Combinatorial Methods for Enumerating Maps in Surfaces of Arbitrary Genus

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The problem of map enumeration is one that has been studied intensely for the past half century. Early work on this subject included the works of Tutte for various types of rooted planar maps (e.g. [9, 36, 37]) and the works of Brown [7, 8] for non-planar maps. Furthermore, the works of Bender, Canfield, and Richmond [2, 3] as well as Bender and Gao [4] give asymptotic results for the enumeration of various types of maps.

This subject also attracted the attention of physicists when they independently discovered that map enumeration can be applied to quantum field theory. The results of 't Hooft [35] established the connection between matrix integration and map enumeration, which allowed algebraic techniques to be used. Other examples of this application can be found in the papers of Itzykson and Zuber [5, 10, 20].

One result of particular significance is the Harer-Zagier formula [19], which gives the genus series for maps with one vertex. This result has been proved many times in the literature, a selection of which includes the proofs of Goulden and Nica [17], Itzykson and Zuber [21], Jackson [23], Kerov [24], Kontsevich [25], Lass [27], Penner [29], and Zagier [42]. An extension of this result to locally orientable maps on one vertex can be found in Goulden and Jackson [16], while another extension to two vertex maps can be found in Goulden and Slofstra [18].

In this thesis, we will extend the combinatorial techniques used in the papers of Goulden and Nica [17] and Goulden and Slofstra [18], so that they can be applied to maps with an arbitrary number of vertices, when the graph being embedded is a tree with loops and multiple edges. This involves defining a new set of combinatorial objects that extends the ones used in Goulden and Slofstra, and develop new techniques for handling these objects. Furthermore, we will simplify some of the techniques and results in the existing literature. Finally, we seek to relate the techniques used in this thesis to techniques in other map enumeration problems, and briefly discuss the potential of applying our techniques to those problems.

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Dedication

This thesis is dedicated to my parents, Bond and Susanna Chan. Without their support and nurture, I would not be where I am today. Thank you for everything you have done for me.

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$\langle\cdot\rangle_X$	Average with respect to the measure X, 37
$\mathcal{A}$	The set of columns in an arrowed array that have both row 1 and row 2 unmarked, 95
$\overline{\mathcal{A}}, \overline{\mathcal{C}}, \overline{\mathcal{X}}$	The sets of columns in an arrowed array that have row 2 unmarked, and point to a column of $\mathcal{A}$, $\mathcal{C}$, and $\mathcal{X}$, respectively, 95
$\widetilde{\mathcal{A}}, \widetilde{\mathcal{C}}, \widetilde{\mathcal{X}}$	The sets of columns in an arrowed array that have row 2 marked, and point to a column of $\mathcal{A}$, $\mathcal{C}$, and $\mathcal{X}$, respectively, 95
$\mathcal{A}_{n,L}^{(\mathbf{q};\mathbf{s})}$	Subset of pairings in $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$, such that for $\mu \in \mathcal{A}_{n,L}^{(\mathbf{q};\mathbf{s})}$, $\mu\gamma_{p_1,\ldots,p_n}^{-1}$ has exactly L cycles, 28
$a_{n,L}^{(\mathbf{q};\mathbf{s})}$	Cardinality of $\mathcal{A}_{n,L}^{(\mathbf{q};\mathbf{s})}$, 32
$A_n^{(\mathbf{q};\mathbf{s})}(x)$	Generating series for the number of elements in $\mathcal{A}_{n,L}^{(\mathbf{q};\mathbf{s})}$, 32
α, β, σ	Generically, a paired array, 73
$(\alpha,\phi), (\sigma,\phi)$	Generically, an arrowed array, 86
$\mathcal{AR}_{K;R_1,R_2}^{(s)}$	Set of arrowed arrays that satisfies the forest condition with K columns, R_1 marked cells in row 1, R_2 marked cells in row 2, and s mixed pairs, 87
$\mathcal{B}$	The set of columns in an arrowed array that have row 1 marked and row 2 unmarked, 95
B	Positive definite matrix that defines the quadratic form of a measure, 38
$\mathcal{C}$	The set of columns in an arrowed array that have row 1 unmarked and row 2 marked, 95

$c_{n,K}^{(\mathbf{q};\mathbf{s})}$	Cardinality of the set $\mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$ of canonical arrays, 74
$\mathcal{C}_{n,L}^{\mathbf{p}}$	Subset of pairings in $\mathcal{P}_{p_1,\ldots,p_n}$, such that for $\mu \in \mathcal{C}_{n,L}^{\mathbf{p}}$, $\mu\gamma_{p_1,\ldots,p_n}^{-1}$ has exactly L cycles, 26
$\mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$	The set of canonical arrays with n rows, K columns, q_i non-mixed pairs in row i, and $s_{i,k}$ mixed pairs between rows i and k, 74
$\mathcal{D}$	The set of columns in an arrowed array that have both row 1 and row 2 marked, 95
d	Total number of edges in a combinatorial map, 24
Δ	Substructure of the form $(\mathbf{w}, \mathcal{R}_1, \phi)$, 115
δ	Face permutation of a map, 22
Δ_A	Substructure that describes the set of arrowed arrays that satisfies substructure Δ, and have exactly A columns of type $\mathcal{A}$, 115
$\delta_{m,n}$	Delta function of m and n, which equals to 1 if $m = n$, and 0 otherwise, 42
$d\mu\left(x\right)$	Gaussian measure, defined as $d\mu\left(x\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$, 37
$d\mu\left(H\right)$	Gaussian measure defined by the quadratic form $\mathrm{tr}\left(H^2\right)$, 40
$d\mu\left(H, G\right)$	Gaussian measure defined by the quadratic form $\mathrm{tr}\left(H^2\right) + \mathrm{tr}\left(G^2\right) - 2c\,\mathrm{tr}\left(HG\right)$, 48
$f\colon X \nrightarrow Y$	Partial function from the set X to the set Y, 7
$\mathcal{F}_i$	The set of columns in a paired array that have at least one vertex in row i, 73
$\mathcal{F}_{n,K}^{(\mathbf{q};\mathbf{s})}$	The set of paired functions (μ, ϕ), where $\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ and $\pi\colon\ [p_1, \ldots, p_n] \to [K]$ is a function that preserves the cycles of $\mu\gamma_{p_1,p_2,\ldots,p_n}^{-1}$, 35
$f_{n,K}^{(\mathbf{q};\mathbf{s})}$	Cardinality of the set $\mathcal{F}_{n,K}^{(\mathbf{q};\mathbf{s})}$ of paired functions, 35
$\mathcal{FVA}_{n,K;\mathbf{R}}^{(\mathbf{s})}$	The set of full, proper vertical arrays with n rows, K columns, R_i marked cells in row i, and $s_{i,j}$ mixed pairs between rows i and j, 180

G	Generically, a graph, 7
g	Genus of a map, 19
$g^{(\mathbf{s})}_{n,K;\mathbf{R}}$	Cardinality of the set $\mathcal{FVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$ of full, proper vertical arrays, 180
Γ	Substructure of the form $(\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$, 89
$\Gamma(x)$	Gamma function of x, 8
(γ, μ)	A combinatorial map, 24
$\Gamma \hookrightarrow \Delta$	Substructure Γ is a refinement of substructure Δ, 116
γ_n	Canonical long cycle of $\mathcal{S}_n$, given by $(1, 2, \dots, n)$, 18
$\gamma_{p_1,p_2,\dots,p_n}$	Canonical long permutation of $\mathcal{S}_{p_1,\dots,p_n}$, given by $\gamma^{\underline{1}}_{p_1}\gamma^{\underline{2}}_{p_2}\cdots\gamma^{\underline{n}}_{p_n}$, 18
$\gamma^{\underline{i}}_{p_i}$	Canonical long cycle of $\mathcal{S}_{[p_i]^{\underline{i}}}$, given by $\left(1^{\underline{i}}, 2^{\underline{i}}, \dots, p_i^{\underline{i}}\right)$, 18
Γ_{vu}	Substructure denoting the set of arrowed arrays that satisfies the substructure Γ, and have a fixed vertex pair $\{v, u\}$, 96
$\Gamma_{\mathcal{XY}}$	Substructure obtained by having the critical vertex in a column of type $\mathcal{X}$ pair with a non-critical vertex in a column of type $\mathcal{Y}$ in Γ, 104
$\Gamma_{\mathcal{XY}_c}$	Substructure obtained by having the critical vertex in a column of type $\mathcal{X}$ pair with the critical vertex in a column of type $\mathcal{Y}$ in Γ, 104
$\mathcal{H}$	The set of columns in an arrowed array that contain arrow-tails, 86
$\overline{h}$	Complex conjugate of h, 40
$\mathcal{H}_K$	Space of all $K \times K$ Hermitian matrices, 40
$H_n(x)$	The n'th Hermite polynomial, 41
$h^{(\mathbf{s})}_{n,K;\mathbf{R}}$	Cardinality of the set $\mathcal{NVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$ of non-empty, proper vertical arrays, 168
i, k	Generically, row index of a paired array, 73
$i^{\underline{n}}$	An element of the set $[p]^{\underline{n}}$, 6
j, ℓ	Generically, column index of a paired array, 73

$\mathcal{K}$	The set of all columns in a paired array, 73
K	Number of columns in a paired array, 72
$\mathcal{K}_{p_1,\ldots,p_r}$	Conjugacy class of the permutations with cycle type $\{p_1,\ldots,p_r\}$, 16
$\mathcal{K}_\pi$	Conjugacy class of the permutations with the same cycle type as π, 16
L	Total number of faces in a combinatorial map, 24
Λ	Substructure of the form $(\mathbf{x},\mathcal{P},\phi)$, 132
$\Lambda_{\alpha,i,\mathcal{W}}$	Substructure of the form $(\mathbf{x},\mathcal{R}_i,\psi_i)$, where x_j is the number of vertices inserted into cell (i,j) of α by $\mathcal{W}$ with the insertion procedure, 147
$m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$	Cardinality of the set $\mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ of minimal arrays, 79
$\mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$	The set of minimal arrays with n rows, K columns, R_i marked cells in row i, q_i non mixed pairs in row i, and $s_{i,k}$ mixed pairs between rows i and k, 79
μ	Generically, a pairing on a set S, 4
$[n]$	The set of integers $\{1,\ldots,n\}$, 4
$n!!$	Double factorial of n, given by $n\,(n-2)\cdots 3\cdot 1$ if n is odd, and $n\,(n-2)\cdots 4\cdot 2$ if n is even, 4
n	*combinatorial map*: Total number of edges in a combinatorial map, 24
n	*paired array*: Number of rows in a paired array, 72
$[n;k]$	The set of all k-subsets of $[n]$, 4
$[n]^k$	The set $[n]\times[n]\times\cdots\times[n]$, k times, 4
$n^{(k)}$	Rising factorial, given by $n\,(n+1)\cdots(n+k-1)$, 5
$\mathcal{NVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$	The set of non-empty, proper vertical arrays with n rows, K columns, R_i marked cells in row i, and $s_{i,j}$ mixed pairs between rows i and j, 168
$\mathbf{p}$	Vector of positive integers $(p_1,\ldots,p_n)$, where p_i is the number of elements of the form $x^{\underline{i}}$, 6
$[p]^{\underline{n}}$	The set $\left\{1^{\underline{n}},2^{\underline{n}},\ldots,p^{\underline{n}}\right\}$, 6

$[p_1, \ldots, p_n]$	The set $[p_1]^{\underline{1}} \cup \cdots \cup [p_n]^{\underline{n}}$, 6
${}_pF_q$	Generalized hypergeometric series with p parameters on top and q parameters on the bottom, 10
p_i	Total number of vertices in row i of a paired array, 72
$\mathcal{P}_n$	The set of all pairings on $[n]$, 4
$\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$	The set of pairings on $[p_1, \ldots, p_n]$ with q_i non-mixed pairs of the form $\{x^{\underline{i}}, y^{\underline{i}}\}$ and $s_{i,k}$ mixed pairs of the form $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$, 7
$\mathcal{P}_{p_1,\ldots,p_n}$	The set of all pairings on $[p_1, \ldots, p_n]$, 6
$\mathcal{PA}_{n,K;\mathbf{R}}^{(\mathbf{q};\mathbf{s})}$	The set of paired arrays with n rows, K columns, R_i marked cells in row i, q_i non-mixed pairs in row i, and $s_{i,k}$ mixed pairs between rows i and k, 72
$\mathcal{P}$	The subset of columns in an arrowed array that contains either a critical vertex or a marked cell in row 1, 129
ϕ	Generically, a partial function from $\mathcal{H}$ to $\mathcal{K}$ that represents the arrows of an arrowed array, 86
π, ρ, σ	Generically, a permutation of the symmetric group $\mathcal{S}_X$, 15
ψ_i	The forest condition function for row i of a paired array, 74
$\mathcal{PVA}_{n,K;\mathbf{R}}^{(\mathbf{s})}$	The set of proper vertical arrays with n rows, K columns, R_i marked cells in row i, and $s_{i,k}$ mixed pairs between rows i and k, 79
$\mathbf{q}$	*paired array*: The vector $(q_1, \ldots, q_n)$, where q_i is the number of non-mixed pair in row i of a paired array, 72
$\mathbf{q}$	*pairing*: The vector $(q_1, \ldots, q_n)$, where q_i is the number of non-mixed pairs of the form $\{x^{\underline{i}}, y^{\underline{i}}\}$, 6
$\mathbf{R}$	The vector $(R_1, \ldots, R_n)$, where R_i is the number of marked cells in row i of a paired array, 72
$\mathcal{R}_i$	The set of columns in a paired array that are marked in row i, 73
$\mathbf{s}$	*paired array*: Strictly upper triangular matrix $(s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$, where $s_{i,k}$ is the number of mixed pairs between rows i and k of a paired array, 72

$\mathbf{s}$	*pairing*: Strictly upper triangular matrix $(s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$, where $s_{i,k}$ is the number of mixed pairs of the form $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$, 6
s_i	*paired array*: The total number of mixed vertices in row i of a paired array, 73
s_i	*pairing*: Number of elements of the form $x^{\underline{i}}$ that are part of a mixed pair, 6
$s_{i,k,j}$	Number of vertices in cell (i, j) of a paired array are that paired with a vertex in row k, 75
$\mathcal{S}_{p_1,\ldots,p_n}$	The set of permutations of $[p_1, \ldots, p_n]$, 17
$\mathcal{S}_X$	Symmetric group on the set X, 15
$T(\cdot)$	The number of arrowed arrays that satisfies the given substructure, 99
$\mathcal{T}_{n,k}$	The set of all k-partial pairing on $[n]$, 4
Θ	Substructure of the form $(\mathbf{y}, \mathcal{P}, \phi)$, 129
$\Theta_{\alpha,i,\mathcal{W}}$	Substructure of the form $(\mathbf{y}, \mathcal{R}_i, \psi_i)$, where y_j is the number of vertices inserted into cell (i, j) of α by $\mathcal{W}$ with the insertion procedure, 136
$\{u, v\}$	*paired array*: A pair of vertices joined by an edge in a paired array, 72
$\{u, v\}$	*pairing*: A pair of elements in a pairing, 5
u, v	Generically, vertices of a paired array, 72
$\mathcal{V}$	A subset of the unpaired vertices in some row i of a partially-paired array α, either to be extracted by the extraction procedure, or is the result of the insertion procedure, 80
$v^{(\mathbf{s})}_{n,K;\mathbf{R}}$	Cardinality of the set $\mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$ of proper vertical arrays, 79
$\mathcal{VA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$	The set of vertical arrays with n rows, K columns, R_i marked cells in row i, and $s_{i,k}$ mixed pairs between rows i and k, 79
$\mathbf{w}$	$n \times K$ matrix $(w_{1,1}, \ldots, w_{n,K})$, where $w_{i,j}$ is the number of vertices in cell (i, j) of a paired array, 73
$w(\mu)$	Weight of μ, given by the number of cycles in $\mu\gamma^{-1}_{p_1,\ldots,p_n}$, 32

x — The vector $(x_1, \ldots, x_n)$, where x_j is the number of non-critical vertices in cell $(1, j)$ of substructure Λ, 132

$(\mathbf{x}, \mathbf{y})$ — Inner product of **x** and **y**, defined as $x_1y_1 + \cdots + x_ky_k$, 38

$\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ — Generically, a column or a set of columns of a paired array, 73

y — The vector $(y_1, \ldots, y_n)$, where y_j is the number of vertices in cell $(2, j)$ of substructure Θ, 129

Chapter 1

Introduction and Background

1.1 Thesis Outline

The study of maps, or graphs embedded on surfaces, is as old as graph theory itself. The early results on maps are mainly topological, such as the classic formula by Euler that relates the number of vertices, edges, and faces of a map with the genus of the surface in which it is embedded. However, much work on the combinatorial aspect of maps has been done in the past half century, with the enumeration of maps. Various types of rooted planar maps have been enumerated by Tutte, e.g. [9, 36, 37], and several types of non-planar maps were enumerated by Brown [7, 8]. A survey of the enumeration of maps of arbitrary genus can be found in the census paper by Walsh and Lehman [40]. Furthermore, there has been much research done on the asymptotics of map enumeration, such as the papers by Bender, Canfield, and Richmond [2, 3], or Bender and Gao [4]. The result of Edmonds [13] allowed maps to be encoded combinatorially as permutations, using objects called combinatorial maps. This allows for the use of various algebraic and combinatorial techniques.

The popularity of this subject grew when physicists independently discovered that map enumeration can be applied to quantum field theory. Some examples of such application can be found in the papers of Itzykson and Zuber [5, 10, 20], and more recently Zograf [43] and Eynard [14]. In particular, 't Hooft [35] established the connection between matrix integration and map enumeration.

One result derived from the matrix integration technique is the Harer-Zagier formula [19], which gives the genus series for maps with one vertex. The result has been proved many times in the literature, using various algebraic and combinatorial techniques. A selection of the proofs can be found in Goulden and Nica [17], Itzykson and Zuber [21], Jackson [23], Kerov [24], Kontsevich [25], Lass [27], Penner [29], and Zagier [42]. An extension of this result to locally orientable maps on one vertex can be found in Goulden and Jackson [16]. The particular proof we are interested in is the combinatorial proof by Goulden and Nica, as their proof was extended to two vertex maps in the paper of Goulden and Slofstra [18]. This thesis will generalize Goulden and Slofstra combinatorial approach to maps with an arbitrary number of vertices, as well as provide simplifications to both the approach and objects used.

This thesis is organized as follows.

In Chapter 1, we discuss the basic background and notation used in this thesis. We start by introducing the notation necessary to cover the other sections. Next, we introduce generalized hypergeometric series, which are important functions in combinatorics as they can be used to express most summations. We will present the tools for manipulating and simplifying these series, and show how they can be used to simplify algebraic expressions involving sums. Then, we introduce the symmetric group and the notation we will use for describing permutations. Finally, we introduce maps, first as a topological object, then as a combinatorial object. We will show how the two objects are related to each other, and how the combinatorial map can be represented using the symmetric group. This allows us to present the main focus of this thesis, as a problem about enumerating combinatorial maps by genus.

In Chapter 2, we discuss the background and historical context surrounding map enumeration. We start by formally defining the problem introduced in Chapter 1 using the notation of the symmetric group, then give some elementary results. Next, we introduce the paired function, which is also called the N-coloured map in some parts of the literature. The paired function is a combinatorial object that is used in both the algebraic and combinatorial approaches to map enumeration, and counting these objects is sufficient for giving the generating series for our problem. Then, we discuss one of the algebraic techniques used in the literature, known as the matrix model. The matrix model is the integral on Gaussian measure over the space of Hermitian matrices. We are interested in the one matrix model, for which we will introduce the necessary background and notation, as well as the theorems we can use to evaluate the integrals that result from using this model. In the following section, we apply the one matrix model to the one vertex case of our problem, which gives us the results of the Harer-Zagier formula [19]. Subsequently, we extend the one matrix model into the two matrix model, which allows us to derive the result of Goulden and Slofstra [18] algebraically. Finally, we give further context to our problem by discussing several related problems in enumerating maps and permutations.

Chapter 3 is an extension to the work in Goulden and Slofstra [18], modified so that it fits our definition of the paired function. First, we give a pictorial description of the paired functions, which are presented in the form of labelled arrays. Then, we can strip the labels from the labelled arrays, which gives us our main combinatorial object, the paired arrays. We introduce the various terminology and lemmas for describing paired arrays, including the balance and forest conditions. In particular, we will discuss some of the differences between our definition of paired arrays and the definition in Goulden and Slofstra. By using these lemmas, we can provide a bijection between paired arrays and labelled arrays, which shows that this preserves the necessary information for reconstruction. Finally, we develop some of the tools needed to decompose paired arrays, and start the first step of our decomposition by decomposing paired arrays into minimal arrays.

Chapter 4 introduces the arrowed array, which is an new extension to the paired array defined in Chapter 3. First, we define the arrowed array as a combinatorial object, then extend the notation used for paired arrays to arrowed arrays. Next, we introduce the arrow simplification lemmas, which is a set of lemmas that allow us to reduce arrowed arrays to specific forms. This allows us to partition the set of arrowed arrays into substructures, and describe these substructures using a number of parameters. Furthermore, the arrow simplification lemmas can be used to reduce one substructure to another. This allows us to count the numbers of arrowed arrays that satisfy the substructures we have defined, by using induction and the arrow simplification lemmas on substructures.

Chapter 5 continues the discussion on arrowed arrays by introducing more types of substructures. Each new type of substructure introduced is the aggregate of the substructures of the previous type. This allow us to derive formulas for the new types of substructures by summing over the formulas for the previous ones. For each type of substructure, we will present two results, which correspond to the two decompositions we will give in the next chapter. This culminates in a pair of formulas that can be used for further decomposing paired arrays.

Chapter 6 continues the decomposition started in Chapter 3, where we will use the results derived in the previous two chapters to completely decompose the paired array. We start by decomposing minimal arrays into vertical arrays, using an alternate proof to Goulden and Slofstra that does not rely on the forest completion algorithm. This gives us a formula for the number of paired arrays in terms of the number of vertical arrays, which is applicable regardless of whether the underlying graph is a tree. In cases where the graph is a tree with loops and multiple edges, we can recursively decompose vertical arrays, and use induction to derive an expression for the number of vertical arrays that can be substituted into the previous formula. To end this chapter, we demonstrate that the expression we have derived for paired arrays is a polynomial, which allows us to use this expression as the generating series to the main problem.

Finally, Chapter 7 discusses the application of results derived in this thesis, as well as other miscellaneous results. We start off by showing how the result in this thesis can be specialized into the one and two vertex cases covered in Chapter 2. Then, we give a further simplification to the formula of Goulden and Slofstra by using Pfaff's identity. Next, we discuss some results for when the underlying graph is not a tree, and show how they can be used to derive the series computationally for the main problem in this thesis. Finally, we talk about the possible directions to go forward, as well as the potential for applying the techniques in this thesis to other enumeration problems.

Among the many results that appear in this thesis, the main new contributions are given by Definition 4.1, Definition 4.9, Theorem 4.13, Theorem 6.7, and Theorem 6.9. Together, these give our extension of the paired array, called the arrowed array, the partitioning of arrowed arrays into irreducible substructures, and the key steps for using arrowed arrays to enumerate the number of vertical arrays. We can then combine these results with previous work to obtain the generating series for the main problem.

1.2 Basic Notation

In this section, we will describe basic combinatorial notation and results used in this thesis. In particular, we will focus on those relating to sets, functions, and graphs. Let n and k be integers such that $0 \leq k \leq n$. We use $[n]$ to denote the set $\{1, \dots, n\}$, $[n]^k$ to denote the Cartesian product of $[n]$ with itself k times, and $[n; k]$ to denote the set of all k-subsets of $[n]$. Suppose S is a set of size n, where n is even. A *pairing* μ of S is a partition of S into disjoint subsets of size 2. In this context, the set S is called the *support* of μ. Furthermore, the set of all pairings of $[n]$ is denoted as $\mathcal{P}_n$. Next, suppose k is a non-negative integer, and S is a set of size at least $2k$. A *partial pairing* T of S is a pairing on a subset $S' \subseteq S$ of even cardinality. If $|S'| = 2k$, then T is called a *k-partial pairing* of S. As with pairings, the set S' is called the *support* of the partial pairing T. Finally, the set of all k-partial pairings of $[n]$ is denoted as $\mathcal{T}_{n,k}$.

Next, we will introduce a number of standard notations for expressing the cardinalities of the above sets. Let $n \geq 0$ be an integer. The *factorial* of n, denoted $n!$, is defined by

$$n! = n\,(n-1)\cdots 3 \cdot 2 \cdot 1$$

for $n \geq 1$, with $0! = 1$ by convention. Similarly, the *double factorial* of n, denoted as $n!!$, is defined by

$$n!! = \begin{cases} n\,(n-2)\cdots 5\cdot 3\cdot 1 & n \text{ odd} \\ n\,(n-2)\cdots 6\cdot 4\cdot 2 & n \text{ even} \end{cases}$$

for $n \geq 1$, with $0!! = (-1)!! = 1$ by convention. By rewriting the double factorial in terms of normal factorials, we see that for $n = 2k-1$ odd, we have $(2k-1)!! = \frac{(2k)!}{2^k k!}$, and for $n = 2k$ even, we have $(2k)!! = 2^k k!$

Now, for a complex number n and an integer $k \geq 0$, the *rising factorial* $n^{(k)}$ is defined by

$$n^{(k)} = n\,(n+1)\cdots(n+k-1)$$

for $k \geq 1$, with $n^{(0)} = 1$. Note that for $n \geq 0$ an integer, we have $n! = 1^{(n)}$. Furthermore, for a fixed integer k, $n^{(k)}$ is a polynomial in n of degree k. With the rising factorial defined, the *binomial coefficient* $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{(n-k+1)^{(k)}}{k!}$$

with $\binom{n}{k} = 0$ for integer $n \geq 0$ and $k < 0$ by convention. If n and k are integers and $0 \leq k \leq n$, we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. As with rising factorials, for a fixed integer k, $\binom{n}{k}$ is a polynomial in n of degree k. Unlike in most of the combinatorial literature, we will use the rising factorial instead of the falling factorial defined by $(n)_k = n\,(n-1)\cdots(n-k+1)$. The reason will become apparent when we define the hypergeometric series in Section 1.3. Furthermore, we will also elaborate on how we use the factorial function and rising factorial in that section.

With the values defined above, we can now give the cardinalities of the sets introduced earlier in this section. Let $s_1, \ldots, s_n$ be the elements of S. Notice that in a pairing μ of S, the element s_1 must be paired with some element s_i, where $2 \leq i \leq n$. Suppose μ' is μ with the pair $\{s_1, s_i\}$ removed, then μ' is a pairing of $S\setminus\{s_1, s_i\}$. Conversely, any pairing μ' of $S\setminus\{s_1, s_i\}$ can be made into a pairing μ of S by adding the pair $\{s_1, s_i\}$. Therefore, by doing some elementary counting, we obtain the following cardinalities.

$$\begin{aligned}
|[n;k]| &= \binom{n}{k} \\
|\mathcal{P}_n| &= (n-1)!! \\
|\mathcal{T}_{n,k}| &= \binom{n}{2k}(2k-1)!! \\
&= \frac{n!}{2^k k!\,(n-2k)!}
\end{aligned}$$

Next, we will describe various notation involving sets described by multiple integers. Let p and n be positive integers. We use $[p]^{\underline{n}}$ to denote the set $\left\{1^{\underline{n}}, 2^{\underline{n}}, \ldots, p^{\underline{n}}\right\}$, whose elements $i^{\underline{n}}$, $i = 1, \ldots, p$, are regarded as a labelled version of the integer i, labelled by the "$\underline{n}$" in the superscript position. Then, suppose $\mathbf{p} = (p_1, \ldots, p_n)$ is a vector of length n of positive integers, we let $[p_1, \ldots, p_n]$ to be the set $[p_1]^{\underline{1}} \cup \cdots \cup [p_n]^{\underline{n}}$. For example, $[3, 5, 2]$ is the set $\left\{1^{\underline{1}}, 2^{\underline{1}}, 3^{\underline{1}}, 1^{\underline{2}}, 2^{\underline{2}}, 3^{\underline{2}}, 4^{\underline{2}}, 5^{\underline{2}}, 1^{\underline{3}}, 2^{\underline{3}}\right\}$. Furthermore, if $p_1 + \cdots + p_n$ is even, then the set of all pairings of $[p_1, \ldots, p_n]$ is denoted as $\mathcal{P}_{p_1,\ldots,p_n}$. Now, if μ is a pairing of $[p_1, \ldots, p_n]$, then a pair $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ in μ is a *mixed pair* if $i \neq k$, and a *non-mixed pair* otherwise. To describe the number of mixed and non-mixed pairs in a pairing μ, we introduce the parameters $\mathbf{q}$ and $\mathbf{s}$ defined as follows. Let $\mathbf{q} = (q_1, \ldots, q_n)$ be a vector of length n, where q_i is the number of non-mixed pairs of the form $\{x^{\underline{i}}, y^{\underline{i}}\}$ in μ, and $\mathbf{s} = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$ be an $n \times n$ strictly upper triangular matrix, where $s_{i,k}$ is the number of mixed pairs of the form $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ in μ. For ease of notation, we let $s_{i,k} = s_{k,i}$ for $i > k$, and let $s_i = \sum_{k \neq i} s_{i,k}$. Equivalently, $\mathbf{s}$ is a symmetric matrix with zeroes on its diagonal, and s_i is the sum of row i of that matrix. For convenience, we will also sometimes treat $\mathbf{s}$ as a vector of length $\frac{n(n-1)}{2}$. As each non-mixed pair $\{x^{\underline{i}}, y^{\underline{i}}\}$ has 2 elements of $[p_i]^{\underline{i}}$, and each mixed pair $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ has 1 element of $[p_i]^{\underline{i}}$, we see that $p_i = 2q_i + s_i$ for $1 \leq i \leq n$. Finally, given a strictly upper triangular matrix $\mathbf{s}$, the *support graph* of $\mathbf{s}$ is the graph G with the vertex set $[n]$, such that $\{i, k\}$ is an edge of G if and only if $s_{i,k} > 0$.

For example, $\mu = \left\{\left\{1^{\underline{1}}, 3^{\underline{1}}\right\}, \left\{2^{\underline{1}}, 2^{\underline{2}}\right\}, \left\{1^{\underline{2}}, 2^{\underline{3}}\right\}, \left\{3^{\underline{2}}, 4^{\underline{2}}\right\}, \left\{5^{\underline{2}}, 1^{\underline{3}}\right\}\right\}$ is a pairing of $[3, 5, 2]$. The parameters for this pairing are $(q_1, q_2, q_3) = (1, 1, 0)$ and $(s_{1,2}, s_{1,3}, s_{2,3}) = (1, 0, 2)$, which gives us $(p_1, p_2, p_3) = (3, 5, 2)$. The support graph of $\mathbf{s}$ is a graph with vertices $\{1, 2, 3\}$, and edges $\{1, 2\}$ and $\{2, 3\}$, as seen in Figure 1.1.

Our next step is to partition the pairings of $\mathcal{P}_{p_1,\ldots,p_n}$ according to the parameters $\mathbf{q}$ and $\mathbf{s}$. Let $\mathbf{q} = (q_1, \ldots, q_n)$ and $\mathbf{s} = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$ be vectors of non-negative integers

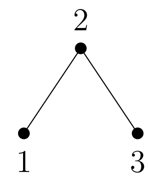


Figure 1.1: Support graph of $\mathbf{s} = (1, 0, 2)$

such that $p_i = 2q_i + s_i$ is positive for $1 \leq i \leq n$. We define $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})} \subseteq \mathcal{P}_{p_1,\ldots,p_n}$ to be the subset of the pairing such that for $\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$, μ has q_i non-mixed pairs of the form $\{x^{\underline{i}}, y^{\underline{i}}\}$ and $s_{i,k}$ mixed pairs of the form $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$. As the parameters p_i's are now redundant, we can drop them from the definition of $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$. However, we will keep the parameter n, to be consistent with other objects defined later in this thesis, where the parameter will be used to decompose those objects.

To conclude this section, we will discuss some of the auxiliary notation that we will be using in this thesis. A *graph* $G = (V, E)$ is a pair consisting of a vertex set V and edge set E, where each edge $e \in E$ is a pair of vertices $u, v \in V$. A graph is *directed* if each pair $e \in E$ is ordered, denoted as $e = (u, v)$. The *out-degree* of a vertex v in a directed graph is the number of edges e of the form $e = (v, u)$. If G is a directed graph where each vertex has out-degree at most one, then G is a *rooted forest* if it is acyclic, in other words, if G does not contain a directed cycle. The *root vertices* of G are the vertices with out-degree 0. Note that this includes the isolated vertices of G. The rooted forest will be used later to help define one of our combinatorial objects. In general, we will introduce lemmas on graphs as they are needed, since most results are only needed once.

Finally, we will briefly cover the notation we use for partial functions. Let X and Y be two sets. A function f is a *partial function* from X to Y, denoted $f\colon X \nrightarrow Y$, if f is a function from a subset $X' \subseteq X$ to Y. By definition, all functions are partial functions. Now, given a partial function $f\colon X \nrightarrow X$, the *functional digraph* of f is a directed graph G with X as its vertex set, and (u, v) is a directed edge of G if and only if $f(u) = v$. Furthermore, suppose $f\colon X \nrightarrow X$ is a partial function that is defined on the set $X' \subseteq X$, and Y is a subset of X that contains $X' \cup f(X')$. Then, f is a partial function from Y to Y. Therefore, we can take the functional digraph of f with respect to the vertex set Y. This means that we consider f as a function $f\colon Y \nrightarrow Y$, and take the functional digraph of this function.

For example, the function $f(1) = 3$, $f(2) = 5$, $f(5) = 6$, $f(6) = 5$ is a partial function $f\colon [8] \nrightarrow [8]$ that is defined on $X' = \{1, 2, 5, 6\}$. Then, $Y = \{1, 2, 3, 5, 6, 8\}$ is a subset of

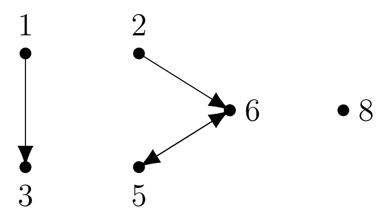


Figure 1.2: Functional digraph of a partial function

[8] that contains X' and $f(X')$. So, $f\colon Y \nrightarrow Y$ is a partial function that has the functional digraph shown in Figure 1.2.

1.3 Generalized Hypergeometric Series

In this section, we begin by describing notation related to generalized hypergeometric series that will be used in this thesis. Most of the definitions in this section are taken from the book Special Functions by Andrews, Askey, and Roy [1], with some notation adjusted to match the notation commonly used in combinatorics. Then, we introduce an elementary but useful technique for manipulating these series that allows us to bypass difficulties arising when the series are undefined. Finally, we introduce four hypergeometric identities that we will be using in the later chapters of the thesis.

Recall that for $n \geq 0$ an integer, the factorial of n is defined by $n! = n(n-1)\cdots 3\cdot 2\cdot 1$ for $n \geq 1$, with $0! = 1$. To extend this definition, we introduce the gamma function, which can be defined as a limit. For x a complex number with $x \neq 0, -1, -2, \ldots$, the *gamma function* $\Gamma(x)$ is defined by

$$\Gamma(x) = \lim_{k\to\infty} \frac{k!k^{x-1}}{x^{(k)}}$$

This function was originally discovered by Euler, and has several equivalent definitions. One alternative definition, also by Euler, is to define the gamma function as an infinite integral for positive real values x, and then extend it analytically to all complex numbers. By taking the ratio of the limits $\frac{\Gamma(x+1)}{\Gamma(x)}$, we can obtain the identity $\Gamma(x+1) = x\Gamma(x)$. Combined with $\Gamma(1) = 1$, the gamma function satisfies $\Gamma(n+1) = n!$ for all non-negative integer n. Also, note that the limit of the reciprocal exists for all complex values x, with $\frac{1}{\Gamma(x)} = 0$ for all non-positive integers x. Therefore, with a slight abuse of notation, we will use factorials and gamma functions interchangeably, and define $\frac{1}{x!}$ to be 0 if x is a negative integer. However, note that a term of $x!$ in the numerator is undefined if x is a negative integer. Furthermore, the reciprocal of the gamma function is entire, which means it has a complex derivative everywhere. This implies that $\frac{1}{\Gamma(x)}$ is continuous at all complex numbers, and $\Gamma(x)$ is continuous except at the points $x \neq 0, -1, -2, \ldots$.

Our first application of the gamma function is to rewrite rising factorials as normal factorials. Recall that for a complex number n and an integer $k \geq 0$, the rising factorial $n^{(k)}$ is defined by $n^{(k)} = n(n+1)\cdots(n+k-1)$. If n is an integer, then we can rewrite the rising factorial to obtain

$$n^{(k)} = \begin{cases} \frac{(n+k-1)!}{(n-1)!} & n > 0 \\ (-1)^k \frac{(-n)!}{(-n-k)!} & n \leq 0 \end{cases}$$

In the case where $-k < n \leq 0$, the denominator of the second expression contains the factorial of a negative integer, which gives $n^{(k)} = 0$, as desired. Furthermore, by replacing the factorials in these expressions with their equivalent gamma functions and using the identity $\Gamma(x+1) = x\Gamma(x)$, we see that the expressions remain valid for non-integer values of n. Note that despite the piecewise representation, $n^{(k)}$ is continuous as a function of n, since $n^{(k)}$ is a polynomial in n for fixed k.

Considering again the case of n an integer and $-k < n \leq 0$, note that we can rewrite $n^{(k)}$ as $(-1)^k(-n-k+1)^{(k)}$. This gives $n^{(k)} = \frac{(n+k-1)!}{(n-1)!}$ with the above formula. Hence, if we have integers n and k such that $n \geq 0$, we can write

$$\frac{n!}{(n-k)!} = (n-k+1)^{(k)}$$

regardless of whether $n-k$ is non-negative. This will be useful later in the thesis.

Another reason for defining $\frac{1}{x!}$ to be 0 when x is a negative integer is that this matches up well with the values of the binomial coefficient. Recall that for a complex number n and an integer $k \geq 0$, the binomial coefficient $\binom{n}{k}$ is defined by $\binom{n}{k} = \frac{(n-k+1)^{(k)}}{k!}$. By rewriting it as a ratio of gamma functions, we have

$$\begin{aligned} \binom{n}{k} &= \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \\ &= \frac{n!}{k!(n-k)!} \end{aligned}$$

if we are to write the gamma functions as factorials. Notice that if n is an integer, the denominator contains the factorial of a negative integer if and only if $k < 0$ or $k > n$, precisely when $\binom{n}{k} = 0$.

With the preliminaries defined, we can now introduce the key tool which we will use to manipulate summations. This is the hypergeometric series, which is given by the following definition

Definition 1.1. A *generalized hypergeometric series* is a series $\sum_{k=0}^{\infty} c_k$ such that $\frac{c_{k+1}}{c_k}$ is a rational function of k. Each of the terms c_k is referred to as a *hypergeometric term*. By factoring the numerator and denominator of $\frac{c_{k+1}}{c_k}$ as polynomials in k, we obtain

$$\frac{c_{k+1}}{c_k} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)\,x}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}$$

for some non-negative integers p and q, and some constant x independent of k. If the factor $k+1$ does not occur naturally in the denominator, we add it to both the numerator and the denominator of $\frac{c_{k+1}}{c_k}$. Then, the series can be normalized by factoring out c_0, which gives

$$\sum_{k=0}^{\infty} c_k = c_0 \sum_{k=0}^{\infty} \frac{a_1^{(k)} \cdots a_p^{(k)}}{b_1^{(k)} \cdots b_q^{(k)}} \cdot \frac{x^k}{k!} =: c_0 \cdot {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; x\right)$$

where we define ${}_pF_q$ as the sum in the middle. From the definition, we see that the b_i cannot be non-positive integers.

Historically, the term (ordinary) hypergeometric series refers to hypergeometric series of the form ${}_2F_1\left(\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right)$. This was first coined by John Wallis in the work Arithmetica Infinitorum (1655), and was later studied by Euler, Gauss, and Kummer [32]. For convenience, we will refer to both ordinary and generalized hypergeometric series simply as hypergeometric series. Many of the special functions and theorems can be expressed in terms of hypergeometric series. For example, we can write $e^x = {}_0F_0\left(\begin{smallmatrix} \text{—} \\ \text{—} \end{smallmatrix}; x\right)$, $\sin x = x{}_0F_1\left(\begin{smallmatrix} \text{—} \\ 3/2 \end{smallmatrix}; \frac{-x^2}{4}\right)$, and $\cos x = {}_0F_1\left(\begin{smallmatrix} \text{—} \\ 1/2 \end{smallmatrix}; \frac{-x^2}{4}\right)$. Further examples of expressing common functions as hypergeometric series can be found in the book Special Functions [1]. However, one particular example we will consider here is the *binomial theorem*, as this helps to explain some of the techniques used later in this thesis. For $n \geq 0$, $(1-x)^n$ can be expressed as

$$\begin{aligned}(1-x)^n &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cdot x^k \\ &= \sum_{k\geq 0} \frac{(-1)^k n!}{k!\,(n-k)!} x^k \\ &= {}_1F_0\left(\begin{matrix}-n \\ \text{—}\end{matrix}; x\right)\end{aligned}$$

In the second line, we take advantage of the fact that $\frac{1}{(n-k)!} = 0$ for $k > n$, which allows us to raise the upper bound of the sum to infinity. To arrive at the third line, we first substitute in $k = 0$ to obtain $c_0 = 1$. Then, by taking the ratio of successive terms, we obtain $\frac{c_{k+1}}{c_k} = \frac{k-n}{k+1}$. This shows that the sum is a ${}_1F_0$, with $a_1 = -n$, as desired.

Note that if one of the a_i in ${}_pF_q\left(\begin{smallmatrix}a_1,\ldots,a_p \\ b_1,\ldots,b_q\end{smallmatrix}; x\right)$ is a non-positive integer $-n$, then the series is a polynomial in x, and is a finite sum with $n+1$ terms. This means that if we have a finite sum $\sum_{k=0}^{n} c_k$ such that the ratio $\frac{c_{k+1}}{c_k}$ has $n-k$ as one of its factors in the numerator, we can write it as the hypergeometric series ${}_pF_q\left(\begin{smallmatrix}-n,a_2,\ldots,a_p \\ b_1,\ldots,b_q\end{smallmatrix}; x\right)$, without first raising the upper bound of the sum to infinity as shown above. Conversely, if n is a non-negative integer, a series $\sum_{k\geq 0} c_k$ that contains $\frac{1}{(n-k)!}$ in its denominator has an implicit upper bound of n. The requirement that the b_i cannot be non-positive integers poses a problem when dealing with combinatorics, as most combinatorial parameters are integers. There are several methods of bypassing this issue. One method is to define an alternative series that allows the b_i to be non-positive integers. For example, we can divide the series by $\Gamma(b_1) \cdots \Gamma(b_q)$, which transforms the denominator into $\Gamma(b_1+k) \cdots \Gamma(b_q+k)$. This approach can be found in the book Generalized Hypergeometric Functions by Slater [32]. A second method is to modify our definition of hypergeometric series to allow for non-positive integers b_i if there are corresponding non-positive integers a_j that are smaller in absolute value. Computer Algebra Systems (CAS) such as Maple use this technique [28]. However, both methods require us to rederive the hypergeometric identities presented in this section, so that they hold for these alternate definitions. Therefore, instead of using one of these methods, we will apply the following technique, adapted from Section 2.7 of Special Functions. This allows us to use the identities in Special Functions as stated.

Fact 1.2. *Let $A\colon \mathbb{R}^k \to \mathbb{R}$ and $B\colon \mathbb{R}^k \to \mathbb{R}$ be functions continuous at a point $\mathbf{t} = (t_1, \ldots, t_k)$, and $\alpha \in \mathbb{R}^k$. If there exists $r \in \mathbb{R}$ such that $A(\mathbf{t} + \epsilon\alpha) = B(\mathbf{t} + \epsilon\alpha)$ for all $0 < \epsilon < r$, then $A(\mathbf{t}) = B(\mathbf{t})$.*

While Fact 1.2 is elementary, it is extremely useful. Let $\mathbf{t}, \alpha \in \mathbb{R}^k$, $r \in \mathbb{R}$, and $\mathcal{N} = \{\mathbf{t} + \epsilon\alpha \mid 0 < \epsilon < r\}$ be a path approaching $\mathbf{t}$. Suppose we have functions $A(\mathbf{x})$ and $B(\mathbf{x})$ that are continuous at $\mathbf{t}$, and functions $\hat{A}(\mathbf{x})$ and $\hat{B}(\mathbf{x})$ such that $A(\mathbf{x}) = \hat{A}(\mathbf{x}) = \hat{B}(\mathbf{x}) = B(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{N}$, then $A(\mathbf{t}) = B(\mathbf{t})$. In particular, we can let $A(\mathbf{x})$ and $B(\mathbf{x})$ be functions that are expressed as sums, and are continuous at $\mathbf{t}$. Then, let $\hat{A}(\mathbf{x})$ and $\hat{B}(\mathbf{x})$ be $A(\mathbf{x})$ and $B(\mathbf{x})$, respectively, but written as hypergeometric series. While $\hat{A}(\mathbf{x})$ and $\hat{B}(\mathbf{x})$ may not be defined at $\mathbf{t}$, we can choose α in such a way that the bottom parameters of the hypergeometric series in $\hat{A}(\mathbf{x})$ and $\hat{B}(\mathbf{x})$ are non-integers. This gives us $A(\mathbf{x}) = \hat{A}(\mathbf{x})$ and $B(\mathbf{x}) = \hat{B}(\mathbf{x})$ for all points $\mathbf{x} \in \mathcal{N}$. Finally, we can prove that $\hat{A}(\mathbf{x}) = \hat{B}(\mathbf{x})$ for $\mathbf{x} \in \mathcal{N}$ using a hypergeometric transformation, and use Fact 1.2 to deduce that $A(\mathbf{t}) = B(\mathbf{t})$.

As an example of this technique, we next present the Chu-Vandermonde identity in its hypergeometric form, then show how to derive the combinatorial form from it. Both forms of the identity will be used later in the thesis for simplifying certain summations.

Proposition 1.3. *Let $N \geq 0$ be a non-negative integer, and $a, c \in \mathbb{C}$ where c is not a non-positive integer. Then the Chu-Vandermonde identity is given by*

$$ {}_2F_1\left(\begin{matrix} -N, a \\ c \end{matrix}; 1\right) = \frac{(c-a)^{(N)}}{c^{(N)}} $$

Example 1.4. Let a, b, and n be non-negative integers, and consider the identity $A(b) := \sum_{k=0}^{n} \binom{a}{k}\binom{b}{n-k} = \binom{a+b}{n} =: B(b)$, where we consider both sides of the identity as a function of b. By using the Chu-Vandermonde identity, we obtain

$$\begin{aligned}
& \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} \\
= & \sum_{k=0}^{n} \frac{a!b!}{k!\,(a-k)!\,(n-k)!\,(b-n+k)!} \\
= & \frac{b!}{n!\,(b-n)!}\,{}_2F_1\left(\begin{matrix}-n,-a\\ b-n+1\end{matrix};1\right) \\
= & \frac{b!}{n!\,(b-n)!} \cdot \frac{(a+b-n+1)^{(n)}}{(b-n+1)^{(n)}} \\
= & \frac{b!}{n!\,(b-n)!} \cdot \frac{(a+b)!\,(b-n)!}{(a+b-n)!b!} \\
= & \frac{(a+b)!}{n!\,(a+b-n)!} \\
= & \binom{a+b}{n}
\end{aligned}$$

By the conventions described at the beginning of this section, we are using $x!$ to represent $\Gamma(x+1)$ and $\frac{1}{x!} = 0$ for integers $x < 0$. This choice of notation makes lines 1, 2, 6, and 7 well defined for all values of a, b, and n in consideration. However, if $n > b$, then $\frac{b!}{n!(b-n)!} = 0$ and ${}_2F_1\left(\begin{smallmatrix}-n,-a\\ b-n+1\end{smallmatrix};1\right)$ is undefined in line 3. To remedy this, we replace b with $b+\epsilon$ and let ϵ tend to 0. At $b+\epsilon$, each line of the equation is well defined. As $A(b)$ and $B(b)$ as expressed in lines 2 and 6 are continuous on $b \in (-1, \infty)$, letting b tend to a non-negative integer value gives us $A(b) = B(b)$, even when n is an integer greater than b, as desired.

This example can be generalized to other identities involving hypergeometric series, and in the thesis we will implicitly assume the application of this fact when we carry out hypergeometric manipulations. Furthermore, this example shows that hypergeometric series and their transforms are in general incredibly robust. The above proof holds even if A or B are series whose initial terms are 0. The only issue that we need to watch out for in proving hypergeometric identities is to ensure that the hypergeometric terms in the initial and final series are defined. In particular, we need to ensure that there are no terms with negative factorials in the numerator. Even in cases where this factor can be cancelled out by a factor in the denominator, the bounds implied by the summation may change, rendering the identity invalid. This is the other reason why we avoid allowing the b_i's to be non-positive integers. Normally, if there exists a_i and b_j such that $a_i = b_j$ in a ${}_pF_q$ series, we can remove both a_i and b_j to obtain a ${}_{p-1}F_{q-1}$ series. However, if there exists some a_i and b_j such that $a_i = b_j$, and they are the smallest negative integer in absolute value, then cancelling out a_i and b_j will change where the series terminates. In most cases, the hypergeometric functions that we use satisfy the following conditions. For our purposes, x is generally 1, -1, or $\frac{1}{2}$, and the a_i and b_i are generally integers. As our series are generally finite, there will be at least one a_i that is a non-positive integer.

To end this section, we present two hypergeometric identities that we will use later in this thesis, both of which can be found in Andrews, Askey, and Roy [1]. The first is a ${}_3F_2$ identity that holds when $x = 1$ and $a_1 = -N$ is a non-positive integer.

Theorem 1.5. *Let N be a non-negative integer, and $a, b, c, d \in \mathbb{C}$. Then, the identity*

$$ {}_3F_2\left(\begin{matrix}-N, b, c\\ d, e\end{matrix}; 1\right) = \frac{(d-c)^{(N)}}{d^{(N)}} {}_3F_2\left(\begin{matrix}-N, e-b, c\\ 1-N-d+c, e\end{matrix}; 1\right) $$

holds when both sides are well defined.

This identity can be found as a part of the proof for Sheppard's identity, on pg. 142 of Andrews, Askey, and Roy. By applying this identity to itself repeatedly, we can arrive at a group of 18 transformations, including the identity transform. The second identity is Pfaff's identity, found on pg. 68 of the same book.

Theorem 1.6. *Let $a, b, c, z \in \mathbb{C}$. Then, Pfaff's identity is given by*

$$ {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}; x\right) = (1-x)^{-a} {}_3F_2\left(\begin{matrix}a, c-b\\ c\end{matrix}; \frac{x}{x-1}\right) $$

when both sides are well defined and converge.

Note that if a is a non-positive integer and $x \neq 1$, then both sides of Pfaff's identity are polynomials in x. Furthermore, Pfaff's identity belongs to a group of 24 transformations (including the identity), known as Kummer's 24 solutions. These two identities will be instrumental in transforming our summations into forms where we can apply the Chu-Vandermonde identity.

1.4 Symmetric Group

This section covers the background of the symmetric group, as well as some of the notation and elementary results used in this thesis. The exposition of the symmetric group is taken from the book The Symmetric Group by Sagan [30]. Let $X = \{x_1, \ldots, x_n\}$ be a set with n elements, a *permutation* π of X is a bijective function from X to X. The *symmetric group* on X, denoted $\mathcal{S}_X$, is the set of all permutations of X, with the group action being function composition. In the case $X = [n]$, we will use $\mathcal{S}_n$ to denote $\mathcal{S}_{[n]}$. In this thesis, we will multiply elements of $\mathcal{S}_X$ from right to left. That is, given permutations π and σ, the permutation $\pi\sigma$ is the bijection obtained from first applying σ, then π.

Traditionally, there are three notations for describing a permutation $\pi \in \mathcal{S}_X$. The first of these is the *two-line notation*, given by the array

$$\begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ \pi(x_1) & \pi(x_2) & \cdots & \pi(x_n) \end{array}$$

The second notation, applicable when $X = [n]$, is called the *one-line notation*. For this notation, we implicitly let $x_i = i$, which allows us remove the first row, leaving $\pi(1), \pi(2), \cdots, \pi(n)$ as the result. The last notation, which for our purposes is the most important, is the *cycle notation*. As π is a bijection from X to X, each vertex of the functional digraph G of π has in-degree and out-degree 1. Therefore, G is the union of a number of directed cycles. Now, let C be a directed cycle in G of length p, and i be an element of C. Then, C contains the elements $i, \pi(i), \pi^2(i), \ldots, \pi^{p-1}(i)$ in order, with $\pi^p(i) = i$. We can write this as

$$\left(i, \pi(i), \pi^2(i), \ldots, \pi^{p-1}(i)\right)$$

By writing each cycle of G in this manner and joining the results, we arrive at the cycle notation. Note that the representation of a permutation in cycle notation is not unique. In particular, neither changing the order of the cycles, nor changing the elements which start the cycles, changes the permutation. For example, the permutation

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 1 & 5 & 2 & 6 \end{array}$$

is written as $3, 4, 1, 5, 2, 6$ in the one-line notation, and $(13)(245)(6)$ in the cycle notation. This permutation can also be written as $(452)(31)(6)$ in the cycle notation.

Now, one advantage of the cycle notation is that permutations of an arbitrary set X can be written compactly, as the elements of X are implicitly defined. Another advantage is that the cycles of a permutation can be assigned combinatorial meaning, as we shall see in the next section. Furthermore, if a permutation π is given by the set of cycles $C_1, \ldots, C_r$, then π can be broken up into the product of its cycles, each taken as a permutation by itself. That is, we can write π as the product $\pi = \pi_1 \cdots \pi_r$, where for $1 \leq t \leq r$, π_t is the permutation such that $\pi_t(i) = \pi(i)$ if $i \in C_t$, and $\pi_t(i) = i$ otherwise. Note that the permutations $\pi_1, \ldots, \pi_r$ commute. Furthermore, given a subset $\{i_1, \ldots, i_t\} \subseteq [r]$, the product $\pi' = \pi_{i_1} \cdots \pi_{i_t}$ consists of the cycles $C_{i_1}, \ldots, C_{i_t}$, and can be treated as a permutation on the elements of $C_{i_1} \cup \cdots \cup C_{i_t}$ alone. For example, by taking the first and third cycle of $(13)(245)(6)$, we have that $\pi' = (13)(6)$ is a permutation of $\{1, 3, 6\}$.

If a permutation π has r cycles of length $p_1, \ldots, p_r \geq \mathbf{1}$ in the cycle notation, then we say that π has *cycle type* $\{p_1, \ldots, p_r\}$. Furthermore, if $(i, \pi(i), \pi^2(i), \ldots, \pi^{p-1}(i))$ is a cycle of π, and $\sigma \in \mathcal{S}_X$ is any permutation, we have

$$\sigma C \sigma^{-1} = \left(\sigma(i), \sigma(\pi(i)), \sigma\left(\pi^2(i)\right), \ldots, \sigma\left(\pi^{p-1}(i)\right)\right)$$

This means that conjugating by σ relabels each element i of π by $\sigma(i)$ in the cycle notation, so the cycle type of π is invariant under conjugation. Furthermore, if we have two permutations π and ρ that have the same cycle type, we can match the cycles in the two permutations by their cycle lengths as follows

$$\begin{aligned} \pi &= (\pi_{1,1}, \ldots, \pi_{1,\ell_1}) \cdots (\pi_{r,1}, \ldots, \pi_{r,\ell_r}) \\ \rho &= (\rho_{1,1}, \ldots, \rho_{1,\ell_1}) \cdots (\rho_{r,1}, \ldots, \rho_{r,\ell_r}) \end{aligned}$$

and let $\sigma(\pi_{i,j}) = \rho_{i,j}$ be the permutation such that $\sigma \pi \sigma^{-1} = \rho$. Therefore, the subset of all permutations in $\mathcal{S}_X$ that has the same cycle type as π forms a *conjugacy class*, which we denote $\mathcal{K}_\pi$, or alternatively $\mathcal{K}_{p_1, \ldots, p_r}$ if the cycle type of π is $\{p_1, \ldots, p_r\}$. Now, the *centralizer* of π, denoted $\mathcal{Z}_\pi$, are the elements $\sigma \in \mathcal{S}_X$ such that $\sigma \pi \sigma^{-1} = \pi$. Furthermore, there is a bijection between the cosets of $\mathcal{Z}_\pi$ and the elements of $\mathcal{K}_\pi$, so that

$$|\mathcal{K}_\pi| = \frac{|\mathcal{S}_X|}{|\mathcal{Z}_\pi|}$$

Now, let π be a permutation with cycle type $\{p_1, \ldots, p_r\}$, and for $i \geq 1$, let m_i be the number of elements p_j such that $p_j = i$. Then, we have

$$|\mathcal{Z}_\pi| = 1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!$$

where $n = p_1 + \cdots + p_r$. This result follows from the cycle notation. Any permutation $\sigma \in \mathcal{Z}_\pi$ must send a cycle of length i to a cycle of the same length. So, there are $m_i!$ ways to permute the m_i cycles of length i in π. Then, for each of those cycles, we can perform a cyclic rotation in i ways, independent of the other cycles. Combining this result with the formula for $\mathcal{K}_\pi$, we have

$$|\mathcal{K}_{p_1,\dots,p_r}| = \frac{n!}{1^{m_1}m_1!2^{m_2}m_2!\cdots n^{m_n}m_n!}$$

where $n = p_1 + \cdots + p_r$, as desired.

A permutation π of a set X is a *transposition* if there exists $x, y \in X$ such that $\pi(x) = y$, $\pi(y) = x$, and $\pi(z) = z$ for all $z \in X$, $z \neq x, y$. In other words, π has exactly one cycle of length 2, namely (x, y), and all other cycles of π have length 1. To simplify the notation, we will drop the cycles of length 1 if the context for X is clear. As the remaining cycle of π is simply an unordered pair of X, we will write the transposition π as $\pi = \{x, y\}$. We now have the following elementary proposition.

