pi \mapsto \pi\alpha$ *where* $\pi \in \widehat{C_0(X)}$*.*

Proof. For continuity of Φ, suppose that $(\alpha_i)_{i\in I}$ is a net in $\operatorname{Aut} C_0(X)$ which strong converges to a point $\alpha \in \operatorname{Aut} C_0(X)$. Given any net $(\varphi_i)_{i\in I}$ in $X = \widehat{C_0(X)}$ which converges to a point $\varphi \in \widehat{C_0(X)}$, the net $\varphi_i\alpha_i$ in X then converges to $\varphi\alpha$ pointwise. Similarly, since α_i^{-1} converges to α^{-1} pointwise, the net $\varphi_i\alpha_i^{-1}$ converges to the point $\varphi\alpha^{-1}$ pointwise. This proves continuity.

The inverse of Φ is given by the map

$$\Psi : \operatorname{Homeo} X \to \operatorname{Aut} C_0(X) : \Psi(f)(g) = gf$$

and this can be shown to be continuous by the same line of reasoning as well. ☕

Notice that under the isomorphism Φ, our action $G \curvearrowright_\sigma X$ is given by

$$\widehat{\sigma} : G \to \operatorname{Aut} C_0(X) : (\widehat{\sigma}_s f)(x) = f(\sigma_s^{-1} x) \ .$$

This action is strong continuous. Generalizing this gives rise to the notion of a C^*-dynamical system:

Definition 1.5. A (C^*-)dynamical system is a triple (A, G, α) where A is a C^*-algebra, G is a locally compact group, and $\alpha : G \to \operatorname{Aut} A$ is a strong continuous group morphism. We will call A a G-space.

For example, $(C_0(X), G, \widehat{\sigma})$ with $G \curvearrowright_\sigma X$ as before is a dynamical system. Before we get our hands dirty with some dynamical systems, we should introduce two important notions from dynamics.

Definition 1.6. Given a transformation group (G, X, σ) and a point $x \in X$, the orbit of x is defined as the set

$$\sigma_G x := \{\sigma_s x : s \in G\} \ .$$

The stabilizer subgroup of G at x is

$$G_x := \{s \in G : \sigma_s x = x\} \ .$$

As the name suggests, G_x is a subgroup of G. It is closed if X is Hausdorff.

Observe two basic facts about orbits and stabilizers:

Lemma 1.7. *If (G, X, σ) is a transformation group then*

I. The relation $\sim$ on X given by $x \sim y$ if $x \in \sigma_G y$ is an equivalence relation on X.

II. For any $x \in X$, there is a continuous bijection

$$G/G_x \to \sigma_G x : sG_x \mapsto \sigma_s x \ .$$

Proof. The proof is immediate. ☕

Remark 1.8. It would be nice if the bijection $G/G_x \to \sigma_G x$ is a homeomorphism, however this is not true in general (see Example 1.10). We will come back to a condition that will make this map a homeomorphism soon.

Let's see some examples!

Example 1.9. Take any group G. We saw that Cayley's theorem gave rise to a dynamical system. What about other actions? For instance, we have the action

$$G \xrightarrow{\tau} \operatorname{Homeo} G : \tau_s(g) = sgs^{-1} .$$

The orbits correspond to conjugacy classes of G. How about the stabilizers? If $x \in G$, then

$$G_x = \{s \in G : sxs^{-1} = x\} = C_G(x) .$$

This is the centralizer of x. By Lemma 1.7, we know that $|\sigma_G x| = |G : C_G(x)|$. We therefore get the identity

$$|G| = \sum_i |G : C_G(x_i)|$$

where the x_i are distinct representatives of each conjugacy class. This is the class equation!

Example 1.10. Let $X = \mathbf{T}$ be the unit circle as a subset of $\mathbf{C}$. Fix an irrational $\theta \in \mathbf{R}$. Let us define $\mathbf{Z} \curvearrowright_\sigma \mathbf{T}$ by $\sigma_1 : z \mapsto e^{2\pi i\theta} z$ (this is enough to tell us what σ_s is). Since $\mathbf{Z}$ is discrete, this action is automatically continuous. What do the orbits look like? If $z \in \mathbf{T}$ then the orbit $\sigma_{\mathbf{Z}} z$ is dense in $\mathbf{T}$. Let's show this using the following Lemma:

Lemma 1.11. *If H is a proper closed subgroup of $\mathbf{T}$ then H is finite.*

Proof. Here is a cute proof using some abelian Harmonic analysis (you can also do this by hard work if you wish). Since $\mathbf{T} = \widehat{\mathbf{Z}}$, we know

$$\mathbf{Z}/H^\perp \simeq \widehat{H}$$

where $H^\perp$ is the annihlator of H:

$$H^\perp := \{\gamma \in \widehat{\mathbf{T}} : \gamma|_H = 1\} .$$

Since $H \neq \mathbf{T}$, we know that $H^\perp$ cannot be 1. Therefore, $H^\perp = n\mathbf{Z}$ for some positive integer n. But this means that $\widehat{H}$ is finite. The dual of a finite group is finite. By Pontryagin duality, H is finite. ☕

In particular, if we consider the orbit $\sigma_{\mathbf{Z}}1$, it is a subgroup of $\mathbf{T}$. If it was not dense, then it would be finite. But then there would be some integer n for which

$$e^{2\pi i n\theta}1 = 1 \ .$$

So, θ would have to be rational, which is absurd! Since $\sigma_{\mathbf{Z}}z = (\sigma_Z 1)z$, $\sigma_{\mathbf{Z}}z$ is dense in $\mathbf{T}$ as well.

What about the orbits? Notice that for any $z \in \mathbf{T}$, $\mathbf{Z}_z = \{0\}$ since otherwise we can conclude θ is rational again.

Definition 1.12. If $G \curvearrowright_\sigma X$, then we say that the action σ is free if the stabilizer subgroups are always singletons.

Coming back to our Lemma 1.7, we have a continuous bijection

$$\mathbf{Z}(= \mathbf{Z}/\mathbf{Z}_1) \to \sigma_{\mathbf{Z}}1 \ .$$

But this map could not be a homeomorphism; the space $\mathbf{Z}$ is discrete while the space $\sigma_{\mathbf{Z}}1$ is dense in $\mathbf{T}$.

Example 1.13. Let's look at more groups. Take two groups H and K. If

$$K \xrightarrow{\sigma} \operatorname{Aut} H$$

is a (continuous) group action on H, then we can form the semidirect product $H \rtimes_\sigma K$. It is a topological group under the product topology. Here is a nice fact: if we think of K and H as subgroups of $H \rtimes_\sigma K$, then the action σ is given by an inner automorphism

$$\sigma_k : h \mapsto khk^{-1} \ .$$

What can we say about the orbits? We can at least say that since we assumed that the σ_k are group automorphisms, $\sigma_k 1 = 1$ must always hold: $\sigma_K 1 = \{1\}$. In particular, for all semi-direct products $H \rtimes_\sigma K$ with $K \neq 1$, the action is never free: $K_1 = \{k \in K : \sigma_k 1 = 1\} = K$.

Some examples of semi-direct products are

$$\begin{aligned} D_{2n} &= \mathbf{Z}_n \rtimes_{\sigma[n]} \mathbf{Z}_2 \\ D_\infty &= \mathbf{Z} \rtimes_{\sigma[\infty]} \mathbf{Z}_2 \\ \operatorname{Isom} \mathbf{R}^d &= \mathbf{R}^d \rtimes_\alpha O_d \end{aligned}$$

where O_d is the group of $d \times d$ orthogonal matrices, $\sigma[n]_1 : k \mapsto -k$ for $n \in \mathbf{N} \cup \{\infty\}$, and $\alpha_A : x \mapsto Ax$.

Remark 1.14. In all of the examples of dynamical systems that I've presented so far, the underlying topological space X has always been a group. One may wonder if it is *always* the case that X has a continuous group structure. The answer is negative due to the following theorem:

Theorem 1.15. *(See [8] 3.20) If G is a topological group then $\pi_1(G, e)$ is abelian.*

Example 1.16. Classical examples of dynamical systems arise as solutions to ordinary differential equations. Dare we try one? Here is one of my favourites: consider the system

$$\frac{d^2}{dt^2}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & \frac{-2k}{m} \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

in $\mathbf{R}^2$ where k and m are constants. The physical system is two blocks (whose positions at time t are given by $x(t)$ and $y(t)$) each of mass m in a well connected to the walls and each other by springs of coefficient k. Since our goal is not physics, let's set $k = m = 1$, and since we will only deal with complex systems, we may as well let $x(t)$ and $y(t)$ take values in $\mathbf{C}$. The resulting system is

$$\frac{d^2}{dt^2}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

The solution is an eigenvalue problem. We have the identity

$$\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

Set

$$W = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

and let $[u(t)\ v(t)]^T = W[x(t)\ y(t)]^T$. Under this change of coordinates we get the ODE

$$\frac{d^2}{dt^2}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} u(t) \\ v(t) \end{bmatrix}.$$

This differential equation has solutions

$$\begin{bmatrix} u(t) \\ v(t) \end{bmatrix} = \begin{bmatrix} a_{11}\cos(t) + a_{12}\sin(t) \\ a_{21}\cos(\sqrt{3}t) + a_{22}\sin(\sqrt{3}t) \end{bmatrix}$$

for constants $a_{ij} \in \mathbf{C}$. If we specify initial values $u(0) = u_0, v(0) = v_0, u'(0) = u'_0, v'(0) = v'_0$, one finds that $a_{11} = u_0$, $a_{12} = v_0$, $u'_0 = a_{21}$, and $v'_0 = a_{22}\sqrt{3}$. Changing our variables back to the original state, the solution to our ODE is

$$\begin{aligned} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} \cos(t) - \cos(\sqrt{3}t) & \cos(t) + \cos(\sqrt{3}t) \\ \cos(t) + \cos(\sqrt{3}t) & \cos(t) - \cos(\sqrt{3}t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \sin(t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) & \sin(t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) \\ \sin(t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) & \sin(t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) \end{bmatrix} \begin{bmatrix} x'_0 \\ y'_0 \end{bmatrix} \end{aligned}$$

given initial conditions $(x_0, y_0, x'_0, y'_0) \in \mathbf{C}^4$. Let us call

$$\begin{aligned} A(t) &:= \begin{bmatrix} \cos(t) - \cos(\sqrt{3}t) & \cos(t) + \cos(\sqrt{3}t) \\ \cos(t) + \cos(\sqrt{3}t) & \cos(t) - \cos(\sqrt{3}t) \end{bmatrix}, \\ B(t) &:= \begin{bmatrix} \sin(t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) & \sin(t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) \\ \sin(t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) & \sin(t) - \frac{1}{\sqrt{3}}\sin(\sqrt{3}t) \end{bmatrix}, \text{ and} \\ \sigma_t &:= \begin{bmatrix} A(t) & B(t) \\ A'(t) & B'(t) \end{bmatrix}. \end{aligned}$$

Since our initial conditions require four values, instead of considering the pair $(x(t), y(t))$, lets instead consider the quadruple $(x(t), y(t), x'(t), y'(t))$. In this case, the solution to our ODE is

$$[x(t)\ y(t)\ x'(t)\ y'(t)]^T = \sigma_t [x_0\ y_0\ x'_0\ y'_0]^T$$

where $\mathbf{R} \to U_4 : t \mapsto \sigma_t$ is a strong continuous group morphism. Our dynamical system in this case is $(\mathbf{C}^4, \mathbf{R}, \sigma)$. The space $\mathbf{C}^4$ is called phase space and the orbits correspond to the solution curves of our differential equation (indeed we get such a dynamical system for any second order linear differential equation). Note that we are not thinking of $\mathbf{C}^4$ as a C^*-algebra here but rather as a topological space.

Example 1.17. Here is a variation on Example 1.16. Suppose that we have a linear differential equation

$$\psi'(t) = H\psi(t)$$

for a matrix $H \in M_d$ and $\psi : \mathbf{R} \to \mathbf{C}^d$. As before, points on $\mathbf{C}^d$ correspond to different states of the system. The solution is given by a strong continuous unitary

$$\sigma : \mathbf{R} \to U_d\ .$$

In the case of finite quantum systems, self-adjoint matrices in M_d correspond to properties of the space. Taking a self-adjoint $A \in M_d$, and a state $\psi \in \mathbf{C}^d$ (of norm 1), the expected value of A in state ψ is given by

$$E[A|\psi] := \langle \psi, A\psi \rangle \ .$$

Evolving ψ under our differential equation, we have $\psi(t) = \sigma_t \psi$. Notice that

$$\begin{aligned} E[A|\psi(t)] &= \langle \psi(t), A\psi(t) \rangle = \langle \psi, \sigma_t^* A \sigma_t \psi \rangle \\ &= E[A|\psi] \end{aligned}$$

so $E[A|\psi]$ is t-invariant. On the other hand, since the system is determined by the self-adjoint matrices, instead of evolving the state over time, we can evolve our matrix A by the rule

$$\begin{aligned} \alpha &: \mathbf{R} \to \operatorname{Aut} M_d \\ \alpha_t &: A \mapsto \sigma_t^* A \sigma_t \ . \end{aligned}$$

This map α is a strong continuous group morphism and so $(M_d, \mathbf{R}, \alpha)$ is a C^*-dynamical system.

Example 1.18. This last example is for people with knowledge of graph algebras [5]. If E is a row-finite graph, then it has a gauge-action $\gamma : \mathbf{T} \to C^*(E)$ making $(C^*(E), \mathbf{T}, \gamma)$ into a C^*-dynamical system.

1.2 Orbits and Proper actions

Given an action $G \curvearrowright_\sigma X$, we can form the space of orbits:

$$G\backslash X := \{\sigma(G)x : x \in X\}$$

given the quotient topology. The canonical quotient map $X \xrightarrow{p} G\backslash X$ is called the orbit map. If the action is given on the right, we will write X/G instead. Here are some basic facts about the orbit space:

Lemma 1.19. *If $G \curvearrowright X$ then the orbit map p is continuous and open.*

Proof. Continuity is by definition. To see p is open, let $U \subset_{\text{open}} X$. We see

$$p^{-1}(p(U)) = \bigcup_{s \in G} s \cdot U$$

and such a set is open, whence $p(U)$ is open. ☕

Lemma 1.20. *The orbit space $G\backslash X$ is locally compact (but not necessarily Hausdorff).*

Proof. Is immediate by Lemma 1.19. ☕

Example 1.21. The orbit space $\mathbf{Z}\backslash\mathbf{T}$ of irrational rotation (as in Example 1.10) is not Hausdorff. In fact, since orbits are dense, it has the trivial topology.

Lemma 1.22. *Suppose that X is a locally compact G-space. If $T \subset G\backslash X$ is compact, then there is a $D \Subset X$ for which $p(D) \supset T$. If $G\backslash X$ is Hausdorff, then T is compact if and only if there is a $D \Subset X$ for which $p(D) = T$.*

Proof. Suppose that Gx is an arbitrary point in T. Then, there is some precompact neighbourhood V_x of x in X. Since p is continuous and open, $p(V_x)$ is a precompact neighbourhood of Gx. As T is compact and

$$\bigcup_{Gx \in T} p(V_x) \supset T\ ,$$

we can pick $x_1, \ldots, x_n$ so that $p(\bigcup_i V_{x_i}) \supset T$. Set $D := \overline{\bigcup_i V_{x_i}}$.

If $G\backslash X$ is also Hausdorff, then $p^{-1}(T)$ is closed. In this case, take $D = \overline{\bigcup_i V_{x_i}} \cap p^{-1}(T)$. The converse is clear. ☕

Lemma 1.23. *Suppose that (x_i) is a net in X for which $(p(x_i))$ converges to a point $p(x)$ in $G\backslash X$. There is a subnet $(x_{i_j})_j$ of (x_i) and elements $s_{i_j} \in G$ for which $s_{i_j} \cdot x_{i_j} \to_j x$.*

Proof. The proof of this Lemma follows once you draw a picture (take $X = [0,1] \times [0,1]$ and take p to be the projection onto the first component), so I leave it to the reader. ☕

Let's come back to remark 1.8. Here is a sufficient condition to get homeomorphism:

Definition 1.24.
1. We will call a continuous map $X \xrightarrow{f} Y$ between two locally compact spaces X and Y *proper* if $f^{-1}(K)$ is compact whenever $K \subset Y$ is compact.
2. We will call a locally compact G-space P proper if the map

$$G \times P \to P \times P : (s, x) \mapsto (s \cdot x, x)$$

is proper. We say $G \curvearrowright P$ properly in this case.

Example 1.25. Take $H \leq_{\text{closed}} G$. Then, $H \curvearrowright G$ by $h \cdot g = hg$. The map $H \times G \xrightarrow{\varphi} G \times G : (s, x) \mapsto (sx, x)$ is proper. Notice that we can extend φ to a map

$$\Phi : G \times G \to G \times G : (s, x) \mapsto (sx, x)$$

and Φ has a continuous inverse Ψ. If $K \Subset G \times G$, $\Psi(K)$ is compact. Since $\varphi^{-1}(K) = \Psi(K) \cap H \times G$, φ is proper. Notice as well that $H \curvearrowright G$ freely.

Proper maps will satisfy a number of nice properties:

Proposition 1.26. *Suppose that $G \curvearrowright P$ properly. Then, $G\backslash P$ is a locally compact Hausdorff space.*

Proof. That $G\backslash P$ is locally compact follows from the fact that P is locally compact. To see that $G\backslash P$ is Hausdorff, we prove the following Lemma:

Lemma 1.27. *If $G \curvearrowright P$, then $G \curvearrowright P$ properly if and only if, whenever we have a net $(x_i)_{i \in I}$ in P and a net $(s_i)_{i \in I}$ in G for which $x_i \to x$ and $s_i \cdot x_i \to y$, there is a subnet of s_i which converges.*

Proof of Lemma. Suppose first that $G \curvearrowright P$ properly. Let $\Phi : (s, x) \mapsto (s \cdot x, x)$. Suppose that K is a compact neighbourhood of x and y. Eventually, both x_i and $s_i \cdot x_i$ are in K. Since $\Phi^{-1}(K \times K)$ is compact, and $(s_i, x_i) \in \Phi^{-1}(K \times K)$, we can find a convergent subnet of s_i.

Conversely, to show that Φ is a compact map, suppose that $K \subset P \times P$ is compact. Suppose that $((s_i, x_i))$ is a net in $\Phi^{-1}(K)$, then, by taking subnets, we can assume $x_i \to x$ and $s_i \cdot x_i \to y$ for some $x, y \in K$. By assumption, we can take a further subnet so that s_i converges as well. ☕

With this Lemma, suppose that a net (w_i) in $G\backslash P$ converges to two points $G \cdot x$ and $G \cdot y$. We want to show $G \cdot x = G \cdot y$. Since $p : P \to G\backslash P$ is open, continuous, and surjective, we may take a subnet of w_i to find a net (x_i) in P for which $G \cdot x_i = w_i$ for all i and for which $x_i \to x$. Since $G \cdot x_i \to G \cdot y$, we can again lift and take a subnet to find s_i in G for which $s_i \cdot x_i \to y$. By our Lemma, we can take a third subnet so that s_i converges to some point s. This means that $s \cdot x = \lim_i s_i \cdot x_i = y$. That is, $G \cdot x = G \cdot y$. ☕

We now come to the result that we wish to establish:

Proposition 1.28. *If $G \curvearrowright P$ properly, then for each $x \in P$, there is a homeomorphism $G/G_x \to G \cdot x : sG_x \mapsto s \cdot x$.*

Proof. By Lemma 1.7, we have a continuous group morphism $G/G_x \to G \cdot x$. It remains to show that this map is open. Suppose that $N \subset_{\text{open}} G$. To see that $N \cdot x \subset_{\text{open}} G \cdot x$, suppose that $s_i \cdot x$ is a sequence not in $N \cdot x$ for which $s_i \cdot x \to n \cdot x$ for some $n \in N$ to derive a contradiction. By our Lemma, we may take a subnet so that s_i converges to some point $s \in G$. Since N is open, we know that $s \notin N$. On the other hand,

$$s \in nG_x \subset NG_x \subset_{\text{open}} G$$

This tells us s_i is eventually in N, which is a contradiction. ☕

Notice that if $G \curvearrowright_{\text{free}} P$ is a proper action, then the above Proposition tells us that G is homeomorphic to $G \cdot x$.

Chapter 2

Crossed products

In the theory of groups, there are few properties more influential on the structure of a group than commutativity. I posit that, much like the theory of abelian groups, having a discrete group in a dynamical system produces a theory of crossed products with a flavour of its own. To demonstrate this, the first section will explore discrete crossed products, as motivation for the general case. The reader who is comfortable with crossed products may wish to skip this chapter.

2.1 The discrete case

Note: All groups will be assumed to be discrete and all C^*-algebras will be assumed to be unital in this section.

2.1.1 Twisted polynomials and the algebraic structure of crossed products

Let us first explore a fundamental result of group representations. If G is a group, we have the algebra

$$\mathbf{C}[G] = \left\{ \sum_{g \in G} \alpha_g u_g : \alpha_g \in \mathbf{C}, \alpha_g = 0 \text{ for all but finitely many } g \right\}$$

with mutliplication $*$ specified by distributivity and the rule

$$u_g * u_h = u_{gh}$$

for all $g, h \in G$. The product on $\mathbf{C}[G]$ is called the convolution. Unlike other choices for the base field, in choosing $\mathbf{C}$, the algebra $\mathbf{C}[G]$ also has an involution specified by the rule

$$(\alpha_g u_g)^* = \overline{\alpha_g} u_{g^{-1}} \ .$$

With this map, $(\mathbf{C}[G], *, ^*)$ is a *-algebra.

Proposition 2.1. *There is a bijection between*

$$\textit{Representations } G \xrightarrow{\rho} U(\mathcal{H})$$

for $U(\mathcal{H})$ the group of unitaries on a Hilbert space $\mathcal{H}$ and *-morphisms

$$\mathbf{C}[G] \xrightarrow{\varphi} \mathcal{B}(\mathcal{H})$$

given by sending ρ to $1 \rtimes \rho$ where

$$1 \rtimes \rho \left(\sum_{g \in G} \alpha_g u_g \right) = \sum_{g \in G} \alpha_g \rho(g) \ .$$

(the '1' in $1 \rtimes \rho$ is in reference to the fact that it does nothing to the constant α_g.)

Proof. The place where one may get stuck is finding the inverse map. The inverse of this bijection is given as follows: if $\mathbf{C}[G] \xrightarrow{\varphi} \mathcal{B}(\mathcal{H})$ is a *-morphism, then $G \xrightarrow{\rho} U(\mathcal{H}) : g \mapsto \varphi(u_g)$ will be the associated representation. ☕

In other words, the result tells us that understanding representations of G can be reduced to understanding the *-morphisms of $\mathbf{C}[G]$.

Our goal is to understand group dynamics. Is there a similar correspondence for dynamical systems? What should we mean when we talk about a representation of (A, G, α)? A clue is provided in Example 1.17.

Definition 2.2. If $\mathscr{A} := (A, G, \alpha)$ is a C^*-dynamical system, then we say that a pair (π, u) is a covariant representation of $\mathscr{A}$ if

$$A \xrightarrow{\pi} \mathcal{B}(\mathcal{H})$$

is a *-morphism,

$$G \xrightarrow{u} U(\mathcal{H})$$

is a unitary representation, and the identity

$$u_s \pi(a) = \pi(\alpha_s(a)) u_s$$

holds for all $s \in G$ and $a \in A$.

Example 2.3. Covariant representations generalize the notion of a group representation: if $1 : G \to \operatorname{Aut} \mathbf{C}$ is the trivial morphism then $(\mathbf{C}, G, 1)$ is a dynamical system. Given a representation $G \xrightarrow{\rho} U(\mathcal{H})$, we can define $1 : \mathbf{C} \to \mathcal{B}(\mathcal{H}) : x \mapsto x 1_{\mathcal{H}}$ which make $(1, \rho)$ a covariant representation for $(\mathbf{C}, G, 1)$.

Example 2.4. Covariant representations also generalize the notion of a *-representation: if A is a C^*-algebra then $(A, \{1\}, 1)$ is a dynamical system, where $1 : \{1\} \to \operatorname{Aut}(A) : 1 \mapsto 1_A$. If π is a representation of A, then $1 : \{1\} \to U(\mathcal{H}_\pi) : 1 \mapsto 1_{\mathcal{H}_\pi}$ will make $(\pi, 1)$ into a covariant representation.

Example 2.5. For a group G, define the left-regular representation as

$$\lambda : G \to U(\ell^2(G)) : \lambda_s \delta_r = \delta_{sr} \ .$$

In this case, $(1, \lambda)$ is a covariant pair for $(\mathbf{C}, G, 1)$, where $1 : \mathbf{C} \to \mathcal{B}(\ell^2(G)) : 1 \mapsto 1_{\ell^2(G)}$. We therefore get a *-morphism

$$\mathbf{C}[G] \xrightarrow{1 \rtimes \lambda} \mathcal{B}(\ell^2(G)) \ .$$

This *-morphism must be injective: if

$$\sum_s a_s u_s \in \ker(1 \rtimes \lambda) \ ,$$

then

$$\left(\sum_s a_s \lambda_s \right) \delta_e = \sum_s a_s \delta_s = 0 \ .$$

Therefore, $a_s = 0$ for all s. Indeed, in our construction of discrete crossed products, we will show that $A \rtimes_\alpha^{\text{disc.}} G$ always embeds into some $\mathcal{B}(\mathcal{H})$.

Example 2.6. If we are given a compact Hausdorff space X and a homeomorphism σ on X, we can define the C^*-dynamical system $(C(X), \mathbf{Z}, \widehat{\sigma})$ as in Lemma 1.4. This lets us form the twisted polynomials $C(X) \rtimes_{\widehat{\sigma}}^{\text{disc.}} \mathbf{Z}$.

We now have the appropriate notion of a representation for a dynamical system. We now want to construct a *-algebra $A \rtimes_{\alpha}^{\text{disc.}} G$ for which we have a correspondence between *-representations of $A \rtimes_{\alpha}^{\text{disc.}} G$ and covariant representations of (A, G, α).[1] We'll define the algebra as

$$A \rtimes_{\alpha}^{\text{disc.}} G = \left\{ \sum_{g \in G} a_g u_g : a_g \in A, a_g = 0 \text{ for all but finitely many } g \right\}$$

as a $\mathbf{C}$-vector space. We define the convolution $*$ on $A \rtimes_{\alpha}^{\text{disc.}} G$ by distributivity and the rules

1. $u_g * u_h = u_{gh}$
2. $u_g a = \alpha_g(a) u_g$

for any $a \in A$ and $g, h \in G$. The involution * is given by the rule

$$(a_g u_g)^* = u_g^{-1} a_g^* = \alpha_g^{-1}(a_g)^* u_g^{-1} \ .$$

One checks that $A \rtimes_{\alpha}^{\text{disc.}} G$ is a *-algebra. We'll call $A \rtimes_{\alpha}^{\text{disc.}} G$ the twisted polynomials over (A, G, α).[2] Let's see some computations.

Example 2.7. In the dynamical system $(\mathbf{C}, G, 1)$ as in Example 2.3, we get $\mathbf{C} \rtimes_{1}^{\text{disc.}} G = \mathbf{C}[G]$.

Example 2.8. In the dynamical system $(A, 1, 1)$ as in Example 2.4, we get $A \rtimes_{1}^{\text{disc.}} 1 = A$.

Example 2.9. If A is a C^*-algebra and G is a group and we consider the trivial action

$$1 : G \to \operatorname{Aut} A : s \mapsto 1_A \ ,$$

then what is $A \rtimes_{1}^{\text{disc.}} G$? The rule $u_g * u_h = u_{gh}$ is still relevant. The identities $u_g a = a u_g$ and $(a u_g)^* = a^* u_g^{-1}$ now hold as well. These rules suggest that $A \rtimes_{1}^{\text{disc.}} G$ is a formal product of A and $\mathbf{C}[G]$.

---

[1] The notation $A \rtimes_{\alpha}^{\text{disc.}} G$ is my own. I think $A[G]$ or even $C_c(G, A)$ (which I will use when we deal with the general case) is more standard, but I wanted notation that alludes to crossed products.

[2] Again, this is my own terminology, and comes from Hilbert's skew polynomial rings. Maybe the biggest problem with this notation is that I would rather call $A \rtimes_{\alpha}^{\text{disc.}} G$ the discrete crossed product, but this would go against already standard terminology.

Proposition 2.10. *In the dynamical system* $(A, G, 1)$*, we have the* **-isomorphism*

$$A \rtimes_1^{\text{disc.}} G \simeq A \otimes_{\mathbf{C}} \mathbf{C}[G]$$

where * *on* $A \otimes \mathbf{C}[G]$ *is defined by* $^* : a \otimes u_g \mapsto a^* \otimes u_g^*$.

Proof. We use the universal property of $A \otimes_{\mathbf{C}} \mathbf{C}[G]$ to get the isomorphism. First let us define the bilinear map

$$\Phi : A \times \mathbf{C}[G] \to A \rtimes_1^{\text{disc.}} G : (a, u_g) \mapsto a u_g \ .$$

Given an abelian group M with bilinear map $A \times \mathbf{C}[G] \xrightarrow{b} M$, let us define

$$A \rtimes_1^{\text{disc.}} G \xrightarrow{\varphi} M : \sum_g a_g u_g \mapsto \sum_g b(a_g, u_g) \ .$$

It is clear that the map φ is additive and makes the diagram

$$\begin{tikzcd} A \times \mathbf{C}[G] \arrow[r, "b"] \arrow[d, "\Phi"] & M \\ A \rtimes_1^{\text{disc.}} G \arrow[ur, "\varphi"] & \end{tikzcd}$$

commute. Indeed, the commutativity of this diagram guarantees that φ is the unique such map. The map

$$A \rtimes_1^{\text{disc.}} G \to A \otimes_{\mathbf{C}} \mathbf{C}[G] : \sum_g a_g u_g \mapsto \sum_g a_g \otimes u_g$$

is therefore an isomorphism by the universal property of $\otimes_{\mathbf{C}}$. It is easy to check that multiplication and involution is preserved under this map. ☕

Example 2.11. When one has a tensor product $A \otimes_{\mathbf{C}} B$, we know that we can find a copy of A and B embedded inside by the maps

$$A \hookrightarrow A \otimes_{\mathbf{C}} B : a \mapsto a \otimes 1 \text{ and}$$
$$B \hookrightarrow A \otimes_{\mathbf{C}} B : b \mapsto 1 \otimes b \ .$$

Example 2.9 then tell us that there are natural embeddings of A and $\mathbf{C}[G]$ into $A \rtimes_1^{\text{disc.}} G$. Since there is a natural embedding of G into $\mathbf{C}[G]$ by $g \mapsto u_g$, we get a pair of embeddings

$$\pi : A \hookrightarrow A \rtimes_1^{\text{disc.}} G : a \mapsto au_1 \text{ and}$$
$$\rho : G \hookrightarrow U(A \rtimes_1^{\text{disc.}} G) : g \mapsto u_g\ .$$

Notice that (π, ρ) satisfies the covariance relation $\rho_s \pi(a) = \pi(a)\rho_s$ for $(A, G, 1)$. Is this phenomenon exhibited in general?

Proposition 2.12. *If (A, G, α) is a dynamical system, then there is a pair*

$$A \xrightarrow{i_A} A \rtimes_\alpha^{\text{disc.}} G : a \mapsto au_1 \ \textit{and}$$
$$G \xrightarrow{i_G} U(A \rtimes_\alpha^{\text{disc.}} G) : s \mapsto u_s$$

where i_A is a *-morphism and i_G is a group morphism for which we have the covariance relation

$$i_{G,s} i_A(a) = i_A(\alpha_s(a)) i_{G,s}$$

for all $s \in G$ and $a \in A$.

Proof. Is a straightforward calculation. ☕

So twisted polynomials over a dynamical system (A, G, α) with A unital and G discrete always contain a copy of A and G.

Example 2.13. Let (K, H, σ) be a dynamical system with K, H groups and $K \xrightarrow{\sigma} \operatorname{Aut} H$ a group action as in Example 1.13. On the one hand we have the *-algebra $\mathbf{C}[H \rtimes_\sigma K]$. On the other hand, if we loosen our notion of dynamical system to deal with *-algebras rather than C^*-algebras (ignoring the continuity conditions), we can form the dynamical system $(\mathbf{C}[H], K, \alpha)$ where

$$\alpha_k(u_h) = u_{\sigma_k h}\ .$$

We then have the algebra $\mathbf{C}[H] \rtimes_\alpha^{\text{disc.}} K$. What is the relation between $\mathbf{C}[H \rtimes_\sigma K]$ and $\mathbf{C}[H] \rtimes_\alpha^{\text{disc.}} K$?

Proposition 2.14. *If (K, H, σ) is a dynamical system as above, then we have the *-isomorphism*

$$\mathbf{C}[H \rtimes_\sigma K] \simeq \mathbf{C}[H] \rtimes_\alpha^{\text{disc.}} K\ .$$

Proof. In order to clean up notation, let's set

$$\begin{aligned} \mathbf{C}[H] &= \operatorname{span}_{\mathbf{C}}\{u_h : h \in H\} \text{ and} \\ \mathbf{C}[H] \rtimes^{\text{disc.}}_{\alpha} K &= \operatorname{span}_{\mathbf{C}}\{a_k v_k : k \in K, a_k \in \mathbf{C}[H]\} \ . \end{aligned}$$

The proof is a Currying argument.[3] Define

$$\begin{aligned} &\mathbf{C}[H \rtimes_\sigma K] \xrightarrow{\Phi} \mathbf{C}[H] \rtimes^{\text{disc.}}_{\alpha} K \\ &: u_{(h,k)} \mapsto u_h v_k \end{aligned}$$

extended by linearity. To see that Φ is a *-morphism, let $(h,k),(h',k') \in H \rtimes_\sigma K$. Then,

$$\begin{aligned} \Phi(u_{(h,k)} * u_{(h',k')}) &= \Phi(u_{(h\sigma_k h', kk')}) = u_{h\sigma_k h'} v_{kk'} \\ &= u_h u_{\sigma_k h'} v_k v_{k'} = u_h \alpha_k(u_{h'}) v_k v_{k'} \\ &= (u_h v_k) * (u_{h'} v_{k'}) = \Phi(u_h v_k) * \Phi(u_{h'} v_{k'}) \end{aligned}$$

and

$$\begin{aligned} \Phi(u^*_{(h,k)}) &= \Phi(u_{(\sigma_k^{-1} h^{-1}, k^{-1})}) = \alpha_k^{-1}(u_h^*) v_{k^{-1}} = v_k^* u_h^* = (u_h v_k)^* \\ &= \Phi(u_{(h,k)})^* \ . \end{aligned}$$

Since the basis elements map to basis elements, this *-morphism is a bijection. ☕

For example, we know now that $\mathbf{C}[\operatorname{Isom} \mathbf{R}^d] \simeq \mathbf{C}[\mathbf{R}^d] \rtimes^{\text{disc.}}_{\alpha} O_d$, and $\mathbf{C}[D_{2n}] \simeq \mathbf{C}[\mathbf{Z}_n] \rtimes^{\text{disc.}}_{\alpha} \mathbf{Z}_2$.

Before we go through more examples, let us establish the correspondence between covariant representations of a C^*-dynamical system (A, G, α) and *-representations of $A \rtimes^{\text{disc.}}_{\alpha} G$. Before we do this, let us define the notion of an irreducible covariant representation.

Definition 2.15. Let (π, U) be a covariant representation for (A, G, α). We say that (π, U) is irreducible if there are no (closed) subspaces $\mathcal{V} \subset \mathcal{H}_\pi$ for which $\mathcal{V}$ is both π and U-invariant.

[3]Currying (named after Haskell Curry) is the principle that functions $(A \times B) \to C$ can be written as a composition of functions $A \to (B \to C)$ and conversely.

Theorem 2.16 (Correspondence Principle)**.** *Let (A, G, α) be a C^*-dynamical system. There is a 1-1 correspondence between covariant representations of (A, G, α) and *-representations of $A \rtimes_\alpha^{\text{disc.}} G$ given by sending a covariant pair (π, U) to*

$$\begin{aligned} \pi \rtimes U &: A \rtimes_\alpha^{\text{disc.}} G \to \mathcal{B}(\mathcal{H}_\pi) \\ &: au_s \mapsto \pi(a)U_s \ . \end{aligned}$$

This correspondence preserves unitary equivalence and invariant subspaces. In particular, the correspondence preserves irreducibility. The representation $\pi \rtimes U$ is called the integrated form of (π, U).

Proof. Covariance guarantees that $\pi \rtimes U$ is a *-representation. To see such a correspondence is surjective, suppose that Φ is a *-representation on $A \rtimes_\alpha^{\text{disc.}} G$ and consider the covariant morphism (i_A, i_G) as in Proposition 2.12. Set $\pi = \Phi i_A$ and $U = \Phi i_G$. I claim that (π, U) is a covariant pair for (A, G, α). Let $s \in G$ and $a \in A$. A computation shows

$$\begin{aligned} U_s\pi(a) &= \Phi(i_{G,s})\Phi(i_A(a)) = \Phi(i_{G,s}i_A(a)) \\ &= \Phi(i_A(\alpha_s(a))i_{G,s}) = \pi(\alpha_s(a))U_s \ . \end{aligned}$$

For any $s \in G$ and $a \in A$,

$$\pi \rtimes U(au_s) = \Phi(i_A(a)i_{G,s}) = \Phi(au_s)$$

and so $\Phi = \pi \rtimes U$. For injectivity, suppose that (π, U) and (τ, V) are two covariant representations for which $\pi \rtimes U = \tau \rtimes V$. Since composing with i_A and i_G recovers the covariant pairs, we get the identity $(\pi, U) = (\tau, V)$. Therefore, we get the correspondence. It is immediate that this correspondence preserves unitary equivalence. It remains to check that the correspondence preserves invariant subspaces.

Let $\mathcal{V} \subset \mathcal{H}_\pi$ be an invariant subspace for (π, U). For any $au_s \in A \rtimes_\alpha^{\text{disc.}} G$ and for any $h \in \mathcal{V}$,

$$\pi \rtimes U(au_s)h = \pi(a)U_sh \in \mathcal{V} \ .$$

Conversely, if $\mathcal{V} \subset \mathcal{H}_\pi$ is $\pi \rtimes U$-invariant, then for any $a \in A$ and any $h \in \mathcal{V}$,

$$\pi \rtimes U(au_1)h = \pi(a)U_1h = \pi(a)h$$

and so $\pi(a)h \in \mathcal{V}$. Similarly, since $1 \in A$, for any $s \in G$ and for any $h \in \mathcal{V}$, $U_sh \in \mathcal{V}$. ☕

Remark 2.17. Notice that the correspondence principle can be established between *-morphisms $A \rtimes_\alpha^{\text{disc.}} G \to B$ to a unital *-algebra B and pairs (π, U) where $\pi : A \to B$ is a *-morphism, $U : G \to U(B)$ is a group morphism, and (π, U) satisfies the covariance relation for (A, G, α). We call pairs (π, U) which satisfy the above conditions a covariance morphism.

2.1.2 Discrete crossed products

In group representation theory, often one looks for an object with a nice analytic structure that contains $\mathbf{C}[G]$ as a dense subalgebra. Two possible candidates are $\ell^1(G)$ and the group von Neumann algebra $L(G)$. In our case, we will work instead with a smaller algebra $C^*(G)$.

Instead of constructing $C^*(G)$, let's instead use the fact that $\mathbf{C}[G] = \mathbf{C} \rtimes_1^{\text{disc.}} G$ as in Example 2.7 and construct an analytic extension $A \rtimes_\alpha G$ of the twisted polynomials $A \rtimes_\alpha^{\text{disc.}} G$. We will write $A \rtimes_\alpha G$ as the completion of $A \rtimes_\alpha^{\text{disc.}} G$ under a norm. What should we expect $A \rtimes_\alpha G$ to satisfy? We would at least want theorem 2.16 to hold on $A \rtimes_\alpha G$ and we would want $A \rtimes_\alpha G$ to be large enough to satisfy a universal property. To this end, define for any $f \in A \rtimes_\alpha^{\text{disc.}} G$

$$\begin{aligned} \|f\| &= \sup\{\|\Phi(f)\| : \Phi \text{ is a }^*\text{-representation of } A \rtimes_\alpha^{\text{disc.}} G\} \\ &= \sup\{\|\pi \rtimes U(f)\| : (\pi, U) \text{ is a covariant representation of } (A, G, \alpha)\} \,. \end{aligned}$$

Notice first that for any integrated form $\pi \rtimes U$ and for $f = \sum_g a_g u_g$,

$$\begin{aligned} \|\pi \rtimes U(f)\| &\leq \sum_g \|\pi \rtimes U(a_g u_g)\| \\ &\leq \sum_g \|\pi(a_g)\| \\ &\leq \sum_g \|a_g\| = \|f\|_1 \,. \end{aligned}$$

This tells us for any $f \in A \rtimes_\alpha^{\text{disc.}} G$, $\|f\| \leq \|f\|_1 < \infty$.

The fact that this is a C^*-semi-norm follows from the fact that we are working in C^*-norms on the representations. But we would like this to be a norm so that we have a legitimate copy of $A \rtimes_\alpha^{\text{disc.}} G$ in $A \rtimes_\alpha G$. It would certainly suffice to find a *-representation Φ for which $\Phi(f) \neq 0$ if $f \neq 0$. Let $\rho : A \hookrightarrow \mathcal{B}(\mathcal{H})$ be a faithful representation of A. Set

$$\ell^2(G, H) = \ell^2(G) \otimes \mathcal{H} = \overline{\text{span}}\{h_s \delta_s : s \in G, h_s \in \mathcal{H}\}$$

where $(\delta_s)_{s \in G}$ is the standard basis for $\ell^2(G, \mathcal{H})$. Define

$$\pi : A \to \mathcal{B}(\ell^2(G, \mathcal{H})) : \pi(a)(h\delta_r) = \rho(\alpha_r^{-1} a)(h)\delta_r$$

and define

$$U : G \to U(\ell^2(G, \mathcal{H})) : U_s(h\delta_r) = h\delta_{sr} \,.$$

Let's see that (π, U) is a covariant representation of (A, G, α). Let $a \in A$, $s, r \in G$, and $h \in \mathcal{H}$. A calculation:

$$\begin{aligned} U_s \pi(a) h \delta_r &= U_s(\rho(\alpha_r^{-1} a) h) \delta_r \\ &= (\rho(\alpha_r^{-1} a) h) \delta_{sr} \\ &= \pi(\alpha_s(a))(h \delta_{sr}) \\ &= \pi(\alpha_s(a)) U_s(h \delta_r) \ . \end{aligned}$$

As well, let us show $\pi \rtimes U(f) \neq 0$ if $f \neq 0$. Say $f = \sum_s a_s u_s$ where $a_s \neq 0$ for some s. Since ρ is faithful, we can choose some $h \in \mathcal{H}$ for which $\rho(a_s)h \neq 0$. We see

$$\begin{aligned} \pi \rtimes U(f) h \delta_{s^{-1}} = \sum_r \pi(a_r) U_r h \delta_{s^{-1}} &= \sum_r \pi(a_r)(h \delta_{rs^{-1}}) \\ &= \sum_r (\rho(\alpha_{sr^{-1}}^{-1} a_s) h) \delta_{rs^{-1}} \ . \end{aligned}$$

So since when $r = s$, we have $(\rho(\alpha_{ss^{-1}}^{-1} a_s) h) = \rho(a_s) h \neq 0$. Therefore, $\pi \rtimes U(f) \neq 0$. We therefore can conclude that the norm on $A \rtimes_\alpha^{\text{disc.}} G$ is a C^*-norm and $A \rtimes_\alpha G := \overline{A \rtimes_\alpha^{\text{disc.}} G}^{\|\cdot\|}$ is a C^*-algebra.[4] We will set $C^*(G) := \mathbf{C} \rtimes_1 G$.

Remark 2.18. In the case when $A = C(X)$ for some compact Hausdorff space X and $G = \mathbf{Z}$ with the action α given by some homeomorphism σ on X, a representation

$$\rho : C(X) \to \mathbf{C} : f \mapsto f(x)$$

for a fixed $x \in X$ would induce the covariant pair

$$\begin{aligned} &\pi : C(X) \to \mathcal{B}(\ell^2(\mathbf{Z})) : \pi(f)\delta_r = f(\sigma^r(x))\delta_r \text{ and} \\ &U : G \to U(\ell^2(G)) : U_s \delta_r = \delta_{s+r} \ . \end{aligned}$$

So in this case, π can be thought of as a diagonal $\mathbf{Z}$-indexed matrix

$$\pi(f) = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & f(\sigma^{-1}(x)) & 0 & 0 & \cdots \\ \cdots & 0 & f(\sigma^0(x)) & 0 & \cdots \\ \cdots & 0 & 0 & f(\sigma^1(x)) & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

[4] One quirk of discrete crossed products is that we can establish theorem 2.16 for the dense subalgebra $A \rtimes_\alpha^{\text{disc.}} G$. This tells us that *all* *-morphisms on $A \rtimes_\alpha^{\text{disc.}} G$ into a C^*-algebra extend to a *-morphism on $A \rtimes_\alpha G$. This will not be true in the general case, but we shall establish a criterion which is just as easy to check.

and U is the left regular representation, which is given by the backward shift

$$U_1 = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & \cdots \\ \cdots & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

All of the examples on $A \rtimes_\alpha^{\text{disc.}} G$ will extend to $A \rtimes_\alpha G$ once we replace $\rtimes^{\text{disc.}}$ with $\rtimes$ and $\mathbf{C}[G]$ with $C^*(G)$ but let's first see which examples extend immediately. Example 2.8 tell us that $A \rtimes_1 1 = A \rtimes_1^{\text{disc.}} 1 = A$. Proposition 2.12 work automatically on $A \rtimes_\alpha G$. For theorem 2.16, we should make sure that given a covariant pair (π, U) of (A, G, α), the integrated form

$$\pi \rtimes U : A \rtimes_\alpha^{\text{disc.}} G \to B(\mathcal{H}_\pi)$$

extends to a *-morphism on $A \rtimes_\alpha G$. That is to say, we need to guarantee that for any $f \in A \rtimes_\alpha^{\text{disc.}} G$, $\|f\| \geq \|\pi \rtimes U(f)\|$. But this is exactly how the norm is defined! Theorem 2.16 follows.

Technology needed to extend the other examples to $A \rtimes_\alpha G$ will be easier once we have a universal property for $A \rtimes_\alpha G$.

2.1.3 The universal property

The crossed product is universal with respect to covariant maps. That is:

Theorem 2.19. *Suppose that (B, π, U) is a triple where B is a C^*-algebra, $A \xrightarrow{\pi} B$ is a *-morphism and $G \xrightarrow{U} U(B)$ is a unitary morphism for which (π, U) is a covariant morphism. Then there is a unique *-morphism*

$$A \rtimes_\alpha G \xrightarrow{\Phi} B$$

for which the diagrams

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "i_A"'] & B \\ A \rtimes_\alpha G \arrow[ur, "\Phi"] & \end{tikzcd}$$

and

$$\begin{tikzcd} G \arrow[r, "U"] \arrow[d, "i_G"'] & B \\ A \rtimes_\alpha G \arrow[ur, "\Phi"] & \end{tikzcd}$$

commute.

Proof. Let (B, π, U) be as above. The map $\Phi = \pi \rtimes U$ will be one such map. Since we have a correspondence between *-morphisms and covariant morphisms, uniqueness follows. ☕

Let's come back to our examples. The extension of Proposition 2.10 is

Proposition 2.20. *In the dynamical system $(A, G, 1)$, we have the *-isomorphism*

$$A \rtimes_1 G \simeq A \otimes C^*(G)$$

where here and elsewhere, $\otimes := \otimes_{\max}$.

Proof. Let's show that $A \otimes C^*(G)$ satisfies the universal property for $A \rtimes_1 G$. Let $j_A : A \to A \otimes C^*(G)$ and $j_{C^*(G)} : C^*(G) \to A \otimes C^*(G)$ be the usual commuting pair for $A \otimes C^*(G)$. Let $k_G : G \to C^*(G) : s \mapsto u_s$ be the standard map. Set $U := j_{C^*(G)} k_G$ and set $\pi := j_A$. Since $(j_A, j_{C^*(G)})$ is a commuting pair in $A \otimes C^*(G)$, (π, U) is covariant: for any $a \in A$ and $s \in G$, $\pi(a)U_s = U_s \pi(a)$.

To see that $(A \otimes C^*(G), \pi, U)$ satisfies the universal property, let (B, τ, V) satisfy the conditions of theorem 2.43. Define

$$\omega : C^*(G) \to B$$

to be $\omega = 1 \rtimes U$. The pair (τ, ω) is a commuting pair for B: for any $a \in A$ and $bu_s \in C^*(G)$,

$$\tau(a)\omega(bu_s) = b\tau(a)V_s = bV_s\tau(a) = \omega(bu_s)\tau(a)\ .$$

Therefore, we get a *-morphism

$$\tau \otimes \omega : A \otimes C^*(G) \to B$$

for which $(\tau \otimes \omega)j_A = \tau$ and $(\tau \otimes \omega)j_{C^*(G)} = \omega$. Correspondence on covariant morphisms of $(\mathbf{C}, G, 1)$ and *-morphisms of $C^*(G)$ show us that $\tau \otimes \omega$ is the map Φ that we desire. ☕

The reader may exercise their calculational prowess and show that the appropriate analogue of Example 2.14 holds. We will prove the appropriate analogue in the general case later so we will not do it here.

Here is a new example:

Example 2.21. Suppose that G is an abelian group. Then, $C^*(G) \simeq C(\widehat{G})$ where $\widehat{G}$ is the Pontryagin dual of G.[5] To see this, consider the Fourier transform

$$\mathbf{C}[G] \xrightarrow{\mathscr{F}} C(\widehat{G}) : \mathscr{F}(u_s) : \gamma \mapsto \gamma(s) \ .$$

By definition of the topology on $\widehat{G}$, $\mathscr{F}$ is a well-defined linear map. Let's call $\widehat{s} := \mathscr{F}(u_s)$ in keeping with standard notation. Since for any $s \in G$, $\widehat{s^{-1}}(\gamma) = \gamma(s^{-1}) = \overline{\gamma(s)} = \widehat{s}^*$, $\mathscr{F}$ is *-preserving. It preserves multiplication since $\mathscr{F}(u_s * u_t) = \widehat{st} = \widehat{s}\widehat{t}$. There are two things to check to see that $\mathscr{F}$ will extend to an isomorphism on $C^*(G)$: the first is that $\mathscr{F}$ is an isometry and the second is that ran $\mathscr{F}$ is a dense *-subalgebra of $C(\widehat{G})$. The latter is just an application of the Stone-Weierstrass theorem. That ran $\mathscr{F}$ separates points follows from the fact that elements of $\widehat{G}$ which agree on all $\widehat{s}$ must be equal.

To see that ran $\mathscr{F}$ is an isometry, suppose that π is an irreducible representation of $C^*(G)$. We may write $\pi = 1 \rtimes U$ for an irreducible unitary representation U of G by correspondence. Since G is abelian, up-to-unitary equivalence, U is equivalent to a 1-dimensional representation. That is, up-to-unitary, $U = \gamma$ for some $\gamma \in \widehat{G}$. We see for any $f = \sum_s a_s u_s \in \mathbf{C}[G]$,

$$\begin{aligned}\|\pi(f)\| = \|1 \rtimes \gamma(f)\| &= \|\sum_s a_s \gamma(s)\| \\ &= \|\sum_s a_s \widehat{s}(\gamma)\| = \|\mathscr{F}(f)(\gamma)\| \ .\end{aligned}$$

A calculation gets us the identity

$$\|f\| = \sup_{\pi \in \widehat{C^*(G)}} \|\pi(f)\| = \sup_{\gamma \in \widehat{G}} \|\mathscr{F}(f)(\gamma)\| = \|\mathscr{F}(f)\|_\infty$$

whence we conclude that we have an isometry.

In particular, $C^*(\mathbf{Z}) \simeq C(\mathbf{T})$ by the map which sends the unitary $u_1 \in \mathbf{C}[\mathbf{Z}]$ to the identity map $z \mapsto z$ in $C(\mathbf{T})$.

[5] Notice that as G is discrete, $\widehat{G}$ is compact. In fact, this isomorphism proves that $\widehat{G}$ must be compact since $C(\widehat{G})$ is unital.

Example 2.22. Consider the dynamical system $(\mathbf{T}, \mathbf{Z}, \sigma)$ where $\sigma_1 : z \mapsto e^{2\pi i\theta}z$ for some irrational θ as in Example 1.10. Let's set α to be the induced action on $C(\mathbf{T})$. What is the structure of $C(\mathbf{T}) \rtimes_\alpha \mathbf{Z}$? Since by Stone-Weierstrass, $C(\mathbf{T})$ is generated by the unitary $u : z \mapsto z$, the space

$$C(\mathbf{T}) \rtimes_\alpha \mathbf{Z} = \overline{\operatorname{span}}\{f v_k : f \in C(\mathbf{T}), k \in \mathbf{Z}\}$$

is generated by the unitaries u and $v := v_1$. The covariance relation is translated into

$$vu = u_1 v = \alpha_1(u) v_1 = \rho^{-1} uv$$

where $\rho := e^{2\pi i\theta}$. This leads to

Proposition 2.23. *Suppose that U and V are two unitaries in a space $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$ which satisfy the relation*

$$UV = \rho VU \ .$$

There is an isomorphism $C^*(U, V) \simeq C(\mathbf{T}) \rtimes_\alpha \mathbf{Z}$ which maps u to U and v to V. Futhermore, the algebra $C(\mathbf{T}) \rtimes_\alpha \mathbf{Z}$ is simple.

Proof. Let us show that $C^*(U, V)$ satisfies the universal property of $C(\mathbf{T}) \rtimes_\alpha \mathbf{Z}$. First we want to find a copy of $C(\mathbf{T})$ in $C^*(U, V)$. By the continuous functional calculus, $C^*(U) \simeq C(\sigma(U))$ by an isomorphism which sends U to the identity map $z \mapsto z$. Since U is a unitary, $\sigma(U)$ is a closed subset of $\mathbf{T}$. Let us show that $\sigma(U)$ is dense in $\mathbf{T}$ to get the equality $\sigma(U) = \mathbf{T}$. Suppose that $\lambda \in \sigma(U)$. By definition, this means that $U - \lambda I$ is not invertible. Since V is a unitary, $V(U - \lambda I)$ is not invertible. But by the identity $VU = \rho UV$, by interchanging U and V, we get that $(U - \rho^{-1}\lambda I)\rho V$ is not invertible. Therefore, $\rho^{-1}\lambda \in \sigma(U)$. This argument tells us in particular that the orbit $\sigma(\mathbf{Z})\lambda$ is contained in $\sigma(U)$. But we know that $\sigma(\mathbf{Z})\lambda$ is dense in $\mathbf{T}$. This gets us the equality.

The same argument shows that a copy of $C(\mathbf{T})$ in $C^*(U, V)$ coming from $C^*(V)$ as well. Let

$$\pi : C(\mathbf{T}) \to C^*(U, V) : u \mapsto U$$

be the first map and let

$$V : \mathbf{Z} \to U(C^*(U, V))$$

be given by composing the maps

$$\mathbf{Z} \to C^*(\mathbf{Z}) \xrightarrow{\simeq} C(\mathbf{T}) \xrightarrow{\simeq} C^*(V) \xrightarrow{\subseteq} C^*(U,V)$$

where $C^*(\mathbf{Z}) \to C(\mathbf{T})$ is the map which sends $u_1 = v$ to $z \mapsto z$. By construction V is a group morphism and π is a *-morphism. Notice that $V_1 = V$ and that this tells us (π, V) is a covariant pair. This gets us a surjective *-morphism $\pi \rtimes V : C(\mathbf{T}) \rtimes_\alpha \mathbf{Z} \to C^*(U,V)$ which maps u to U and v to V. If we can show that $C(\mathbf{T}) \rtimes_\alpha \mathbf{Z}$ is simple, then we would be done. The proof of this fact will use vector-valued integration and the use of dual actions so we will come back to it once the necessary technology is developed.

Example 2.24. If G is a finite group then

$$C(G) \rtimes_{lt}^{\text{disc.}} G \simeq M_n \ .$$

Define

$$\begin{gathered} M : C(G) \to B(\ell^2(G)) : f \mapsto g \mapsto f \cdot g \text{ and} \\ \lambda : G \to U(\ell^2(G)) : (\lambda_s f)t = f(s^{-1}t) \ . \end{gathered}$$

The idea here is to use the fact that $B(\ell^2(G)) \simeq M_n$. Since (M, λ) is a covariant pair, we get the integrated form

$$M \rtimes \lambda : C(G) \rtimes_{\text{lt}}^{\text{disc.}} G \to B(\ell^2(G)) \ .$$

We just need to check that $M \rtimes \lambda$ is a bijection. Thinking of $C(G) \rtimes_{\text{lt}}^{\text{disc.}} G = C(G \times G)$ as a set, for any $f \in C(G \times G)$, $g \in C(G)$, and for any $t \in G$,

$$\begin{aligned} (M \rtimes \lambda(f)g)t &= \sum_{s \in G} M(f(s,\cdot))(\lambda_s g)t = \sum_{s \in G} f(s,t) g(s^{-1}t) \\ &= \left\langle \text{setting } r = s^{-1}t \right\rangle \\ &\quad \sum_{r \in G} f(tr^{-1}, t) g(r) \\ M \rtimes \lambda(f)g &= \begin{bmatrix} \\ \quad f(tr^{-1},t) \quad \\ \\ \end{bmatrix}_{r,t} \begin{bmatrix} g(s_1) \\ \vdots \\ g(s_n) \end{bmatrix} . \end{aligned}$$

where we choose some enumeration $G = \{s_1, \dots, s_n\}$. This tells us that under this enumeration,

$$M \rtimes \lambda(f) = \begin{bmatrix} & & \\ & f(tr^{-1}, t) & \\ & & \end{bmatrix}_{r,t} .$$

But $\langle f(tr^{-1}, t) : t, r \in G \rangle$ is an enumeration of all of the values of f. Therefore, we get the bijectivity.

Example 2.25. Example 2.24 has a far reaching extension: first note that we could build the crossed product $A \rtimes_\alpha G$ after dropping the condition that A needs to be unital.

corollary 2.26. *If G is a discrete group, then $C_0(G) \rtimes_{lt} G \simeq \mathcal{K}(\ell^2(G))$.*

Proof. With M, λ as in Example 2.24, we have

$$M \rtimes \lambda : C_0(G) \rtimes_{\mathrm{lt}}^{\text{disc.}} G \to B(\ell^2(G)) .$$

The above example shows that ran $M \rtimes \lambda$ is the collection of finite rank operators on $\ell^2(G)$. Indeed, doing the same computation shows

$$M \rtimes \lambda(f) : g \mapsto \begin{bmatrix} & & \\ & f(tr^{-1}, t) & \\ & & \end{bmatrix}_{t \in G, r \in \operatorname{supp} f} \begin{bmatrix} | \\ g(t) \\ | \end{bmatrix}_{t \in G}$$

and from this it follows that the kernel of $M \rtimes \lambda$ is trivial. We therefore have a surjective *-morphism

$$M \rtimes \lambda : C_0(G) \rtimes_{lt} G \to \mathcal{K}(\ell^2(G)) .$$

It remains to check that $M \rtimes \lambda$ is isometric. Let $\pi : C_0(G) \rtimes_{lt} G \hookrightarrow \mathcal{B}(\mathcal{H})$ be a representation. We know that $C_c(G) \rtimes_{\mathrm{lt}}^{\text{disc.}} G$ is dense in $C_0(G) \rtimes_{\mathrm{lt}}^{\text{disc.}} G$ and we can think of $C_c(G) \rtimes_{\mathrm{lt}}^{\text{disc.}} G = C_c(G \times G)$. All this is to say that $M \rtimes \lambda(C_c(G \times G))$ is dense in $\mathcal{K}(\ell^2(G))$. Let us define $\tau := M \rtimes \lambda|_{C_c(G \times G)}$. For any $A \in$ ran τ, there is a finite $G_0 \subset G$ for which, after assigning an ordering on G_0, we can say

$$A = \begin{bmatrix} & & \\ & A_{i,j} & \\ & & \end{bmatrix}_{i,j \in G_0} .$$

More precisely, let

$$\iota : M_{|G_0|} \hookrightarrow \text{ran } \tau : X \mapsto \begin{bmatrix} & & \\ & X_{i,j} & \\ & & \end{bmatrix}_{i,j \in G_0} .$$

This is a *-embedding. In particular, $\pi.\tau^{-1}\iota : M_{|G_0|} \hookrightarrow \mathcal{B}(\mathcal{H})$ is a *-embedding and therefore an isometry. We now have the following commutative diagram:

$$\begin{tikzcd}
 & & \text{ran } \tau \arrow[rr, "\tau^{-1}"] \arrow[dl, color=red] \arrow[ddrr, hook, "\varphi"] & & C_c(G \times G) \arrow[dl, color=red] \arrow[dd, "\pi"] \\
M_{|G_0|} \arrow[urr, hook, "\iota"] \arrow[drrrr, hook] & \mathcal{K}(\ell^2(G)) \arrow[drrr, hook, color=blue, "\Phi"] & & C_0(G) \rtimes_{lt} G \arrow[ll, color=blue, "M \rtimes \lambda"'] \arrow[dr, hook, color=blue, "\pi"] & \\
 & & & & \mathcal{B}(\mathcal{H})
\end{tikzcd}$$

The red arrows correspond to inclusions. The map Φ is obtained since φ is an isometry. Looking at the blue diagram, for any $f \in C_0(G) \rtimes_{lt} G$,

$$\|M \rtimes \lambda(f)\| = \|\Phi.M \rtimes \lambda(f)\| = \|\pi(f)\| = \|f\| .$$

This gets us the isomorphism. ☕

2.2 The general case

Note: groups are not assumed to be discrete and C^*-algebras are not assumed to be unital in this section.

2.2.1 Facts about vector-valued integration and multipliers

Vector-valued integration

We want to now consider the case when G has any locally compact (Hausdorff) topology and A does not necessarily have a unit. Let us fix a Haar measure $\mu = \mu_G$ on G. In this case, the twisted polynomials $A \rtimes_{\alpha}^{\text{disc.}} G$ should be replaced by the space $C_c(G, A)$ of continuous compactly supported functions on G with values in A.[6] The convolution is then given by

$$f * g : G \to A : s \mapsto \int_G f(t)\alpha_t(g(t^{-1}s))d\mu(s)$$

and the involution should be given by

$$f^* : s \mapsto \Delta_G(s^{-1})\alpha_s(f(s^{-1})^*)$$

where $\Delta_G : G \to (0, \infty)$ is the modular function on G. In the discrete case, we had no $\Delta_G(s^{-1})$ term in our involution, but this is because in the discrete case $\Delta_G = 1$. Before we can establish that these are indeed the right operations which get us correspondence, we should make sense of the integral

$$\int_G f(s)d\mu(s)$$

for $f \in C_c(G, A)$. What properties should we expect this integral to have? Here are some properties we expect of this integral (see [9] for details):

I. the map $\int_G \cdot \, d\mu(s) : C_c(G, A) \to A$ is linear.

II. We have the bound

$$\left\| \int_G f(s)d\mu(s) \right\| \leq \int_G \|f(s)\| d\mu(s)$$

for any $f \in C_c(G, A)$.

[6] Note that if G is discrete then $A \rtimes_{\alpha}^{\text{disc.}} G = C_c(G, A)$.

III. If $\psi : A \to \mathbf{C}$ is a linear functional then for any $f \in C_c(G, A)$,

$$\psi\left(\int_G f(s)d\mu(s)\right) = \int_G \psi(f(s))d\mu(s)$$

where the right side of the equation is the usual Haar-integral.

IV. If B is a C^*-algebra and $\pi : A \to B$ is a bounded linear map then

$$\pi\left[\int_G f(s)d\mu(s)\right] = \int_G \pi(f(s))d\mu(s)$$

for any $f \in C_c(G, A)$.

V. If $f(s) = \varphi(s)a$ for $\varphi \in C_c(G)$ and $a \in A$, then

$$\int_G f(s)d\mu(s) = \left[\int_G \varphi(s)d\mu(s)\right] a \ .$$

VI. Fubini's theorem should hold: if $F \in C_c(G \times H, A)$ then

$$s \mapsto \int_G F(s,t)d\mu_H(t) \text{ and } t \mapsto \int_G F(s,t)d\mu_G(s)$$

are in $C_c(G, A)$ and $C_c(H, A)$ respectively and we have the identity

$$\int_G \int_H F(s,t)d\mu_H(t)d\mu_G(s) = \int_H \int_G F(s,t)d\mu_G(s)d\mu_H(t) \ .$$

VII. This final property will be relevant later on, once the crossed product is defined: suppose that (A, H, α) is a dynamical system. If $F \in C_c(G \times H, A)$ and $f : G \to C_c(H, A)$ is defined by $f(s)(p) := F(s,p)$. Say that $A \rtimes_\alpha H$ is the crossed product extension of $C_c(H, A)$. We then have the identity

$$\left[\int_G^{A \rtimes_\alpha H} f(s)d\mu(s)\right](t) = \int_G^A F(s,t)d\mu(s)$$

for any $t \in H$ where the superscript $\int^X$ is to denote the fact that the integral takes values in X.

Indeed, such an integral exists, but we shall not prove it here. Let's see some consequences of the above properties. This tells us that

$$\left\| \int_G f(s) d\mu(s) \right\| \leq \int_G \|f(s)\| d\mu(s) \leq \|f\|_\infty \mu(\text{supp } f) \ .$$

Let's set $\|f\|_1 = \int_G \|f(s)\| d\mu(s)$.

If π is a *-representaion of A, then

$$\left\langle \left(\int_G f(s) d\mu(s) \right) h, k \right\rangle = \int_G \langle f(s)h, k \rangle \, d\mu(s)$$

holds for all $f \in C_c(G, A)$, $h, k \in \mathcal{H}_\pi$. This follows from the fact that the integral commutes with linear functionals. In particular, this shows us

$$\left[\int_G f(s) d\mu(s) \right]^* = \int_G f(s)^* d\mu(s) \ .$$

Our Fubini integral also tells us that since for any $f, g \in C_c(G, A)$,

$$(s, t) \mapsto f(t)\alpha_t(g(t^{-1}s))$$

is in $C_c(G \times G, A)$, the convolution $f * g$ is a member of $C_c(G, A)$. One can now check that the operations $*$ and * make $C_c(G, A)$ into a *-algebra.

The Multiplier algebra

We now have a *-algebra $C_c(G, A)$ and so we will see later that we can make $A \rtimes_\alpha G$ as before. As in Proposition 2.12, we would like to have a covariant pair (i_A, i_G) on (A, G, α) to $A \rtimes_\alpha G$. However, $A \rtimes_\alpha G$ is not necessarily unital. A unitary morphism into $A \rtimes_\alpha G$ does not make sense in this case. Our trick will be to take the multiplier algebra $M(A \rtimes_\alpha G)$ so that we get a covariant pair

$$\begin{aligned} i_A &: A \to M(A \rtimes_\alpha G) \\ i_G &: G \to UM(A \rtimes_\alpha G) \ . \end{aligned}$$

To refresh the reader and to set some notation, let us state some facts about the multiplier algebra (see [6] for details).

Recall that A can be thought of as a right Hilbert C^*-module over itself with action given by right multiplication: $a \cdot b := ab$ for any $a, b \in A$ and the A-valued inner product given by $\langle a, b \rangle_A := a^* b$. In order to shorten terminology, I will just denote by an A-module a Hilbert C^*-module over A.

The space $M(A)$ is the C^*-algebra $\mathcal{L}(A_A)$ of bounded adjointable operators over A_A. It is necessarily unital since $\mathcal{L}(X_A)$ is always untial whenever X_A is an A-module. A copy of A lives in $M(A)$ by left-multiplication:

$$L : A \hookrightarrow M(A) : L_a(b) = ab \ .$$

In fact, this copy of A is equal to the compact operators $\mathcal{K}(A_A)$:

$$L_A = \mathcal{K}(A_A) := \overline{\operatorname{span}}\{\Theta_{a,b} : a, b \in A\}$$

where $\Theta_{a,b} : A \to A : x \mapsto a \langle b, x \rangle$.[7] To see this, notice first that $\Theta_{a,b} = L_{ab^*}$ for all $a, b \in A$ so $L_A \supset \mathcal{K}(A_A)$. Conversely, if $a \in A$, then taking an approximate identity (e_i) in A, we see for any $b \in A$,

$$\|L_a(b) - \Theta_{a,e_i}(b)\| = \|(a - ae_i^*)b\| \leq \|a - ae_i^*\|\|b\| \ .$$

Therefore, $\|L_a - \Theta_{a,e_i}\| \leq \|a - ae_i^*\| \to_i 0$. This gets us equality. Since $\mathcal{K}(A)$ is an ideal in $M(A)$, A can be embedded into an ideal of $M(A)$.[8] It turns out that $M(A)$ is the largest unital C^*-algebra with A as an essential ideal. However, instead of using the universal property of $M(A)$ for our analysis, we will work instead with the concrete representation of $M(A)$ as adjointables.

Let's do two concrete examples of a multiplier algebra:

Example 2.27. If $\mathcal{H}$ is a Hilbert space (thought of as a $\mathbf{C}$-module), then $M(\mathcal{K}(\mathcal{H})_{\mathcal{K}(\mathcal{H})}) = \mathcal{B}(\mathcal{H})$.

Example 2.28. If X is a locally compact Hausdorff space then $M(C_0(X)) \simeq C_b(X)$ by the *-morphism

$$\Phi : C_b(X) \to M(C_0(X)) : \Phi(f) : g \mapsto fg \ .$$

It is easy to show that Φ is a *-morphism. That this is an embedding is immediate as well. To show surjectivity, suppose that T is a multiplier on $C_0(X)$. Let $f_T : X \to \mathbf{C}$ be

[7] Another notation for $\Theta_{a,b}$ could be $|a\rangle\langle b|$ or even just ab^*.

[8] This is because $T\Theta_{x,y} = \Theta_{Tx,y}$ for any adjointable T and any elements x, y.

defined as follows: given any point $x \in X$, let $g \in C_0(X)$ be such that $g(x) = 1$. Define $f_T(x) = (Tg)(x)$. To see that this is well-defined, it suffices to show that if $g(x) = 0$ for some $x \in X$ then $(Tg)(x) = 0$. This is because

$$|(Tg)(x)|^2 = \langle Tg, Tg \rangle (x) = \langle g, T^*Tg \rangle (x) = 0 \ .$$

That this is well-defined also shows us f_T is continuous. It is bounded because T is bounded. It remains to check that $\Phi(f_T) = T$. Given any $g \in C_0(X)$, for any $x \in X$ for which $g(x) \neq 1$,

$$((\Phi f_T)g)(x) = T(g/g)(x)g(x) = (Tg)(x) \ .$$

Whenever $g(x) = 0$, $(\Phi(f_T)g)(x) = 0$ so this shows that $\Phi(f_T)g = Tg$ everywhere.