Proposition 1.7. *Let π be a permutation of a set X with L cycles, and σ be a transposition $\{x, y\}$ of X. Then, $\sigma\pi$ has $L + 1$ cycles if x and y are in the same cycle of π, and $L - 1$ cycles otherwise.*

Proof. Suppose x and y are in the same cycle C. Without loss of generality, let $C = (x, \pi(x) \dots, y, \dots)$. Then, applying σ to π will break C into the cycles $(x, \dots, \pi^{-1}(y))$ and $(y, \dots, \pi^{-1}(x))$, while leaving the other cycles unchanged. Therefore, the permutation $\sigma\pi$ has $L + 1$ cycles.

Similarly, suppose x is in the cycle C_1 and y is in the cycle C_2, where $C_1 = (x, \pi(x), \dots)$ and $C_2 = (y, \pi(y), \dots)$. Then, applying σ to π will merge C_1 and C_2 into the cycle $(x, \dots, \pi^{-1}(x), y, \dots, \pi^{-1}(y))$, again leaving other cycles unchanged. Therefore, the permutation $\sigma\pi$ has $L - 1$ cycles. □

A permutation μ of a set X is an *involution* if μ^2 is the identity, and is a *fixed-point free involution* if μ is an involution that does not contain a fixed point. That is, μ is a fixed-point free involution of if for all elements $i \in X$, $\mu^2(i) = i$ and $\mu(i) \neq i$. As all cycles in a fixed-point free involutions have length 2, fixed-point free involutions of a set X are in direct bijection with pairings of X. Similarly, any involutions of a set X with k cycles of length 2 are in direct bijection with k-partial pairings of X. Therefore, in this thesis, we will often refer to transpositions as pairs, involutions as partial pairings, and fixed-point free involutions as pairings. Furthermore, we will generally use μ to denote fixed-point free involutions. This terminology with pairings will be particularly helpful when we use combinatorial objects to enumerate permutations later in this thesis.

Finally, we will look at the case $X = [p_1, \dots, p_n]$ and introduce several permutations of interest. Let $p_1, \dots, p_n$ be positive integers, let $\mathcal{S}_{p_1,\dots,p_n}$ to be the symmetric group over

$[p_1, \ldots, p_n]$. If the sum of the p_i's is even, then the set of pairings $\mathcal{P}_{p_1,\ldots,p_n}$ is the set of fixed-point free involutions of $\mathcal{S}_{p_1,\ldots,p_n}$. Recall from Section 1.2 that for vectors of non-negative integers $\mathbf{q} = (q_1, \ldots, q_n)$ and $\mathbf{s} = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$, $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ is the subset of pairings such that for $\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$, μ has q_i non-mixed pairs of the form $\{x^{\underline{i}}, y^{\underline{i}}\}$ and $s_{i,k}$ mixed pairs of the form $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$. As discussed previously, $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ can also be considered as the set of fixed-point free involutions satisfying the parameters $\mathbf{q}$ and $\mathbf{s}$.

A permutation γ is a *long cycle* if γ contains only one cycle in its cycle notation. The *canonical long cycle* of $\mathcal{S}_n$, denoted as γ_n, is given by $\gamma_n = (1, 2, \ldots, n)$. Similarly, the canonical long cycle of $\mathcal{S}_{[p_i]^{\underline{i}}}$ is denoted as $\gamma^{\underline{i}}_{p_i}$, and is given by $\gamma^{\underline{i}}_{p_i} = \left(1^{\underline{i}}, 2^{\underline{i}}, \ldots, p_i^{\underline{i}}\right)$. We can now define the *canonical permutation* of $\mathcal{S}_{p_1,\ldots,p_n}$. Given positive integers $p_1, \ldots, p_n$, the canonical permutation $\gamma_{p_1,\ldots,p_n}$ of $\mathcal{S}_{p_1,\ldots,p_n}$ is defined by

$$\begin{aligned}\gamma_{p_1,\ldots,p_n} &= \gamma^{\underline{1}}_{p_1}\gamma^{\underline{2}}_{p_2}\cdots\gamma^{\underline{n}}_{p_n} \\ &= \left(1^{\underline{1}}, 2^{\underline{1}}, \ldots, p_1^{\underline{1}}\right)\left(1^{\underline{2}}, 2^{\underline{2}}, \ldots, p_2^{\underline{2}}\right)\cdots\left(1^{\underline{n}}, 2^{\underline{n}}, \ldots, p_n^{\underline{n}}\right)\end{aligned}$$

The permutations $\mu \in \mathcal{P}_{p_1,\ldots,p_n}$ and the canonical permutation $\gamma_{p_1,\ldots,p_n}$ will be of particular importance when we discuss maps in surfaces in the next section.

1.5 Maps in Surfaces

In this section, we will be describing maps and surfaces. The background and definitions are generally taken from the survey papers of Walsh and Lehman [40], Zvonkin [44], and the book Graphs on Surfaces and Their Applications by Lando and Zvonkin [26]. For a more rigorous treatment of combinatorial maps, as well as a more general construction applicable to maps in non-orientable surfaces, see Chapter 10 of Graph Theory by Tutte [38]. We will first define the map as a topological object, then describe a way to transform it into a combinatorial object. This will allow us to relate the problem of embedding maps in surfaces to that of counting permutations in the symmetric group. As the problem will become combinatorial in nature, we will be sketchy with the topological definitions, and only define what is necessary.

For the purpose of describing maps, we allow graphs to contain loops and multiple edges. In this context, a graph $G = (V, E, I)$ is a triple consisting of a vertex set V, an edge set E, and an incidence relation I between the vertex set and the edge set. Each edge $e \in E$ is either incident to 2 vertices $u, v \in V$, or is incident to a single vertex $v \in V$. In the latter case, e is a loop edge of v, and is considered to be incident to v twice. This is different from the definition of graphs used in Section 1.2, where the incidence relation is implicitly defined by E. The separation of the edges and incidence relation is needed here to allow for loops and multiple edges.

Definition 1.8. Let $G = (V, E, I)$ be a connected graph. A *map* M is an embedding of G in an *orientable surface* X without boundary such that

- The vertices are distinct points of X.
- The edges are curves on X that only intersect at the vertices they are incident to.
- $X \backslash M$ is a set of regions each homeomorphic to an open disc, which are called *faces*. The set of faces is denoted F.

Given a map M, the *degree* of a vertex $v \in V$, denoted $\deg(v)$, is the number of edges incident to v, where loop edges incident to v are counted twice. Similarly, the *degree* of a face $f \in F$, denoted $\deg(f)$, is the number of edges incident to the f, where an edge e is counted twice if f is incident to both sides of e. In graph theoretic terms, bridges are counted twice for the face they are contained in. Observe that unlike the vertex degrees, both the number of faces and the face degrees of a map are dependent upon the surface X and the way the graph is embedded. Then, by counting the number of edges incident to each vertex, we have $\sum_{v \in V} \deg(v) = 2|E|$. Similarly, by counting the number of edges incident to each face, we have $\sum_{f \in F} \deg(f) = 2|E|$.

One of the most important attributes of a map is the *genus*, denoted g, which is defined to be the genus of the underlying surface. The genus of an orientable surface X is a non-negative integer, given by the maximum number of closed curves that can be cut on X without disconnecting it. Equivalently, it is the number of handles on the surface. For example, the sphere is a surface of genus 0, while the torus is a surface of genus 1. Note that a map embedded in a surface of genus zero can be presented as a map embedded in the plane, and graphs that have plane embeddings are called *planar graphs*. An example of a planar graph and two possible plane embeddings can be found in Figure 1.3 and Figure 1.4. Note that the face degrees of the two embeddings are different, and that the exterior of these maps constitute faces. Furthermore, one can check that the embeddings satisfy the vertex and face degree formulas.

To relate the above concepts together, we have the Euler characteristic, which is given by

$$\chi(M) = |V| - |E| + |F|$$

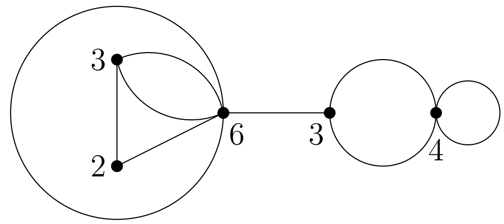


Figure 1.3: A graph with its vertex degrees labelled

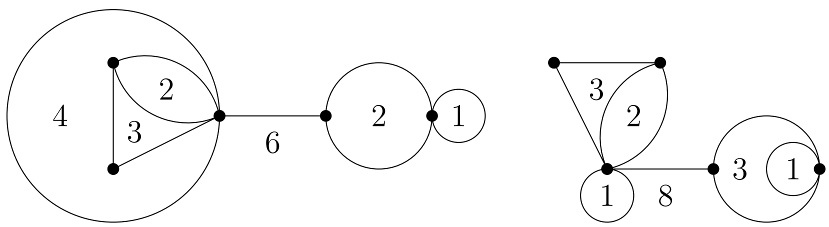


Figure 1.4: Two embeddings of the graph in Figure 1.3, with face degrees labelled

For a map M embedded in an orientable surface of genus g, $\chi(M)$ is equal to $2-2g$. Note that the maps in Figure 1.4 satisfy the Euler characteristic with $g=0$.

Now, two maps $M_1 \subset X_1$ and $M_2 \subset X_2$ are *isomorphic* if and only if there exists an orientation preserving homeomorphism $u\colon X_1 \to X_2$ such that the restriction of u on M_1 and M_2 is a graph isomorphism from G_1 to G_2. Note that this is a more general definition than continuous deformation. In Figure 1.5, we have two maps that are isomorphic, even though they cannot be continuously deformed from one to the other. Furthermore, as the surface of the map is orientated, a map and its reflection are in general not isomorphic. To avoid complications arising from maps with non-trivial automorphisms, combinatorialists generally count *rooted maps* instead. A *rooted map* is a map with a distinguished edge $e \in E$ and a direction associated with that edge. Two rooted maps $M_1 \subset X_1$ and $M_2 \subset X_2$ with distinguished edges e_1 and e_2 are isomorphic if and only if there exists an orientation preserving homeomorphism $u\colon X_1 \to X_2$ that is a map isomorphism between M_1 and M_2, and u maps e_1 to e_2 in such a way that preserves the directions associated with them. As we shall see, assigning a root edge to a map removes all non-trivial automorphisms. In Figure 1.6, we have the three non-isomorphic rooted maps with 1 vertex and 2 edges. The first two maps are embedded in the plane and have genus 0, while the third one is embedded in a torus and has genus 1.

Before discussing rooted maps, we will first motivate and describe labelled maps. Given a map M and a vertex v in M of degree p_v, observe that in the neighbourhood of v, there

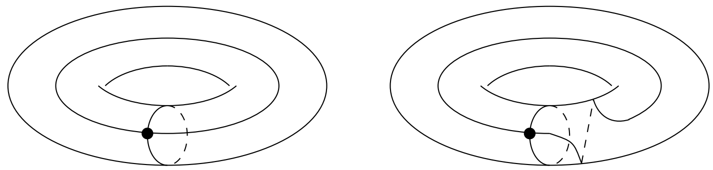


Figure 1.5: Two pictures of the same map

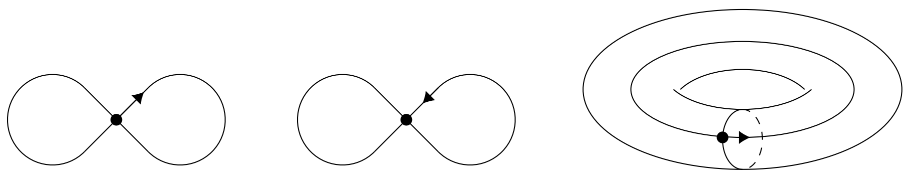


Figure 1.6: Non-isomorphic rooted maps with 1 vertex, 2 edges

are p_v pieces of edges coming out of v. We call these edge pieces *half-edges*, and the vertex v with its half-edges a *star*. As M is embedded in an orientable surface, these half-edges are arranged in some cyclic order. Furthermore, each half-edge in M must be joined to some other half-edge, dictated by the underlying graph of M. Therefore, if we label half-edges of M with a set S, we can describe both the edges and their ordering around the vertices by a pair of permutations.

Formally, let M be a map, and $D = E \times \{+, -\}$ be the set of half-edges of M. That is, each element $(e, \pm) \in D$ represents a distinct end of $e \in E$. A *labelled map* is a map M and a labelling of D with a set S of size $2|E|$. As the half-edges incident to each vertex v are in cyclic order, we can write the labels of those half-edges as a cycle in a permutation of S. By doing this for each vertex $v \in V$ and combining these cycles, we obtain a permutation γ of S, which we call the *vertex permutation* of M. By convention, the cycles of γ describe the cyclic orderings of the half-edges in counterclockwise order. Also, note that each half-edge in M is paired with another half-edge to form the edges of the map. This pairing of half-edges gives a pairing on S, which can be viewed as a fixed-point free involution. We call this involution the *edge permutation* of M, which we denote as μ. Finally, observe that the faces of M can be read off from the product $\delta = \mu\gamma^{-1}$. Consider a half-edge labelled s that is incident to a vertex v in M. Then, $\gamma^{-1}(s)$ is the half-edge incident to v that is to the right of s, and $\mu\gamma^{-1}(s)$ is the other end of that half-edge. As seen in Figure 1.7, both s and $\mu\gamma^{-1}(s)$ are incident to the same face of M on their right, labelled as F in the diagram. Furthermore, the edge containing $\mu\gamma^{-1}(s)$ is counterclockwise to the edge containing s with respect to F. Therefore, by successively applying $\mu\gamma^{-1}$ to s, we can trace out the face F. As this can be done with all half-edges of M, the cycles of $\mu\gamma^{-1}$ represent

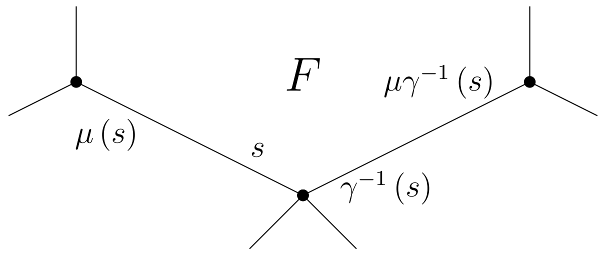


Figure 1.7: Applying $\mu\gamma^{-1}$ on a half-edge e

the faces of M. Hence, we call $\mu\gamma^{-1}$ the *face permutation* of M, and we denote it with δ.

Now, given a permutation γ and a fixed-point free involution μ on a set S, we can attempt to reconstruct the labelled map M as follows. Let each cycle of γ describes a vertex, where a cycle C of length p_v represents a vertex v with degree p_v. This is graphically depicted as a star with p_v half-edges, labelled with elements C in counterclockwise order. Then, for each pair of μ, we join together the two half-edges labelled with the elements of the pair, which becomes an edge of the graph. Note that this always constructs a graph G with all its half-edges labelled with elements of S, regardless of whether γ and μ were extracted from a map. Furthermore, the orbits of the subgroup generated by γ and μ describes the half-edges reachable from a given half-edge by moving along the vertices and edges of G. That is, each orbit is a component of G, so G is connected if and only if the subgroup generated by γ and μ is transitive. Finally, given two permutation pairs (γ_1, μ_1) and (γ_2, μ_2), the two half-edge labelled graphs constructed in this manner will be the same if and only if $\mu_1 = \mu_2$, and the cycles of γ_1 and γ_2 contain the same elements.

Suppose that the subgroup generated by γ and μ is transitive, then this reconstruction always produces a unique, valid map. This follows from Edmonds [13], which states that given a graph with its half-edges labelled, any cyclic ordering of half-edges around each vertex uniquely determines an embedding of the graph into a surface. Furthermore, we note that two distinct pairs (γ_1, μ_1) and (γ_2, μ_2) cannot produce the same labelled map. This is because any label preserving map homeomorphism must preserve neighbourhoods of the vertices, as well as the pairings of the half-edges. Hence, the cycles in the permutations γ and μ must remain the same. Therefore, each pair (γ, μ) represents one distinct labelled map and vice-versa, so we can count pairs of permutations instead of maps on surfaces.

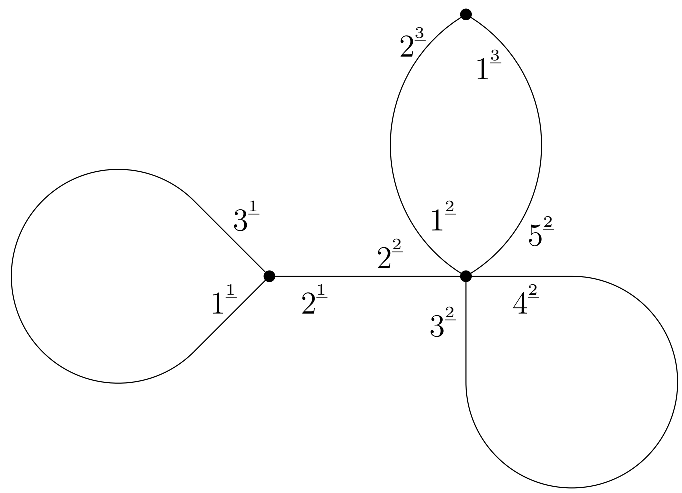


Figure 1.8: Labelled map with 3 vertices and 5 edges

For example, consider the group $\mathcal{S}_{3,5,2}$ and permutations $\gamma_{3,5,2}$ and μ where

$$\begin{aligned}\mu &= \left(1^{\underline{1}}3^{\underline{1}}\right)\left(2^{\underline{1}}2^{\underline{2}}\right)\left(1^{\underline{2}}2^{\underline{3}}\right)\left(3^{\underline{2}}4^{\underline{2}}\right)\left(5^{\underline{2}}1^{\underline{3}}\right)\\ \gamma_{3,5,2} &= \left(1^{\underline{1}}2^{\underline{1}}3^{\underline{1}}\right)\left(1^{\underline{2}}2^{\underline{2}}3^{\underline{2}}4^{\underline{2}}5^{\underline{2}}\right)\left(1^{\underline{3}}2^{\underline{3}}\right)\end{aligned}$$

This pair of permutation describes a map with 3 vertices and 5 edges, where the vertices have degrees 2, 5, and 3. The faces of this map are given by the product $\delta = \mu\gamma_{3,5,2}^{-1}$, which is

$$\mu\gamma_{3,5,2}^{-1} = \left(1^{\underline{1}}\right)\left(2^{\underline{1}}3^{\underline{1}}2^{\underline{2}}2^{\underline{3}}5^{\underline{2}}3^{\underline{2}}\right)\left(1^{\underline{2}}1^{\underline{3}}\right)\left(4^{\underline{2}}\right)$$

Therefore, this map has 4 faces, with face degrees 1, 1, 2, and 6. A diagram of this map can be found in Figure 1.8. The labels of the half-edges are placed on the right hand side of their respective half-edges, to make it easier to trace the face permutation.

Remark 1.9. Some authors may use a different direction for the permutation γ, or a different combination of μ and γ, such as $\delta = \gamma\mu$, to describe the faces of the map M. In terms of enumerating maps, these are all equivalent. The only differences between these alternate conventions are the relative positions of the faces with respect to the edges they are incident to, and the direction with which we trace the edges incident to the faces. By graph duality, the roles of γ and δ can be inverted as well, with γ describing the faces and δ the vertices. Pictorially, this can be seen as gluing faces along their edges together, as opposed to the gluing of half-edges we have described here. A more detailed description of this alternate picture can be found in Chapter 3 of Lando and Zvonkin [26], where they discuss the enumeration of one vertex maps.

Given that we can describe labelled maps uniquely as pairs of permutations, we will now dispense with the topology and describe maps in purely combinatorial terms.

Definition 1.10. Let S be a set of size $2d$ for some $d \geq 1$. A *combinatorial map* with n vertices, d edges, and L faces is a pair of permutations (γ, μ) such that

- γ is a permutation of S with n cycles.
- μ is a fixed-point free involution of S.
- $\delta = \mu\gamma^{-1}$ is a permutation of S with L cycles.
- The subgroup generated by γ and μ is transitive.

As noted above, a labelled map is equivalent to a combinatorial map. Furthermore, the vertices of the combinatorial map have degrees $p_1, \ldots, p_n$ if the cycles of γ have length $p_1, \ldots, p_n$, and the faces of the combinatorial map have face degrees $h_1, \ldots, h_L$ if the cycles of δ have length $h_1, \ldots, h_L$. For convenience, we will sometimes include pairs (γ, μ) such that the subgroup generated by γ and μ is not transitive in our discussion of combinatorial maps. When we need to distinguish between the two, we will call (γ, μ) a *connected* combinatorial map if the subgroup generated by γ and μ is transitive, and a *disconnected* combinatorial map otherwise. In general, disconnected combinatorial maps correspond to graphs that have multiple components, so they do not correspond to rooted maps. One way to view disconnected maps topologically is to view each component as a map embedded in its own surface, and the genus of the map is given by the sum of the genera of the components. This way of defining the genus for disconnected maps is consistent with the Euler characteristic, when applied to all components of the map as a whole.

Next, we will show the relationship between rooted maps and combinatorial maps. Let M be a rooted map with d edges. We want to label M with a set S of size $2d$ in $(2d-1)!$ ways. Note that in constructing the rooted map M, we are choosing an edge $e \in E$ and a direction for that edge. This is equivalent to picking a half-edge of M as the root half-edge, which we will take to be the half-edge on e that is away from the direction of the root. Now, to label the map M with the set S, we will label the root half-edge with 1 if $S = \mathcal{S}_{2d}$, or 1^1 if $S = \mathcal{S}_{p_1,\ldots,p_n}$. Then, we can label the remaining $2d-1$ half-edges arbitrarily in $(2d-1)!$ ways. Each of these labellings corresponds to a combinatorial map (γ, μ). Furthermore, if two labellings M_1 and M_2 of M give the same combinatorial map (γ, μ), then M_1 and M_2

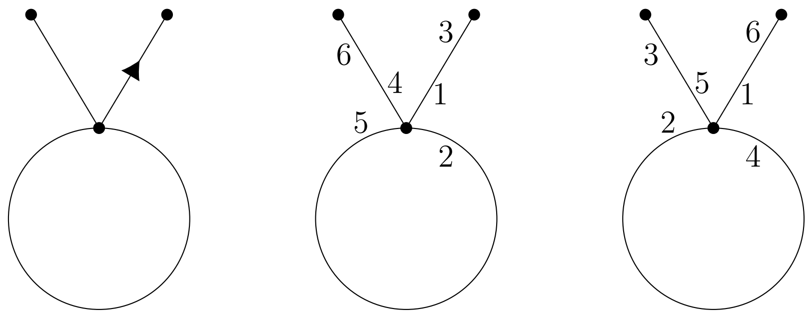


Figure 1.9: A rooted map and two ways of labelling its half-edges

must in fact be the same. By construction, the root half-edge is labelled with 1 in both M_1 and M_2. Next, if a half-edge of M is labelled by the same element $s \in S$ in M_1 and M_2, then the half-edge counterclockwise to it must be labelled $\gamma(s)$ in both M_1 and M_2. Similarly, the half-edge that forms the other end of this half-edge must be labelled $\mu(s)$ in both M_1 and M_2. As the subgroup generated by γ and μ is transitive, all of the half-edges of M must be labelled the same between M_1 and M_2. Conversely, given any combinatorial map, we can obtain a rooted map by first creating a labelled map using Edmonds' result, then use the label 1 to recover the root edge and its direction. Therefore, for every root map M with d edges, there are exactly $(2d-1)!$ combinatorial maps corresponding to it. As an example, we have a rooted map and two ways to label the half-edges in Figure 1.9.

Note that this also shows that a rooted map cannot have any non-trivial automorphisms. To prove this fact, we arbitrary label the rooted map to obtain a labelling M_1. Then, we can apply an automorphism to the rooted map to obtain another labelling M_2. As the root edge is preserved, both M_1 and M_2 have the same label on the root half-edge. Furthermore, the automorphism preserves the neighbourhoods of the vertices and the pairings of the half-edges, so both M_1 and M_2 must give the same combinatorial map (γ, μ). Therefore, we have that $M_1 = M_2$, so the automorphism must in fact be trivial.

In the literature of combinatorial maps, we generally count maps where the number of vertices and their degrees are given. Furthermore, we generally enumerate maps according to their genus, or equivalently, their number of faces. Formally, for $p_1, \ldots, p_n \geq 1$, let $\mathcal{M}_n^{\mathbf{p}}$ be the set of possibly disconnected combinatorial maps (γ, μ) such that γ has cycle type $\mathbf{p} = \{p_1, \ldots, p_n\}$. Then, for $L \geq 1$, let $\mathcal{M}_{n,L}^{\mathbf{p}} \subseteq \mathcal{M}_n^{\mathbf{p}}$ be the subset of maps such that for $(\gamma, \mu) \in \mathcal{M}_{n,L}^{\mathbf{p}}$, $\mu\gamma^{-1}$ has exactly L cycles. Finally, let $\overline{\mathcal{M}_n^{\mathbf{p}}} \subseteq \mathcal{M}_n^{\mathbf{p}}$ and $\overline{\mathcal{M}_{n,L}^{\mathbf{p}}} \subseteq \mathcal{M}_n^{\mathbf{p}}$ be the subsets of maps that are connected. Now, one method to make the counting of these maps easier is to fix the permutation γ, then count over the permutations μ subjected to certain restrictions, depending on the type of maps that is being counted. To this end,

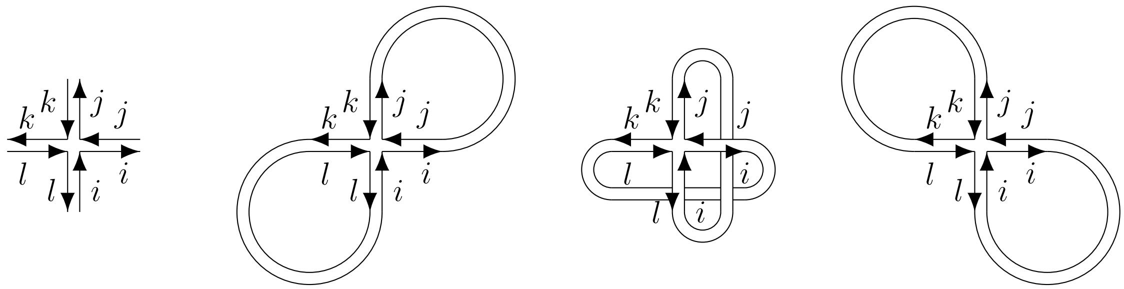


Figure 1.10: Feynman diagrams of 1 vertex maps with 2 edges

we say that a combinatorial map is *canonical* if $\gamma = \gamma_{p_1,\dots,p_n}$ for some $p_1,\dots,p_n \geq \mathbf{1}$. As μ ranges over all pairings in $\mathcal{P}_{p_1,\dots,p_n}$, there are exactly $d = \frac{p_1+\cdots+p_n}{2}$ possibly disconnected maps with these parameters. We let $\mathcal{C}^{\mathbf{p}}_{n,L} \subseteq \mathcal{P}_{p_1,\dots,p_n}$ be the subset of pairings such that for $\mu \in \mathcal{C}^{\mathbf{p}}_{n,L}$, $\mu\gamma^{-1}_{p_1,\dots,p_n}$ has exactly L cycles, and let $\overline{\mathcal{C}^{\mathbf{p}}_{n,L}} \subseteq \mathcal{C}^{\mathbf{p}}_{n,L}$ be the subset of pairings such that for $\mu \in \overline{\mathcal{C}^{\mathbf{p}}_{n,L}}$, the map $(\gamma_{p_1,\dots,p_n}, \mu)$ is connected. For example, the labelled map in Figure 1.8 corresponds to a canonical combinatorial map with its elements in $\mathcal{S}_{3,5,2}$. For simplicity, all references to combinatorial maps in later chapters implicitly refer to canonical combinatorial maps.

One way to represent canonical combinatorial maps is to use *Feynman diagrams*, also known as ribbon graphs. Instead of representing each half-edge with a single line, we represent them with a ribbon. For each edge, we label the adjacent corner using the element of S that represents the edge, and glue the ribbons together without twisting. For example, a 4-star and all three of its possible gluings are represented in Figure 1.10. Note that in this example, we take $S = \{i, j, k, \ell\}$. This set of labels and the method of labelling corners will be useful when we discuss the algebraic method of enumerating maps in Chapter 2. Furthermore, we will later show that the Feymann diagrams can be used to describe and enumerate maps in locally oriented surfaces in Section 2.5.

Next, we will show the relationship between the number of canonical combinatorial maps and the number of combinatorial maps in general using the following proposition

Proposition 1.11. *Let (γ, μ) be a pair of permutations on a set S such that μ is a fixed-point free involution. Then, $(\gamma', \mu') = (\sigma\gamma\sigma^{-1}, \sigma\mu\sigma^{-1})$ is also a pair of permutations on S, where γ, μ, and $\mu\gamma^{-1}$ have the same cycle types as γ', μ', and $\mu'(\gamma')^{-1}$, respectively. Furthermore, (γ, μ) represents a connected combinatorial map if and only if (γ', μ') represents one.*

Proof. From our discussion in Section 1.4, conjugating a permutation π by σ does not change the cycle type of π, so γ and μ have the same cycle types as γ' and μ', respectively. Then, we see that $\mu'(\gamma')^{-1} = \sigma\mu\gamma^{-1}\sigma^{-1}$, so $\mu\gamma^{-1}$ have the same cycle type as $\mu'(\gamma')^{-1}$. Now, let $\pi_1, \ldots, \pi_t \in \{\gamma, \mu\}$ be a sequence of permutations. Then,

$$\begin{aligned} \pi_1\pi_2\cdots\pi_t(i) &= j \\ \sigma\pi_1\pi_2\cdots\pi_t\sigma^{-1}(\sigma(i)) &= \sigma(j) \\ \sigma\pi_1\sigma^{-1}\sigma\pi_2\sigma^{-1}\cdots\sigma\pi_t\sigma^{-1}(\sigma(i)) &= \sigma(j) \end{aligned}$$

This means that if a sequence of permutations using γ and μ maps i to j, then the same sequence of permutations using γ' and μ' maps $\sigma(i)$ to $\sigma(j)$, and vice-versa. Therefore, the subgroup generated by γ and μ is transitive if and only if the subgroup generated by γ' and μ' is transitive, so (γ, μ) is connected if and only if (γ', μ') is also connected. □

In the cycle notation, conjugating γ and μ by σ is the same as replacing each element $s \in S$ in their cycles by $\sigma(s)$. In topological terms, this is the same as relabelling each half-edge s of the labelled map represented by (γ, μ) with $\sigma(s)$. Hence, (γ, μ) and (γ', μ') represent the same map if we ignore the labels.

For each fixed permutation $\gamma_0 \in \mathcal{K}_{p_1,\ldots,p_r}$, we know from Section 1.4 that there exists $\sigma \in \mathcal{S}_{p_1,\ldots,p_n}$ such that $\sigma\gamma_0\sigma^{-1} = \gamma_{p_1,\ldots,p_n}$, as γ_0 and $\gamma_{p_1,\ldots,p_n}$ have the same cycle type. So, for each pairing $\mu \in \mathcal{P}_{p_1,\ldots,p_n}$, the pair (γ_0, μ) is a combinatorial map if and only if $(\gamma_{p_1,\ldots,p_n}, \sigma\mu\sigma^{-1})$ is also a combinatorial map. Furthermore, by Proposition 1.11, $\mu\gamma_0^{-1}$ has the same number of cycles as $(\sigma\mu\sigma^{-1})\gamma_{p_1,\ldots,p_n}^{-1}$. This gives a genus preserving bijection between canonical combinatorial maps and combinatorial maps of the form (γ_0, μ). Therefore, if we are to fix a permutation $\sigma_\gamma \in \mathcal{S}_{p_1,\ldots,p_n}$ for each $\gamma \in \mathcal{K}_{p_1,\ldots,p_n}$ such that $\sigma_\gamma\gamma(\sigma_\gamma)^{-1} = \gamma_{p_1,\ldots,p_n}$, then every pairing in $\overline{\mathcal{C}^{\mathbf{p}}_{n,L}}$ corresponds to $|\mathcal{K}_{p_1,\ldots,p_n}|$ combinatorial maps in $\overline{\mathcal{M}^{\mathbf{p}}_{n,L}}$. So, if we let $\mathcal{R}^{\mathbf{p}}_{n,L}$ be the number of rooted maps with degree sequence $\{p_1, \ldots, p_n\}$ and L faces, then

$$\begin{aligned} (2d-1)!\left|\overline{\mathcal{R}^{\mathbf{p}}_{n,L}}\right| &= \left|\overline{\mathcal{M}^{\mathbf{p}}_{n,L}}\right| = \frac{(2d)!}{\left(\prod_j m_j!\right)\left(\prod_i i^{m_i}\right)}\left|\overline{\mathcal{C}^{\mathbf{p}}_{n,L}}\right| \\ \left|\overline{\mathcal{R}^{\mathbf{p}}_{n,L}}\right| &= \frac{2d}{\left(\prod_j m_j!\right)\left(\prod_i i^{m_i}\right)}\left|\overline{\mathcal{C}^{\mathbf{p}}_{n,L}}\right| \end{aligned}$$

where $d = \frac{p_1+\cdots+p_n}{2}$, and for $i \geq 1$, m_i is the number of elements p_j such that $p_j = i$. Furthermore, the same proof shows that this relation holds for $\mathcal{R}^{\mathbf{p}}_{n,L}$ and $\mathcal{C}^{\mathbf{p}}_{n,L}$ as well. A brief discussion of how to enumerate canonical combinatorial maps of this type by genus can be found in Section 2.5.

Our main focus in this thesis is the enumeration of rooted maps for a fixed underlying graph. Given a graph G with vertices labelled $1, \ldots, n$ that allows for loops and multiple edges, we can uniquely describe it with a vector $\mathbf{q} = (q_1, \ldots, q_n)$ of length n and a strictly upper triangular matrix $\mathbf{s} = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$ of size $n \times n$. We do this by simply letting q_i be the number of loop edges on vertex i, and $s_{i,k}$ be the number of edges between vertices i and k. For convenience, we let $s_{i,k} = s_{k,i}$ if $i > k$, and $p_i = 2q_i + \sum_{k \neq i} s_{i,k}$ as in Section 1.2. By construction, p_i is the degree of vertex i in G. Furthermore, the set of pairings $\mathcal{P}_n^{(\mathbf{q},\mathbf{s})}$ can be used to represent the set of canonical combinatorial maps that satisfies these conditions. Hence, we can let $\mathcal{A}_{n,L}^{(\mathbf{q},\mathbf{s})} \subseteq \mathcal{P}_n^{(\mathbf{q},\mathbf{s})}$ be the subset of pairings such that for $\mu \in \mathcal{A}_{n,L}^{(\mathbf{q},\mathbf{s})}$, $\mu\gamma_{p_1,\ldots,p_n}^{-1}$ has L cycles. Note that it is unnecessary to specify whether we count disconnected maps, as connectivity of the maps is given by the connectivity of G.

Now, let $\mathcal{M}_n^{(\mathbf{q},\mathbf{s})}$ be the set of combinatorial maps such that for $(\gamma, \mu) \in \mathcal{M}_n^{(\mathbf{q},\mathbf{s})}$, the map represent by (γ, μ) is an embedding of G. In other words, if $(\gamma, \mu) \in \mathcal{M}_n^{(\mathbf{q},\mathbf{s})}$, then there exists a labelling ϕ of the cycles of γ with $1, \ldots, n$ such that the following holds: For $1 \leq i \leq n$, there are q_i pairs $\{x, y\}$ in μ where both x and y are in the i'th cycle of γ. Also, for $i < k$, there are $s_{i,k}$ pairs $\{x, y\}$ in μ such that x is in the i'th cycle of γ and y is in the k'th cycle of γ. In topological terms, $\mathcal{M}_n^{(\mathbf{q},\mathbf{s})}$ is the set of maps such that for $(\gamma, \mu) \in \mathcal{M}_n^{(\mathbf{q},\mathbf{s})}$, there is a labelling ϕ of the vertices so that (γ, μ) represents an embedding of the labelled graph G. Note that we can view ϕ as a function $\phi\colon [p_1, \ldots, p_n] \to [n]$ that maps two elements to the same output if and only if they belong to the same cycle. Additionally, we can deduce that if a cycle is labelled i by ϕ, then it must have length p_i regardless of the value of μ.

Next, let $\mathcal{M}_{n,L}^{(\mathbf{q},\mathbf{s})} \subseteq \mathcal{M}_n^{(\mathbf{q},\mathbf{s})}$ to be the subset of maps such that for $(\gamma, \mu) \in \mathcal{M}_{n,L}^{(\mathbf{q},\mathbf{s})}$, $\mu\gamma^{-1}$ has L cycles. Furthermore, let $\mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}$ be the set of triples (γ, μ, ϕ) such that $(\gamma, \mu) \in \mathcal{M}_{n,L}^{(\mathbf{q},\mathbf{s})}$, and ϕ is a labelling of the cycles of γ that makes (γ, μ) satisfy the conditions of $\mathbf{q}$ and $\mathbf{s}$ in the previous paragraph. Finally, let $\mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})} \subseteq \mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}$ be the subset of triples (γ, μ, ϕ) such that $\gamma = \gamma_{p_1,\ldots,p_n}$. Note that each graph automorphism of G is a labelling of the vertices, so it corresponds to a permutation of the output of ϕ that preserves the conditions given by $\mathbf{q}$ and $\mathbf{s}$. Therefore, each map $(\gamma, \mu) \in \mathcal{M}_{n,L}^{(\mathbf{q},\mathbf{s})}$ corresponds to $|\mathrm{aut}\,(G)|$ triples $(\gamma, \mu, \phi) \in \mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}$, which gives the relation $\left|\mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}\right| = |\mathrm{aut}\,(G)| \left|\mathcal{M}_{n,L}^{(\mathbf{q},\mathbf{s})}\right|$.

As with counting maps with fixed vertex degrees, we know that for each fixed permutation $\gamma_0 \in \mathcal{K}_{p_1,\ldots,p_r}$, there exists $\sigma \in \mathcal{S}_{p_1,\ldots,p_n}$ such that $\sigma\gamma_0\sigma^{-1} = \gamma_{p_1,\ldots,p_n}$. This means that for a given triple $(\gamma_0, \mu, \phi) \in \mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}$, we can conjugate γ and μ with σ to replace each element s of the map (γ_0, μ) with $\sigma(s)$. To preserve the labelling of the cycles, we apply σ^{-1}

to undo the effect of the conjugation before applying ϕ, which gives us $\phi\sigma^{-1}$. Combining these together gives us $(\sigma\gamma_0\sigma^{-1}, \sigma\mu\sigma^{-1}, \phi\sigma^{-1})$, and we can use the elements $\sigma(s)$ to verify that it satisfies the conditions of $\mathbf{q}$ and $\mathbf{s}$. We can also verify that both the labelling of the cycles and the number of cycles in $\mu\gamma^{-1}$ are preserved. Furthermore, by replacing σ with σ^{-1}, we see that a triple (γ_0, μ, ϕ) is in $\mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}$ if and only if $(\gamma_{p_1,\ldots,p_n}, \sigma\mu\sigma^{-1}, \phi\sigma^{-1})$ is in $\mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}$. This gives a bijection between the set $\mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}$ and the subset of $\mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}$ such that $\gamma = \gamma_0$. Therefore, if we are to fix a permutation $\sigma_\gamma \in \mathcal{S}_{p_1,\ldots,p_n}$ for each $\gamma \in \mathcal{K}_{p_1,\ldots,p_n}$ such that $\sigma_\gamma\gamma(\sigma_\gamma)^{-1} = \gamma_{p_1,\ldots,p_n}$, then every triple in $\mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}$ corresponds to $|\mathcal{K}_{p_1,\ldots,p_n}|$ triples in $\mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}$. In other words, we have $\left|\mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}\right| = |\mathcal{K}_{p_1,\ldots,p_n}| \left|\mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}\right|$.

Let $\phi_0\colon [p_1,\ldots,p_n] \to [n]$ and suppose $\phi(x^{\underline{i}}) = k_i$, where $1 \le i, k_i \le n$, is a labelling of the cycles of $\gamma_{p_1,\ldots,p_n}$. For there to exist a $\mu \in \mathcal{P}_{p_1,\ldots,p_n}$ such that $(\gamma_{p_1,\ldots,p_n}, \mu, \phi_0) \in \mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}$, the cycle labelled i must have length p_i for $1 \le i \le n$. Now, let $\rho \in \mathcal{S}_{p_1,\ldots,p_n}$ be the permutation such that $\rho(x^{\underline{i}}) = x^{\underline{k_i}}$. Then, we get that $\phi\rho^{-1}\left(x^{\underline{k_i}}\right) = k_i$ for $1 \le i \le n$, which can also be expressed as the function $\phi_I\colon [p_1,\ldots,p_n] \to [n]$ such that $\phi_I(x^{\underline{i}}) = i$ for all $x^{\underline{i}} \in [p_1,\ldots,p_n]$. By the same reason as in the previous paragraph, we can apply ρ to get that $(\gamma_{p_1,\ldots,p_n}, \mu, \phi_0) \in \mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}$ if and only if $(\gamma_{p_1,\ldots,p_n}, \rho\mu\rho^{-1}, \phi_I) \in \mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}$. Furthermore, note that if we fix $\gamma = \gamma_{p_1,\ldots,p_n}$ and ϕ_I to be such that $\phi_I(x^{\underline{i}}) = i$ for all i, then $\mu \in \mathcal{A}_{n,L}^{(\mathbf{q},\mathbf{s})}$ if and only if $(\gamma_{p_1,\ldots,p_n}, \mu, \phi_I) \in \mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}$ by definition. Therefore, we have a bijection between the set $\mathcal{A}_{n,L}^{(\mathbf{q},\mathbf{s})}$ and the subset of $\mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}$ such that $\phi = \phi_0$. Now, if we let m_i to be the number of elements p_j such that $p_j = i$, then there are $m_1! \cdots m_n!$ functions $\phi\colon [p_1,\ldots,p_n] \to [n]$ to label $\gamma_{p_1,\ldots,p_n}$ such that the cycle labelled i has length p_i. Therefore, every pairing $\mu \in \mathcal{A}_{n,L}^{(\mathbf{q},\mathbf{s})}$ corresponds to $m_1! \cdots m_n!$ triples in $\mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}$. Combining this with the previous result gives $\left|\mathcal{D}_{n,L}^{(\mathbf{q},\mathbf{s})}\right| = \frac{(2d)!}{p_1\cdots p_n}\left|\mathcal{A}_{n,L}^{(\mathbf{q},\mathbf{s})}\right|$.

Finally, if $\mathcal{R}_{n,L}^{(\mathbf{q},\mathbf{s})}$ is the number of rooted embeddings of G with L faces, then we have these two relationships

$$\begin{aligned}(2d-1)!\left|\mathcal{R}_{n,L}^{(\mathbf{q},\mathbf{s})}\right| &= \left|\mathcal{M}_{n,L}^{(\mathbf{q},\mathbf{s})}\right| \\ |\mathrm{aut}(G)|\left|\mathcal{M}_{n,L}^{(\mathbf{q},\mathbf{s})}\right| &= \left|\mathcal{B}_{n,L}^{(\mathbf{q},\mathbf{s})}\right| = \frac{(2d)!}{p_1\cdots p_n}\left|\mathcal{A}_{n,L}^{(\mathbf{q},\mathbf{s})}\right|\end{aligned}$$

Combining these gives

$$\left|\mathcal{R}_{n,L}^{(\mathbf{q},\mathbf{s})}\right| = \frac{2d}{p_1\cdots p_n\cdot|\mathrm{aut}(G)|}\left|\mathcal{A}_{n,L}^{(\mathbf{q},\mathbf{s})}\right|$$

as desired. This gives the number of rooted maps that are embeddings of the graph G in terms of the number of canonical combinatorial maps. In the next chapter, we will formally state this problem purely in the language of permutations, and show some of the techniques used to solve restricted cases of this problem.

Chapter 2

Techniques in Map Enumeration

In the last chapter, we saw that we can describe the problem of enumerating maps on surfaces as one about multiplying permutations. We will therefore begin this chapter by stating the problem of enumerating maps corresponding to specific graphs formally, in terms of multiplying permutations together. We will then provide some elementary results that allow us to restrict our attention to certain sets of permutations, as well as set up a framework that is used in the multiple approaches to tackling this problem. In the subsequent sections, we will discuss the special cases corresponding to maps with one or two vertices, as well as surveying some of the previous techniques used in deriving these results. The techniques presented here are mainly algebraic, in contrast to the main approach used later in this thesis. In the final section, we will briefly cover some of the other map enumeration problems that can be solved by encoding them as problems of multiplying permutations.

2.1 Problem Statement

Let n be a positive integer, $\mathbf{q} = (q_1, \ldots, q_n)$ be a vector of length n, and $\mathbf{s} = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n})$ be a strictly upper triangular matrix of size $n \times n$, where the q_i's and $s_{i,k}$'s are non-negative integers for $1 \leq i, k \leq n$, with $i < k$. For convenience, let $s_{k,i} = s_{i,k}$, $s_i = \sum_{k \neq i} s_{i,k}$, and $p_i = 2q_i + s_i$, as in Chapter 1. Let $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ be the set of pairings of $[p_1, \ldots, p_n]$ with q_i non-mixed pairs of the form $\{x^{\underline{i}}, y^{\underline{i}}\}$ and $s_{i,k}$ mixed pairs of the form $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$. Let $\gamma_{p_1,\ldots,p_n}$ be the canonical cycle permutation of $\mathcal{S}_{p_1,\ldots,p_n}$, given by $\gamma_{p_1,\ldots,p_n} = \left(1^{\underline{1}}, \ldots, p_1^{\underline{1}}\right) \cdots \left(1^{\underline{n}}, \ldots, p_n^{\underline{n}}\right)$. For $L \geq 1$, we define $\mathcal{A}_{n,L}^{(\mathbf{q};\mathbf{s})} \subseteq \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ to be the subset of pairings such that for $\mu \in \mathcal{A}_{n,L}^{(\mathbf{q};\mathbf{s})}$, $\mu\gamma_{p_1,\dots,p_n}^{-1}$ has exactly L cycles, and let $a_{n,L}^{(\mathbf{q};\mathbf{s})} = \left|\mathcal{A}_{n,L}^{(\mathbf{q};\mathbf{s})}\right|$. Then, our goal is to determine an expression for the generating series

$$A_n^{(\mathbf{q};\mathbf{s})}(x) = \sum_{L\geq 1} a_{n,L}^{(\mathbf{q};\mathbf{s})} x^L$$

for given values of n, $\mathbf{q}$, and $\mathbf{s}$. Equivalently, the series can also be written as

$$A_n^{(\mathbf{q};\mathbf{s})}(x) = \sum_{\mu\in\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}} x^{w(\mu)}$$

where w is the weight function on $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$, defined such that for $\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$, $w(\mu)$ is the number of disjoint cycles in the permutation $\mu\gamma_{p_1,\dots,p_n}^{-1}$.

In the language of enumerating maps, this generating series counts the number of combinatorial maps with n vertices and L faces, such that there are q_i loop edges incident to vertex i, and $s_{i,j}$ edges between vertices i and j. Furthermore, the combinatorial maps counted in this series are connected if and only if the support graph of $\mathbf{s}$ is connected. As in Chapter 1, we let $d = \frac{1}{2}\sum_{i=1}^n p_i$ be the total number of pairs of μ, which also represents the total number of edges in the combinatorial map. By Proposition 1.7, the number of cycles of a permutation changes parity whenever it is multiplied by a transposition, so $\mu\gamma_{p_1,\dots,p_n}^{-1}$ has the same parity as $n+d$. Therefore, $A_n^{(\mathbf{q};\mathbf{s})}(x)$ is a polynomial with non-negative integer coefficients, which is an even polynomial if $n+d$ is even, and is an odd polynomial if $n+d$ is odd.

We will now state a few elementary propositions related to the problem statement above, which will in turn allow us to state an assumption that we will use for the rest of the thesis. Although these results are more easily proved within the context of maps and graph theory, we will prove them in the context of multiplying permutations, so as to abstract the problem from its graph theoretical roots.

Proposition 2.1. *Let μ be a pairing in $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$, where the support graph of $\mathbf{s}$ has r components. Then, $\mu\gamma_{p_1,\dots,p_n}^{-1}$ has at most $2r-n+d$ cycles.*

Proof. Let G be the support graph of $\mathbf{s}$, and $C_1,\dots,C_r$ be the components of G. Let $T_1,\dots,T_r$ be the spanning trees of $C_1,\dots,C_r$, and observe that the r trees have $n-r$ edges in total. If $e=\{i,k\}$ is an edge of T_i, then μ must contain at least one pair of the form $\{x^{\underline{i}}, y^{\underline{k}}\}$. Hence, for each edge $e=\{i,k\}$ in $T_1\cup\cdots\cup T_r$, we can take an arbitrary pair of the form $\{x^{\underline{i}}, y^{\underline{k}}\}$, and denote the transposition consisting of that pair as μ_e. As discussed in Section 1.4, we can decompose μ into a product of transpositions consisting of the d pairs of μ, which commute with each other. Therefore, we can write $\mu = \bar{\mu}\mu_{e_{n-r}} \cdots \mu_{e_1}$, where the transpositions $\mu_{e_1}, \ldots, \mu_{e_{n-r}}$ are given by the edges of $T_1 \cup \cdots \cup T_r$, and $\bar{\mu} = \mu\mu_{e_{n-r}} \cdots \mu_{e_1}$ is an involution consisting of the $d - n + r$ remaining pairs of μ.

Let G_t be the graph on n vertices and edge set $e_1, \ldots, e_t$ for $0 \leq t \leq n - r$. We will now show inductively that each cycle of $\mu_{e_t} \cdots \mu_{e_1}\gamma^{-1}_{p_1,\ldots,p_n}$ corresponds to a component of G_t. Recall that $\gamma^{-1}_{p_1,\ldots,p_n}$ contains n cycles, each of the form $\left(1^{\underline{i}}, \ldots, p_i^{\underline{i}}\right)$. As G_0 has no edges, each component of G_0 is a single vertex, so the base case holds. Assume that this holds for $\mu_{e_{t-1}} \cdots \mu_{e_1}\gamma^{-1}_{p_1,\ldots,p_n}$ and G_{t-1}, and let $e_t = \{i, k\}$. Then, μ_{e_t} is a transposition of the form $\left(x^{\underline{i}}, y^{\underline{k}}\right)$ by construction. As the edges $e_1, \ldots, e_{n-r}$ form a forest, i and k must be in two different components of G_{t-1}, so $x^{\underline{i}}$ and $y^{\underline{k}}$ must be in different cycles of $\mu_{e_t} \cdots \mu_{e_1}\gamma^{-1}_{p_1,\ldots,p_n}$. By Proposition 1.7, multiplying by μ_{e_t} merges these two cycles into one. Similarly, adding e_t to G_{t-1} merges these two components into one in G_t. As all the other cycles and components are unchanged, this proves our statement.

From this, we deduce that $\mu_{e_{n-r}} \cdots \mu_{e_1}\gamma^{-1}_{p_1,\ldots,p_n}$ has r cycles. By Proposition 1.7, multiplying $\mu_{e_{n-r}} \cdots \mu_{e_1}\gamma^{-1}_{p_1,\ldots,p_n}$ by each of the $d-n+r$ transpositions in $\bar{\mu}$ can at most increase the number of cycles by 1. Therefore, $\mu\gamma^{-1}_{p_1,\ldots,p_n}$ contains at most $2r - n + d$ cycles, as desired. ∎

This gives an upper bound on the degree of $A_n^{(\mathbf{q};\mathbf{s})}(x)$, which can be useful in computing the polynomial $A_n^{(\mathbf{q};\mathbf{s})}(x)$. In graph theoretic terms, permutations μ such that $\mu\gamma^{-1}_{p_1,\ldots,p_n}$ has $2r - n + d$ cycles are the maps of genus 0. As such, $A_n^{(\mathbf{q};\mathbf{s})}(x)$ may not necessary attain its maximal degree, since doing so requires the underlying graph to be planar.

Proposition 2.2. *Let μ be a pairing in $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$, such that the support graph of $\mathbf{s}$ has r components $C_1, \ldots, C_r$. Suppose the component C_t contains the vertices $c_{t-1}+1, \ldots, c_t$ for $1 \leq t \leq r$, where $0 = c_0 < c_1 < \cdots < c_r = n$. Then, $A_n^{(\mathbf{q};\mathbf{s})} = A_{c_1-c_0}^{(\mathbf{q}_1;\mathbf{s}_1)} \times \cdots \times A_{c_r-c_{r-1}}^{(\mathbf{q}_k;\mathbf{s}_k)}$, where $\mathbf{q}_t = \left(q_{c_{t-1}+1}, \ldots, q_{c_t}\right)$ is the vector of length $c_t - c_{t-1}$ containing the $c_{t-1}+1$ to c_t entries of $\mathbf{q}$, and $\mathbf{s}_t$ is the submatrix consisting of the diagonal block of $\mathbf{s}$ between rows and columns $c_{t-1}+1$ to c_t.*

Proof. Let $S_t = \left[p_{c_{t-1}+1}\right]^{\underline{c_{t-1}+1}} \cup \cdots \cup \left[p_{c_t}\right]^{\underline{c_t}}$ for $1 \leq t \leq r$, and $\mathcal{P}_t$ be the set of pairings of S_t such that for $\mu_t \in \mathcal{P}_t$, μ_t has q_i non-mixed pairs of the form $\{x^{\underline{i}}, y^{\underline{i}}\}$ and $s_{i,k}$ mixed pairs of the form $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$, where $c_{t-1}+1 \leq i, k \leq c_t$ and $i < k$. Then, if $\mu_e = \left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ is a pair of $\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$, then i and k must be in the component of the support graph of $\mathbf{s}$, as we have either $i = k$ or $s_{i,k} \neq 0$. Therefore, both elements of μ_e are in the same S_t for some t. Now, let μ_t be the product of all transpositions where both elements are in S_t for $1 \le t \le r$. Then, we have $\mu_t \in \mathcal{P}_t$ by counting the number of mixed and non-mixed pairs, and μ can be decomposed as $\mu = \mu_1 \mu_2 \cdots \mu_r$. Conversely, given $\mu_t \in \mathcal{P}_t$ for $1 \le t \le r$, we have that $\mu = \mu_1 \mu_2 \cdots \mu_r$ is an element of $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$, as the pairs of $\mu_1, \ldots, \mu_r$ do not have any element in common. Together, we conclude that $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_r$.

Similarly, we let $\gamma^{\underline{i}} = \left(1^{\underline{i}} \cdots p_i^{\underline{i}}\right)$ for $1 \le i \le n$, and let $\gamma_t = \gamma^{\underline{c_{t-1}+1}} \cdots \gamma^{\underline{c_t}}$ be the canonical permutation of S_t. By multiplying the permutations together, we see that $\gamma_{p_1,\ldots,p_n}^{-1} = \gamma_1^{-1} \gamma_2^{-1} \cdots \gamma_r^{-1}$. Now, as the non-trivial cycles μ_t and $\gamma_{t'}$ are disjoint unless $t = t'$, we have

$$
\begin{aligned}
\mu \gamma_{p_1,\ldots,p_n}^{-1} &= \mu_1 \mu_2 \cdots \mu_r \gamma_1^{-1} \gamma_2^{-1} \cdots \gamma_r^{-1} \\
&= \mu_1 \gamma_1^{-1} \mu_2 \gamma_2^{-1} \cdots \mu_r \gamma_r^{-1}
\end{aligned}
$$

where each $\mu_t \gamma_t^{-1}$ is a permutation of the subset S_t. Since the sets S_t partition $[p_1, \ldots, p_n]$, the number of cycles in $\mu \gamma_{p_1,\ldots,p_n}^{-1}$ is the sum of the number of cycles in $\mu_t \gamma_t^{-1}$. That is, if we let $w_t(\mu_t)$ be the number of cycles in $\mu_t \gamma_t^{-1}$ in S_t, then $w(\mu) = w_1(\mu_1) + \cdots + w_r(\mu_r)$. By noting that $\mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ decomposes into the product $\mathcal{P}_1 \times \cdots \times \mathcal{P}_r$, we conclude that $A_n^{(\mathbf{q};\mathbf{s})}(x) = A_1(x) A_2(x) \cdots A_r(x)$, where $A_t(x) = \sum_{\mu_t \in \mathcal{P}_t} x^{w_t(\mu_t)}$. Finally, by relabelling the elements of S_t with elements of $\left[p_{c_{t-1}+1}, \ldots, p_{c_t}\right] = \left[p_{c_{t-1}+1}\right]^{\underline{1}} \cup \cdots \cup \left[p_{c_t}\right]^{\underline{c_t - c_{t-1}}}$, we see that $A_t(x) = A_{c_t - c_{t-1}}^{(\mathbf{q}_t;\mathbf{s}_t)}(x)$, which proves the proposition. □

Proposition 2.2 allows us to assume that the support graph of $\mathbf{s}$ is connected. While some of the results do not depend on this assumption, we will assume this throughout the rest of the thesis for consistency, as there are no drawbacks in doing so. In particular, this means that $p_i \ge s_i > 0$ for $1 \le i \le n$. Later, we will focus our attention on cases where the support graph of $\mathbf{s}$ is a tree, but we will explicitly point out when we need the tree assumption.

From this point on, we will assume that the support graph of $\mathbf{s}$ is connected.

As we shall see in the following sections, the one and two vertex cases of this problem have been studied by numerous people, with both algebraic and combinatorial methods.

The common technique that is used in both styles of proof is to colour each cycle of $\mu\gamma_{p_1,p_2,\ldots,p_n}^{-1}$ with one of K colours. For this, we will use a combinatorial object called paired functions, which is related to the paired surjections introduced in Goulden and Solfstra [18]. The reason for choosing this combinatorial object over the more conventional treatments is so that we can define the colouring without referring to maps, and this it fits well with the combinatorial approach we will use in later chapters. See Remark 2.5 for more details.

Definition 2.3. Let $n, K \geq 1$, $\mathbf{q} = (q_1, \ldots, q_n) \geq \mathbf{0}$, $\mathbf{s} = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n}) \geq \mathbf{0}$, and $p_i = 2q_i + \sum_{k \neq i} s_{k,i}$ for $1 \leq i \leq n$. An ordered pair (μ, π) is a *paired function* if $\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ and $\pi\colon [p_1, \ldots, p_n] \to [K]$ is a function satisfying

$$\pi\left(\mu\left(v\right)\right) = \pi\left(\gamma_{p_1,p_2,\ldots,p_n}\left(v\right)\right) \qquad \text{for all } v \in [p_1, \ldots, p_n]$$

We denote the set of paired functions satisfying the parameters n, K, $\mathbf{q}$, and $\mathbf{s}$ as $\mathcal{F}_{n,K}^{(\mathbf{q};\mathbf{s})}$, and we let $f_{n,K}^{(\mathbf{q};\mathbf{s})} = \left|\mathcal{F}_{n,K}^{(\mathbf{q};\mathbf{s})}\right|$.

An example of a paired function, as well as its graphical representation, can be found at the beginning of Section 3.1, where we go into detail on how to represent such an object. By substituting in $u = \gamma_{p_1,p_2,\ldots,p_n}(v)$, we have $\pi\left(u\right) = \pi\left(\mu\gamma_{p_1,p_2,\ldots,p_n}^{-1}\left(u\right)\right)$ for all $u \in [p_1, \ldots, p_n]$. This implies that the cycles of $\mu\gamma_{p_1,p_2,\ldots,p_n}^{-1}$ are preserved by π. Hence, for any given pairing $\mu \in \mathcal{A}_{n,L}^{(\mathbf{q};\mathbf{s})}$, there are K^L functions $\pi\colon [p_1, \ldots, p_n] \to [K]$ such that (μ, π) is a paired function. Furthermore, by applying the definition to all pairs $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ of μ, we have that (μ, π) is a paired function if and only if

$$\begin{aligned}\left(\pi\left(\mu\left(y^{\underline{k}}\right)\right), \pi\left(\gamma_{p_1,p_2,\ldots,p_n}\left(x^{\underline{i}}\right)\right)\right) &= \left(\pi\left(\gamma_{p_1,p_2,\ldots,p_n}\left(y^{\underline{k}}\right)\right), \pi\left(\mu\left(x^{\underline{i}}\right)\right)\right)\\ \left(\pi\left(x^{\underline{i}}\right), \pi\left(\left(x+1\right)^{\underline{i}}\right)\right) &= \left(\pi\left(\left(y+1\right)^{\underline{k}}\right), \pi\left(y^{\underline{k}}\right)\right)\end{aligned} \tag{2.1}$$

holds for all pairs $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ of μ, where addition is done modulo p_i and p_k on the left and right hand side, respectively.

We will now demonstrate that the generating series $A_n^{(\mathbf{q};\mathbf{s})}\left(x\right)$ can be used to describe the number of paired functions $f_{n,K}^{(\mathbf{q};\mathbf{s})}$. Recall that $a_{n,L}^{(\mathbf{q};\mathbf{s})}$ is the number of pairings $\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ such that $\mu\gamma_{p_1,p_2,\ldots,p_n}^{-1}$ has exactly L cycles. For each of these pairings, there are K^L functions

π such that (μ, π) is a paired function. Therefore, for $K \geq 1$, we have

$$\begin{aligned} A_n^{(\mathbf{q};\mathbf{s})}(K) &= \sum_{L \geq 1} a_{n,L}^{(\mathbf{q};\mathbf{s})} K^L \\ &= \sum_{\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}} K^{w(\mu)} \\ &= f_{n,K}^{(\mathbf{q};\mathbf{s})} \end{aligned} \tag{2.2}$$

where $w(\mu)$ is the number of cycles in $\mu\gamma_{p_1,p_2,\ldots,p_n}^{-1}$. Conversely, if we can find an expression for $f_{n,K}^{(\mathbf{q};\mathbf{s})}$ that is a polynomial in K, then this expression agrees with $A_n^{(\mathbf{q};\mathbf{s})}(x)$ for all positive integer values of K. As $A_n^{(\mathbf{q};\mathbf{s})}(x)$ is also a polynomial, they must in fact be the same. Therefore, we can substitute $K = x$ into the expression for $f_{n,K}^{(\mathbf{q};\mathbf{s})}$ to obtain $A_n^{(\mathbf{q};\mathbf{s})}(x)$. This is summarized by the following fact.

Fact 2.4. *Let $n \geq 1$, $\mathbf{q} \geq \mathbf{0}$, and $\mathbf{s} \geq \mathbf{0}$. If there exists an expression $p(K)$ such that $f_{n,K}^{(\mathbf{q};\mathbf{s})} = p(K)$ for all $K \geq 1$, and p is a polynomial in K, then $A_n^{(\mathbf{q};\mathbf{s})}(x) = p(x)$.*

As we shall see, Fact 2.4 will be the basis of all approaches used to compute $A_n^{(\mathbf{q};\mathbf{s})}(x)$ in this thesis. In the next few sections of this chapter, we will compute $f_{n,K}^{(\mathbf{q};\mathbf{s})}$ for $n = 1$ and $n = 2$ using algebraic techniques involving the integration of Gaussian measures. Then, from Chapter 3 to Chapter 6, we will compute $f_{n,K}^{(\mathbf{q};\mathbf{s})}$ for general n using combinatorial methods, focusing on cases where the support graph of $\mathbf{s}$ is a tree. This corresponds to graphs that are trees with loops and multiple edges.

Remark 2.5. In the literature where this problem is treated using algebraic techniques, paired functions are often referred to as *N-coloured maps*, which we will call K-coloured maps to match with variable K in this section. A K-coloured map is a map where each face is assigned one of K colours, without restrictions on the colours of adjacent faces. An alternative way of counting K-coloured maps is to colour each corner of a map with one of K colours, then only count the coloured maps where colouring of the corners is the same for each face. By associating each corner of the map with the half-edge incident to its left, this method of colouring is equivalent to using a function π to assign the elements $x^{\underline{i}}$ of $\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ with elements of $[K]$. As the faces of a map are given by the cycles of $\mu\gamma_{p_1,p_2,\ldots,p_n}^{-1}$, the condition that the colouring is consistent is equivalent to (μ, π) satisfying (2.1) for all pairs $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ of μ.

2.2 Background on Matrix Integrals

In this section, we introduce the background to the matrix integral techniques that we will be using in Section 2.3 and Section 2.4, where we will survey the algebraic techniques used to approach the main problem stated in Section 2.1 for $n = 1, 2$. Our presentation will mostly follow that of Lando and Zvonkin [26], with facts related to the Hermite polynomials modified from Szegö [34]. Furthermore, as our approach for larger n in this thesis is combinatorial, we will again only define what is necessary. Consequently, the terminologies defined in this section are only relevant for this chapter, so we will be reusing some of our notations in other parts of this thesis.

We start off by defining the *standard Gaussian measure*, denoted μ, which is the measure with the density

$$d\mu\left(x\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

For convenience of notation, for any function $f\colon X \to \mathbb{R}$, we let $\langle f \rangle_X = \frac{\int_X f(x) d\mu(x)}{\int_X d\mu(x)}$ denote the *mean*, or average, of f with respect to the measure μ on X. Note that if $\int_X d\mu\left(x\right) = 1$, then μ is called a *probability measure*, and $\langle f \rangle_X = \int_X f\left(x\right) d\mu\left(x\right)$. As in the physics literature, μ and X are usually omitted if the context is clear. We can check that μ as defined above is a probability measure, and with respect to $d\mu\left(x\right)$, we have $\langle 1 \rangle = 1$, $\langle x \rangle = 0$, and $\langle x^2 \rangle = 1$. Therefore, μ is a measure with mean 0 and variance 1. Also, using integration by parts, we obtain

$$\begin{aligned}
\left\langle x^k \right\rangle &= \int_{-\infty}^{\infty} x^k \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= -x^{k-1} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + (k-1) \int_{-\infty}^{\infty} x^{k-2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&= (k-1) \left\langle x^{n-2} \right\rangle
\end{aligned}$$

This gives $\langle x^{2n} \rangle = (2n-1)!!$ and $\langle x^{2n+1} \rangle = 0$ for all integers $n \geq 0$. Hence, $\langle p\left(x\right) \rangle$ converges for any polynomial p. Finally, we can check by substitution that

$$\int e^{-\frac{bx^2}{2}} dx = \frac{1}{\sqrt{b}} \int e^{-\frac{x^2}{2}} dx$$

holds for any positive real number b.

Next, we define the *Gaussian measure on a vector space* as follows. Let B be a positive definite matrix of size $k \times k$ and $dv(x) = dx_1 dx_2 \cdots dx_k$. We define

$$d\mu(x) = (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}(Bx, x)\right\} dv(x)$$

where $(x, y) = x^T y = x_1 y_1 + \cdots + x_k y_k$ is the inner product on $\mathbb{R}^n$. Note that if we take i'th entry of x to be x_i and the entries of B to be b_{ij} for $1 \leq i \leq j \leq N$, then expanding the inner product gives

$$\begin{aligned} (Bx, x) &= x^T B^T x \\ &= \sum_{i=1}^{N} b_{ii} x_{ii}^2 + \sum_{i<j} 2 b_{ij} x_i x_j \end{aligned} \tag{2.3}$$

To show that $d\mu(x)$ is a probability measure, we apply an orientation preserving orthogonal transformation $x = Oy$ to diagonalize B. That is, O is an orthogonal matrix with $\det O = 1$ such that $D = O^{-1}BO$ is a diagonal matrix. Applying this transform gives us

$$\begin{aligned} \int_{\mathbb{R}^k} d\mu(x) &= \int_{\mathbb{R}^k} (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(O^{-1}BOy, y\right)\right\} dv(Oy) \\ &= \int_{\mathbb{R}^k} (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} \prod_{i=1}^{k} \exp\left\{-\frac{1}{2} d_i y_i^2\right\} dv(y) \\ &= (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} \prod_{i=1}^{k} \left(\frac{2\pi}{d_i}\right)^{\frac{1}{2}} \\ &= 1 \end{aligned}$$

where the d_i's are the diagonal entries of D, which are also the eigenvalues of B. Integrating each variable separately and noting that $\det B = \det D = \prod_{i=1}^{k} d_i$ finishes the proof.

In probability theory, the matrix $C = B^{-1}$ is called the covariance matrix, and for any x_i and x_j, we have $\langle x_i \rangle = 0$ and $\langle x_i x_j \rangle = c_{ij}$. To prove these results, we first note that it holds true for diagonal matrices B and C, as we can integrate each variable separately.

Then, by writing x_i as $e_i^T x$ and applying the transformation $x = Oy$ as above, we have

$$\begin{aligned} \langle x_i \rangle &= \int_{\mathbb{R}^k} (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} e_i^T x \exp\left\{-\frac{1}{2}(Bx, x)\right\} dv(x) \\ &= \int_{\mathbb{R}^k} (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} e_i^T O y \prod_{n=1}^{k} \exp\left\{-\frac{1}{2}(Dy, y)\right\} dv(y) \end{aligned}$$

where e_i is the i'th standard basis vector. As $e_i^T O y$ is a linear combination of the y_n's, $\langle x_i \rangle$ is a linear combination of the averages $\langle y_n \rangle$, taken with respect to D. Therefore, we can use the result for diagonal matrices to get $\langle x_i \rangle = \sum_{n=1}^{k} c_n \langle y_n \rangle = 0$ for some constants c_n, as desired.

Now, let P_{ij} be the matrix that has 1 at position (i, j), and 0 elsewhere. Then, by writing $x_i x_j$ as $x^T P_{ij} x$ and applying the transformation $x = Oy$ as above, we have

$$\begin{aligned} \langle x_i x_j \rangle &= \int_{\mathbb{R}^k} (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} x^T P_{ij} x \exp\left\{-\frac{1}{2}(Bx, x)\right\} dv(x) \\ &= \int_{\mathbb{R}^k} (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} y^T O^T P_{ij} O y \exp\left\{-\frac{1}{2}(Dy, y)\right\} dv(y) \\ &= \int_{\mathbb{R}^k} (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} \left(\sum_{m,n=1}^{k} y_m o_{im} o_{jn} y_n\right) \exp\left\{-\frac{1}{2}(Dy, y)\right\} dv(y) \end{aligned}$$

since $\left(O^T P_{ij} O\right)_{mn} = o_{im} o_{jn}$. As the off-diagonal entries of D are zero, we can use the result for diagonal matrices to get that $\langle y_m y_n \rangle = 0$ for $m \neq n$. Therefore, the only terms which can survive the integration are the square terms, which reduces the above sum to

$$\begin{aligned} \langle x_i x_j \rangle &= \int_{\mathbb{R}^k} (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} \sum_{n=1}^{k} o_{in} o_{jn} y_n^2 \exp\left\{-\frac{1}{2}(Dy, y)\right\} dv(y) \\ &= (2\pi)^{-\frac{k}{2}} (\det B)^{\frac{1}{2}} \sum_{n=1}^{k} \frac{o_{in} o_{jn}}{d_n} \prod_{i=1}^{k} \left(\frac{2\pi}{d_i}\right)^{\frac{1}{2}} \\ &= \sum_{n=1}^{k} \frac{o_{in} o_{jn}}{d_n} \end{aligned}$$

where we obtain $\langle y_{ii} \rangle = \frac{1}{d_i}$ using the results on diagonal matrices. Finally, note that $B^{-1} = OD^{-1}O^{-1}$, so we have $c_{ij} = \sum_{n=1}^{k} \frac{o_{in} o_{jn}}{d_n} = \langle x_i x_j \rangle$, as desired.

Finally, we define the *Gaussian measure on the space of Hermitian matrices* as follows. A *Hermitian matrix* H is an $K \times K$ matrix such that for all $1 \leq i, j \leq K$, $h_{ij} = \overline{h_{ji}}$, where $\overline{h_{ji}}$ is the complex conjugate of h_{ji}. Now, let $\mathcal{H}_K$ be the space of all $K \times K$ Hermitian matrices. Since the diagonal entries of a Hermitian matrix must be real, we can let $h_{ii} = x_{ii}$ for all diagonal entries of H, and $h_{ij} = \overline{h_{ji}} = x_{ij} + iy_{ij}$ for all $i < j$, where $x_{ii}, x_{ij}, y_{ij} \in \mathbb{R}$. Hence, we can treat $\mathcal{H}_K$ as a vector space of dimension K^2, which allows us to define the ordinary measure of $\mathcal{H}_K$ as $dv(H) = \prod dx_{ii} \prod dx_{ij} dy_{ij}$. Now, let $d\mu(H)$, commonly referred to as the *one matrix model*, be the measure defined by the quadratic form

$$\begin{aligned}
\operatorname{tr}\left(H^2\right) &= \sum_{i,j=1}^{K} h_{ij} h_{ji} \\
&= \sum_{i,j=1}^{K} \left(x_{ij} + iy_{ij}\right)\left(x_{ij} - iy_{ij}\right) \\
&= \sum_{i=1}^{K} x_{ii}^2 + 2 \sum_{i<j} \left(x_{ij}^2 + y_{ij}^2\right)
\end{aligned}$$

That is, we let $\operatorname{tr}(H^2)$ represent a matrix B such that $(Bx, x) = \operatorname{tr}(H^2)$, where x is the vector containing the variables x_{ii}, x_{ij}, and y_{ij} in order, with $1 \leq i, j \leq K$ and $i < j$. By comparing the coefficients with (2.3), we can deduce that B is a diagonal matrix with 1's for the first K diagonal entries, and 2's for the remaining $K^2 - K$ entries. This gives us $\det B = 2^{K^2 - K}$, which gives

$$d\mu(H) = (2\pi)^{-K^2/2} \, 2^{\left(K^2 - K\right)/2} \exp\left\{-\frac{1}{2} \operatorname{tr}\left(H^2\right)\right\} dv(H)$$

By computing $C = B^{-1}$, we see that $\langle x_{ii}^2 \rangle = 1$ and $\left\langle x_{ij}^2 \right\rangle = \left\langle y_{ij}^2 \right\rangle = \frac{1}{2}$. From this, we deduce that $\langle h_{ii}^2 \rangle = \langle x_{ii}^2 \rangle = 1$, $\left\langle h_{ij}^2 \right\rangle = \left\langle x_{ij}^2 + 2ix_{ij}y_{ij} - y_{ij}^2 \right\rangle = 0$, and $\langle h_{ij} h_{ji} \rangle = \left\langle x_{ij}^2 + y_{ij}^2 \right\rangle = 1$, for all $i < j$. For all other $i, j, k, l \in [K]$, we have $(i, j) \neq (k, l)$. This gives $\langle h_{ij} h_{kl} \rangle = 0$, as the terms in the product only involve off-diagonal entries of the covariance matrix.

Next, we introduce two theorems related to integrating over the space of Hermitian matrices. We will state these theorems without proof as they be found in Lando and Zvonkin [26].

Theorem 2.6. *(Wick's formula) Let $f_1, f_2, \ldots, f_{2n}$ be a set of (not necessarily distinct) linear functions of $x_1, \ldots, x_k$. Then*

$$\langle f_1 f_2 \cdots f_{2n} \rangle = \sum_{\mu \in \mathcal{P}_{2n}} \langle f_{p_1} f_{q_1} \rangle \langle f_{p_2} f_{q_2} \rangle \cdots \langle f_{p_n} f_{q_n} \rangle$$

where the sum is taken over all $(2n-1)!!$ pairings $\mu = \{\{p_1, q_1\}, \ldots, \{p_n, q_n\}\}$ of $[2n]$.

This theorem allows us to reduce the integral of a product into a sum of the products of quadratic terms, which we can evaluate using results from the above discussion of the one matrix model. Conversely, if we have linear functions $f_1, \ldots, f_{2n}$ such that the sum of their averages over all pairings counts some meaningful quantity, we can convert it into a single integral over the set of Hermitian matrices, which can then be evaluated using algebraic techniques.

Theorem 2.7. *Suppose F is a unitary invariant function on $\mathcal{H}_K$. That is, suppose $F(U^{-1}HU) = F(H)$ holds for any unitary matrix U and Hermitian matrix H. Then*

$$\int_{\mathcal{H}_K} F(H)\, d\mu(H) = c_K \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} F(\Lambda) \prod_{1 \le i < j \le K} (\lambda_i - \lambda_j)^2 \, d\mu(\lambda_1) \cdots d\mu(\lambda_K)$$

where $c_K = \frac{1}{K!(K-1)!\cdots 1!}$, Λ is the diagonal $K \times K$ matrix with entries $\lambda_1, \ldots, \lambda_K$, and $d\mu(\lambda_i)$ is the standard Gaussian measure.

Theorem 2.7 was originally stated by Weyl, and the proof of this can be found in Section 3.2 of Lando and Zvonkin. The computation of the constant c_K can be found in Section 3.5 of the same book. Note in particular that the dimension of the integral is reduced from K^2 to K.

To facilitate our integration over the measure $d\mu$, we introduce the *Hermite polynomials*, defined by

$$H_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{k!\,(n-2k)!} \cdot \frac{x^{n-2k}}{2^k}$$

The Hermite polynomials are monic polynomials of degree n that are orthogonal with respect to the measure $d\mu$. For $m, n \ge 0$, they satisfy the relation

$$\int H_m H_n \, d\mu(x) = \delta_{m,n} n!$$

where $\delta_{m,n} = 1$ if $m = n$, and 0 otherwise. The Hermite polynomials also have the exponential generating series given by

$$\sum_{i\geq 0} \frac{H_i(x)\, w^i}{i!} = \exp\left\{xw - \frac{w^2}{2}\right\}$$

Using this exponential generating series, we can obtain a formula that allows us to rewrite a product of Hermite polynomials as a sum. This relation was first discovered by Feldheim, but the technique shown here is by Watson [41]. By taking the coefficient of $w^m z^n$ in $\exp\left\{xw - \frac{w^2}{2} + xz - \frac{z^2}{2}\right\}$, we obtain

$$\begin{aligned}
\frac{H_m(x)\, H_n(x)}{m!n!} &= [w^m z^n] \exp\left\{xw - \frac{w^2}{2}\right\} \exp\left\{xz - \frac{z^2}{2}\right\} \\
&= [w^m z^n] \exp\left\{x(w+z) - \frac{1}{2}(w+z)^2\right\} \exp\{wz\} \\
&= [w^m z^n] \left(\sum_{i\geq 0}\sum_{j\geq 0} \frac{H_i(x)}{i!}\binom{i}{j} w^j z^{i-j}\right)\left(\sum_{k\geq 0}\frac{w^k z^k}{k!}\right) \\
&= \sum_{k=0}^{\min(m,n)} \frac{H_{m+n-2k}(x)}{(m-k)!\,(n-k)!k!}
\end{aligned}$$

where we take $j = m-k$ and $i = m+n-2k$ to arrive at the coefficient of $w^m z^n$. In particular, by letting $m = n$ and reversing the summation order, we obtain

$$H_n(x)^2 = n!^2 \sum_{k=0}^{n} \frac{H_{2k}(x)}{k!^2\,(n-k)!} \tag{2.4}$$

Next, we will develop an inversion formula that allows us to write the monomial x^n in terms of the Hermite polynomials. Combined with the above formula, this will allow us to evaluate the integral $\langle x^{2n} H_{2m}(x)\rangle$. The technique presented here is sketched out in pg. 385-386 of Szegö.

Lemma 2.8. *Let g_n and f_n be two sequences of integers. Then, one of the relations*

$$g_n = \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor} \frac{f_{n-2i}}{i!} \qquad\qquad f_n = \sum_{i=0}^{\lfloor \frac{n}{2}\rfloor} \frac{(-1)^i g_{n-2i}}{i!}$$

holds for all n if and only if the other holds for all n.

Proof. Note that the odd and even indexed terms are independent of each other. Hence, we can let $d_{\lfloor \frac{n}{2} \rfloor} = g_n$ and $c_{\lfloor \frac{n}{2} \rfloor} = f_n$ for all n of a given parity. Then, by inclusion-exclusion (see pg. 66 of Enumerative Combinatorics, Volume 1, Stanley [33]), we have

$$\begin{aligned}
b_m = \sum_{i=0}^{m} \binom{m}{i} a_i \qquad &\Longleftrightarrow \qquad a_m = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} b_i \\
\frac{b_m}{m!} = \sum_{i=0}^{m} \frac{1}{i!} \frac{a_{m-i}}{(m-i)!} \qquad &\Longleftrightarrow \qquad \frac{a_m}{m!} = \sum_{i=0}^{m} \frac{(-1)^i}{i!} \frac{b_{m-i}}{(m-i)!} \\
d_m = \sum_{i=0}^{m} \frac{c_{m-i}}{i!} \qquad &\Longleftrightarrow \qquad c_m = \sum_{i=0}^{m} \frac{(-1)^i d_{m-i}}{i!}
\end{aligned}$$

where we let $c_m = \frac{a_m}{m!}$ and $d_m = \frac{b_m}{m!}$ for all m. Substituting back in $m = \lfloor \frac{n}{2} \rfloor$ gives the result as desired. □

Now, note that we can rewrite the Hermite polynomial as

$$\frac{H_n(x)\sqrt{2^n}}{n!} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k}{k!} \cdot \frac{x^{n-2k}\sqrt{2^{n-2k}}}{(n-2k)!}$$

for all $n \geq 0$. Using Lemma 2.8, we can obtain

$$\begin{aligned}
\frac{x^n\sqrt{2^n}}{n!} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} \cdot \frac{H_{n-2k}(x)\sqrt{2^{n-2k}}}{(n-2k)!} \\
x^n &= n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{H_{n-2k}(x)}{2^k k!\,(n-2k)!}
\end{aligned}$$

Finally, we can evaluate the value of $\langle x^{2n} H_{2m}(x) \rangle$. For $n \geq m$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} x^{2n} H_{2m}(x)\, d\mu(x) &= \int_{-\infty}^{\infty} (2n)! \sum_{k=0}^{n} \frac{H_{2n-2k}(x)}{2^k k!\,(2n-2k)!} H_{2m}(x)\, d\mu(x) \\ &= \int_{-\infty}^{\infty} (2n)! \frac{H_{2m}(x)^2}{2^{n-m}(n-m)!m!} d\mu(x) \\ &= \frac{(2n)!}{2^{n-m}(n-m)!} \end{aligned} \tag{2.5}$$

as only the summation term containing $H_{2n}(x)$ can survive the integration. This also shows that the integral is zero for $n < m$.

Remark 2.9. Further discussion of the Hermite polynomials can be found in Orthogonal Polynomials by Szegö [34]. However, the reader should be aware that the Hermite polynomial defined here is called $He_n(x)$ in some texts (for example, see Chihara [12]), and $H_n(x)$ is instead defined as $H_n(x) = n! \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k (2x)^{n-2k}}{k!(n-2k)!}$. The two functions are related by $H_n(x) = 2^{\frac{n}{2}} He_n\left(\sqrt{2} \cdot x\right)$. The difference in definition stems from the field of study. In particular, our definition of $H_n(x)$ is common in probability theory, as it is consistent with the measure $d\mu(x)$.

2.3 Enumeration of One Vertex Maps

In this section, we will examine the simplest non-trivial case of the problem statement in Section 2.1. This is when $n = 1$, which is the enumeration of one vertex maps. The problem was first solved by Harer and Zagier [19], using a matrix integral technique. In this case, there are no mixed pairs, so we have $p_1 = 2q_1$, and $\mathbf{s}$ is the 1×1 matrix $[0]$, which we can omit. Using the notation we have developed, the Harer-Zagier formula can be written as follows.

Theorem 2.10. *(Harer-Zagier [19]) Let q be a positive integer, and $\mathcal{A}_L^{(q)}$ be the subset of pairings of $\mathcal{P}_{2q}$ such that for $\mu \in \mathcal{A}_L^{(q)}$, $\mu\gamma_{2q}^{-1}$ has exactly L cycles. If we let $a_L^{(q)} = \left|\mathcal{A}_L^{(q)}\right|$, then the generating series for $a_L^{(q)}$ is given by*

$$A^{(q)}(x) = (2q-1)!! \sum_{k \geq 1} 2^{k-1} \binom{q}{k-1} \binom{x}{k}$$

There are numerous proof of this formula in the literature, both algebraic and combinatorial. A selection of the proofs can be found in the papers by Goulden and Nica [17], Itzykson and Zuber [21], Jackson [23], Kerov [24], Kontsevich [25], Lass [27], Penner [29], and Zagier [42]. As we will be giving a combinatorial proof in the later chapters on a generalized result, we will not pursue it here. Instead, we will be surveying an algebraic proof to demonstrate some of the techniques used in the literature. Again, our presentation here will mostly follow Chapter 3 of Lando and Zvonkin [26], but we will also be using parts of Jackson [23] for certain computations. This approach encodes the pairing of half edges as an integral over Hermitian matrices, and then evaluates the integral using techniques presented in Section 2.2. Note that in our presentation, we will be using paired functions instead of N-coloured maps. See Remark 2.5 for further details.

Let $H \in \mathcal{H}_K$ be the Hermitian matrix such that $h_{ii} = x_{ii}$ for all diagonal entries of H, and $h_{ij} = \overline{h_{ji}} = x_{ij} + iy_{ij}$ for all $i < j$, where $x_{ii}, x_{ij}, y_{ij} \in \mathbb{R}$. By considering the integral of $\mathrm{tr}H^{2q}$ over the measure $d\mu(H)$ and expanding the product, we have

$$\begin{aligned}\left\langle \mathrm{tr}H^{2q} \right\rangle &= \left\langle \sum_{i_1,\ldots,i_{2q}=1}^{K} h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_{2q} i_1} \right\rangle \\ &= \sum_{i_1,\ldots,i_{2q}=1}^{K} \left\langle h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_{2q} i_1} \right\rangle \\ &= \sum_{\pi\colon [2q]\to[K]} \left\langle h_{\pi(1)\pi(2)} h_{\pi(2)\pi(3)} \cdots h_{\pi(2q)\pi(1)} \right\rangle\end{aligned}$$

where we treat the multi-sum of $i_1, \ldots, i_{2q}$ as the sum over all functions $\pi\colon [2q] \to [K]$, with $\pi(u) = i_u$ for $1 \le u \le 2q$. By Wick's formula in Theorem 2.6, we have

$$\left\langle h_{\pi(1)\pi(2)} h_{\pi(2)\pi(3)} \cdots h_{\pi(2q)\pi(1)} \right\rangle = \sum_{\mu\in\mathcal{P}_{2q}} \prod_{\{u_j,v_j\}\in\mu} \left\langle h_{\pi(u_j)\pi(u_j+1)} h_{\pi(v_j)\pi(v_j+1)} \right\rangle$$

where the sum is taken over all pairings $\mu = \{\{u_1, v_1\}, \ldots, \{u_q, v_q\}\}$ of $[2q]$, with addition being taken modulo $2q$. As described in Section 2.2 with Gaussian measures on Hermitian matrices, each term $\left\langle h_{\pi(u_j)\pi(u_j+1)} h_{\pi(v_j)\pi(v_j+1)} \right\rangle$ is 1 if and only if $(\pi(u_j), \pi(u_j+1)) = (\pi(v_j+1), \pi(v_j))$ for $1 \le j \le q$. Since this is the same condition as (2.1), the summation term is 1 if and only if (μ, π) is a paired function. Therefore, $\langle \mathrm{tr}H^{2q} \rangle$ counts the number of paired functions, so by (2.2), we have $\langle \mathrm{tr}H^{2q} \rangle = f_{1,K}^{(q)} = A^{(q)}(K)$.

Example 2.11. Let $q = 2$, and let i, j, k, ℓ represent $i_1, \ldots, i_4$. Then by Wick's formula, we have

$$\begin{aligned}\left\langle \mathrm{tr} H^4 \right\rangle &= \sum_{i,j,k,\ell=1}^{K} \langle h_{ij} h_{jk} h_{k\ell} h_{\ell i} \rangle \\ &= \sum_{i,j,k,\ell=1}^{K} \langle h_{ij} h_{jk} \rangle \langle h_{k\ell} h_{\ell i} \rangle + \langle h_{ij} h_{k\ell} \rangle \langle h_{jk} h_{\ell i} \rangle + \langle h_{ij} h_{\ell i} \rangle \langle h_{jk} h_{k\ell} \rangle\end{aligned}$$

The three terms here correspond to the three maps in Figure 1.10, from left to right respectively.

Notice that for any unitary matrix U, $\mathrm{tr}\left(U^{-1} H U\right)^{2q} = \mathrm{tr}\left(U^{-1} H^{2q} U\right) = \mathrm{tr} H^{2q}$, so $\mathrm{tr} H^{2q}$ is unitary invariant. By applying Theorem 2.7, we obtain

$$\left\langle \mathrm{tr} H^{2q} \right\rangle = c_K \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left(\lambda_1^{2q} + \cdots + \lambda_K^{2q}\right) \prod_{1 \leq i < j \leq K} \left(\lambda_i - \lambda_j\right)^2 d\mu\left(\lambda_1\right) \cdots d\mu\left(\lambda_K\right)$$

where $c_K = \frac{1}{K!(N-1)!\cdots 1!}$.

Now,

$$\prod_{1 \leq i < j \leq K} \left(\lambda_i - \lambda_j\right)^2 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_K \\ \vdots & \vdots & & \vdots \\ \lambda_1^{K-1} & \lambda_2^{K-1} & \cdots & \lambda_K^{K-1} \end{vmatrix}^2$$

is the square of the *Vandermonde determinant*. By taking linear combinations of the rows of this matrix, we can replace each λ_i^j by the Hermite polynomials $H_j\left(\lambda_i\right)$ without changing the value of the determinant. If we then expand the determinant using the cofactor expansion, we can arrive at

$$\left\langle \mathrm{tr} H^{2q} \right\rangle = c_K K \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \lambda_1^{2q} \left(\sum_{\sigma \in \mathcal{S}_K} H_0\left(\lambda_{\sigma(1)}\right) \cdots H_{K-1}\left(\lambda_{\sigma(K)}\right) \right)^2 d\mu\left(\lambda_1\right) \cdots d\mu\left(\lambda_K\right)$$

where by symmetry we have replaced λ_i with λ_1 for $1 \leq i \leq K$. Note that the only terms which can survive the integration are ones where we take the same permutation in both copies of $\sum_{\sigma \in \mathcal{S}_K} H_0\left(\lambda_{\sigma(1)}\right) \cdots H_{K-1}\left(\lambda_{\sigma(K)}\right)$, as $\int_{-\infty}^{\infty} H_i\left(\lambda_k\right) H_j\left(\lambda_k\right) d\mu\left(\lambda_k\right) = 0$ if $i \neq j$.

Therefore, we can simplify the expression to

$$\begin{aligned}
\left\langle \mathrm{tr} H^{2q} \right\rangle &= c_K K \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \lambda_1^{2q} \sum_{\sigma \in \mathcal{S}_K} H_0 \left(\lambda_{\sigma(1)}\right)^2 \cdots H_{K-1} \left(\lambda_{\sigma(K)}\right)^2 d\mu\left(\lambda_1\right) \cdots d\mu\left(\lambda_K\right) \\
&= c_K K \int_{-\infty}^{\infty} \lambda_1^{2q} \sum_{\sigma \in \mathcal{S}_K} \frac{(K-1)! \cdots 1!0!}{\left(\sigma^{-1}(1) - 1\right)!} \cdot H_{\sigma^{-1}(1)-1} \left(\lambda_1\right)^2 d\mu\left(\lambda_1\right) \\
&= \sum_{i=0}^{K-1} \int_{-\infty}^{\infty} \frac{H_i \left(\lambda_1\right)^2 \lambda_1^{2q}}{i!} d\mu\left(\lambda_1\right)
\end{aligned}$$

by integrating all the variables except λ_1. The subsequent simplification then comes from noting that for $1 \leq i \leq K$, there are $(K-1)!$ permutations $\sigma \in \mathcal{S}_K$ such that $\sigma^{-1}(1) = i$.

To integrate this directly, we use the approach presented in Jackson [23]. Using (2.4) and (2.5), we have

$$\begin{aligned}
\left\langle \mathrm{tr} H^{2q} \right\rangle &= \sum_{i=0}^{K-1} \int_{-\infty}^{\infty} \sum_{k=0}^{i} \frac{i! H_{2k}\left(\lambda_1\right) \lambda_1^{2q}}{k!^2 (i-k)!} d\mu\left(\lambda_1\right) \\
&= \sum_{i=0}^{K-1} \sum_{k=0}^{i} \frac{i! (2q)!}{k!^2 (i-k)! 2^{q-k} (q-k)!} \\
&= \sum_{k=0}^{K-1} \sum_{j=0}^{K-k-1} \frac{(2q)!}{k! 2^{q-k} (q-k)!} \binom{j+k}{k} \\
&= \sum_{k=0}^{K-1} \frac{(2q)!}{k! 2^{q-k} (q-k)!} \binom{K}{k+1} \\
&= (2q-1)!! \sum_{k \geq 1} 2^{k-1} \binom{K}{k} \binom{q}{k-1}
\end{aligned}$$

where in the third line we switch the two sums and let $j = i - k$. We can also drop the upper summation bound as $\binom{K}{k} = 0$ for $k > K$. Since this is a polynomial expression in K of degree $q+1$, by Fact 2.4, we can substitute $K = x$ to obtain $A^{(q)}(x)$. Hence, we have proved the Harer-Zagier formula stated in Theorem 2.10.

2.4 Enumeration of Two Vertex Maps

The next non-trivial case of the problem statement in Section 2.1 is $n = 2$, which is the enumeration of two vertex maps. This was first given in Goulden and Slofstra [18] by a combinatorial technique that we will be extending later. Therefore, in this section, we will survey an algebraic proof that uses a similar technique as the one in Section 2.3. This was carried out by Carrell [11] and communicated to me privately via e-mail. Note that for $n = 2$, the matrix $\mathbf{s}$ contains only one non-zero entry, which we denote s. Using the notation we have developed, the theorem of Goulden and Slofstra can be written as follows.

Theorem 2.12. *(Goulden-Slofstra [18]) Let q_1 and q_2 be non-negative integers, and s be a positive integer. Let $\mathcal{A}_L^{(q_1,q_2;s)}$ be the subset of pairings of $\mathcal{P}^{(q_1,q_2;s)}$ such that for $\mu \in \mathcal{A}_L^{(q_1,q_2;s)}$, $\mu\gamma_{2q_1+s,2q_2+s}^{-1}$ has exactly L cycles. If we let $a_L^{(q_1,q_2;s)} = \left|\mathcal{A}_L^{(q_1,q_2;s)}\right|$, then the generating series for $a_L^{(q_1,q_2;s)}$ is given by*

$$A^{(q_1,q_2;s)}(x) = p_1!p_2! \sum_{k=1}^{d+1} \sum_{i=0}^{\left\lfloor \frac{1}{2}p_1 \right\rfloor} \sum_{j=0}^{\left\lfloor \frac{1}{2}p_2 \right\rfloor} \frac{1}{2^{i+j} i! j! (d-i-j)!} \binom{x}{k} \binom{d-i-j}{k-1} \Delta_k^{(q_1,q_2;s)}$$

where $p_1 = 2q_1 + s$, $p_2 = 2q_2 + s$, $d = q_1 + q_2 + s$, and

$$\Delta_k^{(q_1,q_2;s)} = \binom{k-1}{q_1 - i}\binom{k-1}{q_2 - j} - \binom{k-1}{q_1 + s - i}\binom{k-1}{q_2 + s - j}$$

In this expression, p_1 and p_2 are the degrees of vertices 1 and 2, respectively, and d is the total number of pairs in the pairing.

As in the case $n = 1$, we want find a matrix integral that encodes (2.1), so that we can count the number of paired functions in $\mathcal{F}_{2,K}^{(q_1,q_2;s)}$. Since there are two vertices in this version of the problem, our matrix model will contain two matrices, with one representing each vertex. Let $(G, H) \in \mathcal{H}_K \times \mathcal{H}_K$ be a pair of Hermitian matrices. As in Section 2.2, we let $h_{ii} = x_{ii}$ for all diagonal entries of H, and $h_{ij} = \overline{h_{ji}} = x_{ij} + iy_{ij}$ for all $i < j$, where $x_{ii}, x_{ij}, y_{ij} \in \mathbb{R}$. Additionally, we let $g_{ii} = z_{ii}$ for all diagonal entries of G, and $g_{ij} = \overline{g_{ji}} = z_{ij} + iw_{ij}$ for all $i < j$, where $z_{ii}, z_{ij}, w_{ij} \in \mathbb{R}$. Similar to the $n = 1$ case, we can treat $\mathcal{H}_K \times \mathcal{H}_K$ as a vector space of dimension $2K^2$, and the ordinary measure of $\mathcal{H}_K \times \mathcal{H}_K$ can be defined as $dv(H, G) = \prod dx_{ii}dz_{ii} \prod dx_{ij}dy_{ij}dz_{ij}dw_{ij}$. However, we need to choose a quadratic form that can encode the number of loop and non-loop edges on each vertex. To this end, we let c be an indeterminate, and let $\mathrm{tr}(H^2) + \mathrm{tr}(G^2) - 2c\,\mathrm{tr}(HG)$ be our quadratic form. Expanding the traces yield

$$\begin{aligned}
\operatorname{tr}\left(H^2\right) &= \sum_{i=1}^{K} x_{ii}^2 + 2\sum_{i<j}\left(x_{ij}^2 + y_{ij}^2\right) \\
\operatorname{tr}\left(G^2\right) &= \sum_{i=1}^{K} z_{ii}^2 + 2\sum_{i<j}\left(z_{ij}^2 + w_{ij}^2\right) \\
\operatorname{tr}\left(HG\right) &= \sum_{i=1}^{K} x_{ii}z_{ii} + 2\sum_{i<j}\left(x_{ij}z_{ij} + y_{ij}w_{ij}\right)
\end{aligned}$$

Note that in the quadratic form, the x_{ii} terms only appear together with the z_{ii} terms, the x_{ij} with the z_{ij} terms, and the y_{ij} with the w_{ij} terms. Therefore, to make it easier to describe the matrix B for this quadratic form, we arrange the variables of the vector x in the order $x_{11}, z_{11}, x_{22}, z_{22}, \ldots, x_{KK}, z_{KK}$, followed by $x_{12}, z_{12}, y_{12}, w_{12}, x_{13}, z_{13}, y_{13}, w_{13}, \ldots, y_{K-1,K}, w_{K-1,K}$, which groups correlated terms together. Since we need $(Bx, x) = \operatorname{tr}\left(H^2\right) + \operatorname{tr}\left(G^2\right) - 2c\operatorname{tr}\left(HG\right)$, by comparing the coefficients with (2.3), we have that B is a $2K^2 \times 2K^2$ matrix with blocks of size 2×2 on the diagonal, and 0 everywhere else. Furthermore, the first $2K$ diagonal entries of B correspond to the coefficients of x_{ii}^2 and z_{ii}^2, which are both 1. Likewise, the coefficients of $x_{ii}z_{ii}$ are $-2c$, so their corresponding entries in B are $-c$. Therefore, the first K blocks of B are given by

$$\begin{bmatrix} 1 & -c \\ -c & 1 \end{bmatrix}$$

Similarly, the coefficients of x_{ij}^2, y_{ij}^2, z_{ij}^2, and w_{ij}^2 are all 2, and the coefficients of $x_{ij}z_{ij}$ and $y_{ij}w_{ij}$ are $-4c$. Therefore, the remaining $K^2 - K$ blocks of B are given by

$$\begin{bmatrix} 2 & -2c \\ -2c & 2 \end{bmatrix}$$

This gives us $\det B = 2^{4\left(K^2-K\right)}\left(1-c^2\right)^{K^2}$, which gives

$$\begin{aligned}
d\mu\left(H, G\right) &= (2\pi)^{-K^2}\, 2^{2\left(K^2-K\right)}\left(1-c^2\right)^{K^2} \times \\
&\quad \exp\left\{-\frac{1}{2}\left(\operatorname{tr}\left(H^2\right) + \operatorname{tr}\left(G^2\right) - c\operatorname{tr}\left(HG\right)\right)\right\} dv\left(H, G\right)
\end{aligned}$$

We can then compute the covariance matrix $C = B^{-1}$ by taking the inverse of each block. By treating c as an indeterminate, we see that C is a matrix with K^2 blocks, where the first K blocks are of the form

$$\frac{1}{1-c^2}\begin{bmatrix}1 & c\\ c & 1\end{bmatrix}$$

while the other blocks are of the form

$$\frac{1}{2}\cdot\frac{1}{1-c^2}\begin{bmatrix}1 & c\\ c & 1\end{bmatrix}$$

This gives us the set of averages as follows

$$\begin{aligned}
\left\langle x_{ii}^2\right\rangle &= \frac{1}{1-c^2} & \left\langle x_{ij}^2\right\rangle = \left\langle y_{ij}^2\right\rangle &= \frac{1}{2}\cdot\frac{1}{1-c^2}\\
\left\langle z_{ii}^2\right\rangle &= \frac{1}{1-c^2} & \left\langle z_{ij}^2\right\rangle = \left\langle w_{ij}^2\right\rangle &= \frac{1}{2}\cdot\frac{1}{1-c^2}\\
\left\langle x_{ii}z_{ii}\right\rangle &= \frac{c}{1-c^2} & \left\langle x_{ij}z_{ij}\right\rangle = \left\langle y_{ij}w_{ij}\right\rangle &= \frac{1}{2}\cdot\frac{c}{1-c^2}
\end{aligned}$$

From this, we deduce that

$$\begin{aligned}
\left\langle h_{ii}^2\right\rangle = \left\langle x_{ii}^2\right\rangle &= \frac{1}{1-c^2} & \left\langle g_{ii}^2\right\rangle = \left\langle z_{ii}^2\right\rangle &= \frac{1}{1-c^2}\\
\left\langle h_{ij}^2\right\rangle = \left\langle x_{ij}^2 + 2ix_{ij}y_{ij} - y_{ij}^2\right\rangle &= 0 & \left\langle g_{ij}^2\right\rangle = \left\langle z_{ij}^2 + 2iz_{ij}w_{ij} - w_{ij}^2\right\rangle &= 0\\
\left\langle h_{ij}h_{ji}\right\rangle = \left\langle x_{ij}^2 + y_{ij}^2\right\rangle &= \frac{1}{1-c^2} & \left\langle g_{ij}g_{ji}\right\rangle = \left\langle z_{ij}^2 + w_{ij}^2\right\rangle &= \frac{1}{1-c^2}
\end{aligned}$$

$$\begin{aligned}
\left\langle h_{ii}g_{ii}\right\rangle = \left\langle x_{ii}z_{ii}\right\rangle &= \frac{c}{1-c^2}\\
\left\langle h_{ij}g_{ij}\right\rangle = \left\langle x_{ij}z_{ij} + iy_{ij}z_{ii} + ix_{ij}w_{ij} - y_{ij}w_{ii}\right\rangle &= 0\\
\left\langle h_{ij}g_{ji}\right\rangle = \left\langle x_{ij}z_{ij} + iy_{ij}z_{ii} - ix_{ij}w_{ij} + y_{ij}w_{ii}\right\rangle &= \frac{c}{1-c^2}
\end{aligned}$$

holds for all $1 \le i, j \le n$, and $i < j$. As with the case $n = 1$, all other correlations are zero, as those correlations only involve entries not on the 2×2 diagonal blocks of the covariance matrix.

By considering the integral of $\mathrm{tr}H^{p_1}\mathrm{tr}G^{p_2}$ over the measure $d\mu\left(H, G\right)$ and expanding the product, we have

$$\begin{aligned}\langle \mathrm{tr} H^{p_1} \mathrm{tr} G^{p_2}\rangle &= \left\langle \sum_{i_1,\ldots,i_{p_1}=1}^{K} \sum_{j_1,\ldots,j_{p_2}=1}^{K} h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_{p_1} i_1} g_{j_1 j_2} g_{j_2 j_3} \cdots g_{j_{p_2} j_1} \right\rangle \\ &= \sum_{i_1,\ldots,i_{p_1}=1}^{K} \sum_{j_1,\ldots,j_{p_2}=1}^{K} \left\langle h_{i_1 i_2} h_{i_2 i_3} \cdots h_{i_{p_1} i_1} g_{j_1 j_2} g_{j_2 j_3} \cdots g_{j_{p_2} j_1} \right\rangle \\ &= \sum_{\pi\colon [p_1,p_2]\to[K]} \left\langle h_{\pi\left(1^{\underline{1}}\right)\pi\left(2^{\underline{1}}\right)} h_{\pi\left(2^{\underline{1}}\right)\pi\left(3^{\underline{1}}\right)} \cdots h_{\pi\left(p_1^{\underline{1}}\right)\pi\left(1^{\underline{1}}\right)} \times \right. \\ &\qquad \left. g_{\pi\left(1^{\underline{2}}\right)\pi\left(2^{\underline{2}}\right)} g_{\pi\left(2^{\underline{2}}\right)\pi\left(3^{\underline{2}}\right)} \cdots g_{\pi\left(p_2^{\underline{2}}\right)\pi\left(1^{\underline{2}}\right)} \right\rangle\end{aligned}$$

where we treat the multi-sums of $i_1, \ldots, i_{p_1}$ and $j_1, \ldots, j_{p_2}$ as the sum over all functions $\pi\colon [p_1, p_2] \to [K]$, with $\pi\left(x^{\underline{1}}\right) = i_x$ for $1 \le x \le p_1$ and $\pi\left(y^{\underline{2}}\right) = j_y$ for $1 \le y \le p_2$. Then, to simplify our notation, we let $t\colon [p_1, p_2] \to \{h_{ij}, g_{ij} \mid 1 \le i \le j \le K\}$ be defined as

$$t\left(u_j\right) = \begin{cases} h_{\pi\left(x^{\underline{1}}\right),\pi\left(\left(x+1\right)^{\underline{1}}\right)} & \text{if } u_j = x^{\underline{1}} \text{ for some } 1 \le x \le p_1 \\ g_{\pi\left(y^{\underline{2}}\right),\pi\left(\left(y+1\right)^{\underline{2}}\right)} & \text{if } u_j = y^{\underline{2}} \text{ for some } 1 \le y \le p_2 \end{cases}$$

where addition of the $x^{\underline{1}}$ and $y^{\underline{2}}$ are being taken modulo p_1 and p_2, respectively. By using Wick's formula in Theorem 2.6 on the product, we have

$$\left\langle h_{\pi\left(1^{\underline{1}}\right)\pi\left(2^{\underline{1}}\right)} \cdots g_{\pi\left(p_2^{\underline{2}}\right)\pi\left(1^{\underline{2}}\right)} \right\rangle = \sum_{\mu\in\mathcal{P}_{p_1,p_2}} \prod_{\{u_j,v_j\}\in\mu} \langle t\left(u_j\right) t\left(v_j\right)\rangle$$

where the sum is taken over all pairings $\mu = \{\{u_1, v_1\}, \ldots, \{u_d, v_d\}\}$ in $\mathcal{P}_{p_1,p_2}$.

Let $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ be a pair in μ. By enumerating the possibilities of i and k, we have

$$\left\langle t\left(x^{\underline{i}}\right) t\left(y^{\underline{k}}\right)\right\rangle = \begin{cases} \left\langle h_{\pi\left(x^{\underline{1}}\right)\pi\left(\left(x+1\right)^{\underline{1}}\right)} h_{\pi\left(y^{\underline{1}}\right)\pi\left(\left(y+1\right)^{\underline{1}}\right)} \right\rangle & \text{if } i = k = 1 \\ \left\langle g_{\pi\left(x^{\underline{2}}\right)\pi\left(\left(x+1\right)^{\underline{2}}\right)} g_{\pi\left(y^{\underline{2}}\right)\pi\left(\left(y+1\right)^{\underline{2}}\right)} \right\rangle & \text{if } i = k = 2 \\ \left\langle h_{\pi\left(x^{\underline{1}}\right)\pi\left(\left(x+1\right)^{\underline{1}}\right)} g_{\pi\left(y^{\underline{2}}\right)\pi\left(\left(y+1\right)^{\underline{2}}\right)} \right\rangle & \text{if } i = 1, k = 2 \end{cases}$$

In all three cases, $\left\langle t\left(x^{\underline{i}}\right) t\left(y^{\underline{k}}\right)\right\rangle$ is non-zero if and only if $\left(\pi\left(x^{\underline{i}}\right), \pi\left(\left(x+1\right)^{\underline{i}}\right)\right) =$

$\left(\pi\left((y+1)^{\underline{k}}\right),\pi\left(y^{\underline{k}}\right)\right)$, which is the same condition as if (μ,π) is a paired function, as shown in (2.1). Furthermore, if (μ,π) is a paired function, then $\left\langle t\left(x^{\underline{i}}\right)t\left(y^{\underline{k}}\right)\right\rangle=\frac{c}{1-c^2}$ if $i\neq k$, and $\left\langle t\left(x^{\underline{i}}\right)t\left(y^{\underline{k}}\right)\right\rangle=\frac{1}{1-c^2}$ otherwise. Hence, if μ have s mixed pairs, then the summation term is precisely $\frac{c^s}{(1-c^2)^d}$. Therefore,

$$f_{2,K}^{(q_1,q_2;s)}=[c^s]\left(1-c^2\right)^d\langle\mathrm{tr}H^{p_1}\mathrm{tr}G^{p_2}\rangle$$

counts the number of paired functions (μ,π) such that $\mu\in\mathcal{P}_2^{(q_1,q_2;s)}$ and $\pi\in[p_1,p_2]\to[K]$. As in the $n=1$ case, this also gives an expression for $A^{(q_1,q_2;s)}(K)$.

To evaluate this expression, we use what is commonly called the *two matrix model*. Let r_1,r_2 be positive integers, the two matrix model, denoted by $\mathcal{M}_{r_1,r_2}$, is given by

$$Z_K(s,t,g)=\int_{\mathcal{H}_K^2}\exp\{\mathrm{tr}U\}\,dv(G,H)$$

where $U=V_1+V_2+gHG$, $V_1=\sum_{i=1}^{r_1}t_iH^i$, and $V_2=\sum_{i=1}^{r_2}s_jG^j$. In our case, we take $r_1=r_2=2$, which yields $V_1=t_1H+t_2H^2$ and $V_2=s_1G+s_2G^2$. Then, let $m,n\geq 0$ be integers. By Appendix A of Bonora, Constantinidis, and Xiong [6], we have

$$\begin{aligned}&\langle\mathrm{tr}H^n\mathrm{tr}G^m\rangle\\&=\sum_{\ell=0}^{n}\sum_{k=0}^{\ell/2}\sum_{r=0}^{(\ell-2k)/2}\sum_{p=0}^{m}\sum_{q=0}^{p/2}\sum_{j=0}^{\min(\ell-2k,p-2q)}\frac{n!m!}{2^{k+q}(n-\ell)!(m-p)!k!q!}\times\\&\left(\frac{1}{r!j!\left(r+k-q+\frac{1}{2}(p-\ell)\right)!(\ell-2k-r-j)!\left(\frac{1}{2}(\ell+p)-k-q-r-j\right)!}-\right.\\&\left.\frac{1}{j!(\ell-2k-r)!(r-j)!\left(\frac{1}{2}(\ell+p)-k-q-r\right)!\left(\frac{1}{2}(p-\ell)-q+k+r-j\right)}\right)\times\\&\left(\frac{1}{2}(\ell+p)-k-q-j\right)!\binom{K}{\frac{1}{2}(\ell+p)-k-q-j+1}\times\\&\alpha_2^{(p-\ell)/2+k+r}\beta_1^{n-\ell}\beta_2^{m-p}\gamma_1^{k+r}\gamma_2^{(p+\ell)/2-k-r}\end{aligned}$$

where $\langle\cdot\rangle$ is the average with respect to the model $Z_K(s,t,g)$, and

$$\alpha_2 = -\frac{2t_2}{g}, \beta_1 = \frac{gs_1 - 2s_2t_1}{4s_2t_2 - g^2}, \beta_2 = \frac{gt_1 - 2t_2s_1}{4s_2t_2 - g^2}, \gamma_1 = \frac{2s_2}{g^2 - 4s_2t_2}, \gamma_2 = \frac{g}{4s_2t_2 - g^2}$$

This is called the 2 point correlation function of the model $\mathcal{M}_{2,2}$. Note that if we let U be $-\frac{1}{2}\left(\mathrm{tr}\left(H^2\right) + \mathrm{tr}\left(G^2\right) - c\,\mathrm{tr}\left(HG\right)\right)$, then the model $Z_K(s,t,g)$ differs only from the model $d\mu(H,G)$ in the normalization factor $\det B$. Therefore, the averages are the same between the two models, since the normalization factor cancels out. By equating $d\mu(H,G)$ with U, we see that $t_1 = s_1 = 0$, $t_2 = -\frac{1}{2}$, $s_2 = -\frac{1}{2}$, and $g = c$. This gives

$$\alpha_2 = \frac{1}{c}, \beta_1 = \beta_2 = 0, \gamma_1 = \frac{1}{1-c^2}, \gamma_2 = \frac{c}{1-c^2}$$

Since $\beta_1 = \beta_2 = 0$, the only terms that can contribute to the sum are the ones where $\ell = n$ and $p = m$. Specializing the above equation and substituting in $n = p_1$ and $m = p_2$ gives

$$\begin{aligned}
&\langle \mathrm{tr}H^{p_1}\mathrm{tr}G^{p_2}\rangle \\
=&\sum_{k=0}^{p_1/2}\sum_{r=0}^{(p_1-2k)/2}\sum_{q=0}^{p_2/2}\sum_{j=0}^{\min(p_1-2k,p_2-2q)} \frac{p_1!p_2!}{2^{k+q}k!q!}\times \\
&\left(\frac{1}{r!j!\left(r+k-q+\frac{1}{2}(p_2-p_1)\right)!(p_1-2k-r-j)!\left(\frac{1}{2}(p_1+p_2)-k-q-r-j\right)!}-\right. \\
&\left.\frac{1}{j!(p_1-2k-r)!(r-j)!\left(\frac{1}{2}(p_1+p_2)-k-q-r\right)!\left(\frac{1}{2}(p_2-p_1)-q+k+r-j\right)!}\right)\times \\
&\left(\frac{1}{2}(p_1+p_2)-k-q-j\right)!\binom{K}{\frac{1}{2}(p_1+p_2)-k-q-j+1}\times \\
&\frac{c^{p_1-2k-2r}}{(1-c^2)^{(p_1+p_2)/2}}
\end{aligned}$$

To obtain $f_{2,K}^{(q_1,q_2;s)}$, we take the coefficient $[c^s]\left(1-c^2\right)^d\langle \mathrm{tr}H^{p_1}\mathrm{tr}G^{p_2}\rangle$, where we recall that $d = \frac{p_1+p_2}{2}$. By comparing the exponent on c, we must have $k + r = \frac{p_1-s}{2}$. Doing these substitutions and removing the summation index r gives

$$
\begin{aligned}
&f_{2,K}^{(q_1,q_2;s)} \\
&=\sum_{k=0}^{p_1/2}\sum_{q=0}^{p_2/2}\sum_{j=0}^{\min(p_1-2k,p_2-2q)} \frac{p_1!p_2!}{2^{k+q}k!q!j!}\times \\
&\left(\frac{1}{\left(\frac{1}{2}(p_1-s)-k\right)!\left(\frac{1}{2}(p_2-s)-q\right)!\left(\frac{1}{2}(p_1+s)-k-j\right)!\left(\frac{1}{2}(p_2+s)-q-j\right)!}-\right. \\
&\left.\frac{1}{\left(\frac{1}{2}(p_1+s)-k\right)!\left(\frac{1}{2}(p_1-s)-k-j\right)!\left(\frac{1}{2}(p_2+s)-q\right)!\left(\frac{1}{2}(p_2-s)-q-j\right)!}\right)\times \\
&(d-k-q-j)!\binom{K}{d-k-q-j+1}
\end{aligned}
$$

Next, we rewrite p_1 and p_2 using the fact that $p_1 = 2q_1 + s$ and $p_2 = 2q_2 + s$. The above expression then simplifies to

$$
\begin{aligned}
f_{2,K}^{(q_1,q_2;s)} \quad = \quad & \sum_{k=0}^{p_1/2}\sum_{q=0}^{p_2/2}\sum_{j=0}^{\min(p_1-2k,p_2-2q)} \frac{p_1!p_2!}{2^{k+q}k!q!j!}\times \\
&\left(\frac{1}{(q_1-k)!\,(q_2-q)!\,(q_1+s-k-j)!\,(q_2+s-q-j)!}-\right. \\
&\left.\frac{1}{(q_1+s-k)!\,(q_1-k-j)!\,(q_2+s-q)!\,(q_2-q-j)!}\right)\times \\
&(d-k-q-j)!\binom{K}{d-k-q-j+1}
\end{aligned}
$$

For the penultimate step, we replace k by i, q by j, and j by r, so that it better matches the formula given by Goulden and Slofstra, thus obtaining

$$
\begin{aligned}
f_{2,K}^{(q_1,q_2;s)} &= \sum_{i=0}^{p_1/2}\sum_{j=0}^{p_2/2}\sum_{r=0}^{\min(p_1-2i,p_2-2j)} \frac{p_1!p_2!}{2^{i+j}i!j!r!} \times \\
&\left(\frac{1}{(q_1-i)!\,(q_2-j)!\,(q_1+s-i-r)!\,(q_2+s-j-r)!} - \right. \\
&\left.\frac{1}{(q_1+s-i)!\,(q_1-i-r)!\,(q_2+s-j)!\,(q_2-j-r)!}\right) \times \\
&(d-i-j-r)!\binom{K}{d-i-j-r+1}
\end{aligned}
$$

Observe that for the inner terms to be non-zero, the factorials in the denominator must be non-negative. Therefore, we must have $q_1 - i \geq 0$ and $q_1 + s - i - r \geq 0$ for the first term, and $q_1 + s - i \geq 0$ and $q_1 - i - r \geq 0$ for the second. In both cases, this adds up to $2q_1 + s - 2i - r \geq 0$, or $r \leq p_1 - 2i$. The same argument shows that we must have $r \leq p_2 - 2j$ for the inner terms to be non-zero. Therefore, the upper bound of r can be raised to $d - i - j$ without changing the sum. By raising the bound and doing the substitution $r = d - i - j - k + 1$, we obtain

$$
\begin{aligned}
f_{2,K}^{(q_1,q_2;s)} &= \sum_{i=0}^{p_1/2}\sum_{j=0}^{p_2/2}\sum_{k=1}^{d-i-j+1} \frac{p_1!p_2!}{2^{i+j}i!j!\,(d-i-j-k+1)!} \times \\
&\left(\frac{1}{(q_1-i)!\,(q_2-j)!\,(k-1-q_2+j)!\,(k-1-q_1+i)!} - \right. \\
&\left.\frac{1}{(q_1+s-i)!\,(k-1-q_2-s+j)!\,(q_2+s-j)!\,(k-1-q_1-s+i)!}\right) \times \\
&(k-1)!\binom{K}{k}
\end{aligned}
$$

From here, we can raise the summation bound on k to $d + 1$, as we have the term $(d - i - j - k + 1)!$ in the denominator. Writing the expression using binomial coefficients yields

$$
\begin{aligned}
f_{2,K}^{(q_1,q_2;s)} &= \sum_{i=0}^{p_1/2}\sum_{j=0}^{p_2/2}\sum_{k=1}^{d+1}\binom{K}{k}\binom{d-i-j}{k-1}\frac{p_1!p_2!}{2^{i+j}i!j!\,(d-i-j)!}\times \\
&\left(\binom{k-1}{q_1-i}\binom{k-1}{q_2-j}-\binom{k-1}{q_1+s-i}\binom{k-1}{q_2+s-j}\right)
\end{aligned}
$$

which is a polynomial expression in K that matches the formula of Goulden and Slofstra. Using Fact 2.4 and substituting $K = x$ completes the proof.