The important thing about multiplier algebras is the following fact:

Lemma 2.29. *Suppose that B is a unital C^*-algebra and that there is a non-degenerate *-morphism $\alpha : A \to B$. There is then a unique extension of α to a *-morphism $\overline{\alpha} : M(A) \to B$.*

We will always denote by $\overline{\alpha}$ for the extension of α to its multiplier algebra.

When we establish correspondence for the crossed product $A \rtimes_\alpha G$, we will want to make sense of the integrated form $\pi \rtimes U$ whenever (π, U) is a covariant representation of (A, G, α). This will mean making sense of

$$\int_G \pi(f(s))U_s d\mu(s)$$

whenever $f \in C_c(G, A)$. However, the function $s \mapsto \pi(f)U_s$ is only assumed to be strong continuous and not norm continuous. To deal cases like this, we will have to establish a notion of integral for weaker topologies.

The multiplier algebra can be endowed with a topology weaker than the norm topology called the strict topology. We will define it as follows: a net $(T_i)_{i \in I}$ in $M(A)$ converges to a point $T \in M(A)$ in the strict topology if for any $a \in A$, $aT_i \to_i aT$ and $T_i a \to_i Ta$ in A. We write $M_s(A)$ for $M(A)$ endowed with the strict topology. The important fact about strict toplogy is:

Lemma 2.30. *If $u : G \to UM(A)$ is a group morphism then u is strict continuous if and only if u is strong continuous.*

Therefore, we need only establish an integral whenever we have a function $f \in C_c(G, M_s(A))$. There is always an integral

$$\int_G \cdot \, d\mu(s) : C_c(G, M_s(A)) \to M(A)$$

which satisfy the above properties of the integral. Therefore, as $s \mapsto \pi(f(s))U_s$ is a member of $C_c(G, M_s(A))$, we get an integral. So much for multipliers.

2.2.2 The general crossed product

Let (A, G, α) be a dynamical system. Our first goal is to construct $A \rtimes_\alpha G$. In our discussion of the multiplier algebra, we saw that once we have a covariant representation (π, U) of (A, G, α), for any $f \in C_c(G, A)$, since $s \mapsto \pi(f(s))U_s$ is a member of $C_c(G, \mathcal{B}_s(\mathcal{H}))$, we can define the integrated form

$$\pi \rtimes U(f) := \int_G \pi(f(s))U_s ds$$

where to declutter notation, I have taken ds to stand for $d\mu(s)$. Let's first check that $\pi \rtimes U$ is a *-morphism on $C_c(G, A)$. For any $f, g \in C_c(G, A)$,

$$\begin{aligned}
\pi \rtimes U(f * g) &= \int_G \pi((f * g)(s))U_s ds \\
&= \int_G \int_G \pi(f(t))\pi(\alpha_t(g(t^{-1}s)))U_s dt ds \\
&= \int_G \int_G \pi(f(t))U_t \pi(g(t^{-1}s))U_{t^{-1}s} dt ds \\
&= \int_G \int_G \pi(f(t))U_t \pi(g(r))U_r dr ds \\
&= \pi \rtimes U(f) \; \pi \rtimes U(g)
\end{aligned}$$

where in the second line we make use of covariance of (π, U) and in the penultimate line we make the change of variables $r = t^{-1}s$ for s. For involution, let $h, k \in \mathcal{H}_\pi$. A calculation shows

$$\begin{aligned}
\langle \pi \rtimes U(f^*)h, k\rangle &= \int_G \left\langle \pi(\alpha_s(f(s^{-1})^*))U_s h, k\right\rangle \Delta(s^{-1})ds \\
&= \int_G \left\langle U_s \pi(f(s^{-1})^*))h, k\right\rangle \Delta(s^{-1})ds \\
&= \int_G \left\langle h, \pi(f(s^{-1}))U_{s^{-1}}k\right\rangle \Delta(s^{-1})ds \ .
\end{aligned}$$

Under the change of variables $s \to s^{-1}$, we rid of the modular function to get

$$\begin{aligned}
\langle \pi \rtimes U(f^*)h, k\rangle &= \int_G \langle h, \pi(f(s))U_s k\rangle \, ds \\
&= \langle h, \pi \rtimes U(f)k\rangle \ .
\end{aligned}$$

By definition, the identity $\pi \rtimes U(f^*) = \pi \rtimes U(f)^*$ holds.

We want to define the norm of $C_c(G, A)$ to be

$$\|f\| := \sup\{\|\pi \rtimes U(f)\| : (\pi, U) \text{ is a covariant pair for } (A, G, \alpha)\}$$

whenever $f \in C_c(G, A)$. We should first make sure that this supremum exists in $\mathbf{R}$. Whenever (π, U) is a covariant pair for (A, G, α), as in the discrete case:

$$\begin{aligned}
\|\pi \rtimes U(f)\| &= \left\| \int_G \pi(f(s))U_s ds \right\| \\
&\leq \int_G \|\pi(f(s))\| ds \leq \|f\|_1 \ .
\end{aligned}$$

Since $\|f\|_1 \leq \mu(\text{supp } f)\|f\|_\infty$, we may conclude that the supremum is finite, so long as a covariant representation for (A, G, α) exists. Like the discrete case, modulo the existence of a covariant representation, we know that we get a C^*-semi-norm on $C_c(G, A)$ from this definition. Let us show that a covariant representation exists and that this is a norm by finding a covariant pair (π, U) for which $\pi \rtimes U(f) \neq 0$ if $f \neq 0$ just as in the discrete case. As before, we let $\rho : A \hookrightarrow \mathcal{B}(\mathcal{H})$ be a faithful representation. On $L^2(G, \mathcal{H}) = L^2(G) \otimes \mathcal{H}$, define

$$\pi : A \to \mathcal{B}(L^2(G, \mathcal{H})) : \pi(a)(f)(r) = \rho(\alpha_r^{-1} a)(f(r))$$

for $f \in C_c(G)$ and define

$$U : G \to U(L^2(G, \mathcal{H})) : U_s(f)(r) = f(s^{-1}r) \ .$$

For any non-zero $f \in C_c(G,A)$, we would like to show as in the discrete case that $\pi \rtimes U(f) \neq 0$. Let $s \in G$ be such that $f(s) \neq 0$. Let $g : G \to A : t \mapsto \alpha_t^{-1}(f(t))$. We know that $g \in C_c(G,A)$ and that $g(s) \neq 0$.[9] Since ρ is faithful, we can find $h, k \in \mathcal{H}$ for which $\langle \rho(g(s))h, k\rangle \neq 0$. Say $\epsilon > 0$ is such that $|\langle \rho(g(s))h, k\rangle| > 2\epsilon$. We would like to find a bump function $\varphi \in C_c^+(G)$ around r for which

$$|\langle \rho(g(t))h, k\rangle - \langle \rho(g(s))h, k\rangle| \leq \epsilon$$

and for which

$$\int_G \int_G \varphi(t^{-1}r)\overline{\varphi(r)} dt dr = 1 \ .$$

If we can do this then

$$\begin{aligned}
&\left| \left\langle \left(\int_G \rho(g(t)) U_t dt \right) \varphi h, \varphi k \right\rangle - \langle \rho(g(s))h, k\rangle \right| \\
=&\left| \int_G \int_G (\langle \rho(g(t))h, k\rangle - \langle \rho(g(s))h, k\rangle) \varphi(t^{-1}r)\overline{\varphi(r)} dt dr \right| \\
\leq& \epsilon
\end{aligned}$$

and we can conclude that $\int_G \pi(f(t))U_t dt \neq 0$ (lest $2\epsilon < \epsilon$). To find such a φ, let V be a neighbourhood of s for which $|\langle \rho(g(t))h, k\rangle - \langle \rho(g(s))h, k\rangle| < \epsilon$ for all $t \in V$. Let W be a neighbourhood of e for which $W^2 \subset s^{-1}V$. Let $\psi \in C_c^+(G)$ be such that $\psi(s) \neq 0$ and for which supp $\psi \subset sW$.[10] By construction, for any $t \in$ supp ψ,

$$|\langle \rho(g(t))h, k\rangle - \langle \rho(g(s))h, k\rangle| < \epsilon \ .$$

As well, if we define the function $\eta : G \times G \to \mathbf{C} : (t,r) \mapsto \varphi(t^{-1}r)\overline{\varphi(r)}$, $\eta \in C_c^+(G)$ and $\eta(e,s) \neq 0$. Therefore, $\int_{G\times G} \eta(t,r) dt dr \neq 0$. By renormalizing ψ to a new function φ, we get the identity

$$\int_G \int_G \varphi(t^{-1}s)\overline{\varphi(s)} dt ds = 1 \ .$$

We therefore get a C^*-norm on $C_c(G,A)$ and our extension $A \rtimes_\alpha G := \overline{C_c(G,A)}^{\|\cdot\|}$ is a C^*-algebra.

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[9] Continuity of g is a standard trick: assume that $\epsilon > 0$ and a net (t_i) which converges to a point t exists for which $\|\alpha_{t_i}^{-1}(f(t_i)) - \alpha_t^{-1}(f(t))\| \geq \epsilon$ cofinally many times. Take a subnet to assume this inequality holds for all i and derive a contradiction using strong continuity of α and continuity of f.

[10] This is possible due to Uryshon's Lemma.

When we have an integrated form

$$\pi \rtimes U : C_c(G, A) \to \mathcal{B}(\mathcal{H}_\pi) \ ,$$

for many reasons, including wanting an extension of this map to $M(A \rtimes_\alpha G)$, we will want $\pi \rtimes U$ to be non-degenerate.[11] Let us first show that it suffices for π to be non-degenerate to guarantee $\pi \rtimes U$ is non-degenerate.

Lemma 2.31. *If (π, U) is a covariant representation on (A, G, α) and π is a non-degenerate representation of A, the integrated form $\pi \rtimes U$ is non-degenerate.*

Proof. Let $\epsilon > 0$ and fix an approximate identity $(e_i)_{i \in I}$ on A. Let $h, k \in \mathcal{H}$ and, using strong continuity of U, suppose that V is a neighbourhood of e in G for which $|\langle U_s h, k\rangle - \langle h, k\rangle| < \epsilon$ for all $s \in G$. A calculation shows us that for any $\varphi \in C_c^+(G)$ with $\operatorname{supp} \varphi \subset V$ and $\int_G \varphi(s)ds = 1$, and for any $a \in A$,

$$\begin{aligned}
|\langle \pi \rtimes U(\varphi a)h, k\rangle - \langle h, k\rangle| &= \left| \int_G \langle \pi(a)U_s h, k\rangle \varphi(s)ds - \langle h, k\rangle \right| \\
&= \left| \int_G \langle \pi(a)U_s h - h, k\rangle \varphi(s)ds \right| \\
&\le \int_G (|\langle U_s h, \pi(a^*)k\rangle - \langle U_s h, k\rangle| \\
&\quad + |\langle U_s h, k\rangle - \langle h, k\rangle|)\varphi(s)ds \\
&\le \int_G (|\langle U_s h, \pi(a^*)k - k\rangle| + \epsilon)\varphi(s)ds \\
&\le \int_G (\|h\|\|\pi(a^*)k - k\| + \epsilon)\varphi(s)ds \ .
\end{aligned}$$

Choose an $i_0 \in I$ for which $\|\pi(e_i^*)k - k\| \le \epsilon$ for all $i \ge i_0$. We get the bound

$$|\langle \pi \rtimes U(\varphi e_i)h, k\rangle - \langle h, k\rangle| \le (\|h\| + 1)\epsilon$$

for all $i \ge i_0$. In particular, we get $\pi \rtimes U(\varphi e_i)h \to_i h$ whenever $h \in \mathcal{H}$ and we get non-degeneracy of $\pi \rtimes U$. ☕

Let us show that it suffices to only consider non-degenerate covariant representations of (A, G, α). That is, for any $f \in C_c(G, A)$, we have the identity

$$\|f\| = \sup_{(\pi, U)} \|\pi \rtimes U(f)\|$$

[11] That is, $(\operatorname{ran} \pi \rtimes U)\mathcal{H}$ spans a dense subspace of $\mathcal{H}$.

where (π, U) is taken over all covariant representations of (A, G, α). The proof is as follows: suppose that (π, U) is a covariant representation of (A, G, α). Define $\mathcal{S} \subset \mathcal{H}_\pi$ to be the subspace spanned by $\pi(A)\mathcal{H}_\pi$. Let $\tau := \pi|_\mathcal{S}$ and let $V := U|_\mathcal{S}$. By the covariance relation $U_s\pi(a) = \pi(\alpha_s(a))U_s$, $U(G)\mathcal{S} \subset \mathcal{S}$. Therefore, (τ, V) is a (necessarily non-degenerate) covariant representation of (A, G, α). For any $f \in C_c(G, A)$, and for any $h \in \mathcal{H}_\pi$, writing $h = x + y$ for $x \in \mathcal{S}$ and $y \in \mathcal{S}^\perp$, since

$$\pi \rtimes U(f)h = \int_G \pi(f(s))U_s h \, ds = \int_G \pi(f(s))U_s x \, ds = \tau \rtimes V(f)x \ ,$$

we conclude that $\|\pi \rtimes U(f)\| = \|\tau \rtimes V(f)\|$. This gets us our identity.

Let us now construct our canonical covariant pair (i_A, i_G) as in Proposition 2.12. Mimicking the discrete case, define

$$i_A : A \to M(A \rtimes_\alpha G)$$

as follows: let $i_{0,A}(a) : C_c(G, A) \to C_c(G, A) : f \mapsto af$ for each $a \in A$. This map is bounded in the universal norm since for any covariant pair (π, U) of (A, G, α),

$$\|\pi \rtimes U(af)\| = \left\| \int_G \pi(af(s))U_s ds \right\| = \|\pi(a)\pi \rtimes U(f)\| \leq \|a\|\|f\| \ .$$

We therefore get a linear map $i_A(a) : A \rtimes_\alpha G \to A \rtimes_\alpha G$ as the extension of $i_{0,A}(a)$ to $A \rtimes_\alpha G$. This map is adjointable since for any $f, g \in C_c(G, A)$,

$$\langle i_A(a)f, g \rangle_{A \rtimes_\alpha G} = (af)^* * g = f^* * (a^* g) = \langle f, i_A(a^*)g \rangle_{A \rtimes_\alpha G} \ .$$

This also shows us i_A preserves involutions. That i_A is mutliplicative and linear is immediate. It is easy to see that i_A is faithful as well.

For the map i_G, $i_{G,s}$ should act like multiplication on the left by an element u_s. To this end, let us define for $s \in G$,

$$i_{G,0,s} : C_c(G, A) \to C_c(G, A) : i_{G,0,s}(f)(r) = \alpha_s(f(s^{-1}r)) \ .$$

To see that this linear map is isometric, take any covariant pair (π, U) in (A, G, α). For any $f \in C_c(G, A)$,

$$\begin{aligned}
\pi \rtimes U(i_{G,0,s}(f)) &= \int_G \pi(i_{G,0,s}(f)(t))U_t dt \\
&= \int_G \pi(\alpha_s(f(s^{-1}t)))U_t dt \\
&= U_s \int_G \pi(f(s^{-1}t))U_{s^{-1}t} dt \\
&= Us \int_G \pi(f(t))U_t dt \\
&= U_s(\pi \rtimes U)(f)
\end{aligned}$$

where in the penultimate line we use the change of variables $st \leftarrow t$. Since U_s is a unitary, we get our isometry. We therefore get an extension of $i_{G,0,s}$ to a map $i_{G,s}$ on $A \rtimes_\alpha G$. To see that i_G is adjointable, let $s \in G$, and let $f, g \in C_c(G, A)$. Since we expect i_G to be a unitary representation, we should hope $i_{G,s}^* = i_{G,s^{-1}}$. A calculation shows

$$\begin{aligned}
\langle i_{G,s} f, g \rangle_{A \rtimes_\alpha G} &= [r \mapsto \alpha_s(f(s^{-1}r))]^* * g \\
&= [r \mapsto \Delta(r^{-1})\alpha_r \alpha_s(f(s^{-1}r^{-1}))^*] * g \\
&= r \mapsto \int_G \Delta(t^{-1})\alpha_{ts}(f(s^{-1}t^{-1})^*)\alpha_t(g(t^{-1}r))dt
\end{aligned}$$

while

$$\begin{aligned}
\langle f, i_{G,s^{-1}} g \rangle_{A \rtimes_\alpha G} &= f^* * [r \mapsto \alpha_s^{-1} g(sr)] \\
&= r \mapsto \int_G \Delta(t^{-1})\alpha_t(f(t^{-1})^*)\alpha_{ts^{-1}} g(st^{-1}r)dt\ .
\end{aligned}$$

If we make the change of variables $ts \leftarrow t$, then at the gain of a $\Delta(s)$ term, we get

$$\begin{aligned}
\langle f, i_{G,s^{-1}} g \rangle_{A \rtimes_\alpha G} &= r \mapsto \int_G \Delta((ts)^{-1})\alpha_{ts}(f(s^{-1}t^{-1})^*)\alpha_t g(t^{-1}r)\Delta(s)dt \\
&= r \mapsto \int_G \Delta(t^{-1})\alpha_{ts}(f(s^{-1}t^{-1})^*)\alpha_t g(t^{-1}r)dt \\
&= \langle i_{G,s} f, g \rangle_{A \rtimes_\alpha G}
\end{aligned}$$

as we expect. That $i_{G,s}$ is a group morphism is an easy calculation.

Let's check that i_G is strong continuous. Take any $f \in C_c(G, A)$, and any covariant pair (π, U), and take any $\epsilon > 0$. It suffices to show that $\|i_{G,s}f - f\|_1 < \epsilon$ for all s in a neighbourhood of e. For any $s, r \in G$,

$$\begin{aligned}
\|i_{G,s}(f)(r) - f(r)\| &= \|\alpha_s(f(s^{-1}r)) - f(r)\| \\
&\leq \|\alpha_s(f(s^{-1}r) - f(r))\| + \|\alpha_s(f(r)) - f(r)\| \\
&= \|f(s^{-1}r) - f(r)\| + \|\alpha_s(f(r)) - f(r)\|.
\end{aligned}$$

We know if we fix an r, by strong continuity of α, we can make the above term smaller than ϵ. We use compactness of supp f to get the result for any r: let K be a compact neighbourhood of e in G. Suppose in order to derive a contradiction that ϵ is such that for all neighbourhoods $V \subset K$ of e, there is some $s_V \in V$ and some $r_V \in K\text{supp } f$ for which

$$\|f(s_V^{-1}r_V) - f(r_V)\| \geq \epsilon \ .$$

(s_V) and (r_V) are nets ordered by reverse inclusion. Since we can take subnets to make s_V and r_V converge to points s and r respectively, we reach a contradiction on continuity of f. There is therefore a precompact neighbourhood W of e for which given any $s \in W$ and any $r \in G$,

$$\|f(s^{-1}r) - f(r)\| < \epsilon \ .$$

The same argument as the above shows that there is some precompact neighbourhood $V \subset W$ of e for which the identity

$$\|\alpha_s(f(r)) - f(r)\| < \epsilon$$

holds for all $s \in V$ and $r \in G$. Therefore, for any $s \in V$,

$$\begin{aligned}
\int_G \|i_{G,s}(f)(r) - f(r)\| dr &\leq \int_{V\text{supp } f} (\|f(s^{-1}r) - f(r)\| + \|\alpha_s(f(r)) - f(r)\|) dr \\
&\leq \int_{V\text{supp } f} 2\epsilon dr \\
&= 2\epsilon\mu(V\text{supp } f) \ .
\end{aligned}$$

Since $V\text{supp } f$ is precompact, $\mu(V\text{supp } f) < \infty$. Therefore, eventually $\|i_{G,s}(f) - f\|_1 < \epsilon$. Checking the covariance relation for the pair (i_A, i_G) is an easy calculation.

Approximation techniques for $A \rtimes_\alpha G$

Before we prove correspondence for $A \rtimes_\alpha G$, we would like some approximation techniques for $C_c(G, A)$. To this end, we have the following definition:

Definition 2.32. We say that a net $(f_i)_{i \in I}$ in $C_c(G, A)$ converges to a point $f \in C_c(G, A)$ in the inductive limit topology if f_i converges to f uniformly and there is a compact $K \subset G$ for which eventually supp $f_i \subset K$.

The first important fact about inductive limit convergence is

Lemma 2.33. *Suppose that (f_i) is a net in $C_c(G, A)$ that inductive limit converges to a point f. The net (f_i) then L^1 converges to f.*

Proof. Let K be compact for which eventually supp $f_i \subset K$. We may as well assume that supp $f \subset K$ by taking $K \cup \text{supp } f \leftarrow K$ if necessary. We then get the inequality:

$$\int_G \|f_i(s) - f(s)\| ds = \int_K \|f_i(s) - f(s)\| ds \le \|f_i - f\|_\infty \mu(K)$$

and we know the term on the right tends to 0 for large i. ☕

Since the L^1-norm dominates the universal norm, this tells us inductive limit convergence implies convergence in the universal norm. Here is an application:

Lemma 2.34. *If $A_0 \subset A$ is a dense subset of a C^*-algebra A and G is a locally compact group then the space*

$$C_c(G)A_0 := \text{span}\{\varphi a : \varphi \in C_c(G), a \in A_0\}$$

of elementary tensors is inductive limit dense in $C_c(G, A)$.

Proof. The proof is a partition of unity argument. We will assume $A_0 = A$ since it is easy to estimate an element of $C_c(G)A$ by an element of $C_c(G)A_0$. Let $f \in C_c(G, A)$. We showed that f is uniformly continuous in our argument of strong continuity of i_G. Let $\epsilon > 0$ and let $V_1^\epsilon, \dots, V_n^\epsilon$ be a precompact open cover of supp f for which given any $r, s \in V_i^\epsilon$,

$$\|f(r) - f(s)\| < \epsilon .$$

Let $\varphi_1, \dots, \varphi_n$ be a partition of unity for $(V_1^\epsilon, \dots, V_n^\epsilon)$ over supp f. That is, the φ_i are functions in $C_c^+(G)$ for which

1. ran $\varphi_i \subset [0,1]$,
2. $\sum_i \varphi_i = 1$ on supp f, and
3. supp $\varphi_i \subset V_i^\epsilon$ for all i.

Pick any $s_i \in V_i^\epsilon$ for each i. The function $g_\epsilon := \sum_i \varphi_i f(s_i)$ is in $C_c(G)A$. As well, for any $r \in G$,

$$\|g_\epsilon(r) - f(r)\| < \epsilon$$

by definition of V_i^ϵ. We therefore have a sequence $(g_{1/n})_{n\in\mathbf{N}}$ which uniformly converges to f. We can inductively assume that $V_i^{1/(n+1)}$ is a subset of some $V_j^{1/n}$ to make sure that we get a decreasing chain

$$\text{supp } g_1 \supset \text{supp } g_{1/2} \supset \text{supp } g_{1/3} \supset \dots$$

to get inductive limit convergence of $g_{1/n}$ to f. ☕

corollary 2.35. *If (A, G, α) is a dynamical system then the canonical map i_A is non-degenerate.*

Proof. We know how that $C_c(G)A$ is dense in $A \rtimes_\alpha G$. Fix an approximate identity $(e_i)_{i\in I}$ in A. For any $\varphi \in C_c(G)$ and $a \in A$, $\lim_i i_A(e_i)(\varphi a) = \lim_i \varphi(e_i a) = \varphi a$. Since $(i_A(e_i))_{i\in I}$ strong converges to $1 \in M(A \rtimes_\alpha G)$ in the dense subalgebra $C_c(G)A$, it strong converges on $A \rtimes_\alpha G$. ☕

For a non-degenerate covariant representation (π, U) of (A, G, α), the identities

$$\pi \rtimes U(i_A(a)f) = \pi(a)\pi \rtimes U(f) \text{ and}$$
$$\pi \rtimes U(i_{G,s}f) = U_s \pi \rtimes U(f)$$

derived in the construction of the covariant pair (i_A, i_G) extend by density to the following identites:

Lemma 2.36. *If (A, G, α) is a dynamical system and (π, U) is a non-degenerate covariant representation, then*

$$\overline{\pi \rtimes U} i_A(a) = \pi(a)\pi \rtimes U \text{ and}$$
$$\overline{\pi \rtimes U} i_{G,s} = U_s \pi \rtimes U$$

hold on $A \rtimes_\alpha G$.

The reader may wish to check what this says in the discrete case.

We know that $i_A(a)$ should be thought of as a and $i_{G,s}$ should be thought of as u_s as elements of $M(A \rtimes_\alpha G)$. We then expect that identities like

$$\int_G i_A(f(s)) i_{G,s} ds = f$$

should hold. The next Lemma codifies this intuition. Notice first that as i_G is a unitary morphism of G, we can form the integrated form

$$1 \rtimes i_G : C^*(G) \to M(A \rtimes_\alpha G) : f \in C_c(G) \mapsto \int_G f(s) i_{G,s} ds \ .$$

Lemma 2.37. *Suppose that (A, G, α) is a dynamical system. For any $a \in A$, $f \in C_c(G, A) \subset M(A \rtimes_\alpha G)$, and $\varphi \in C_c(G)$, the following identites hold:*

$$i_A(a)(1 \rtimes i_G)(\varphi) = \varphi a \ \textit{and}$$
$$\int_G i_A(f(s)) i_{G,s} ds = f \ .$$

In particular, for any $g \in C_c(G, A)$,

$$\left[\int_G i_A(f(s)) i_{G,s} ds \right] g = \int_G i_A(f(s)) i_{G,s} g \ ds = f * g \ .$$

Proof. Let (π, U) be a non-degenerate covariant representation of (A, G, α). By Lemma 2.36,

$$\overline{\pi \rtimes U} \int_G i_A(f(s)) i_{G,s} ds = \int_G \pi(f(s)) U_s ds = \pi \rtimes U(f) \ .$$

Since non-degenerate representations determine the norm on $A \rtimes_\alpha G$, this calculation shows us

$$\pi \rtimes U \left[\left(\int_G i_A(f(s)) i_{G,s} ds - f \right) g \right] = 0$$

for all $g \in A \rtimes_\alpha G$. Therefore,

$$\left(\int_G i_A(f(s)) i_{G,s} ds - f \right) g = 0$$

for all $g \in A \rtimes_\alpha G$ from which we get the second identity. The case when $f = \varphi a$ for $\varphi \in C_c(G)$ and $a \in A$ get us the first identity. ☕

The following result, the argument which [9] attributes to Raeburn tells us that inductive limit continuity of a *-morphism $C_c(G,A) \to B$ is enough to extend to a *-morphism on the crossed product.

Lemma 2.38. *If $C_c(G,A) \xrightarrow{\pi} \mathcal{B}(\mathcal{H})_{\mathrm{WOT}}$ is a *-morphism which is inductive limit continuous, then π is bounded by the universal norm: $\|\pi(f)\| \leq \|f\|$ for all $f \in C_c(G,A)$.*

Proof. The idea of this Lemma is as follows: We construct a Hilbert space $\mathcal{V}$, which is a completion of the space $C_c(G,A) \odot \mathcal{H}$ with an appropriate quotient for which we have a unitary $U : \mathcal{V} \to \mathcal{H}$. We construct a covariant pair (M,V) on $\mathcal{V}$ for which $M \rtimes V$ is equivalent to π by U. This tells us $\|f\| \geq \|M \rtimes V(f)\| = \|\pi(f)\|$.

We may as well assume π is non-degenerate by taking a subspace of $\mathcal{H}$ if necessary. First to construct $\mathcal{V}$, let us define the pre-inner product

$$\langle f \otimes h, g \otimes k\rangle = \langle \pi(g^* * f)h, k\rangle$$

on $C_c(G,A) \odot \mathcal{H}$ for $f, g \in C_c(G,A)$, $h, k \in \mathcal{H}$. This is a sesquilinear form and it is positive since

$$\begin{aligned}\left\langle \sum_i f_i \otimes h_i, \sum_j f_j \otimes h_j \right\rangle &= \sum_{i,j} \langle \pi(f_j^* * f_i)h_i, h_j\rangle \\ &= \sum_{i,j} \langle \pi(f_i)h_i, \pi(f_j)h_j\rangle \\ &= \left\langle \sum_i \pi(f_i)h_i, \sum_j \pi(f_j)h_j \right\rangle \geq 0 \ .\end{aligned}$$

Let $\mathcal{V}$ be the closure of $C_c(G,A) \odot \mathcal{H}$ by this inner product. The above calculatoin shows that the map

$$u_0 : C_c(G,A) \odot \mathcal{H} \to \mathcal{H} : f \otimes h \mapsto \pi(f)h$$

(which has dense range by non-degeneracy of π) extends to a unitary

$$u : \mathcal{V} \to \mathcal{H} \ .$$

The advantage of $\mathcal{V}$ is that it contains a copy of $C_c(G,A)$. From this, we can define the covariant pair (M,V) of (A,G,α) on $\mathcal{V}$ as follows: for M, let $a \in A$. Define

$$M(a) : f \otimes h \mapsto i_A(a)(f) \otimes h \ .$$

It is clear from the fact that i_A is a *-morphism that M is a *-moprhism. To see that $M(a)$ extends to an element of $\mathcal{B}(\mathcal{V})$, notice that as $\|a\|^2 1 - a^*a \geq 0$, we get the bound

$$\begin{aligned}
&\|a\|^2 \langle f \otimes h, f \otimes h \rangle - \langle M(a) f \otimes h, M(a) f \otimes h \rangle \\
&= \left\langle (\|a\|^2 - M(a^*)M(a)) f \otimes h, f \otimes h \right\rangle \\
&= \left\langle M(\|a\|^2 - a^*a) f \otimes h, f \otimes h \right\rangle \\
&= \left\langle M(\sqrt{\|a\|^2 - a^*a}) f \otimes h, M(\sqrt{\|a\|^2 - a^*a}) f \otimes h \right\rangle \geq 0 \ .
\end{aligned}$$

From this it follows that $\|M(a)\| \leq \|a\|$.

As you might expect, a unitary $V : G \to U(\mathcal{V})$ is defined by

$$V_s : f \otimes h \mapsto i_{G,s}(f) \otimes h \ .$$

A quick check shows that V is a unitary-valued group morphism. For $s \in G$, the calculation:

$$\|V_s(f \otimes h) - f \otimes h\|^2 = 2 \langle \pi(f^* * f)h, h \rangle - 2\mathrm{Re} \langle \pi(f^* * i_{G,s}(f))h, h \rangle$$

and the fact that π is WOT-continuous get us strong continuity.

It remains to show that $M \rtimes V$ interwines π. For $f, g \in C_c(G, A)$ and $h, k \in \mathcal{H}$, we see

$$\begin{aligned}
\langle U(M \rtimes V(f))(g \otimes h), k \rangle &= \langle M \rtimes V(f)(g \otimes h), U^* k \rangle \\
&= \int_G \langle M(f(s)) V_s(g \otimes h), U^* k \rangle \, ds \\
&= \int_G \langle f(s) i_{G,s}(g) \otimes h, U^* k \rangle \, ds \\
&= \int_G \langle U(f(s) i_{G,s}(g) \otimes h), k \rangle \, ds \\
&= \int_G \langle \pi(f(s) i_{G,s}(g))h, k \rangle \, ds \ .
\end{aligned}$$

On the other hand,

$$\langle \pi(f) U(g \otimes h), k \rangle = \langle \pi(f * g)h, k \rangle \ .$$

If we knew that π is a *-morphism on $A \rtimes_\alpha G$, then we would be done by an application of the previous Lemma. Since this is what we are trying to show, we will have to work harder.

The function

$$G \times G \to A : (s, r) \mapsto f(s) i_{G,s}(g)(r)$$

is continuous with support in supp $f \times$ (supp f)(supp g). Let $U \supset$ (supp f)(supp g) be a precompact open set. Since

$$s \mapsto f(s) i_{G,s}(g)$$

defines a function $G \to C_0(U, A)$, we can define the integral

$$\int_G^{C_0(U,A)} f(s) i_{G,s}(g) ds \ .$$

Since for $r \in G$, evaluation at r is a linear functional $C_0(U, A) \to A$,

$$\left[\int_G^{C_0(U,A)} f(s) i_{G,s}(g) ds \right] (r) = \int_G^A f(s) i_{G,s}(g)(r) ds = f * g(r) \ .$$

Since the map

$$L : C_0(U, A) \to \mathbf{C} : f \mapsto \langle \pi(f) h, k \rangle$$

is a linear functional, we get the identity

$$\begin{aligned} \langle \pi(f) U(g \otimes h) h, k \rangle &= L(f * g) \\ &= \int_G L(f(s) i_{G,s}(g)) ds \\ &= \langle U(M \rtimes V(f))(g \otimes h), k \rangle \ . \end{aligned}$$

This gets us the proof. ☕

In particular

corollary 2.39. *If (A, G, α) and (B, H, β) are dynamical systems and if $\pi : C_c(G, A) \to C_c(H, B)$ is an inductive limit continuous *-morphism, then π extends to a *-morphism $A \rtimes_\alpha G \xrightarrow{\pi} B \rtimes_\beta H$.*

The correspondence principle for $A \rtimes_\alpha G$

We are now ready to prove correspondence:

Theorem 2.40. *Let (A, G, α) be a dynamical system. There is a bijection between non-degenerate covariant pairs (π, U) on (A, G, α) and non-degenerate representations Φ on $A \rtimes_\alpha G$ given by sending (π, U) to $\pi \rtimes U$. This correspondence preserves unitary equivalence and invariant subspaces. Consequently irreducibility is preserved.*

Proof. Lemma 2.36 tell us that our correspondence is injective. To see that this correspondence is surjective, let $A \rtimes_\alpha G \xrightarrow{\Phi} \mathcal{B}(\mathcal{H})$ be a non-degenerate representation. Let

$$\pi := \overline{\Phi} i_A \text{ and } U := \overline{\Phi} i_G \ .$$

Since i_A is non-degenerate, π is non-degenerate. To see that (π, U) is a covariant pair: for any $a \in A$ and $s \in G$,

$$\pi(a)U_s = \overline{\Phi}(i_A(a)i_{G,s}) = \overline{\Phi}(i_{G,s}i_A(\alpha_s(a)) = U_s\pi(\alpha_s(a)) \ .$$

For any $f = \int_G i_A(f(s))i_{G,s}ds$ in $C_c(G, A)$,

$$\begin{aligned} \pi \rtimes U(f) &= \int_G \pi(f(s))U_s ds \\ &= \overline{\Phi} \int_G i_A(f(s))i_{G,s}ds \\ &= \Phi(f) \end{aligned}$$

whence we get surjectivity.

To see that the correspondence preserves unitary representations, suppose that (π, U) is equivalent (τ, V) by a unitary W. For any $f \in C_c(G, A)$,

$$\begin{aligned} W\pi \rtimes U(f)W^* &= W\left[\int_G \pi(f(s))U_s ds\right]W^* \\ &= \int_G \tau(f(s))V_s ds = \tau \rtimes V(f) \ . \end{aligned}$$

Conversely, if W is a unitary equivalence from $\pi \rtimes U$ to $\tau \rtimes V$, then W is a unitary equivalence from $\overline{\pi \rtimes U}$ to $\overline{\tau \rtimes V}$.[12] Therefore,

$$W\pi(a)W^* = W\overline{\pi \rtimes U}(i_A(a))W^* = \tau(a)$$

[12] That is, given $T \in M(A \rtimes_\alpha G)$, $W\overline{\pi \rtimes U}(T)W^* = \overline{\tau \rtimes V}(T)$. Use an approximate identity in $A \rtimes_\alpha G$ to get this identity.

and $WU_sW^* = V_s$ for any $a \in A$ and $s \in G$.