2.5 Other Map Enumeration Problems

In this section, we will briefly examine some related map enumeration problems that exist in the literature, as well as the techniques used to solve them. The first two problems are in some ways extensions to the Harer-Zagier formula in Section 2.3, and we will treat them using the algebraic techniques previously discussed in this chapter. The third problem is the enumeration of bicoloured maps, which we will treat using a combinatorial approach similar to the one used later in this thesis. In all three cases, we are more interested in the approaches used to solve these problems, rather than the technicalities. Therefore, we will omit proofs and details that are not important to understanding the approaches used.

The first problem we will discuss is the enumeration of maps by genus, according to vertex degrees. This is essentially the same as the main problem of this thesis, with the key difference being that there are no restrictions on how the edges are connected between vertices. The way which we will approach this problem is taken from Section 3.3 of Lando and Zvonkin [26], as well as from the earlier sections of this chapter. When we set this problem up as a problem about multiplying permutations, it is similar to the one in Section 2.1. However, instead of having the parameters $\mathbf{q}$ and $\mathbf{s}$, we simply have the parameter $\mathbf{p}$, as we do not need to keep track of the number of edges between each pair of vertices. Formally, let n be a positive integer, and $\mathbf{p} = (p_1, \ldots, p_n) \geq \mathbf{1}$ be a vector of length n. Then, let $\mathcal{P}_{p_1,\ldots,p_n}$ be the set of pairings of $[p_1, \ldots, p_n]$, and $\gamma_{p_1,\ldots,p_n}$ be the canonical cycle permutation of $\mathcal{S}_{p_1,\ldots,p_n}$, as defined in Chapter 1. For $L \geq 1$, we define $\mathcal{C}^{\mathbf{p}}_{n,L} \subseteq \mathcal{P}_{p_1,\ldots,p_n}$ to be the subset of pairings such that for $\mu \in \mathcal{C}^{\mathbf{p}}_{n,L}$, $\mu\gamma^{-1}_{p_1,\ldots,p_n}$ has exactly L cycles, and let $c^{\mathbf{p}}_{n,L} = \left|\mathcal{C}^{\mathbf{p}}_{n,L}\right|$. Then, our goal is to determine an expression for the generating series

$$\begin{aligned} C_n^{\mathbf{p}}(x) &= \sum_{L\geq 1} c_{n,L}^{\mathbf{p}} x^L \\ &= \sum_{\mu\in\mathcal{P}_{p_1,\ldots,p_n}} x^{w(\mu)} \end{aligned}$$

for given values of n and $\mathbf{p}$, where w is the weight function on $\mathcal{P}_{p_1,\ldots,p_n}$, defined such that for $\mu \in \mathcal{P}_{p_1,\ldots,p_n}$, $w(\mu)$ is the number of disjoint cycles in the permutation $\mu\gamma_{p_1,\ldots,p_n}^{-1}$.

One key difference between this problem and the one in Section 2.1 is that $C_n^{\mathbf{p}}(x)$ counts all permutations $\mu \in \mathcal{P}_{p_1,\ldots,p_n}$, regardless of whether the subgroup generated by μ and $\gamma_{p_1,\ldots,p_n}$ is transitive. Therefore, in the language of enumerating maps, this generating series counts the number of possibly disconnected combinatorial maps with n vertices and L faces, such that there are exactly p_i edges incident to vertex i. As in Chapter 1, we let $d = \frac{1}{2}\sum_{i=1}^{n} p_i$ be the total number of pairs of μ, which also represents the total number of edges in the combinatorial map. The fact that $C_n^{\mathbf{p}}(x)$ also counts disconnected combinatorial maps is undesirable from a topological standpoint, and while this issue can be remedied in some cases, there is no known method in general. For now, we will view this problem simply as one about multiplying permutations, without concerning ourselves with whether the permutations we are enumerating represent combinatorial maps.

Analogous to Definition 2.3, for $\mu \in \mathcal{P}_{p_1,\ldots,p_n}$ and $\pi\colon [p_1,\ldots,p_n] \to [K]$, we can define an ordered pair (μ,π) to be a paired function if

$$\pi(\mu(v)) = \pi(\gamma_{p_1,p_2,\ldots,p_n}(v))$$

holds for all $v \in [p_1,\ldots,p_n]$, and let $\mathcal{D}_{n,K}^{\mathbf{p}}$ to be the set of all paired functions satisfying the parameters n, K, and $\mathbf{p}$, with $d_{n,K}^{\mathbf{p}} = \left|\mathcal{D}_{n,K}^{\mathbf{p}}\right|$. Then, by the same logic used in Section 2.1, we can obtain results analogous to (2.1), (2.2), and Fact 2.4 for this definition of paired function. In particular, we have that (μ,π) is a paired function if and only if

$$\left(\pi\left(x^{\underline{i}}\right), \pi\left((x+1)^{\underline{i}}\right)\right) = \left(\pi\left((y+1)^{\underline{k}}\right), \pi\left(y^{\underline{k}}\right)\right) \tag{2.6}$$

holds for all pairs $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ of μ, where addition is done modulo p_i and p_k on the left and right hand side, respectively. Then, we have $C_n^{\mathbf{p}}(K) = d_{n,K}^{\mathbf{p}}$ for all $k \geq 1$. Finally, we can conclude that finding a polynomial expression for $d_{n,K}^{\mathbf{p}}$ is sufficient for determining the generating series $C_n^{\mathbf{p}}(x)$.

As with the case $n = 1$ in Section 2.3, we will use the one matrix model, described in Section 2.2. Let $H \in \mathcal{H}_K$ be the Hermitian matrix such that $h_{ii} = x_{ii}$ for all diagonal entries of H, and $h_{ij} = \overline{h_{ji}} = x_{ij} + iy_{ij}$ for all $i < j$, where $x_{ii}, x_{ij}, y_{ij} \in \mathbb{R}$. By expanding $\mathrm{tr}H^{p_i}$, we have

$$\mathrm{tr}H^{p_i} = \sum_{\pi\colon [p_i]\to[K]} h_{\pi\left(1^{\underline{i}}\right)\pi\left(2^{\underline{i}}\right)} h_{\pi\left(2^{\underline{i}}\right)\pi\left(3^{\underline{i}}\right)} \cdots h_{\pi\left(p_i^{\underline{i}}\right)\pi\left(1^{\underline{i}}\right)}$$

Therefore, if we consider the integral of $\mathrm{tr}H^{p_1}\cdots\mathrm{tr}H^{p_n}$ over the measure $d\mu(H)$, we have

$$\begin{aligned}\left\langle \mathrm{tr}H^{p_1}\cdots\mathrm{tr}H^{p_n}\right\rangle &= \sum_{\pi\colon [p_1,\ldots,p_n]\to[K]} \Big\langle h_{\pi\left(1^{\underline{1}}\right)\pi\left(2^{\underline{1}}\right)} \cdots h_{\pi\left(p_1^{\underline{1}}\right)\pi\left(1^{\underline{1}}\right)} \times \\ &\qquad h_{\pi\left(1^{\underline{2}}\right)\pi\left(2^{\underline{2}}\right)} \cdots h_{\pi\left(p_2^{\underline{2}}\right)\pi\left(1^{\underline{2}}\right)} \cdots h_{\pi\left(p_n^{\underline{n}}\right)\pi\left(1^{\underline{n}}\right)} \Big\rangle \\ &= \sum_{\pi\colon [p_1,\ldots,p_n]\to[K]} \sum_{\mu\in\mathcal{P}_{p_1,\ldots,p_n}} \prod_{\left\{x^{\underline{i}},y^{\underline{k}}\right\}\in\mu} \left\langle h_{\pi\left(x^{\underline{i}}\right),\pi\left((x+1)^{\underline{i}}\right)} h_{\pi\left(y^{\underline{k}}\right),\pi\left((y+1)^{\underline{k}}\right)} \right\rangle\end{aligned}$$

where we used Wick's formula in Theorem 2.6 to expand the product. As in the case $n = 1$, each term $\left\langle h_{\pi\left(x^{\underline{i}}\right),\pi\left((x+1)^{\underline{i}}\right)} h_{\pi\left(y^{\underline{k}}\right),\pi\left((y+1)^{\underline{k}}\right)} \right\rangle$ is 1 if and only if (2.6) holds. Therefore, $\left\langle \mathrm{tr}H^{p_1}\cdots\mathrm{tr}H^{p_n}\right\rangle$ counts the number of paired functions. In other words, we have $\left\langle \mathrm{tr}H^{p_1}\cdots\mathrm{tr}H^{p_n}\right\rangle = d^{\mathbf{p}}_{n,K} = C^{\mathbf{p}}_n(K)$. Note that this formula is a direct generalization of the one in Section 2.3, though unlike the $n = 1$ case, there is no known method for evaluating this in general.

Now, in the case where each vertex has degree r, we can write the generating series for the number of connected combinatorial maps in terms of the generating series for the number of possibly disconnected combinatorial maps. Recall from Section 1.5 that one of the conditions for μ being a combinatorial map is that the subgroup generated by μ and $\gamma_{p_1,\ldots,p_n}$ is transitive. Let $\overline{\mathcal{C}^r_{n,L}} \subseteq \mathcal{C}^{(r,\ldots,r)}_{n,L}$ be the subset of pairings that satisfies this condition, and $\overline{c^r_{n,L}} = \left|\overline{\mathcal{C}^r_{n,L}}\right|$ is the number of combinatorial maps with n vertices and L faces, with each vertex having degree r. Then, the generating series for the number of combinatorial maps is given by

$$\overline{C^r_n}(x) = \sum_{L\geq 1} \overline{c^r_{n,L}} x^L$$

For convenience, let $C^r_n(x) = C^{(r,\ldots,r)}_n(x)$ be the generating series for the number of possibly disconnected combinatorial maps with n vertices and L faces, with each vertex having degree r, as described at the beginning of this section.

Now, let $\mathcal{D}^r_{n,K} = \mathcal{D}^{(r,\ldots,r)}_{n,K}$ be the set of paired functions (μ, π) such that μ is a pairing on $[r, \ldots, r]$ (n times) and π is the colouring function $\pi\colon [r, \ldots, r] \to [K]$. The set of *connected paired functions* $\overline{\mathcal{D}^r_{n,K}} \subseteq \mathcal{D}^r_{n,K}$ is the subset of paired function such that the subgroup generated by μ and $\gamma^r_n = \gamma_{r,\ldots,r}$ is transitive. By following the same proof as Section 2.1, we see that $\overline{C^r_n}(K) = \left|\overline{\mathcal{D}^r_{n,K}}\right|$. Furthermore, let $\mathcal{D}^r_K = \bigcup_{n\geq 0} \mathcal{D}^r_{n,K}$ and $\overline{\mathcal{D}^r_K} = \bigcup_{n\geq 1} \overline{\mathcal{D}^r_{n,K}}$ be the unions of these sets of paired functions. Then, our objective is to provide a relation between these two sets, and express that using exponential generating series. Background on exponential generating series can be found in Combinatorial Enumeration by Goulden and Jackson [15].

For $i \geq 1$, we let $S^{\underline{i}} = \left\{1^{\underline{i}}, \ldots, r^{\underline{i}}\right\}$ be a set labelled i, which corresponds to vertex i of a map. Now, a pairing μ is a permutation on $S^{\underline{1}} \cup \cdots \cup S^{\underline{n}}$ for some n, so μ can be treated as an object with labels $1, \ldots, n$, with the weight of μ being n, denoted $v(\mu) = n$, to distinguish it from the weight given by the number of cycles in $\mu\gamma^{-1}$. In terms of graph theory, a pairing μ has weight n if the possibly disconnected map that μ represents has n vertices. As the product and composition lemmas for labelled objects use exponential generating series, we will use these series to enumerate the sets $\mathcal{D}^r_K$ and $\overline{\mathcal{D}^r_K}$, where the weight of $(\mu, \pi) \in \mathcal{D}^r_K$ is given by $v(\mu)$. Let $C^r(t, K)$ and $\overline{C^r}(t, K)$ be the exponential generating series for $\mathcal{D}^r_K$ and $\overline{\mathcal{D}^r_K}$, respectively. Then, we have

$$
\begin{aligned}
C^r(t, K) &= \sum_{(\mu,\pi)\in\mathcal{D}^r_K} \frac{t^{v(\mu)}}{n!} \\
&= \sum_{n\geq 0} C^r_n(K) \cdot \frac{t^n}{n!}
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{C^r}(t, K) &= \sum_{(\mu,\pi)\in\overline{\mathcal{D}^r_K}} \frac{t^{v(\mu)}}{n!} \\
&= \sum_{n\geq 1} \overline{C^r_n}(K) \cdot \frac{t^n}{n!}
\end{aligned}
$$

Note that $\overline{C^r}(t, K)$ does not contain the term $\overline{C^r_0}(K)$, which is important for the composition of exponential generating series.

Consider a paired function $(\mu, \pi) \in \mathcal{D}_K^r$, observe that each orbit $\mathcal{O}$ of the subgroup generated by μ and γ_n^r is a union of some sets $S^{\underline{j_1}} \cup \cdots \cup S^{\underline{j_t}}$. In the language of maps, $\mathcal{O}$ represents a component containing the vertices $j_1, \ldots, j_t$. By noting that each pair $\{x^{\underline{i}}, y^{\underline{k}}\}$ of μ must either be contained in or disjoint from $\mathcal{O}$, we see the paired function condition, given by (2.6), still holds if we restrict μ and π to $S^{\underline{j_1}} \cup \cdots \cup S^{\underline{j_t}}$. Therefore, (μ, π) restricted to the subset $S^{\underline{j_1}} \cup \cdots \cup S^{\underline{j_t}}$ is a paired function. We can relabel the superscripts $j_1, \ldots, j_t$ of the elements in μ and π to $1, \ldots, t$, which gives us the paired function $(\mu', \pi') \in \mathcal{D}_{t,K}^r$. Furthermore, the subgroup generated by μ' and γ_t^r is transitive, so $(\mu', \pi') \in \overline{\mathcal{D}_K^r}$. This means that paired functions $(\mu, \pi) \in \mathcal{D}_K^r$ can be decomposed into zero or more connected paired functions, each given by the restriction of (μ, π) to an orbit of the subgroup generated by μ and γ_n^r.

Conversely, given connected paired functions $(\mu_1, \pi_1), \ldots, (\mu_k, \pi_k)$ and a partition ϕ of $[n]$ with k parts, such that the i'th part of ϕ has size $v(\mu_i)$, we can construct $(\mu, \pi) \in \mathcal{D}_K^r$ as follows. For each (μ_i, π_i), if the i'th part of ϕ is $\{j_1, \ldots, j_t\}$, where $t = v(\mu_i)$, we relabel the set of elements $S^{\underline{1}} \cup \cdots \cup S^{\underline{t}}$ in μ_i and π_i with $S^{\underline{j_1}} \cup \cdots \cup S^{\underline{j_t}}$. As ϕ is a partition of $[n]$, the connected paired function (μ_i, π_i) gets labelled with different sets of elements, and together they contain all of $S^{\underline{1}} \cup \cdots \cup S^{\underline{n}}$. Therefore, we can obtain a pairing μ on $[r, \ldots, r]$ (n times) and a colouring function $\pi\colon [r, \ldots, r] \to [K]$ by combining $\mu_1, \ldots, \mu_k$ and $\pi_1, \ldots, \pi_k$ together. Furthermore, we can check the paired function condition holds for (μ, π) by noting that (2.6) holds for all pairs $\{x^{\underline{i}}, y^{\underline{k}}\}$ of μ, as each pair is in some paired function (μ_i, π_i). This gives us $(\mu, \pi) \in \mathcal{D}_K^r$.

Using the language of Goulden and Jackson, we can write this decomposition as $\mathcal{D}_K^r = \{\emptyset, [1], [2], \ldots\} \circledast \overline{\mathcal{D}_K^r}$. As the generating series for $\{\emptyset, [1], [2], \ldots\}$ is e^x, and the generating series for the composition of labelled objects is given by the composition of functions, we have

$$C^r(t, K) = \exp\left\{\overline{C^r}(t, K)\right\}$$

Note that $\overline{C_0^r}(K) = 0$ is required for $C^r(t, K)$ to be well defined. By using $C_n^{\mathbf{p}}(K) = \langle \mathrm{tr} H^{p_1} \cdots \mathrm{tr} H^{p_n} \rangle$ and simplifying the expression, we obtain

$$\begin{aligned}
\overline{C^r}(t, K) &= \log\{C^r(t, K)\} \\
&= \log\left\{\sum_{n \geq 0} \langle (\mathrm{tr} H^r)^n \rangle \cdot \frac{t^n}{n!}\right\} \\
&= \log\left\{\left\langle \sum_{n \geq 0} \frac{(t \cdot \mathrm{tr} H^r)^n}{n!} \right\rangle\right\} \\
&= \log\{\langle \exp\{t \cdot \mathrm{tr} H^r\} \rangle\}
\end{aligned}$$

Note that the variables in the summand are independent of the integral, so we can move them inside the sum as above. Also, if we can take the $\frac{t^n}{n!}$ coefficient of this series, then the substitution $K = x$ gives us $\overline{C_n^r}(x)$. Further discussion of the application of this result can be found in Lando and Zvonkin [26].

The second problem we will discuss is the enumeration of *maps in locally orientable surfaces* by their genus. As with the first problem, we will place no restrictions on how the edges are connected between vertices. One way to represent these is to represent each edge with 4 elements of a permutation, instead of the 2 we have for maps in orientable surfaces. This approach was used in Graph Theory by Tutte [38], and we can rigourously define the problem in terms of multiplying permutations as in Section 1.5 and Section 2.1 if we so desire. However, for the sake of brevity, we will be less formal and instead describe maps in locally orientable surfaces using Feymann diagrams and K-coloured maps. The method used here is taken from the exercises in Chapter 3 of Lando and Zvonkin [26], as well as the paper by Goulden and Jackson [16].

In our discussion in Section 1.5, we know that labelled maps can be described as a set of vertices with half-edges attached to them. Furthermore, we know that the pairing of these half-edges uniquely determines a labelled map. To visualize labelled maps, we introduced Feymann diagrams, which are diagrams of maps where the edges are represented as ribbons. Each corner in the diagram is labelled with the half-edge adjacent to it, which helps to visualize which corners of the map belong to the same faces when the half-edges are glued together.

Note that for maps in orientable surfaces, there is an orientation for the stars, ribbons, and spaces between the ribbons. Therefore, we have to glue the ribbons representing half-edges together without twisting them, to preserve their orientations. Now, to construct maps in locally orientable surfaces, we allow each ribbon to have zero or one twist, where the direction of the twist is irrelevant. Since our definition of map isomorphism is based on having an orientation preserving homeomorphism of the faces, it is sufficient to only consider these two ways of twisting the ribbons.

Analogous to Remark 2.5, a *twisted K-coloured map* is a map in a locally orientable surface, where each face is assigned one of K colours, without restrictions on the colours of adjacent faces. As with K-coloured maps, we can count twisted K-coloured maps by colouring each corner of a locally orientable map, then only count the maps where the colouring of the corners is the same for each face. To represent twisted K-coloured maps as combinatorial objects, we will use the symmetric group, similar to what we did with K-coloured maps. Given an n vertex map such that vertex i has degree p_i, we use the canonical cycle permutation $\gamma_{p_1,\ldots,p_n}$ of $\mathcal{S}_{p_1,\ldots,p_n}$ to represent the vertices, and we use the pairing $\mu \in \mathcal{P}_{p_1,\ldots,p_n}$ to represent the half-edges. Furthermore, we will introduce the twisting function $\phi\colon \mu \to \{0,1\}$, such that for a pair $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ in μ, $\phi\left(\left\{x^{\underline{i}}, y^{\underline{k}}\right\}\right)$ is 1 if the ribbon joining $x^{\underline{i}}$ and $y^{\underline{k}}$ is twisted, and 0 otherwise. Finally, the colouring of the corners will be represented by $\pi\colon [p_1,\ldots,p_n] \to [K]$. By observing how the ribbons are joined together, we see that the colouring of the corners is consistent if and only if

$$\left(\pi\left(x^{\underline{i}}\right), \pi\left((x+1)^{\underline{i}}\right), \phi\left(\left\{x^{\underline{i}}, y^{\underline{k}}\right\}\right)\right) = \begin{cases} \left(\pi\left((y+1)^{\underline{k}}\right), \pi\left(y^{\underline{k}}\right), 0\right) & \text{OR} \\ \left(\pi\left(y^{\underline{k}}\right), \pi\left((y+1)^{\underline{k}}\right), 1\right) \end{cases} \tag{2.7}$$

holds for all pairs $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ of μ. We call the triples (μ,π,ϕ) that satisfy this condition *triple functions*. Furthermore, if for a given μ and π there exist at least one ϕ such that (μ,π,ϕ) is a triple function, then the number of ϕ's such that (μ,π,ϕ) is a triple function is 2^t, where t is the number of pairs $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ in μ such that $\pi\left(x^{\underline{i}}\right) = \pi\left((x+1)^{\underline{i}}\right) = \pi\left(y^{\underline{k}}\right) = \pi\left((y+1)^{\underline{k}}\right)$.

To enumerate triple functions algebraically, we take the approach in Section 2.2 and Section 2.3. However, instead of using a Hermitian matrix H, we use a symmetric matrix M with entries $m_{ij} = m_{ji}$. By computing the determinant, we get

$$\operatorname{tr}\left(M^2\right) = \sum_{i=1}^{K} m_{ii}^2 + 2\sum_{i<j} m_{ij}^2$$

That is, we let $\operatorname{tr}\left(M^2\right)$ represent a matrix B such that $(Bx,x) = \operatorname{tr}\left(M^2\right)$, where x is the vector containing the variables m_{ii} and m_{ij} in order, with $1 \le i,j \le K$ and $i<j$. By comparing the coefficients with (2.3), we can deduce that B is a diagonal matrix with 1's for the first K diagonal entries, and 2's for the remaining $\frac{K^2-K}{2}$ entries. This gives us $\det B = 2^{\left(K^2-K\right)/2}$, which gives

$$d\mu\left(M\right) = (2\pi)^{-\left(K^2+K\right)/4}\, 2^{\left(K^2-K\right)/4} \exp\left\{-\frac{1}{2}\operatorname{tr}\left(M^2\right)\right\} dv\left(M\right)$$

By computing $C = B^{-1}$, we see that $\left\langle m_{ii}^2\right\rangle = 1$ and $\left\langle m_{ij}^2\right\rangle = \left\langle m_{ij}m_{ji}\right\rangle = \frac{1}{2}$ for all $i<j$. For all other $i,j,k,l \in [K]$, we have $(i,j) \ne (k,l)$. This gives $\left\langle m_{ij}m_{kl}\right\rangle = 0$, as the terms in the product only involve off-diagonal entries of the covariance matrix.

As with the problem of enumerating maps in orientable surfaces, if we consider the integral of $\text{tr}M^{p_1}\cdots\text{tr}M^{p_n}$ over the measure $d\mu(M)$, we have

$$\langle \text{tr}M^{p_1}\cdots\text{tr}M^{p_n}\rangle = \sum_{\pi:\,[p_1,\ldots,p_n]\to[K]}\sum_{\mu\in\mathcal{P}_{p_1,\ldots,p_n}}\prod_{\left\{x^{\underline{i}},y^{\underline{k}}\right\}\in\mu}\left\langle m_{\pi(x^{\underline{i}}),\pi\left((x+1)^{\underline{i}}\right)}m_{\pi\left(y^{\underline{k}}\right),\pi\left((y+1)^{\underline{k}}\right)}\right\rangle$$

by using Wick's formula in Theorem 2.6 to expand the product. Also, each term $\left\langle m_{\pi(x^{\underline{i}}),\pi\left((x+1)^{\underline{i}}\right)}m_{\pi\left(y^{\underline{k}}\right),\pi\left((y+1)^{\underline{k}}\right)}\right\rangle$ is $\frac{1}{2}$ if $\left(\pi\left(x^{\underline{i}}\right),\pi\left((x+1)^{\underline{i}}\right)\right)$ is equal to exactly one of $\left(\pi\left(y^{\underline{k}}\right),\pi\left((y+1)^{\underline{k}}\right)\right)$ or $\left(\pi\left((y+1)^{\underline{k}}\right),\pi\left(y^{\underline{k}}\right)\right)$, and is 1 if it is equal to both. By comparing with (2.7), we see that $2^d\left\langle m_{\pi(x^{\underline{i}}),\pi\left((x+1)^{\underline{i}}\right)}m_{\pi\left(y^{\underline{k}}\right),\pi\left((y+1)^{\underline{k}}\right)}\right\rangle$ counts the number of ϕ's such that (μ,π,ϕ) is a triple function, where $d=\frac{p_1+\cdots+p_n}{2}$ is the total number of edges. Therefore, $2^d\langle\text{tr}M^{p_1}\cdots\text{tr}M^{p_n}\rangle$ counts the number of triple functions. As with the case of maps in orientable surfaces, there is no known method of evaluating this in general. However, in the case of $n=1$, we can evaluate the integral and obtain the generating series

$$B^{(p)}(x)=p!\sum_{k=0}^{p}2^{2p-k}\sum_{r=0}^{p}\binom{p-\frac{1}{2}}{p-r}\binom{k+r-1}{k}\binom{\frac{1}{2}(x-1)}{r}+\frac{(2p)!}{2^p p!}\sum_{k=0}^{p}2^k\binom{p}{k}\binom{x-1}{k+1}$$

that counts the number of one vertex maps with p edges in locally orientable surfaces, sorted by genus. Notice that the second part of the sum is $A^{(p)}(x-1)$, where $A^{(p)}(x)$ is given by the Harer-Zagier formula in Theorem 2.10. As the derivation for $B^{(p)}(x)$ is quite involved, we will not cover it in this thesis. Any reader interested in this result is referred to Goulden and Jackson [16].

The third problem we will discuss is the enumeration of *unicellular bicoloured maps*. The approach we use here is taken from Schaeffer and Vassilieva [31], and is a combinatorial proof that has much in common with Goulden and Nica [17]. By extension, this proof is related to the work in this thesis, with which we will draw several comparisons later. Recall from Section 1.5 that the set of all one vertex maps with q edges can be encoded using the pairings of $\mathcal{P}_{2q}$. For $\mu\in\mathcal{P}_{2q}$, we can let $\gamma_{2q}=(1,\ldots,2q)$ represent the vertex, μ represent the edges, and $\alpha=\mu\gamma_{2q}^{-1}$ represent the faces of the map. By using map duality, we can instead let γ_{2q} represent the single face of the map, μ represent the edges, and $\alpha=\mu\gamma_{2q}^{-1}$ represent the vertices. Therefore, the set of pairings of $\mathcal{P}_{2q}$ also encode all maps with one face and q edges. Hence, the enumeration of one vertex maps is also known as the enumeration of *unicellular maps* in some parts of the literature.

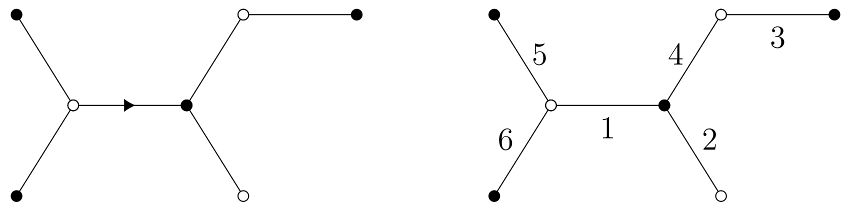


Figure 2.1: Rooted unicellular bicoloured map with $p = 3$, $q = 4$, and 6 edges

Now, a unicellular bicoloured map is an embedding of a bipartite graph in a surface such that the embedding has exactly one face, and a rooted unicellular bicoloured map is a unicellular bicoloured map with a distinguished edge e. Unlike in our description of rooted maps from Section 1.5, we do not choose a direction for this edge ourselves, but instead orient the edge going from the white vertex to the black vertex. Without loss of generality, we can label the edges around the face of the map with a canonical labelling, in a similar manner to how we labelled the half edges in a rooted map. This is done by tracing the face of the map and labelling every other edge with the labels $1, \ldots, d$, starting from the right hand side of the root edge. As there is only one face, each edge is traversed exactly twice, once from each direction. Furthermore, this procedure can only label an edge on the white to black direction, so each edge is labelled exactly once.

To enumerate unicellular bicoloured maps, we will use the following encoding. Let M be a rooted unicellular bicoloured map with m white vertices, n black vertices, and d edges. We represent M as a pair of permutations $(\alpha, \beta) \in \mathcal{S}_d \times \mathcal{S}_d$ such that $\gamma_d = \alpha\beta$. Each cycle of α represents a white vertex, and the elements of the cycle are the edges incident to it, in counterclockwise order. Similarly, each cycle of β represents a black vertex, and the elements of the cycle are the edges incident to it, also in counterclockwise order. Note that $\beta = \alpha^{-1}\gamma_d$, so β is actually determined by α. Furthermore, this form of the expression suggests that unicellular bicoloured maps are in some way related to unicellular maps. Further discussion on the relationship between maps and bipartite maps can be found in the paper by Schaeffer and Vassilieva, as well as Section 1.5 of Lando and Zvonkin [26].

As an example, consider the rooted unicellular bicoloured map on the left diagram of Figure 2.1. This is a map with 3 white vertices, 4 black vertices, and 6 edges. By giving it a canonical labelling using the procedure described above, we obtain the labelled map on the right. Then, by letting each vertex be represented as a cycle in a permutation, we obtain $\alpha = (156)\,(2)\,(34)$ and $\beta = (124)\,(3)\,(5)\,(6)$. We can verify that $\alpha\beta = \gamma_6$ indeed holds.

In the remainder of this section, we will give a sketch of a combinatorial proof to the following theorem, which allows us to enumerate unicellular bicoloured maps.

Theorem 2.13. *(Schaeffer-Vassilieva [31]) Let d be a positive integer, and for $m, n > 0$, let $B(m,n,d)$ be the number of permutation pairs $(\alpha,\beta) \in \mathcal{S}_d \times \mathcal{S}_d$ such that α and β have m and n cycles, respectively, and $\gamma_d = \alpha\beta$. Then, the generating series for the numbers $B(m,n,d)$ is given by*

$$\sum_{m,n\geq 1} B(m,n,d)\, y^m z^n = d! \sum_{p,q\geq 1} \frac{(d-1)!}{(p-1)!\,(q-1)!\,(d-p-q+1)!} \binom{y}{p}\binom{z}{q}$$

The general case of this problem was studied earlier by Jackson, using an algebraic approach with the evaluation of characters of the symmetric group. Using our notation, the result can be written as follows.

Theorem 2.14. *(Jackson [22]) Let d be a positive integer, and $B(m_1,\ldots,m_k,d)$ be the number of factorizations of $\gamma_d = (1,2,\ldots,d)$ into a product of k permutations with respectively $m_1,\ldots,m_k$ cycles. Then, the generating series for the numbers $B(m_1,\ldots,m_k,d)$ is given by*

$$\begin{aligned}&\sum_{m_1,\ldots,m_k\geq 1} B(m_1,\ldots,m_k,d)\, z_1^{m_1}\cdots z_k^{m_k}\\ =&d!\Phi\left\{z_1\cdots z_k\left((1+z_1)\cdots(1+z_k)-z_1\cdots z_k\right)^{d-1}\right\}\end{aligned}$$

where Φ is the linear operator on polynomials defined by

$$\Phi\left(z_1^{\ell_1}\cdots z_k^{\ell_k}\right) = \binom{z_1}{\ell_1}\cdots\binom{z_k}{\ell_k}$$

The proof of Schaeffer and Vassilieva begins by partitioning the cycles of α and β into blocks. Let π_1 and π_2 be partitions of $[d]$ with p and q blocks respectively, and $\alpha \in \mathcal{S}_n$. The triple (π_1,π_2,α) is a *partitioned unicellular bicolored map* if

- Each block of π_1 is a union of cycles of α, and
- Each block of π_2 is a union of cycles of $\beta = \alpha^{-1}\gamma_d$

By letting $\mathcal{C}_{p,q,d}$ be the set of such triples, and $C(p,q,d) = |\mathcal{C}_{p,q,d}|$, we have

$$C(p,q,d) = \sum_{m\geq p, n\geq q} S(m,p)\, S(n,q)\, B(m,n,d)$$

where $S(m,p)$ and $S(n,q)$ are the Stirling numbers of the second kind, which satisfies $\sum_{b=1}^{a}\binom{x}{b}b!S(a,b)=x^a$. By summing over p and q, we can obtain

$$\sum_{m,n\geq 1} B(m,n,d)\,y^m z^n = \sum_{p,q\geq 1} C(p,q,d)\binom{y}{p}\binom{z}{q}p!q! \tag{2.8}$$

This technique of partitioning the cycles of α and β, then using Stirling numbers to write the generating series, is the same as the one used in Goulden and Nica. It is also similar to the function π we have defined in Section 2.1 for paired functions. However, we do not have the implicit requirement that π is a surjection, unlike the partitions π_1 and π_2 used here. A further discussion on this non-empty condition with respect to paired functions can be found in Section 7.1.

Now, let $BT(p,q)$ be the set of ordered rooted bicoloured trees with p white vertices, q black vertices, and a white root. Then, the cardinality of $BT(p,q)$ is given by

$$|BT(p,q)| = \frac{p+q-1}{pq}\binom{p+q-2}{p-1}^2$$

Let $PP(d,d-1,d-p-q+1)$ be the set of partial permutations from a $(d-p-q+1)$-subset of $[d]$ to a $(d-p-q+1)$-subset of $[d-1]$. That is, $\sigma\in PP(d,d-1,d-p-q+1)$ is an injective partial function $\sigma\colon [d]\to[d-1]$ such that σ is defined on $d-p-q+1$ elements of $[d]$. Then, the cardinality of $PP(d,d-1,d-p-q+1)$ is given by

$$|PP(d,d-1,d-p-q+1)| = \binom{d}{d-p-q+1}\binom{d-1}{d-p-q+1}(d-p-q+1)!$$

If we can show that there exists a bijection

$$\zeta\colon \mathcal{C}_{p,q,d}\to BT(p,q)\times PP(d,d-1,d-p-q+1)$$

between the set of partitioned unicellular bicoloured maps and the product of ordered rooted bicoloured trees and partial permutations, then

$$\begin{aligned} C(p,q,d) &= \frac{p+q-1}{pq}\binom{p+q-2}{p-1}^2\binom{d}{d-p-q+1}\binom{d-1}{d-p-q+1}(d-p-q+1)! \\ &= \frac{d!\,(d-1)!}{p!q!\,(p-1)!\,(q-1)!\,(d-p-q+1)!} \end{aligned}$$

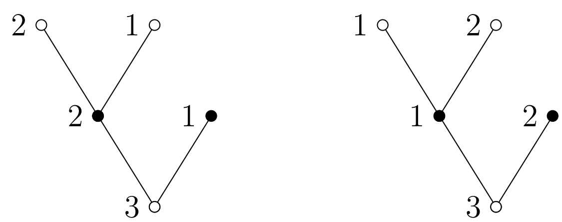


Figure 2.2: Ordered rooted bicolour tree t from decomposing Figure 2.1

Substituting this into (2.8) yields the generating series in Theorem 2.13, as desired.

With the necessary objects defined, we can finally provide the decomposition. As this is meant to be a brief survey, we will only provide the decomposition, without proving that the resulting objects are well defined, or that the decomposition is a bijection.

Let $\pi_1^1, \ldots, \pi_1^p$ be the white blocks of π_1 and $\pi_2^1, \ldots, \pi_2^q$ be the black blocks of π_2, such that $1 \in \pi_1^p$, and the other blocks are arbitrarily labelled. Then, let the maximum elements of π_1^i be m_1^i for $1 \leq i \leq p$, and the maximum elements of π_2^j be m_2^j for $1 \leq j \leq q$. We can then construct a labelled bicoloured tree T with p white vertices and q black vertices, where each block is represented by a vertex of the same colour.

First, the white block π_1^p is taken to be the root of the tree. Then, for each block π_2^j, the corresponding black vertex j is a child of the white vertex i if $\beta\left(m_2^j\right)$ belongs to white block π_1^i. If two black vertices j and k are both children of the white vertex i, then j is left of k if $\beta\left(m_2^j\right) < \beta\left(m_2^k\right)$. Similarly, for each block π_1^i except π_1^p, the white vertex i is a child of the black vertex j if m_1^i, or equivalently $\beta^{-1}\left(m_1^i\right)$, belongs to black block π_2^j. If two white vertices i and ℓ are both children of the black vertex j, then i is left of ℓ if $\beta^{-1}\left(m_1^i\right) < \beta^{-1}\left(m_1^\ell\right)$. By removing the labels, we can obtain the tree $t \in BT(p,q)$.

Continuing the above example, we let $\left(\pi_1^1, \pi_1^2, \pi_1^3\right) = (34, 2, 156)$ and $\left(\pi_2^1, \pi_2^2\right) = (356, 124)$. By computing $\beta^{-1}\left(m_1^i\right)$, we see that $\left(\beta^{-1}\left(m_1^1\right), \beta^{-1}\left(m_1^2\right)\right) = (2,1)$, so π_1^1 and π_1^2 are both children of π_2^1, with π_1^2 to the left of π_1^1. Similarly, by computing $\beta\left(m_2^j\right)$, we see that $\left(\beta\left(m_2^1\right), \beta\left(m_2^2\right)\right) = (6,1)$, so π_2^1 and π_2^2 are both children of π_1^3, with π_2^1 to the left of π_2^3. Putting these together gives us the tree in the left diagram of Figure 2.2. Removing the labels from this tree gives us the ordered rooted bicolour tree t for our decomposition.

The construction of the partial permutation σ is significantly more involved. First, we relabel the tree T from bottom to top, right to left, with $p, p-1, \ldots, 1$ on white vertices, and $q, q-1, \ldots, 1$ on black vertices. Then, we let π_1^i be the white block labelled i for $1 \leq i \leq p$, and π_2^j be the black block labelled j for $1 \leq j \leq q$, with m_1^i and m_2^j being the maximum elements of their respective blocks. Note that the root vertex retains the label p, so $1 \in \pi_1^p$. By writing out the blocks $\pi_1^1, \ldots, \pi_1^p$ and $\pi_2^1, \ldots, \pi_2^q$ in order, we can create the two row permutations

$$\lambda = \left(\begin{array}{c|c|c} \pi_1^1 & \cdots & \pi_1^p \\ 1, 2, & \cdots & d-1, d \end{array} \right) \qquad \nu = \left(\begin{array}{c|c|c} \pi_2^1 & \cdots & \pi_2^q \\ 1, 2, & \cdots & d-1, d \end{array} \right)$$

where the elements of each block are written in increasing order. Finally, let $S = [d] \setminus \left\{ m_1^1, \ldots, m_1^{p-1}, \beta\left(m_2^1\right), \ldots, \beta\left(m_2^q\right) \right\}$, and create the partial permutation

$$\sigma = \nu \beta^{-1} \lambda^{-1} \mid_{\lambda(S)}$$

where the domain is restricted to $\lambda(S)$. Together (t, σ) is a decomposition of the partitioned unicellular bicoloured map into the product of an ordered rooted bicoloured tree and a partial permutation, as desired.

Continuing the example above, we relabel the tree in the left diagram of Figure 2.2, which gives us the tree on the right. We can then write the blocks down in order to obtain the permutations λ and ν, given by

$$\lambda = \left(\begin{array}{c|cc|ccc} 2 & 3 & 4 & 1 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \right) \qquad \nu = \left(\begin{array}{ccc|ccc} 1 & 2 & 4 & 3 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \right)$$

With this, we can compute $\nu\beta^{-1}\lambda^{-1}$, which is

$$\nu \beta^{-1} \lambda^{-1} = \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 3 & 5 & 6 \end{array} \right)$$

As $S = [6] \setminus \{2, 4, \beta(4), \beta(6)\} = \{3, 5\}$, we have $\lambda(S) = \{2, 5\}$ and $\sigma = \left(\begin{array}{cc} 2 & 5 \\ 4 & 5 \end{array} \right)$, which is a partial permutation $PP(6, 5, 2)$, as desired.

Conversely, given an ordered rooted bicoloured tree t and a partial permutation σ, we will sketch how to reconstruct (π_1, π_2, α). If we write σ in the two row notation, then the first row of σ is missing the elements $\lambda\left(m_1^i\right)$ and $\lambda\left(\beta\left(m_2^j\right)\right)$. By construction, we have $\lambda\left(m_1^1\right) < \cdots < \lambda\left(m_1^{p-1}\right)$, and for any child vertex j of π_1^i, we have $\lambda\left(m_1^{i-1}\right) < \lambda\left(\beta\left(m_2^j\right)\right) <$

$\lambda(m_1^i)$. Similarly, the second row of σ is missing the elements $\nu\left(m_2^j\right)$ and $\nu\left(\beta^{-1}\left(m_1^i\right)\right)$. Again, by construction, we have $\nu\left(m_2^1\right) < \cdots < \nu\left(m_2^q\right)$, and for any child vertex i of π_2^j, we have $\nu\left(m_2^{j-1}\right) < \nu\left(\beta^{-1}\left(m_1^i\right)\right) < \nu\left(m_2^j\right)$. Together with the fact that $\beta^{-1}\left(m_1^i\right) < \beta^{-1}\left(m_1^\ell\right)$ if vertex i is to the left of vertex ℓ, and $\beta\left(m_2^j\right) < \beta\left(m_2^k\right)$ if vertex j is to the left of vertex k, we can determine λ and ν on the elements of $[d] \setminus S$. We can use this information to extend σ to $\bar{\sigma} = \nu\beta^{-1}\lambda^{-1}$, which is a permutation on the whole set $[d]$.

By construction, the blocks $\lambda\left(\pi_1^i\right)$ and $\nu\left(\pi_2^j\right)$ contain consecutive elements, so our knowledge of the elements $\lambda\left(m_1^i\right)$ and $\nu\left(m_2^j\right)$ allows us to recover the size of the blocks. Furthermore, we can use $\bar{\sigma}$ to deduce $\nu\left(\pi_1^i\right)$ and $\lambda\left(\pi_2^j\right)$. This allows us to fully recover λ and ν by using a variant of the label recovery procedure for paired functions, which we will not cover here, as a similar procedure is covered in Theorem 3.7. In summary, if we know what is $\lambda(k)$, and which block of π_1 contains it, we can use $\bar{\sigma}$ to determine $\nu(k)$, and which block of π_2 contains it. In turn, this allows us to determine $\lambda(k+1)$, and the block of π_1 that contains it. Combined with the fact that $\lambda(1)$ is the smallest element of π_1^p, we can inductively recover λ and ν. This also allows us to recover π_1 and π_2. Finally, we have $\alpha = \gamma_d\beta^{-1} = \gamma_d\nu^{-1}\bar{\sigma}\lambda$. Since the permutations on the right hand side are now known, we have successfully recovered α as well.

As the authors have noted in their paper, this decomposition is similar to Goulden and Nica. Furthermore, it is related to the first step of our decomposition of paired functions, in the form of paired arrays, which we will give in Section 3.1. Like the forest condition function ψ for paired arrays, the tree t involves the largest elements of each block, while σ determines the relationship of the rest of the elements. Further parallels can be seen between this construction and that of Theorem 3.13. Finally, one minor difference between the construction here and the one for paired functions is that π is a function, so we have no need to relabel the partitions during the decomposition step.

Chapter 3

Paired Arrays

In this chapter, we will give a graphical representation of the paired functions introduced in Chapter 2, then show that we can remove the labels from this representation without losing any information. This allows us to define a stand-alone combinatorial object that does not refer to permutations. We will then introduce a number of notations, conventions, and lemmas for this combinatorial object, and show that it is in bijection with paired functions. Finally, we will start the first stage of the decomposition, where we remove vertex pairs that are non-critical to the combinatorial object. The approach and techniques used in this chapter was first given in Goulden and Nica [17], and further extended in Goulden and Slofstra [18]. However, we will introduce certain changes that allow the results to be generalized. Critically, we remove one of the conditions for the combinatorial object used in their paper, as the use of paired functions instead of paired surjections made that condition unnecessary.

3.1 Definitions and Terminology of Paired Arrays

To represent the paired functions in the set $\mathcal{F}_{n,K}^{(\mathbf{q};\mathbf{s})}$ introduced in Chapter 2, we use a graphical representation introduced in Goulden and Slofstra, called the *labelled array*. This is an $n \times K$ array of cells arranged in a grid. Each element $x^{\underline{i}}$ of μ is represented as a vertex, where the vertex labelled $x^{\underline{i}}$ is placed into cell (i, j) if $\pi\left(x^{\underline{i}}\right) = j$. The vertices are arranged horizontally within a cell, in increasing order of the labels. Furthermore, for each pair $\left\{x^{\underline{i}}, y^{\underline{k}}\right\}$ in μ, an edge is drawn between their corresponding vertices.

For example, let $(\mu, \pi) \in \mathcal{F}_{3,4}^{(\mathbf{q};\mathbf{s})}$, where $\mathbf{q} = (2, 2, 3)$, and $\mathbf{s} = (1, 3, 1)$. Suppose μ and π

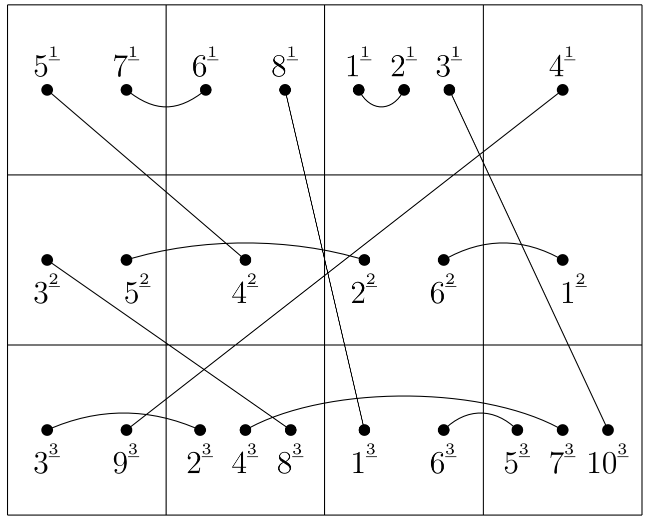


Figure 3.1: A labelled array with 3 rows and 4 columns

are given by

$$
\begin{aligned}
\mu &= \Big\{\Big\{1^{\underline{1}},2^{\underline{1}}\Big\},\Big\{3^{\underline{1}},10^{\underline{3}}\Big\},\Big\{4^{\underline{1}},9^{\underline{3}}\Big\},\Big\{5^{\underline{1}},4^{\underline{2}}\Big\},\Big\{6^{\underline{1}},7^{\underline{1}}\Big\},\Big\{8^{\underline{1}},1^{\underline{3}}\Big\}\\
&\qquad \Big\{1^{\underline{2}},6^{\underline{2}}\Big\},\Big\{2^{\underline{2}},5^{\underline{2}}\Big\},\Big\{3^{\underline{2}},8^{\underline{3}}\Big\},\Big\{2^{\underline{3}},3^{\underline{3}}\Big\},\Big\{4^{\underline{3}},7^{\underline{3}}\Big\},\Big\{5^{\underline{3}},6^{\underline{3}}\Big\}\Big\}\\
\pi^{-1}(1) &= \Big\{5^{\underline{1}},7^{\underline{1}},3^{\underline{2}},5^{\underline{2}},3^{\underline{3}},9^{\underline{3}}\Big\}\\
\pi^{-1}(2) &= \Big\{6^{\underline{1}},8^{\underline{1}},4^{\underline{2}},2^{\underline{3}},4^{\underline{3}},8^{\underline{3}}\Big\}\\
\pi^{-1}(3) &= \Big\{1^{\underline{1}},2^{\underline{1}},3^{\underline{1}},2^{\underline{2}},6^{\underline{2}},1^{\underline{3}},6^{\underline{3}}\Big\}\\
\pi^{-1}(4) &= \Big\{4^{\underline{1}},1^{\underline{2}},5^{\underline{3}},7^{\underline{3}},10^{\underline{3}}\Big\}
\end{aligned}
$$

Then, the labelled array representing (μ,π) is given by Figure 3.1.

Note that an $n \times K$ array with paired and labelled vertices as described above uniquely represents a pairing $\mu \in \mathcal{P}_n^{(\mathbf{q};\mathbf{s})}$ and a function $\pi\colon [p_1,\dots,p_n] \to [K]$. The condition $\pi(\mu(v)) = \pi(\gamma_{p_1,p_2,\dots,p_n}(v))$ is fulfilled if and only if for every pair $\{x^{\underline{i}}, y^{\underline{k}}\}$ in the array, the vertex $(x+1)^{\underline{i}}$ is in the same column as the vertex of $y^{\underline{k}}$, where the addition $x+1$ is taken modulo p_i.

Next, we will show that this condition is sufficient to reconstruct the array if the labels are removed and replaced by marked cells. We do this by defining paired arrays as abstract combinatorial objects, then creating a bijection between paired arrays and labelled arrays. Furthermore, we will extend the definition of paired arrays to cover a larger class of objects, so that we can decompose and enumerate them easily. One major difference in our definition compared to the one in Goulden and Slofstra is that we will decouple the conditions that allow paired arrays to be in bijection with labelled arrays. This allows for greater flexibility in the chapters to come, as we will be violating these conditions when we further generalize paired arrays.

Definition 3.1. Let $n, K \geq 1$, $\mathbf{q} = (q_1, \ldots, q_n) \geq \mathbf{0}$, $\mathbf{s} = (s_{1,2}, s_{1,3}, \ldots, s_{n-1,n}) \geq \mathbf{0}$, and $\mathbf{R} = (R_1, \ldots, R_n) \in [K]^n$. We define $\mathcal{PA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ to be the set of *paired arrays*, which are arrays of cells and vertices subjected to the following conditions.

- A paired array is an array of cells, arranged in n rows and K columns.
- Each cell (i, j) contains an ordered list of vertices, arranged left to right, so that row i contains $p_i \coloneqq 2q_i + \sum_{k<i} s_{k,i} + \sum_{k>i} s_{i,k}$ vertices in total.
- Each vertex u is paired with exactly one other vertex v, which is called the *partner* of u. Exactly $2q_i$ vertices of row i are paired with other vertices of row i, and for $i < k$, exactly $s_{i,k}$ vertices of row i are paired with vertices of row k. Graphically, the pairings are denoted as edges between vertices.
- Each row i has exactly R_i marked cells, which are denoted by marking the cell with a box in its upper right corner.
- A vertex v is *critical* if it is the rightmost vertex of a cell, and the cell it belongs to is not marked. A pair $\{u, v\}$ that contains a critical vertex is a *critical pair*.
- A pair of vertices $\{u, v\}$ is *redundant* if both u and v belong to the same row, and neither u nor v is critical. The vertices u and v are called *redundant vertices*.
- A pair of vertices $\{u, v\}$ is a *mixed pair* if u and v belong to different rows. The vertices u and v are called *mixed vertices*.
- An *object* of a paired array refers to either a vertex, or the box used to indicate that a cell is marked. If a cell both contains vertices and a box, the box is to be taken as the rightmost object of the cell.

Generally, we use $\alpha \in \mathcal{PA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ to denote a paired array. Before introducing the conditions used in Goulden and Slofstra, we will first introduce a number of useful notations and conventions.

Convention 3.2. *For notational convenience, we introduce the following:*

- *We use calligraphic letters to denote columns or sets of columns. For generic columns or sets of columns, we use the letters $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$.*
- *For each calligraphic letter, we use the corresponding upper case letter to denote the number of columns in the set. For example, $X = |\mathcal{X}|$.*
- *For each calligraphic letter, we use the corresponding lower case letter, subscripted by the row number, to denote the total number of vertices in those columns for a given row. For example, x_i is the total number of vertices in row i of the columns of $\mathcal{X}$.*
- *We generally use i, j, k, ℓ as index variables, with i and k for rows, and j and ℓ for columns. Furthermore, we use cell (i, j) to denote the cell in row i, column j of the array.*
- *We use $\mathcal{K}$ to denote the set of all columns, and K to denote the number of columns.*
- *We use $\mathcal{R}_i$ to denote the set of columns that are marked in row i, and R_i to denote the number of columns that are marked in row i.*
- *We use $\mathcal{F}_i$ to denote the set of columns that have at least one vertex in row i, and F_i to denote the number of columns that are marked in row i.*
- *We use $w_{i,j}$ to denote the number of vertices in cell (i, j), and $\mathbf{w}$ to denote a matrix of $w_{i,j}$ describing the number of vertices in each cell of row i.*
- *We let $s_{i,k} = s_{k,i}$ for $i > k$, and $s_i = \sum_{k \neq i} s_{i,k}$ be the total number of mixed vertices of row i. This means that row i contains $p_i = 2q_i + s_i$ vertices.*

With these conventions, we are ready to define the two conditions that allow us to create a bijection between labelled arrays and paired arrays.

Definition 3.3. Let $\alpha \in \mathcal{PA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ be a paired array.

- α is said to satisfy the *balance condition* if for each cell (i, j), the number of mixed vertices in cell (i, j) is equal to the number of mixed pairs $\{u, v\}$ such that u is in row i and v is in column j (but not row i).

- For each row i, the *forest condition function* $\psi_i\colon \mathcal{F}_i \backslash \mathcal{R}_i \mapsto \mathcal{K}$ is defined as follows: For each column $j \in \mathcal{F}_i \backslash \mathcal{R}_i$, if the rightmost vertex v is paired with a vertex u in column ℓ, then $\psi_i(j) = \ell$. α is said to satisfy the *forest condition* if for each row i, the functional digraph of ψ_i on the vertex set $\mathcal{F}_i \cup \psi_i(\mathcal{F}_i) \cup \mathcal{R}_i$ is a forest with root vertices $\mathcal{R}_i$. That is, for each column $j \in \mathcal{F}_i \backslash \mathcal{R}_i$, there exists some positive integer t such that $\psi_i^t(j) \in \mathcal{R}_i$. Note that we always include $\mathcal{R}_i$ in the vertex set of the functional digraph of ψ_i, regardless of whether they are in the domain or range of ψ_i.

Note that permuting the columns of a paired array does not change whether the array satisfies the balance or forest conditions, as all this action does is to relabel the vertices of the functional digraph. By convention, given paired arrays α and α', we use ψ_i and ψ_i' to denote the forest condition functions for row i of α and α', respectively. A paired array is *proper* if it satisfies the balance and forest conditions. A paired array is called a *canonical array* if it is proper and $\mathbf{R} = \mathbf{1}$. We denote the set of canonical arrays as $\mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$, and we let $c_{n,K}^{(\mathbf{q};\mathbf{s})} = \left|\mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}\right|$.

In Definition 3.3, the balance condition is expressed in terms of the numbers of mixed vertices and mixed pairs. Equivalently, the balance condition can also be expressed with respect to the total number of vertices in cell (i,j), as shown in the following proposition.

Proposition 3.4. *Let $\alpha \in \mathcal{PA}_{n,K;\mathbf{R}}^{(\mathbf{q};\mathbf{s})}$ be a paired array. Then, α satisfies the balance condition if and only if for each cell (i,j), the number of vertices in cell (i,j) is equal to the number of vertices u in row i such that its partner v is in column j (and possibly in row i).*

Proof. Let $p_{i,j}$ be the number of vertices in cell (i,j), and $q_{i,j}$ be the number of non-mixed vertices in cell (i,j). Let $p_{i,j}'$ be the number of vertices u in row i such that its partner v is in column j, and $q_{i,j}'$ be the number of vertices u in row i such that its partner v is in cell (i,j). Note that if both vertices of a pair are in cell (i,j), they are counted twice in both $p_{i,j}'$ and $q_{i,j}'$. Now, the number of mixed vertices in cell (i,j) is given by $p_{i,j} - q_{i,j}$, and the number of mixed pairs $\{u,v\}$ such that u is in row i and v is in column j is given by $p_{i,j}' - q_{i,j}'$. Therefore, it suffices to show that $q_{i,j} = q_{i,j}'$.

Let u be a vertex in row i. Then, its partner v is counted in $q_{i,j}$ if and only if it is in cell (i,j). Likewise, u is counted in $q_{i,j}'$ if and only if v is in cell (i,j). As both $q_{i,j}$ and $q_{i,j}'$ require $\{u,v\}$ to be a non-mixed pair, it is sufficient to only consider vertices u in row i. This shows that $q_{i,j} = q_{i,j}'$. Therefore, α satisfies the balance condition as described in Definition 3.3 if and only if it satisfies the balance condition as described in Proposition 3.4. □

Of the two conditions in Definition 3.3, the forest condition is the more fundamental one, and all the arrays we define in this thesis will satisfy some form of this condition. However, it is relatively row independent, as we shall see in later chapters. The balance condition is in general difficult to handle, but can be radically simplified if the support of $\mathbf{s}$ forms a tree. This gives rise to the following definition.

Definition 3.5. A paired array $\alpha \in \mathcal{PA}_{n,K;\mathbf{R}}^{(\mathbf{q};\mathbf{s})}$ is called *tree-shaped* if the support graph of $\mathbf{s}$ forms a tree.

With tree-shaped arrays, we can reduce the balance condition to a condition that only depends on the number of mixed vertices in a cell, essentially allowing us to ignore it. This is the main reason why we focus on counting maps where the support graph of $\mathbf{s}$ forms a tree.

Lemma 3.6. *Let $\alpha \in \mathcal{PA}_{n,K;\mathbf{R}}^{(\mathbf{q};\mathbf{s})}$ be a tree-shaped paired array, and suppose that $s_{i,k,j}$ is the number of vertices in cell (i,j) that are paired with a vertex in row k for all $1 \leq i,k \leq n$ and $1 \leq j \leq K$. Then, α satisfies the balance condition if and only if $s_{i,k,j} = s_{k,i,j}$ for all $i \neq k$.*

Proof. We will first present some preliminary facts about the $s_{i,k,j}$'s. Let G be the support graph of $\mathbf{s}$. By Definition 3.5, G is a tree. Furthermore, $s_{i,k}$ is the number of mixed pairs $\{u,v\}$ with u in row i and v in row k, so $s_{i,k} = \sum_j s_{i,k,j}$. Also, let $x_{i,j}$ be the number of mixed vertices in cell (i,j). As each mixed vertex in cell (i,j) must be joined to a vertex in some other row k, we have $x_{i,j} = \sum_{k \neq i} s_{i,k,j}$.

Now, suppose $s_{i,k,j} = s_{k,i,j}$ for all $1 \leq i,k \leq n$ and $1 \leq j \leq K$. Then, by summing over all $k \neq i$, we have $x_{i,j} = \sum_{k \neq i} s_{i,k,j} = \sum_{k \neq i} s_{k,i,j}$. As $s_{k,i,j}$ is the number of mixed vertices in cell (k,j) that are paired with a vertex in row i, the latter sum counts the number of mixed pairs $\{u,v\}$ such that u is in row i and v is in row k. Therefore, α satisfies the balance condition.

Conversely, suppose α satisfies the balance condition. By the same reasoning, we have $x_{i,j} = \sum_{k \neq i} s_{i,k,j} = \sum_{k \neq i} s_{k,i,j}$. We will show by induction that if the support of $s_{i,k} = \sum_j s_{i,k,j}$ forms a tree and $x_{i,j} = \sum_{k \neq i} s_{i,k,j} = \sum_{k \neq i} s_{k,i,j}$, then $s_{i,k,,j} = s_{k,i,j}$ for all $i \neq k$.

Let G be the support graph of $\mathbf{s}$ and suppose G is a tree. Without loss of generality, let the vertex n be a leaf of G, and assume that it is adjacent to the vertex $n-1$. As n is not joined to other vertices in G, we have $s_{n,k} = s_{k,n} = 0$ for $1 \leq k \leq n-2$. This implies that $\sum_j s_{n,k,j} = \sum_j s_{k,n,j} = 0$, so $s_{n,k,j} = s_{k,n,j} = 0$ for all $1 \leq k \leq n-2$ and $1 \leq j \leq K$. Substituting this into $\sum_{k \neq n} s_{n,k,j} = \sum_{k \neq i} s_{k,n,j}$, we obtain $s_{n,n-1,j} = s_{n-1,n,j}$. Together with $s_{n,k,j} = s_{k,n,j} = 0$, we have that $s_{n,k,j} = s_{k,n,j}$ for $1 \leq k \leq n-1$ and $1 \leq j \leq K$.

Now, let $s'_{i,k} = \sum_j s_{i,k,j}$ and $x'_{i,j} = \sum_{k \neq i,n} s_{i,k,j}$ for $1 \leq i,k \leq n-1$, $i \neq k$, and $1 \leq j \leq K$. That is, we have effectively removed the last row of α. Then,

$$\begin{aligned} \sum_{k \neq i,n} s_{i,k,j} &= \sum_{k \neq i} s_{i,k,j} - s_{i,n,j} \\ &= \sum_{k \neq i} s_{k,i,j} - s_{n,i,j} \\ &= \sum_{k \neq i,n} s_{k,i,j} \end{aligned}$$

by using the fact that $s_{i,n,j} = s_{n,i,j}$, and substituting in the identity for $x_{i,j}$. Furthermore, as $s'_{i,k} = s_{i,k}$ for $1 \leq i,k \leq n-1$, the support graph given by the $s'_{i,k}$'s is $G \backslash \{n\}$. As n is a leaf of G, $G \backslash \{n\}$ is also a tree. By the inductive hypothesis, $s_{i,k,j} = s_{k,i,j}$ for all $1 \leq i,k \leq n-1$ and $1 \leq j \leq K$, where $i \neq k$.

Therefore, α satisfies the balance condition if and only if $s_{i,k,j} = s_{k,i,j}$ for all $i \neq k$, as desired. □

Now that we have defined the necessary framework for paired arrays, we will prove that canonical arrays are in bijection with labelled arrays.

Theorem 3.7. *For $n, K \geq 1$, $\mathbf{q} \geq \mathbf{0}$, and $\mathbf{s} \geq \mathbf{0}$, $f_{n,K}^{(\mathbf{q};\mathbf{s})} = c_{n,K}^{(\mathbf{q};\mathbf{s})}$.*

Proof. We will prove the theorem using a modified version of the *label recovery procedure* introduced in Goulden and Solfstra. This provides a bijection between paired functions and canonical arrays. Recall our assumption that the support graph of $\mathbf{s}$ is connected, so each row of the paired array contains at least one vertex. This is required for the bijection to work.

Let $(\mu, \pi) \in \mathcal{F}_{n,K}^{(\mathbf{q};\mathbf{s})}$ be a paired function. We can obtain the paired array α from (μ, π) by first representing (μ, π) as a labelled array, denoted β. For each row i, we mark the cell of β containing the label $1^{\underline{i}}$ with a box. Then, we remove all labels from β. This gives us the desired paired array, which we denote as α. Note that exactly 1 cell in each row is marked, which gives $\mathbf{R} = \mathbf{1}$.

Now, each pair $\{x^{\underline{i}}, y^{\underline{k}}\}$ of μ is represented as a pair of vertices in rows i and k, and contributes to the same parameter in both α and μ. Hence, α satisfies the parameters $\mathbf{q}$ and $\mathbf{s}$. Also, recall that if $\{x^{\underline{i}}, y^{\underline{k}}\}$ is a pair such that $y^{\underline{k}}$ is in some column j, then $(x+1)^{\underline{i}}$ must be in cell (i,j). Therefore, the number of vertices $x^{\underline{i}}$ in row i such that its partner $y^{\underline{k}}$ is in column j is equal to the number of vertices $(x+1)^{\underline{i}}$ in cell (i,j). By using Proposition 3.4, we see that α satisfies the balance condition.

To show that α satisfies the forest condition, we need to show that the functional digraph of ψ_i, denoted as G_i, is acyclic, and has root vertex in the marked cell. Let j be a column such that $\psi_i(j)$ is defined and suppose $\psi_i(j) = \ell$. Then, cell (i,j) must be non-empty and unmarked. Let the rightmost vertex of cell (i,j) be $x^{\underline{i}}$. By the definition of ψ_i, $x^{\underline{i}}$ must have its partner $y^{\underline{k}}$ in column ℓ. Therefore, $(x+1)^{\underline{i}}$ must be in cell (i,ℓ). If $(x+1)^{\underline{i}} = 1^{\underline{i}}$, then cell (i,ℓ) is marked and is in $\mathcal{R}_i$. Otherwise, it is unmarked and $\psi_i(\ell)$ must be defined. Therefore, the only possible root vertex of G_i is $1^{\underline{i}}$, which is marked.

Furthermore, note that if $(x+1)^{\underline{i}} \neq 1^{\underline{i}}$, then the rightmost vertex of cell (i,ℓ) must have a larger label than that of cell (i,j). Therefore, if G_i contains a directed cycle $(j_1, j_2, \ldots, j_t)$ of length t, then the rightmost label of cell (i, j_{r+1}) must be larger than the rightmost label of cell (i, j_r) for $1 \leq r \leq t$, with addition taken modulo t. However, this gives a cycle of strictly increasing labels, which is a contradiction. Therefore, ψ_i is acyclic. Together, this shows that α is a canonical array in $\mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$.

To describe the inverse, if $\alpha \in \mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$ is a canonical array, then we can recover the labels of α as follows. We label the vertices in each row i in increasing order, from $1^{\underline{i}}$ to $p_i^{\underline{i}}$. As the labels within a cell are arranged in ascending order, we will always put the label on the leftmost unlabelled vertex. First, suppose cell (i,j) is the marked cell in row i. We label the leftmost vertex of cell (i,j) with $1^{\underline{i}}$. Then, for $1 \leq x \leq p_i - 1$, we place the label $(x+1)^{\underline{i}}$ by looking at the partner v of the vertex labelled $x^{\underline{i}}$. Suppose v is in some column ℓ, then $(x+1)^{\underline{i}}$ must be in cell (i,ℓ) for $\pi(\mu(v)) = \pi(\gamma_{p_1,p_2,\ldots,p_n}(v))$ to be satisfied. Therefore, we label the leftmost remaining vertex of cell (i,ℓ) with $(x+1)^{\underline{i}}$. We now need to prove that this procedure can only terminate after all the labels have been placed.

First, note that with the exception of $1^{\underline{i}}$, for every label $x^{\underline{i}}$ placed in cell (i,j), we must have already labelled some vertex u in row i with $(x-1)^{\underline{i}}$, whose partner v is in column j. By Proposition 3.4, the number of such vertices is equal to the number of vertices in cell (i,j). Hence, with the possible exception of the cell containing $1^{\underline{i}}$, our procedure cannot place more labels in a cell than the number of vertices in it. Therefore, it suffices to show that the procedure cannot terminate early on the column containing $1^{\underline{i}}$.

Suppose for contradiction that this is not the case. Note that at termination, if all vertices of cell (i,j) are labelled, then all vertices u whose partner v is in column j must also be labelled. This includes the cell containing $1^{\underline{i}}$, as the last step of the procedure must label a vertex u whose partner v is in the same column as $1^{\underline{i}}$. Therefore, if the rightmost vertex u of cell (i,j) is not labelled, and it is paired with a vertex v in column ℓ, then $\psi_i(j) = \ell$. Furthermore, the rightmost vertex of column ℓ is also not labelled. Hence, if

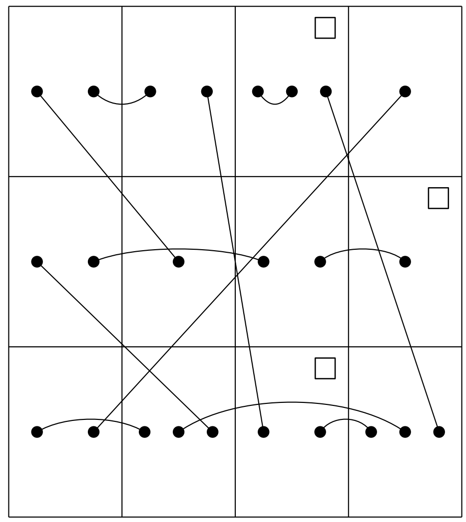


Figure 3.2: A canonical paired array with 3 rows and 4 columns

$\mathcal{X}$ is the set of all columns such that for $j \in \mathcal{X}$, the rightmost vertex of cell (i, j) is not labelled, then $\psi_i(j)$ is defined, and $\psi_i(j) \in \mathcal{X}$. However, this means that the functional digraph of ψ_i with the vertex set restricted to $\mathcal{X}$ is a directed graph with $|\mathcal{X}|$ vertices and $|\mathcal{X}|$ edges, so it must contain a directed cycle. Therefore, this violates the forest condition, which is a contradiction.

Finally, to show that this is a bijection, we only need to start with a canonical array $\alpha \in \mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$, apply the label recovery procedure, then strip off the labels via the inverse described above. As the marked cell in each row is the same as the cell containing the vertex $1^{\underline{i}}$ in both procedures, the positions of the marked cells of α are preserved. Furthermore, the positions of the vertices and edges do not change during either procedures. Therefore, these procedures are inverses of each other. This shows that $\mathcal{F}_{n,K}^{(\mathbf{q};\mathbf{s})}$ is in bijection with $\mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$, as desired. $\square$

As an example of the label recovery procedure and its inverse, we have transformed the labelled array depicted in Figure 3.1 into the canonical array depicted in Figure 3.2.

3.2 Decomposition of Canonical Arrays

Now that we have shown that canonical arrays are in bijection with labelled arrays with the same parameters, the problem of enumerating maps on surfaces reduces to that of enumerating canonical arrays. To solve the latter problem, we will first decompose canonical arrays by removing redundant pairs. Then, we will remove vertex pairs where both vertices are in the same row. Finally, we will decompose the resulting paired arrays via induction, removing one row at a time. This motivates us to define subsets of paired arrays describing each stage of the procedure. At the same time, we will also define notations for these subsets and their cardinalities.

Definition 3.8. A paired array is called a *minimal array* if it is proper and does not contain redundant pairs. We denote the set of minimal arrays as $\mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$, and we let $m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}} = \left|\mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}\right|$. Similarly, a paired array is called a *vertical array* if for every pair $\{u, v\}$, u and v are in different rows. As with paired arrays, a vertical array is *proper* if it satisfies the balance and forest conditions. We denote the set of vertical arrays as $\mathcal{VA}^{(\mathbf{s})}_{n,K;\mathbf{R}} = \mathcal{PA}^{(\mathbf{0};\mathbf{s})}_{n,K;\mathbf{R}}$ and the set of proper vertical arrays as $\mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$. Finally, we let $v^{(\mathbf{s})}_{n,K;\mathbf{R}} = \left|\mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}\right|$. For notational convenience, we extend our definition of $m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ and $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ to all $\mathbf{R} \geq \mathbf{1}$ by letting $m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}} = v^{(\mathbf{s})}_{n,K;\mathbf{R}} = 0$ if $R_i > K$ for some $1 \leq i \leq n$.

Note that we will generally not work directly with paired arrays that do not satisfy the forest condition. However, as vertical arrays not satisfying the forest condition are vital for extending paired arrays, we have separated the forest condition from the definition of vertical arrays itself. Next, we will introduce our first extension of paired arrays, where instead of requiring every vertex to be paired with another vertex, we only require critical vertices to be paired.

Definition 3.9. If $n, K \geq 1$, then a *partially-paired array* α is an $n \times K$ array of cells, where each cell contains zero or more vertices, and is either marked or unmarked. Furthermore, each vertex of the array may be paired with another vertex. However, only the rightmost vertices of unmarked cells are required to be paired with another vertex, and we call the vertices not paired with any other vertices *unpaired vertices*. As with paired arrays, we can define the terms *critical vertices*, *redundant pairs*, *mixed pairs*, and *objects* in the same manner as in Definition 3.1. Likewise, a partially-paired array is *proper* if it satisfies the balance and forest conditions. Additionally, we use p_i to denote the total number of vertices in row i, q_i to denote the number of non-mixed pairs in row i, R_i to denote the number of marked cells in row i, and $s_{i,k}$ to denote the number of mixed pairs with one vertex in row i and one vertex in row k.

By definition, all paired arrays are partially-paired arrays. Also, as the unpaired vertices are not critical vertices, they do not affect the forest condition. Furthermore, as they are not mixed pairs, they do not affect the balance condition. Hence, we can essentially ignore these vertices when discussing the two conditions. However, note that we do consider unpaired vertices to be objects of a partially-paired array.

Now, our main reason for using partially-paired arrays is so that we can unpair vertices of a paired array. That is, if u and v are non-critical vertices that are paired together in a partially-paired array α, we can unpair them to create a new partially-paired array α' that is otherwise identical to α, but with u and v unpaired. Then, we can remove u and v separately, perhaps using different procedures, without impacting the balance and forest conditions. First, however, we need to show that we can unpair vertices without violating these conditions. In the case where $\{u, v\}$ forms a redundant pair, we have the following proposition.

Proposition 3.10. *Let $\{u, v\}$ be a redundant pair in a partially-paired array α, then the partially-paired array β formed by unpairing u and v satisfies the balance and forest conditions if and only if α satisfies them, respectively.*

Proof. As redundant pairs consist of vertices in the same row, unpairing them does not change the number of mixed pairs. Also, as redundant vertices are not the rightmost objects of their cells, they are not used in the forest condition function. Hence, α satisfies the balance and forest conditions if and only if β satisfies them, respectively. □

In particular, given partially-paired array α and a redundant pair $\{u, v\}$, the partially-paired array β formed unpairing u and v is proper if and only if α is proper. Now, one recurrent theme in the proofs of the theorems that follow is the labelling of objects in a row of a partially-paired array with a set of positive integers. This allows us to remove a subset of the unpaired vertices while keeping track of their positions. Conversely, we can insert unpaired vertices into a row of a partially-paired array, again using a subset of positive integers to denote the positions of insertion.

Procedure 3.11. *Let α be a partially-paired array with p_i vertices and R_i marked cells in row i, where $1 \leq i \leq n$. We describe the following three procedures:*

1. *Let $\mathcal{S}$ be a set of positive integers of size $p_i + R_i$. To* label row i of α with $\mathcal{S}$ *is to assign from left to right elements of $\mathcal{S}$ to the objects of row i, from smallest to largest. As described in Definition 3.1, in a cell that contains both vertices and a box, the box is to be taken as the rightmost object of the cell.*

2. *Let $\mathcal{V}$ be a subset of the unpaired vertices in row i. To* extract $\mathcal{V}$ from α *is to create a partially-paired array α' and a set of positive integers $\mathcal{W}$, where α' is α with $\mathcal{V}$*

deleted, and $\mathcal{W}$ is a $|\mathcal{V}|$-subset of $[p_i + R_i - 1]$. This is done by labelling row i of α with $[p_i + R_i]$, then deleting $\mathcal{V}$ from α. We let $\mathcal{W}$ be the labels of the vertices deleted. As the deleted vertices are non-critical, none of them can be the rightmost object of a cell. Therefore, they cannot acquire the label $p_i + R_i$. Hence, $\mathcal{W}$ is a $|\mathcal{V}|$-subset of $[p_i + R_i - 1]$, as desired.

3. *Let $\mathcal{W}$ be a y-subset of $[p_i + R_i + y - 1]$, where $y \geq 0$. To* insert $\mathcal{W}$ into row i of α *is to add y unpaired vertices to row i of α to create a partially-paired array α'. This is done by labelling row i of α with $[p_i + R_i + y] \setminus \mathcal{W}$. Then, for each $w \in \mathcal{W}$, we find the smallest $w' \notin \mathcal{W}$ such that $w' > w$, and place a vertex to the left of and in the same cell as the object labelled w'. As the new vertex is not the rightmost object of a cell, it is non-critical. Furthermore, if there is more than one vertex to be inserted to the left of an object, they should be inserted in increasing order from left to right. In the end, row i of α' contains $p_i + R_i + y$ objects, labelled from left to right by 1 to $p_i + R_i + y$ in increasing order. Finally, we let $\mathcal{V}$ denote the set of vertices inserted, to mirror the extraction procedure.*

Notice that in both the extraction and insertion procedures, the vertices involved are unpaired. The reason for this is that the processes using these procedures require different ways of pairing the vertices. Furthermore, the use of the same variables $\mathcal{V}$ and $\mathcal{W}$ between procedure 2 and 3 is deliberate, as we shall now show that the extraction and insertion procedures are inverses of each other.

Proposition 3.12. *Let α be a partially-paired array with p_i vertices and R_i marked cells in row i, and $\mathcal{V}$ be a subset of the unpaired vertices in row i, where $1 \leq i \leq n$. Let β be the partially-paired array and $\mathcal{W}$ be the subset of $[p_i + R_i - 1]$ created from extracting $\mathcal{V}$ from α. Suppose α' is the partially-paired array formed by reinserting $\mathcal{W}$ into row i of β, and $\mathcal{V}'$ is the set of vertices inserted, then $\alpha = \alpha'$ and $\mathcal{V} = \mathcal{V}'$. Conversely, let β be a partially-paired array with p_i vertices and R_i marked cells in row i, where $1 \leq i \leq n$, and suppose $\mathcal{W}$ is a y-subset of $[p_i + R_i + y - 1]$, with $y \geq 0$. Let α be the partially-paired array formed by inserting $\mathcal{W}$ into row i of β, and $\mathcal{V}$ be the set of inserted vertices. Suppose β' and $\mathcal{W}'$ is the pair of objects created from extracting $\mathcal{V}$ from α, then $\beta = \beta'$ and $\mathcal{W} = \mathcal{W}'$. In both cases, α satisfies the balance and forest conditions if and only if β satisfies them, respectively.*

Proof. Note that when we extract $\mathcal{V}$ from α, we obtain the partially-paired array β and the set $\mathcal{W}$ that is a $|\mathcal{V}|$-subset of $[p_i + R_i - 1]$. Furthermore, β is a partially-paired array with $p_i + R_i - |\mathcal{V}|$ objects in row i, so $\mathcal{W}$ is a subset of $[p_i + R_i - |\mathcal{V}| + |\mathcal{W}| - 1]$. Therefore, we can insert $\mathcal{W}$ into β to obtain the partially-paired array α' and the set $\mathcal{V}'$ of inserted vertices. Notice that the objects remaining in β are labelled with $[p_i + R_i - 1] \setminus \mathcal{W}$ during the extraction, and that they acquire the same labels when we insert $\mathcal{W}$ into row i of β. Consequently, the relative positions of $\mathcal{V}'$ compared to the objects remaining in β are the same as that of $\mathcal{V}$. All that is left to check is that the vertices of $\mathcal{V}'$ are in the same cells as the ones they are extracted from. Consider a vertex $v \in \mathcal{V}$. As v is a non-critical vertex, there must be another object in the same cell and to the right of v. Let u be the leftmost of such an object, and suppose v is labelled w_v and u is labelled w_u by the extraction process. When inserting $\mathcal{W}$ into row i of β, a vertex with the label w_v will be inserted to the left of and in the same cell as the object labelled w_u. This means that a vertex is inserted into the same cell as u. As this holds for every vertex of $\mathcal{V}$, the vertices of $\mathcal{V}'$ are in the same cells as the vertices of $\mathcal{V}$. Therefore, $\alpha = \alpha'$ and $\mathcal{V} = \mathcal{V}'$.

Conversely, when we insert $\mathcal{W}$ into row i of β, each vertex being inserted is to the left of and in the same cell as another object. Therefore, the vertices inserted by $\mathcal{W}$ are non-critical vertices. After the insertion, the objects in row i of α are labelled from left to right with $[p_i + R_i + |\mathcal{W}|]$, where by construction the set of inserted vertices is labelled with $\mathcal{W}$. Therefore, when we extract $\mathcal{V}$ from α, we label these same vertices with $\mathcal{W}$ before removing them from the array. This implies that $\beta = \beta'$ and $\mathcal{W} = \mathcal{W}'$.

Now, let α be a partially-paired array, $\mathcal{V}$ be a subset of the unpaired vertices in row i, and β be the resulting partially-paired array when we extract $\mathcal{V}$ from α. As the vertices of $\mathcal{V}$ are unpaired, the extraction does not impact the balance condition. Similarly, as the vertices of $\mathcal{V}$ are non-critical, they are not used in the forest condition function ψ_i, so ψ_i remains unchanged between α and β. Finally, as the extraction procedure and insertion procedure are inverses of each other, the converse statements also hold. Hence, α satisfies the balance and forest conditions if and only if β satisfies them, respectively. □

One immediate corollary is that given a partially-paired array α, and a partially-paired array β formed by extracting some vertex set $\mathcal{V}$ from row i of α, α is proper if and only if β is proper. Furthermore, the result holds when we simply remove $\mathcal{V}$ instead of extracting it, as we do not have to keep track of $\mathcal{W}$. Conversely, the result also holds when we insert a set of unpaired vertices $\mathcal{V}$ into row i of α, regardless of the means of insertion. Now that we have the extraction and insertion procedures, we will provide the first stage of our decomposition. This composition takes a canonical array, and removes all its redundant pairs. In its place, we are left with a minimal array, and a set of partial pairings describing the positions and pairings of the vertices removed.

Theorem 3.13. *Let $n, K \geq 1$, $\mathbf{q} \geq \mathbf{0}$, and $\mathbf{s} \geq \mathbf{0}$. We have*

$$c_{n,K}^{(\mathbf{q};\mathbf{s})} = \sum_{\mathbf{t}=\mathbf{0}}^{\mathbf{q}} \binom{2q_1 + s_1}{2t_1} \cdots \binom{2q_n + s_n}{2t_n} (2t_1 - 1)!! \cdots (2t_n - 1)!! m_{n,K;\mathbf{1}}^{(\mathbf{q}-\mathbf{t};\mathbf{s})}$$

where $s_i = \sum_{j \neq i} s_{i,j}$ and $\mathbf{t} = (t_1, \ldots, t_n)$.

Proof. For each row i, let $p_i = 2q_i + s_i$, and recall that $\mathcal{T}_{p_i,t_i}$ is the set of t_i-partial pairings on $[p_i]$. We will provide a mapping

$$\zeta \colon \mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})} \to \bigcup_{\mathbf{t}=\mathbf{0}}^{\mathbf{q}} \mathcal{T}_{p_1,t_1} \times \cdots \times \mathcal{T}_{p_n,t_n} \times \mathcal{MA}_{n,K;\mathbf{1}}^{(\mathbf{q}-\mathbf{t};\mathbf{s})}$$

and show that this mapping is a bijection.

Let $\alpha \in \mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$, and suppose that α has t_i redundant pairs in row i. Now, for each row i, let $\mathcal{V}_i$ be the set of redundant vertices. By unpairing and extracting each of the $\mathcal{V}_i$'s from α as described in Procedure 3.11, we can obtain sets $\mathcal{W}_1, \ldots, \mathcal{W}_n$, and a paired array β that contains no redundant vertices. Furthermore, we can keep track of the pairing of redundant vertices by creating a pairing T_i on $\mathcal{W}_i$. That is, for each redundant pair $\{u, v\}$ in row i of α, we let $\{w_u, w_v\}$ be a pair of T_i, where w_u and w_v are the labels corresponding to u and v in $\mathcal{W}_i$. As $R_i = 1$, $\mathcal{W}_i$ is a $2t_i$ subset of $[p_i]$. Therefore, T_i is a t_i-partial pairing on $[p_i]$, so $T_i \in \mathcal{T}_{p_i,t_i}$. Now, as the pairs of $\mathcal{V}_i$ are non-mixed pairs, β contains $q_i - t_i$ non-mixed pairs in row i. In addition, the number of mixed vertices and the number of marked cells remains unchanged between α and β. Since β contains no redundant vertices, and is proper by Proposition 3.10 and Proposition 3.12, it is a minimal array. Hence, $\beta \in \mathcal{MA}_{n,K;\mathbf{1}}^{(\mathbf{q}-\mathbf{t};\mathbf{s})}$ as desired.

Conversely, let $\beta \in \mathcal{MA}_{n,K;\mathbf{1}}^{(\mathbf{q}-\mathbf{t};\mathbf{s})}$ and $T_i \in \mathcal{T}_{p_i,t_i}$ for $1 \leq i \leq n$. We can recover α by doing the following. For each i, let $\mathcal{W}_i$ be the support of T_i. By inserting each of the $\mathcal{W}_i$'s into β as described in Procedure 3.11, we can obtain a partially-paired array α' and sets $\mathcal{V}_1, \ldots, \mathcal{V}_n$ of unpaired vertices in α'. Now, note that each T_i records a pairing of vertices of the corresponding $\mathcal{V}_i$, which we can use to reconstruct α. For each pair $\{w_u, w_v\}$ in T_i, we let $\{u, v\}$ be a pair of α, where u and v are the vertices labelled w_u and w_v in the insertion procedure. As these vertices are non-critical, the inserted pairs are redundant pairs. Therefore, α contains q_i non-mixed pairs in row i. In addition, the number of mixed vertices and the number of marked cells remain unchanged between α and β. Again, by Proposition 3.10 and Proposition 3.12, both α' and α are proper paired arrays. Therefore, α is a canonical array. Hence, $\alpha \in \mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$ as desired.

By Proposition 3.12, the extraction procedure and insertion procedure are inverses of each other. Furthermore, if we extract the $\mathcal{V}_i$'s and reinsert them, they acquire the same labels as before the extraction. Hence, the redundant pairs of α can be recovered from the T_i's. Therefore, ζ as described, is a bijection. By taking the cardinality of both sides, we obtain our result as desired. □

For example, if for each row i we label the rows of the canonical array in Figure 3.2 with $[p_i + R_i]$, we obtain the diagram in Figure 3.3. Then, by decomposing the paired array using the bijection described in Theorem 3.13, we obtain the tuple (T_1, T_2, T_3, α'), where $T_1 = \{\{5,6\}\} \in \mathcal{T}_{8,1}$, $T_2 = \emptyset \in \mathcal{T}_{6,0}$, $T_3 \in \{\{1,3\}, \{4,10\}, \{7,9\}\} \in \mathcal{T}_{10,3}$, and β is the minimal array in Figure 3.4.

Note that this decomposition works regardless of whether the support of $\mathbf{s}$ forms a tree. Furthermore, as $\mathbf{s}$ does not change, the paired arrays in $\mathcal{CA}_{n,K}^{(\mathbf{q};\mathbf{s})}$ are tree-shaped if and only if the paired arrays in $\mathcal{MA}_{n,K;\mathbf{1}}^{(\mathbf{q}-\mathbf{t};\mathbf{s})}$ are tree-shaped. Now that we have decomposed canonical arrays into minimal arrays, it suffices to decompose minimal arrays and find a formula for the number of them. However, our present tools are inadequate for the task. In Goulden and Slofstra's paper, they introduced the *forest completion algorithm*, which is a method for constructing rooted forests that contain a given subforest. Here, we will use an alternative method that generalizes two-row paired arrays by the introduction of arrows. This new method allows us to not only decompose minimal arrays into vertical arrays, but also to recursively decompose vertical arrays into smaller vertical arrays.

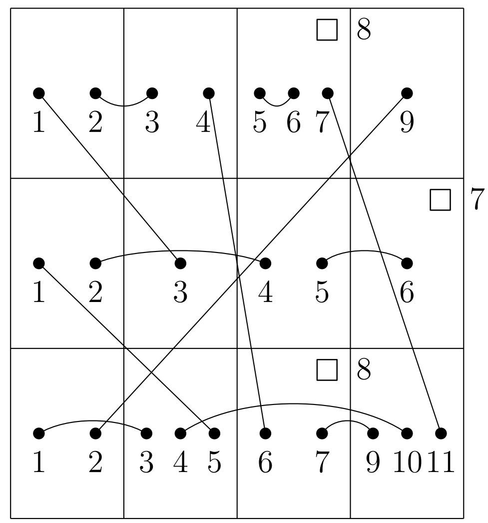


Figure 3.3: Figure 3.2 with objects labelled with $[p_i + R_i]$

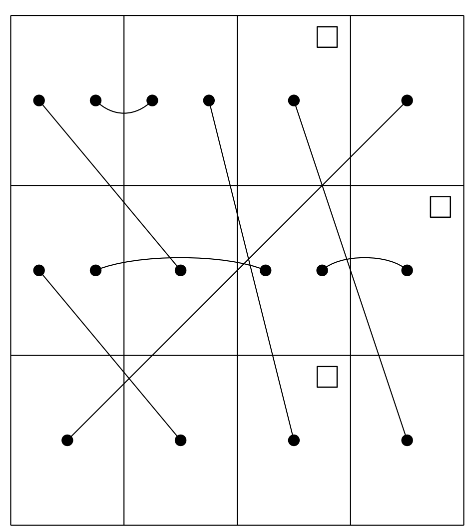


Figure 3.4: Minimal array corresponding to Figure 3.2

Chapter 4

Arrowed Arrays

In this chapter, we will extend two-row paired arrays by the addition of arrows, which represent hypothetical critical vertices. This will allow us to decouple the forest condition with the vertex pairings, which allows for the deletion of vertices and pairings from paired arrays. Next, we will discuss the use of substructures to further partition the set of paired arrays with arrows into subsets that can be enumerated separately. These substructures will fix the positions of the marked cells, arrows, and vertices. We will then derive several reduction lemmas to limit the possible forms of these arrowed arrays, and introduce parameters to describe substructures. Finally, we will give an inductive proof on the number of arrowed arrays based on these parameters, by deleting edges one at a time.

4.1 Definitions and Terminology of Arrowed Arrays

We start off by defining the following extension of paired arrays.

Definition 4.1. Let $K \geq 1$, $s \geq 0$, and $1 \leq R_1, R_2 \leq K$. An *arrowed array* is a pair (α, ϕ), where $\alpha \in \mathcal{VA}^{(s)}_{2,K;R_1,R_2}$ is a two-row vertical array, and $\phi\colon \mathcal{K}\backslash\mathcal{R}_1 \to \mathcal{K}$ is a partial function from $\mathcal{H} \subseteq \mathcal{K}\backslash\mathcal{R}_1$ to $\mathcal{K}$, with $\mathcal{R}_1$ being the set of marked columns in row 1 of α. Graphically, ϕ is denoted by arrows drawn above row 1, where an arrow from j to j' is drawn if $j \in \mathcal{H}$ and $\phi(j) = j'$. For convenience, the two ends of the arrow belonging to columns j and j' are called the *arrow-tail* and *arrow-head* respectively, and column j is said to *point to* column j'. Furthermore, both the arrow-tail and arrow-head belong to row 1 of their respective columns.

With the generalization of paired arrays to arrowed arrays, there are corresponding generalizations of the terms and conventions used to describe paired arrays. These generalizations will be compatible with the conventions for paired arrays if the partial function ϕ is empty.

- An *object* of (α, ϕ) refers to either a vertex, a box, or an arrow-tail. If a cell both contains vertices and a box, or vertices and an arrow-tail, either the box or the arrow-tail is to be taken as the rightmost object of the cell.
- A vertex v of an arrowed array is *critical* if it is the rightmost vertex of a cell, and the cell it belongs to is neither marked nor contains an arrow-tail.
- (α, ϕ) is said to satisfy the *non-empty condition* if for each column j, there exists at least one cell that contains an object.
- (α, ϕ) is said to satisfy the *balance condition* if for each column j, the number of vertices in cell $(1, j)$ is equal to the number of vertices in cell $(2, j)$.
- Let $\mathcal{F}_i$ be the set of columns in row i that contain at least one vertex. The *forest condition function* $\psi_1\colon (\mathcal{H} \cup \mathcal{F}_1) \setminus \mathcal{R}_1 \mapsto \mathcal{K}$ for row 1 is defined as follows: For each column $j \in \mathcal{H}$, let $\psi_1(j) = \phi(j)$; for $j \in \mathcal{F}_1 \setminus (\mathcal{H} \cup \mathcal{R}_1)$, if the rightmost vertex v is paired with a vertex u in column j', let $\psi_1(j) = j'$. The forest condition function ψ_2 for row 2 is defined to be the same as the one for paired arrays in Definition 3.3. (α, ϕ) is said to satisfy the *forest condition* if the functional digraph of ψ_1 on the vertex set $\mathcal{H} \cup \mathcal{F}_1 \cup \psi_1(\mathcal{H} \cup \mathcal{F}_1) \cup \mathcal{R}_1$ is a forest with root vertices $\mathcal{R}_1$, and the functional digraph of ψ_2 on the vertex set $\mathcal{F}_2 \cup \psi_2(\mathcal{F}_2) \cup \mathcal{R}_2$ is a forest with root vertices $\mathcal{R}_2$. That is, for each column $j \in (\mathcal{H} \cup \mathcal{F}_1) \setminus \mathcal{R}_1$, there exists some positive integer t such that $\psi_1^t(j) \in \mathcal{R}_1$, and for each column $j \in \mathcal{F}_2 \setminus \mathcal{R}_2$, there exists some positive integer t such that $\psi_2^t(j) \in \mathcal{R}_2$.
- Additionally, (α, ϕ) is said to satisfy the *full condition* if every cell contains at least one object.

The set of arrowed arrays that satisfies the forest condition is denoted $\mathcal{AR}^{(s)}_{K;R_1,R_2}$.

Notice in particular that a cell cannot contain both an arrow-tail and be marked at the same time. Unless otherwise stated, we will continue to use the conventions for paired arrays defined in Convention 3.2 for arrowed arrays. However, we will be using the definition of critical vertex defined here instead of the one in Definition 3.1. As with paired

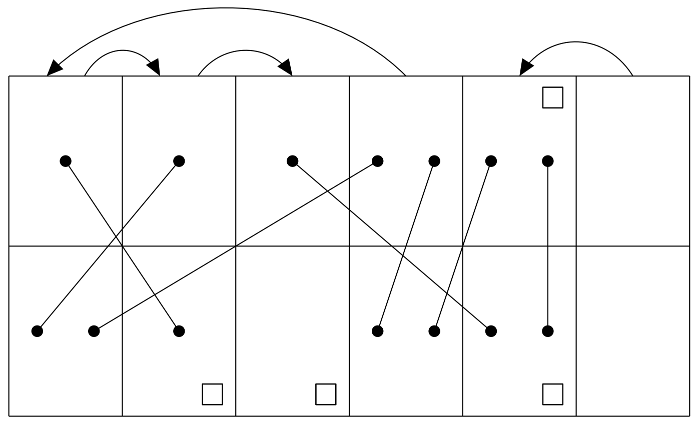


Figure 4.1: A arrowed array in $\mathcal{AR}^{(7)}_{6;1,3}$

arrays, we will always include the columns $\mathcal{R}_i$ in the vertex set for the functional digraph of ψ_i, regardless of whether they are in the range of ψ_i. Similarly, given arrowed arrays (α, ϕ) and (α', ϕ'), we will use ψ_i to denote the forest condition function for row i of (α, ϕ), and ψ'_i to denote the forest condition function for row i of (α', ϕ'). Another parallel is that permuting the columns of an arrowed array does not change whether the array satisfies the balance or forest conditions, as all this action does is to relabel the vertices of the functional digraph. Furthermore, to reduce cluttering, we will draw the boxes for row 2 at the lower right corner instead of the upper right. An example of an arrowed array that satisfies the forest condition can be found in Figure 4.1.

Remark 4.2. Notice that the definition of critical vertices for both paired arrays and arrowed arrays refers to vertices that contribute the forest condition. Also, the balance condition for arrowed arrays is the result of restricting the balance condition of paired arrays to two rows. As arrowed arrays are generalized vertical arrays, there are no redundant pairs, and all pairs are mixed pairs. One thing that differs is the notation used to describe the set of paired arrays compared with the set of arrowed arrays. With paired arrays, $\mathcal{PA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ does not require the forest condition to be satisfied, while $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ assumes that it is so. The reason for this difference is so that we can use paired arrays to define arrowed arrays.

While the parameters used for defining the set of arrowed arrays is natural with respect to paired arrays, it does not easily lend itself to a formula. To make it manageable for summation, we need to partition the set of arrowed arrays by adding further constraints.

Definition 4.3. Let $K \geq 1$, $s \geq 0$, and $1 \leq R_1, R_2 \leq K$. A *substructure* Γ of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ is a set of constraints that defines a subset of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$. For convenience, an arrowed array

(α, ϕ) is said to satisfy Γ if (α, ϕ) satisfies the constraints given by Γ. In particular, let $\mathbf{w}$ be a non-negative matrix of size $2 \times K$, $\mathcal{R}_1, \mathcal{R}_2$ be R_1 and R_2 subsets of $\mathcal{K}$, and ϕ be a partial function from $\mathcal{H} \subseteq \mathcal{K} \backslash \mathcal{R}_1$ to $\mathcal{K}$. The substructure $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ is defined to be the subset of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$, such that for each pair $(\alpha', \phi') \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$, the marked cells in row 1 and 2 of α' are $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively, α' contains $w_{i,j}$ vertices in cell (i, j), and $\phi' = \phi$.

Note that knowing $\mathbf{w}$, $\mathcal{R}_1$, $\mathcal{R}_2$ and ϕ is enough to determine whether an arrowed array satisfies the balance, non-empty, or full conditions. It is also sufficient to determine whether a vertex is critical, regardless of the actual pairing of the vertices. Therefore, we can use these terms, and terms such as arrow-head, arrow-tail, and points to with respect to Γ.

4.2 Arrowed Array Simplification Lemmas

Next, we will lay the ground work for the enumeration of arrowed arrays satisfying a given substructure Γ. This involves introducing several lemmas that limit the number of possibilities we have to consider, as well as lemmas that allow us to remove pairings from arrowed arrays. This allows us to categorize Γ based on a number of parameters that serve as invariants for the number of arrowed arrays that satisfy Γ.

Lemma 4.4. *Let $\mathcal{G}$ and $\mathcal{R}$ be disjoint subsets of $\mathcal{K}$. Let $\psi\colon \mathcal{G} \to \mathcal{K}$ and $G = (V, E)$ be the functional digraph of ψ on the vertex set $\mathcal{G} \cup \psi(\mathcal{G}) \cup \mathcal{R}$.*

1. *If (u, v) is an edge of G and $v \in \mathcal{R}$, then G is a forest with root vertices $\mathcal{R}$ if and only if $G' = (V, E \backslash (u, v))$ is a forest with root vertices $\mathcal{R} \cup \{u\}$.*

2. *If (u, v) and (v, w) are edges of G, then G is a forest with root vertices $\mathcal{R}$ if and only if $G' = (V, E \cup (u, w) \setminus (u, v))$ is a forest with root vertices $\mathcal{R}$.*

3. *If (u, v) is an edge G and u is a leaf vertex, then G is a forest with root vertices $\mathcal{R}$ if and only if $G' = (V \backslash \{u\}, E \backslash (u, v))$ is a forest with root vertices $\mathcal{R}$.*

Proof. Note that in all three cases, aside from the component(s) containing u and v, G is a forest with root vertices in $\mathcal{R}$ if and only if G' is a forest with root vertices in $\mathcal{R}$. Let C be the component of G that contains u, and T be the subgraph of C that has a directed path to u. If C is a tree, then T is a tree with root u.

Now, let (u,v) be an edge of G and $v \in \mathcal{R}$. Suppose G is a rooted forest, and C is the component containing u. Then, both T and $C\backslash T$ are rooted trees with roots u and v respectively, so G' is a rooted forest with root vertices $\mathcal{R} \cup \{u\}$. Conversely, suppose T is a tree rooted at u and C is a tree rooted at $v \in \mathcal{R}$. Then, adding the edge (u,v) joins T to C, forming a tree with root v. Therefore, G is a rooted forest with root vertices $\mathcal{R}$.

Similarly, let (u,v) and (v,w) be edges of G. Suppose G is a rooted forest, and C is the component containing u. Deleting (u,v) gives us the trees T and $C\backslash T$, with w in $C\backslash T$. Hence, adding the edge (u,w) gives us a new tree C', with the same root as C. Conversely, suppose G' is a rooted forest, and C' is the component containing u. Deleting (u,w) gives us the trees T and $C'\backslash T$, with v in $C\backslash T$. Hence, adding the edge (u,v) gives us back the tree C. In either case, the root vertices remain unchanged, hence G is a rooted forest with root vertices $\mathcal{R}$ if and only if G' is a rooted forest with root vertices $\mathcal{R}$.

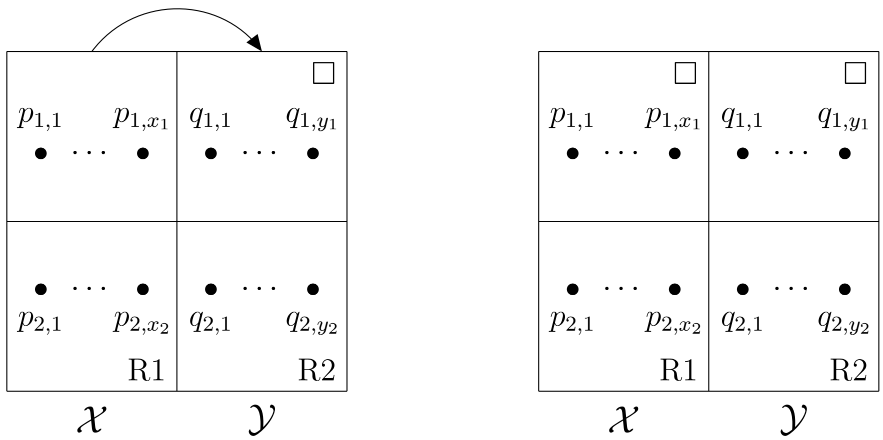
Finally, let (u,v) be an edge of G and u be a leaf vertex. Suppose G is a rooted forest, and C is the component containing u. As u is a leaf, C is a tree if and only if $C\backslash\{u\}$ is a tree. As the root vertices remain unchanged, G is a rooted forest with root vertices $\mathcal{R}$ if and only if G' is a rooted forest with root vertices $\mathcal{R}$. □

Note that in Item 2, the distance between u and its root vertex in $\mathcal{R}$ is closer in G' than it is in G. This means that if G is a forest with root vertices $\mathcal{R}$, by repeatively applying Item 2, we can reduce G to a graph where all edges are from $\mathcal{G}$ to $\mathcal{R}$. This lemma allows us to modify the forest condition functions ψ_i of an arrowed array in certain ways that preserve the forest condition. In particular, by applying the first two points, we obtain the following lemmas.

Lemma 4.5. *Let (α,ϕ) be an arrowed array, and suppose that ϕ contains a column $\mathcal{X}$ that points to a column $\mathcal{Y}$, where cell $(1,\mathcal{Y})$ of α is marked. Let (α',ϕ') be an arrowed array, such that α' is a vertical array otherwise identical to α, but with cell $(1,\mathcal{X})$ marked, and ϕ' is such that*

$$\phi'(j) \quad = \quad \begin{cases} \text{undefined} & j = \mathcal{X} \\ \phi(j) & j \in \mathcal{H}\backslash\mathcal{X} \end{cases},$$

that is, instead of having an arrow pointing from $\mathcal{X}$ to $\mathcal{Y}$, we mark $(1,\mathcal{X})$ of (α',ϕ'). Then, (α,ϕ) is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ if and only if (α',ϕ') is in $\mathcal{AR}^{(s)}_{K;R_1+1,R_2}$. Furthermore, (α,ϕ) satisfies the balance, non-empty, and full conditions if and only if (α',ϕ') satisfies them, respectively.



By applying the arrow simplification procedure to the left figure, we arrive at the right figure. R1 and R2 can be arbitrary in whether they are marked, but they must be the same between the two figures.

Figure 4.2: Arrow Simplification 1

Proof. As we have not changed the vertex pairings, ψ_2 remains unchanged between (α, ϕ) and (α', ϕ'). On the other hand, $\psi_1' (\mathcal{X})$ is now undefined and $\mathcal{X} \in \mathcal{R}_1'$. By taking the functional digraph and applying Item 1 of Lemma 4.4, ψ_1 satisfies the forest condition if and only if ψ_1' satisfies it. Furthermore, K, R_2, and s remain the same between the two arrowed arrays, and $|\mathcal{R}_1'| = |\mathcal{R}_1| + 1$. Therefore, (α, ϕ) is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ if and only if (α', ϕ') is in $\mathcal{AR}^{(s)}_{K;R_1+1,R_2}$.

Note that the only change between (α, ϕ) and (α', ϕ') is the replacement of an arrow-tail by a box in cell $(1, \mathcal{X})$, so cell $(1, \mathcal{X})$ contains at least one object in both (α, ϕ) and (α', ϕ'). As all other objects of (α', ϕ') remain unchanged, including the positions of the vertices, (α, ϕ) satisfies the balance, non-empty, and full conditions if and only if (α', ϕ') satisfies them, respectively. $\square$

Lemma 4.6. *Let (α, ϕ) be an arrowed array, and suppose that ϕ contains a column $\mathcal{X}$ that points to a column $\mathcal{Y}$, and the column $\mathcal{Y}$ points to another column $\mathcal{Z}$. Let (α, ϕ') be an arrowed array, where ϕ' is such that*

$$\phi' (j) \quad = \quad \begin{cases} \mathcal{Z} & j = \mathcal{X} \\ \phi (j) & j \in \mathcal{H} \backslash \mathcal{X} \end{cases},$$

that is, instead of pointing to $\mathcal{Y}$, $\mathcal{X}$ now points to $\mathcal{Z}$ in ϕ'. Then, (α, ϕ) is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ if and only if (α', ϕ') is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$. Furthermore, (α, ϕ) satisfies the balance, non-empty, and full conditions if and only if (α', ϕ') satisfies them, respectively.

Proof. Again, as we have not changed the vertex pairings, ψ_2 remains unchanged between (α, ϕ) and (α', ϕ'). On the other hand, $\psi'_1(\mathcal{X}) = \mathcal{Z}$ is the only change in row 1 between ψ_1 and ψ'_1. By taking the functional digraph and applying Item 2 of Lemma 4.4, ψ_1 satisfies the forest condition if and only ψ'_1 satisfies it. Furthermore, K, R_1, R_2, and s remain the same between the two arrowed arrays. Therefore, (α, ϕ) is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ if and only if (α, ϕ') is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$.

Note that the only change between (α, ϕ) and (α, ϕ') is the position of an arrow-head, so all objects of (α', ϕ') remain unchanged, as an arrow-head is not an object of an arrowed array. Since this includes the positions of all vertices, (α, ϕ) satisfies the balance, non-empty, and full conditions if and only if (α', ϕ') satisfies them, respectively. □

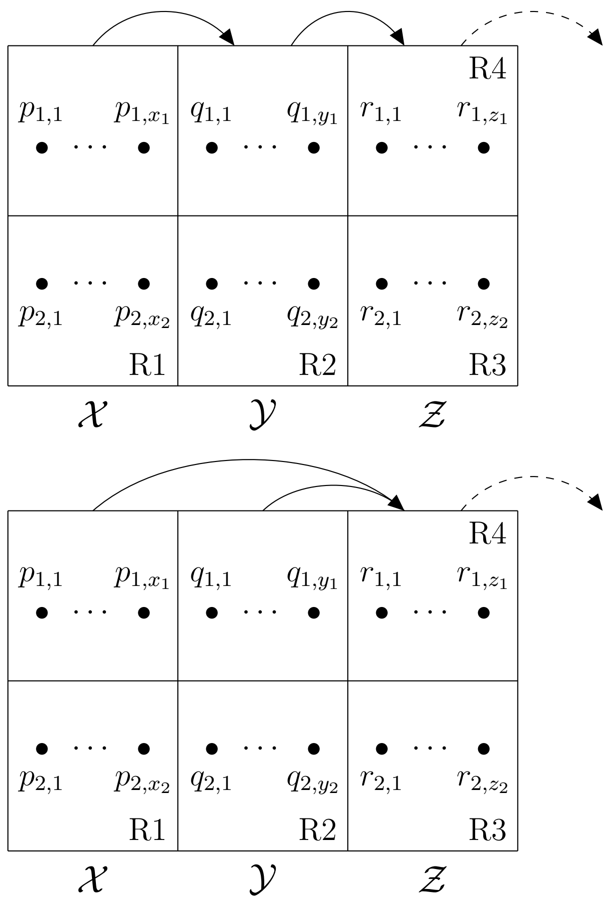
Collectively, Lemma 4.5 and Lemma 4.6 are the *arrow simplification lemmas* for arrowed arrays, and pictures describing the applications of these lemmas can be found in Figure 4.2 and Figure 4.3. Note that these lemmas can be applied repeatedly to simplify an arrowed array, until either all arrow-heads are in cells that are unmarked and have no arrow-tails, or an arrow-head is in the same cell as its own arrow-tail. Furthermore, we can extend these lemmas to substructures of the form $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$. This gives us the following lemmas.

Lemma 4.7. *Let $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$, and suppose that ϕ contains a column $\mathcal{X}$ that points to a column $\mathcal{Y}$, with cell $(1, \mathcal{Y})$ marked. Let $\Gamma' = (\mathbf{w}, \mathcal{R}_1 \cup \{\mathcal{X}\}, \mathcal{R}_2, \phi')$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1+1,R_2}$, such that*

$$\phi'(j) \quad = \quad \begin{cases} \textit{undefined} & j = \mathcal{X} \\ \phi(j) & j \in \mathcal{H} \backslash \mathcal{X} \end{cases},$$

that is, instead of pointing to $\mathcal{Y}$, we mark cell $(1, \mathcal{X})$ of Γ'. Then, the number of arrowed arrays satisfying Γ and the number of arrowed arrays satisfying Γ' are equal. Furthermore, Γ satisfies the balance, non-empty, and full conditions if and only if Γ' satisfies them, respectively.

Proof. Let $\alpha \in \mathcal{VA}^{(s)}_{2,K;R_1,R_2}$ be a two-row vertical array, and α' be a vertical array otherwise identical to α, but with cell $(1, \mathcal{X})$ marked. By Lemma 4.5, (α, ϕ) is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ if and



By applying the arrow simplification procedure to the top figure, we arrive at the bottom figure. R1, R2, R3, and R4 can be arbitrary in whether they are marked, but they must be the same between the two figures. The same holds for the optional arrow with $\mathcal{Z}$ as its tail.

Figure 4.3: Arrow Simplification 2

only if (α', ϕ') is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$. Furthermore, (α, ϕ) satisfies the remaining constraints of Γ if and only if (α', ϕ') satisfies them for Γ' by construction. Therefore, the number of arrowed arrays satisfying Γ and Γ' are equal.

As with Lemma 4.5, the only change between Γ and Γ' is the replacement of an arrow-tail by a box in cell $(1, \mathcal{X})$, so cell $(1, \mathcal{X})$ contains at least one object in both Γ and Γ'. As all other objects of Γ' remain unchanged, including the positions of the vertices, Γ satisfies the balance, non-empty, and full conditions if and only if Γ' satisfies them, respectively. □

Lemma 4.8. *Let $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$, and suppose that ϕ contains a column $\mathcal{X}$ that points to a column $\mathcal{Y}$, and the column $\mathcal{Y}$ points to another column $\mathcal{Z}$. Let $\Gamma' = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi')$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ such that*

$$\phi'(j) \;=\; \begin{cases} \mathcal{Z} & j = \mathcal{X} \\ \phi(j) & j \in \mathcal{H} \backslash \mathcal{X} \end{cases},$$

that is, instead of pointing to $\mathcal{Y}$, $\mathcal{X}$ now points to $\mathcal{Z}$ in ϕ'. Then, the number of arrowed arrays satisfying Γ and the number of arrowed arrays satisfying Γ' are equal. Furthermore, Γ satisfies the balance, non-empty, and full conditions if and only if Γ' satisfies them, respectively.

Proof. Let $\alpha \in \mathcal{VA}^{(s)}_{2,K;R_1,R_2}$ be a two-row vertical array. By Lemma 4.6, (α, ϕ) is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ if and only if (α, ϕ') is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$. Furthermore, (α, ϕ) satisfies the remaining constraints of Γ if and only if (α, ϕ') satisfies them for Γ' by construction. Therefore, the number of arrowed arrays satisfying Γ and Γ' are equal.

As with Lemma 4.6, the only change between Γ and Γ' is the position of an arrow-head, so all objects of Γ' remain unchanged, as an arrow-head is not an object of an arrowed array. Since this includes the positions of all vertices, Γ satisfies the balance, non-empty, and full conditions if and only if Γ' satisfies them, respectively. □

Correspondingly, Lemma 4.7 and Lemma 4.8 are the *arrow simplification lemmas for substructures* $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$. As with individual arrowed arrays, these lemmas can be applied repeatedly to simplify a substructure, until either all arrow-heads are in cells that are unmarked and have no arrow-tails, or an arrow-head is in the same cell as its own arrow-tail. We are only interested in the former, as the latter implies that there is a cycle in the functional digraph of ϕ, which violates the forest condition. This gives rise to the following definition.