Finally, we need to check that the correspondence preserves invariant subspaces. Suppose first that $\mathcal{V} \subset \mathcal{H}$ is an invariant subspace for a covariant representation (π, U). Then, for any $h \in \mathcal{V}$ and $k \in \mathcal{V}^\perp$, given any $f \in C_c(G, A)$,

$$\langle \pi \rtimes U(f)h, k\rangle = \int_G \langle \pi(f(s))U_s h, k\rangle \, ds \ .$$

Since $\pi(f(s))U_s h \in \mathcal{V}$, the integrand, and hence the integral, is zero. It follows that $\pi \rtimes U(f)h \in \mathcal{V}^{\perp\perp} = \mathcal{V}$.

Conversely, if $\mathcal{V}$ is an invariant subspace for $\pi \rtimes U$, then by non-degeneracy, $\mathcal{V}$ is an invariant subspace for $\overline{\pi \rtimes U}$. Therefore, $\pi = \overline{\pi \rtimes U} i_A$ and $U = \overline{\pi \rtimes U} i_G$ are invariant under $\mathcal{V}$.

Remark 2.41. By Raeburn's result (Lemma 2.38), we could have constructed correspondence for $C_c(G, A)$ instead as we did for $A \rtimes_\alpha^{\text{disc.}} G$ so long as we chose inductive limit continuous representations on $C_c(G, A)$. Notice in the discrete case, this inductive limit continuous assumption is not needed! The reader can verify that inductive limit continuity is immediate for representations of $A \rtimes_\alpha^{\text{disc.}} G$. As a corollary, we get

corollary 2.42. *If (A, G, α) is a dynamical system and $f \in C_c(G, A)$,*

$$\|f\| = \sup_\Phi \|\Phi(f)\|$$

where Φ is taken over all inductive limit continuous representations of $C_c(G, A)$.

2.2.3 The universal property

Now that we have correspondence, we can prove the universal property for crossed products. After this, (finally!) we will get back to some more examples.

Theorem 2.43. *Suppose that (B, π, U) is a triple where B is a C^*-algebra, $A \xrightarrow{\pi} M(B)$ is a non-degenerate *-morphism and $G \xrightarrow{U} UM(B)$ is a strong continuous unitary morphism for which*

1. *(π, U) is a covariant morphism and*
2. *$\text{span}\{\pi(a)(1 \rtimes U)(\varphi) : a \in A, \varphi \in C_c(G)\} \subset B$.*

Then there is a unique non-degenerate **-morphism*

$$A \rtimes_\alpha G \xrightarrow{\Phi} B (\subset M(B))$$

for which the diagrams

$$\begin{tikzcd} A \arrow[r, "\pi"] \arrow[d, "i_A"'] & M(B) \\ M(A \rtimes_\alpha G) \arrow[ur, "\overline{\Phi}"] & \end{tikzcd}$$

and

$$\begin{tikzcd} G \arrow[r, "U"] \arrow[d, "i_G"'] & M(B) \\ M(A \rtimes_\alpha G) \arrow[ur, "\overline{\Phi}"] & \end{tikzcd}$$

commute.

Proof. The proof is similar to the discrete case. Set $\Phi = \pi \rtimes U$. For $f \in C_c(G, A)$, since $s \mapsto \pi(f(s))U_s$ takes values in B, the integrated form $\pi \rtimes U(f)$ belongs to B. The diagrams commute by Lemma 2.36. Uniqueness follows by correspondence. ☕

Example 2.44. If G is an abelian group then $C^*(G) \simeq C_0(\widehat{G})$. The proof is almost exactly as in the discrete case (see Example 2.44).

The next result is a generalization of the identity $A \rtimes_1 G \simeq A \otimes C^*(G)$.

Proposition 2.45. *Let* (A, G, α) *be a dynamical system and let* B *be a* C^**-algebra. We have the isomorphism*

$$(A \otimes B) \rtimes_{\alpha \otimes 1} G \simeq (A \rtimes_\alpha G) \otimes B$$

where $(\alpha \otimes 1)_s(a \otimes b) = \alpha_s(a) \otimes b$.

Before we begin the proof, one should check that given dynamical systems (A, G, α) and (B, G, β), that $\alpha \otimes \beta : G \to UM(A \otimes B)$ is strong continous. Notice that for $t \in A \otimes B$, $s, r \in G$, if $\sum a_i \otimes b_i$ is an approximation for t, then

$$\begin{aligned}
\|(\alpha \otimes \beta)_s(t) - (\alpha \otimes \beta)_r(t)\| \leq & \|(\alpha \otimes \beta)_s(t - \sum a_i \otimes b_i)\| \\
& + \|(\alpha \otimes \beta)_r(t - \sum a_i \otimes b_i)\| \\
& + \|(\alpha \otimes \beta)_s(\sum a_i \otimes b_i) - (\alpha \otimes \beta)_r(\sum a_i \otimes b_i)\| \\
\leq & 2\|t - \sum a_i \otimes b_i\| \\
& + \sum \|\alpha_s(a_i) \otimes \beta_s(b_i) - \alpha_r(a_i) \otimes \beta_r(b_i)\| \\
\leq & 2\|t - \sum a_i \otimes b_i\| \\
& + \sum \|(\alpha_s(a_i) - \alpha_r(a_i)) \otimes \beta_s(b_i)\| \\
& + \|\alpha_r(a_i) \otimes (\beta_s(b_i) - \beta_r(b_i))\| \ .
\end{aligned}$$

This estimate is enough to get continuity.

Proof. We want to show that $(A \rtimes_\alpha G) \otimes B$ satisfies the universal property for $(A \otimes B, G, \alpha \otimes 1)$. Lots of this proof is bookkeeping. Let $(j_{A \rtimes_\alpha G}, j_B)$ be the canonical commuting pair for $(A \rtimes_\alpha G) \otimes B$. Let (i_A, i_G) be the canonical covariant pair for $A \rtimes_\alpha G$. We want to construct a covariant pair $(k_{A \otimes B}, k_G)$ on $(A \otimes B, G, \alpha \otimes 1)$ into $M((A \rtimes_\alpha G) \otimes B)$ using the pairs $(j_{A \rtimes_\alpha G}, j_B)$ and (i_A, i_G). To this end, we will set

$$k_G = \overline{j_{A \rtimes_\alpha G}}.i_G$$

and we will set

$$k_{A \otimes B} = (\overline{j_{A \rtimes_\alpha G}}.i_A) \otimes j_B \ .$$

The map $k_{A \otimes B}$ is only well-defined if $(\overline{j_{A \rtimes_\alpha G}}.i_A, j_B)$ is a commuting pair for $A \otimes B$, but this follows from the fact that $(j_{A \rtimes_\alpha G}, j_B)$ is a commuting pair for $(A \rtimes_\alpha G) \otimes B$.

We should now verify that $(k_{A \otimes B}, k_G)$ is a covariant pair. Let $a \otimes b$ be an elementary tensor in $A \otimes B$ and let $s \in G$. Then,

$$\begin{aligned}
k_{A \otimes B}(a \otimes b) k_{G,s} &= \overline{j_{A \rtimes_\alpha G}}(i_A(a)) j_B(b) \overline{j_{A \rtimes_\alpha G}}(i_{G,s}) \\
&= \overline{j_{A \rtimes_\alpha G}}(i_A(a) i_{G,s}) j_B(b) \\
&= \overline{j_{A \rtimes_\alpha G}}(i_{G,s} i_A(\alpha_s(a))) j_B(b) \\
&= k_{G,s} k_{A \otimes B}(\alpha_s(a) \otimes b) \ .
\end{aligned}$$

It remains to check universality. Let (C, π, U) be as in the universal property for $(A \otimes B, G, \alpha \otimes 1)$. Let (l_A, l_B) be the canonical commuting pair for $A \otimes B$. Set $\pi_A = \overline{\pi}.l_A$ and $\pi_B = \overline{\pi}.l_B$. Since (π, U) is a covariant pair, (π_A, U) is a covariant pair for (A, G, α). Let $\Phi = (\pi_A \rtimes U) \otimes \pi_B$. Again, to see Φ is well-defined, we should make sure that $(\pi_A \rtimes U, \pi_B)$ is a commuting pair for $(A \rtimes_\alpha G) \otimes B$. Let $f \in C_c(G, A)$ and let $b \in B$. First notice that for any $s \in G$, $b \in B$, and any elementary tensor $x \otimes y \in A \otimes B$,

$$\begin{aligned} U_s \overline{\pi}(l_B(b)) \pi(x \otimes y) &= U_s \pi(l_B(b)(x \otimes y)) \\ &= U_s \pi(x \otimes by) \\ &= \pi(\alpha_s(x) \otimes by) U_s \\ &= \overline{\pi}(l_B(b)) \pi((\alpha \otimes 1)_s (x \otimes y)) U_s \ . \end{aligned}$$

By continuity, for any $t \in A \otimes B$, the identity

$$U_s \overline{\pi}(l_B(b)) \pi(t) = \overline{\pi}(l_B(b)) \pi((\alpha \otimes 1)_s t) U_s$$

holds. We can pick an approximate identity (e_i) on $A \otimes B$, and using non-degeneracy of π, get the identity

$$U_s \overline{\pi}(l_B(b)) = \overline{\pi}(l_B(b)) U_s \ .$$

From this calculation, we get

$$\begin{aligned} \pi_A \rtimes U(f) \pi_B(b) &= \left(\int_G \overline{\pi}.l_A(f(s)) U_s ds \right) \overline{\pi}.l_B(b) \\ &= \int_G \overline{\pi}.l_A(f(s)) U_s \overline{\pi}.l_B(b) ds \\ &= \int_G \overline{\pi}(l_A(f(s)) l_B(b)) U_s ds \\ &= \pi_B(b) \pi_A \rtimes U(f) \end{aligned}$$

and so Φ is well-defined. The fact that the associated diagrams commute is an easy calculation. We just need to make sure that ran $\Phi \subset C$. It suffices to show that ran $\pi_B \subset C$ and ran $(\pi_A \rtimes U) \subset C$. But this follows from the fact that elements of the form $\pi(a)(1 \rtimes U)(\varphi)$ belong in C. ☕

Generalizing what we have seen before, the following identity holds for semi-direct products:

Proposition 2.46. *If we have a semi-direct product $H \rtimes_\sigma K$ then*

$$C^*(H \rtimes_\sigma K) \simeq C^*(H) \rtimes_\alpha K$$

where $\alpha_k(f)(h) = \delta(k^{-1})f(\sigma_k h)$ (for δ defined in the proof) for any $f \in C_c(H)$, $k \in K$, and $h \in H$.

Proof. Before we begin, let us establish an integral on $H \rtimes_\sigma K$. To ease notation, let's take $H \leq H \rtimes_\sigma K$ and $K \leq H \rtimes_\sigma K$ so that $\sigma_k h = khk^{-1}$. A Haar integral can be given by

$$\int_{H \rtimes_\sigma K} f(h,k)d(h,k) := \int_K \int_H f(kh)dhdk \ .$$

As in the proof of the existence of a modular function Δ_G of a topological group G, there is a continuous group morphism $\delta : K \to (0, \infty)$ for which the identity

$$\delta(k) \int_H f(khk^{-1})dh = \int_H f(h)dh \ .$$

Let us show that $C^*(H \rtimes_\sigma K)$ satisfies the universal property for $(C^*(H), K, \alpha)$. Let $i_{H \rtimes_\sigma K}$ be the canonical unitary for $C^*(H \rtimes_\sigma K)$ and let $i_H : H \hookrightarrow H \rtimes_\sigma K$ and $i_K : K \hookrightarrow H \rtimes_\sigma K$ be the canonical embeddings. Define

$$\begin{aligned} j_{C^*(H)} &= 1 \rtimes (i_{H \rtimes_\sigma K}.i_H) \text{ and} \\ j_K &= i_{H \rtimes_\sigma K}.i_k \ . \end{aligned}$$

A calculation shows that $(j_{C^*(H)}, j_K)$ is a (non-degenerate) covariant pair for the system $(C^*(H), K, \alpha)$.

Let (B, π, U) be as in the universal property for $(C^*(H), K, \alpha)$. Since π is a *-morphism on $C^*(H)$, by correspondence, there is a unitary $\rho : H \to UM(B)$ for which $\pi = 1 \rtimes \rho$. Define the strong continuous *-morphism

$$\tau : H \rtimes_\sigma K \to UM(B) : (h,k) \mapsto \rho_h U_k \ .$$

This induces a map $\Phi := 1 \rtimes \tau : C^*(H \rtimes_\sigma K) \to M(B)$. It is an easy calculation to see that the associated diagrams commute. It remains to check that $\operatorname{ran} \Phi \subset B$. Let $f \in C_c(H \rtimes_\sigma K)$. Using the construction of our integral,

$$\begin{aligned} \int_{H \rtimes_\sigma K} f(h,k)\tau_{(h,k)}d(h,k) &= \int_H \int_K f(kh)\rho_h U_k dkdh \\ &= \int_K \pi(h \mapsto f(kh))U_k dk \in B \ . \end{aligned}$$

Example 2.47. Let E be a row-finite graph. Consider the system $(C^*(E), \mathbf{T}, \gamma)$ as in Example 1.18. In [5] chapter 6, it is shown using the Cuntz-Kreiger uniqueness theorem that

$$C^*(E) \rtimes_\gamma \mathbf{T} \simeq C^*(F)$$

where F is a graph with no cycles. In particular, $C^*(E) \rtimes_\gamma \mathbf{T}$ is an AF-algebra. Indeed, in the case when E is the graph with one vertex and exactly one edge, $C^*(F)$ is the compact operators $\mathcal{K}$ on a separable Hilbert space. Since $C^*(E) \simeq C(\mathbf{T})$ in this case, and γ becomes left translation in this case, we get the isomorphism

$$C(\mathbf{T}) \rtimes_{\mathrm{lt}} \mathbf{T} \simeq \mathcal{K} .$$

Compare this with Example 2.24 and Example 2.25.

Example 2.48. We will say that two dynamical systems (A, G, α) and (B, G, β) are equivariantly isomorphic if there is an isomorphism

$$\varphi : A \to B$$

for which the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_s"] & B \arrow[d, "\beta_s"] \\ A \arrow[r, "\varphi"] & B \end{tikzcd}$$

commutes for all $s \in G$. I claim:

Lemma 2.49. *If $(A, G, \alpha) \xrightarrow{\varphi} (B, G, \beta)$ is an equivariant isomorphism, then φ induces an isomorphism*

$$\varphi \rtimes 1 : A \rtimes_\alpha G \to B \rtimes_\beta G$$

where $\varphi \rtimes 1(f)(s) = \varphi(f(s))$ for all $f \in C_c(G, A)$.

Proof. This follows from the universal property for crossed products. ☕

Chapter 3

Morita equivalence

3.1 Imprimitivity Bimodules

We first start with the notion of an imprimitivity bimodule. Let A and B be C^*-algebras. We say that a $\mathbf{C}$-vector space X is an (A, B)-imprimitivity bimodule if X is a left (Hilbert) A-module and a right B-module for which

I. the module ${}_AX$ is full and X_B is full in the sense that we have the identities

$$A = \overline{\operatorname{span}}_A \langle X, X \rangle \text{ and } B = \overline{\operatorname{span}} \langle X, X \rangle_B \ ;$$

II. for any $a \in A$, $b \in B$, and $x, y \in X$,

$${}_A \langle xb, y \rangle = {}_A \langle x, yb^* \rangle \text{ and } \langle ax, y \rangle_B = \langle x, a^*y \rangle \ ; \text{ and}$$

III. for any $x, y, z \in X$, we have the identity

$${}_A \langle x, y \rangle z = x \langle y, z \rangle_B \ .$$

One thing that is immediate from our definition is that an imprimitivity bimodule ${}_AX_B$ is indeed a bimodule: for any $a \in A$, $x, y, z \in X$, and $b \in B$,

$$\begin{aligned} {}_A \langle (ax)b, y \rangle z = {}_A \langle ax, yb^* \rangle z = a({}_A \langle x, yb^* \rangle z) = a(x \langle yb^*, z \rangle_B) \\ = a(xb \langle y, z \rangle_B) = a({}_A\langle xb, y \rangle z) = {}_A \langle a(xb), y \rangle z \ . \end{aligned}$$

Therefore, $(ax)b = a(xb)$. Similarly, for any $\lambda \in \mathbf{C}$, $(\lambda a)xb = ax(\lambda b)$.

Example 3.1. We will assume that all Hilbert spaces are right $\mathbf{C}$-modules for this section. In this case, $\mathcal{H}$ is a $(\mathcal{K}(\mathcal{H}), \mathbf{C})$-imprimitivity bimodule with inner product ${}_{\mathcal{K}(\mathcal{H})}\langle x, y\rangle := xy^*$.

Example 3.2. Generalizing the previous example, let T be a locally compact Hausdorff space and let $\mathcal{H}_{\mathbf{C}}$ be a Hilbert space. The space $C_0(T, \mathcal{H})$ is a $(C_0(T, \mathcal{K}(\mathcal{H})), C_0(T))$-imprimitivity bimodule with actions and inner products given by:

$$f \cdot x : t \mapsto f(t)(x(t)) \ , \ x \cdot \varphi : t \mapsto x(t)\varphi(t)$$
$${}_A\langle x, y\rangle : t \mapsto x(t)y(t)^* \ , \ \langle x, y\rangle_B : t \mapsto \langle x(t), y(t)\rangle$$

where $f \in A := C_0(T, \mathcal{K}(\mathcal{H}))$, $x, y \in C_0(T, \mathcal{H})$ and $\varphi \in C_0(T)$.

Example 3.3. Any C^*-algebra A is an (A, A)-imprimitivity bimodule. This fact is encoded in the next Proposition.

Proposition 3.4. *Every full B-module X_B is a $(\mathcal{K}(X), B)$-imprimitivity bimodule with action of $\mathcal{K}(X) \curvearrowright X$ given by evaluation and inner product given by ${}_K\langle x, y\rangle := xy^*$.*

Conversely, if X is an (A, B)-imprimitivity bimodule then there is an isomorphism

$$\varphi : A \to \mathcal{K}(X) : {}_A\langle x, y\rangle \mapsto {}_K\langle x, y\rangle \ .$$

Proof. That X is a left $\mathcal{K}(X)$-module is mostly straightforward. The only tricky part to check is that xx^* is positive for any $x \in X$. The usual trick for inner products shows us that it suffices to show that $\langle xx^*y, y\rangle_B \geq 0$ for any $y \in X$. This is immediate. Our ${}_KX$ is full is immediate from the definition of $\mathcal{K}(X)$. To check the second property, we have

$$\begin{aligned} {}_K\langle xb, y\rangle : z \mapsto xb\langle y, z\rangle_B = x\langle yb^*, z\rangle_B = {}_K\langle x, yb^*\rangle z \text{ and} \\ \langle zw^*x, y\rangle_B = \langle z\langle w, x\rangle_B, y\rangle_B = \langle x, w\rangle_B\langle z, y\rangle_B \\ = \langle x, w\langle z, y\rangle_B\rangle_B = \langle x, wz^*y\rangle_B \ . \end{aligned}$$

The third property is by definition.

For the converse, if we can show φ is well-defined then this morphism will be isometric because there is a *-embedding $A \hookrightarrow \mathcal{L}(X_B)$ given by our action [1] and

$$\begin{aligned} \|{}_A\langle x, y\rangle\| &= \sup_{\|z\|\leq 1} \|{}_A\langle x, y\rangle z\| = \sup_{\|z\|\leq 1} \|x\langle y, z\rangle_B\| \\ &= \sup_{\|z\|\leq 1} \|{}_K\langle x, y\rangle z\| = \|{}_K\langle x, y\rangle\| \ . \end{aligned}$$

[1] That this map is injective takes some care to show: suppose that $x, y \in X$ and $a \in A$ is such that $az = 0$ for all $z \in X$. Then,

$$0 = {}_A\langle ax, y\rangle = a{}_A\langle x, y\rangle$$

Since $A \curvearrowright X$ is full, we conclude that $a = 0$.

To see that this φ is well-defined, suppose that $\sum_i {}_A\langle x_i, y_i\rangle = 0$. For any $z \in X$,

$$0 = \sum_i {}_A\langle x_i, y_i\rangle z = \sum_i x_i \langle y_i, z\rangle_B = \sum_i {}_K\langle x_i, y_i\rangle z \ .$$

Since this is true for arbitrary z, we conclude that $\sum_i {}_K\langle x_i, y_i\rangle = 0$. It is immediate from the definition of $\mathcal{K}(\mathcal{H})$ that this map is surjective. ☕

Example 3.5. Suppose that $p \in M(A)$ is a projection. Then, Ap is a full right pAp-module and a full left $\overline{ApA}$-module where Ap is a right Hilbert pAp-module by the usual right multiplication and inner product $\langle x, y\rangle_{pAp} = x^*y$ and the action of $\overline{ApA}$ on Ap is given by

$$r \cdot x = r^*x$$

for $r \in \overline{ApA}$ and $x \in Ap$ and the inner product is given by ${}_{\overline{ApA}}\langle x, y\rangle = xy^*$. Since $(pA^*)Ap$ is dense in Ap and $Ap(Ap)^*$ is dense in $\overline{ApA}$, we have fullness. The algebraic conditions for being an imprimitivity bimodule is a calculation.

Remark 3.6. We can replace our condition II in our definition of an (A, B) imprimitivity bimodule with the condition:

II'. If $a \in A$ and $b \in B$ and $x \in X$, then

$$\langle ax, ax\rangle_B \leq \|a\|^2 \langle x, x\rangle_B \ \text{ and } \ {}_A\langle xb, xb\rangle \leq \|b\|^2 {}_A\langle x, x\rangle \ .$$

We may as well define Morita equivalence now:

Definition 3.7. We will say that two C^*-algebras A and B are Morita equivalent (denoted $A \sim_M B$) if there is an imprimitivity bimodule ${}_AX_B$.

Notice that our examples show that if $A \simeq B$ then $A \sim_M B$, $\mathcal{K}(\mathcal{H}) \sim_M \mathbf{C}$, and $\overline{ApA} \sim_M pAp$. By flipping our actions, we can show that if $A \sim_M B$ then $B \sim_M A$. The harder part is transitivity. To this end, suppose that ${}_AX_B$ and ${}_BY_C$ are imprimitivity bimodules. We can define the algebraic B-tensor product

$$X \odot_B Y := X \odot Y / \operatorname{span}\{(xb) \otimes y - x \otimes (by) : b \in B, x \in X, y \in Y\}$$

which comes naturally with an A-action and a C-action. We would like to complete this vector space into an imprimitivity bimodule. To do this, we will want a notion of a preimprimitivity bimodule.

Definition 3.8. Suppose that A and B are C^*-algebras. We say that an (A, B)-bimodule X_0 is a preimprimitivity bimodule if

I. X_0 is a full pre-inner product left A-module and a full pre-inner product right B-module and

II. X_0 satisfies the algebraic conditions II and III for imprimitivity bimodules as before.

The natural thing to do, given a preimprimitivity bimodule ${}_AX_{0,B}$ is to complete it with a norm induced from the inner product. However, we have two inner products in our bimodule. This turns out not to be an issue.

Proposition 3.9. *If X_0 is an (A, B) preimprimitivity bimodule then for any $x \in X$, one has the identity*

$$\|_A \langle x, x\rangle \| = \| \langle x, x\rangle_B \| .$$

Proof. It suffices by symmetry to show $\|_A \langle x, x\rangle \| \leq \| \langle x, x\rangle_B \|$. A preliminary calculation shows

$${}_A \langle x, x\rangle^2 = {}_A \langle {}_A \langle x, x\rangle x, x\rangle = {}_A \langle x \langle x, x\rangle_B , x\rangle .$$

Let us first assume that B is unital. In this case, since $\|x\|_B^2 1 - \langle x, x\rangle_B \geq 0$, we get

$${}_A \langle x \langle x, x\rangle_B , x\rangle \leq \|x\|_B^2 {}_A \langle x, x\rangle .$$

By the C^*-identity, we get $\|_A \langle x, x\rangle \| \leq \| \langle x, x\rangle_B \|$. If B is nonunital, then on its unitization $B^1 = B \oplus \mathbf{C}1$, we can define a right B^1-action by $x \cdot (b + \lambda 1) \mapsto xb + \lambda x$. We still have the identity

$${}_A \langle xb, y\rangle = {}_A \langle x, yb^*\rangle$$

for any $x, y \in X$ and $b \in B^1$. The proof in the unital case then goes through. ☕

In particular, for imprimitivity bimodules, we needn't worry about which of the two norms we complete by.

corollary 3.10. *If X_0 is an (A, B)-preimprimitivity bimodule, then there is an (A, B) imprimitivity bimodule X along with a linear morphism $q : X_0 \to X$ for which $q(X_0)$ is dense and*

$$\langle q(x), q(y)\rangle_B = \langle x, y\rangle_B \text{ and } {}_A \langle q(x), q(y)\rangle = {}_A \langle x, y\rangle$$

for any $x, y \in X_0$.

I note as in remark 3.6 that we could have worked with the alternate condition II' as stated before in the definition of a preimprimitivity bimodule. Condition II' has a particular advantage in that we could have defined preimprimitivity bimodules over a pair (A_0, B_0) of dense *-subalgebras of (A, B) instead and then have taken a completion to an (A, B)-imprimitivity bimodule.

We are now in a position to prove that Morita equivalence is an equivalence relation.

Proposition 3.11. *If X is an (A, B)-imprimitivity bimodule and Y is a (B, C)-imprimitivity bimodule, then on $X \odot_B Y$, there are A- and C-linear inner products ${}_A\langle \mid \rangle$ and $\langle \mid \rangle_C$ given by the identities*

$$\begin{aligned} {}_A\langle x \otimes y \mid z \otimes w\rangle &= {}_A\langle x, z\langle w, y\rangle_B\rangle \\ \langle x \otimes y \mid z \otimes w\rangle_C &= \langle\langle z, x\rangle_B\, y, w\rangle_C \end{aligned}$$

for which $X \odot_B Y$ is an (A, C)-preimprimitivity bimodule.

Proof. That our inner products are well-defined follow from the usual trick for defining bilinear forms on a tensor product using the universal property. It is immediate that this is a sesquilinear form. The other pre-inner product properties: that ${}_A\langle x \mid x\rangle \geq 0$, that ${}_A\langle ax \mid y\rangle = a\,{}_A\langle x \mid y\rangle$, and that ${}_A\langle x \mid y\rangle^* = {}_A\langle y \mid x\rangle$ follow by checking on elementary tensors. The same is true for $\langle \mid \rangle_C$. It remains to check the preimprimitivity bimodule conditions. Conditions II and III are algebraic conditions and follow immediately from a check on the elementary tensors. The only thing left to check is fullness of the actions.

We check fullness of C acting on $X \odot_B Y$. The proof is the same for fullness of A. Suppose that $q \in C$. Let $\epsilon > 0$. By density of $\langle Y, Y\rangle_C$, we can choose $y_1, \ldots, y_n, z_1, \ldots, z_n$ in Y for which

$$\|\sum_i \langle y_i, z_i\rangle - q\| < \epsilon\ .$$

Since B acts on Y non-degenerately, by Cohen factorization,[2] we may write y_i in the form $b_i w_i$ for $b_i \in B$ and $w_i \in Y$. Since $\operatorname{span}\langle X, X\rangle_B$ is dense in B, we may choose $x_i \in \operatorname{span}\langle X, X\rangle_B$ for which

$$\|x_i - b_i\|\|y_i\|\|z_i\| < \frac{\epsilon}{n}\ .$$

[2] Cohen factorization states

Theorem 3.12 (Cohen)**.** *If X is a non-degenerate A-module then every element of X is of the form xa for some $a \in A$ and $x \in X$.*

See Proposition 2.33 of [6] for a proof.

By Cauchy-Schwarz, we get the inequality

$$\| \sum_i \langle x_i w_i, z_i \rangle_C - q \| < 2\epsilon \ .$$

The sum on the left is a sum of elements of $\langle X \odot_B Y \mid X \odot_B Y \rangle_C$ and so we get density. ☕

3.2 Induced representations

The set-up is as follows: we have a (Hilbert) B-module X_B along with a *-morphism $A \to \mathcal{L}(X_B)$ (say A acts on X_B by adjointables) giving X the structure of an (A,B)-bimodule. On the other hand, we may have a non-degenerate representation $\pi : B \to \mathcal{B}(\mathcal{H})$ making H into a $(B,\mathbf{C})$-bimodule. As before, we can define a preinner product on $X \odot_B \mathcal{H}$:

$$\langle x \otimes h | y \otimes k \rangle := \langle \pi(\langle y, x\rangle_B)h, k\rangle \ .$$

The usual calculations show that this is a pre-inner product and we produce a new Hilbert space $X \otimes_B \mathcal{H}$ by taking the completion.

The representation π now can be *induced up* to a representation $X\text{-}\pi \uparrow_B^A$ of A as adjointables on $X \otimes_B H$. This is the content of the first Proposition.

Proposition 3.13. *For a fixed $a \in A$, the map*

$$X\text{-}\pi \uparrow_B^A (a) : X \odot_B \mathcal{H} \to X \odot_B \mathcal{H} : x \otimes h \mapsto (ax) \otimes h$$

extends to an adjointable on $X \otimes_B \mathcal{H}$.

Remark 3.14. If we have this Proposition, notice that because $X\text{-}\pi \uparrow_B^A (a^*) = X\text{-}\pi \uparrow_B^A (a)^*$, $X\text{-}\pi \uparrow_B^A$ is a *-representation of A. If $A \to \mathcal{L}(X_B)$ is a non-degenerate action, then the induced representation is non-degenerate.

Proof. The boundedness is a cute trick with matrices. Let $\sum_i x_i \otimes h_i$ be an arbitrary element of $X \odot_B \mathcal{H}$. Then,

$$\begin{aligned}\|\sum_{i\le n} ax_i \otimes h_i\|^2 &= \sum_{i,j} \langle ax_j \otimes h_j | ax_i \otimes h_i\rangle = \sum_{i,j} \left\langle \pi(\langle ax_i, x_j\rangle_B)h_j, h_i\right\rangle \\ &= \langle \pi_n[\langle ax_i, ax_j\rangle]_{i,j} h, h\rangle\end{aligned}$$

where h is the vector $(h_1, \ldots, h_n) \in \mathcal{H}^n$. Notice that by taking a unitization of B, $[\langle ax_i, ax_j\rangle] \le \|a\|^2[\langle x_i, x_j\rangle]$. Therefore, we get by positivity of π_n that

$$\|\sum_{i\le n} ax_i \otimes h_i\|^2 \le \|a\|^2 \langle \pi_n[\langle x_i, x_j\rangle]h, h\rangle = \|a\|^2 \|\sum_{i\le n} x_i \otimes h_i\|^2$$

We therefore get the appropriate extension. Notice that the above calculation shows that an induced representation is non-degenerate if the action $A \to \mathcal{L}(X)$ is non-degenerate. ☕

If the overarching X is clear, I will endeavor to write $\pi\uparrow_B^A$ or even $\pi\uparrow$ for the induced representation instead.

Example 3.15. Suppose that A is any C^*-algebra. Then, A is a $(M(A),A)$-bimodule with adjointables acting on the left. If we have any non-degenerate representation $\pi : A \to \mathcal{B}(\mathcal{H})$ then what does the induced action $\pi\uparrow_A^{M(A)}$ look like? First we see that the map

$$\varphi : A \odot_A \mathcal{H} \to \mathcal{H} : a \otimes h \mapsto ah$$

is an isometry:

$$\langle \varphi(a\otimes h), \varphi(b\otimes k)\rangle = \langle ah, bk\rangle = \langle \pi(\langle b,a\rangle_A)h, k\rangle = \langle a\otimes h | b\otimes k\rangle \ .$$

It follows from π being non-degenerate that φ has dense range. Therefore, the extension $U : A \otimes_A \mathcal{H} \to \mathcal{H}$ is a unitary. We see that for any $m \in M(A)$,

$$U\pi\uparrow(m)(a\otimes h) = U(ma\otimes h) = \pi(ma)h = \overline{\pi}(m)(a\cdot h) = \overline{\pi}(m)U(a\otimes h) \ .$$

That is, $\pi\uparrow$ is unitarily equivalent to the usual extension $\overline{\pi} : M(A) \to \mathcal{B}(\mathcal{H})$.

The next Proposition will show that induced representations preserve unitary equivalence and direct sums.

Proposition 3.16. *Let $\pi_i : B \to \mathcal{B}(\mathcal{H}_i)$ $(i = 1,2)$ be two non-degenerate representations of B and let ${}_AX_B$ be an (A,B)-bimodule with A acting as non-degenerate adjointables on X. If $T : \mathcal{H}_1 \to \mathcal{H}_2$ is a bounded linear operator which intertwines π_1 and π_2, that is, $T\pi_1(b) = \pi_2(b)T$, then the linear map defined by*

$$1 \otimes_B T : X \otimes_B \mathcal{H}_1 \to X \otimes_B \mathcal{H}_2 : x \otimes h \mapsto x \otimes Th$$

intertwines $\pi_1\uparrow$ and $\pi_2\uparrow$. The correspondence $T \mapsto 1\otimes_B T$ is *-linear. Furthermore, if π_3 is a third representation of B and S intertwines π_2 and π_3, then $1\otimes_B ST = (1\otimes_B S)(1\otimes_B T)$ interwines π_1 and π_3. In particular, X-$\uparrow_B^A$ preserves unitary equivalence and direct sums.

Proof. That $1\otimes_B T$ is well-defined is exactly the same matrix trick as in the calculation that $\pi\uparrow$ is well-defined. This shows X-$\uparrow_B^A$ preserves unitary equivalence. To see that inducing up preserves direct sums, we let $\pi : B \to B(\bigoplus_s H_s)$ be a representation with $\pi = \bigoplus_s \pi_s$. We first need to verify a Lemma.

Lemma 3.17. *If π is as above, then one has the identity*

$$X \otimes_B \bigoplus_s \mathcal{H}_s \simeq \bigoplus_s (X \otimes_B \mathcal{H}_s) \ .$$

Proof. Let $\iota_t : \mathcal{H}_t \hookrightarrow \bigoplus_s \mathcal{H}_s$ and $j_t : X \otimes_B \mathcal{H}_t \to \bigoplus (X \otimes_B \mathcal{H}_s)$ be the canonical embeddings and let us define

$$\begin{aligned}\Psi : X \odot \bigoplus_s^c \mathcal{H}_s &\to \bigoplus_s (X \otimes_B \mathcal{H}_s)\\ : x \otimes \iota_t(h) &\mapsto j_t(x \otimes h)\end{aligned}$$

where $\bigoplus_s^c \mathcal{H}_s$ denotes the set of finite linear combinations of elements of $\mathcal{H}_s$. The usual tensor product trick shows that Ψ is a well-defined linear map. To see that Ψ is an isometry,

$$\begin{aligned}\|\Psi(\sum_i x_i \otimes \iota_{t_i}(h_i))\|^2 = \|\sum_i j_{t_i}(x_i \otimes h_i)\|^2 &= \sum_{i,j} \left\langle j_{t_i}(x_i \otimes h_i), j_{t_j}(x_j \otimes h_j)\right\rangle \\ &= \sum_{i,j} \left\langle x_i \otimes \iota_{t_i} h_i, x_j \otimes \iota_{t_j} h_j \right\rangle = \|\sum_i x_i \otimes \iota_{t_i} h_i\|^2\end{aligned}$$

where in the penultimate equality, we are using the fact that the collection $(j_t)_t$ are orthogonal isometries so that if $t_i = t_j$, then

$$\left\langle j_{t_i}(x_i \otimes h_i), j_{t_j}(x_j \otimes h_j)\right\rangle = \langle x_i \otimes h_i, x_j \otimes h_j\rangle = \left\langle x_i \otimes \iota_{t_i}(h_i), x_j \otimes \iota_{t_j}(h_j)\right\rangle$$

and if $t_i \neq t_j$, then

$$\begin{aligned}\left\langle x_i \otimes \iota_{t_i} h_i, x_j \otimes \iota_{t_j} h_j\right\rangle &= \iota_{t_i}(\left\langle \pi_{t_i}(\langle x_j, x_i\rangle_B) h_i), \iota_{t_j}(h_j)\right\rangle \\ &= 0 = \left\langle j_{t_i}(x_i \otimes h_i), j_{t_j}(x_j \otimes h_j)\right\rangle \ .\end{aligned}$$

It is easy to verify that Ψ has dense range (we need only check that $\bigoplus_s^c (X \odot \mathcal{H}_s)$ is dense in $\bigoplus_s (X \otimes_B \mathcal{H}_s)$). Therefore, Ψ is the desired isomorphism. ☕

Let ι_s and j_s be as in the Lemma. We need only verify that $X\text{-}\pi \uparrow_B^A (a) j_s = j_s X\text{-}\pi_s \uparrow_B^A (a)$. However, notice that $j_s = 1 \otimes_B \iota_s$ under the isomorphism so we do indeed have this identity. ☕

Definition 3.18. We say that a representation ρ of A is weakly contained in a family of representations Π of A if $\bigcap_{\pi \in \Pi} \ker \pi \subset \ker \rho$.