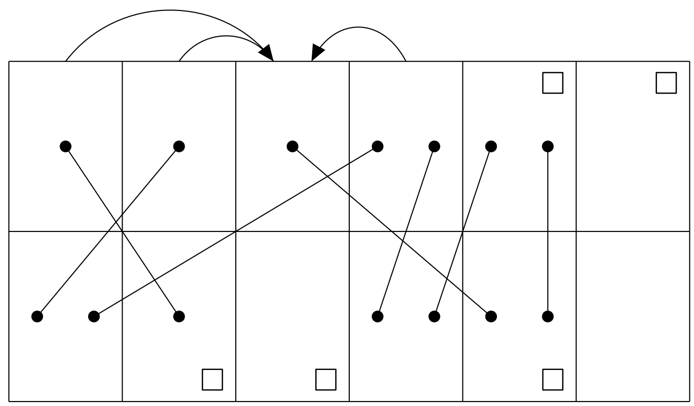


Figure 4.4: Simplification of the arrowed array in Figure 4.1 into an irreducible array

Definition 4.9. A substructure $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ is *irreducible* if the functional digraph of ϕ is acyclic, and Γ cannot be further simplified with the application of the arrow simplification lemmas. Any cell of an irreducible substructure containing an arrow-head must be unmarked in row 1, and cannot contain an arrow-tail. Furthermore, it follows from definition that if an irreducible substructure satisfies the full condition, then any cell containing an arrow-head must also contain a critical vertex in row 1.

As we can use the arrow simplification lemmas to simplify arrowed arrays, and we can call an arrowed array *irreducible* if cannot be further simplified. In particular, an example of an irreducible arrowed array can be found in Figure 4.4. This corresponds to the arrowed array in Figure 4.1.

Definition 4.10. If $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ is an irreducible substructure, then we can categorize the columns of Γ as follows: Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be a partition of the columns of $\mathcal{K} \backslash \mathcal{H}$, where

- Columns in $\mathcal{A}$ have both row 1 and row 2 unmarked
- Columns in $\mathcal{B}$ have row 1 marked and row 2 unmarked
- Columns in $\mathcal{C}$ have row 1 unmarked and row 2 marked
- Columns in $\mathcal{D}$ have both row 1 and row 2 marked

Furthermore, if $\mathcal{X}$ is a column or a set of columns, let $\overline{\mathcal{X}}$ and $\widetilde{\mathcal{X}}$ be the sets of columns that have arrows pointing to $\mathcal{X}$, and that have row 2 unmarked and marked, respectively. In

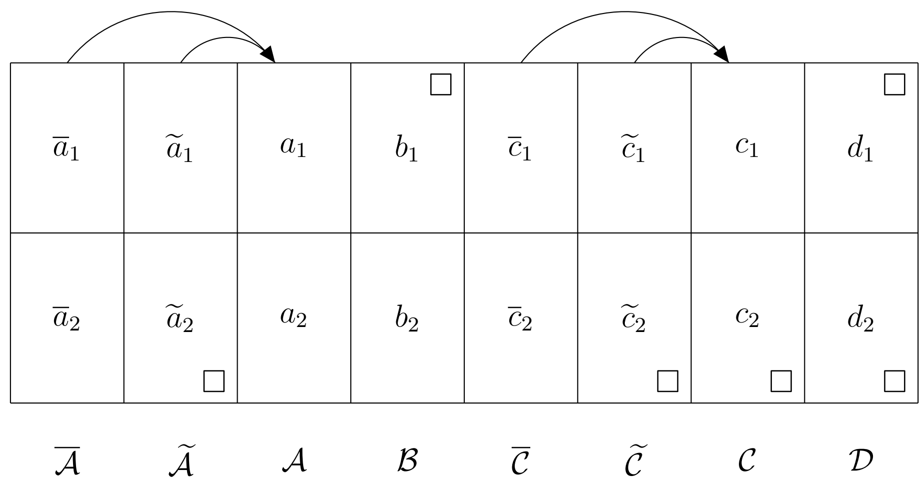


Figure 4.5: Column types and variables for the number of vertices

particular, $\overline{\mathcal{A}}$ and $\widetilde{\mathcal{A}}$ denotes the sets of columns pointing to $\mathcal{A}$, and $\overline{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ denotes the sets of columns pointing to $\mathcal{C}$, with row 2 unmarked and marked, respectively. These sets of columns implicitly defined by Γ are referred to as *column types*, and a diagram with all the column types can be found in Figure 4.5.

As with irreducibility, we can also apply these column types to individual arrowed arrays, as long as they are irreducible. Now, these eight column types form a partition of $\mathcal{K}$ on irreducible substructures. Furthermore, we shall see that knowing the number of columns and the number of vertices for each column type of Γ is sufficient to count the number of arrowed arrays satisfying it. However, before proving the main theorem of this chapter, we will need another two lemmas for simplifying arrowed arrays that contain a fixed pair of vertices.

Lemma 4.11. *(column pointing) Let $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$, v be a critical vertex in cell $(1, \mathcal{X})$, u be a non-critical vertex in cell $(2, \mathcal{Y})$, and $\mathcal{X} \neq \mathcal{Y}$. Let the substructure Γ_{vu} be the set of arrowed arrays that satisfies Γ and contains the pair $\{v, u\}$, and $\Gamma' = (\mathbf{w}', \mathcal{R}_1, \mathcal{R}_2, \phi')$ be a substructure of $\mathcal{AR}^{(s-1)}_{K;R_1,R_2}$ such that*

$$w'_{i,j} = \begin{cases} w_{i,j} - 1 & \text{cell } (i,j) \text{contains } u \text{ or } v \\ w_{i,j} & \text{otherwise} \end{cases}$$

$$\phi'(j) = \begin{cases} \phi(j) & j \in \mathcal{H} \\ \mathcal{Y} & j = \mathcal{X} \end{cases}$$

Note that ϕ' contains one more element in its domain than ϕ. Then, the number of arrowed arrays satisfying Γ_{vu} and the number of arrowed arrays satisfying Γ' are equal. Furthermore, Γ_{vu} satisfies the non-empty and full conditions if and only if Γ' satisfies them.

Proof. To prove that the number of arrowed arrays are equal, we provide a bijection between arrowed arrays satisfying Γ_{vu} and arrowed arrays satisfying Γ'. Let (α, ϕ) be an arrowed array that satisfies Γ and contains the pair $\{v, u\}$. As u is not critical, removing the pair $\{v, u\}$ does not affect ψ_2. Therefore, we can obtain an arrowed array (α', ϕ') by removing $\{v, u\}$ and replacing it by an arrow pointing from $\mathcal{X}$ to $\mathcal{Y}$, while keeping all the other pairs intact. This reduces the number of vertices in $(1, \mathcal{X})$ and $(2, \mathcal{Y})$ by 1, and leaves ψ_1 unchanged. Hence, the forest condition is preserved, and (α', ϕ') satisfies Γ'.

Conversely, given an arrowed array (α', ϕ') that satisfies Γ', we can remove the arrow pointing from $\mathcal{X}$ to $\mathcal{Y}$ and replace it by the pair $\{v, u\}$ given by Γ_{vu}. Since the positions of v and u are fixed in Γ_{vu}, there is no ambiguity as to where to add them. Again, the forest condition is preserved as ψ_1 and ψ_2 are unchanged by this substitution. Finally, both cells $(1, \mathcal{X})$ and $(2, \mathcal{Y})$ contain at least one object in both Γ_{vu} and Γ'. Cell $(1, \mathcal{X})$ contains either a critical vertex or an arrow-tail, and cell $(2, \mathcal{Y})$ contains at least one other object as u is not critical. Since all other cells remain unchanged, Γ_{vu} satisfies the non-empty and full conditions if and only if Γ' satisfies them. □

Lemma 4.12. *(column merging) Let $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$, v be a critical vertex in cell $(1, \mathcal{X})$, u be a critical vertex in cell $(2, \mathcal{Y})$, and $\mathcal{X} \neq \mathcal{Y}$. Suppose that Γ satisfies the full condition, and without loss of generality, assume that $\mathcal{Y}$ is the last column of Γ for purposes of column indexing. Let the substructure Γ_{vu} be the set of arrowed arrays that satisfies Γ and contains the pair $\{v, u\}$, and $\Gamma' = (\mathbf{w}', \mathcal{R}'_1, \mathcal{R}'_2, \phi')$ be a substructure of $\mathcal{AR}^{(s-1)}_{K-1;R_1,R_2}$ such that*

$$\mathcal{R}'_i = \begin{cases} \mathcal{R}_i \cup \mathcal{X} \backslash \mathcal{Y} & \mathcal{Y} \in \mathcal{R}_i \\ \mathcal{R}_i & \textit{otherwise} \end{cases}$$

$$w'_{i,j} = \begin{cases} w_{i,j} + w_{i,\mathcal{Y}} - 1 & j = \mathcal{X} \\ w_{i,j} & \textit{otherwise} \end{cases}$$

$$\phi'(j) = \begin{cases} \phi(\mathcal{Y}) & j = \mathcal{X}, \phi(\mathcal{Y}) \textit{ is defined} \\ \mathcal{X} & j \in \mathcal{H}, \phi(j) = \mathcal{Y} \\ \phi(j) & j \in \mathcal{H}, \phi(j) \neq \mathcal{Y} \end{cases}$$

Then, the number of arrowed arrays satisfying Γ_{vu} and the number of arrowed arrays *satisfying* Γ' *are equal. Furthermore,* Γ' *also satisfies the full condition.*

Proof. To prove that the number of arrowed arrays are equal, we provide a bijection between arrowed arrays satisfying Γ_{vu} and arrowed arrays satisfying Γ'. The idea behind this bijection is to merge the columns $\mathcal{X}$ and $\mathcal{Y}$ in such a way that keeps the rightmost objects of cell $(2, \mathcal{X})$ and $(1, \mathcal{Y})$ intact. By construction, the rightmost objects of cells $(1, \mathcal{Y})$ and $(2, \mathcal{X})$ in α are the same as the rightmost object of cells $(1, \mathcal{X})$ and $(2, \mathcal{X})$ in α'. As all other cells remain unchanged, Γ' satisfies the full condition.

Let (α, ϕ) be an arrowed array that satisfies Γ and contains the pair $\{v, u\}$. To obtain α', we take the vertices of cell $(2, \mathcal{Y})$ except u and place them in cell $(2, \mathcal{X})$ in order, before the vertices originally in $(2, \mathcal{X})$. Then, for any column j that points to $\mathcal{Y}$, we change them to point to $\mathcal{X}$ instead. For convenience, let the forest condition function for row 1 at this stage be ψ_1''. Next, we take the vertices of cell $(1, \mathcal{Y})$ and place them in cell $(1, \mathcal{X})$ before v. Here, we let the forest condition function for row 2 be ψ_2''. Furthermore, if cell $(1, \mathcal{Y})$ is marked, we mark cell $(1, \mathcal{X})$, and if column $\mathcal{Y}$ points to some column $\mathcal{Z}$, we make $\mathcal{X}$ point to $\mathcal{Z}$. Finally, we remove the pair $\{v, u\}$ and the column $\mathcal{Y}$.

Conversely, given an arrowed array (α', ϕ') that satisfies Γ', we can recover (α, ϕ) by splitting the column $\mathcal{X}$. Since we only use the forward direction to show that the forest condition is preserved, we will describe the recovery in a more convenient order. We first add the column $\mathcal{Y}$ to (α', ϕ'). Then, if cell $(1, \mathcal{X})$ is marked, we mark cell $(1, \mathcal{Y})$ and unmarked cell $(1, \mathcal{X})$. Furthermore, if column $\mathcal{X}$ points to some column $\mathcal{Z}$, we make column $\mathcal{Y}$ point to $\mathcal{Z}$ and remove the arrow from $\mathcal{X}$. Afterwards, we move the last $w_{1,\mathcal{Y}}$ vertices of cell $(1, \mathcal{X})$ of α' to cell $(1, \mathcal{Y})$, and move the first $w_{2,\mathcal{Y}} - 1$ vertices of cell $(2, \mathcal{X})$ of α' to cell $(2, \mathcal{Y})$, keeping all pairings intact. Finally, we add the vertices u and v to cells $(1, \mathcal{X})$ and $(2, \mathcal{Y})$ respectively, and pair them to obtain (α, ϕ). This is unambiguous, as the column $\mathcal{Y}$ and the quantities $w_{i,\mathcal{Y}}$ are given by Γ, which is fixed.

By construction, (α, ϕ) satisfies Γ_{vu} if and only if (α', ϕ') satisfies Γ', with the possible exception of the forest condition. Now, consider ψ_2 during the transformation from (α, ϕ) to (α', ϕ'). Note that moving the vertices of cell $(2, \mathcal{Y})$ and moving the arrow-heads has no impact on ψ_2. Then, when a vertex of $(1, \mathcal{Y})$ is moved, there is either no impact, or the vertex is paired with some critical vertex in cell $(2, j)$. In the latter case, we have $\psi_2''(j) = \mathcal{X}$. As $\psi_2(\mathcal{Y}) = \mathcal{X}$, by the repeated application of Item 2 of Lemma 4.4, we have that ψ_2 satisfies the forest condition if and only if ψ_2'' satisfies it. Finally, from ψ_2'' to ψ_2', we deleted the pair $\{v, u\}$ and the column $\mathcal{Y}$. As $\mathcal{X}$ contains at least 1 object, it remains in the forest condition function ψ_2. Therefore, we can apply Item 3 of Lemma 4.4 to show that ψ_2 satisfies the forest condition if and only if ψ_2' satisfies it.

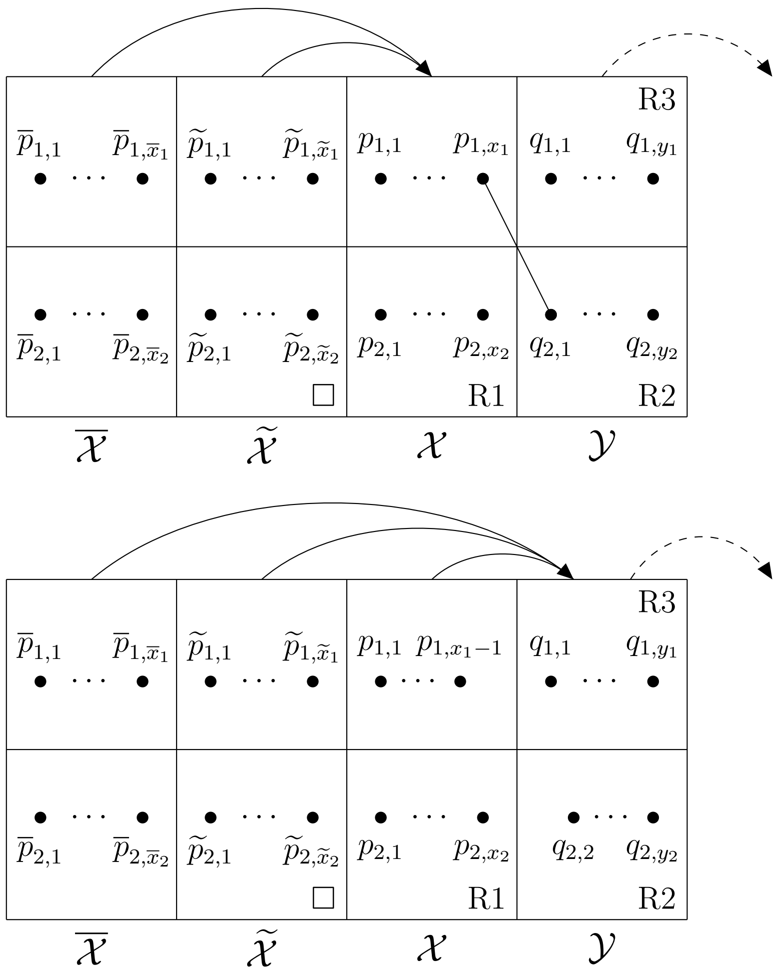
The proof for row 1 is similar, though slightly more complicated. By moving the vertices of cell $(2, \mathcal{Y})$ and moving the arrow-heads, we are changing ψ_1, so that if j is a column where $\psi_1(j) = \mathcal{Y}$, then $\psi_1''(j) = \mathcal{X}$. As in the row 2 case, we can repeatedly apply Item 2 of Lemma 4.4, but with the roles of G and G' reversed. Then, the movement of the non-critical vertices in cell $(1, \mathcal{Y})$ does not change ψ_1''. If cell $(1, \mathcal{Y})$ contains a critical vertex or an arrow-tail, we obtain that $\psi_1(\mathcal{Y}) = \mathcal{Z}$ for some column $\mathcal{Z}$. This implies $\psi_1'(\mathcal{X}) = \mathcal{Z}$, due to either moving the arrow-tail or the critical vertex. By Item 2 and 3 of Lemma 4.4, we can remove the column $\mathcal{Y}$, and ψ_1'' satisfies the forest condition if and only if ψ_1' satisfies it. Otherwise, cell $(1, \mathcal{Y})$ is marked in α as (α, ϕ) satisfies the full condition, which translates to cell $(1, \mathcal{X})$ being marked in α'. Therefore, we can use Item 1 of Lemma 4.4 to show that ψ_1'' satisfies the forest condition if and only if ψ_1' satisfies it. This allows us to safely delete $\mathcal{Y}$, as it is now an isolated root vertex in $\mathcal{R}_1$. Consequently, ψ_1 satisfies the forest condition if and only if ψ_1' satisfies it. This shows that the numbers of arrowed arrays satisfying Γ_{vu} and Γ' are equal. □

The application of Lemma 4.11 to replace Γ_{vu} with Γ' is called the *column pointing procedure*, and a diagram of this procedure can be found in Figure 4.6. Similarly, the application of Lemma 4.12 to replace Γ_{vu} with Γ' is called the *column merging procedure*, and a diagram of this procedure can be found in Figure 4.7. After applying either procedure, we can apply the arrow simplification lemmas to Γ' to further simplify the substructure.

Note that unlike the other simplification lemmas, column merging requires the substructure to satisfy the full condition. In particular, it requires each cell of the columns being merged to be non-empty. Otherwise, the resulting column will completely drop out of the forest condition, which can break the the bijection. Namely, it is possible to have a substructure Γ such that the substructure Γ_{vu} cannot be satisfied by any arrowed array, while the substructure Γ' is satisfied by some arrowed arrays. An example of this can be found in Figure 4.8.

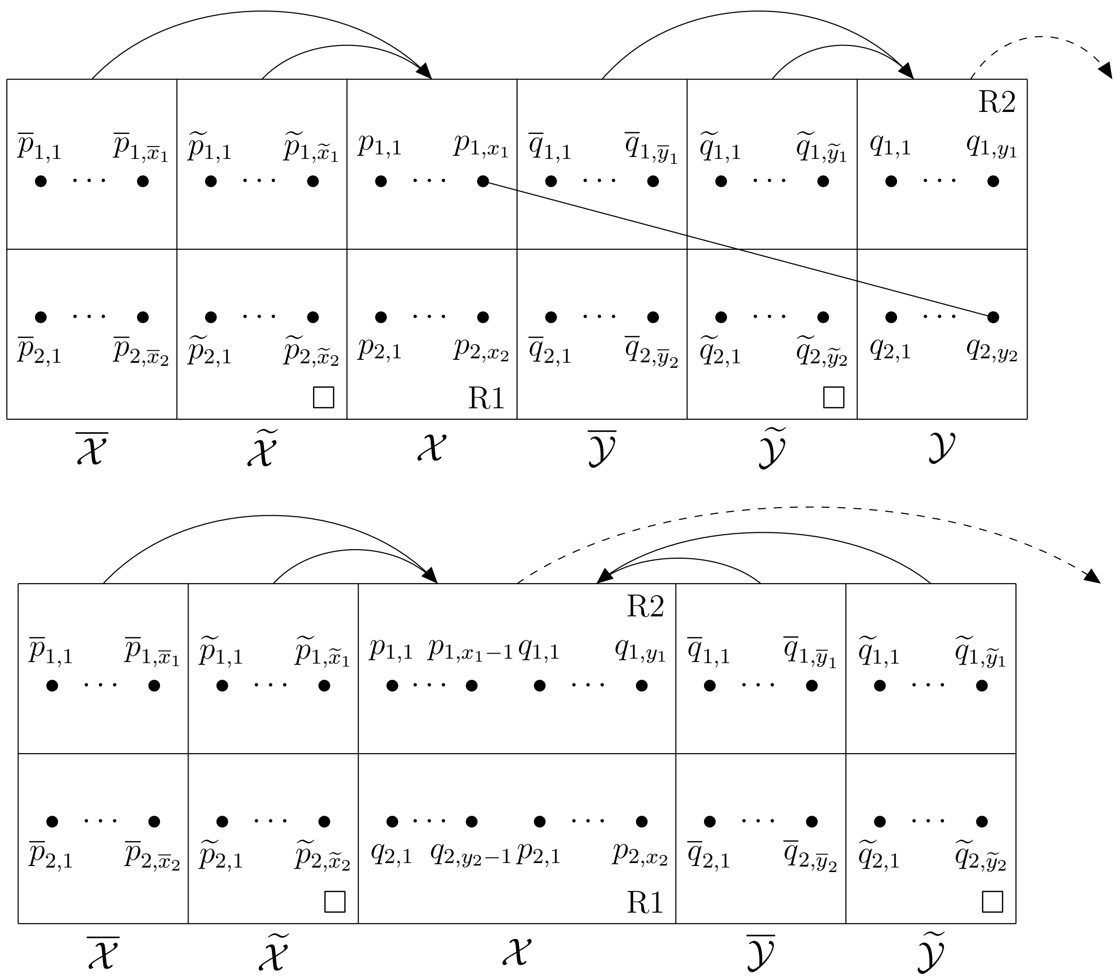
4.3 Enumeration of Substructure $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$

Now, we have everything we need to provide a formula for the number of arrowed arrays satisfying the substructure $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$, where Γ is an irreducible substructure satisfying the full condition. The formula will be given by the number of vertices in each column type, as well as the number of columns of type $\mathcal{A}$. Let $T(\Gamma)$ be the number of arrowed arrays that satisfy the substructure Γ, then the following are two theorems for the formulas of $T(\Gamma)$, one for the case $s \geq A + 2$, and one for the case $s = A + 1$.



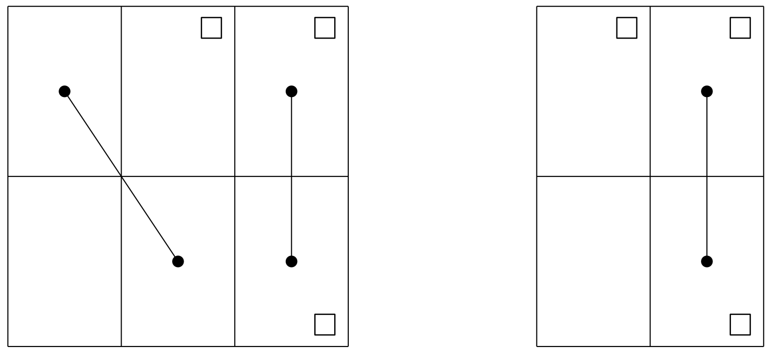
By applying the column pointing procedure to the top figure, we arrive at the bottom figure. Here, $u = p_{1,x_1}$ and $v = q_{2,1}$. R1, R2, and R3 can be arbitrary in whether they are marked, but they must be the same between the two figures. The same holds for the optional arrow with $\mathcal{Y}$ as its tail.

Figure 4.6: Column pointing



By applying the column merging procedure to the top figure, we arrive at the bottom figure. Here, $u = p_{1,x_1}$ and $v = q_{2,y_2}$. R1 and R2 can be arbitrary in whether they are marked, but they must be the same between the two figures. The same holds for the optional arrow with $\mathcal{Y}$ as its tail.

Figure 4.7: Column merging



A failed attempt to merge columns 1 and 2 when the full condition is not satisfied. Note that the first arrowed array fails the forest condition for row 2, while the second arrowed array satisfies it.

Figure 4.8: Column merging without full condition

Theorem 4.13. *Given an irreducible substructure* $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ *that satisfies the full condition with* $s \geq A + 2$*, the number of arrowed arrays* $(\alpha, \phi) \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$ *that satisfy* Γ *is given by the formula*

$$T(\Gamma) = (s-1)! \left[\frac{(b_2 + d_2)(\widetilde{a}_1 + c_1 + \widetilde{c}_1 + d_1)}{s - A} + \frac{b_1 (c_2 + \overline{c}_2 + \widetilde{c}_2) - \overline{c}_1 (b_2 + d_2)}{(s-A)(s-A-1)} \right]$$

By the convention set out in Convention 3.2, we let a lower case variable x_i represent the total number of points in row i of the columns of type $\mathcal{X}$, and A represent the number of columns of type $\mathcal{A}$.

Proof. We prove this via induction on the total number of vertices, and tiebreak by the number of critical vertices in the row 2. There are two base cases and three inductive cases to consider, depending on whether Γ contains a column of type $\mathcal{A}$, a column of type $\mathcal{C}$ and no columns of type $\mathcal{A}$, or no columns of type $\mathcal{A}$ or $\mathcal{C}$.

Base case 1:

Suppose Γ has no critical vertex. As Γ is irreducible, each cell must either be marked or have an arrow-tail. However, the latter cannot happen as an arrow-head of an irreducible substructure must be in an unmarked cell. Hence, every cell of Γ must be marked, so the forest condition is trivially satisfied. Therefore, there are $s!$ ways to pair the vertices of the array. By substituting $d_1 = d_2 = s$ into $T(\Gamma)$, and setting all other variables to 0, we see that $T(\Gamma) = s!$ as desired.

Base case 2:

If $s = 2$, $\mathcal{A} = \emptyset$, and $\mathcal{C} \neq \emptyset$, then

$$\begin{aligned} T(\Gamma) &= \left[\frac{(b_2 + d_2)(c_1 + \widetilde{c}_1 + d_1)}{2} + \frac{b_1(c_2 + \overline{c}_2 + \widetilde{c}_2) - \overline{c}_1(b_2 + d_2)}{2}\right] \\ &= b_1 + (1 - b_1 - \overline{c}_1)(b_2 + d_2) \end{aligned}$$

by substituting in $2 = b_i + \overline{c}_i + \widetilde{c}_i + c_i + d_i$. This case is needed as the inductive step for Γ containing no columns of type $\mathcal{A}$ but at least one column of type $\mathcal{C}$ requires that $T(\Gamma)$ be true for $s-1$. However, if $s = 1$, then $s < A+2$, and this creates a zero in the denominator of our formula.

Suppose $\overline{c}_1 \neq 0$, then $\overline{c}_1 = 1$ and $b_1 = 0$, as a column of type $\mathcal{C}$ contains a critical vertex in row 1, and $\mathcal{C}$ is non-empty. Furthermore, this implies that there is only one column of type $\mathcal{C}$. In this case, our formula gives $T(\Gamma) = 0$. Combinatorially, if there is a column of type $\overline{\mathcal{C}}$, then it has a critical vertex in row 2, as the array is full. Now, if this vertex is matched with the vertex in row 1 of $\mathcal{C}$, then the forest condition for row 1 is violated. Otherwise, it is matched with the vertex in row 1 of $\overline{\mathcal{C}}$, and the forest condition in row 2 is violated. Therefore, no such arrowed array exists, and so $T(\Gamma) = 0$ as desired.

Suppose $\overline{c}_1 = 0$ and $b_1 \neq 0$, then $b_1 = 1$, as again there is exactly one column of type $\mathcal{C}$. In this case, our formula gives $T(\Gamma) = 1$. Let the column contributing to b_1 be $\mathcal{X}$, and note $\mathcal{X}$ is a column of type $\mathcal{B}$, which is unmarked in row 2. Therefore, $\mathcal{X}$ must contain a critical vertex in row 2, and this vertex must be joined with the critical vertex in $\mathcal{C}$ to not violate the forest condition for row 2. Doing so satisfies the forest condition in row 1, as $\psi_1(\mathcal{C}) = \mathcal{X}$, which is in $\mathcal{R}_1$. Now, The other vertex in row 2 is either a non-critical vertex, or a critical vertex that is paired with the vertex of $\mathcal{X}$ in row 1. In either case, the forest condition for row 2 is satisfied as $\psi_2(\mathcal{X}) = \mathcal{C}$, which is in $\mathcal{R}_2$. This gives $T(\Gamma) = 1$ as desired.

For the last case, suppose $\overline{c}_1 = 0$ and $b_1 = 0$. In this case, our formula gives $T(\Gamma) = b_2 + d_2$. Note that all vertices of row 1 are in $\widetilde{\mathcal{C}} \cup \mathcal{C} \cup \mathcal{D}$, which are all marked in row 2. Therefore, no matter where the vertices in row 2 are positioned, they are paired with vertices in row 1 whose columns are marked in row 2. This means that the forest condition for row 2 is automatically satisfied.

Now, there are 2 subcases for row 1. If there is only one column of type $\mathcal{C}$, then the other vertex in row 1 is in either $\widetilde{\mathcal{C}}$ or $\mathcal{D}$. In the former case, that column points to $\mathcal{C}$, and in the latter case, row 1 of that column is marked. In both cases, the array satisfies the forest condition if and only if the column that the vertex in $\mathcal{C}$ pairs to is marked in row 1.

As the columns marked in row 1 are $\mathcal{B}$ and $\mathcal{D}$, we have $T(\Gamma) = b_2 + d_2$ as desired.

Otherwise, there are 2 columns of type $\mathcal{C}$, each containing 1 critical vertex. We denote the two columns as $\mathcal{C}_1$ and $\mathcal{C}_2$. If both vertices in row 2 are in $\mathcal{B} \cup \mathcal{D}$, then we can pair them with the vertices of $\mathcal{C}$ arbitrary, so $T(\Gamma) = 2$. Suppose one vertex in row 2 is in some column $\mathcal{X}$ in $\overline{\mathcal{C}} \cup \widetilde{\mathcal{C}} \cup \mathcal{C}$, and the other vertex is in some column $\mathcal{Y}$ in $\mathcal{B} \cup \mathcal{D}$. Without loss of generality, let $\mathcal{X}$ be $\mathcal{C}_1$, or a column that points to $\mathcal{C}_1$. In this case, the vertex in row 1 of $\mathcal{C}_1$ must be paired with the vertex of $\mathcal{B} \cup \mathcal{D}$ to not violate the forest condition. The vertex in row 1 of $\mathcal{C}_2$ is consequently paired with the vertex in $\mathcal{X}$. As $\mathcal{X}$ is either $\mathcal{C}_1$ itself, or points to $\mathcal{C}_1$, the entire component of the functional digraph has $\mathcal{Y}$ as its root, which is in $\mathcal{R}_1$. This gives $T(\Gamma) = 1$. Finally, if both vertices of row 2 are in $\overline{\mathcal{C}} \cup \widetilde{\mathcal{C}} \cup \mathcal{C}$, then no matter how they are paired, the component(s) of the functional digraph containing $\mathcal{C}$ is entirely in $\overline{\mathcal{C}} \cup \widetilde{\mathcal{C}} \cup \mathcal{C}$. Therefore, it cannot have roots in $\mathcal{B} \cup \mathcal{D}$, so the forest condition for row 1 can never be satisfied. This gives $T(\Gamma) = 0$. In all three cases, we have $T(\Gamma) = b_2 + d_2$ as desired.

Case 1:

Suppose Γ contains at least one column of type $\mathcal{A}$, and $\mathcal{X}$ is one such column. Let $\overline{\mathcal{X}}$ and $\widetilde{\mathcal{X}}$ be columns pointing to $\mathcal{X}$ as defined in Definition 4.10, and note that they are columns of type $\overline{\mathcal{A}}$ and $\widetilde{\mathcal{A}}$, respectively. Then, the critical vertex v of cell $(1, \mathcal{X})$ must be paired with some vertex u in a cell $(2, \mathcal{Y})$. To satisfy the forest condition for row 1, $\mathcal{Y}$ cannot be a column of $\mathcal{X}$, $\overline{\mathcal{X}}$, or $\widetilde{\mathcal{X}}$. By fixing u, we can pair vertices u and v to obtain the substructure Γ_{uv}. Then, we simplify Γ_{uv} using the column pointing and column merging procedures described in Lemma 4.11 and Lemma 4.12, which makes the columns of $\mathcal{X}$, $\overline{\mathcal{X}}$, and $\widetilde{\mathcal{X}}$ point to $\mathcal{Y}$. Now, $\mathcal{Y}$ cannot point to $\mathcal{X}$, $\overline{\mathcal{X}}$, or $\widetilde{\mathcal{X}}$, as that would either imply that $\mathcal{Y} \in \overline{\mathcal{X}} \cup \widetilde{\mathcal{X}}$, or that Γ is not irreducible. Therefore, $\mathcal{Y}$ must either not contain an arrow-tail, or be pointing to some other column $\mathcal{Z}$ that has a critical vertex in row 1. Therefore, the functional digraph of ϕ is acyclic, and by using the arrow simplification procedures described in Lemma 4.5 and Lemma 4.6, we obtain an irreducible substructure Γ' that has one less vertex per row than Γ. Furthermore, both s and A decrease by 1, so the inequality $s \geq A + 2$ holds. Depending on the column type of $\mathcal{Y}$ and whether u is critical, we can use the inductive hypothesis to determine $T(\Gamma')$ in terms of existing parameters given by the column types of Γ. The full list of substitutions can be found in Table 4.1, where an entry $\mathcal{Z}$ in the table means that $\mathcal{Y}$ is a column of type $\mathcal{Z}$ and u is a non-critical, while an entry $\mathcal{Z}c$ means that $\mathcal{Y}$ is a column of type $\mathcal{Z}$, and u is critical.

For example, let $\mathcal{Y}$ be a column of type $\mathcal{D}$. Then, after applying the column pointing procedure, $\mathcal{X}$ becomes a column of type $\mathcal{B}$, the columns of $\overline{\mathcal{X}}$ become columns of type $\mathcal{B}$, and the columns of type $\widetilde{\mathcal{X}}$ become columns of type $\mathcal{D}$. Hence, in the resulting substructure

Column type of $\mathcal{Y}$	a'_i	$\overline{a}'_i$	$\widetilde{a}'_i$	b'_i
$\mathcal{A}$	$a_1 - x_1$ $a_2 - x_2$	$\overline{a}_1 + x_1 - 1$ $\overline{a}_2 + x_2 - 1$		
$\mathcal{A}c$	$a_1 - 1$ $a_2 - 1$			
$\overline{\mathcal{A}}$ or $\overline{\mathcal{A}}c$	$a_1 - x_1$ $a_2 - x_2$	$\overline{a}_1 + x_1 - 1$ $\overline{a}_2 + x_2 - 1$		
$\widetilde{\mathcal{A}}$	$a_1 - x_1$ $a_2 - x_2$	$\overline{a}_1 + x_1 - 1$ $\overline{a}_2 + x_2$	 $\widetilde{a}_2 - 1$	
$\mathcal{B}$ or $\mathcal{B}c$	$a_1 - x_1$ $a_2 - x_2$	$\overline{a}_1 - \overline{x}_1$ $\overline{a}_2 - \overline{x}_2$	$\widetilde{a}_1 - \widetilde{x}_1$ $\widetilde{a}_2 - \widetilde{x}_2$	$b_1 + x_1 + \overline{x}_1 - 1$ $b_2 + x_2 + \overline{x}_2 - 1$
$\mathcal{C}$	$a_1 - x_1$ $a_2 - x_2$	$\overline{a}_1 - \overline{x}_1$ $\overline{a}_2 - \overline{x}_2$	$\widetilde{a}_1 - \widetilde{x}_1$ $\widetilde{a}_2 - \widetilde{x}_2$	
$\overline{\mathcal{C}}$ or $\overline{\mathcal{C}}c$	$a_1 - x_1$ $a_2 - x_2$	$\overline{a}_1 - \overline{x}_1$ $\overline{a}_2 - \overline{x}_2$	$\widetilde{a}_1 - \widetilde{x}_1$ $\widetilde{a}_2 - \widetilde{x}_2$	
$\widetilde{\mathcal{C}}$	$a_1 - x_1$ $a_2 - x_2$	$\overline{a}_1 - \overline{x}_1$ $\overline{a}_2 - \overline{x}_2$	$\widetilde{a}_1 - \widetilde{x}_1$ $\widetilde{a}_2 - \widetilde{x}_2$	
$\mathcal{D}$	$a_1 - x_1$ $a_2 - x_2$	$\overline{a}_1 - \overline{x}_1$ $\overline{a}_2 - \overline{x}_2$	$\widetilde{a}_1 - \widetilde{x}_1$ $\widetilde{a}_2 - \widetilde{x}_2$	$b_1 + x_1 + \overline{x}_1 - 1$ $b_2 + x_2 + \overline{x}_2$

Table 4.1: Table of substitution when Γ contains a column of type $\mathcal{A}$. In all cases, $s' = s - 1$ and $A' = A - 1$. Table continues at Table 4.2

Column type of $\mathcal{Y}$	c'_i	$\overline{c}_i$	$\widetilde{c}_i$	d_i
$\mathcal{A}$				
$\mathcal{A}c$				
$\overline{\mathcal{A}}$ or $\overline{\mathcal{A}}c$				
$\widetilde{\mathcal{A}}$				
$\mathcal{B}$ or $\mathcal{B}c$				$d_1 + \widetilde{x}_1$ $d_2 + \widetilde{x}_2$
$\mathcal{C}$	$c_2 - 1$	$\overline{c}_1 + x_1 + \overline{x}_1 - 1$ $\overline{c}_2 + x_2 + \overline{x}_2$	$\widetilde{c}_1 + \widetilde{x}_1$ $\widetilde{c}_2 + \widetilde{x}_2$	
$\overline{\mathcal{C}}$ or $\overline{\mathcal{C}}c$		$\overline{c}_1 + x_1 + \overline{x}_1 - 1$ $\overline{c}_2 + x_2 + \overline{x}_2 - 1$	$\widetilde{c}_1 + \widetilde{x}_1$ $\widetilde{c}_2 + \widetilde{x}_2$	
$\widetilde{\mathcal{C}}$		$\overline{c}_1 + x_1 + \overline{x}_1 - 1$ $\overline{c}_1 + x_1 + \overline{x}_1$	$\widetilde{c}_1 + \widetilde{x}_1$ $\widetilde{c}_2 + \widetilde{x}_2 - 1$	
$\mathcal{D}$				$d_1 + \widetilde{x}_1$ $d_2 + \widetilde{x}_2 - 1$

Table 4.2: Continuation of Table 4.1

$\Gamma' = \Gamma_{\mathcal{AD}}$ after simplification, we have

- $a_i' = a_i - x_i$
- $\overline{a}_i' = \overline{a}_i - \overline{x}_i$
- $\widetilde{a}_i' = \widetilde{a}_i - \widetilde{x}_i$
- $b_i' = b_i + x_i + \overline{x}_i - \delta_{1,i}$
- $d_i' = d_i + \widetilde{x}_i - \delta_{2,i}$

where $\delta_{i,j} = 1$ if $i = j$, and 0 otherwise. Substituting this into the inductive hypothesis, we have

$$\begin{aligned} T(\Gamma_{\mathcal{AD}}) &= (s-2)! \left[\frac{(b_2 + x_2 + \overline{x}_2 + d_2 + \widetilde{x}_2 - 1)(\widetilde{a}_1 + c_1 + \widetilde{c}_1 + d_1)}{s - A} + \right. \\ & \left. \frac{(b_1 + x_1 + \overline{x}_1 - 1)(c_2 + \overline{c}_2 + \widetilde{c}_2) - \overline{c}_1 (b_2 + x_2 + \overline{x}_2 + d_2 + \widetilde{x}_2 - 1)}{(s-A)(s-A-1)} \right] \end{aligned}$$

We repeat this computation for all possible column types of $\mathcal{Y}$, and whether u is critical. The results of this are listed in Table 4.3, where $T(\Gamma_{\mathcal{AZ}})$ is the number of arrowed arrays with substructure Γ, and the vertex v is joined to a non-critical vertex u in a column of type $\mathcal{Z}$; $T(\Gamma_{\mathcal{AZ}c})$ is the number of arrowed arrays with substructure Γ, and the vertex v is joined to a critical vertex u in a column of type $\mathcal{Z}$. The $\mathcal{A}$ here denotes that v is a column of type $\mathcal{A}$, to separate it from a similar table in Case 2.

By letting u range across all vertices of row 2, we obtain all possible pairings of the critical vertex v in column $\mathcal{X}$. Therefore, by counting the number of vertices of each column type, we obtain the number of occurrences of each Γ'. Adding everything together, we have

$$\begin{aligned} T(\Gamma) &= (a_2 - x_2 + \overline{a}_2 - \overline{x}_2 + \widetilde{a}_2 - \widetilde{x}_2) T(\Gamma_{\mathcal{AA}}) + \\ & (c_2 + \overline{c}_2 + \widetilde{c}_2) T(\Gamma_{\mathcal{AC}}) + (b_2 + d_2) T(\Gamma_{\mathcal{AB}}) \end{aligned}$$

By substituting in $s = a_i + \overline{a}_i + \widetilde{a}_i + c_i + \overline{c}_i + \widetilde{c}_i + b_i + d_i$ and simplifying, we can show that $T(\Gamma)$ satisfies the inductive hypothesis. This proves the case where Γ contains a column of type $\mathcal{A}$.

Case 2:

$$
\begin{aligned}
T(\Gamma_{\mathcal{A}\mathcal{A}}) &= (s-2)!\left[\frac{(b_2+d_2)(\widetilde{a}_1+c_1+\widetilde{c}_1+d_1)}{s-A}+\right.\\
&\left.\frac{b_1(c_2+\overline{c}_2+\widetilde{c}_2)-\overline{c}_1(b_2+d_2)}{(s-A)(s-A-1)}\right]\\
T(\Gamma_{\mathcal{A}\mathcal{A}c}) &= T(\Gamma_{\mathcal{A}\mathcal{A}})\\
T(\Gamma_{\mathcal{A}\overline{\mathcal{A}}}) &= T(\Gamma_{\mathcal{A}\mathcal{A}})\\
T(\Gamma_{\mathcal{A}\overline{\mathcal{A}}c}) &= T(\Gamma_{\mathcal{A}\mathcal{A}})\\
T\left(\Gamma_{\mathcal{A}\widetilde{\mathcal{A}}}\right) &= T(\Gamma_{\mathcal{A}\mathcal{A}})\\
T(\Gamma_{\mathcal{A}\mathcal{B}}) &= (s-2)!\left[\frac{(b_2+x_2+\overline{x}_2+d_2+\widetilde{x}_2-1)(\widetilde{a}_1+c_1+\widetilde{c}_1+d_1)}{s-A}+\right.\\
&\left.\frac{(b_1+x_1+\overline{x}_1-1)(c_2+\overline{c}_2+\widetilde{c}_2)-\overline{c}_1(b_2+x_2+\overline{x}_2+d_2+\widetilde{x}_2-1)}{(s-A)(s-A-1)}\right]\\
T(\Gamma_{\mathcal{A}\mathcal{B}c}) &= T(\Gamma_{\mathcal{A}\mathcal{B}})\\
T(\Gamma_{\mathcal{A}\mathcal{C}}) &= (s-2)!\left[\frac{(b_2+d_2)(\widetilde{a}_1+c_1+\widetilde{c}_1+d_1)}{s-A}+\right.\\
&\left.\frac{b_1(c_2+\overline{c}_2+x_2+\overline{x}_2+\widetilde{c}_2+\widetilde{x}_2-1)-(\overline{c}_1+x_1+\overline{x}_1-1)(b_2+d_2)}{(s-A)(s-A-1)}\right]\\
T(\Gamma_{\mathcal{A}\overline{\mathcal{C}}}) &= T(\Gamma_{\mathcal{A}\mathcal{C}})\\
T(\Gamma_{\mathcal{A}\overline{\mathcal{C}}c}) &= T(\Gamma_{\mathcal{A}\mathcal{C}})\\
T\left(\Gamma_{\mathcal{A}\widetilde{\mathcal{C}}}\right) &= T(\Gamma_{\mathcal{A}\mathcal{C}})\\
T(\Gamma_{\mathcal{A}\mathcal{D}}) &= T(\Gamma_{\mathcal{A}\mathcal{B}})
\end{aligned}
$$

Table 4.3: Case 1: Formula for $T(\Gamma')$ after simplification

Suppose that Γ does not contain columns of type $\mathcal{A}$, but contains at least one column of type $\mathcal{C}$. The formula simplifies to

$$T(\Gamma) = (s-1)!\left[\frac{(b_2+d_2)(c_1+\widetilde{c}_1+d_1)}{s} + \frac{b_1(c_2+\overline{c}_2+\widetilde{c}_2)-\overline{c}_1(b_2+d_2)}{s(s-1)}\right]$$

While the formula is simpler in this case, the proof is slightly more involved. Let $\mathcal{X}$ be a fixed column of type $\mathcal{C}$, and let $\overline{\mathcal{X}}$ and $\widetilde{\mathcal{X}}$ be columns pointing to $\mathcal{X}$ as defined in Definition 4.10. Note that they are columns of type $\overline{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$, respectively. As in Case 1, the critical vertex v of cell $(1, \mathcal{X})$ must be paired with some vertex u in a cell. Again, to satisfy the forest condition for row 1, $\mathcal{Y}$ cannot be a column of $\mathcal{X}$, $\overline{\mathcal{X}}$ or, $\widetilde{\mathcal{X}}$. Therefore, we pair u and v to obtain the substructure Γ_{uv}, which we simplify using the same lemmas used in Case 1 to obtain an irreducible substructure Γ'. As the case $s = 2$ is already handled, we can assume $s \geq 3$, so $s \geq A + 2$ still holds. Depending on the column type of $\mathcal{Y}$ and whether u is critical, we can use the inductive hypothesis to determine $T(\Gamma')$ in terms of existing parameters given by column types of Γ. The full list of substitutions can be found in Table 4.4, where an entry $\mathcal{Z}$ in the table means that $\mathcal{Y}$ is a column of type $\mathcal{Z}$ and u is a non-critical, while an entry $\mathcal{Z}c$ means that $\mathcal{Y}$ is a column of type $\mathcal{Z}$, and u is critical. The major difference in this case is that if u is a critical vertex, then both $\mathcal{X}$ and $\mathcal{Y}$ become columns of a different type, so we must introduce the parameters y_i for the number of vertices in column i of $\mathcal{Y}$.

As in Case 1, we can compute $T(\Gamma')$ for all possible column types of $\mathcal{Y}$, and whether u is critical. Doing this gives us the formulas listed in Table 4.5. Again note that y_1 depends on the column $\mathcal{Y}$, so some of the formulas are dependent on which particular column u is in.

By letting u range across all vertices of row 2, we obtain all possible pairings of the critical vertex v in column $\mathcal{X}$. Notice that as we pair v each vertex of $\mathcal{B}$, we add $y_1 T_{\mathcal{CB}c}$ if and only if u is the rightmost vertex of $\mathcal{Y}$. Since each column of $\mathcal{B}$ has exactly one rightmost vertex, $\sum_{\mathcal{Y}\in\mathcal{B}} y_1 = b_1$. Similarly, $\sum_{\mathcal{Y}\in\overline{\mathcal{C}}} y_1 = \overline{c}_1 - \overline{x}_1$. Therefore, by counting the number of vertices of each column type, we obtain the number of occurrences of each Γ'. Adding everything together, we have

$$\begin{aligned} T(\Gamma) &= (c_2 - x_2 + \overline{c}_2 - \overline{x}_2 + \widetilde{c}_2 - \widetilde{x}_2)\, T(\Gamma_{\mathcal{CC}}) + \\ &\quad T_{\mathcal{C}\overline{\mathcal{C}}c}(\overline{c}_1 - \overline{x}_1) + (b_2 + d_2)\, T(\Gamma_{\mathcal{CB}}) + b_1 T_{\mathcal{CB}c} \end{aligned}$$

By substituting in $s = c_2 + \overline{c}_2 + \widetilde{c}_2 + b_2 + d_2$ and simplifying, we can show that $T(\Gamma)$ satisfies the inductive hypothesis. This proves the case where Γ contains a column of type $\mathcal{C}$, but

Column type of $\mathcal{Y}$	b_i	c_i	$\overline{c}_i$
$\mathcal{B}$	$b_1 + \overline{x}_1$ $b_2 + \overline{x}_2 - 1$	$c_1 - x_1$ $c_2 - x_2$	$\overline{c}_1 - \overline{x}_1$ $\overline{c}_2 - \overline{x}_2$
$\mathcal{B}c$	$b_1 + \overline{x}_1 - y_1$ $b_2 + \overline{x}_2 - y_2$	$c_1 - x_1$ $c_2 - x_2$	$\overline{c}_1 - \overline{x}_1$ $\overline{c}_2 - \overline{x}_2$
$\mathcal{C}$		$c_1 - x_1$ $c_2 - x_2 - 1$	
$\overline{\mathcal{C}}$		$c_1 - x_1$ $c_2 - x_2$	$\overline{c}_2 - 1$
$\overline{\mathcal{C}}c$		$c_1 - x_1$ $c_2 - x_2$	$\overline{c}_1 - y_1$ $\overline{c}_2 - y_2$
$\widetilde{\mathcal{C}}$		$c_1 - x_1$ $c_2 - x_2$	
$\mathcal{D}$	$b_1 + \overline{x}_1$ $b_2 + \overline{x}_2$	$c_1 - x_1$ $c_2 - x_2$	$\overline{c}_1 - \overline{x}_1$ $\overline{c}_2 - \overline{x}_2$

Column type of $\mathcal{Y}$	$\widetilde{c}_i$	d_i
$\mathcal{B}$	$\widetilde{c}_1 - \widetilde{x}_1$ $\widetilde{c}_2 - \widetilde{x}_2$	$d_1 + x_1 + \widetilde{x}_1 - 1$ $d_2 + x_2 + \widetilde{x}_2$
$\mathcal{B}c$	$\widetilde{c}_1 - \widetilde{x}_1$ $\widetilde{c}_2 - \widetilde{x}_2$	$d_1 + x_1 + \widetilde{x}_1 + y_1 - 1$ $d_2 + x_2 + \widetilde{x}_2 + y_2 - 1$
$\mathcal{C}$	$\widetilde{c}_1 + x_1 - 1$ $\widetilde{c}_2 + x_2$	
$\overline{\mathcal{C}}$	$\widetilde{c}_1 + x_1 - 1$ $\widetilde{c}_2 + x_2$	
$\overline{\mathcal{C}}c$	$\widetilde{c}_1 + x_1 + y_1 - 1$ $\widetilde{c}_2 + x_2 + y_2 - 1$	
$\widetilde{\mathcal{C}}$	$\widetilde{c}_1 + x_1 - 1$ $\widetilde{c}_2 + x_2 - 1$	
$\mathcal{D}$	$\widetilde{c}_1 - \widetilde{x}_1$ $\widetilde{c}_2 - \widetilde{x}_2$	$d_1 + x_1 + \widetilde{x}_1 - 1$ $d_2 + x_2 + \widetilde{x}_2 - 1$

Table 4.4: Table of substitution when Γ contains no columns of type $\mathcal{A}$, but a column of type $\mathcal{C}$. In all cases, $s' = s - 1$

$$
\begin{aligned}
T\left(\Gamma_{\mathcal{CB}}\right) &= (s-2)!\left[\frac{\left(b_2+\overline{x}_2+d_2+x_2+\widetilde{x}_2-1\right)\left(c_1+\widetilde{c}_1+d_1-1\right)}{s-1}+\right.\\
&\quad \frac{\left(b_1+\overline{x}_1\right)\left(c_2-x_2+\overline{c}_2-\overline{x}_2+\widetilde{c}_2-\widetilde{x}_2\right)}{(s-1)(s-2)}-\\
&\quad \left.\frac{\left(\overline{c}_1-\overline{x}_1\right)\left(b_2+\overline{x}_2+d_2+x_2+\widetilde{x}_2-1\right)}{(s-1)(s-2)}\right]\\
T\left(\Gamma_{\mathcal{CB}c}\right) &= (s-2)!\left[\frac{\left(b_2+x_2+d_2+\overline{x}_2+\widetilde{x}_2-1\right)\left(c_1+\widetilde{c}_1+d_1+y_1-1\right)}{s-1}+\right.\\
&\quad \frac{\left(b_1+\overline{x}_1-y_1\right)\left(c_2-x_2+\overline{c}_2-\overline{x}_2+\widetilde{c}_2-\widetilde{x}_2\right)}{(s-1)(s-2)}-\\
&\quad \left.\frac{\left(\overline{c}_1-\overline{x}_1\right)\left(b_2+\overline{x}_2+d_2+x_2+\widetilde{x}_2-1\right)}{(s-1)(s-2)}\right]\\
&= T\left(\Gamma_{\mathcal{CB}}\right)+\\
&\quad y_1(s-2)!\left[\frac{\left(b_2+x_2+d_2+\overline{x}_2+\widetilde{x}_2-1\right)}{s-1}-\frac{\left(c_2-x_2+\overline{c}_2-\overline{x}_2+\widetilde{c}_2-\widetilde{x}_2\right)}{(s-1)(s-2)}\right]\\
&= T\left(\Gamma_{\mathcal{CB}}\right)+y_1T_{\mathcal{CB}c}\\
T\left(\Gamma_{\mathcal{CC}}\right) &= (s-2)!\left[\frac{\left(b_2+d_2\right)\left(c_1+\widetilde{c}_1+d_1-1\right)}{s-1}+\right.\\
&\quad \left.\frac{b_1\left(c_2+\overline{c}_2+\widetilde{c}_2-1\right)-\overline{c}_1\left(b_2+d_2\right)}{(s-1)(s-2)}\right]\\
T\left(\Gamma_{\mathcal{C}\overline{\mathcal{C}}}\right) &= T\left(\Gamma_{\mathcal{CC}}\right)\\
T\left(\Gamma_{\mathcal{C}\overline{\mathcal{C}}c}\right) &= (s-2)!\left[\frac{\left(b_2+d_2\right)\left(c_1+\widetilde{c}_1+d_1+y_1-1\right)}{s-1}+\right.\\
&\quad \left.\frac{b_1\left(c_2+\overline{c}_2+\widetilde{c}_2-1\right)-\left(\overline{c}_1-y_1\right)\left(b_2+d_2\right)}{(s-1)(s-2)}\right]\\
&= T\left(\Gamma_{\mathcal{CC}}\right)+y_1(s-3)!\left(b_2+d_2\right)\\
&= T\left(\Gamma_{\mathcal{CC}}\right)+y_1T_{\mathcal{C}\overline{\mathcal{C}}c}\\
T\left(\Gamma_{\mathcal{C}\widetilde{\mathcal{C}}}\right) &= T\left(\Gamma_{\mathcal{CC}}\right)\\
T\left(\Gamma_{\mathcal{CD}}\right) &= T\left(\Gamma_{\mathcal{CB}}\right)
\end{aligned}
$$

Table 4.5: Case 2: Formula for $T\left(\Gamma'\right)$ after simplification

no columns of type $\mathcal{A}$.

Case 3:

If Γ does not contain any column of type $\mathcal{A}$ or $\mathcal{C}$, then every cell in row 1 is marked, leaving us only with columns of type $\mathcal{B}$ and $\mathcal{D}$. In this case, the formula simplifies to

$$T(\Gamma) = d_1 (s-1)!$$

as $s = b_2 + d_2$. Since Γ does not contain any arrows, we can switch the two rows and invert the roles of $\mathcal{B}$ and $\mathcal{C}$ to obtain Γ'. Furthermore, at least one cell in row 2 is unmarked, as otherwise we would have the base case. Therefore, the number of critical vertices in row 2 decreases in Γ', and we can continue the induction using Case 2. Furthermore, neither s nor A changed, so $s \geq A + 2$ still holds. Now, Γ' only have columns of type $\mathcal{C}$ and $\mathcal{D}$, so by the inductive hypothesis,

$$T(\Gamma') = d_2 (s-1)!$$

as $s = c_1 + d_1$ in Γ'. This completes the induction and proves our formula for $T(\Gamma)$. □

In the case where $s = A+1$, the second term of the aforementioned formula is undefined. Fortunately, we can simply set it to zero and pretend it does not exist.

Theorem 4.14. *Given an irreducible substructure* $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ *that satisfies the full condition with* $s = A + 1$*, the number of arrowed arrays* $(\alpha, \phi) \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$ *with substructure* Γ *is given by the formula*

$$T(\Gamma) = (s-1)! \, (b_2 + d_2) \, (\widetilde{a}_1 + c_1 + \widetilde{c}_1 + d_1)$$

Proof. We prove this via induction on the number of columns of type $\mathcal{A}$.

Base case:

Suppose Γ has no columns of type $\mathcal{A}$. Since $s = A + 1$, we have $A = 0$, $\widetilde{a}_1 = 0$, and $s = 1$. Furthermore, $\widetilde{c}_1 = 0$ as a column of type $\mathcal{C}$ requires a critical vertex in row 1. Therefore,

$$T(\Gamma) = (b_2 + d_2)(c_1 + d_1)$$

Now, if the vertex in row 1 is in $\mathcal{C} \cup \mathcal{D}$, and the vertex in row 2 is in $\mathcal{B} \cup \mathcal{D}$, then pairing them satisfies the forest condition for those two cells. All other cells are either marked or have an arrow on them. If they are marked, they satisfy the forest condition for the row

$$
\begin{aligned}
T\left(\Gamma_{\mathcal{A}\mathcal{A}}\right) &= (s-2)!\left(b_2+d_2\right)\left(\widetilde{a}_1+c_1+\widetilde{c}_1+d_1\right)\\
T\left(\Gamma_{\mathcal{A}\mathcal{A}c}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{A}}\right)\\
T\left(\Gamma_{\mathcal{A}\overline{\mathcal{A}}}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{A}}\right)\\
T\left(\Gamma_{\mathcal{A}\overline{\mathcal{A}}c}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{A}}\right)\\
T\left(\Gamma_{\mathcal{A}\widetilde{\mathcal{A}}}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{A}}\right)\\
T\left(\Gamma_{\mathcal{A}\mathcal{B}}\right) &= (s-2)!\left(b_2+x_2+\overline{x}_2+d_2+\widetilde{x}_2-1\right)\left(\widetilde{a}_1+c_1+\widetilde{c}_1+d_1\right)\\
T\left(\Gamma_{\mathcal{A}\mathcal{B}c}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{B}}\right)\\
T\left(\Gamma_{\mathcal{A}\mathcal{C}}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{A}}\right)\\
T\left(\Gamma_{\mathcal{A}\overline{\mathcal{C}}}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{A}}\right)\\
T\left(\Gamma_{\mathcal{A}\overline{\mathcal{C}}c}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{A}}\right)\\
T\left(\Gamma_{\mathcal{A}\widetilde{\mathcal{C}}}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{A}}\right)\\
T\left(\Gamma_{\mathcal{A}\mathcal{D}}\right) &= T\left(\Gamma_{\mathcal{A}\mathcal{B}}\right)
\end{aligned}
$$

Table 4.6: Case $s = A + 1$: Formula for $T(\Gamma')$ after simplification

they belong in. Otherwise, the cells must be in row 1, and must be pointing at a column of type $\mathcal{C}$. As the vertex in $\mathcal{C}$ is matched with a vertex in $\mathcal{B} \cup \mathcal{D}$, row 1 satisfies the forest condition.

Suppose that the vertex in row 1 is not in $\mathcal{C} \cup \mathcal{D}$. It cannot be in $\overline{\mathcal{C}}$ or $\widetilde{\mathcal{C}}$, as those require a column of type $\mathcal{C}$. Therefore, it must be in a column of type $\mathcal{B}$. This column has a critical vertex in row 2, which if paired will violate the forest condition in row 2. Similarly, if the vertex in row 2 is not in $\mathcal{B} \cup \mathcal{D}$, it must be in a column of type $\mathcal{C}$, $\overline{\mathcal{C}}$, or $\widetilde{\mathcal{C}}$. In all such cases, a column of type $\mathcal{C}$ exists, and contains a critical vertex in row 1. If these vertices are paired together, the forest condition in row 1 is again violated. In both cases we have $T(\Gamma) = 0$ as desired.

Inductive step:

Suppose Γ contains at least one column of type $\mathcal{A}$. The proof here is exactly the same as in Case 1 of the proof for Theorem 4.13. We also end with the same substitutions as the ones in Table 4.1 and Table 4.2. By substituting this into the inductive hypothesis, we obtain the results for $T(\Gamma')$ as listed in Table 4.6.

By letting u range across all vertices of row 2, we obtain all possible pairings of the critical vertex v in column $\mathcal{X}$. Therefore, by counting the number of vertices of each column type, we obtain the number of occurrences of each Γ'. Adding everything together, we have

$$\begin{aligned} T(\Gamma) &= (a_2 - x_2 + \overline{a}_2 - \overline{x}_2 + \widetilde{a}_2 - \widetilde{x}_2 + c_2 + \overline{c}_2 + \widetilde{c}_2)\, T(\Gamma_{\mathcal{A}\mathcal{A}}) + \\ & (b_2 + d_2)\, T(\Gamma_{\mathcal{A}\mathcal{B}}) \end{aligned}$$

By substituting in $s = a_i + \overline{a}_i + \widetilde{a}_i + c_i + \overline{c}_i + \widetilde{c}_i + b_i + d_i$ and simplifying, we can show that $T(\Gamma)$ satisfies the inductive hypothesis. This completes the induction and proves our formula for $T(\Gamma)$. □

Note that if Γ satisfies the full condition and $s \leq A$, then $T(\Gamma) = 0$, as each column of type $\mathcal{A}$ requires one critical vertex for each row. Furthermore, as those vertices can only be paired with each other, $\psi_i(\mathcal{X}) \in \mathcal{A}$ for all $\mathcal{X} \in \mathcal{A}$. This violates the forest condition for row i. Together, Theorem 4.13 and Theorem 4.14 give the number of arrowed arrays satisfying a substructure Γ, where Γ satisfies the full condition. In the next chapter, we will sum over $T(\Gamma)$ to obtain formulas for more general substructures, where the positions of the marked cells are no longer fixed.

Chapter 5

Enumeration of Arrowed Arrays

In this chapter, we will define increasingly coarse substructures that capture larger subsets of arrowed arrays, where the positions of the marked cells are not fixed. Then, by summing over the formulas on the number of arrowed arrays satisfying the refined substructures, we can provide formulas for the number of arrowed arrays satisfying these coarser substructures. Furthermore, we will use hypergeometric identities to simplify the resulting formulas, and transform the formulas so that they can be readily used in the next chapter. By the end of the chapter, we will obtain two formulas, corresponding to the two different ways we can use arrowed arrays to extend vertical arrays.

5.1 Substructure $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$

Our first substructure allows us to mark the cells of row 2 arbitrarily, while keeping the positions of the marked cells in row 1 and the vertices fixed. Additionally, we also define a substructure that fixes the number of columns of type $\mathcal{A}$, as $T(\Gamma)$ is dependent on A.

Definition 5.1. Let $\mathbf{w}$ be a non-negative matrix of size $2 \times K$, $\mathcal{R}_1$ be an R_1-subset of $\mathcal{K}$, and $\phi\colon \mathcal{K}\backslash\mathcal{R}_1 \to \mathcal{K}$ be a partial function from $\mathcal{H} \subseteq \mathcal{K}\backslash\mathcal{R}_1$ to $\mathcal{K}$. The substructure $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ is defined to be the subset of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$, such that for each pair $(\alpha', \phi') \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$, the marked cells in row 1 of α' is $\mathcal{R}_1$. Furthermore, α' contains $w_{i,j}$ vertices in cell (i,j), and $\phi' = \phi$. For a given substructure $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ and $A \geq 0$, we define Δ_A to be the substructure that describes the set of arrowed arrays that satisfies Δ, and have exactly A columns of type $\mathcal{A}$. For convenience, we say a substructure Γ is a *refinement* of another substructure Δ if the set of arrowed arrays satisfying Γ is a subset of the arrowed arrays satisfying Δ. We denote it as $\Gamma \hookrightarrow \Delta$. Furthermore, if $\Gamma_1, \ldots, \Gamma_t$ is a set of substructures that are refinements of a substructure Δ, we say that $\Gamma_1, \ldots, \Gamma_t$ *partitions* Δ if the sets of arrowed arrays satisfying the Γ_i's are mutually disjoint, and their union is the set of arrowed arrays that satisfy Δ. Finally, we say $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ satisfies the *non-empty condition* if each cell in row 1 has at least one object in it.

Note that the non-empty condition for substructure $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ is more stringent than it is for arrowed arrays. This is to account for the fact that we do not have control of the positions of the marked cells in row 2. By considering all possible R_2-subsets $\mathcal{R}_2$, we see that the set of substructures of the form $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ partitions the substructure $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$. Furthermore, the subset of substructures with exactly A columns of type $\mathcal{A}$ partitions Δ_A, which in turn partitions Δ by taking A from 0 to $s-1$. Now, as the arrow simplification lemmas only act on row 1, we can obtain arrow simplification lemmas for substructures $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ similar to those of Lemma 4.7 and Lemma 4.8.

Lemma 5.2. *Let $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$, and suppose that ϕ contains a column $\mathcal{X}$ that points to a column $\mathcal{Y}$, with cell $(1, \mathcal{Y})$ marked. Let $\Delta' = (\mathbf{w}, \mathcal{R}_1 \cup \{\mathcal{X}\}, \phi)$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1+1,R_2}$, such that*

$$\phi'(j) \;=\; \begin{cases} \text{undefined} & j = \mathcal{X} \\ \phi(j) & j \in \mathcal{H}\backslash\mathcal{X} \end{cases},$$

that is, instead of pointing to $\mathcal{Y}$, we mark cell $(1, \mathcal{X})$ of Δ'. Then, the number of arrowed arrays satisfying Δ and the number of arrowed arrays satisfying Δ' are equal. Furthermore, Δ satisfies the balance and non-empty condition if and only if Δ' satisfies them, respectively.

Proof. Let $\alpha \in \mathcal{VA}^{(s)}_{2,K;R_1,R_2}$ be a two-row vertical array, and α' be a vertical array otherwise identical to α, but with cell $(1, \mathcal{X})$ marked. By Lemma 4.5, (α, ϕ) is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ if and only if (α', ϕ') is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$. Furthermore, (α, ϕ) satisfies the remaining constraints of Δ if and only if (α', ϕ') satisfies them for Δ' by construction. Therefore, the number of arrowed arrays satisfying Δ and Δ' are equal.

As with Lemma 4.5, the only change between Δ and Δ' is the replacement of an arrow-tail by a box in cell $(1, \mathcal{X})$, so cell $(1, \mathcal{X})$ contains at least one object in both Δ and Δ'. As all other objects of Δ' remain unchanged, including the positions of the vertices, Δ satisfies the balance and non-empty conditions if and only if Δ' satisfies them, respectively. □

Lemma 5.3. *Let $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$, and suppose that ϕ contains a column $\mathcal{X}$ that points to a column $\mathcal{Y}$, and the column $\mathcal{Y}$ points to another column $\mathcal{Z}$. Let $\Delta' = (\mathbf{w}, \mathcal{R}_1, \phi')$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ such that*

$$\phi'(j) = \begin{cases} \mathcal{Z} & j = \mathcal{X} \\ \phi(j) & j \in \mathcal{H} \backslash \mathcal{X} \end{cases}$$

That is, instead of pointing to $\mathcal{Y}$, $\mathcal{X}$ now points to $\mathcal{Z}$ in ϕ'. Then, the number of arrowed arrays satisfying Δ and the number of arrowed arrays satisfying Δ' are equal. Furthermore, Δ satisfies the balance and non-empty condition if and only if Δ' satisfies them, respectively.

Proof. Let $\alpha \in \mathcal{VA}^{(s)}_{2,K;R_1,R_2}$ be a two-row vertical array. By Lemma 4.6, (α, ϕ) is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ if and only if (α, ϕ') is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$. Furthermore, (α, ϕ) satisfies the remaining constraints of Δ if and only if (α, ϕ') satisfies them for Δ' by construction. Therefore, the number of arrowed arrays satisfying Δ and Δ' are equal.

As with Lemma 4.6, the only change between Δ and Δ' is the position of an arrow-head, so all objects of Δ' remain unchanged, as an arrow-head is not an object of an arrowed array. Since this includes the positions of all vertices, Δ satisfies the balance and non-empty conditions if and only if Δ' satisfies them, respectively. □

Correspondingly, Lemma 5.2 and Lemma 5.3 are the *arrow simplification lemmas for substructures* $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$. While we will not be using these directly until Section 5.3, they will serve as the motivation for the following definition.

Definition 5.4. A substructure $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ is *irreducible* if the functional digraph of ϕ is acyclic, and Δ cannot be further simplified with the application of the arrow simplification lemmas. As with Definition 4.9, any cell of an irreducible substructure containing an arrow-head must be unmarked in row 1, and cannot contain an arrow-tail. Furthermore, if the substructure satisfies the non-empty condition, then any cell containing an arrow-head must also contain a critical vertex in row 1.

With the substructure $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ defined, we will now provide two formulas for it, corresponding to the two ways we can use it to extend vertical arrays.

Lemma 5.5. *Given an irreducible substructure $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ that satisfies the non-empty condition and with $R_2 = K$, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$ satisfying substructure Δ is given by*

$$T(\Delta) = r_2 (s-1)!$$

where r_2 is the total number of vertices in row 2 that are in the columns of $\mathcal{R}_1$.

Proof. As $R_2 = K$, there is only one substructure $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ that is a refinement of Δ. Namely, we have $\mathcal{R}_2 = \mathcal{K}$, so each cell of row 2 also contains at least one object. Therefore, Γ satisfies the full condition, and we can use the formula of $T(\Gamma)$ given by Theorem 4.13. As all cells of the row 2 are marked, there are no columns of type $\mathcal{A}$, $\mathcal{B}$, $\overline{\mathcal{A}}$, or $\overline{\mathcal{C}}$. This also implies that there are no columns of type $\widetilde{\mathcal{A}}$. Hence,

$$\begin{aligned} T(\Delta) &= (s-1)! \left[\frac{d_2 (c_1 + \widetilde{c}_1 + d_1)}{s} \right] \\ &= d_2 (s-1)! \end{aligned}$$

as $s = c_1 + \widetilde{c}_1 + d_1$. Since all cells in row 2 are marked, a vertex is in a column of type $\mathcal{D}$ if and only if its column is marked in row 1. This gives $d_2 = r_2$, and our formula, as desired. □

This formula will be useful for decomposing minimal arrays into vertical arrays. However, to decompose vertical arrays into arrowed arrays, we need a much more substantial theorem.

Theorem 5.6. *Let $R_1, R_2 \geq 1$, and let $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ be an irreducible substructure that satisfies the balance and non-empty conditions. Furthermore, suppose $w_{2,j} > 0$ for all cells $(2, j)$. Then, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$ with substructure Δ is given by the formula*

$$T(\Delta) = s! \sum_{A=0}^{s-1} \frac{r}{s-A} \binom{M}{M-A} \binom{K-M-1}{R_2 - M + A - 1}$$

where r is the total number of vertices in row 1 of the columns of $\mathcal{R}_1$, and M is the number of columns that contain a critical vertex in row 1.

Proof. To prove this theorem, we sum $T(\Gamma)$ over all substructures $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ that are refinements of Δ. Since $w_{2,j} > 0$ for all cells $(2, j)$, all substructures Γ satisfy the full condition, so we can use the formulas of $T(\Gamma)$ given by Theorem 4.13 and Theorem 4.14. Note that $T(\Gamma)$ only depends on the number of columns of type $\mathcal{A}$, even though it depends on the number of vertices of other column types. Therefore, we first sum over all Γ with A columns of type $\mathcal{A}$ to obtain $T(\Delta_A)$, then we sum A from 0 to $s-1$ to obtain $T(\Delta)$. As Δ satisfies the balance condition, so must all Γ that are refinements of Δ. This implies that we can drop the subscripts from $T(\Gamma)$. For convenience, we will refer to the number of vertices of row 1 in a set of column $\mathcal{X}$ simply as the number of vertices in $\mathcal{X}$, as that number is the same between row 1 or row 2.

Let $\mathcal{M}$ be the set of columns that contains a critical vertex in row 1, and $\mathcal{H}$ be the set of columns that contains an arrow-tail. Then, $\mathcal{R}_1$, $\mathcal{M}$, and $\mathcal{H}$ partitions $\mathcal{K}$. As $R_1 \geq 1$, we have $M < K$. In the case where $M = 0$, we have $\mathcal{M} = \mathcal{H} = \emptyset$ and $\mathcal{R}_1 = \mathcal{K}$. By inverting the two rows and applying Lemma 5.5, we have

$$T(\Delta) = \sum_{\Gamma \hookrightarrow \Delta} r(s-1)!$$

where r is the number of vertices in $\mathcal{R}_2$. Note that a vertex v in row 1 of a column $\mathcal{X}$ contributes to r if $\mathcal{X}$ is marked in row 2. As there are $\binom{K-1}{R_2-1}$ ways to mark the columns of $\mathcal{K}$ in row 2 with $\mathcal{X}$ marked, and s vertices in row 1, we have

$$\begin{aligned} T(\Delta) &= r(s-1)! \cdot \frac{s}{r}\binom{K-1}{R_2-1} \\ &= s!\binom{K-1}{R_2-1} \end{aligned}$$

This result agrees with substituting $M = A = 0$ into the formula of $T(\Delta)$.

In the case where $1 \leq M \leq K-1$, we have $\mathcal{R}_1 = \mathcal{B} \cup \mathcal{D}$. This gives us $r = b + d$, and allows us to rewrite $T(\Gamma)$ as

$$\begin{aligned} T(\Gamma) &= \left[\frac{(b+d)(\widetilde{a}+c+\widetilde{c}+d)}{s-A} + \frac{b(c+\bar{c}+\widetilde{c}) - \bar{c}(b+d)}{(s-A)(s-A-1)}\right] \\ &= (s-1)!\,(T_1(\Gamma) + T_2(\Gamma) + T_3(\Gamma) + T_4(\Gamma)) \end{aligned}$$

where

$$\begin{aligned} T_1(\Gamma) &= \frac{rc}{s-A} \\ T_2(\Gamma) &= \frac{r(\widetilde{a}+\widetilde{c}+d)}{s-A} \\ T_3(\Gamma) &= \frac{b(c+\bar{c}+\widetilde{c})}{(s-A)(s-A-1)} \\ T_4(\Gamma) &= -\frac{r\bar{c}}{(s-A)(s-A-1)} \end{aligned}$$

for $0 \leq A \leq s-2$. For $A = s-1$, we let $T_1(\Gamma)$ and $T_2(\Gamma)$ be defined as above, and let $T_3(\Gamma) = T_4(\Gamma) = 0$. As the substructures $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ with A columns of type $\mathcal{A}$ partitions Δ_A, we can let $T_i(\Delta_A) = \sum_{\Gamma \hookrightarrow \Delta_A} T_i(\Gamma)$ for $i = 1, 2, 3, 4$, which gives us

$$T(\Delta_A) = (s-1)!\,(T_1(\Delta_A) + T_2(\Delta_A) + T_3(\Delta_A) + T_4(\Delta_A))$$

and

$$T(\Delta) = (s-1)!\left(\sum_{A=0}^{s-1} (T_1(\Delta_A) + T_2(\Delta_A)) + \sum_{A=0}^{s-2} (T_3(\Delta_A) + T_4(\Delta_A))\right)$$

Note that the second summation for A goes from 0 to $s-2$ as $T_3(\Delta_{s-1}) = T_4(\Delta_{s-1}) = 0$.

To evaluate each of the $T_i(\Delta_A)$, we look at the number of substructures Γ such that a vertex or a pair of vertices contributes to the numerator of $T_i(\Delta_A)$. Note that we can ignore r since it is the number of vertices in $\mathcal{R}_1$, which is a constant with respect to Δ. Of the three sets of columns, only the columns of $\mathcal{M}$ can become columns of type $\mathcal{A}$, as the columns of type $\mathcal{A}$ require both the top and bottom cells to be unmarked. Therefore, if a substructure Γ is a refinement of Δ_A, it must have exactly $M - A$ marked cells in row 2 of $\mathcal{M}$, where $M = |\mathcal{M}|$. It must also have exactly $R_2 - M + A$ marked cells in row 2 of $\mathcal{R}_1 \cup \mathcal{H}$. This means in total, there are $\binom{M}{M-A}\binom{K-M}{R_2-M+A}$ substructures of the form $\Gamma = (\mathbf{w}, \mathcal{R}_1, \mathcal{R}_2, \phi)$ that are refinements of Δ_A. Observe that by letting $1 \leq M \leq K-1$, we ensure that the top terms of the binomial coefficients are never negative, and that we do not divide by zero later on.

Now, a vertex v in row 1 of a column $\mathcal{X}$ contributes to c if $\mathcal{X} \in \mathcal{M}$ and $\mathcal{X}$ is marked in row 2. As there are $\binom{M-1}{M-A-1}$ ways to mark the columns of $\mathcal{M}$ in row 2 with $\mathcal{X}$ marked, and $\binom{K-M}{R_2-M+A}$ ways to mark the columns of $\mathcal{K} \backslash \mathcal{M}$, v contributes $\binom{M-1}{M-A-1}\binom{K-M}{R_2-M+A}$ times to c. Let m be the total number of vertices in $\mathcal{M}$, we have

$$\begin{aligned} T_1(\Delta_A) &= T_1(\Gamma) \cdot \frac{m}{c}\binom{M-1}{M-A-1}\binom{K-M}{R_2-M+A} \\ &= \frac{rm}{s-A}\binom{M-1}{M-A-1}\binom{K-M}{R_2-M+A} \end{aligned}$$

Next, a vertex v in row 1 of a column $\mathcal{X}$ contributes to $\widetilde{a} + \widetilde{c} + d$ if $\mathcal{X} \in \mathcal{K} \backslash \mathcal{M}$ and $\mathcal{X}$ is marked in row 2. As there are $\binom{K-M-1}{R_2-M+A-1}$ ways to mark the columns of $\mathcal{K} \backslash \mathcal{M}$ in row 2 with $\mathcal{X}$ marked, and $\binom{M}{M-A}$ ways to mark the columns of $\mathcal{M}$, v contributes $\binom{M}{M-A}\binom{K-M-1}{R_2-M+A-1}$

times to $\widetilde{a}+\widetilde{c}+d$. Given that there are $s-m$ vertices in $\mathcal{K}\backslash\mathcal{M}$, we have

$$\begin{aligned} T_2(\Delta_A) &= T_2(\Gamma)\cdot\frac{s-m}{\widetilde{a}+\widetilde{c}+d}\binom{M}{M-A}\binom{K-M-1}{R_2-M+A-1} \\ &= \frac{r(s-m)}{s-A}\binom{M}{M-A}\binom{K-M-1}{R_2-M+A-1} \end{aligned}$$

Similarly, let $\{v,u\}$ be a pair of vertices with v in row 1 of a column $\mathcal{X}$ and u in row 2 of a column $\mathcal{Y}$. Then, $\{v,u\}$ contributes to $b(c+\overline{c}+\widetilde{c})$ if the following conditions hold. First, we have $\mathcal{X}\in\mathcal{R}_1$, $\mathcal{Y}\in\mathcal{K}\backslash\mathcal{R}_1$, and $\mathcal{X}$ unmarked in row 2. Furthermore, let $\mathcal{Z}$ be the column $\mathcal{Y}$ if $\mathcal{Y}\in\mathcal{M}$, and $\mathcal{Z}$ be the column that $\mathcal{Y}$ points to if $\mathcal{Y}\in\mathcal{H}$. Then, $\mathcal{Z}$ must be a column of $\mathcal{M}$ and must also be marked. Now, as there are $\binom{M-1}{M-A-1}$ ways to mark the columns of $\mathcal{M}$ with $\mathcal{Z}$ marked, and $\binom{K-M-1}{R_2-M+A}$ ways to mark the columns of $\mathcal{K}\backslash\mathcal{M}$ in row 2 with $\mathcal{X}$ unmarked, $\{v,u\}$ contributes $\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A}$ times to $b(c+\overline{c}+\widetilde{c})$. Given that there are $r(s-r)$ such pairs of $\{v,u\}$, we have

$$\begin{aligned} T_3(\Delta_A) &= T_3(\Gamma)\cdot\frac{r}{b}\cdot\frac{s-r}{c+\overline{c}+\widetilde{c}}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A} \\ &= \frac{r(s-r)}{(s-A)(s-A-1)}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A} \end{aligned}$$

Finally, a vertex v in row 1 of a column $\mathcal{X}$ contributes to $\overline{c}$ if $\mathcal{X}\in\mathcal{H}$, $\mathcal{X}$ is unmarked in row 2, and the column $\mathcal{Z}$ that $\mathcal{X}$ points to is marked in row 2. As there are $\binom{K-M-1}{R_2-M+A}$ ways to mark the columns of $\mathcal{K}\backslash\mathcal{M}$ in row 2 with $\mathcal{X}$ unmarked, and $\binom{M-1}{M-A-1}$ ways to mark the columns of $\mathcal{M}$ with $\mathcal{Z}$ marked, v contributes $\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A-1}$ times to $\overline{c}$. Given that there are $s-m-r$ vertices in $\mathcal{H}$, we have

$$\begin{aligned} T_4(\Delta_A) &= T_4(\Gamma)\cdot\frac{s-m-r}{\overline{c}}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A} \\ &= -\frac{r(s-m-r)}{(s-A)(s-A-1)}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A} \end{aligned}$$

Now, let $T_{3+4}(\Delta_A)=T_3(\Delta_A)+T_4(\Delta_A)$, and observe that

$$T_{3+4}(\Delta_A) = \frac{rm}{(s-A)(s-A-1)}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A}$$

and

$$\begin{aligned}
T_1(\Delta_A) + T_2(\Delta_A) &= \frac{r\left(sMR_2 - sM^2 + sMA - mMR_2 + mMK - mAK\right)}{M(s-A)(K-M)} \times \\
&\quad \binom{M}{M-A}\binom{K-M}{R_2-M+A} \\
&= T_r(\Delta_A) + T_{m1}(\Delta_A) + T_{m2}(\Delta_A)
\end{aligned}$$

where

$$\begin{aligned}
T_r(\Delta_A) &= \frac{rs}{s-A}\binom{M}{M-A}\binom{K-M-1}{R_2-M+A-1} \\
T_{m1}(\Delta_A) &= \frac{rm}{s-A}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A} \\
T_{m2}(\Delta_A) &= -\frac{rm}{s-A}\binom{M-1}{M-A}\binom{K-M-1}{R_2-M+A-1}
\end{aligned}$$

By substituting these formulas into $T(\Delta)$, we have

$$T(\Delta) = (s-1)!\left(\sum_{A=0}^{s-1}\left(T_r(\Delta_A) + T_{m1}(\Delta_A) + T_{m2}(\Delta_A)\right) + \sum_{A=0}^{s-2} T_{3+4}(\Delta_A)\right)$$

Next, we will show that $\sum_{A=0}^{s-1}\left(T_{m1}(\Delta_A) + T_{m2}(\Delta_A)\right) + \sum_{A=0}^{s-2} T_{3+4}(\Delta_A) = 0$. By combining $T_{m1}(\Delta_A)$ with $T_{3+4}(\Delta_A)$, we have

$$\begin{aligned}
T_{m1}(\Delta_A) + T_{3+4}(\Delta_A) &= \frac{rm}{s-A}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A} + \\
&\quad \frac{rm}{(s-A)(s-A-1)}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A} \\
&= \frac{rm}{s-A-1}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A}
\end{aligned}$$

for $0 \leq A \leq s-2$. Therefore,

$$\begin{aligned}
& \sum_{A=0}^{s-1} \left(T_{m1}\left(\Delta_A\right) + T_{m2}\left(\Delta_A\right)\right) + \sum_{A=0}^{s-2} T_{3+4}\left(\Delta_A\right) \\
= & \sum_{A=0}^{s-2} \frac{rm}{s-A-1} \binom{M-1}{M-A-1} \binom{K-M-1}{R_2-M+A} + \\
& rm \binom{M-1}{M-s} \binom{K-M-1}{R_2-M+s-1} - \\
& \sum_{A=0}^{s-1} \frac{rm}{s-A} \binom{M-1}{M-A} \binom{K-M-1}{R_2-M+A-1} \\
= & rm \binom{M-1}{M-s} \binom{K-M-1}{R_2-M+s-1}
\end{aligned}$$

by shifting the index of the second sum and noting that $\binom{M-1}{M} = 0$. Now, for $\binom{M-1}{M-s}$ to be non-zero, we require $M \geq s$. However, this implies that there are at least s columns of $\mathcal{M}$, each requiring a critical vertex. As there are only s vertices in row 1, r is forced to be 0. Therefore, the entire sum is equal to zero regardless of the value of M. Substituting this result back into $T(\Delta)$, we obtain

$$\begin{aligned}
T(\Delta) &= (s-1)! \left(\sum_{A=0}^{s-1} \left(T_1\left(\Delta_A\right) + T_2\left(\Delta_A\right)\right) + \sum_{A=0}^{s-2} \left(T_3\left(\Delta_A\right) + T_4\left(\Delta_A\right)\right) \right) \\
&= (s-1)! \sum_{A=0}^{s-1} T_r\left(\Delta_A\right) \\
&= s! \sum_{A=0}^{s-1} \frac{r}{s-A} \binom{M}{M-A} \binom{K-M-1}{R_2-M+A-1}
\end{aligned}$$

This proves our formula for $T(\Delta)$. □

Remark 5.7. The proof for Theorem 5.6 works even if Δ does not satisfy the balance condition, as long as $w_{2,j} > 0$ remain satisfied for all j. However, we will have to retain the row subscripts from both the formula and the proof of $T(\Delta)$. By following the same proof, we obtain that

$$\begin{aligned} T(\Delta) &= s! \sum_{A=0}^{s-1} \frac{r_2}{s-A} \binom{M}{M-A} \binom{K-M-1}{R_2-M+A-1} + \\ & s! \sum_{A=0}^{s-2} \frac{r_1 - r_2}{(s-A)(s-A-1)} \binom{M-1}{M-A-1} \binom{K-M-1}{R_2-M+A} \end{aligned}$$

when Δ satisfies the non-empty condition, but does not necessary satisfy the balance condition. The second term of this expression is given by the simplification of $T_{3+4}(\Delta_A)$, where r_i is the number of vertices in row i of the columns of $\mathcal{R}_1$. Note that both r_i's refer to the columns with marked cells in row 1, as the marked cells in row 2 are not fixed.

5.2 Admissible Substructures

Recall that our overall strategy in proving a formula for the number of tree-shaped n-row vertical arrays is to decompose each n-row vertical array into an $(n-1)$-row vertical array and an arrowed array, then provide the inverse to establish a bijection. Then, for each $(n-1)$-row vertical array we count the number of arrowed arrays that are compatible with it, and sum this over all $(n-1)$-row vertical arrays. However, the resulting vertical array and arrowed array from the decomposition may not necessarily satisfy the non-empty condition, even if the original vertical array satisfies it. Furthermore, there is also no guarantee that there will be a vertex in each cell of row 2 of the resulting arrowed array, which is a crucial condition for Theorem 5.6. With the vertical array, we can bypass this issue either by temporarily removing columns with no vertices, or by allowing vertical arrays to have empty columns and using inclusion-exclusion to remove them. However, these approaches do not work on arrowed arrays as arrowed arrays may have arrows in columns that are otherwise empty, and may not have at least one vertex per cell. Therefore, we need to extend Theorem 5.6 to cover a wider range of arrowed arrays.

Definition 5.8. An irreducible substructure $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ is *admissible* if it satisfies the following conditions

1. For each cell in row 1 containing an arrow-head, it contains at least one vertex.
2. For each cell in row 1 containing a vertex, the corresponding cell in row 2 contains at least one vertex.

3. For each cell in row 2 containing a vertex, the corresponding cell in row 1 contains at least one object.

Let Δ be an admissible substructure, and (α, ϕ) be an arrowed array that satisfies Δ. Suppose cell (i, j) contains a critical vertex or an arrow-tail, then $\psi_i(j)$ is either in $\mathcal{R}_i$, or $\psi_i(\psi_i(j))$ is defined. In other words, there does not exist a column j such that cell $(i, \psi_i(j))$ contains no object. This means that the only way for (α, ϕ) to violate the forest condition of row i is for there to be a cycle in the functional digraph of ψ_i. Note that conditions 2 and 3 are not symmetric, and this discrepancy stems from the fact that we are permuting the marked cells in row 2, which means we cannot guarantee that an empty cell is marked for the forest condition function ψ_2. Now, if Δ satisfies the balance condition, then the second and third point of the definition are automatically satisfied.

Theorem 5.9. *Let $R_1, R_2 \geq 1$, and let $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ be an admissible substructure. Then, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$ with substructure Δ is given by the same formula as in Theorem 5.6 and Remark 5.7. That is,*

$$\begin{aligned} T(\Delta) &= s! \sum_{A=0}^{s-1} \frac{r_2}{s-A} \binom{M}{M-A} \binom{K-M-1}{R_2-M+A-1} + \\ & \quad s! \sum_{A=0}^{s-2} \frac{r_1-r_2}{(s-A)(s-A-1)} \binom{M-1}{M-A-1} \binom{K-M-1}{R_2-M+A} \end{aligned}$$

with the latter term naturally being zero if Δ satisfies the balance condition.

Proof. As permuting the columns of an arrowed array does not change whether it satisfies the forest condition, we can without loss of generality assume that the first k of the K columns of Δ are the ones that contain at least one vertex in row 2. By condition 2 of Definition 5.8, all vertices in row 1 are in these k columns. In particular, it means that $\phi_i(j) \in [k]$ as each cell containing an arrow-head must contain at least one vertex. Now, let Δ^R be the subset of arrowed arrays that satisfies Δ, and have exactly R marked cells in the first k columns of row 2. Furthermore, let $\Delta^{R;k} = (\mathbf{w}', \mathcal{R}_1 \cap [k], \phi')$ be the restriction of Δ^R to the first k columns. In other words, $\Delta^{R;k} = (\mathbf{w}', \mathcal{R}_1 \cap [k], \phi')$ is a substructure of $\mathcal{AR}^{(s)}_{k;|\mathcal{R}_1 \cap [k]|,R}$, where $w'_{i,j} = w_{i,j}$ and $\phi'_i(j) = \phi_i(j)$ for $1 \leq j \leq k$. Note that $\phi_i(j) \in [k]$ implies that $\phi'_i(j) \in [k]$, so this is well defined. We will show that there is a $\binom{K-k}{R_2-R}$ to 1 correspondence between arrowed arrays satisfying Δ^R and arrowed arrays satisfying $\Delta^{R;k}$.

Let (α, ϕ) be an arrowed array satisfying Δ^R and consider the cell (i, j), where $k+1 \leq j \leq K$. By condition 1, it cannot contain an arrow-head. Furthermore, as the column contains no vertices, there cannot be another column j' such that $\psi_i(j') = j$. Now, this cell can either be unmarked, marked, or contain an arrow-tail. If the cell is unmarked, then it does not factor into the forest condition of row i, as it is in neither the domain nor range of ψ_i. If the cell is marked, then it is in $\mathcal{R}_i$, so it is an isolated root in the functional digraph of ψ_i. Finally, if it contains an arrow-tail, then the column it points to must contain at least one vertex, and must be unmarked. Therefore, cell (i, j) is a leaf in the functional digraph of ψ_i. In all three cases, we can remove the column j from the array without violating the forest condition, using Lemma 4.4 for the third case. As this holds for all $j > k$, we can simply cut off the rightmost $K - k$ columns of (α, ϕ) to obtain an arrowed array (α', ϕ') that satisfies $\Delta^{R;k}$.

Conversely, given an arrowed array (α', ϕ') satisfying $\Delta^{R;k}$, we can add $K - k$ columns with no vertices to obtain an arrowed array (α, ϕ) satisfying Δ^R. Note that the positions of arrows and marked cells in row 1 is completely fixed by Δ^R. However, only the first k columns of (α, ϕ) are predetermined in row 2, as given by (α', ϕ'). For the remaining $K - k$ columns, we can mark $R_2 - R$ cells arbitrarily and satisfy the forest condition, as adding columns with no vertices does not change ψ_2. Therefore, for each arrowed array (α', ϕ') satisfying $\Delta^{R;k}$, there are exactly $\binom{K-k}{R_2-R}$ arrowed arrays satisfying Δ^R.

By construction, each of the $\Delta^{R;k}$ satisfies the non-empty condition, and has $w_{2,j} \geq 1$ for $1 \leq j \leq k$. Therefore, we can use Theorem 5.6 and Remark 5.7 to obtain $T\left(\Delta^{R;k}\right)$, with K and R_2 being substituted by k and R respectively. Furthermore, $\Delta^1, \ldots, \Delta^{\min(k,R_2)}$ partitions Δ, and for $R = 0$ or $R > k$, we have $T\left(\Delta^{R;k}\right) = 0$. This is given by the factors r_1 and r_2 in the former case, and the binomial term in $\binom{k-M-1}{R-M+A-1}$ the latter. Therefore, we can change the bounds of k to $0 \leq k \leq R_2$, and use the Chu-Vandermonde identity introduced in Proposition 1.3 to obtain

$$\begin{aligned}
T(\Delta) &= \sum_{R=0}^{\min(k,R_2)} T\left(\Delta^{R;k}\right)\binom{K-k}{R_2-R} \\
&= s!\sum_{A=0}^{s-1}\sum_{R=0}^{R_2}\frac{r_2}{s-A}\binom{M}{M-A}\binom{k-M-1}{R-M+A-1}\binom{K-k}{R_2-R}+ \\
&\quad s!\sum_{A=0}^{s-2}\sum_{R=0}^{R_2}\frac{r_1-r_2}{(s-A)(s-A-1)}\binom{M-1}{M-A-1}\binom{k-M-1}{R-M+A}\binom{K-k}{R_2-R} \\
&= s!\sum_{A=0}^{s-1}\frac{r_2}{s-A}\binom{M}{M-A}\binom{K-M-1}{R_2-M+A-1}+ \\
&\quad s!\sum_{A=0}^{s-2}\frac{r_1-r_2}{(s-A)(s-A-1)}\binom{M-1}{M-A-1}\binom{K-M-1}{R_2-M+A}
\end{aligned}$$

which is the formula for $T(\Delta)$ as given by Theorem 5.6 and Remark 5.7. □

Before defining another substructure and further generalizing this formula, we will first rewrite it using hypergeometric transformations, as that will simplify our work later. From here on, we will only consider the case when Δ satisfies the balance condition. One reason for this assumption is that the balance condition holds for tree-shaped arrays. Another reason is that we will permute the marked cells in row 1, which allows us to cancel out r_1 and r_2.

Theorem 5.10. *Let $R_1, R_2 \geq 1$, and let $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ be an admissible substructure that satisfies the balance condition. Then, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$ with substructure Δ is given by the formula*

$$T(\Delta) = r\sum_{A=0}^{\min(s,K)-1}\frac{M!\,(K-A-1)!\,(s-A-1)!}{(M-A)!\,(K-R_2-A)!\,(R_2-1)!}$$

where r is the total number of vertices in row 1 of the columns of $\mathcal{R}_1$, and M is the number of columns that contain a critical vertex in row 1.

Proof. First, we rewrite $T(\Delta)$ using factorials to obtain

$$T(\Delta) = r\sum_{A=0}^{s-1} \frac{s!M!(s-A-1)!(K-M-1)!}{(s-A)!(M-A)!A!(R_2-M+A-1)!(K-R_2-A)!}$$

Since $R_2 \geq 1$, we have $M < K$, so the numerator is always defined. Furthermore, as discussed in Section 1.3, we can take $1/x!$ to be zero if x is a negative integer. Note that $M \geq s$ implies $r = 0$, as each column of $\mathcal{M}$ requires a critical vertex, and there are only s vertices in row 1. In this case, the theorem is true as both the original formula and the new formula imply that $T(\Delta) = 0$. Otherwise, we have $M \leq s-1$. Since $1/(M-A)!$ is zero for $A > M$, we can lower the upper bound of the summation to M. We can then write it as a generalized hypergeometric function with $-M$ as one of the parameters, matching the upper bound of the sum. This gives us

$$\begin{aligned}
T(\Delta) &= r\sum_{A=0}^{M} \frac{s!M!(s-A-1)!(K-M-1)!}{(s-A)!(M-A)!A!(R_2-M+A-1)!(K-R_2-A)!} \\
&= r\cdot {}_3F_2\left(\begin{matrix}-M,-s,-K+R_2\\ R_2-M,-s+1\end{matrix};1\right)\frac{(s-1)!(K-M-1)!}{(R_2-M-1)!(K-R_2)!} \\
&= r\cdot {}_3F_2\left(\begin{matrix}-M,1,-K+R_2\\ 1-K,-s+1\end{matrix};1\right)\frac{(K-M)^{(M)}(s-1)!(K-M-1)!}{(R_2-M)^{(M)}(R_2-M-1)!(K-R_2)!} \\
&= r\sum_{A=0}^{M}\frac{M!(K-A-1)!(s-A-1)!}{(M-A)!(K-R_2-A)!(R_2-1)!} \\
&= r\sum_{A=0}^{\min(s,K)-1}\frac{M!(K-A-1)!(s-A-1)!}{(M-A)!(K-R_2-A)!(R_2-1)!}
\end{aligned}$$

where we use the ${}_3F_2$ identity described in Theorem 1.5. As $1/(M-A)!$ is again part of the new summation, we can raise the summation index without changing the value of the sum. Note that we know $M \leq s-1$, and we can deduce that $M \leq K-1$ as $R_1 \geq 1$. This allows us to raise the upper bound to $\min(s,K)-1$, while keeping the numerator well defined. □

The benefit of this new formula is that we are no longer required to keep $M \leq \min(s,K)-1$. While taking $M \geq \min(s,K)$ for Δ makes no sense combinatorially, the value for $T(\Delta)$ is well defined and finite. This frees up M for manipulation and summation if we can multiply $T(\Delta)$ with an expression that is zero if $M \geq s$ or $M \geq K$. When we do the induction on the number of vertical arrays, this fact will become extremely useful.

5.3 Substructures $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$ and $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$

In this final section, we will turn our focus to specific substructures that can be directly used to extend vertical arrays, instead of generic substructures with generalized parameters. To this end, we want to define a pair of substructures Θ and Λ such that Δ is a partition to both of them. These two substructures will correspond to the two ways we will use arrowed arrays to extend vertical arrays. For both substructures, instead of taking the set of marked cells in row 1 as being fixed, we take the set of marked cells as a R_1-subset of a set $\mathcal{P} \subseteq \mathcal{K}$. Furthermore, instead of fixing the number of vertices in each cell, we fix the number of non-critical vertices in each cell of a given row. Finally, we will provide formulas for both substructures, using the two formulas for $T(\Delta)$.

Definition 5.11. Let $\mathcal{P}$ be a subset of $\mathcal{K}$ with $|\mathcal{P}| \geq R_1 \geq 1$, $\mathbf{y}$ be non-negative vectors of size K, and $\phi\colon \mathcal{K}\backslash\mathcal{P} \to \mathcal{K}$ be a partial function from $\mathcal{H} \subseteq \mathcal{K}\backslash\mathcal{P}$ to $\mathcal{H} \cup \mathcal{P}$. Suppose that $y_j = 0$ for all $j \notin \mathcal{H} \cup \mathcal{P}$ and $\sum_j y_j = |\mathcal{P}| - R_1$. The substructure $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$ is defined to be the subset of $\mathcal{AR}_{K;R_1,K}^{(|\mathcal{P}|-R_1)}$, such that for each pair $(\alpha', \phi') \in \mathcal{AR}_{K;R_1,K}^{(|\mathcal{P}|-R_1)}$, the set of marked cells in row 1 of α' is a subset of $\mathcal{P}$ and $\phi' = \phi$. Furthermore, for each $j \in \mathcal{P}$ such that cell $(1, j)$ is unmarked, α' contains a vertex in that cell. Finally, for each $1 \leq j \leq K$, α' contains y_j vertices in cell $(2, j)$.

Note that all cells in row 2 are marked in this substructure, corresponding to Lemma 5.5. Furthermore, all the vertices are in the columns of $\mathcal{H} \cup \mathcal{P}$. This allows us to later remove the columns not in $\mathcal{H} \cup \mathcal{P}$, as they contain no vertices, arrow-heads, or arrow-tails. The motivation behind this definition is to convert marked cells into critical vertices, so that the number of marked cells and non-critical vertices remain constant. As $\mathbf{y}$ represents the number of vertices in row 2, we have $s = |\mathcal{P}| - R_1$.

Lemma 5.12. *Let $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$ be a substructure of $\mathcal{AR}_{K;R_1,K}^{(|\mathcal{P}|-R_1)}$, and suppose that ϕ contains a column $\mathcal{X}$ that points to a column $\mathcal{Y}$, and the column $\mathcal{Y}$ points to another column $\mathcal{Z}$. Let $\Theta' = (\mathbf{y}, \mathcal{P}, \phi')$ be a substructure of $\mathcal{AR}_{K;R_1,K}^{(|\mathcal{P}|-R_1)}$ such that*

$$\phi'(j) \;=\; \begin{cases} \mathcal{Z} & j = \mathcal{X} \\ \phi(j) & j \in \mathcal{H}\backslash\mathcal{X} \end{cases},$$

that is, instead of pointing to $\mathcal{Y}$, $\mathcal{X}$ now points to $\mathcal{Z}$ in ϕ'. Then, the number of arrowed arrays satisfying Θ and the number of arrowed arrays satisfying Θ' are equal.

Proof. First, we need to check that Θ' is a proper substructure of form $\Theta' = (\mathbf{y}, \mathcal{P}, \phi')$. Note that none of $\mathbf{y}$, $\mathcal{P}$, or R_1 have changed, so we have $\sum_j y_j = |\mathcal{P}| - R_1$. Then, note that $\mathcal{H}$ remains the same as the domain of ϕ', so we have $y_j = 0$ for all $j \notin \mathcal{H} \cup \mathcal{P}$ in Θ'. Finally, we have that $\phi'(j) = \phi(j) \in \mathcal{H} \cup \mathcal{P}$ for all $j \neq \mathcal{X}$, and $\phi'(\mathcal{X}) = \mathcal{Z} = \phi(\mathcal{Y}) \in \mathcal{H} \cup \mathcal{P}$. Therefore, Θ' satisfies the conditions for a substructure of type $\Theta' = (\mathbf{y}, \mathcal{P}, \phi')$.

Let $\alpha \in \mathcal{VA}_{2,K;R_1,K}^{(|\mathcal{P}|-R_1)}$ be a two-row vertical array. By Lemma 4.6, (α, ϕ) is in $\mathcal{AR}_{K;R_1,K}^{(|\mathcal{P}|-R_1)}$ if and only if (α, ϕ') is in $\mathcal{AR}_{K;R_1,K}^{(|\mathcal{P}|-R_1)}$. Furthermore, as the sets of marked cells and vertices are unchanged, (α, ϕ) satisfies the remaining constraints of Θ if and only if (α, ϕ') satisfies them for Θ' by construction. Therefore, the number of arrowed arrays satisfying Θ and Θ' are equal. □

As the marked cells in row 1 are not fixed, there is only one *arrow simplification lemma for substructures* $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$, corresponding to Lemma 4.6. However, we can still repeatedly use this lemma to simplify substructures of the form $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$, which gives rise to the following definition.

Definition 5.13. A substructure $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$ is *irreducible* if the functional digraph of ϕ is acyclic, and Θ cannot be further simplified with the application of Lemma 5.12. As the arrow-heads of an irreducible substructure must be in cells of $\mathcal{H} \cup \mathcal{P}$, and cannot be in $\mathcal{H}$, a substructure $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$ is irreducible if and only if ϕ is a function from $\mathcal{H}$ to $\mathcal{P}$.

Note that the definition of *irreducible* is compatible with Definition 5.4. That is, if $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$ is an irreducible substructure and $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ is a refinement of Θ, then Δ can be reduced to an irreducible substructure Δ' by the application of Lemma 5.2. With the substructure $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$ defined, we will now provide a formula for it, corresponding to the subset of substructures Δ such that all cells in row 2 are marked.

Theorem 5.14. *Given an irreducible substructure* $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$*, the number of arrowed arrays* $(\alpha, \phi) \in \mathcal{AR}_{K;R_1,K}^{(|\mathcal{P}|-R_1)}$ *satisfying substructure* Θ *is given by the formula*

$$T(\Theta) = \frac{(P-1)!}{(R_1-1)!}$$

where P *is the number of columns of* $\mathcal{P}$*.*

Proof. We prove this by substituting into the formula for $T(\Delta)$ given by Lemma 5.5. Without loss of generality, assume that $\mathcal{H} \cup \mathcal{P}$ are the first k of the K columns of Θ. Let $\mathcal{R}_1$ be an R_1-subset of $\mathcal{P}$, and consider the substructure $\Delta = ([\mathbf{x}', \mathbf{y}], \mathcal{R}_1, \phi)$, where $x_j' = 1$

if $x \in \mathcal{P}\backslash\mathcal{R}_1$, and $x'_j = 0$ otherwise. That is, $\mathbf{w} \coloneqq [\mathbf{x}', \mathbf{y}]$ is a $2 \times K$ matrix containing the number of vertices in each cell. By construction, the number of vertices in cell (i, j) is 0 if $j \notin \mathcal{H} \cup \mathcal{P}$. As all arrow-tails of ϕ are contained in $\mathcal{H}$, and all arrow-heads of ϕ are contained in $\mathcal{P}$, the columns of $\mathcal{K}\backslash(\mathcal{H} \cup \mathcal{P})$ contain only a marked cell in row 2, and no other objects. Now, let Δ' be the substructure Δ restricted to the columns of $\mathcal{H} \cup \mathcal{P}$. Notice that the set of arrowed arrays satisfying Δ' is in bijection with the set of arrowed arrays satisfying Δ, as we can add or remove the columns of $\mathcal{K}\backslash(\mathcal{H} \cup \mathcal{P})$ without violating the forest condition. The forest condition of row 2 is always satisfied, as all cells in row 2 are marked. Furthermore, for any arrowed array satisfying Δ, both the domain and range of the forest condition function ψ_1 for row 1 are in $\mathcal{H} \cup \mathcal{P}$. Therefore, ψ_1 remains the same when we transform an arrowed array satisfying Δ to an arrowed array satisfying Δ' by restricting the set of columns to $\mathcal{H} \cup \mathcal{P}$, and vice-versa.

Now, Δ' may not be irreducible, as there can be arrows pointing to the columns of $\mathcal{R}_1$. Therefore, we have to reduce Δ' using the arrow simplification lemma defined in Lemma 5.2. This gives us an irreducible substructure $\Delta'' = ([\mathbf{x}', \mathbf{y}], \mathcal{R}_1 \cup \mathcal{H}_1, \phi')$, where $\mathcal{H}_1 \subseteq \mathcal{H}$ is the set of columns that points to $\mathcal{R}_1$, and ϕ' is ϕ restricted to the columns of $\mathcal{H}\backslash\mathcal{H}_1$. That is, we have changed all the cells that point to $\mathcal{R}_1$ in Δ' into marked cells. Observe that in Δ'', the columns of $\mathcal{R}_1 \cup \mathcal{H}_1$ are marked in row 1, the columns of $\mathcal{H}\backslash\mathcal{H}_1$ contain arrow-tails, and the columns of $\mathcal{P}\backslash\mathcal{R}_1$ contain critical vertices in row 1. Therefore, Δ'' satisfies the non-empty condition, so we can use the formula for $T(\Delta)$ given by Lemma 5.5. Now, a vertex u in row 2 of a column $\mathcal{X}$ contributes to r_2 of the formula for $T(\Delta)$ if $\mathcal{X} \in \mathcal{R}_1$, or $\mathcal{X} \in \mathcal{H}$ and $\mathcal{X}$ points to a column in $\mathcal{R}_1$. In either case, there are $\binom{P-1}{R_1-1}$ different subsets $\mathcal{R}_1$ such that $\mathcal{X}$ is marked in Δ'', out of the $\binom{P}{R_1}$ possible R_1-subsets of $\mathcal{P}$. Given that all vertices of row 2 are in $\mathcal{P} \cup \mathcal{H}$, we have

$$\begin{aligned} T(\Theta) &= T(\Delta'') \cdot \frac{s}{r_2}\binom{P-1}{R_1-1} \\ &= s!\binom{P-1}{R_1-1} \\ &= \frac{(P-1)!}{(R_1-1)!} \end{aligned}$$

as desired. □

As a final step, we want to remove the restriction that Θ is irreducible. This can be done by repeatedly applying Lemma 5.12, which gives us the following corollary.

Corollary 5.15. *Given a substructure $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$ such that the functional digraph of ϕ on $\mathcal{H} \cup \mathcal{P}$ is a rooted forest with root vertices $\mathcal{P}$, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{AR}_{K;R_1,K}^{(|\mathcal{P}|-R_1)}$ satisfying substructure Θ is given by the formula*

$$T(\Theta) = \frac{(P-1)!}{(R_1-1)!}$$

where P is the number of columns of $\mathcal{P}$.

Proof. We use Lemma 5.12 to reduce Θ to an irreducible substructure $\Theta' = (\mathbf{y}, \mathcal{P}, \phi')$. As R_1 and $\mathcal{P}$ remain the same, we have $T(\Theta) = T(\Theta')$. The result then follows from Theorem 5.14. □

As we will see in the next chapter, this theorem is useful for decomposing minimal arrays into vertical arrays. To decompose vertical arrays, we will need to introduce another substructure and a corresponding theorem to go with it.

Definition 5.16. Let $\mathcal{P}$ be a subset of $\mathcal{K}$ with $|\mathcal{P}| \geq R_1 \geq 1$, $\mathbf{x}$ be a non-negative vector of size K, and $\phi\colon \mathcal{K}\backslash\mathcal{P} \to \mathcal{K}$ be a partial function from $\mathcal{H} \subseteq \mathcal{K}\backslash\mathcal{P}$ to $\mathcal{H} \cup \mathcal{P}$. Suppose that $x_j = 0$ for all $j \notin \mathcal{H} \cup \mathcal{P}$ and s be such that $\sum_j x_j = s - |\mathcal{P}| + R_1$. The substructure $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$ is defined to be the subset of $\mathcal{AR}_{K;R_1,R_2}^{(s)}$, such that for each pair $(\alpha', \phi') \in \mathcal{AR}_{K;R_1,R_2}^{(s)}$, (α', ϕ') satisfies the balance condition, the set of marked cells in row 1 of α' is a subset of $\mathcal{P}$, and $\phi' = \phi$. Furthermore, for each column $j \in \mathcal{H} \cup \mathcal{P}$, cell $(1, j)$ contains $x_j + 1$ vertices if $j \in \mathcal{P}$ and is unmarked, and x_j vertices otherwise.

By the balance condition, cell $(2, j)$ also contains either x_j or $x_j + 1$ vertices, depending on whether cell $(1, j)$ is marked. As with the previous definition, the motivation behind this definition is to convert marked cells into critical vertices. However, we want to do it in such a way that the balance condition is preserved. This corresponds to Theorem 5.6 and Theorem 5.10, where the balance condition is also satisfied. Note that $\mathbf{x}$ represents the number of non-critical vertices in row 1, as the columns of $\mathcal{H}$ contain an arrow-tail, and columns of $\mathcal{P}$ are either marked or contain one extra vertex. Therefore, we have $\sum_j x_j = s - |\mathcal{P}| + R_1$. Note that this also implies $P \leq s + R_1$. As with the definition of Θ, this definition also restricts all vertices to be in the columns of $\mathcal{H} \cup \mathcal{P}$. This will allow us to use admissible substructures when providing a formula for substructures of type Λ.

Lemma 5.17. *Let $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$ be a substructure of $\mathcal{AR}_{K;R_1,R_2}^{(s)}$, and suppose that ϕ contains a column $\mathcal{X}$ that points to a column $\mathcal{Y}$, and the column $\mathcal{Y}$ points to another* *column $\mathcal{Z}$. Let $\Lambda' = (\mathbf{y}, \mathcal{P}, \phi')$ be a substructure of $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ such that*

$$\phi'(j) = \begin{cases} \mathcal{Z} & j = \mathcal{X} \\ \phi(j) & j \in \mathcal{H} \backslash \mathcal{X} \end{cases},$$

that is, instead of pointing to $\mathcal{Y}$, $\mathcal{X}$ now points to $\mathcal{Z}$ in ϕ'. Then, the number of arrowed arrays satisfying Λ and the number of arrowed arrays satisfying Λ' are equal.

Proof. First, we need to check that Λ' is a proper substructure of form $\Lambda' = (\mathbf{x}, \mathcal{P}, \phi')$. Note that none of $\mathbf{x}$, $\mathcal{P}$, R_1, or s have changed, so we have $\sum_j x_j = s - |\mathcal{P}| + R_1$. Then, note that $\mathcal{H}$ remains the same as the domain of ϕ', so we have $x_j = 0$ for all $j \notin \mathcal{H} \cup \mathcal{P}$ in Λ'. Finally, we have that $\phi'(j) = \phi(j) \in \mathcal{H} \cup \mathcal{P}$ for all $j \neq \mathcal{X}$, and $\phi'(\mathcal{X}) = \mathcal{Z} = \phi(\mathcal{Y}) \in \mathcal{H} \cup \mathcal{P}$. Therefore, Λ' satisfies the conditions for a substructure of type $\Lambda' = (\mathbf{y}, \mathcal{P}, \phi')$.

Let $\alpha \in \mathcal{VA}^{(s)}_{2,K;R_1,R_2}$ be a two-row vertical array. By Lemma 4.6, (α, ϕ) is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$ if and only if (α, ϕ') is in $\mathcal{AR}^{(s)}_{K;R_1,R_2}$. Furthermore, as the sets of marked cells and vertices are unchanged, (α, ϕ) satisfies the remaining constraints of Λ if and only if (α, ϕ') satisfies them for Λ' by construction. Therefore, the number of arrowed arrays satisfying Λ and Λ' are equal. □

As with substructure Θ, there is only one *arrow simplification lemma for substructures* $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$, corresponding to Lemma 4.6. However, we can still repeatedly use this lemma to simplify substructures of the form $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$, which gives rise to the following definition.

Definition 5.18. A substructure $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$ is *irreducible* if the functional digraph of ϕ is acyclic, and Λ cannot be further simplified with the application of Lemma 5.17. As the arrow-heads of an irreducible substructure must be in cells of $\mathcal{H} \cup \mathcal{P}$, and cannot be in $\mathcal{H}$, a substructure $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$ is irreducible if and only if ϕ is a function from $\mathcal{H}$ to $\mathcal{P}$.

Note that the definition of *irreducible* is compatible with the definition of *admissible* in Definition 5.8. That is, if $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$ is an irreducible substructure and $\Delta = (\mathbf{w}, \mathcal{R}_1, \phi)$ is a refinement of Λ, then Δ can be reduced to an admissible substructure Δ' by the application of Lemma 5.2. With the substructure $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$ defined, we will now provide a formula for it, corresponding to the subset of substructures Δ such that the balance condition is satisfied.

Theorem 5.19. *Given an irreducible substructure $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$ satisfying substructure Λ is given by the formula*

$$T(\Lambda) = \sum_{A=0}^{\min(s,K)-1} \frac{(s-P+R_1)\,(K-A-1)!\,(s-A-1)!\,(P-1)!}{(P-R_1-A)!\,(K-R_2-A)!\,(R_1-1)!\,(R_2-1)!}$$

where P is the number of columns of $\mathcal{P}$.

Proof. The proof of this theorem is similar to that of Theorem 5.14. However, we will be substituting into the formula for $T(\Delta)$ given by Theorem 5.10 instead. Let $\mathcal{R}_1$ be an R_1-subset of $\mathcal{P}$, and consider the substructure $\Delta = ([\mathbf{x}', \mathbf{x}'], \mathcal{R}_1, \phi)$, where $x'_i = x_i + 1$ if $x \in \mathcal{P} \backslash \mathcal{R}_1$, and $x'_i = x_i$, otherwise. That is, $\mathbf{w} \coloneqq [\mathbf{x}', \mathbf{x}']$ is a $2 \times K$ matrix containing the number of vertices in each cell. Now, note that Δ may not be irreducible, as there can be arrows pointing to the columns of $\mathcal{R}_1$. Therefore, we have to reduce Δ using the arrow simplification lemma defined in Lemma 5.2. This gives us an irreducible substructure $\Delta' = ([\mathbf{x}', \mathbf{x}'], \mathcal{R}_1 \cup \mathcal{H}_1, \phi')$, where $\mathcal{H}_1 \subseteq \mathcal{H}$ is the set of columns that points to $\mathcal{R}_1$, and ϕ' is ϕ restricted to the columns of $\mathcal{H} \backslash \mathcal{H}_1$.