Proposition 3.19. *Suppose that $A \to \mathcal{L}(X_B)$ acts non-degenerately and Π is a collection of non-degenerate representations of B. If ρ is a non-degenerate representation of B which is weakly contained in Π then $\rho\uparrow$ is weakly contained in $\{\pi \uparrow : \pi \in \Pi\}$.*

Proof. Since $\ker(\bigoplus_{\pi\in\Pi}\pi) = \bigcap_{\pi\in\Pi}\ker\pi$ and direct sums commute with inducing up, we may as well assume that Π is composed of a singleton π. Then,

$$\begin{aligned}
\pi\uparrow(a) = 0 &\iff \forall x \in X, h \in \mathcal{H}, \pi\uparrow(a)(x\otimes h) = 0 \\
&\iff \forall x, h,\ \|ax\otimes h\| = 0 \\
&\iff \forall x, h,\ \langle \pi(\langle ax, ax\rangle_B)h, h\rangle = 0 \\
&\iff \forall x,\ \pi(\langle ax, ax\rangle_B) = 0 \\
&\implies \forall x,\ \rho(\langle ax, ax\rangle_B) = 0 \\
&\iff \forall x, h,\ \rho\uparrow(a)(x\otimes h) = 0 \\
&\iff \rho\uparrow(a) = 0\ .
\end{aligned}$$

☕

corollary 3.20. *Inducing up is well-defined on PrimB: if π and ρ are two representations with $\ker\pi = \ker\rho$, then $\ker\pi\uparrow = \ker\rho\uparrow$.*

corollary 3.21. *Suppose that $A \hookrightarrow \mathcal{L}(X_B)$ and π is a faithful representation of B. Then, $\pi\uparrow_B^A$ is a faithful representation of A.*

Proof. The proof of the Proposition shows us $\pi\uparrow(a) = 0$ if and only if $\pi(\langle ax, ax\rangle_B) = 0$ for all $x \in X$. ☕

3.3 The Rieffel correspondence

For Morita equivalent C^*-algebras A and B, the Rieffel correspondence gives us an explicit lattice isomorphism between the class $I(A)$ of ideals of A and the class $I(B)$. In particular, that our C^*-algebra is simple is preserved under Morita equivalence. The first theorem gets us the lattice isomorphism.

Theorem 3.22. *Suppose that X is an (A, B)-imprimitivity bimodule. There are lattice isomorphisms between $I(A)$, $I(B)$, and the lattice of closed sub-(A, B)-modules of X. The isomorphism is given by:*

I. *For $J \in I(B)$, the associated sub-(A, B)-module of X is*

$$X_J := \{y \in X : \forall x \in X,\ \langle x, y\rangle_B \in J\} = \bigcap_{x\in X} \{y \in X : \langle x, y\rangle_B \in J\};$$

II. *If Y is a closed sub-(A, B)-module of X, then*

$$\begin{aligned} {}_Y I &= \overline{\langle X, Y\rangle_B} \text{ and} \\ I_Y &= \overline{{}_A\langle Y, X\rangle} \end{aligned}$$

are the associated ideals in $I(B)$ and $I(A)$ respectively; and

III. *Given $K \in I(A)$, the associated sub-(A, B)-module is*

$${}_K X := \bigcap_{x\in X} \{y \in X : {}_A\langle y, x\rangle \in K\} .$$

To prove the theorem, we will be in want of the following Lemma, reminiscent of the analysis of the kernel of the form induced by states in the GNS construction:

Lemma 3.23. *Suppose that X is an (A, B)-imprimitivity bimodule and $J \in I(B)$. Then, X_J is a closed sub-(A, B)-module of X and*

$$\overline{X \cdot J} = X_J = \{y \in X : \langle y, y\rangle_B \in J\} .$$

Proof. It is immediate that $\overline{X \cdot J} \subset X_J \subset \{y : \langle y, y\rangle_B \in J\}$. For what remains, suppose that $y \in X$ is such that $\langle y, y\rangle_B \in J$. Suppose that $(e_i)_i$ is an approximate identity of contractions in J. Then,

$$\|y - y \cdot e_i\|^2 = \| \langle y, y\rangle - e_i \langle y, y\rangle_B - \langle y, y\rangle_B e_i + e_i \langle y, y\rangle_B e_i\| \to_i 0 .$$

Therefore, $y \in \overline{X \cdot J}$. ☕

Proof of theorem. It is easy to see that all of the objects at play are in the appropriate classes. We only show the correspondence between $I(B)$ and sub-(A,B)-modules of X since the correspondence between $I(A)$ and sub-(A,B)-modules of X is similar.

Let $J \in I(B)$. A simple calculation shows

$$_{X_J}I = \overline{\langle X, X_J \rangle_B} = \overline{\left\langle X, \overline{X \cdot J} \right\rangle_B} = \overline{\langle X, X \cdot J \rangle_B} \subset J.$$

The same calculation shows $_{X_J}I \supset \langle X \cdot J, X \cdot J \rangle_B = J \langle X, X \rangle_B J$. Since $\langle X, X \rangle_B$ is dense in B, $J \langle X, X \rangle_B J$ is a dense ideal in J. This gets us the equality.

Conversely, Let Y be a sub-(A,B)-module of X. Again, a calculation:

$$X_{_YI} = \bigcap_{x \in X} \{y \in X \, \langle x, y \rangle \in \overline{\langle X, Y \rangle_B}\} \supset Y \, .$$

Since $X_{_YI} = \overline{X \cdot {_Y}I}$, $X_{_YI}$ is spanned by elements of the form $x \langle y, z \rangle_B$ with $x, y \in X$ and $z \in Y$. By imprimitivity, $x \langle y, z \rangle_B = {}_A \langle x, y \rangle z$ and since $z \in Y$, ${}_A \langle x, y \rangle z \in Y$. This gets us the other equality.

Finally, it remains to check that this correspondence is an order isomorphism. We need only recall the definition. For $I, J \in I(B)$:

$$I \subset J \implies \bigcap_x \{y \in X : \langle x, y \rangle_B \in I\} \subset \bigcap_x \{y \in X : \langle x, y \rangle_B \in J\}$$

and for $Y, Z \trianglelefteq X$,

$$Y \subset Z \implies \overline{\langle X, Y \rangle_B} \subset \overline{\langle X, Z \rangle_B} \, .$$

☕

This leads us to the Rieffel correspondence.

Proposition 3.24 (Rieffel Correspondence)**.** *Let A and B be C^*-algebras. If $X\text{-}\uparrow_B^A$: $I(B) \to I(A)$ is the lattice isomorphism between $I(B)$ and $I(A)$ as given above, then $X\text{-}\uparrow_B^A$ is given explicitly by*

$$X\text{-}J \uparrow_B^A = \overline{{}_A \langle X \cdot J, X \rangle} \, .$$

If $K = X\text{-}J \uparrow_B^A$ then the corresponding submodule is $\overline{K \cdot X} = \overline{X \cdot J}$. If $\pi : B \to \mathcal{B}(\mathcal{H})$ is a representation of B, then one has the identity

$$X\text{-}(\ker \pi) \uparrow_B^A = \ker(X\text{-}\pi \uparrow_B^A) \, .$$

Proof. It remains to show $(\ker\pi)\uparrow = \ker(\pi\uparrow)$. Since $(\ker\pi)\uparrow = \overline{{}_A\langle X\cdot\ker\pi, X\rangle}$, given any $x, y \in X$ and $b \in B$ so that $\pi(b) = 0$,

$$\begin{aligned}\pi\uparrow({}_A\langle xb, y\rangle)(z\otimes h) &= {}_A\langle xb, y\rangle\, z\otimes h\\ &= x\,\langle yb^*, z\rangle_B \otimes h = x\otimes\pi(b\,\langle y, z\rangle_B)h = 0\end{aligned}$$

so $(\ker\pi)\uparrow \subset \ker(\pi\uparrow)$.

Conversely, suppose that $a \in A$ with $\pi\uparrow(a) = 0$. This means that for any $x, y \in X$ and $h, k \in \mathcal{H}$,

$$0 = \langle ax\otimes h, y\otimes k\rangle = \langle \pi(\langle y, ax\rangle_B)h, k\rangle$$

that is, $\langle y, ax\rangle_B \in \ker\pi$ for any $x, y \in X$. In particular, $ax \in X_{\ker\pi}$ for any $x \in X$.

Let $\epsilon > 0$. By fullness of $A \curvearrowright X$, we may suppose that $\sum_i {}_A\langle x_i, y_i\rangle$ is such that

$$\|a(\sum_i {}_A\langle x_i, y_i\rangle) - a\| < \epsilon\ .$$

Since $a(\sum_i {}_A\langle x_i, y_i\rangle) = \sum_i {}_A\langle ax_i, y_i\rangle$ belongs to ${}_A\langle X_{\ker\pi}, X\rangle = {}_A\langle \overline{X\cdot\ker\pi}, X\rangle \subset (\ker\pi)\uparrow$, and this is true of all such approximations of a, we get the other inclusion. ☕

Before we derive more consequences from the Rieffel correspondence, we should talk about an alternative characterisation of Morita equivalence. Recall that for a C^*-algebra A and a projection $p \in M(A)$, if p is *full*, that is, $\overline{ApA} = A$ then $A \sim_M pAp$ by the (A, pAp)-imprimitivity bimodule Ap. In particular, if p, q are complemented (that is, $p + q = 1_A$) full projections in $M(A)$ then the corners pAp and qAq are Morita equivalent. The next Proposition tells us that one can realize every pair of Morita equivalent C^*-algebras as the complemented full corners of a certain C^*-algebra.

Proposition 3.25. *Suppose that $A \sim_M B$ by an imprimitivity bimodule ${}_AX_B$. Then, there is a C^*-algebra C, called the linking algebra, and complemented full projections p and $q = 1 - p$ of $M(C)$ for which $A = pCp$ and $B = qCq$.*

Proof. Let $M := X \oplus B$ – a Hilbert B-module. We define

$$C := \begin{bmatrix} A & X\\ X^\flat & B\end{bmatrix} \subset \mathcal{L}(M)$$

where $X^\flat = X$ as an abelian group and is a (B, A)-imprimitivity bimodule with $bx^\flat = (xb^*)^\flat$ for any $b \in B$ and $x^\flat a = (a^*x)^\flat$ for any $a \in A$. Denote by $\cdot^\flat : X \to X^\flat$ the canonical anti-homomorphism. We should think of $\cdot^\flat$ as an adjoint operation. This means that we should define $x^\flat y := \langle x, y\rangle_B$ and $xy^\flat := {}_A\langle x, y\rangle$. A matrix

$$L = \begin{bmatrix} a & x \\ y^\flat & b \end{bmatrix}$$

in C acts on M by

$$L(z \oplus c) = (az + xc) \oplus (\langle y, z\rangle_B + bc)$$

It is easy to see that C is a *-subalgebra on $\mathcal{L}(M)$. To see that C is a C^*-subalgebra, we have the following Lemma:

Lemma 3.26. *If $L \in \mathcal{L}(M)$ as above then*

$$\max\{\|a\|, \|x\|, \|y\|, \|b\|\} \le \|L\| \le \|a\| + \|x\| + \|y\| + \|b\| \ .$$

Proof. For the right inequality, we just note that

$$L = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ y^\flat & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

and check that each corner does indeed achieve the expected norm. For the left inequality, we see

$$\| \langle 0 \oplus \langle y, y\rangle_B \,, L(y \oplus 0)\rangle_M \| = \| \langle y, y\rangle_B^2 \| = \|y\|^4$$

and by Cauchy Schwarz, $\| \langle 0 \oplus \langle y, y\rangle_B \,, L(y \oplus 0)\rangle_M \| \le \|L\|\|y\|^3$. Similarly,

$$\| \langle x \oplus 0, L(0 \oplus \langle x, x\rangle_B)\rangle_M \| = \| \langle x, x\rangle_B^2 \| = \|x\|^4$$

and $\| \langle x \oplus 0, L(0 \oplus \langle x, x\rangle_B)\rangle_M \| \le \|L\|\|x\|^3$.

To get $\|b\| \le \|L\|$, pick an approximate identity of positive contractions (e_i) of B. Then, $\|e_i b e_i\| \to_i \|b\|$. Therefore,

$$\|e_i b e_i\| = \| \langle 0 \oplus e_i, L(0 \oplus e_i)\rangle_M \| \le \|L\| \ .$$

For $\|a\| \le \|L\|$, recall that $A \hookrightarrow \mathcal{L}(X)$ since A is full. For every $\epsilon > 0$, there is then some $x_\epsilon \in b_1(X)$ for which $|\|ax_\epsilon\| - \|a\|| < \epsilon$. Then,

$$\| \langle ax_\epsilon \oplus 0, L(x_\epsilon \oplus 0)\rangle_M \| = \| \langle ax_\epsilon, ax_\epsilon\rangle_B \| = \|ax_\epsilon\|^2 = \|a\|^2 + O(\epsilon) \ .$$

By Cauchy Schwarz again,

$$\| \langle ax_\epsilon \oplus 0, L(x_\epsilon \oplus 0)\rangle_M \| \le \|L\|\|ax_\epsilon\| = \|L\|\|a\| + O(\epsilon) \ .$$

Letting $\epsilon \to 0$, we get our final inequality. ☕

Consider the complementary projections

$$p = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix},\ q = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix} .$$

Of course, A and B are corners given by these projections. It remains to show that these projections are full. A computation shows us

$$CpC = \left\{ \begin{bmatrix} aa' & az \\ w^\flat a' & \langle w, z\rangle_B \end{bmatrix} : aa' \in A, z, w \in X \right\} \text{ and}$$
$$CqC = \left\{ \begin{bmatrix} {}_A\langle w, z\rangle & xb' \\ bz^\flat & bb' \end{bmatrix} : b, b' \in B, z, w \in X \right\} .$$

By inspection, one sees that one can approximate any corner of the element $L \in C$ as above by an appropriate choice of z, w, a, a', b, b'. ☕

We now come back to the Rieffel correspondence.

Proposition 3.27. *Let A and B be C^*-algebras and let X be an (A, B)-imprimitivity bimodule. If $J \in I(B)$ then X_J is a $(J \uparrow_B^A, J)$-imprimitivity bimodule and $X^J := X/X_J$ is an $(A/J \uparrow_B^A, B/J)$-imprimitivity bimodule. Furthermore, the quotient norm on X^J agrees with the norm induced by B/J.*

Proof. First let us show X_J is a $(J\uparrow, J)$-imprimitivity bimodule. Since $X_J = \overline{X \cdot J}$ J does define a right action on X_J and the J-valued inner product defined by

$$\langle x, y\rangle_J = \langle x, y\rangle_B$$

defines an inner product on X_J. This action is full because

$$\left\langle \overline{X \cdot J}, \overline{X \cdot J} \right\rangle_J \supset J \langle X, X\rangle_B J$$

and $J \langle X, X\rangle_B J$ is dense in J. Since by the Rieffel correspondence, $\overline{J\uparrow \cdot X} = \overline{X \cdot J}$, it follows that the left action $J\uparrow$ on X also defines a full left action and the algebraic conditions of being an imprimitivity bimodule follow since X is an imprimitivity bimodule.

Next we want to show that X^J is a $(A/J\uparrow, B/J)$-preimprimitivity bimodule. Let us write down what the actions and inner products are since it should be clear once we do so that we do have such a structure:

$$\begin{aligned} (a + J\uparrow)(x + X_J) &= ax + X_J \\ (x + X_J)(b + J) &= xb + X_J \\ {}_{A/J\uparrow}\langle x + X_J, y + X_J\rangle &= {}_A\langle x, y\rangle + J_\uparrow \\ \langle x + X_J, y + X_J\rangle_{B/J} &= \langle x, y\rangle_B + J\ . \end{aligned}$$

Since $X_J = \overline{J\uparrow \cdot X} = \overline{X \cdot J}$, the actions and ideals are well-defined. The algebraic conditions of an imprimitivity bimodule go through since X is an imprimitivity bimodule. Fullness also follows.

It remains to check that the quotient norm on X^J agrees with the norm induced by B/J since X^J is complete with the quotient norm. The trick is to use the linking algebra. Let

$$D = \begin{bmatrix} J\uparrow & X_J \\ X_J^\flat & J \end{bmatrix} .$$

The linking algebra D can be thought of as an ideal in

$$C = \begin{bmatrix} A & X \\ X^\flat & B \end{bmatrix} .$$

We check closure under left multiplication only:

$$DC = \begin{bmatrix} J\uparrow & X_J \\ X_J^\flat & J \end{bmatrix} \begin{bmatrix} A & X \\ X^\flat & B \end{bmatrix} = \begin{bmatrix} J\uparrow A + {}_A\langle X_J, X\rangle & J\uparrow X + X_J B \\ (AX_J + XJ)^\flat & \langle X_J, X\rangle_B + JB \end{bmatrix}$$

and one checks that the individual components are subsets of the appropriate components in D. The ideal D is closed in C since the topology on D is determined by its components (this is Lemma 3.26) and each component of D is closed.

On the other hand, if $E \subset \mathcal{L}(\overline{X^J} \oplus B/J)$ is the linking algebra for the completion $\overline{X^J}$ of X^J under the norm $\|\cdot\|_{B/J}$, then one has an embedding

$$C/D \hookrightarrow E$$

given by the first isomorphism theorem. In particular, this map is an isometry. Given any $x \in X$,

$$\begin{aligned} \inf_{z\in X_J} \|x+z\| &= \left\| \begin{bmatrix} 0 & x + X_J \\ 0 & 0 \end{bmatrix} \right\|_E = \inf_{T\in D} \left\| \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} + T \right\|_C \\ &= \inf_{z\in X_J} \left\| \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix} \right\|_C = \|x + X_J\|_{B/J} . \end{aligned}$$

☕

Let $J \in I(B)$. Recall that there are natural lattice isomorphisms

$$\begin{aligned} \{I \in I(B) : I \subset J\} &\xrightarrow{\sim} I(J) : I \mapsto I \text{ and} \\ \{I \in I(B) : I \supset J\} &\xrightarrow{\sim} I(B/J) : I \mapsto I/J . \end{aligned}$$

The Rieffel correspondence will then restrict to isomorphisms $I(J) \simeq I(J\uparrow)$ and $I(B/J) \simeq I(A/J\uparrow)$. Furthermore, if $I \subset J$,

$$X\text{-}I\uparrow = \overline{{}_A\langle X \cdot I, X\rangle} = \overline{{}_A\langle X \cdot I, X \cdot I\rangle} = \overline{{}_{J\uparrow}\langle X \cdot I, X \cdot J\rangle} = X_J\text{-}I\uparrow \ .$$

If $I \supset J$, then

$$X^J\text{-}(I/J)\uparrow = \overline{{}_{A/J\uparrow}\langle X^J \cdot I/J, X^J\rangle} = \overline{{}_A\langle X \cdot I, X\rangle}/J\uparrow = X\text{-}I\uparrow/J\uparrow \ .$$

So, if $I \supset J$ then $X^J\text{-}I\uparrow = X\text{-}I\uparrow$ in the appropriate correspondence.

Our notion of Morita equivalence is stronger than the usual form of Morita equivalence. In general, one says that two rings R and S are Morita equivalent if there is an equivalence of categories

$$\mathrm{Mod}_R \equiv \mathrm{Mod}_S$$

where Mod_A denotes the category of A-modules with morphisms being A-module morphisms. Recall that an equivalence of categories consists of two covariant functors

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[r, bend right, "F"'] & \mathrm{Mod}_S \arrow[l, bend right, "G"'] \end{tikzcd}$$

for which G and F are bijections on objects and morphisms and we have isomorphisms $\alpha_A : A \to FG(A)$, $\beta_B : B \to GF(B)$ for which given any R-module morphism $A \xrightarrow{\varphi} B$, we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\alpha_A"] & B \arrow[d, "\alpha_B"] \\ GF(A) \arrow[r, "GF(\varphi)"] & GF(B) \end{tikzcd}$$

and a similar diagram for morphisms between S-modules should also hold.

In the case when we are dealing with modules over C^*-algebras, we want to consider Hilbert spaces $\mathcal{H}_{\mathbf{C}}$ for which a C^*-algebra A acts on $\mathcal{H}_{\mathbf{C}}$ non-degenerately by adjointables. Given $A \sim_M B$ with imprimitivity bimdoule ${}_AX_B$, the claim then is that

$$X\text{-}\uparrow_B^A : \mathrm{Rep}_A \to \mathrm{Rep}_B : \pi \mapsto X\text{-}\pi\uparrow_B^A$$

induces the equivalence of categories $\mathrm{Mod}_A \equiv \mathrm{Mod}_B$ (where the morphisms are A-linear bounded linear operators $\mathcal{H}_1 \to \mathcal{H}_2$) with inverse $X^\flat\text{-}\uparrow_A^B$.

Before going through the proof, let us note what this equivalence of categories gets us:

corollary 3.28. *If X is an (A,B)-imprimitivity bimodule, then the inverse of $X\text{-}\uparrow_B^A$: $I(B) \to I(A)$ is $X^\flat\text{-}\uparrow_A^B$.*

Proof. The Rieffel correspondence tells us $\ker(X\text{-}\pi\uparrow_B^A) = X\text{-}(\ker\pi)\uparrow_B^A$ and every ideal is the kernel of some representation. ☕

corollary 3.29. *If X is an (A,B)-imprimitivity bimodule and π is a non-degenerate representation of B, then π is irreducible if and only if $\pi\uparrow_B^A$ is.*

Proof. This is because $\cdot\uparrow_B^A$ preserves direct sums. ☕

corollary 3.30. *Suppose ${}_AX_B$ is an imprimitivity bimodule.*

1. *The Rieffel correspondence induces a homeomorphism $h_X : PrimB \to PrimA$.*
2. *We have $h_X|_{PrimJ} = h_{X_J}$ and $h_X|_{PrimB/J} = h_{X^J}$.*

Proof. By the Rieffel correspondence, the map h_X is a bijection. Since the Rieffel correspondence is order preserving and since open sets are given by sets of the form

$$\mathcal{O}_J := \{P \in \mathrm{Prim}B : J \not\subset P\} .$$

The second result is immediate from the remarks made before. ☕

Proof of Morita equivalence. Assume that $A \sim_M B$ and suppose X is a (B,A)-imprimitivity bimodule. Let $\pi : A \to \mathcal{L}(\mathcal{H}_\pi)$ be a non-degenerate representation. We can induce the representation up to

$$X\text{-}\pi\uparrow_A^B : B \to \mathcal{L}(X \otimes_A \mathcal{H}_\pi) .$$

This produces the associated B-module ${}_BX \otimes_A \mathcal{H}_\pi$. On the other hand, we can consider the (A,B)-imprimitivity bimodule ${}_AX^\flat_B$ that will induce a non-degenerate representation $\tau : B \to \mathcal{L}(\mathcal{H}_\tau)$ to

$$X^\flat\text{-}\tau\uparrow_B^A : A \to \mathcal{L}(X^\flat \otimes_A \mathcal{H}_\tau)$$

and this produces an A-module ${}_AX^\flat \otimes_A \mathcal{H}_\tau$. The claim is that these are the maps which produce the equivalence of categories. Since we expect this to be a functor, we better make sure that there is an asosciated map under morphisms. Given a B-linear map

$${}_B\mathcal{H} \xrightarrow{\varphi} {}_B\mathcal{K} ,$$

consider

$$X\text{-}\varphi\uparrow : X \otimes_B \mathcal{H} \to X \otimes_B \mathcal{K} : x \otimes h \mapsto x \otimes \varphi(h) \ .$$

By Proposition 3.16, this map is A-linear and induces a covariant functor $X\text{-}\uparrow$ from A-modules to B-modules. Similarly, we have the map $X^\flat\text{-}\uparrow$ as a functor from B-modules to A-modules. For the equivalence, let us fix a non-degenerate representation $\pi : A \to \mathcal{L}(\mathcal{H})$ on A. We induce up twice to get $X^\flat\text{-}X\text{-}\pi\uparrow\uparrow$ on $X^\flat \otimes (X \otimes \mathcal{H})$. We would like to show that the following map is an A-linear unitary:

$$X^\flat \otimes (X \otimes \mathcal{H}) \xrightarrow{u} \mathcal{H} : x^\flat \otimes (y \otimes h) \mapsto \pi(\langle x, y\rangle_B)h \ .$$

If we can do this, then given any A-linear $\varphi : {}_A\mathcal{H} \to {}_A\mathcal{K}$, supposing that $u : X^\flat \otimes (X \otimes \mathcal{H}) \simeq \mathcal{H}$ and $v : X^\flat \otimes (X \otimes \mathcal{K}) \simeq \mathcal{K}$ are the associated A-linear unitaries, we get the commutative diagram

$$\begin{tikzcd}
\mathcal{H} \arrow[rr, "\varphi"] & & \mathcal{K} \\
X^\flat \otimes (X \otimes \mathcal{H}) \arrow[u, "u"] \arrow[rr, "X^\flat\text{-}X\text{-}\varphi\uparrow\uparrow"] & & X^\flat \otimes (X \otimes \mathcal{K}) \arrow[u, "v"']
\end{tikzcd}$$

and we could do the same argument for B-linear maps and get the equivalence of categories.

Let is define

$$u_0 : X^\flat \odot (X \odot \mathcal{H}) \to \mathcal{H} : x^\flat \otimes (y \otimes h) \mapsto \pi(\langle x, y\rangle_B)h \ .$$

By the universal property of algebraic tensor products, such a map u_0 exists. Since π is non-degenerate, and $\langle\ ,\ \rangle_B$ is full in X, the map u_0 has dense range. To see that u_0 extends to an isomorphism, it therefore suffices to check that u_0 is a unitary. On elementary tensors,

$$\begin{aligned}
\left\langle x^\flat \otimes (y \otimes h), z^\flat \otimes (w \otimes k)\right\rangle &= \langle X\text{-}\pi(\langle z, x\rangle)(y \otimes h), w \otimes k\rangle \\
&= \langle (\langle z, x\rangle\, y) \otimes h, w \otimes k\rangle \\
&= \langle \pi(\langle w, \langle z, x\rangle\, y\rangle)h, k\rangle \\
&= \langle \pi(\langle w, z\,\langle x, y\rangle\rangle)h, k\rangle \\
&= \langle \pi(\langle x, y\rangle)h, \pi(\langle z, w\rangle)k\rangle \\
&= \left\langle u_0(x^\flat \otimes (y \otimes h)), u_0(z^\flat \otimes (w \otimes k))\right\rangle \ .
\end{aligned}$$

It remains to show that u interwines the actions. This is just a little calculation on elementary tensors:

$$
\begin{aligned}
u((a \cdot x^\flat) \otimes (y \otimes h)) &= \pi(\langle x \cdot a^*, y\rangle)h \\
&= \pi(a)\pi(\langle x, y\rangle)h \\
&= \pi(a)u(x^\flat \otimes (y \otimes h)) \; .
\end{aligned}
$$

Remark 3.31. Notice if A and B are commutative C^*-algebras, the Rieffel correspondence tells us that $A \sim_M B$ implies $\mathrm{Prim}(A) \simeq \mathrm{Prim}(B)$. Morita equivalence is therefore the same as isomorphism in the commutative setting.

Remark 3.32. One may wonder if all simple C^*-algebras are Morita equivalent to $\mathbf{C}$. As it turns out, if $A \sim_M B$, then they have isomorphic K-groups (see [2] for a proof). In particular, as $K_1(C(\mathbf{T}) \rtimes_\theta \mathbf{Z}) \neq K_1(\mathbf{C})$, where $C(\mathbf{T}) \rtimes_\theta \mathbf{Z}$ is the irrational rotation algebra with rotation θ, we see that this is not true in general.

Chapter 4

The symmetric imprimitivity theorem

Now that we have the machinery of Morita equivalence, we will prove the symmetric imprimitivity theorem. The proof comes from [9], which is a generalization of the proof given by Rieffel in [7].

4.1 Induced algebras

Let P be a locally compact space. Suppose that P is a proper G-action and (A, G, α) is a dynamical system. We can define the induced algebra $(A, \alpha) \uparrow_G^P$ (also denoted $\alpha \uparrow_G^P$ or $\alpha \uparrow$) as the space

$$\begin{aligned} \alpha \uparrow_G^P = \{f \in C_b(P, A) : [&\forall x \in P\ \forall s \in G\ , f(s \cdot x) = \alpha_s f(x)], \\ &(\|f\| : G\backslash P \to \mathbf{R} : x \mapsto \|f(x)\|) \in C_0(G\backslash P)\} \end{aligned}$$

Notice that the condition that $f(s \cdot x) = \alpha_s f(x)$ guarantees that $\|f\|$ is well-defined (that is, $\|f(x)\| = \|f(s \cdot x)\|$ for any $s \in G$ and $x \in P$). This is a closed $*$-subalgebra of $C_b(P, A)$ so $\alpha \uparrow_G^P$ is a C^*-algebra.

Sometimes our action will be a right action instead of a left action. In this case, we will define

$$\begin{aligned} \alpha \uparrow_G^P = \{f \in C_b(P, A) : [&\forall x \in P\ \forall s \in G\ , f(x \cdot s) = \alpha_s^{-1} f(x)], \\ &(\|f\| : G\backslash P \to \mathbf{R} : x \mapsto \|f(x)\|) \in C_0(P/G)\} \end{aligned}$$

Example 4.1. Suppose that $H \leq_{\text{closed}} G$. Then, $H \curvearrowright G$ on the right. If (A, H, β) is a dynamical system, then we can define the induced algebra $\beta \uparrow_H^G$. If (A, G, α) is a dynamical system for which $\alpha|_H = \beta$ then the *-isomorphism

$$\varphi : C_b(G, A) \to C_b(G, A) : f \mapsto (s \mapsto \alpha_s(f(s)))$$

restricts to a *-isomorphism of $\beta \uparrow_H^G$ onto $C_0(G/H, A)$ (thought of as functions in $C_0(G, A)$ which are constant on cosets).

We see how this example relates to a previous example when $H = \{e\}$ by the following Lemma:

Lemma 4.2. *Suppose that $H \leq_{closed} G$ and (A, G, β) is a dynamical system. Then, there is a dynamical system $(\beta \uparrow_H^G, G, lt)$ where*

$$lt_s f(x) = f(s^{-1}x) \ .$$

It is immediate that lt is a group morphism. That this is a strong continuous map will be proven in Lemma 4.6.

We now want to focus our attention on induced algebras of free and proper actions.

Proposition 4.3. *Suppose that $G \curvearrowright_{free} P$ is proper and (A, G, α) is a dynamical system. Then, $\alpha \uparrow_G^P$ is a $C_0(G\backslash P)$-algebra with action given by*

$$\varphi \cdot f(x) = \varphi(G \cdot x) f(x) \ .$$

Furthermore, $f \mapsto f(x)$ induces an isomorphism $\alpha \uparrow_G^P (G \cdot x) \simeq A$.

It is immediate that $C_0(G\backslash P) \curvearrowright \alpha \uparrow$. From Cohen factorization, it follows that $\ker(\Phi : f \mapsto f(x)) = I_{G \cdot x}$. It remains to check that ran Φ is dense in A. To do this, we want two Lemmas. First, let us define $\alpha \uparrow_c$ to be those elements of $\alpha \uparrow$ with compact support. Notice that $\alpha \uparrow_c$ is dense in $\alpha \uparrow$. We have the following Currying result for induced algebras:

Lemma 4.4. *Suppose that Y is a locally compact space and $G \curvearrowright P$ and (A, G, α) is a dynamical system. Let $C_{cc}(Y, \alpha \uparrow_c)$ be those functions $f \in C(Y \times P, A)$ for which*

I. $f(y, s \cdot x) = \alpha_s f(y, x)$ and

II. there are $C \Subset Y$ and $T \Subset G\backslash P$ for which $f(y, x) = 0$ if $(y, G \cdot x) \notin G \times T$.

The map

$$\Phi : C_{cc}(Y, \alpha\uparrow_c) \to C_c(Y, \alpha\uparrow) : f \mapsto (y \mapsto f(y, \cdot))$$

is well-defined and ran Φ *is inductive limit dense in* $C_c(Y, \alpha\uparrow)$.

Proof. First we want to show $\Phi(f)(y) \in \alpha\uparrow$. Our first condition tells us $\alpha_s(\Phi(f)y)(x) = (\Phi(f)y)(s \cdot x)$. As well, that $\|\Phi(f)y\| \in C_0(G\backslash P)$ follows from our condition II (in fact, $\|\Phi(f)y\| \in C_c(G\backslash P)$). That $\Phi(f)$ has compact support also follows from our condition II. It remains to show that $\Phi(f)$ is continuous. Suppose that $y_i \to y$ in Y. Let $\epsilon > 0$. In order to derive a contradiction, we suppose by taking a subnet that there is a net (x_i) in P for which

$$\|f(y_i, x_i) - f(y, x_i)\| \geq \epsilon$$

for all i. Since $G \cdot x_i \in T$, it converges to some point $G \cdot x$. We may then take a subnet to find a net (s_i) in G for which $s_i \cdot x_i \to x$. Notice

$$\begin{aligned}\|f(y_i, s_i \cdot x_i) - f(y, s_i \cdot x_i)\| &= \|\alpha_s(f(y_i, x_i) - f(y, x_i)\| \\ &= \|f(y_i, x_i) - f(y, x_i)\| \geq \epsilon \,.\end{aligned}$$

Letting $i \to \infty$ gets us our contradiction.

To see that ran Φ is dense in the inductive limit topology, notice that functions of the form zg for $z \in C_c(Y)$ and $g \in \alpha\uparrow$ is inductive limit dense in $C_c(Y, \alpha\uparrow)$. On the other hand, the function

$$f : (y, x) \mapsto z(y)g(x)$$

is in $C_{cc}(Y, \alpha\uparrow_c)$ and $\Phi(f) = zg$. ☕

Lemma 4.5. *Suppose that* Y *is a locally compact space and that* $G \curvearrowright_{\mathit{free}} P$ *is proper and* (A, G, α) *is a dynamical system. If* $F \in C_c(Y \times P, A)$ *then*

$$\psi : Y \times P \to \alpha A : (y, x) \mapsto \int_G \alpha_s(F(y, s^{-1} \cdot x))d\mu(s)$$

is an element of $C_{cc}(Y, \alpha\uparrow_c)$.

Proof. Let $K \Subset Y$ and $C \Subset P$ be such that supp $F \subset K \times C$. Since $s \mapsto \alpha_s(F(y, s^{-1} \cdot x))$ is continuous and has support given by

$$\{s \in G : s^{-1} \cdot x \in C\}$$

which is compact since its inverse corresponds to the compact set C under the homeomorphism $G \xrightarrow{\simeq} G{\cdot}x : s \mapsto s{\cdot}x$, our integral is well-defined. Notice condition II is automatically satisfied for ψ since supp $\psi \subset K \times G \cdot C$ and condition I is a simple calculation. The only thing left to check is that ψ is continuous. To this end, suppose that $((y_i, x_i))_{i \in I}$ is a net in $Y \times P$ which converges to a point (y, x). We want to show $\psi(y_i, x_i) \to \psi(y, x)$. We may as well take a subnet given by a ray in I and throw in a precompact open neighbourhood into C so that $x_i \in C$ for all i. Since our action is proper, the set

$$\{(s, x) \in G \times P : (x, s^{-1} \cdot x) \in C \times C\}$$

is compact. Therefore, by projecting onto the second coordinate, there is some $C' \Subset P$ for which

$$\{s \in G : \exists i \in I,\ s^{-1} \cdot x_i \in C\} \subset C' .$$

Let $\epsilon > 0$. We want to show that eventually,

$$\|F(y_i, s^{-1} \cdot x_i) - F(y, s^{-1} \cdot x)\| < \epsilon$$

for every $s \in G$. We assume in order to derive a contradiction that there is a subnet for which there are $s_i \in G$ with

$$\|F(y_i, s_i^{-1} \cdot x_i) - F(y, s_i^{-1} \cdot x)\| \geq \epsilon$$

for all i. Notice that each s_i is necessarily in C'. Since C' is compact, we may take a subnet to make s_i converge to some point s. Letting $i \to \infty$ gets us a contradiction.

We then see eventually

$$\|\psi(y_i, x_i) - \psi(y, x)\| \leq \int_G \|F(y_i, s^{-1} \cdot x_i) - F(y, s^{-1} \cdot x)\| d\mu(s) \leq \epsilon\mu(C')$$

and so we get continuity. ☕

We now come back to our Proposition: recall we had a map

$$\Phi : \alpha \uparrow_G^P \to A : f \mapsto f(x)$$

with kernel $I_{G\cdot x}$ and we wanted to show ran Φ is dense in A. Let $a \in A$ and $\epsilon > 0$. Since α is continuous, there is some neighbourhood V of e for which

$$\|\alpha_s(a) - a\| < \epsilon$$

for any $s \in V$. Since $G \simeq G \cdot x : g \mapsto s \cdot x$, we conclude $V \cdot x \subset_{\text{open}} G \cdot x$. There is then some open $U \subset P$ for which $U \cap G \cdot x = V \cdot x$. Let $z \in C_c^+(G)$ have supp $z \subset U$ and

$$\int_G z(s^{-1} \cdot x) d\mu(s) = 1$$

by normalizing. By our Lemma, the function

$$f : x \mapsto \int_G z(s^{-1} \cdot x)\alpha_s(a) d\mu(s)$$

is in $\alpha \uparrow$. As well,

$$\|f(x) - a\| \leq \int_G |z(s^{-1} \cdot x)| \|\alpha_s(a) - a\| d\mu(s) < \epsilon\ .$$

This gets us density and completes our Proposition.

Finally, we finish the proof that $(\beta \uparrow_H^G, H, \mathrm{lt})$ is a dynamical system.

Lemma 4.6. *Suppose that we have free, proper actions of locally compact groups K and H on the left and the right respectively on a locally compact space P for which the identity*

$$t \cdot (p \cdot s) = (t \cdot p) \cdot s$$

holds. Suppose that α, β are strongly continuous actions of K and H respectively on a C^*-algebra A. There are strongly continuous actions

$$\sigma : K \to \mathrm{Aut}(\beta \uparrow_H^P) \text{ and } \tau : H \to \mathrm{Aut}(\alpha \uparrow_K^P)$$

given by $\sigma_t(f)(p) = \alpha_t(f(t^{-1} \cdot p))$ and $\tau_s(f)(p) = \beta_s(f(p \cdot s))$. The actions σ and τ are called diagonal actions (the point being that σ is really like restricting the action lt $\otimes\, \alpha$ to $\beta \uparrow$).

Proof. Since the proofs are the same, we need only verify σ is strong continuous. Suppose that $t_i \to t$ in K. Since $\beta \uparrow_c$ is dense in $\beta \uparrow$, we pick any $f \in \beta \uparrow_c$ and show $\sigma_{t_i}(f) \to \sigma_t(f)$ to complete our proof. By lifting, we take $D \subset P$ compact so that supp $f \subset D \cdot H$.

Suppose that N is a compact neighbourhood of t. Then, eventually $t_i \in N$. The functions $p \mapsto \|\sigma_{t_i}(f)(p)\|$ eventually vanish outside of $N \cdot D \cdot H$ and are constant on H-orbits. So, eventually,

$$\|\sigma_{t_i}(f) - \sigma_t(f)\| = \sup_{p \in N \cdot D} \|\sigma_{t_i}(f)(p) - \sigma_t(f)(p)\| \ .$$

Since $N \cdot D$ is compact in P, $f(t^{-1} \cdot N \cdot D)$ is compact in A. Therefore, we can eventually get

$$\begin{aligned} \|f(t_i^{-1} \cdot p) - f(t^{-1} \cdot p)\| < \epsilon \text{ for all } p \in N \cdot D \text{ and} \\ \|\alpha_{t_i}(a) - \alpha_t(a)\| < \epsilon \text{ for all } a \in f(t^{-1} \cdot N \cdot D) \ . \end{aligned}$$

(We can prove both inequalities by contradiction and using the fact that both sets that we quantify over are compact.) In that case, $\|\sigma_{t_i}(f) - \sigma_t(f)\| < 3\epsilon$.

Here are two results which we will make use of later.

Proposition 4.7. *If K is a locally compact group, P is a locally compact G-space and (A, G, α) is a dynamical system, then for any $f \in C_{cc}(K, \alpha \uparrow_c)$, and $\epsilon > 0$, there is some neighbourhood $V \subset K$ of e for which given $kt^{-1} \in V$,*

$$\|f(k, x) - f(t, x)\| < \epsilon$$

for any $x \in P$.

Proof. Is by contradiction.

Lemma 4.8. *Let $G \curvearrowright_{\mathit{free}} P$ be a proper action and let $f \in C_c^+(P)$. Given any $\epsilon > 0$, there is some $g \in C_c^+(P)$ with supp $g \subset$ supp f and*

$$\left| f(x) - g(x) \int_G g(s^{-1}x) ds \right| < \epsilon \ .$$

Proof. Let

$$F : Gx \mapsto \int_G f(s^{-1}x) ds \ .$$

I first claim that this function is in $C_c^+(G \backslash P)$. The function F has compact support because supp $F \subset G \cdot$ supp f. For continuity of F, we suppose that some net (x_i) in P converges to a point x. Let $\eta > 0$. Because $G \curvearrowright P$ freely and properly, it suffices to show that for any $s \in G$, we have

$$|f(s^{-1}x) - f(s^{-1}x_i)| < \eta$$

eventually. This inequality holds because otherwise, we may take a subnet and find some $s_i \in G$ for which

$$|f(s_i^{-1}x) - f(s_i^{-1}x_i)| \geq \eta \ .$$

In this case, either $s_i^{-1}x \in \operatorname{supp} f$ or $s_i^{-1}x_i \in \operatorname{supp} f$. Define $w_i := x$ if $s_i^{-1}x \in \operatorname{supp} f$ and $w_i := x_i$ otherwise. Notice that as $x_i \to x$, we must also have $w_i \to x$. Furthermore, we know that we always have $s_i^{-1}w_i \in \operatorname{supp} f$. Take a subnet of w_i so that $s_i^{-1}w_i \to y$ for some $y \in P$. Since $G \curvearrowright P$ properly, we may take another subnet to make s_i converge to some point $s \in G$. This is a contradiction.

Notice that we really want to find some g for which we have

$$\left| f(x) \int_G f(s^{-1}x)ds - g(x)\sqrt{F(Gx)} \int_G g(s^{-1}x)\sqrt{F(Gx)}ds \right| < \epsilon F(Gx) \ .$$

In other words, we would like to choose a function $h(x)$ so that $|f(x) - h(x)| < \epsilon$ for all $x \in P$ and for which $h(x) = 0$ whenever $F(Gx) = 0$. Setting $g(x) = h(x)/\sqrt{F(Gx)}$ will then do. To this end, let

$$C := \{x \in P : f(x) \geq \epsilon\} \ .$$

This is a compact set with the property that for any $x \in C$, $F(Gx) > 0$. Since GC is compact in $G \backslash P$, we can set

$$m := \min_{x \in C} F(Gx) \ ,$$

which is a positive number. Let $U := \{Gx \in G\backslash P : F(Gx) > m/2\}$ –an open neighbourhood of GC . If we set $Q \in C_c^+(G\backslash P)$ to be a bump function with support in $\overline{U}$, $Q \leq 1$, and $Q|_{GC} = 1$, then $h(x) = f(x)Q(Gx)$ does the job. ☕

4.2 The symmetric imprimitivity theorem

The goal in this section is to prove the following theorem:

Theorem 4.9 (Raeburn)**.** *Suppose that K and H are free and proper left and right actions on a locally compact space P for which K and H commute:*

$$t(ph) = (tp)h$$

for any $t \in K$, $h \in H$, and $p \in P$. Suppose moreover that there are C^*-dynamical systems (A, K, α) and (A, H, β) for which α and β commute.

Since the actions are proper, it makes sense to construct the C^*-dynamical systems

$$(\alpha \uparrow_H^P, K, \sigma) \text{ and } (\beta \uparrow_K^P, H, \tau)\ .$$

The symmetric imprimitivity theorem states that the induced crossed products $\alpha \uparrow \rtimes_\sigma K$ and $\beta \uparrow \rtimes_\tau H$ are Morita equivalent. The Morita equivalence is given by showing that the space $Z_0 := C_c(P, A)$ has a left action from $E_0 := C_{cc}(K, \alpha \uparrow_c)$ and a right action from $B_0 := C_{cc}(H, \beta \uparrow_c)$ so that Z_0 is a (E_0, B_0) preimprimitivity bimodule. Since E_0 is dense in $\alpha \uparrow \rtimes_\sigma H$ and B_0 is dense in $\beta \uparrow \rtimes_\tau K$, this would get us the Morita equivalence.

The actions are given by:

$$\begin{aligned}
cf(p) &= \int_K c(t,p)\alpha_t(f(t^{-1}p))\sqrt{\Delta_K(t)}dt \\
fb(p) &= \int_H \beta_s(f(ps)b(s^{-1}, ps))\frac{ds}{\sqrt{\Delta_H(s)}} \\
{}_{E_0}\langle f, g\rangle\,(t,p) &= \frac{1}{\sqrt{\Delta_K(t)}}\int_H \beta_s(f(ps)\alpha_t(g(t^{-1}ps)^*))ds \\
\langle f, g\rangle_{B_0}(s,p) &= \frac{1}{\sqrt{\Delta_H(s)}}\int_K \alpha_t(f(t^{-1}p)^*\beta_s(g(t^{-1}ps)))dt
\end{aligned}$$

where $c \in E_0$, $b \in B_0$, and $f, g \in Z_0$.

Notice the similarity between the definition of cf and convolution of two elements in $(\alpha \uparrow_H^P, K, \sigma)$. Before we prove that all of these operations are well-defined, let us note the symmetry in the operations and inner products: if we instead consider K acting on the right of P and H acting on the left of P by $h : p = ph^{-1}$ and $p : t = t^{-1}p$ then these would still be commuting free and proper actions and we would instead get a (B_0, E_0)-bimodule Z_0 with actions given by

$$\begin{aligned}
b : f(p) &= \int_H b(s,p)\beta_s(f(s^{-1} : p))\sqrt{\Delta_H(s)}ds \\
f : c(p) &= \int_K \alpha_t(f(p:t)c(t^{-1}, p:t))\frac{dt}{\sqrt{\Delta_K(t)}} \\
{}_{B_0}\langle f, g\rangle (t,p) &= \frac{1}{\sqrt{\Delta_H(s)}}\int_K \alpha_t(f(p:t)\beta_s(g(s^{-1}:p:t)^*))dt \\
\langle f, g\rangle_{E_0}(t,p) &= \frac{1}{\sqrt{\Delta_K(t)}}\int_H \beta_s(f(s^{-1}:p)^*\alpha_t(g(s^{-1}:p:t)))ds \ .
\end{aligned}$$

The significance of swapping the actions is the following: the map

$$\Phi : Z_0 \to Z_0 : f \mapsto f^*$$

behaves like an adjoint on the actions. A calculation shows:

$$\begin{aligned}
\Phi(cf)(p) &= \Phi\left(\int_K c(t,p)\alpha_t(f(t^{-1}p))\sqrt{\Delta_K(t)}dt\right) \\
&= \int_K \alpha_t(f(t^{-1}p)^*\alpha_t^{-1}(c(t,p)^*)\Delta_K(t))\frac{dt}{\sqrt{\Delta_K(t)}} \\
&= \int_K \alpha_t(\Phi(f)(p:t)\alpha_t^{-1}(c(t,p)^*)\Delta_K(t))\frac{dt}{\sqrt{\Delta_K(t)}} \ .
\end{aligned}$$

A further calculation shows us that

$$c^*(t,p) = \Delta_K(t^{-1})\sigma_t(c(t^{-1},p)^*) = \Delta_K(t^{-1})\alpha_t(c(t^{-1}, t^{-1}p)^*) \ .$$

In particular,

$$c^*(t^{-1}, p:t) = c^*(t^{-1}, t^{-1}p) = \Delta_K(t)\alpha_t^{-1}(c(t,p)^*) \ .$$

Therefore, $\Phi(cf) = \Phi(f) : c^*$. Similarly, $\Phi(fb) = b^* : \Phi(f)$. The map Φ is also inner product preserving:

$$\begin{aligned}
\langle \Phi(f), \Phi(g)\rangle_{E_0}(t,p) &= \frac{1}{\sqrt{\Delta_K(t)}}\int_H \beta_s(\Phi(f)(ps)^*\alpha_t(\Phi(g)(t^{-1}ps)))ds \\
&= \frac{1}{\sqrt{\Delta_K(t)}}\int_H \beta_s(f(ps)\alpha_t(g(t^{-1}ps)))ds = {}_{E_0}\langle f, g\rangle (t,p) \ .
\end{aligned}$$

Similarly, ${}_{B_0}\langle \Phi(f), \Phi(g)\rangle = \langle f, g\rangle_{B_0}$.

We now prove that our operations are well-defined, exploiting the symmetry endowed by Φ.

Lemma 4.10. *If $c \in E_0$, $b \in B_0$, and $f, g \in Z_0$, then $cf, fb \in Z_0$, ${}_{E_0}\langle f, g\rangle \in E_0$, and $\langle f, g\rangle_{B_0} \in Z_0$. Moreoever, if f_i and g_i converge to f and g respectively in the inductive limit topology for Z_0, then ${}_{E_0}\langle f_i, g_i\rangle$ converges to ${}_{E_0}\langle f, g\rangle$ in the inductive limit topology and $\langle f_i, g_i\rangle_{B_0}$ converges to $\langle f, g\rangle_{B_0}$ in the inductive limit topology.*

Proof. Since $\varphi : (t, p) \mapsto c(t, p)\alpha_t(f(t^{-1}p))$ is in $C_c(K \times P, A)$, $cf \in Z_0$. To see that this function has compact support, let $C \Subset P$ and $T \Subset P/H$ be such that $\operatorname{supp} c \subset C \times TH$. For $\varphi(t, p) \neq 0$ we need $f(t^{-1}p) \neq 0$ and $c(t, p) \neq 0$. That is, we need p such that there is some $t \in C$ for which $t^{-1}p \in \operatorname{supp} f$. Let $\Psi : K \times P \to P \times P : (t, p) \mapsto (p, t^{-1}p)$. We know that preimages of compacts are compact for Ψ. In particular, the set

$$\Psi^{-1}(C \times \operatorname{supp} f) = \{(t, p) : t \in C \text{ and } t^{-1}p \in \operatorname{supp} f\}$$

is compact. Notice that if $(t, p) \in \operatorname{supp} \varphi$, then $(t, p) \in \Psi^{-1}(C \times \operatorname{supp} f)$. This gets us the compact support. A similar calculation shows that $fb \in Z_0$.

To see that ${}_{E_0}\langle f, g\rangle \in E_0$, notice that if we can show that the map

$$\psi : (t, p) \mapsto f(p)\alpha_t(g(t^{-1}p)^*)$$

is in $C_c(K \times P, A)$, then by Lemma 4.5, the map

$$(t, p) \mapsto \int_H \beta_s(\psi(t, ps))ds = \int_H \beta_s(f(ps)\alpha_t(g(t^{-1}ps)))ds$$

is in E_0 and whence ${}_{E_0}\langle f, g\rangle \in E_0$. But that ψ is in $C_c(K \times P, A)$ is the same kind of calculation as above. Indeed, $\operatorname{supp} \psi \subset C_K \times D_f$ where $D_f = \operatorname{supp} f$ and

$$C_K = \{t \in K : D_f \cap tD_g \neq \varnothing\} = \{t : (p, t^{-1}p) \in D_f \times D_g \text{ for some } p\}\,.$$

The same calculation shows that $\langle f, g\rangle_{B_0} \in B_0$. It remains to check that ${}_{E_0}\langle f_i, g_i\rangle$ inductive limit converges to ${}_{E_0}\langle f, g\rangle$ whenever $f_i \to_{i.l.} f$ and $g_i \to_{i.l.} g$. Before we do this, notice that ${}_{E_0}\langle f, g\rangle(t, p) = 0$ if $(t, p) \notin C_K \times D_f H$. This tells us in particular that

$$\|{}_{E_0}\langle f, g\rangle\|_\infty = \sup_{(t,p)\in C_K\times D_f} \|{}_{E_0}\langle f, g\rangle(t, p)\|\,.$$

Since the action of H on P is proper, the set

$$C_H = \{s \in H : D_f \cap D_f s^{-1} \neq \varnothing\}$$

is compact. Notice that if $(t,p) \in C_K \times D_f$, then whenever $s \notin C_H$, $\psi(ps) = 0$. Therefore,

$$\begin{aligned}\|{}_{E_0}\langle f, g\rangle\|_\infty &= \sup_{(t,p)\in C_K \times D_f} \|{}_{E_0}\langle f, g\rangle(t,p)\| \\ &\leq \left(\sup_{t\in C_K} \frac{1}{\sqrt{\Delta_K(t)}}\right) \|f\|_\infty \|g\|_\infty \mu(C_H) \ .\end{aligned}$$

Finally, the identity

$${}_{E_0}\langle f_i, g_i\rangle - {}_{E_0}\langle f, g\rangle = {}_{E_0}\langle f_i - f, g_i\rangle - {}_{E_0}\langle f, g_i - g\rangle$$

and the fact that C_H and C_K are only dependent on f and g get us the result.

It will be make proving the imprimitivity theorem much easier if we have an approximate identity of a very special form.

Proposition 4.11. *There is a net $(e_m)_{m\in M}$ in E_0 for which*

1. *for any $c \in E_0$, $e_m * c \to c$ in the inductive limit topology,*
2. *for any $f \in Z_0$, $e_m f \to f$ in the inductive limit topology, and*
3. *for every index $m \in M$, there are $f_i^m \in Z_0$ such that*

$$e_m = \sum_{i=1}^{n_m} {}_{E_0}\langle f_i^m, f_i^m\rangle \ .$$

We will come back to the proof of this Proposition after proving the imprimitivity theorem.

Proof of the imprimitivity theorem. Let us first verify that the actions we have make Z_0

into a bimodule.

$$\begin{aligned}
f(b*b')(p) &= \int_H \beta_s(f(ps)(b*b')(s^{-1},ps))\sqrt{\Delta_H(s)}ds \\
&= \int_H \beta_s\left(f(ps)\int_H b(u,ps)\beta_u(b'(u^{-1}s^{-1},psu))\right)\sqrt{\Delta_H(s)}ds \\
&= \int_H\int_H \beta_s(f(ps)b(u,ps))\beta_{su}(b'((su)^{-1},p(su)))\sqrt{\Delta_H(s)}duds \\
&= \int_H\int_H \beta_s(f(ps)b(s^{-1}u,ps))\beta_u(b'(u^{-1},pu))\sqrt{\Delta_H(s)}duds
\end{aligned}$$

while

$$\begin{aligned}
(fb)b'(p) &= \int_H \beta_u(fb(pu)b'(u^{-1},pu))\sqrt{\Delta_H(u)}du \\
&= \int_H \beta_u\left(\int_H \beta_s(f(pus)b(s^{-1},pus))\sqrt{\Delta_H(s)}b'(u^{-1},pu)ds\right)\sqrt{\Delta_H(u)}du \\
&= \int_H\int_H \beta_{us}(f(p(us)b(s^{-1},p(us)))\beta_r(b'(u^{-1},pu))\sqrt{\Delta_H(us)}dsdu \\
&= \int_H\int_H \beta_s(f(ps)b((u^{-1}s)^{-1},ps))\beta_u(b'(u^{-1},pu))\sqrt{\Delta_H(s)}dsdu \\
&= f(b*b')(p)\ .
\end{aligned}$$

To verify that $E_0 \curvearrowright Z_0$, notice for any $c, c' \in E_0$,

$$\begin{aligned}
\Phi((c*c')f) &= \Phi(f):(c*c')^* = (\Phi(f):c'):c \\
&= \Phi(c'f):c = \Phi(c(c'f))\ .
\end{aligned}$$

Finally, we need to check that for any $f \in Z_0$, $b \in B_0$, and $c \in E_0$, $(cf)b = c(fb)$. Again, a calculation shows

$$\begin{aligned}
(cf)b(p) &= \int_H \beta_s((cf)(ps)b(s^{-1},ps))\frac{ds}{\sqrt{\Delta_H(s)}} \\
&= \int_H \beta_s\left(\int_K c(t,ps)\alpha_t(f(t^{-1}ps)b(s^{-1},ps))\sqrt{\Delta_K(t)}dt\right)\frac{ds}{\sqrt{\Delta_H(s)}} \\
&= \int_H\int_K \beta_s(c(t,ps))\beta_s\alpha_t(f(t^{-1}ps))\beta_s\alpha_t(b(s^{-1},ps))\frac{\sqrt{\Delta_K(t)}}{\sqrt{\Delta_H(s)}}dtds \\
&= \int_H\int_K c(t,p)\beta_s\alpha_t(f(t^{-1}ps))\beta_s\alpha_t(b(s^{-1},ps))\frac{\sqrt{\Delta_K(t)}}{\sqrt{\Delta_H(s)}}dtds
\end{aligned}$$

where in the last line we use the fact that because $c \in E_0$, $\beta_s(c(t, ps)) = c(t, p)$ while

$$\begin{aligned}
c(fb)(p) &= \int_K c(t,p)\alpha_t(fb(t^{-1}p))\sqrt{\Delta_K(t)}dt \\
&= \int_K \int_H c(t,p)\alpha_t\beta_s(f(t^{-1}ps))\beta_s(b(s^{-1}, t^{-1}ps))\frac{\sqrt{\Delta_K(t)}}{\sqrt{\Delta_H(s)}}dsdt \\
&= \int_K \int_H c(t,p)\alpha_t\beta_s(f(t^{-1}ps))\beta_s\alpha_t(b(s^{-1}, ps))\frac{\sqrt{\Delta_K(t)}}{\sqrt{\Delta_H(s)}}dsdt \\
&= (cf)b(p)
\end{aligned}$$

where in the penultimate line we use the fact that $\alpha_t(b(s^{-1}, ps)) = b(s^{-1}, t^{-1}ps)$.

We now check that Z_0 is an (E_0, B_0) preimprimitivity bimodule. This boils down to checking the following four conditions:

1. Z_0 is a preinner product E_0-module and a preinner product B_0-module.
2. ${}_{E_0}\langle Z_0, Z_0\rangle$ and $\langle Z_0, Z_0\rangle_{B_0}$ span dense ideals.
3. We have the identities
$$\begin{aligned}
{}_{E_0}\langle fb, fb\rangle &\le \|b\|^2 {}_{E_0}\langle f, f\rangle \text{ and} \\
\langle cf, cf\rangle_{B_0} &\le \|c\|^2 \langle f, f\rangle_{B_0} \ .
\end{aligned}$$
4. For any $f, g, h \in Z_0$, ${}_{E_0}\langle f, g\rangle h = f\langle g, h\rangle_{B_0}$.

The last condition is a little computation:

$$\begin{aligned}
{}_{E_0}\langle f, g\rangle h(p) &= \int_K {}_{E_0}\langle f, g\rangle (t,p)\alpha_t(h(t^{-1}p))\sqrt{\Delta_K(t)}dt \\
&= \int_K \int_H \beta_s(f(ps)\alpha_t(g(t^{-1}ps)^*))\alpha_t(h(t^{-1}p))dsdt
\end{aligned}$$

while

$$\begin{aligned}
f\langle gh\rangle_{B_0}(p) &= \int_H \beta_s(f(ps)\langle g, h\rangle_{B_0}(s^{-1}, ps))\frac{ds}{\sqrt{\Delta_H(s)}} \\
&= \int_H \beta_s(f(ps)) \int_K \beta_s\alpha_t(g(t^{-1}ps)^*\beta_s^{-1}(h(t^{-1}p)))dtds \\
&= \int_H \int_K \beta_s(f(ps)\alpha_t(g(t^{-1}ps)^*))\alpha_t(h(t^{-1}p))dtds \\
&= {}_{E_0}\langle f, g\rangle h(p) \ .
\end{aligned}$$

For the first condition, it suffices to check the following four conditions:

1. ${}_{E_0}\langle , \rangle$ is linear in the first variable.
2. ${}_{E_0}\langle cf, g\rangle = c * {}_{E_0}\langle f, g\rangle$.
3. ${}_{E_0}\langle f, g\rangle^* = {}_{E_0}\langle g, f\rangle$.
4. ${}_{E_0}\langle f, f\rangle \geq 0$ in $\beta \uparrow \rtimes_\sigma K$.

The first condition is easy. For the second condition,

$$\begin{aligned}
&\sqrt{\Delta_K(t)}{}_{E_0}\langle cf, g\rangle(t,p) = \int_H \beta_s(cf(ps)\alpha_t(g(t^{-1}ps)^*))ds \\
&= \int_H\int_K \beta_s(c(u,ps))\beta_s\alpha_u(f(u^{-1}ps))\sqrt{\Delta_K(u)}\beta_s\alpha_t(g(t^{-1}ps)^*)duds \\
&= \int_H\int_K c(u,p)\beta_s\alpha_u(f(u^{-1}ps))\beta_s\alpha_t(g(t^{-1}ps)^*)\sqrt{\Delta_K(u)}duds
\end{aligned}$$

while

$$\begin{aligned}
&c * {}_{E_0}\langle f, g\rangle(t,p)\sqrt{\Delta_K(t)} = \int_K c(u,p)\tau_u({}_{E_0}\langle f, g\rangle(u^{-1}t, p))du \\
&= \int_K c(u,p)\alpha_u({}_{E_0}\langle f, g\rangle(u^{-1}t, u^{-1}p))du \\
&= \int_K\int_H c(u,p)\alpha_u\beta_s(f(u^{-1}ps)\alpha_{u^{-1}t}(g(t^{-1}ps)^*))\frac{\sqrt{\Delta_K(u)}}{\sqrt{\Delta_K(t)}}dsdu \\
&= \int_K\int_H c(u,p)\alpha_u\beta_s(f(u^{-1}ps))\beta_s\alpha_t(g(t^{-1}ps)^*)\frac{\sqrt{\Delta_K(u)}}{\sqrt{\Delta_K(t)}}dsdu \\
&= \sqrt{\Delta_K(t)}{}_{E_0}\langle cf, g\rangle(t,p)\ .
\end{aligned}$$

For the third condition, we have

$$\begin{aligned}
{}_{E_0}\langle f, g\rangle^*(t,p) &= \Delta_K(t^{-1})\alpha_t({}_{E_0}\langle f, g\rangle(t^{-1}, t^{-1}p)^*) \\
&= \sqrt{\Delta(t^{-1})}\left[\int_H \alpha_t\beta_s(f(t^{-1}ps)\alpha_t^{-1}(g(ps)^*))ds\right]^* \\
&= \sqrt{\Delta_K(t^{-1})}\int_H \beta_s(g(ps)^*\alpha_t(f(t^{-1}ps)^*))ds \\
&= {}_{E_0}\langle g, f\rangle(t,p)\ .
\end{aligned}$$

For the fourth condition, we use our assumed Proposition. For any $f \in Z_0$, notice

$$_{E_0}\langle f, f\rangle = \langle \Phi(f), \Phi(f)\rangle_{B_0}$$

and so if we can show that $\langle f, f\rangle_{B_0} \geq 0$ for any $f \in Z_0$ then we would get the corresponding result for E_0. Notice that as $e_m f \rightarrow_{i.l} f$ that $\langle e_m f, f\rangle_{B_0} \rightarrow_{i.l.} \langle f, f\rangle_{B_0}$. Therefore,

$$\begin{aligned}
\langle f, f\rangle_{B_0} &= \lim_m \langle e_m f, f\rangle_{B_0} \\
&= \lim_m \sum_i \left\langle \langle f_i^m, f_i^m\rangle_{E_0} f, f\right\rangle_{B_0} \\
&= \lim_m \sum_i \left\langle f_i^m \langle f_i^m, f\rangle_{B_0}, f\right\rangle_{B_0} \\
&= \lim_m \sum_i \langle f_i^m, f\rangle_{B_0}^* * \langle f_i^m, f\rangle_{B_0} \geq 0 \ .
\end{aligned}$$

Our second imprimitivity condition follows using the Proposition as well: for any $c \in E_0$, since $e_m * c \rightarrow_{i.l} c$, it suffices to show that $e_m * c$ is in the span of $_{E_0}\langle Z_0, Z_0\rangle$. However,

$$e_m * c = \sum_i {}_{E_0}\langle f_i^m, c^* f_i^m\rangle \ .$$

and the right hand side is certainly a sum of elements of $_{E_0}\langle Z_0, Z_0\rangle$. The corresponding density result for $\langle Z_0, Z_0\rangle_{B_0}$ follows since $_{B_0}\langle \Phi(Z_0), \Phi(Z_0)\rangle = \langle Z_0, Z_0\rangle_{B_0}$ and the left hand side is dense in B_0 by the same argument.

It remains to check that the actions are bounded. Using the map Φ again, we will be content to show that

$$\langle cf, cf\rangle_{B_0} \leq \|c\|^2 \langle f, f\rangle_{B_0} \ .$$

Let $Z = \overline{Z_0}$ where we take the closure as a Hilbert $\alpha \uparrow \rtimes_\tau H$-module. We want a covariant pair

$$M : \beta \uparrow \rightarrow \mathcal{L}(Z), \ v : K \rightarrow U\mathcal{L}(Z)$$

so that

$$M \rtimes v(c)(f) = cf$$

for any $c \in E_0$ and any $f \in Z_0$. If we can do this, then we are done since

$$\begin{aligned}\langle cf, cf\rangle_{B_0} &= \langle M \rtimes v(c)(f), M \rtimes v(c)(f)\rangle_{B_0} \leq \|M \rtimes v(c)\|^2 \langle f, f\rangle_{B_0} \\ &\leq \|c\|^2 \langle f, f\rangle_{B_0} \ .\end{aligned}$$

Notice for any $p \in P$,

$$M \rtimes v(c)(f)(p) = \int_K M(c(t,\cdot))v_t(f)(p)dt$$

while

$$cf(p) = \int_K c(t,p)\alpha_t(f(t^{-1}p))\sqrt{\Delta_K(t)}dt \ .$$

We might suspect that

$$v_t : Z_0 \to Z_0 : v_t(f)(p) = \alpha_t(f(t^{-1}p))\sqrt{\Delta_K(t)} \ .$$

For the definition of M, let us first define

$$E = \{\varphi \in C_b(P, M(A)) : \varphi(ps) = \overline{\beta}_s^{-1}(\varphi(p))\} \ ,$$

a closed unital *-subalgebra of $C_b(P, M(A))$ containing $\beta\uparrow$. Then we will define $M : E \to \mathcal{L}(Z)$ by

$$M(\varphi)(f)(p) = \varphi(p)f(p) \ .$$

If (M, v) is indeed a covariant pair (when M is restricted to $\beta\uparrow$), we will be done. Let us start by showing M is a non-degenerate *-morphism. To see that $M(\varphi)$ is an adjointable in Z_0, a calculation shows:

$$\begin{aligned}\sqrt{\Delta_H(s)}\langle M(\varphi)f, g\rangle_{B_0}(s,p) &= \int_K \alpha_t(M(\varphi)(f)(t^{-1}p)^*\beta_s(g(t^{-1}ps)))dt \\ &= \int_K \alpha_t(f(t^{-1}p)^*\varphi(t^{-1}p)^*\beta_s(g(t^{-1}ps)))dt \\ &= \int_K \alpha_t(f(t^{-1}p)^*\beta_s(\varphi^*(t^{-1}ps)g(t^{-1}ps)))dt \\ &= \sqrt{\Delta_H(s)}\langle f, M(\varphi^*)g\rangle_{B_0} \ .\end{aligned}$$

As well, since $\|\varphi\|^2 - \varphi^*\varphi \geq 0$ in E, we can set $\|\varphi\|^2 - \varphi^*\varphi = \psi^*\psi$ for some $\psi \in E$. Notice then that

$$\begin{aligned}\|\varphi\|^2 \langle f, f\rangle_{B_0} - \langle M(\varphi)f, M(\varphi)f\rangle_{B_0} &= \left\langle M(\|\varphi\|^2 - \varphi^*\varphi)f, f\right\rangle_{B_0} \\ &= \langle M(\psi)f, M(\psi)f\rangle_{B_0} \geq 0 \;.\end{aligned}$$

Therefore, $M(\varphi)$ is bounded and hence extends to an adjointable on Z. It follows that M is a *-morphism (we already checked the adjoint condition and the other conditions are easier).

For non-degeneracy of M, notice that if we apply our assumed Proposition in the case when $K = \{e\}$, then we can find a net (φ_i) in $\beta\uparrow_c$ for which, given any $f \in Z_0$, $\varphi_i \circ f \to_{i.l} f$ where $\circ$ is the action $E_0 = \beta\uparrow_c \curvearrowright Z_0$ in this case. Notice that $\varphi_i \circ f(p) = \varphi_i(p)f(p) = M(\varphi_i)f(p)$. This tells us that $(M(\varphi_i))$ is strong continuous in Z_0. Therefore, M is non-degenerate.

We now want to show that v is a strong continuous group morphism. First let us check that $v_t \in U\mathcal{L}(Z)$. For any $f, g \in Z_0$,

$$\begin{aligned}\sqrt{\Delta_H(s)}\,\langle v_t(f), v_t(g)\rangle_{B_0}(s,p) &= \int_K \alpha_u(v_t(f)(u^{-1}s)^*\beta_s(v_t(g)(u^{-1}ps)))du \\ &= \int_K \alpha_{ut}(f(t^{-1}u^{-1}s)^*\beta_s(g(t^{-1}u^{-1}ps)))\Delta_K(t)du \\ &= \sqrt{\Delta_H(s)}\,\langle f, g\rangle_{B_0}\end{aligned}$$

so indeed, v_t extends to a unitary on $\mathcal{L}(Z)$. It is clear that v is a group morphism.

The only thing left to check is that v is strong continuous. We fix an $f \in Z_0$. We want to show that as $k \to e$, $v_k(f) \to_{i.l} f$. Let W be a precompact neighbourhood of e in K and let D_f be the support of f. Notice that eventually supp $v_k(f) \subset W \cdot D_f$ –a compact set. So we only need to verify that $v_k(f) \to f$ uniformly. An estimate gets us

$$\|v_k(f) - f\| \leq |\sqrt{\Delta_K(k)} - 1|\|f\|_\infty + \|lt_k(f) - f\|_\infty + \|\alpha_k(f) - f\|_\infty \;.$$

The first term goes to zero because Δ_K is a continuous group morphism, the middle term goes to zero because f is uniformly continuous and the last term goes to zero because α is strong continuous.

With convergence in the inductive limit topology, we see

$$\|v_k(f) - f\|^2_{B_0} = \|2\langle f, f\rangle_{B_0} - \langle v_k(f), f\rangle_{B_0} - \langle f, v_k(f)\rangle_{B_0}\|$$

and each term goes to zero as $k \to e$ so we get strong continuity.

Covariance is a little computation: for $\varphi \in \beta\uparrow$, $f \in Z_0$,

$$\begin{aligned} v_t(M(\varphi)(f))(p) &= \alpha_t(M(\varphi)(f)(t^{-1}p))\sqrt{\Delta_K(t)} \\ &= \alpha_t(\varphi(t^{-1}p)\alpha_t(f(t^{-1}p))\sqrt{\Delta_K(t)} \\ &= \sigma_t(\varphi)(p)v_t(f)(p) \\ &= M(\sigma_t(\varphi))v_t(f)(p) \ . \end{aligned}$$

☕

We now come back to Proposition 4.11.

The Proposition will be easier to prove once we have the following related result:

Proposition 4.12. *Let $(b_l)_{l\in L}$ be an approximate identity for $\beta\uparrow_H^P$. Consider the set*

$$M = \{(T, U, l, \epsilon) : T \Subset P/H, e \in U \subset_{open} K, \overline{U} \Subset K, l \in L, \text{ and } \epsilon > 0\}$$

ordered by decreasing in U and ϵ and increasing in T and l. Whenever there is a net $e = e_{(T,U,l,\epsilon)} \in E_0$ for which

1. *$e(t,p) = 0$ whenever $t \notin U$,*
2. *$\int_K \|e(t,p)\| dt \le 4$ if $pH \in T$, and*
3. *$\left\| \int_K e(t,p)dt - b_l(p) \right\| < \epsilon$ if $pH \in T$,*

the net $(e_{(T,U,l,\epsilon)})$ satisfies conditions 1 and 2 in Proposition 4.11.

The proof of this Proposition will want two Lemmas.

Lemma 4.13. *Suppose $z \in Z_0 = C_c(P, A)$. Fix an approximate identity $(a_j)_{j\in J}$ in A, a compact $C_H \subset H$, and $\epsilon > 0$. Then eventually for all $s \in C_H$ and all $p \in P$,*

$$\|\beta_s(a_j)z(p) - z(p)\| < \epsilon \ .$$

Proof. The proof is by contradiction. ☕

Lemma 4.14. *Suppose that $(b_l)_{l\in L}$ is an approximate identity for $\beta\uparrow_H^P$. Given any $z \in Z_0$ and $c \in E_0$, define $b_l z(p) = b_l(p)z(p)$ and $b_l c(t,p) = b_l(p)c(t,p)$. The following hold:*

a. *For every* $z \in Z_0$*,* $b_l z \to z$ *in the inductive limit topology in* Z_0*.*

b. *For every* $c \in E_0$*,* $b_l c \to c$ *in the inductive limit topology in* $\beta \uparrow \rtimes_\sigma K$*.*

Proof. Notice that supp $b_l c \subset$ supp c and supp $b_l z \subset$ supp z so it suffices to show uniform convergence. Showing uniform convergence of $b_l c$ is easier so let us start with that. Given $\epsilon > 0$, we want to show that eventually we have

$$\|b_l c(t, \cdot) - c(t, \cdot)\|_\infty < \epsilon$$

for every $t \in K$. To this end, we suppose not. Then there are $t_l \in K$ for which, after taking a subnet,

$$\|b_l c(t_l, \cdot) - c(t_l, \cdot)\| \geq \epsilon$$

for every l. By taking another subnet, since $c(t_l, \cdot) \neq 0$, we can assume that $t_l \to t$. We then have

$$\|b_l c(t_l, \cdot) - c(t_l, \cdot)\| \leq \|b_l(c(t_l, \cdot) - c(t, \cdot))\| + \|b_l c(t, \cdot) - c(t, \cdot)\| + \|c(t_l, \cdot) - c(t, \cdot)\| \ .$$

Since $\|b_l\| \leq 1$, the right hand side goes to zero and we reach a contradiction.

To get uniform convergence of $(b_l z)$ to z, I claim that there is a $w \in C_c^+(P)$ such that

$$\int_H w(ps)ds = 1$$

whenever $p \in$ supp z. Since supp $z =$ supp $\|z\|$, let us set $f = \|z\| \in C_c^+(P)$. Let us first define

$$F : pH \to \int_H f(ps)ds \ .$$

This is an element of $C_c(P/H)$. Since supp z is compact and whenever $p \in$ supp z, $F(pH) > 0$, we let

$$m := \inf_{p \in \text{supp } z} F(pH) > 0 \ .$$

Consider the open neighbourhood $U = \{pH : F(pH) > m/2\}$ of (supp z)H in P/H. We let $Q \in C_c(P/H)$ be such that $Q|_{(\text{supp } z)H} = 1$ and $Q = 0$ outside of U. We can now define

$$w : p \mapsto f(p)Q(pH)/F(pH) \ .$$

By construction w is positive and has compact support. Integrating we see for any $p \in$ supp z,

$$\int_H w(ps)ds = \frac{Q(pH)}{F(pH)} \int_H f(ps)ds = 1 \ .$$

Let $C_H = \{s \in H : p \in \text{supp } z, ps \in \text{supp } w\}$. This is a compact set by the fact that H acts on P properly. Let $\epsilon > 0$. By the previous Lemma, we have some $j(\epsilon) \in J$ and some $a \in A$ for which given any $j \geq j(\epsilon)$ and given any $s \in C_H$ and any $p \in P$,

$$\|\beta_s(a)z(p) - z(p)\| < \epsilon \ .$$

Let us define

$$b : p \mapsto \int_H w(ps)\beta_s(a)ds \ .$$

It is easy to see that this is an element of $\beta \uparrow_H^P$. Our first estimate shows us for any p,

$$\|b(p)z(p) - z(p)\| = \left\| \int_H w(ps)(\beta_s(a)z(p) - z(p))ds \right\| \leq \epsilon \ .$$

Since b_l is an approximate identity, we eventually have $\|b_l b - b\| < \epsilon$. Therefore,

$$\begin{aligned} \|b_l z(p) - z(p)\| \leq & \|b_l(p)(z(p) - b(p)z(p))\| + \|(b_l b - b)(p)z(p)\| \\ & + \|b(p)z(p) - z(p)\| \\ \leq & \epsilon(\|z\|_\infty + 2) \ . \end{aligned}$$

Since z is fixed, we get uniform convergence. ☕

With these Lemmas, we can prove our Proposition 4.12.

Proof of Proposition 4.12. Let us first show that $(e_m)_{m \in M}$ acts like an approximate identity for $E_0 \curvearrowright Z_0$. That is, if $z \in Z_0$, we want to show that there is a fixed constant L for which given any $\epsilon > 0$, we have the inequality

$$\|e_m z(p) - z(p)\| < L\epsilon$$

for all $p \in P$ and $m \geq (T(\epsilon), U(\epsilon), l(\epsilon), \delta(\epsilon))$. To find such a quadruple, let U_0 be any precompact neighbourhood of e in K. Since U_0supp z is precompact, we can let $T(\epsilon)$ be any compact set containing $(U_0\text{supp } z)H$.

Let us first note that there is some neighbourhood $U(\epsilon) \subset U_0$ for which

$$\|\alpha_t(z(t^{-1}p))\sqrt{\Delta_K(t)} - z(p)\| < \epsilon$$

for all $p \in P$ and $t \in U(\epsilon)$. The proof of this is as usual by contradiction.

We set $\delta(\epsilon) = \epsilon$ and use the previous Lemma to get some $l(\epsilon)$ such that for all $l \geq l(\epsilon)$,

$$\|b_l(p)z(p) - z(p)\| < \epsilon$$

for all $p \in P$. Let $m = (T, U, l, \epsilon) \geq (T(\epsilon), U(\epsilon), l(\epsilon), \epsilon)$. Notice that if $p \notin U\text{supp } z$, then $e_m z(p) - z(p) = 0$. It then suffices to only consider $p \in U\text{supp } z$. Since $U \subset U(\epsilon)$, $pH \in T(\epsilon) \subset T$. We then have

$$\begin{aligned}
\|e_m z(p) - z(p)\| \leq & \left\| \int_K e_m(t,p)(\alpha_t(z(t^{-1}p))\sqrt{\Delta_K(t)} - z(p))dt \right\| \\
& + \left\| z(p) \left(\int_K e_m(t,p)dt - b_l(p) \right) \right\| \\
& + \|b_l(p)z(p) - z(p)\| \\
\leq & 4\epsilon + \|z\|_\infty \epsilon + \epsilon
\end{aligned}$$

and this gets what we require when $L = 4 + \|z\|_\infty + 1$.

The proof that (e_m) is a left approximate identity for E_0 in the inductive limit topology is almost verbatim the above proof, where in the end we get the bound

$$\begin{aligned}
\|e_m * c(v,p) - c(v,p)\| \leq & \left\| \int_K e_m(t,p)(\alpha_t(c(t^{-1}v, t^{-1}p)) - c(v,p))dt \right\| \\
& + \left\| c(v,p) \left(\int_K e_m(t,p)dt - b_l(p) \right) \right\| \\
& + \|b_l(p)c(v,p) - c(v,p)\| \\
\leq & 4\epsilon + \|c\|_\infty \epsilon + \epsilon
\end{aligned}$$

and the quadruple $(T(\epsilon), U(\epsilon), l(\epsilon), \epsilon)$ is given by taking $T(\epsilon)$ to be any compact set containing $(U_0\text{supp } c)H$, $U(\epsilon) \subset U_0$ is some neighbourhood for which

$$\|\alpha_t(c(t^{-1}v, t^{-1}p)) - c(v,p)\| < \varepsilon$$

for all $p \in P$, $v \in K$, and $t \in U(\epsilon)$, and $l(\epsilon)$ is such that for all $l \geq l(\epsilon)$,

$$\|b_l(p)c(v,p) - c(v,p)\| < \epsilon$$

for all $p \in P$ and $v \in K$. ☕

Before we prove Proposition 4.11, we want three more results:

corollary 4.15. *Let $(\widehat{e}_m)_{m\in M}$ be a net as in Proposition 4.12. Given any $m \in M$, define*

$$e_m(t,p) = \frac{1}{\sqrt{\Delta_K(t)}}\widehat{e}_m(t,p)\ .$$

The net $(e_m)_{m\in M}$ satisfies conditions 1 and 2 in Proposition 4.11.

Proof. Suppose that U_0 is a neighbourhood of e in K for which $|1 - 1/\sqrt{\Delta_K(t)}| < \epsilon$ for any $t \in U_0$. Whenever $m = (T, U, l, \epsilon)$ has $U \subset U_0$, we have

$$\|(\widehat{e}_m - e_m) * c\| \leq 4\epsilon\|c\|_\infty \text{ and } \|(\widehat{e}_m - e_m)z\|_\infty \leq 4\epsilon\|z\|_\infty\ .$$

Since $\widehat{e}_m z \to z$ and $\widehat{e}_m * c \to c$, we get the result for e_m as well. ☕

Lemma 4.16. *If $K \curvearrowright P$ freely and properly and if N is a neighbourhood of e in K then for every $p \in P$ there is a neighbourhood U of p for which*

$$\{t \in K : tU \cap U \neq \varnothing\} \subset N\ .$$

Proof. The proof is by contradiction on the freeness of $K \curvearrowright P$. ☕

Lemma 4.17. *Let $G \curvearrowright_{\mathit{free}} P$ be a proper action and let $f \in C_c^+(P)$. Given any $\epsilon > 0$, there is some $g \in C_c^+(P)$ with supp $g \subset$ supp f and*

$$\left| f(x) - g(x)\int_G g(s^{-1}x)ds \right| < \epsilon\ .$$

Proof. Let

$$F : Gx \mapsto \int_G f(s^{-1}x)ds\ .$$

I first claim that this function is in $C_c^+(G\backslash P)$. The function F has compact support because $\operatorname{supp} F \subset G \cdot \operatorname{supp} f$. For continuity of F, we suppose that some net (x_i) in P converges to a point x. Let $\eta > 0$. Because $G \curvearrowright P$ freely and properly, it suffices to show that for any $s \in G$, we have

$$|f(s^{-1}x) - f(s^{-1}x_i)| < \eta$$

eventually. This inequality holds because otherwise, we may take a subnet and find some $s_i \in G$ for which

$$|f(s_i^{-1}x) - f(s_i^{-1}x_i)| \geq \eta \ .$$

In this case, either $s_i^{-1}x \in \operatorname{supp} f$ or $s_i^{-1}x_i \in \operatorname{supp} f$. Define $w_i := x$ if $s_i^{-1}x \in \operatorname{supp} f$ and $w_i := x_i$ otherwise. Notice that as $x_i \to x$, we must also have $w_i \to x$. Furthermore, we know that we always have $s_i^{-1}w_i \in \operatorname{supp} f$. Take a subnet of w_i so that $s_i^{-1}w_i \to y$ for some $y \in P$. Since $G \curvearrowright P$ properly, we may take another subnet to make s_i converge to some point $s \in G$. This is a contradiction.

Notice that we really want to find some g for which we have

$$\left| f(x) \int_G f(s^{-1}x)ds - g(x)\sqrt{F(Gx)} \int_G g(s^{-1}x)\sqrt{F(Gx)}ds \right| < \epsilon F(Gx) \ .$$

In other words, we would like to choose a function $h(x)$ so that

$$\left| f(x) \int_G f(s^{-1}x)ds - h(x) \int_G h(s^{-1}x)ds \right| < \epsilon F(Gx)$$

for all $x \in P$. Setting $g(x) = h(x)/\sqrt{F(Gx)}$ will then do. To this end, let

$$C := \{x \in P : f(x) \geq \epsilon\} \ .$$

This is a compact set with the property that for any $x \in C$, $F(Gx) > 0$. Since GC is compact in $G\backslash P$, we can set

$$m := \min_{x \in C} F(Gx) \ ,$$

which is a positive number. Let $U := \{Gx \in G\backslash P : F(Gx) > m/2\}$ –an open neighbourhood of GC . If we set $Q \in C_c^+(G\backslash P)$ to be a bump function with support in $\overline{U}$, $Q \leq 1$, and $Q|_{GC} = 1$, then $h(x) = f(x)Q(Gx)$ does the job. ☕

We can now prove Proposition 4.11.

Proof of Proposition 4.11. Suppose that $(b_l)_{l \in L}$ is an approximate identity for $\beta \uparrow_H^P$. Fix a precompact neighbourhood U of e, $T \Subset P/H$, and $\epsilon > 0$. We want to find an $e = e_{(T,U,l,\epsilon)}$ which is the sum of inner products and for which $\tilde{e} : (t,p) \mapsto \sqrt{\Delta_K(t)}e(t,p)$ satisfies the conditions of Proposition 4.12. Let $\delta = \min(\epsilon/3, 1)$.

Let $D \Subset P$ be a lift of T to P. Let $C \supset D$ be a compact neighbourhood of D. Suppose that $\varphi \in C_c^+(P)$ have $\varphi|_C = 1$. Since $A^+ \to A : a \mapsto \sqrt{a}$ is continuous by the continuous functional calculus, the map

$$z : P \to A : p \mapsto \varphi(p)\sqrt{b_l(p)}$$

is an element of Z_0 with $z|_C = \sqrt{b_l}|_C$.

By contradiction, we can prove that there is a neighbourhood $W \subset U$ of e for which

$$\|z(p)\alpha_t(z(t^{-1}p)) - b_l(p)\| < \delta$$

for all $t \in W$ and all $p \in C$.

By Lemma 4.16, every point in D has a neighbourhood $V \subset C$ for which

$$\{t \in K : tV \cap V \neq \varnothing\} \subset W \ .$$

Let $V_1, \ldots, V_n$ be such neighbourhoods that cover D. Let $h_1, \ldots, h_n \in C_c^+(P)$ be an associated partition of unity. It turns out we want a smoother collection of functions to represent our partition of unity. To this end, let us define

$$h : P \to \mathbf{C} : p \mapsto \sum_{i \le n} \int_H h_i(ps)ds \ .$$

Notice that h is constant on cosets and that $h(p) > 0$ if $p \in D$. Define

$$m := \inf_{p \in D} h(p) = \inf_{p \in DH} h(p) \ .$$

Since D is compact, $m > 0$. Define $G : p \mapsto \max(h(p), m/2)$ –this is a continuous function which is never zero. In particular, the map

$$k_i : p \mapsto h_i(p)/G(p)$$

is an element of $C_c^+(P)$ with supp $k_i \subset V_i$ for all i. Notice as well that

$$\sum_{i \le n} \int_H k_i(ps)ds \begin{cases} = 1 & p \in DH \\ \le 1 & \text{otherwise} \end{cases} \ .$$

Since $H \curvearrowright P$ properly, there is a compact neighbourhood

$$C_H \supset \{s \in H : Cs \cap C \neq \varnothing\} \ .$$

The previous Lemma tells us there are $g_i \in C_c^+(P)$ with supp $g_i \subset V_i$ for which

$$\left| k_i(p) - g_i(p) \int_K g_i(t^{-1}p)dt \right| < \frac{\delta}{n\mu_H(C_H)} \ .$$

By definition of C_H and since the supports of the g_i and z_i are contained in C, for any $p \in C$,

$$\left| \int_H k_i(ps)ds - \int_H \int_K g_i(ps)g_i(t^{-1}ps)dtds \right| < \frac{\delta}{n} \ .$$

By left-invariance, this inequality holds for all $p \in CH$. If $p \notin CH$, the g_i and z_i will all vanish so the above inequality is trivially true. Let

$$F : K \times P \to \mathbf{C} : (t,p) \mapsto \sum_{i \leq n} g_i(p)g_i(t^{-1}p) \ .$$

Notice that $F(t,p) = 0$ if $t \notin W$ or $p \notin C$ by our choice of V_i. Furthermore, we get the inequality

$$\left| \sum_{i \leq n} \int_H k_i(ps)ds - \int_H \int_K F(t,ps)dtds \right| < \delta \ .$$

This, and the fact that $\delta \leq 1$, gets us the following two inequalities:

$$\int_H \int_K F(t,ps)dtds \leq 2 \text{ for all } p \in P$$

$$\left| \int_H \int_K F(t,ps)dtds - 1 \right| < \delta \text{ for all } p \in DH \ .$$

We are now ready to construct our $e := e_{(T,U,l,\epsilon)}$. Define

$$f_i : p \mapsto g_i(p)z(p) \ .$$

A calculation shows us

$$\begin{aligned} e(t,p) &= \sum_{i \leq n} {}_{E_0}\langle f_i, f_i \rangle (t,p) \\ &= \frac{1}{\sqrt{\Delta_K(t)}} \sum_{i \leq n} \int_H \beta_s(f_i(ps)\alpha_t(f_i(t^{-1}ps)^*))ds \\ &= \frac{1}{\sqrt{\Delta_K(t)}} \sum_{i \leq n} \int_H F(t,ps)\beta_s(\theta(t,ps))ds \end{aligned}$$

where $\theta(t,p) = z(p)\alpha_t(z(t^{-1}p))$. Since we want estimates on integrals of $\tilde{e}(t,p)$, we shall want the following estimates: $\theta(e,p) = b_l(p)$ whenever $p \in C$. Whenever $t \in W$ and $p \in C$,

$$\|\theta(t,p) - \theta(e,p)\| = \|\theta(t,p) - b_l(p)\| < \delta \ ,$$

and because $\delta \leq 1$ and b_l is a contraction,

$$\|\theta(t,p)\| \leq 2$$

whenever $t \in W$ and $p \in C$.

We now check the three properties required of $\tilde{e}$. That $\tilde{e}(t,p) = 0$ if $t \notin U$ follows because $F(t,ps) = 0$ if $t \notin W$ and $W \subset U$. Two calculations show us that whenever $pH \in T = DH$,

$$\begin{aligned}
\int_K \|\tilde{e}(t,p)\|dt &\leq \sum_{i\leq n} \int_K \int_H F(t,ps)\|\beta_s(\theta(t,ps))\|dsdt \\
&\leq 2\sum_{i\leq n} \int_K \int_H F(t,ps)dsdt \\
&\leq 4 \ .
\end{aligned}$$

Next, using the fact that $\beta_s(b_l(p)) = b_l(ps)$,

$$\begin{aligned}
\left\| \int_K \tilde{e}(t,p)dt - b_l(p) \right\| &= \left\| \sum_{i\leq n} \left[\int_K \int_H F(t,ps)\beta_s(\theta(t,ps))dsdt \right] - b_l(p) \right\| \\
&\leq \left\| \sum_{i\leq n} \int_K \int_H F(t,ps)(\beta_s(\theta(t,ps)) - b_l(ps))dsdt \right\| \\
&\quad + \left| \sum_{i\leq n} \int_K \int_H F(t,ps)dsdt - 1 \right| \|b_l(p)\| \ .
\end{aligned}$$

The first term is bounded by 2δ and the second term is bounded by δ. Since $3\delta \leq \epsilon$, we get the result. ☕

Remark 4.18. There are imprimitivity theorems for graph C^*-algebras as well (see [3] and [4]). As of [4], it is unknown whether these imprimitivity theorems can be derived from the imprimitivity theorems provided in this section.

Chapter 5

Two applications

5.1 Green's imprimitivity theorem

If we have the trivial actions

$$1 : H \to \operatorname{Aut} \mathbf{C}\,, 1 : K \to \operatorname{Aut} \mathbf{C}\,,$$

then with $K, H \curvearrowright P$ from the left and the right respectively, the induced algebras are

$$\begin{aligned}
1 \uparrow_H^P &= \{P \xrightarrow{f} \mathbf{C} : f(hp) = f(p) \text{ for all } h \in H, p \in P, \|f\| \in C_0(P/H)\} \\
&= C_0(P/H) \text{ and} \\
1 \uparrow_K^P &= C_0(K\backslash P)\ .
\end{aligned}$$

The induced dynamical systems are then $(1 \uparrow_H^P, K, \mathrm{lt})$ and $(1 \uparrow_K^P, H, \mathrm{rt})$. The imprimitivity theorem tells us that we have a Morita equivalence

$$C_0(P/H) \rtimes_{\mathrm{lt}} K \sim_M C_0(K\backslash P) \rtimes_{\mathrm{rt}} H$$

given by a preimprimitivty bimodule $Z_0 = C_c(P)$ over the dense subalgebras

$$\begin{aligned}
B_0 &:= C_{cc}(K, 1 \uparrow_c) = C_c(K \times P/H) \text{ and} \\
E_0 &:= C_{cc}(H, 1 \uparrow_c) = C_c(H \times K\backslash P)\ .
\end{aligned}$$

The actions and inner products are then given by

$$\begin{aligned} cf(p) &= \int_K c(t,pH)f(t^{-1}p)\sqrt{\Delta_K(t)}dt \\ fb(p) &= \int_H f(ps)b(s^{-1},Kps)\frac{ds}{\sqrt{\Delta_H(s)}} \\ {}_{E_0}\langle f,g\rangle(t,pH) &= \frac{1}{\sqrt{\Delta_K(t)}}\int_H f(ps)\overline{g(t^{-1}ps)}ds \\ \langle f,g\rangle_{B_0}(s,Kp) &= \frac{1}{\sqrt{\Delta_H(s)}}\int_K \overline{f(t^{-1}p)}g(t^{-1}ps)dt\ . \end{aligned}$$

This imprimitivity theorem is called Green's imprimivity theorem. Notice in particular that when $H = \{e\}$, if $K \curvearrowright P$ freely and properly, then $C_0(P) \rtimes_{\mathrm{lt}} K \sim_M C_0(K\backslash P)$. In particular, $\mathrm{Prim}(C_0(P) \rtimes_{\mathrm{lt}} K) = K\backslash P$.

A natural case is when H and K are closed subgroups of a group G. We get $C_0(G/H) \rtimes_{\mathrm{lt}} H \sim_M C_0(K\backslash G) \rtimes_{\mathrm{rt}} H$ immediately.

Example 5.1. In the case when $\alpha \in \mathbf{R}$, $G = \mathbf{R}$, $H = \alpha\mathbf{Z}$, and $K = \mathbf{Z}$, we get $C(\mathbf{R}/\alpha\mathbf{Z}) \rtimes_{\mathrm{lt}} \mathbf{Z} \sim_M C(\mathbf{Z}\backslash\mathbf{R}) \rtimes_{\mathrm{rt}} \alpha\mathbf{Z}$. What actions σ and τ would we get if we wanted

$$\begin{aligned} (\mathbf{R}/\alpha\mathbf{Z}, \mathbf{Z}, \mathrm{lt}) &\simeq (\mathbf{T}, \mathbf{Z}, \sigma) \text{ and} \\ (\mathbf{Z}\backslash\mathbf{R}, \alpha\mathbf{Z}, \mathrm{rt}) &\simeq (\mathbf{T}, \mathbf{Z}, \tau)\ ? \end{aligned}$$

For σ, we get the group isomorphism

$$\varphi : \mathbf{R}/\alpha\mathbf{Z} \to \mathbf{T} : s + \alpha\mathbf{Z} \mapsto e^{2\pi i s/\alpha}\ .$$

The induced action σ then satisfies the commutative diagram

$$\begin{tikzcd} \mathbf{R}/\alpha\mathbf{Z} \arrow[r, "\mathrm{lt}_1"] \arrow[d, "\varphi"] & \mathbf{R}/\alpha\mathbf{Z} \arrow[d, "\varphi"] \\ \mathbf{T} \arrow[r, "\sigma_1"] & \mathbf{T} \end{tikzcd}$$

which tells us

$$\sigma_1(e^{2\pi i s/\alpha}) = e^{2\pi i(s+1)/\alpha} = e^{2\pi i/\alpha}e^{2\pi i s/\alpha}\ .$$

That is, σ is rotation by $1/\alpha$. For τ, we have the group isomorphism

$$\psi : \mathbf{Z}\backslash\mathbf{R} \to \mathbf{T} : \mathbf{Z} + s \mapsto e^{2\pi s i}\ .$$

The induced action τ then satisfies the commutative diagram

$$\begin{array}{ccc} \mathbf{Z}\backslash\mathbf{R} & \xrightarrow{\mathrm{rt}_\alpha} & \mathbf{Z}\backslash\mathbf{R} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} \\ \mathbf{T} & \xrightarrow{\tau_1} & \mathbf{T} \end{array}$$

which tells us

$$\tau_1(e^{2\pi is}) = e^{2\pi i(s+\alpha)} = e^{2\pi i\alpha}e^{2\pi is} .$$

That is, τ is rotation by α. Thus we have

$$\begin{aligned} C(\mathbf{T}) \rtimes_\sigma \mathbf{Z} &\simeq C(\mathbf{R}/\alpha\mathbf{Z}) \rtimes_{\mathrm{lt}} \mathbf{Z} \\ &\sim_M C(\mathbf{Z}\backslash\mathbf{R}) \rtimes_{\mathrm{rt}} \alpha\mathbf{Z} \\ &\simeq C(\mathbf{T}) \rtimes_\tau \mathbf{Z} . \end{aligned}$$

Here is another corollary to Green's imprimitivity theorem.

corollary 5.2. *Suppose that P is a locally compact G-space and that $H \leq_{closed} G$. If $\sigma : P \to G/H$ is a G-equivariant continuous map, then we have the Morita equivalence*

$$C_0(P) \rtimes_{lt} G \sim_M C_0(Y) \rtimes_{lt} H$$

where $Y = \sigma^{-1}(eH)$.

Proof. Let us define a right H-action on the space $G \times Y$ by

$$(s, x)h = (sh, h^{-1}y) .$$

Let us call $G \times_H Y := (G \times Y)/H$. Let us also define a left G-action by

$$s(t, x) = (st, x) .$$

In this case, it is immediate that $G\backslash(G \times Y) \simeq Y$. Notice that G and H are free and proper commuting actions on $G \times Y$. Green's imprimitivity theorem states

$$C_0(G \times_H Y) \times_{\mathrm{lt}} G \sim_M C_0(Y) \rtimes_{\mathrm{rt}} H .$$

By flipping the right action of H on Y to a left action, we get

$$C_0(G \times_H Y) \rtimes_{\mathrm{lt}} G \sim_M C_0(Y) \rtimes_{\mathrm{lt}} H .$$

It remains to show that $G \times_H Y \simeq P$. To this end, we define

$$\Phi : G \times Y \to P : (s, x) \mapsto sx \ .$$

This is clearly a continuous surjection. To analyze given an element $(s, x) \in G \times Y$, which elements (t, y) satisfy $\Phi(s, x) = \Phi(t, y)$, we see:

$$\begin{aligned}\Phi(s, x) = \Phi(t, y) &\iff sx = ty \iff (t^{-1}s)x = y \\ &\iff (s, x)s^{-1}t = (t, y) \iff (t, y) \in (s, x)H \ .\end{aligned}$$

Therefore, Φ induces a continuous bijection

$$G \times_H Y \xrightarrow{\Phi} P : [s, x] \mapsto sx \ .$$

where $[s, x] := (s, x)H$.

It remains to show that Φ is a homeomorphism. Let $(x_i)_{i \in I}$ be a net in P which converges to a point $x \in P$. We wish to find a net $(s_i)_{i \in I}$ in G and a net $(y_i)_{i \in I}$ in Y for which $\Phi([s_i, y_i]) = x_i$ for all i and $(s_i, y_i) \to (s, y)$ with $\Phi([s, y]) = x$.

Since σ is continuous, $s_i := \sigma(x_i)$ converges to $s := \sigma(x)$. For each $i \in I$, let $y_i := s_i^{-1}x_i$. By G-equivariance of σ, $y_i \in Y$ for all i. As well, since both s_i and x_i converge, $y_i \to_i y := s^{-1}x$. As $\Phi([s_i, y_i]) = x_i$ for all i, we get our result.

☕

The above proof also shows

Example 5.3. If $H \leq_{\text{closed}} G$ and Y is an H-space, then

$$C_0(G \times_H Y) \rtimes_{\text{lt}} G \sim_M C_0(Y) \rtimes_{\text{lt}} H \ .$$

Example 5.4. In the above Example with $Y = \mathbf{T}$ and $\mathbf{Z} \curvearrowright Y$ by irrational rotation θ, if we think of $\mathbf{Z} \leq_{\text{closed}} \mathbf{R}$, we get

$$C_0(\mathbf{R} \times_{\mathbf{Z}} \mathbf{T}) \rtimes_{\text{lt}} \mathbf{R} \sim_M C(\mathbf{T}) \rtimes_\theta \mathbf{Z} \ .$$

Taking $\mathbf{T} = \mathbf{R}/\mathbf{Z}$, notice that the map

$$\Phi : \mathbf{R} \times \mathbf{T} \to \mathbf{T}^2 : (s, t) \mapsto (s, t - \theta s)$$

is a continuous surjection onto a compact space. We see that $\Phi(s, t) = \Phi(a, b)$ if and only if $(a, b) \in (s, t)\mathbf{Z}$. Therefore, we have a homeomorphism $\mathbf{R} \times_{\mathbf{Z}} \mathbf{T} \simeq \mathbf{T}^2$. Let σ be the action $\mathbf{R} \curvearrowright \mathbf{T}^2$ for which we get the isomorphism

$$\Phi : (\mathbf{R} \times_{\mathbf{Z}} \mathbf{T}, \mathbf{R}, \text{lt}) \xrightarrow{\sim} (\mathbf{T}^2, \mathbf{R}, \sigma)$$

of dynamical systems. Because the diagram

$$\begin{array}{ccc} \mathbf{R} \times_{\mathbf{Z}} \mathbf{T} & \xrightarrow{\mathrm{lt}_x} & \mathbf{R} \times_{\mathbf{Z}} \mathbf{T} \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ \mathbf{T}^2 & \xrightarrow{\quad \sigma_x \quad} & \mathbf{T}^2 \end{array}$$

must commute,

$$\begin{aligned} \sigma_x(a,b) &= \sigma_x(a,(b+\theta a)-\theta a) \\ &= \sigma_x \Phi(a, b+\theta a) \\ &= \Phi(a+x, b+\theta a) \\ &= (a+x, b-\theta x)\ . \end{aligned}$$

That is to say, the orbit $\{\sigma_x(a,b) : x \in \mathbf{R}\}$ is the solution to the differential equation

$$dy = -\theta dx$$

with inital value (a,b) on $\mathbf{T}^2$. Although $C(\mathbf{T}^2) \rtimes_\sigma \mathbf{R} \sim_M C(\mathbf{T}) \rtimes_\theta \mathbf{Z}$, notice $C(\mathbf{T}^2) \rtimes_\sigma \mathbf{R}$ is a simple non-unital C^*-algebra while $C(\mathbf{T}) \rtimes_\theta \mathbf{Z}$ is a simple unital C^*-algebra. By Connes' Thom isomorphism (see remark 5.7), we get the equality

$$K_n(C(\mathbf{T}) \rtimes_\theta \mathbf{Z}) = K_{n+1}(C(\mathbf{T}^2))\ .$$

In particular, the groups $K_n(C(\mathbf{T}) \rtimes_\theta \mathbf{Z})$ are independent of the irrational angle θ.

5.2 Stone-von Neumann and Takai duality

5.2.1 The Stone-von Neumann theorem

We start with the theorem of Stone and von Neumann:

Theorem 5.5 (Stone-von Neumann)**.** *Suppose G is a locally compact group. We have the isomorphism*

$$C_0(G) \rtimes_{lt} G \simeq \mathcal{K}(L^2(G)) \ .$$

Proof. We know by the imprimitivity theorem that $C_0(G) \rtimes_{\mathrm{lt}} G$ is simple. Therefore, the usual maps

$$M : C_0(G) \to \mathcal{B}(L^2(G))$$
$$\lambda : G \to U(L^2(G))$$

produces an injective $M \rtimes \lambda$. It remains to check that ran $(M \rtimes \lambda) = \mathcal{K}(L^2(G))$. To this end, suppose that $K \in C_c(G \times G)$. Define

$$f_K(r,s) = \Delta(r^{-1}s)K(s, r^{-1}s) \ .$$

For any $h, k \in C_c(G)$, a calculation shows

$$\langle M \rtimes \lambda(f_K)h, k\rangle = \left\langle \int_G K(\cdot, r)h(r)dr, k \right\rangle \ .$$

That is, $M \rtimes \lambda(f_K) : h \mapsto \int_G K(\cdot, r)h(r)dr$. In particular, when $K(r,s) = f(r)g(s)$, we have

$$M \rtimes \lambda(f_K)h(s) = \left\langle \overline{f}, h \right\rangle g \ ,$$

for which we can get any rank 1 operator. It therefore follows from the density of $C_c(G)$ in $L^2(G)$ that ran $M \rtimes \lambda \supset \mathcal{K}(L^2(G))$. Conversely, if we have any $f \in C_c(G \times G)$, notice that

$$K(r,s) = \Delta(s^{-1})f(rs^{-1}, r)$$

produces $f_K(r,s) = \Delta(s^{-1}r)\Delta(r^{-1}s)f(r,s) = f(r,s)$. It follows that

$$M \rtimes \lambda(C_c(G)C_c(G)) \subset \mathcal{K}(L^2(G))$$

and by density of $C_c(G)C_c(G)$ in $C_0(G) \rtimes_{\mathrm{lt}} G$, we are done. ☕

5.2.2 Takai duality

For this section, assume that G is an abelian locally compact group. Suppose that (A, G, α) is a dynamical system. On the dual group $\widehat{G}$, we may define a new action

$$\widehat{\alpha} : \widehat{G} \to \mathrm{Aut}(A \rtimes_\alpha G)$$

which will make $(A \rtimes_\alpha G, \widehat{G}, \widehat{\alpha})$ into a new dynamical system. The construction is as follows: if $\gamma \in \widehat{G}$, then define

$$\begin{aligned} \widehat{\alpha}_\gamma &: C_c(G, A) \to C_c(G, A) \\ \widehat{\alpha}_\gamma(\varphi) &: s \mapsto \gamma(s^{-1})\varphi(s) \ . \end{aligned}$$

for any $\varphi \in C_c(G, A)$. If $\widehat{\alpha}_\gamma$ is inductive limit continuous, then it will extend to a map on $A \rtimes_\alpha G$ by the fact that the norm on $A \rtimes_\alpha G$ is given by

$$\|\varphi\| = \sup\{\|L\varphi\| : C_c(G, A) \xrightarrow{L} \mathcal{B}(\mathcal{H}) \text{ is an i.l. continuous } ^*\text{-morphism}\}$$

So suppose that φ_i is a net in $C_c(G, A)$ and $\varphi \in C_c(G, A)$ for which $\varphi_i \to_i \varphi$ in the inductive limit topology. Since $\varphi_i \to_i \varphi$ uniformly, and

$$\|\widehat{\alpha}_\gamma(\varphi)(s) - \widehat{\alpha}_\gamma(\varphi_i)(s)\| = \|\varphi_i(s) - \varphi(s)\| \ ,$$

we get uniform convergence of $\widehat{\alpha}_\gamma(\varphi_i)$. Since supp $\widehat{\alpha}_\gamma(\psi) =$ supp ψ for any $\psi \in C_c(G, A)$, we get inductive limit convergence of $(\widehat{\alpha}_\gamma(\varphi_i))$.

We now have a well-defined $\widehat{\alpha}_\gamma$ on $A \rtimes_\alpha G$. A quick calculation shows that $\widehat{\alpha} : \widehat{G} \to \mathrm{Aut}(A \rtimes_\alpha G)$ is a group morphism. To get a dynamical system $(A \rtimes_\alpha G, \widehat{G}, \widehat{\alpha})$, it remains to check that $\widehat{\alpha}$ is strong continuous. Suppose that $\gamma_i \to \gamma$ in $\widehat{G}$, that is, $\gamma_i \to \gamma$ on compacta. Let $\epsilon > 0$. For any $\varphi \in C_c(G, A)$, there is an i_0 for which given any $i \geq i_0$ and any $s \in$ supp φ, we get the estimate

$$\|\gamma_i(s^{-1}) - \gamma(s^{-1})\| \leq \epsilon.$$

From this, we conclude that $\widehat{\alpha}_{\gamma_i}(\varphi) \to_i \widehat{\alpha}_\gamma(\varphi)$ in the inductive limit topology. Since inductive limit convergence implies convergence in the universal norm, we get the desired strong convergence.

With our new dynamical system, we can form the crossed product

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \ .$$

In fact, by Pontryagin duality, we have a new action $\widehat{\widehat{\alpha}}$ on G which makes

$$((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}})$$

into a dynamical system. Takai duality states that there is an equivariant isomorphism between this dynamical system and the dynamical system

$$(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes \operatorname{Ad} \rho)$$

where $\rho : G \to UL^2(G)$ is the right regular representation. To prove Takai duality, we will construct a chain of four *-ismorphisms Φ_k $(k = 1, 2, 3, 4)$ which will compose to give us an isomorphism

$$(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \xrightarrow{\simeq} A \otimes \mathcal{K}(L^2(G))$$

and we will check that this isomorphism is equivariant under our action.

The isomorphism comes as a composition of isomorphisms

$$\begin{aligned} (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} G \xrightarrow{\Phi_1} (A \rtimes_1 \widehat{G}) \rtimes_{\widehat{1}^{-1} \otimes \alpha} G \xrightarrow{\Phi_2} C_0(G, A) \rtimes_{\mathrm{lt} \otimes \alpha} G \\ \xrightarrow{\Phi_3} C_0(G, A) \rtimes_{\mathrm{lt} \otimes 1} G \simeq (C_0(G) \rtimes_{\mathrm{lt}} G) \otimes A \simeq A \otimes \mathcal{K}(L^2(G)) \ . \end{aligned}$$

Our main obstruction to constructing the map Φ_1 is that with crossed products such as $A \rtimes_\alpha G$, we always considered maps on the dense *-subalgebra $C_c(G, A)$ and then used commutative techniques to get a handle on our maps. When we try to do this with the iterated crossed product, we end up with $C_c(\widehat{G}, A \rtimes_\alpha G)$. It would be nicer to consider instead the space $C_c(\widehat{G} \times G, A)$ and embed this space as a dense *-subalgebra of $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$.

To get such as embedding, we step back for a moment and suppose that we have a crossed product of the form

$$(A \rtimes_\beta K) \rtimes_\delta H$$

for some locally compact groups K and H and β, δ the appropriate strong continuous actions. We define

$$C_c(H \times K, A) \xrightarrow{\lambda} C_c(H, C_c(K, A))$$

by setting $\lambda_F(h)(k) := F(h, k)$ for any $F \in C_c(H \times K, A)$. This is a linear embedding. Since $C_c(H) \odot C_c(K, A)$ is dense in $(A \rtimes_\beta K) \rtimes_\delta H$, ran λ is a dense linear subspace of

$(A \rtimes_\beta K) \rtimes_\delta H$. What conditions can we put on δ so that ran λ is closed under convolution and involution?

Suppose that $F \in C_c(H \times K, A)$. Then,

$$\begin{aligned} \lambda_F^*(h)(k) &= \Delta_H(h^{-1})\delta_h(\lambda_F(h)^*)(k) \\ &= \Delta_H(h^{-1})\Delta_K(k^{-1})\beta_k(\delta_h(\lambda_F(h))(k))^* \ . \end{aligned}$$

So, to get closure under involution, we want

$$(h,k) \mapsto \Delta_H(h^{-1})\Delta_K(k^{-1})\beta_k(\delta_h(\lambda_F(h))(k))^*$$

to be an element of $C_c(H \times K, A)$. It would suffice to assume that

1. $C_c(K, A) \subset A \rtimes_\beta K$ is δ-invariant and
2. the function

$$\varphi_F : (h, h', k') \mapsto \delta_h(\lambda(h'))(k')$$

   is continuous and there is a compact $C \subset H \times K$ for which given any $h \in H$, supp $\varphi_F(h, \cdot, \cdot) \subset C$.

We will say that δ is compatible with β if the above two conditions hold. For closure under convolution, let $F_1, F_2 \in C_c(H \times K, A)$. For any $h' \in H$ and $k' \in K$,

$$\begin{aligned} \lambda_{F_1} * \lambda_{F_2}(h')(k') &= \int_H (\lambda_{F_1}(h) * \delta_h(\lambda_{F_2})(h^{-1}h'))(k')dh \\ &= \int_H \int_K \lambda_{F_1}(h)(k)\beta_k(\delta_h(\lambda_{F_2}(h^{-1}h'))(k^{-1}k'))dkdh \ . \end{aligned}$$

By vector-valued integration, it suffices to see that the function

$$(h,k,h',k') \mapsto \lambda_{F_1}(h)(k)\beta_k(\delta_h(\lambda_{F_2}(h^{-1}h'))(k^{-1}k'))$$

is in $C_c(H \times K \times H \times K, A)$. This again follows from our compatibility conditions.

It is easy to see that $\widehat{\alpha}$ is compatible with α. In this case, the convolution and involution on $C_c(\widehat{G} \times G, A)$ are given by

$$F_1 * F_2(\gamma, s) = \int_{\widehat{G}} \int_G F_1(\sigma, t)\sigma(s^{-1}t)\alpha_t(F_2(\sigma^{-1}\gamma, t^{-1}s))dtd\sigma$$

and

$$F^*(\gamma, s) = \Delta_{\widehat{G}}(\gamma^{-1})\Delta_G(s^{-1})\gamma(s^{-1})\alpha_s(F(\gamma, s))^* \ .$$

The first step in getting Takai's duality will be to get an isomorphism between $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ to $(A \rtimes_1 \widehat{G}) \rtimes_\beta G$ for some action β which has the particular advantage that $A \rtimes_1 \widehat{G} \simeq C^*(\widehat{G}) \otimes A \simeq C_0(G) \otimes A \simeq C_0(G, A)$.

From the action $1 : \widehat{G} \to \operatorname{Aut} \mathbf{C}$, we have the action $\widehat{1} : G \to \operatorname{Aut} C^*(\widehat{G})$. Since $\widehat{G}$ is abelian, the map $\widehat{1}^{-1}$ is a group action on G. This induces an action $\widehat{1}^{-1} \otimes \alpha : G \to \operatorname{Aut}(C^*(\widehat{G}) \otimes A)$. Since $C^*(\widehat{G}) \otimes A = A \rtimes_1 \widehat{G}$, we have our action $\beta := \widehat{1}^{-1} \otimes \alpha$. Since β is compatible with 1, we may work with the dense *-subalgebra $C_c(G \times \widehat{G}, A)$. Let us see what the operations are on $C_c(G \times \widehat{G}, A)$ as a subspace of $(A \rtimes_1 \widehat{G}) \rtimes_\beta G$. Given $E_1, E_2 \in C_c(G \times \widehat{G}, A)$, we have

$$\begin{aligned} \lambda_{E_1} * \lambda_{E_2}(s)(\gamma) &= \int_G \int_{\widehat{G}} \lambda_{E_1}(t)(\sigma)(\widehat{1}^{-1} \otimes \alpha)_t(\lambda_{E_2}(t^{-1}s))(\sigma^{-1}\gamma) d\sigma dt \\ &= \int_G \int_{\widehat{G}} \lambda_{E_1}(t)(\sigma)(\sigma^{-1}\gamma)(t)\alpha_t(\lambda_{E_2}(t^{-1}s)(\sigma^{-1}\gamma)) d\sigma dt \ . \end{aligned}$$

Therefore,

$$E_1 * E_2(s, \gamma) = \int_G \int_{\widehat{G}} E_1(t, \sigma)(\sigma^{-1}\gamma)(t)\alpha_t(E_2(t^{-1}s, \sigma^{-1}\gamma)) d\sigma dt \ .$$

As well, for $E \in C_c(G \times \widehat{G}, A)$,

$$E^*(s, \gamma) = \Delta_{\widehat{G}}(\gamma^{-1})\Delta_G(s^{-1})\gamma(s)\alpha_s(E(s, \gamma))^* \ .$$

From this we will define

$$\begin{aligned} \Phi_1 &: C_c(\widehat{G} \times G, A) \to C_c(G \times \widehat{G}, A) \\ \Phi_1(F) &: (s, \gamma) \mapsto \gamma(s)F(\gamma, s) \ . \end{aligned}$$

A calculation shows

$$\begin{aligned} \Phi_1(F_1) * \Phi_1(F_2)(s, \gamma) &= \int_G \int_{\widehat{G}} \sigma(t)F_1(t, \sigma)(\sigma^{-1}\gamma)(t)\alpha_t(F_1(t^{-1}s, \sigma^{-1}\gamma))(\sigma^{-1}\gamma)(t^{-1}s) d\sigma dt \\ &= (\sigma^{-1}\gamma)(s) \int_G \int_{\widehat{G}} \sigma(t)F_1(t, \sigma)\alpha_t(F_1(t^{-1}s, \sigma^{-1}\gamma)) d\sigma dt \\ &= \gamma(s) \int_G \int_{\widehat{G}} \sigma(s^{-1}t)F_1(t, \sigma)\alpha_t(F_1(t^{-1}s, \sigma^{-1}\gamma)) d\sigma dt \\ &= \Phi_1(F_1 * F_2)(s, \gamma) \end{aligned}$$

and

$$\begin{aligned}\Phi_1(F^*)(s,\gamma) &= \gamma(s)\Delta_{\widehat{G}}(\gamma^{-1})\Delta_G(s^{-1})\gamma(s^{-1})\alpha_s(F(\gamma,s))^* \\ &= \gamma(s)\Delta_{\widehat{G}}(\gamma^{-1})\Delta_G(s^{-1})\alpha_s(\gamma(s)F(\gamma,s))^* \\ &= \gamma(s)\Delta_{\widehat{G}}(\gamma^{-1})\Delta_G(s^{-1})\alpha_s(\Phi_1(F)(s,\gamma))^* \ .\end{aligned}$$

It remains to show that Φ_1 is an isometry. To to this, we will compare covariant reprsentations. The idea is that, a representation of $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}$ is going to be spanned by elements of the form aU_sV_γ for $a \in A$, $s \in G$, and $\gamma \in \widehat{G}$. On the other hand, a representation of $(A \rtimes_1 \widehat{G}) \rtimes_{\widehat{1}^{-1}\otimes\alpha} G$ is going to be spanned by elements of the form $aV_\gamma U_s$. The trick will be to show that $U_sV_\gamma = \gamma(s)V_sU_s$ so that our isomorphism falls through.

We will thusly define

Definition 5.6. A pair (U,V), where $U : G \to U(\mathcal{H})$ and $V : \widehat{G} \to U(\mathcal{H})$ is called a Heisenberg pair if it satisfies the identity $U_sV_\gamma = \gamma(s)V_\gamma U_s$.

Suppose that (R,U) is a non-degenerate covariant representation of $(A\rtimes_1\widehat{G}, G, \widehat{1}^{-1}\otimes\alpha)$. Let $R = \pi \rtimes V$ for (π, V) a covariant representation of $(A, \widehat{G}, 1)$. Let $i_{\widehat{G}} : \widehat{G} \to M(C^*(\widehat{G}))$ be the canonical morphism. Notice that $(i_{\widehat{G}}, \widehat{1}^{-1})$ is a Heisenberg pair for G since for any $\varphi \in C_c(\widehat{G})$ and for any $\gamma, \sigma \in \widehat{G}$,

$$\begin{aligned}\widehat{1}_s^{-1} i_{\widehat{G}}(\gamma)\varphi(\sigma) &= \widehat{1}_s^{-1}\varphi(\gamma^{-1}\sigma) = (\gamma\sigma^{-1})(s)\varphi(\gamma^{-1}\sigma) \\ &= \gamma(s) i_{\widehat{G}}(\gamma)(\sigma^{-1}(s)\varphi(\sigma)) = \gamma(s) i_{\widehat{G}}(\gamma)\widehat{1}_s^{-1}\varphi(\sigma) \ .\end{aligned}$$

Let $L = R \rtimes U$ and let $\varphi \otimes f \otimes a \in C_c(\widehat{G}) \odot C_c(G) \odot A$. We then get

$$\begin{aligned}U_sV_\gamma L(\varphi\otimes f\otimes a) &= U_sV_\gamma\pi(a)(1\rtimes V)(\varphi)(1\rtimes U)(f) \\ &= U_s\pi(a)V_\gamma(1\rtimes V)(\varphi)(1\rtimes U)(f) \\ &= U_s\pi(a)(1\rtimes V)(i_{\widehat{G}}(\gamma)\varphi)(1\rtimes U)(f) \\ &= U_s\pi\rtimes V(i_{\widehat{G}}(\gamma)\varphi\otimes a)(1\rtimes U)(f) \\ &= \pi\rtimes V(\widehat{1}_s^{-1} i_{\widehat{G}}(\gamma)\varphi\otimes\alpha_s(a))U_s(1\rtimes U)(f) \\ &= \gamma(s)\pi(\alpha_s(a))(1\rtimes V)(i_{\widehat{G}}(\gamma)\widehat{1}_s^{-1}\varphi)U_s(1\rtimes U)(f) \\ &= \gamma(s)\pi(\alpha_s(a))V_\gamma(1\rtimes V)(\widehat{1}_s^{-1}\varphi)U_s(1\rtimes U)(f) \\ &= \gamma(s)V_\gamma\pi\rtimes V((\widehat{1}^{-1}\otimes\alpha)_s\varphi\otimes a)U_s(1\rtimes U)(f) \\ &= \gamma(s)V_\gamma U_s\pi\rtimes V(\varphi\otimes a)(1\rtimes U)(f) \\ &= \gamma(s)V_\gamma U_sL(\varphi\otimes f\otimes a) \ .\end{aligned}$$

By non-degeneracy of L, we get that (U, V) is a Heisenberg pair. As well, for any $b \in A$,

$$\begin{aligned}
U_s\pi(b)L(\varphi \otimes f \otimes a) &= U_s\pi(ba)(1 \rtimes V)(\varphi)(1 \rtimes U)(f) \\
&= U_s(\pi \rtimes V)(\varphi \otimes ba)(1 \rtimes U)(f) \\
&= \pi(\alpha_s(ba))(1 \rtimes V)(\widehat{1}_s^{-1}\varphi)U_s(1 \rtimes U)(f) \\
&= \pi(\alpha_s(b))U_s(1 \rtimes V)(\varphi \otimes a)(1 \rtimes U)(f) \\
&= \pi(\alpha_s(b))U_sL(\varphi \otimes f \otimes a) \ .
\end{aligned}$$

Therefore, (π, U) is a covariant representation of (A, G, α). As well,

$$\begin{aligned}
V_\gamma \pi \rtimes U(f \otimes a) &= \pi(a)V_\gamma \int_G f(s)U_s ds \\
&= \pi(a) \int_G f(s)\gamma(s^{-1})U_s ds V_\gamma \\
&= \pi \rtimes U(\widehat{\alpha}_\gamma(f \otimes a))V_\gamma \ .
\end{aligned}$$

Therefore, $(\pi \rtimes U, V)$ is a covariant representation of $(A \rtimes_\alpha G, \widehat{G}, \widehat{\alpha})$. We see for any $F \in C_c(\widehat{G} \times G, A)$,

$$\begin{aligned}
L(\Phi_1(F)) &= \int_G \pi \rtimes V(\Phi_1(F)(s, \cdot))U_s ds \\
&= \int_G \int_{\widehat{G}} \pi(\Phi_1(F)(s, \gamma))V_\gamma U_s d\gamma ds \\
&= \int_G \int_{\widehat{G}} \pi(F(\gamma, s))U_s V_\gamma d\gamma ds \\
&= (\pi \rtimes U) \rtimes V(F) \ .
\end{aligned}$$

Therefore, $\|\Phi_1(F)\| \leq \|F\|$.

Conversely, if (R, V) is a non-degenerate covariant pair for $(A \rtimes_\alpha G, \widehat{G}, \widehat{\alpha})$, and $R = \pi \rtimes V$ for a covariant pair (π, U) of (A, G, α),

$$U_s V_\gamma = \gamma(s) V_\gamma U_s \ .$$

To see this, let φ, f, and a be as before. If we set $L = R \rtimes V$,

$$\begin{aligned}
U_s V_\gamma L(\varphi \otimes f \otimes a) &= U_s V_\gamma (\pi \rtimes U)(f \otimes a)(1 \rtimes V)(\varphi) \\
&= U_s (\pi \rtimes U)(\overline{\gamma} f \otimes a) V_\gamma (1 \rtimes V)(\varphi) \\
&= (\pi \rtimes U)(\overline{\gamma} f \otimes a) V_\gamma (1 \rtimes V)(\varphi) \\
&= \gamma(s)(\pi \rtimes U)(\overline{\gamma} f \otimes \alpha_s(a)) V_\gamma (1 \rtimes V)(\varphi) \\
&= \gamma(s) V_\gamma (\pi \rtimes U)(f \otimes \alpha_s(a))(1 \rtimes V)(\varphi) \\
&= \gamma(s) V_\gamma U_s (\pi \rtimes U)(f \otimes a)(1 \rtimes V)(\varphi) \\
&= \gamma(s) V_\gamma U_s L(\varphi \otimes f \otimes a)\ .
\end{aligned}$$

Again, (π, V) is a covariant pair for $(A, \widehat{G}, 1)$. To calculate:

$$\begin{aligned}
\pi(b) V_\gamma L(\varphi \otimes f \otimes a) &= \pi(b)(\pi \rtimes U)(\overline{\gamma} f \otimes a)(1 \rtimes V)(\varphi) \\
&= (\pi \rtimes U)(\overline{\gamma} f \otimes ba)(1 \rtimes V)(\varphi) \\
&= V_\gamma (\pi \rtimes U)(f \otimes ba)(1 \rtimes V)(\varphi) \\
&= V_\gamma \pi(b) L(\varphi \otimes f \otimes a)\ .
\end{aligned}$$

And again $(\pi \rtimes V, U)$ is a covariant pair for $(A \rtimes_1 \widehat{G}, G, \widehat{1}^{-1} \otimes \alpha)$:

$$\begin{aligned}
U_s(\pi \rtimes V)(\varphi \otimes a) &= U_s \pi(a) \int_G \varphi(\sigma) V_\sigma d\sigma \\
&= \pi(\alpha_s(a)) \int_G \sigma(s) \varphi(\sigma) V_\sigma d\sigma U_s \\
&= (\pi \rtimes V)(\widehat{1}_s^{-1}(\varphi) \otimes \alpha_s(a)) U_s \\
&= (\pi \rtimes V)((\widehat{1}^{-1} \otimes \alpha)_s(\varphi \otimes a)) U_s\ .
\end{aligned}$$

As before, we get $\|F\| \leq \|\Phi_1(F)\|$.

To create the isomorphism $(A \rtimes_1 \widehat{G}) \rtimes_{\widehat{1}^{-1} \otimes \alpha} G \xrightarrow{\Phi_2} C_0(G, A) \rtimes_{\mathrm{lt} \otimes \alpha} G$, we just need to show that the isomorphism

$$A \rtimes_1 \widehat{G} \xrightarrow{\varphi_2} C_0(G, A) : \varphi a \mapsto \mathcal{F}(\varphi) a$$

for $\varphi \in C_c(\widehat{G})$ and $a \in A$, where

$$\mathcal{F} : C^*(\widehat{G}) \to C_0(G) : \mathcal{F}(\varphi) : s \mapsto \int_{\widehat{G}} \varphi(\gamma) \overline{\gamma(s)} d\gamma$$

(compare this with the map $\mathscr{F}$ in Example 2.44) is equivariant since then the induced map

$$\Phi_2 : (A \rtimes_1 \widehat{G}) \rtimes_{\widehat{1}^{-1}\otimes\alpha} G \to C_0(G, A) \rtimes_{\mathrm{lt}\otimes\alpha} G : \Phi_2(F)(s, r) = \int_{\widehat{G}} F(s, \gamma)\overline{\gamma(r)}d\gamma$$

for $F \in C_c(G \times \widehat{G}, A)$ as in Example 2.48 is an isomorphism. To establish equivariance, one just follows the term φa through in the diagram

$$\begin{array}{ccc} A \rtimes_1 \widehat{G} & \xrightarrow{(\widehat{1}^{-1}\otimes\alpha)_s} & A \rtimes_1 G \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2} \\ C_0(G, A) & \xrightarrow{(\mathrm{lt}\otimes\alpha)_s} & C_0(G, A) \end{array} .$$

To construct the map

$$C_0(G, A) \rtimes_{\mathrm{lt}\otimes\alpha} \xrightarrow{\Phi_3} C_0(G, A) \rtimes_{\mathrm{lt}\otimes 1} G ,$$

just note that the automorphism

$$C_0(G, A) \xrightarrow{\varphi_3} C_0(G, A) : \varphi_3(f)(r) = \alpha_r^{-1}(f(r))$$

makes the diagram

$$\begin{array}{ccc} C_0(G, A) & \xrightarrow{(\mathrm{lt}\otimes\alpha)_s} & C_0(G, A) \\ \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \varphi_3} \\ C_0(G, A) & \xrightarrow{(\mathrm{lt}\otimes 1)_s} & C_0(G, A) \end{array}$$

commute.

Using Proposition 2.45, we get an isomorphism

$$C_0(G, A) \rtimes_{\mathrm{lt}\otimes 1} G \xrightarrow{\Phi_4} (C_0(G) \rtimes_{\mathrm{lt}} G) \otimes A$$

sending $(\varphi a) \otimes f$ to $(\varphi f) \otimes a$ for $\varphi \in C_0(G)$, $a \in A$, and $f \in C_c(G)$.

Stone-von Neumann gets us an isomorphism

$$(C_0(G) \rtimes_{\mathrm{lt}} G) \otimes A \xrightarrow{(M\rtimes\lambda)\otimes 1} \mathcal{K}(L^2(G)) \otimes A .$$

Diagram chasing shows that the isomorphism $(M \rtimes \lambda) \otimes 1$ is an equivariant isomorphism

$$((C_0(G) \rtimes_{\mathrm{lt}} G) \otimes A, G, (\mathrm{rt} \otimes 1) \otimes \alpha) \to (\mathcal{K}(L^2(G)) \otimes A, G, \mathrm{Ad}\, \rho \otimes \alpha)$$

where $\mathrm{rt} : G \to \mathrm{Aut}\, C_0(G)$ is right translation: $\mathrm{rt}_s(f)(r) = f(rs)$.

Since the isomorphism Φ_4 flips the elementary tensors, the map must be an equivariant isomorphism

$$(C_0(G, A) \rtimes_{\mathrm{lt} \otimes 1} G, G, (\mathrm{rt} \otimes \alpha) \otimes 1) \to ((C_0(G) \rtimes_{\mathrm{lt}} G) \otimes A, G, (\mathrm{rt} \otimes 1) \otimes \alpha)\ .$$

Let $\Phi = \Phi_3\Phi_2\Phi_1$. It remains to check that Φ is an equivariant isomorphism

$$((A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G}, G, \widehat{\widehat{\alpha}}) \to (C_0(G, A) \rtimes_{\mathrm{lt} \otimes 1} G, G, (\mathrm{rt} \otimes \alpha) \otimes 1)\ .$$

This is just chasing more diagrams so I won't do it here but I note that if $F \in C_c(\widehat{G} \times G, A)$, then the identity

$$\widehat{\widehat{\alpha}}_u(F)(\gamma, s) = \overline{\gamma(u)} F(\gamma, s)$$

holds. One just chases F around

$$\begin{array}{ccc} (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} & \xrightarrow{\widehat{\widehat{\alpha}}_u} & (A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \\ \Big\downarrow{\scriptstyle \Phi} & & \Big\downarrow{\scriptstyle \Phi} \\ C_0(G, A) \rtimes_{\mathrm{lt} \otimes 1} G & \xrightarrow{((\mathrm{rt} \otimes \alpha) \otimes 1)_u} & C_0(G, A) \rtimes_{\mathrm{lt} \otimes 1} G \end{array} .$$

Remark 5.7. (See [1] for a proof.) Takai duality is the first step in proving Connes' Thom isomorphism, which states that if $(A, \mathbf{R}, \alpha)$ is a dynamical system, then

$$K_n(A \rtimes_\alpha \mathbf{R}) \simeq K_{n+1}(A)$$

(check what this states in the case when $\alpha = 1$).

Example 5.8. As promised, we return to showing the simplicity of $C(\mathbf{T}) \rtimes_\alpha \mathbf{Z}$, where α is given by rotation by an irrational angle θ. We will adopt all of the notation in Example 2.22. Let's start with a Lemma:

Lemma 5.9. *Let $(A, \mathbf{T}, \gamma)$ be a dynamical system. Define*

$$\Phi : A \to A : a \mapsto \int_{\mathbf{T}} \gamma_z(a) dz$$

and define

$$A^\gamma = \{a \in A : \gamma_z(a) = a \text{ for all } z \in \mathbf{T}\} .$$

Then $\Phi(a) \in A^\gamma$ for all $a \in A$, Φ is positive, linear, and norm-decreasing. Furthermore, Φ is faithful in the sense that $\Phi(a^*a) = 0$ implies that $a = 0$.

Proof. Positivity, linearity, and norm decreasing is easy to see. To see ran $\Phi \subset A^\gamma$, one just checks that $\gamma_z(\Phi(a)) = \Phi(a)$. To see that Φ is faithful, suppose that $\Phi(a^*a) = 0$. Let $\pi : A \hookrightarrow \mathcal{B}(\mathcal{H})$ be a faithful representation of A. If $h \in \mathcal{H}$, then

$$\langle \pi(\Phi(a^*a))h, h\rangle = \int_{\mathbf{T}} \|\pi(\gamma_z(a))h\|^2 dz .$$

This complex integral evaluating to zero implies that $\pi(\gamma_1(a))h = 0$ for all $h \in \mathcal{H}$. In particular, $\pi(a) = 0$. since π is faithful, a must be zero. ☕

We have $\gamma := \widehat{\alpha}$ as a $\mathbf{T}$-action on $A := C(\mathbf{T}) \rtimes_\alpha \mathbf{Z}$. Computing γ, we see $\gamma_z(u^m v^k) = \overline{z}^k u^m v^k$. Our lemma tells us that we have a positive, linear, and faithful contraction

$$\Phi : A \to A^\gamma : a \mapsto \int_{\mathbf{T}} \gamma_z(a) dz .$$

That Φ has range in A^γ tell us that for any $m, k \in \mathbf{N}_0$,

$$\Phi(u^m v^k) = \begin{cases} u^m & k = 0 \\ 0 & k \neq 0 \end{cases} .$$

In particular, $A^\gamma = C(\mathbf{T})$. Let us now show that A is simple. Suppose that I is a non-zero ideal in A. We want to find an invertible element in I. Let $a \in I$ be a non-zero positive element. We know that $\Phi(a) \in C(\mathbf{T})$. I claim if we know $\Phi(a)$ also belongs to I, then we are finished. Notice first that $\Phi(a) > 0$ by faithfulness and positivity of Φ. In particular, there is an open set $U \subset \mathbf{T}$ for which there is a positive ϵ with $\Phi(a)(z) > \epsilon$ whenever $z \in U$. Fix an element $z_0 \in U$ and let $\rho = e^{2\pi i\theta}$ be as in example 2.22. Since $\{\rho^k z_0 : k \in \mathbf{Z}\}$

is dense in $\mathbf{T}$, $\{\rho^k U : k \in \mathbf{Z}\}$ covers $\mathbf{T}$. By compactness of $\mathbf{T}$, we can find $k_1, \ldots, k_n \in \mathbf{Z}$ for which $\bigcup_{i=1}^n \rho^{k_i} U = \mathbf{T}$. Define

$$g = \sum_{i=1}^{n} \alpha_{k_i}(\Phi(a)) \ .$$

Since this is a sum of shifts of our non-negative function $\Phi(a)$ by ρ^{k_i}, the function g is always bounded below by ϵ. In particular, g is an invertible element of $C(\mathbf{T})$. By definition, $\alpha_k(\Phi(a)) = v^k * \Phi(a) * v^{-k}$ for all $k \in \mathbf{Z}$. Since $\Phi(a) \in I$, the invertible element g must also belong to I.

It remains to check that $\Phi(a)$ is a member of I. The trick is to approximate the integral $\Phi(a)$ by Riemann sums. To this end, let us define the linear functionals

$$E_n : A \to A : a \mapsto \frac{1}{2n+1} \sum_{j=-n}^{n} u^{-j} * a * u^j$$

for each $n \in \mathbf{N}$. To see that for all $a \in A$, $\Phi(a) = \lim_{n\to\infty} E_n(a)$, it suffices to check the case when a is a generator $u^m v^k$. Computing:

$$\begin{aligned} E_n(u^m v^k) &= \frac{1}{2n+1} \sum_{|j|\le n} u^{-j} u^m v^k * u^j \\ &= \frac{1}{2n+1} \sum_{|j|\le n} u^{m-j} u^j v^k \rho^{-jk} \\ &= u^m v^k \left[\frac{1}{2n+1} \sum_{|j|\le n} (\rho^k)^j \right] \ . \end{aligned}$$

Using the geometric series on $\sum_{|j|\le n} (\rho^k)^j$, whenever $k \neq 0$,

$$\begin{aligned} \sum_{|j|\le n} \rho^{jk} &= \frac{\rho^{(n+1)k} - \rho^{-nk}}{\rho^k - 1} \\ &= \frac{\rho^{(2n+1)k/2} - \rho^{-(2n+1)k/2}}{\rho^{k/2} - \rho^{-k/2}} \\ &= \frac{\sin((2n+1)\pi\theta k)}{\sin(\pi\theta k)} \ . \end{aligned}$$

Since this sum is bounded above a constant independent of n, $E_n(u^m v^k) \to 0$ whenever $k \neq 0$. If $k = 0$, then $E_n(u^m) = u^m$ and this lets us conclude $\Phi(a) = E_n(a)$ for all $a \in A$. For $a \in I$, since $E_n(a)$ is a sum of elements of the form $u^{-j} * a * u^j$, and I is an ideal, we know $E_n(a) \in I$. We therefore get that $\Phi(a) \in I$. This concludes the proof.

Remark 5.10. As it turns out,

Theorem 5.11. *(See [9] Corollary 7.18.) If (A, G, α) is a dynamical system with G amenable and A nuclear, the crossed product $A \rtimes_\alpha G$ is nuclear.*

In particular, by Example 2.47, whenever we have a dynamical system $(C^*(E), \mathbf{T}, \gamma)$, since $C^*(E) \rtimes_\gamma \mathbf{T}$ is AF (and hence nuclear), we can conclude that

$$C^*(E) \otimes \mathcal{K}(L^2(\mathbf{T})) \simeq (C^*(E) \rtimes_\gamma \mathbf{T}) \rtimes_{\widehat{\gamma}} \mathbf{Z}$$

is nuclear. In particular, $C^*(E)$ is always nuclear.

Bibliography

[1] B. Blackadar, *K-theory for Operator Algebras*, 2nd edition, Cambridge Univ. Press, Cambridge, 1998.

[2] R. Exel, *A Fredholm operator approach to Morita equivalence*, K-theory **7** (1993), 285-308.

[3] A. Kumjian and D. Pask, *C^*-algebras of directed graphs and group actions*, Ergodic Th. & Dynam. Sys. **19** (1999), 1503–1519.

[4] D. Pask and I. Raeburn, *Symmetric Imprimitivity Theorems for Graph C^*-algebras*, International Journal of Mathematics **12**, no. 5 (2001), 609-623.

[5] I. Raeburn, *Graph Algebras*, CBMS regional conference series in mathematicals, no. 103, 2005.

[6] I. Raeburn and D. Williams, *Morita Equivalence and Continuous-Trace C^*-algebras*, Mathematical Surverys and Monographs, Vol. 60, American Mathematical Society, 1998.

[7] M. Rieffel, *Applications of strong Morita equivalence to transformation group C^*-algebras*, Operator algebras and applications, Part I (Kingston, Ont., 1980), Proc. Sympos. Pure Math., Vol. 38, Amer. Math. Soc., Providence, R.I., 1982, 299-310.

[8] J. rotman, *An Introduction to Algebraic Topology*, Graduate texts in mathematics, 119, Springer-Verlang, 1988.

[9] D. Williams, *Crossed products of C^*-algebras*, Mathematical Surverys and Monographs, Vol. 134, American Mathematical Society, 2007.