Now, Δ' satisfies the balance condition by construction. Furthermore, any cell of Δ' that contains an arrow-head must be in $\mathcal{P} \backslash \mathcal{R}_1$, as otherwise Δ' will not be irreducible. Since the columns of $\mathcal{P} \backslash \mathcal{R}_1$ must each contain at least one vertex, Δ' is an admissible substructure, so we can use the formula for $T(\Delta)$ given by Theorem 5.10. As Δ' satisfies the balance condition, we can take r to be the number of vertices in row 1 of $\mathcal{R}_1$. Observe that the $P - R_1$ vertices added to row 1 of $\mathcal{P} \backslash \mathcal{R}_1$ are all critical vertices, regardless of the choice of $\mathcal{R}_1$. Hence, they never contribute to $T(\Delta)$. This means that we only need to consider the non-critical vertices of row 1, which are given by $\mathbf{x}$. Now, a non-critical vertex u in row 1 of a column $\mathcal{X}$ contributes to r of the formula for $T(\Delta)$ if $\mathcal{X} \in \mathcal{R}_1$, or $\mathcal{X} \in \mathcal{H}$ and $\mathcal{X}$ points to a column in $\mathcal{R}_1$. In either case, there are $\binom{P-1}{R_1-1}$ different subsets $\mathcal{R}_1$ such that $\mathcal{X}$ is marked in Δ', out of the $\binom{P}{R_1}$ possible R_1-subsets of $\mathcal{P}$. Given that all non-critical vertices of row 1 are in $\mathcal{P} \cup \mathcal{H}$, and that there are $s - P + R_1$ non-critical vertices in row 1, we have

$$
\begin{aligned}
T(\Lambda) &= T(\Delta') \cdot \frac{s-P+R_1}{r} \binom{P-1}{R_1-1} \\
&= \sum_{A=0}^{\min(s,K)-1} \frac{(s-P+R_1)\, M!\, (K-A-1)!\, (s-A-1)!}{(M-A)!\, (K-R_2-A)!\, (R_2-1)!} \binom{P-1}{R_1-1} \\
&= \sum_{A=0}^{\min(s,K)-1} \frac{(s-P+R_1)\, (K-A-1)!\, (s-A-1)!\, (P-1)!}{(P-R_1-A)!\, (K-R_2-A)!\, (R_1-1)!\, (R_2-1)!}
\end{aligned}
$$

where we substitute in $M = P - R_1$ as the number of critical vertices in row 1. Finally, we simplify the expression using factorials, so that we can apply hypergeometric transformations in the next chapter. □

As with substructure Θ, we want to remove the restriction that Λ is irreducible. This can be done by repeatedly applying Lemma 5.17, which gives us the following corollary.

Corollary 5.20. *Given a substructure $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$ such that the functional digraph of ϕ on $\mathcal{H} \cup \mathcal{P}$ is a rooted forest with root vertices $\mathcal{P}$, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{AR}^{(s)}_{K;R_1,R_2}$ satisfying substructure Λ is given by the formula*

$$
T(\Lambda) = \sum_{A=0}^{\min(s,K)-1} \frac{(s-P+R_1)\, (K-A-1)!\, (s-A-1)!\, (P-1)!}{(P-R_1-A)!\, (K-R_2-A)!\, (R_1-1)!\, (R_2-1)!}
$$

where P is the number of columns of $\mathcal{P}$.

Proof. We use Lemma 5.17 to reduce Λ to an irreducible substructure $\Lambda' = (\mathbf{y}, \mathcal{P}, \phi')$. As s, K, R_1, R_2, and $\mathcal{P}$ all remain the same, we have $T(\Lambda) = T(\Lambda')$. The result then follows from Theorem 5.19. □

It is possible to generalize these two theorems so that the set of vertices does not lie strictly inside $\mathcal{H} \cup \mathcal{P}$. It is also possible to remove the balance condition from the second theorem. The same proofs can be applied to obtain two similar, but more complicated formulas. However, the benefits of doing so is limited, so we shall not pursue it here. With these two substructures and theorems ready, we can proceed to decompose minimal and vertical arrays, which we will do in the next chapter.

Chapter 6

Enumeration of Paired Arrays

In this chapter, we will continue with the decomposition of paired arrays started in Chapter 3. Recall that we have decomposed canonical arrays by removing redundant pairs, which gave us minimal arrays and sets of partial pairings $\mathcal{T}_i$ on $[p_i]$ for each row i. Then, in the previous two chapters, we took a detour to define arrowed arrays and substructures for sets of arrowed arrays. Furthermore, we have developed formulas that count the number of arrowed arrays satisfying these substructures. With these new tools, we can decompose minimal arrays into proper vertical arrays by removing non-mixed pairs, using a q_i-subset of $[s_i + q_i]$, a minimal array, and an arrowed array to record the removed pairs. By doing so, we can give a formula for $m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ in terms of $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$, where $m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}} = \left|\mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}\right|$ and $v^{(\mathbf{s})}_{n,K;\mathbf{R}} = \left|\mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}\right|$ are as defined in Definition 3.8.

6.1 Decomposition of Minimal Arrays

We start with defining a compatibility condition between arrowed array substructures and minimal arrays.

Definition 6.1. Let $\alpha \in \mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ be an n-row minimal array with $q_i = 0$, $\mathcal{R}_i$ as its set of marked cells in row i, and ψ_i as its forest condition function for row i. A substructure $\Theta = (\mathbf{y}, \mathcal{P}, \phi)$ as defined in Definition 5.11 is Θ-*compatible* with row i of α if $\mathcal{P} = \mathcal{R}_i$ and $\phi = \psi_i$. Furthermore, suppose $\mathcal{W}$ is a y-subset of $[s_i + R_i + y - 1]$ for some $y \geq 0$. We define $\Theta_{\alpha,i,\mathcal{W}}$ to be the substructure of $\mathcal{AR}^{(y)}_{K;R_i-y,K}$ with parameters $\Theta_{\alpha,i,\mathcal{W}} = (\mathbf{y}, \mathcal{R}_i, \psi_i)$, where $\mathbf{y} = (y_1, \ldots, y_K)$ and y_j is the number of vertices inserted into cell (i, j) of α if $\mathcal{W}$ is inserted into row i of α by the insertion procedure defined in Procedure 3.11.

By definition, $\Theta_{\alpha,i,\mathcal{W}}$ is Θ-compatible with row i of α. Also, by summing over the number of vertices inserted into cell (i,j), we have $|\mathcal{W}| = \sum_j y_j$. Now, before we go into the actual decomposition, we will first present the following proposition.

Proposition 6.2. *Let $\alpha \in \mathcal{PA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ be a proper paired array. Suppose cell (i,j) of α is an unmarked cell containing at least one vertex, then α' that is formed by marking cell (i,j) of α is also a proper paired array.*

Proof. As $\mathbf{s}$ remains the same between α and α', α' satisfies the balanced condition. Similarly, as α satisfies the forest condition, the functional digraph G of the forest condition function ψ_i is a forest with root vertices $\mathcal{R}_i$. As cell (i,j) is unmarked and non-empty, its rightmost vertex must be paired with some vertex in column $\psi_i(j)$, so $(j, \psi_i(j))$ is an edge of G. By marking cell (i,j), we have removed this edge from G, which splits the component containing j into two components. One component retains its original root vertex, which is in $\mathcal{R}_i$, while the other component has j as its root vertex. Therefore, the functional digraph of the forest condition function ψ_i' is a forest with root vertices $\mathcal{R}_i \cup \{j\}$. Therefore, α' satisfies the forest condition. Together, this implies that α' is a proper paired array. □

Note that the converse of the Proposition 6.2 is not true. For example, if α' has only one marked cell in row i, then unmarking that cell violates the forest condition for that row. This proposition allows us to mark cells containing critical vertices, making those vertices non-critical and removing them from the forest condition. Afterwards, we can unpair and extract those vertices while keeping the resulting paired array proper.

With the preliminaries defined, we will now provide a decomposition of minimal arrays into proper vertical arrays and arrowed arrays. This is done iteratively, by removing the non-mixed vertices one row at a time. For a given row i, we mark the cells containing the critical vertices of non-mixed pairs, then remove these pairs to form a minimal array with only mixed pairs in row i. To keep track of the removed pairs, we use a q_i-subset to represent the non-critical vertices, and an arrowed array to represent the critical vertices and their pairings with the non-critical vertices.

Theorem 6.3. *Let $n, K \geq 1$, $\mathbf{q} \geq \mathbf{0}$, $\mathbf{s} \geq \mathbf{0}$, $\mathbf{R} \in [K]^n$, and $1 \leq i \leq n$. Then, there exists a decomposition*

$$\eta_i \colon \mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}} \to \bigcup_{\substack{\beta \in \mathcal{MA}^{(\mathbf{q}';\mathbf{s})}_{n,K;\mathbf{R}'} \\ \mathcal{W} \in [s_i+R_i+2q_i-1;q_i]}} (\beta, \mathcal{W}, \Theta_{\beta,i,\mathcal{W}})$$

of minimal arrays into triples of smaller arrays, q_i-subsets, and arrowed arrays, where $s_i = \sum_{k \neq i} s_{i,k}$ and

$$R'_k = \begin{cases} R_i + q_i & k = i \\ R_k & \textit{otherwise} \end{cases}$$

$$q'_k = \begin{cases} 0 & k = i \\ q_k & \textit{otherwise} \end{cases}$$

Furthermore, this decomposition is a bijection.

As a side note, the fact that each non-mixed, non-redundant pair in row i contains a critical vertex in row i means that $R_i + q_i \leq K$, so $\mathbf{R}' \in [K]^n$.

Proof. We will provide the decomposition and its inverse, and prove that it is a bijection. Conceptually, we take the non-mixed pairs of a minimal array α, and split them into critical and non-critical vertices. We put the critical vertices into row 1 of an arrowed array (σ, ϕ), and put the non-critical vertices into row 2. Then, we add marked cells and arrows to (σ, ϕ) in such a way that row i of α has the same forest condition function as row 1 of (σ, ϕ). To record the position of the non-critical vertices, we extract and record these vertices as a q_i-subset of $[s_i + R_i + 2q_i - 1]$. Finally, we mark the cells of α containing the removed critical vertices, so as to preserve the forest condition for row i.

Let $\alpha \in \mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ be a minimal array, and suppose $\{u, v\}$ is a non-mixed pair in row i of α. If neither u nor v is critical, then $\{u, v\}$ is a redundant pair. If both u and v are critical, then the functional digraph of ψ_i contains a cycle between the columns containing u and v. As both of these are contradictions, exactly one of the two vertices is critical. Therefore, there are q_i non-critical vertices in row i of α, the set of which we denote as $\mathcal{V}$, and there are q_i critical vertices in row i of α, the set of which we denote as $\mathcal{U}$. Note that the vertices of $\mathcal{U}$ must be in distinct columns, and each vertex of $\mathcal{U}$ must be paired with a vertex of $\mathcal{V}$.

To construct the paired array $\beta \in \mathcal{MA}^{(\mathbf{q}';\mathbf{s})}_{n,K;\mathbf{R}'}$ and the subset $\mathcal{W} \in [s_i + R_i + 2q_i - 1; q_i]$, we first mark the columns containing the vertices of $\mathcal{U}$. This causes the pairs of $\mathcal{U} \cup \mathcal{V}$ to be redundant, so we can unpair them to obtain the partially-paired array α'. By Proposition 6.2 and Proposition 3.10, α' is a proper partially-paired array. Next, we remove the vertices of $\mathcal{U}$ from α' to obtain partially-paired array α'', and extract $\mathcal{V}$ from α'' as described in Procedure 3.11 to obtain the subset $\mathcal{W}$ and the paired array β. As α has $s_i + 2q_i$ vertices and R_i marked cells in row i, α'' has $s_i + q_i$ vertices and $R_i + q_i$

marked cells in row i. Therefore, $\mathcal{W}$ is a q_i-subset of $[s_i + R_i + 2q_i - 1]$. Furthermore, by Proposition 3.12, β is also a proper paired array. Notice that we have not changed any row other than row i, so $q'_k = q_k$ and $R'_k = R_k$ for $k \neq i$. Also, as we have not changed any mixed pairs, $\mathbf{s}$ remains the same between α and β. Now, we have removed q_i non-mixed pairs and marked q_i cells in row i, so $q'_i = 0$ and $R'_i = R_i + q_i$. Finally, as there are no non-mixed pairs in row i of β, and the non-mixed pairs in the other rows are not redundant, β is a minimal array. Together, we have $\beta \in \mathcal{MA}^{(\mathbf{q}';\mathbf{s})}_{n,K;\mathbf{R}'}$ as desired.

To preserve information on the pairs we removed, we construct an arrowed array $(\sigma, \phi) \in \Theta_{\beta,i,\mathcal{W}}$ such that $\psi_i = \psi'_1$, where ψ_i and ψ'_1 are the forest condition functions for row i of α and row 1 of (σ, ϕ), respectively. For each cell (i, j) of α, we do the following

- If cell (i, j) of α is empty, we leave cell $(1, j)$ of σ empty.
- If cell (i, j) of α is a marked cell, we mark cell $(1, j)$ of σ.
- If the rightmost object of cell (i, j) of α is a vertex $u \in \mathcal{U}$, we place a vertex x_u in cell $(1, j)$ of σ.
- If the rightmost object of cell (i, j) of α is a vertex $u \notin \mathcal{U}$, we leave cell $(1, j)$ of σ empty. However, suppose v is the vertex paired with u, and v is in the column j', we let $\phi(j) = j'$.

Next, we mark all cells in row 2 of σ. Then, for each vertex $v \in \mathcal{V}$ that is in cell (i, j) of α, we place a corresponding vertex x_v in cell $(2, j)$ of σ. If we need to place more than one vertex into the same cell, we place them in the same order in σ as they are in α. Finally, for each non-mixed pair $\{u, v\} \in \mathcal{U} \cup \mathcal{V}$ in row i of α, we pair their corresponding vertices x_u and x_v in σ. This completes the construction of (σ, ϕ).

Now, to show that $(\sigma, \phi) \in \Theta_{\beta,i,\mathcal{W}} = (\mathbf{y}, \mathcal{R}'_i, \theta_i)$, we need to show that (σ, ϕ) satisfies the forest condition, as well as the conditions defined by $\mathbf{y}$, $\mathcal{R}'_i$, and θ_i, where $\mathcal{R}'_i$ is the set of marked cells in row i of β, and θ_i is the forest condition function for row i of β. Suppose cell (i, j) of α is empty, then by construction both cell (i, j) of β and cell $(1, j)$ of σ are empty. Therefore, neither $\theta_i(j)$ nor $\phi(j)$ are defined. Alternatively, suppose that the rightmost object of cell (i, j) of α is a mixed vertex $u \notin \mathcal{U}$ and it is paired with some vertex v. Then, cell (i, j) of β remains unmarked, and u remains the rightmost vertex of β. Furthermore, as u is still paired with the same vertex v, we have $\theta_i(j) = \psi_i(j)$. Correspondingly, by construction of (σ, ϕ), we let $\phi(j)$ be the column that v resides in, so $\phi(j) = \psi_i(j) = \theta_i(j)$. Finally, suppose that the rightmost object of cell (i, j) of α is not a mixed vertex. Then, it is either a box or a non-mixed vertex. In either case, the cell would be marked in β, so $\theta_i(j)$ is not defined. Similarly, in our construction of σ, we either mark cell $(1, j)$ of σ or place a vertex in that cell, which leaves $\phi(j)$ undefined. Combining these three cases, we have shown that $\phi = \theta_i$ as desired.

Next, we will show that (σ, ϕ) satisfies the forest condition. As all cells in row 2 are marked, the forest condition of row 2 is trivially satisfied. To show that row 1 also satisfies the forest condition, it suffices to show that $\psi_i = \psi'_1$ and $\mathcal{R}_i = \mathcal{R}'_1$, where $\mathcal{R}_i$ and $\mathcal{R}'_1$ are the marked cells of row i of α and row 1 of σ, respectively. This is because α is a minimal array, so ψ_i must satisfy the forest condition. By construction, we have $\mathcal{R}_i = \mathcal{R}'_1$. Furthermore, cell (i, j) of α is empty if and only if cell $(1, j)$ of σ is empty, and cell (i, j) of α is marked if and only if cell $(1, j)$ of σ is marked. Now, suppose cell (i, j) of α is unmarked, and the rightmost object is a vertex $u \notin \mathcal{U}$. As explained in the previous paragraph, we have $\psi'_1(j) = \phi(j) = \psi_i(j)$. Finally, suppose cell (i, j) of α is unmarked, and the rightmost object is a vertex $u \in \mathcal{U}$. Let $v \in \mathcal{V}$ be the vertex that u is paired to, and suppose v is in column j'. Then, the vertices x_u and x_v corresponding to u and v are paired with each other in σ. Furthermore, x_u and x_v are in the same columns as u and v are in α, respectively. As x_u is the rightmost object of cell $(1, j)$ of σ, we have $\psi'_1(j) = j' = \psi_i(j)$. Combining these results, we have $\psi_i = \psi'_1$ as desired.

Then, if we reinsert $\mathcal{W}$ into row i of β, we recover α'' and the set $\mathcal{V}$ of extracted vertices by Proposition 3.12. Recall that by construction, for each vertex $v \in \mathcal{V}$ in cell (i, j) of α, we have placed a vertex x_v in cell $(2, j)$ of σ. Therefore, the number of vertices inserted into cell (i, j) of β is the same as the number of vertices in cell $(2, j)$ of (σ, ϕ). This proves that (σ, ϕ) satisfies $\mathbf{y}$. Finally, note that the set of marked cells in row 1 of σ is equal to the set of marked cells in row i of α, so it is a subset of the set of marked cells in row i of β. Furthermore, if cell (i, j) is marked in β, then it must either be marked in α, or contain a vertex $u \in \mathcal{U}$. In the former case, cell $(1, j)$ is marked in σ, and in the latter case, it is unmarked and contains the vertex x_u. Therefore, (σ, ϕ) satisfies $\mathcal{R}'_i$. Together, we have $(\sigma, \phi) \in \Theta_{\beta,i,\mathcal{W}}$ as desired.

Conversely, let $\beta \in \mathcal{MA}^{(\mathbf{q}';\mathbf{s})}_{n,K;\mathbf{R}'}$, $\mathcal{W} \in [s_i + R_i + q_i - 1; q_i]$, and $(\sigma, \phi) \in \Theta_{\beta,i,\mathcal{W}}$. We first construct partially-paired array β' by inserting $\mathcal{W}$ into row i of β as described in Procedure 3.11. This gives us a set $\mathcal{V}$ of unpaired vertices in β', which are labelled with the elements of $\mathcal{W}$ by the insertion procedure. By Proposition 3.12, β' is a proper partially-paired array. Furthermore, by the definition of $\Theta_{\alpha,i,\mathcal{W}}$, the vertices of $\mathcal{V}$ are in the same columns as the vertices in row 2 of σ. Therefore, we can create a correspondence between the vertices of $\mathcal{V}$ and the vertices in row 2 of σ. We do this by labelling the vertices in row 2 of σ with $\mathcal{W}$ from left to right, ignoring the boxes used for marking cells. Then, for each vertex $v \in \mathcal{V}$, we let x_v be the vertex in row 2 of σ that acquired the same label as v. Next, consider each cell $(1, j)$ of σ that contains a vertex. Since $\Theta_{\beta,i,\mathcal{W}}$ is Θ-compatible with β, cell (i, j) of β must be marked. Furthermore, as the set of marked cells is the same between β and β', cell (i, j) of β' is also marked. This means that we can add an unpaired vertex u to cell (i, j), which we place to the right of all other vertices in that cell. Similar to the vertices of $\mathcal{V}$, we let the corresponding vertex in cell $(1, j)$ of σ be x_u. After adding these vertices, we let the resulting partially-paired array be β'', and let the set of vertices added to obtain β'' be $\mathcal{U}$. By Proposition 3.12, β'' is a proper partially-paired array. Also, since $\mathcal{W}$ is a q_i-subset, (σ, ϕ) has q_i vertex pairs, so $|\mathcal{U}| = |\mathcal{V}| = q_i$. Finally, to recover α, we unmarked the cells containing the vertices of $\mathcal{U}$, and for each pair $\{x_u, x_v\}$ in σ, we pair their corresponding vertices $u \in \mathcal{U}$ and $v \in \mathcal{V}$ in row i of β''. Note that these pairs are non-redundant, as the vertices of $\mathcal{U}$ are now the rightmost objects of their respective cells, which means that they are critical.

To show that $\alpha \in \mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$, we need to show that α satisfies the parameters $\mathbf{R}$, $\mathbf{q}$, and $\mathbf{s}$, and that it satisfies the balance and forest condition. As the vertices of $\mathcal{U}$ and $\mathcal{V}$ are in the same row, they become non-mixed pairs in α. Hence, they do not affect the balance condition. By the same reasoning, the parameter $\mathbf{s}$ remains unchanged between β, β'', and α. Now, as $|\mathcal{U}| = q_i$, α has q_i non-mixed pairs in row i. Furthermore, this means we have unmarked q_i cells from row i of α, so α has $R'_i - q_i = R_i$ marked cells in row i. As the number of non-mixed pairs and marked cells in the other rows remain unchanged, we have $q'_k = q_k$ and $R'_k = R_k$ for $k \neq i$. Therefore, α satisfies the parameters $\mathbf{R}$ and $\mathbf{q}$ as desired.

What remains to be shown is that α satisfies the forest condition. As the other rows are unchanged, we only have to show that the forest condition holds for row i. To this end, we will show that $\psi_i = \psi'_1$ and $\mathcal{R}_i = \mathcal{R}'_1$, where ψ_i and ψ'_1 are the forest condition functions for row i of α and row 1 of (σ, ϕ), respectively. Similarly, $\mathcal{R}_i$ and $\mathcal{R}'_1$ are the marked cells of row i of α and row 1 of σ, respectively. This is sufficient, as (σ, ϕ) is an arrowed array, which satisfies the forest condition. Note that $(\sigma, \phi) \in \Theta_{\beta,i,\mathcal{W}}$ implies that (σ, ϕ) is Θ-compatible with row i of β, so ϕ is the forest condition for row i of β. Furthermore, if $\mathcal{R}'_i$ is the set of marked cells in row i of β, then by Definition 5.11, $\mathcal{R}'_i$ is also the set of columns that contains the marked cells and vertices in row 1 of σ.

Now, consider cell (i, j) of β. Suppose that the cell is empty, then $\phi(j)$ is undefined. As j is not in $\mathcal{R}'_i$, cell $(1, j)$ of σ is also empty, so $\psi'_1(j)$ is undefined. Then, since the vertices of $\mathcal{U}$ and $\mathcal{V}$ are unpaired in β'', they are added only to non-empty cells of β. Therefore, both cell (i, j) of β'' and cell (i, j) of α remain empty, so $\psi_i(j)$ is also undefined. Next, suppose cell (i, j) of β contains a critical vertex u, paired with some other vertex v. In this case, $\phi(j)$ is defined, and $\psi'_1(j) = \phi(j)$. Again, since the vertices of $\mathcal{U}$ and $\mathcal{V}$ are unpaired in β'', they are not the rightmost objects of their cells, so the set of critical vertices of β and β'' are the same. Furthermore, since cell (i, j) of β'' is already unmarked, unmarking marked cells has no effect on u. In particular, u remains a critical vertex in

α, and is paired with the same vertex v. Therefore, $\psi'_1(j) = \phi(j) = \psi_i(j)$. On the other hand, if cell (i,j) of β and cell $(1,j)$ of σ are both marked, then cell $(1,j)$ of σ does not contain a vertex, so no vertices of $\mathcal{U}$ are added to cell (i,j) of β''. Hence, cell (i,j) of α remains marked when we unmark the columns containing vertices of $\mathcal{U}$. In this case, neither $\psi'_1(j)$ nor $\psi_i(j)$ are defined. Finally, if cell (i,j) of β is marked, but cell $(1,j)$ of σ is not, then cell $(1,j)$ must contain a vertex x_u paired with some vertex x_v in a column j'. In our construction of β'', we have added vertices $u \in \mathcal{U}$ and $v \in \mathcal{V}$ to cell (i,j) and cell (i,j') corresponding to x_u and x_v, respectively. When cells containing the vertices of $\mathcal{U}$ are unmarked, u becomes a critical vertex. Since u is paired with v and x_u is paired with x_v, we have that $\psi'_1(j) = j' = \psi_i(j)$. Combining these results, we have $\psi_i = \psi'_1$ as desired. Furthermore, cell (i,j) of α is marked if and only if cell $(1,j)$ of σ is marked, so we have $\mathcal{R}_i = \mathcal{R}'_1$ as well. This shows that $\alpha \in \mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$.

Finally, we have to show that the two operations presented are inverses of each other. By Proposition 3.12, the extraction and insertion procedures are inverses. Furthermore, if we extract $\mathcal{V}$ and reinsert it, the vertices inserted acquire the same labels as before the extraction. Therefore, we can identify the vertices in row 2 of (σ,ϕ) with the vertices of $\mathcal{V}$. Then, the columns which contain the critical vertices $\mathcal{U}$ are exactly the columns of (σ,ϕ) that contain vertices in row 1. This allows us to recover the columns of $\mathcal{U}$, so that we can add critical vertices and unmark cells. Finally, as we have a correspondence between the vertices of $\mathcal{U}$ and $\mathcal{V}$ with the vertices of (σ,ϕ), the pairing of vertices in (σ,ϕ) allows us to recover the pairing of the removed vertices. Therefore, η_i as described, is a bijection. □

Note that in the proof of Theorem 6.3, α' and β'' corresponds to each other, so does α'' and β'. Now, observe that if we iteratively decompose each of the n rows of a minimal array, the resulting minimal array will have no non-mixed vertices, and hence will be a vertical array. Since minimal arrays are by definition proper, the resulting vertical array is proper. Also note that the decomposition of row i of a minimal array does not change the vertices, boxes, and pairings of the other rows. Therefore, the ordering of the rows in which we decompose the minimal array can be arbitrary. Furthermore, the proper vertical array, the arrowed arrays, and the q_i-subsets resulting from the decomposition remain the same regardless of the order in which we decompose the minimal array. In particular, as the definition of $\Theta_{\beta,i,\mathcal{W}}$ depends only on row i of β, we can replace β with the proper vertical array resulting from the iterated decomposition without changing the set of arrowed arrays satisfing this substructure. In fact, we can simultanenously decompose all n rows at once and obtain the same result. The reason why we have not taken that approach is to keep the proof simple, and also to keep in parallel to the decomposition of proper vertical arrays in the next section.

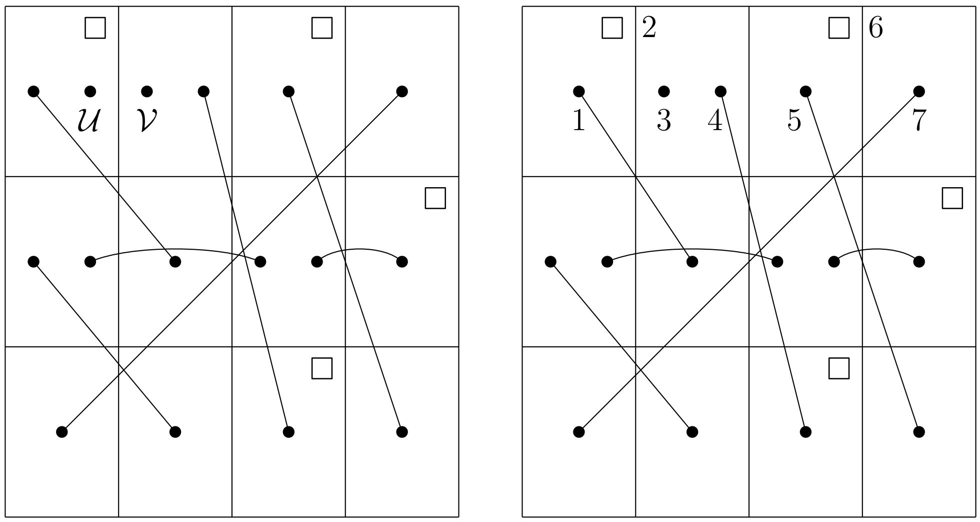


Figure 6.1: Partially-paired array α' and α'' corresponding to the decomposition of row 1 of Figure 3.4

As an example, we will decompose the minimal array in Figure 3.4. By following the decomposition described in Theorem 6.3, we arrive at the partially-paired array α' and α'', as depicted in Figure 6.1. For clarity, we have marked the vertices of $\mathcal{U}$ and $\mathcal{V}$ in α', and labelled the objects in row 1 of α''. After the decomposition, we obtain the minimal array β and the arrowed array (σ, ϕ), depicted in Figure 6.2. We also obtain the subset $\mathcal{W}_1 = \{3\} \in [6; 1]$. We can then continue the decomposition with rows 2 and 3. This gives us the subsets $\mathcal{W}_2 = \{4, 6\} \in [6; 2]$, and $\mathcal{W}_3 = \emptyset \in [4; 0]$, the proper vertical array in Figure 6.3, and the arrowed arrays in Figure 6.4, corresponding to rows 2 and 3, respectively.

Now that we have a decomposition of minimal arrays, we can use it to give an expression for $m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ with respect to $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$. In particular, we are interested in the case $\mathbf{R} = \mathbf{1}$, as that corresponds to the decomposition of canonical arrays. By iterating the decomposition in Theorem 6.3 and taking the cardinality of both sides, we obtain the following corollary.

Corollary 6.4. *Let $n, K \geq 1$, $\mathbf{q} \geq \mathbf{0}$, and $\mathbf{s} \geq \mathbf{0}$. Then,*

$$m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{1}} = \prod_{i=1}^{n} \frac{(s_i + 2q_i)!}{(s_i + q_i)!} \cdot v^{(\mathbf{s})}_{n,K;\mathbf{q}+\mathbf{1}}$$

where $s_i = \sum_{j \neq i} s_{i,j}$.

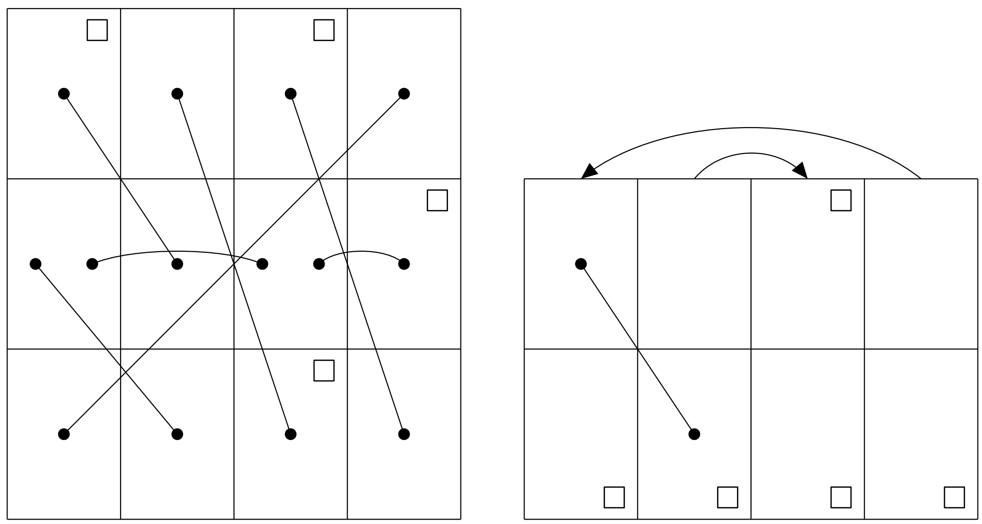


Figure 6.2: Minimal array β and arrowed array (σ, ϕ) corresponding to the decomposition of row 1 of Figure 3.4

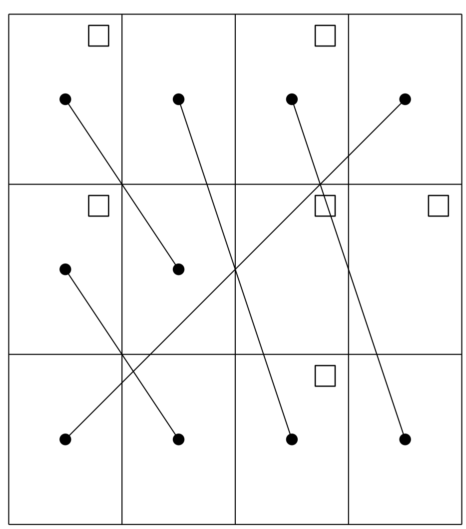


Figure 6.3: Proper vertical array from the iterated decomposition of Figure 3.4

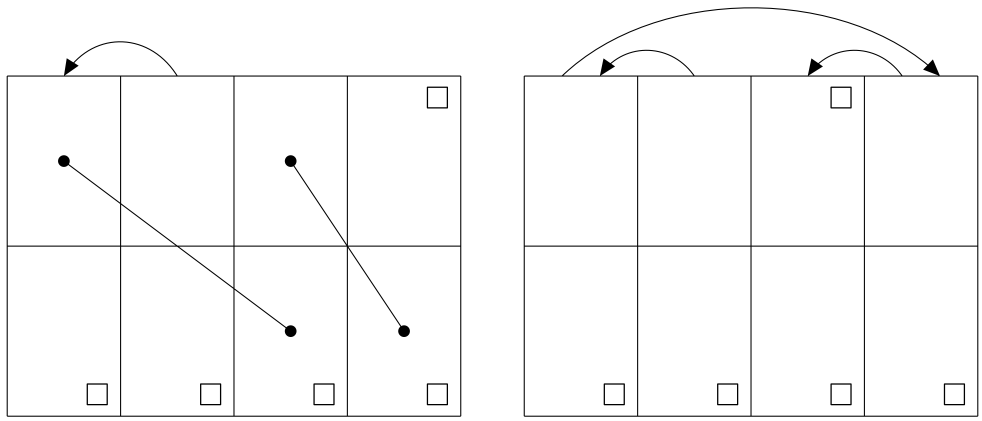


Figure 6.4: Arrowed arrays from the decomposition of rows 2 and 3 of Figure 6.2

Proof. If $q_i \geq K$ for some i such that $1 \leq i \leq n$, then $m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{1}} = 0$, as any minimal array in $\mathcal{MA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{1}}$ must have at least K critical vertices and 1 marked cell in row i, which is a contradiction. On the right hand side, we have that $q_i + 1 > K$, so $v^{(\mathbf{s})}_{n,K;\mathbf{q+1}} = 0$ by convention. Therefore, the identity holds trivially.

Let β be a minimal array with $R'_i = R_i + q_i$ marked cells in row i, and $\mathcal{W}$ be a q_i-subset of $[s_i + R_i + 2q_i - 1]$. As β is a proper minimal array, the forest condition function θ_i for row i satisfies the forest condition, so it is a forest with root vertices $\mathcal{P} = \mathcal{R}'_i$. Therefore, by applying Corollary 5.15, we have

$$T\left(\Theta_{\beta,i,\mathcal{W}}\right) = \frac{(R_i + q_i - 1)!}{(R_i - 1)!}$$

as arrowed arrays satisfying $\Theta_{\beta,i,\mathcal{W}}$ have R_i marked cells in row 1. Note that this is constant for all minimal arrays $\beta \in \mathcal{MA}^{(\mathbf{q}';\mathbf{s})}_{n,K;\mathbf{R}'}$ and all subsets $\mathcal{W} \in [s_i + R_i + 2q_i - 1; q_i]$. Therefore, by taking the cardinality of Theorem 6.3 and substituting in the above formula, we obtain

$$\begin{aligned} m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}} &= \sum_{\substack{\beta \in \mathcal{MA}^{(\mathbf{q}';\mathbf{s})}_{n,K;\mathbf{R}'} \\ \mathcal{W} \in [s_i + R_i + 2q_i - 1; q_i]}} T\left(\Theta_{\beta,i,\mathcal{W}}\right) \\ &= \frac{(s_i + R_i + 2q_i - 1)!}{q_i!\,(s_i + R_i + q_i - 1)!} \cdot \frac{(R_i + q_i - 1)!}{(R_i - 1)!} \cdot m^{(\mathbf{q}';\mathbf{s})}_{n,K;\mathbf{R}'} \end{aligned}$$

where $\mathbf{R}'$ and $\mathbf{q}'$ are as defined in the theorem. By iterating this equation over $1 \leq i \leq n$, we have

$$m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}} = \prod_{i=1}^{n}\left(\frac{(s_i+R_i+2q_i-1)!}{q_i!\,(s_i+R_i+q_i-1)!}\cdot\frac{(R_i+q_i-1)!}{(R_i-1)!}\right)\cdot m^{(\mathbf{0};\mathbf{s})}_{n,K;\mathbf{R}+\mathbf{q}}$$

$$m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{1}} = \prod_{i=1}^{n}\frac{(s_i+2q_i)!}{(s_i+q_i)!}\cdot m^{(\mathbf{0};\mathbf{s})}_{n,K;\mathbf{q}+\mathbf{1}}$$

where in the second row we substitute $R_i = 1$ to simplify the expression. The result of the corollary then follows from noting that $m^{(\mathbf{0};\mathbf{s})}_{n,K;\mathbf{q}+\mathbf{1}} = v^{(\mathbf{s})}_{n,K;\mathbf{q}+\mathbf{1}}$, as the set of minimal arrays with no non-mixed pairs is precisely the set of proper vertical arrays of the same parameter. That is, $\mathcal{MA}^{(\mathbf{0};\mathbf{s})}_{n,K;\mathbf{q}+\mathbf{1}} = \mathcal{VA}^{(\mathbf{s})}_{n,K;\mathbf{q}+\mathbf{1}}$. □

Note that this formula is consistent with and is a direct generalization of Theorem 4.2 of Goulden and Solfstra. Furthermore, we can now express the number of paired functions in terms of the number of proper vertical arrays using this formula.

Corollary 6.5. *Let $n, K \geq 1$, $\mathbf{q} \geq \mathbf{0}$, and $\mathbf{s} \geq \mathbf{0}$. We have*

$$f^{(\mathbf{q};\mathbf{s})}_{n,K} = \sum_{\mathbf{t}=\mathbf{0}}^{\mathbf{q}}\prod_{i=1}^{n}\frac{(2q_i+s_i)!}{2^{t_i}t_i!\,(s_i+q_i-t_i)!}\cdot v^{(\mathbf{s})}_{n,K;\mathbf{q}-\mathbf{t}+\mathbf{1}}$$

Furthermore, if $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ can be written as a polynomial expression in K for all R_i, where $1 \leq R_i \leq q_i + 1$, then $f^{(\mathbf{q};\mathbf{s})}_{n,K}$ can be written as a polynomial expression in K.

Proof. By combining Theorem 3.7, Theorem 3.13, and Corollary 6.4, we have

$$\begin{aligned} f^{(\mathbf{q};\mathbf{s})}_{n,K} &= \sum_{\mathbf{t}=\mathbf{0}}^{\mathbf{q}}\prod_{i=1}^{n}\binom{2q_i+s_i}{2t_i}(2t_i-1)!!\cdot\frac{(s_i+2q_i-2t_i)!}{(s_i+q_i-t_i)!}\cdot v^{(\mathbf{s})}_{n,K;\mathbf{q}-\mathbf{t}+\mathbf{1}} \\ &= \sum_{\mathbf{t}=\mathbf{0}}^{\mathbf{q}}\prod_{i=1}^{n}\frac{(2q_i+s_i)!}{2^{t_i}t_i!\,(s_i+q_i-t_i)!}\cdot v^{(\mathbf{s})}_{n,K;\mathbf{q}-\mathbf{t}+\mathbf{1}} \end{aligned}$$

where we used the fact that $(2t_i-1)!! = \frac{(2t_i)!}{2^{t_i}t_i!}$ to simplify the above expression. Polynomiality of $f^{(\mathbf{q};\mathbf{s})}_{n,K}$ follows from the fact that the summation bounds are independent of K, so $f^{(\mathbf{q};\mathbf{s})}_{n,K}$ as expressed above is a polynomial combination of $v^{(\mathbf{s})}_{n,K;\mathbf{q}-\mathbf{t}+\mathbf{1}}$, with coefficients that are also independent of K. □

6.2 Enumeration of Vertical Arrays

At this point, we are ready to decompose proper vertical arrays. In this section, we will focus on tree-shaped vertical arrays. Recall from Definition 3.5 that a paired array $\alpha \in \mathcal{PA}^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ is tree-shaped if the support graph of $\mathbf{s}$ is a tree. With tree-shaped vertical arrays, we can delete a row that is a leaf in the support graph while keeping the support graph a tree. This allows us to recursively decompose tree-shaped vertical arrays into smaller tree-shaped vertical arrays and arrowed arrays. Then, by using the substructures defined in the previous chapter, we can provide a formula for $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ when the support graph of $\mathbf{s}$ is a tree.

Definition 6.6. Let $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$ be an n-row proper vertical array with $\mathcal{R}_i$ as its set of marked cells in row i, and ψ_i as its forest condition function for row i. A substructure $\Lambda = (\mathbf{x}, \mathcal{P}, \phi)$ as defined in Definition 5.16 is Λ-*compatible* with row i of α if $\mathcal{P} = \mathcal{R}_i$ and $\phi = \psi_i$. Furthermore, let R'_1 and R'_2 be such that $1 \leq R'_1 \leq R_i$ and $1 \leq R'_2 \leq K$, and suppose that $\mathcal{W}$ is a x-subset of $[s_i + R_i + x - 1]$ for some $x \geq 0$. We define $\Lambda_{\alpha,i,\mathcal{W}}$ to be the substructure of $\mathcal{AR}^{(x+R_i-R'_1)}_{K;R'_1,R'_2}$ with parameters $\Lambda_{\alpha,i,\mathcal{W}} = (\mathbf{x}, \mathcal{R}_i, \psi_i)$, where $\mathbf{x} = (x_1, \ldots, x_K)$ and x_j is the number of vertices inserted into cell (i,j) of α if $\mathcal{W}$ is inserted into row i of α by the insertion procedure defined in Procedure 3.11.

By definition, $\Lambda_{\alpha,i,\mathcal{W}}$ is Λ-compatible with row i of α. Note that unlike $\Theta_{\alpha,i,\mathcal{W}}$, defined in Definition 6.1, the parameters of the arrowed array is not predetermined by $\mathbf{x}$. Also, by summing over the number of vertices inserted into cell (i,j), we have $|\mathcal{W}| = \sum_j x_j$.

With this substructure defined, we can now decompose tree-shaped vertical arrays. Let $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n+1,K;\mathbf{R}}$ be an $(n+1)$-row proper vertical array, and without loss of generality assume that row $n+1$ is a leaf vertex adjacent to row n in the support graph of $\mathbf{s}$. To extract row $n+1$ from α, we mark the cells in row n containing the critical vertices matched with vertices in row $n+1$. Then, we remove all pairs between rows n and $n+1$, and delete row $n+1$. To keep track of the removed vertices in row n, we use a $(s_{n+1} - P + R_n)$-subset to represent the positions of the non-critical vertices, and an arrowed array to represent the critical vertices and pairings of the vertices removed.

Theorem 6.7. *Let $n, K \geq 1$, $\mathbf{s} = (s_{1,2}, s_{1,3}, \ldots, s_{n,n+1}) \geq \mathbf{0}$, and $\mathbf{R} = (R_1, \ldots, R_{n+1}) \in [K]^{n+1}$. Suppose the support graph of $\mathbf{s}$ is a tree with the vertex $n+1$ as a leaf adjacent to* *the vertex n. Then, there exists a decomposition*

$$\zeta\colon \mathcal{PVA}^{(\mathbf{s})}_{n+1,K;\mathbf{R}} \to \bigcup_{P=R_n}^{\min(s_{n+1}+R_n,K)} \bigcup_{\substack{\beta\in\mathcal{PVA}^{(\mathbf{s}')}_{n,K;\mathbf{R}'} \\ \mathcal{W}\in[s_n+R_n-1;s_{n+1}-P+R_n]}} (\beta, \mathcal{W}, \Lambda_{\beta,n,\mathcal{W}})$$

of proper vertical arrays into a triple of smaller vertical arrays, $(s_{n+1} - P + R_n)$-subsets, and arrowed arrays. Here, $\Lambda_{\beta,n,\mathcal{W}}$ are substructures of $\mathcal{AR}^{(s_{n+1})}_{K;R_n,R_{n+1}}$, $\mathbf{s}'$ is $\mathbf{s}$ restricted to an $n\times n$ matrix by removing the last row and column, $s_i = \sum_{k\neq i} s_{i,k}$ for $1 \le i \le n+1$, and $\mathbf{R}'$ is a vector of length n given by

$$R'_k = \begin{cases} R_k & k < n \\ P & k = n \end{cases}$$

Furthermore, this decomposition is a bijection.

Note that s_n includes the vertex pairs between rows n and $n+1$, and the marked cells in row n of β are given by $\mathcal{R}'_n$, which is a set of size P that contains $\mathcal{R}_n$ as a subset.

Proof. The proof of this theorem uses techniques similar to those in the proof of Theorem 6.3. That is, we will provide the decomposition and its inverse, and prove that it is a bijection. We take the mixed pairs between row n and row $n+1$ of α, and put them into an arrowed array (σ,ϕ). Then, we add marked cells and arrows to (σ,ϕ) in such a way that rows n and $n+1$ of α have the same forest condition functions as rows 1 and 2 of (σ,ϕ), respectively. To record the position of the non-critical vertices in row n, we extract and record these vertices as a $s_{n+1} - P + R_n$-subset of $[s_n + R_n - 1]$. Finally, we mark the cells of α containing the critical vertices of row n that are paired with vertices of row $n+1$, so as to preserve the forest condition for row n.

Let $\mathcal{V}$ be the set of non-critical vertices that are paired with vertices of row $n+1$, and $\mathcal{U}$ be the set of critical vertices that are paired with vertices of row $n+1$. Note that the vertices of $\mathcal{U}$ and $\mathcal{V}$ must be in row n by our assumption, and that $|\mathcal{U}\cup\mathcal{V}| = s_{n+1}$. Furthermore, the vertices of $\mathcal{U}$ must be in distinct columns, and these columns must be unmarked in row n. Therefore, if we let $P = R_n + |\mathcal{U}|$, we have $R_n \le P \le K$. Furthermore, since $|\mathcal{U}| \le s_{n+1}$, we have $P \le s_{n+1} + R_n$, which combines to give $R_n \le P \le \min(s_{n+1} + R_n, K)$. Then, suppose that $s_{i,k,j}$ is the number of vertices in cell (i,j) that are paired with vertices of row k, where $i \neq k$. By Lemma 3.6, $s_{i,k,j} = s_{k,i,j}$ for all $1 \le i < k \le n+1$ and $1 \le j \le K$.

Now, to construct the proper vertical array $\beta \in \mathcal{PVA}^{(\mathbf{s}')}_{n,K;\mathbf{R}'}$ and the subset $\mathcal{W} \in [s_n + R_n - 1; s_{n+1} - P + R_n]$, we first mark the cells containing the vertices of $\mathcal{U}$, and call this vertical array α'. By Proposition 6.2, α' is also proper. Next, we unpair all vertex pairs with one vertex in row $n+1$, delete row $n+1$, and call the resulting array α''. As this leaves all other mixed pairs unchanged, $\mathbf{s}'$ describes the number of mixed pairs of α'', as it is the restriction of $\mathbf{s}$ to the first n rows and columns. Then, as the support graph of $\mathbf{s}'$ is the support graph of $\mathbf{s}$ with the vertex $n+1$ removed, the support graph of $\mathbf{s}'$ is also a tree. Also, note that deleting row $n+1$ removes the variables $s_{n+1,k,j}$ and $s_{k,n+1,j}$ from α', but leaves the remaining $s_{i,k,j}$ the same for all $1 \leq i, k \leq n$, $i \neq k$. Therefore, we have $s'_{i,k,j} = s_{i,k,j}$ for all $1 \leq i, k \leq n$, so the conditions of Lemma 3.6 remain satisfied in α'', so α'' satisfies the balance condition. In addition, since the vertices of $\mathcal{U} \cup \mathcal{V}$ are non-critical in α', they do not affect the forest condition. Therefore, the forest condition remains satisfied when we unpair the vertices of $\mathcal{U} \cup \mathcal{V}$ and delete row $n+1$. This means that α'' is a proper partially-paired array. Next, we remove the vertices of $\mathcal{U}$ from α'' to obtain the partially-paired array α''', and we extract $\mathcal{V}$ from α''' as described in Procedure 3.11 to obtain the subset $\mathcal{W}$ and the vertical array β. Note that α''' has $R_n + |\mathcal{U}|$ marked cells, $s_n - |\mathcal{U}|$ total vertices, and $|\mathcal{V}| = s_n - P + R_n$ unpaired vertices in row n. Therefore, $\mathcal{W}$ is a $s_n - P + R_n$-subset of $s_n + R_n - 1$. Furthermore, by Proposition 3.12, β is also a proper paired array. Notice that we have not changed any row other than row n and $n+1$, so $R'_k = R_k$ for $k < n$. Then, as row n of β has $R_n + |\mathcal{U}|$ marked cells, we have $R'_n = P$. Also, as with α'', the set of mixed pairs in β is described by $\mathbf{s}'$. Finally, as α has no non-mixed pairs, neither does β, so β is a vertical array. Together, we have $\beta \in \mathcal{PVA}^{(\mathbf{s}')}_{n,K;\mathbf{R}'}$ as desired.

To preserve information on the pairs we removed, we construct an arrowed array $(\sigma, \phi) \in \Lambda_{\beta,n,\mathcal{W}}$ such that $\psi_n = \psi'_1$ and $\psi_{n+1} = \psi'_2$, where ψ_n and ψ_{n+1} are the forest condition functions for rows n and $n+1$ of α, while ψ'_1 and ψ'_2 are the forest condition functions for rows 1 and 2 of (σ, ϕ), respectively. For each vertex $v \in \mathcal{U} \cup \mathcal{V}$ that is in cell (n, j), we place a corresponding x_v into cell $(1, j)$ of σ. Similarly, for each vertex u in cell $(n+1, j)$, we place a corresponding vertex x_u in cell $(2, j)$ of σ. If we need to place more than one vertex into the same cell, we place them in the same order in σ as they are in α. Then, for each pair $\{v, u\}$ between row n and $n+1$, we pair their corresponding vertices x_u and x_v in σ. Next, we mark cell $(1, j)$ of σ if cell (n, j) of α is marked, and we mark cell $(2, j)$ of σ if cell $(n+1, j)$ of α is marked. Finally, suppose (n, j) of α contains a critical vertex $u \notin \mathcal{U}$. Then, it must be paired with some vertex v in some cell (k, j'), where $1 \leq k \leq n-1$. In this case, we let $\phi(j) = j'$. This completes the construction of (σ, ϕ).

Now, to show that $(\sigma, \phi) \in \Lambda_{\beta,n,\mathcal{W}} = (\mathbf{x}, \mathcal{R}'_n, \theta_n)$, we need to first show that (σ, ϕ) is in $\mathcal{AR}^{(s_{n+1})}_{K;R_n,R_{n+1}}$. Then, we need to show (σ, ϕ) satisfies the balance and forest conditions, as well the conditions defined by $\mathbf{x}$, $\mathcal{R}'_n$, and θ_n, where $\mathcal{R}'_n$ is the set of marked cells in row n of β, and θ_n is the forest condition function for row n of β. By construction, the set of marked cells in rows 1 and 2 of σ are the same as the set of marked cells in rows n and $n+1$ of α. Furthermore, there are $|\mathcal{U}| + |\mathcal{V}| = s_{n+1}$ vertices in each row of σ, so $(\sigma, \phi) \in \mathcal{AR}^{(s_{n+1})}_{K;R_n,R_{n+1}}$. Next, note that $\phi(j)$ is defined if and only if cell (n, j) of α contains a critical vertex $u \notin \mathcal{U}$. Suppose u is paired with some vertex v, then cell (n, j) of β remains unmarked, and u remains the rightmost vertex of β. Furthermore, as u is still paired with the same vertex v, we have $\theta_n(j) = \psi_n(j)$. Correspondingly, we have defined $\phi(j)$ to be the column that u resides in by construction of (σ, ϕ), so $\phi(j) = \psi_n(j) = \theta_n(j)$. Therefore, we have that $\phi = \theta_n$ as desired.

Next, we will show that (σ, ϕ) satisfies the balance and forest conditions. Recall that in our construction of σ, we placed a vertex in cell $(1, j)$ of σ for every vertex in cell (n, j) of $\mathcal{U} \cup \mathcal{V}$. Similarly, we placed a vertex into cell $(2, j)$ of σ for every vertex in cell $(n+1, j)$ of α. Therefore, cell $(1, j)$ of σ has $s_{n,n+1,j}$ vertices, and cell $(2, j)$ of σ has $s_{n+1,n,j}$ vertices. By Lemma 3.6, we have $s_{n,n+1,j} = s_{n+1,n,j}$ for all j, so (σ, ϕ) satisfies the balance condition.

To show that (σ, ϕ) satisfies the forest condition, it suffices to show that $\psi_n = \psi'_1$, $\psi_{n+1} = \psi'_2$, $\mathcal{R}_n = \mathcal{R}'_1$, and $\mathcal{R}_{n+1} = \mathcal{R}'_2$, where $\mathcal{R}_n$ and $\mathcal{R}_{n+1}$ are the set of marked cells for rows n and $n+1$ of α, while $\mathcal{R}'_1$ and $\mathcal{R}'_2$ are the marked cells of rows 1 and 2 of σ, respectively. Similarly, ψ_n and ψ_{n+1} are the forest condition functions for rows n and $n+1$ of α, while ψ'_1 and ψ'_2 are the forest condition functions for rows 1 and 2 of (σ, ϕ), respectively. This is because α is a proper vertical array, so ψ_n and ψ_{n+1} must both satisfy the forest condition. By construction, we have $\mathcal{R}_n = \mathcal{R}'_1$. Furthermore, cell (n, j) of α is empty if and only if cell $(1, j)$ of σ is empty, and cell (n, j) of α is marked if and only if cell $(1, j)$ of σ is marked. Now, suppose cell (n, j) of α is unmarked, and the rightmost object is a vertex $u \notin \mathcal{U}$. As explained in the previous paragraph, we have $\psi'_1(j) = \phi(j) = \psi_n(j)$. Finally, suppose cell (n, j) of α is unmarked, and the rightmost object is a vertex $u \in \mathcal{U}$. Let v be the vertex in row $n+1$ that u is paired to, and suppose v is in column j'. Then, the vertices x_u and x_v corresponding to u and v are paired with each other in σ. Furthermore, x_u and x_v are in the same columns as u and v are in α, respectively. Since the vertices in row 1 of σ are in the same relative order as the vertices of $\mathcal{U} \cup \mathcal{V}$, x_u is the rightmost object of cell $(1, j)$ of σ. This gives us $\psi'_1(j) = j' = \psi_n(j)$. Combining these results, we have $\psi_n = \psi'_1$ as desired. The proof for row 2 is identical, except that since all critical vertices of row $n+1$ are paired with vertices of row n, we can omit the case that requires $\phi(j)$.

Note that the set of marked cells in row 1 of σ is equal to the set of marked cells in row n of α, so it is a subset of the set of marked cells in row n of β. Then, if we reinsert $\mathcal{W}$ into row n of β, we recover α''' and the set $\mathcal{V}$ of extracted vertices by Proposition 3.12. Recall that by construction, for each vertex $v \in \mathcal{V}$ in cell (n, j) of α, we have placed a vertex x_v cell $(1, j)$ of σ. Furthermore, if cell (n, j) is marked in β, then it must either be marked in α, or contain a vertex $u \in \mathcal{U}$. In the former case, cell $(1, j)$ is marked in σ, and in the latter case, it is unmarked and contains the vertex x_u. Therefore, cell $(1, j)$ of σ contains $x_j + 1$ vertices if cell (n, j) is marked in β and contains a critical vertex in σ, and x_j vertices, otherwise. Therefore, (σ, ϕ) satisfies $\mathbf{x}$ and $\mathcal{R}'_n$. Together, we have $(\sigma, \phi) \in \Theta_{\beta,i,\mathcal{W}}$ as desired.

Conversely, let $\beta \in \mathcal{PVA}^{(\mathbf{s}')}_{n,K;\mathbf{R}'}$, $\mathcal{W} \in [s_n + R_n - 1; s_{n+1} - P + R_n]$, and $(\sigma, \phi) \in \mathcal{AR}^{(s_{n+1})}_{K;R_n,R_{n+1}}$ that satisfies $\Lambda_{\beta,n,\mathcal{W}}$. We first construct partially-paired array β' by inserting $\mathcal{W}$ into row n of β as described in Procedure 3.11. This gives us a set $\mathcal{V}$ of unpaired vertices in β', which are labelled with the elements of $\mathcal{W}$ by the insertion procedure. By Proposition 3.12, β' is a proper partially-paired array. Furthermore, by the definition of $\Lambda_{\beta,n,\mathcal{W}}$, the vertices of $\mathcal{V}$ are in the same columns as the non-critical vertices in row 1 of σ. Therefore, we can create a correspondence between the vertices of $\mathcal{V}$ and the non-critical vertices in row 1 of σ. We do this by labelling the non-critical vertices in row 1 of σ with $\mathcal{W}$ from left to right, ignoring the critical vertices and boxes used for marking cells. Then, for each vertex $v \in \mathcal{V}$, we let x_v be the non-critical vertex in row 1 of σ that acquired the same label as v. Next, consider each cell $(1, j)$ of σ that contains a critical vertex. Since $\Lambda_{\beta,n,\mathcal{W}}$ is Λ-compatible with β, cell (n, j) of β must be marked. Furthermore, as the set of marked cells is the same between β and β', cell (n, j) of $\beta'$is also marked. This means that we can add an unpaired vertex u to cell (n, j), which we place to the right of all other vertices in that cell. Similarly to the vertices of $\mathcal{V}$, we let the corresponding vertex in cell $(1, j)$ of σ be x_u. After adding these vertices, we let the resulting partially-paired array be β'', and let the set of vertices added to obtain β'' be $\mathcal{U}$. By Proposition 3.12, β'' is a proper partially-paired array. Since row n of β'' has P marked cells, while row 1 of σ has R_n marked cells, we have $|\mathcal{U}| = P - R_n$. Also, since $\mathcal{W}$ is a $(s_{n+1} - P + R_n)$-subset, we have $|\mathcal{U}| + |\mathcal{V}| = s_{n+1}$ as desired. Next, we extend β'' by adding row $n + 1$, and for each cell $(2, j)$ of σ that is marked, we mark cell $(n + 1, j)$ of β''. Similarly, for each vertex x_v in cell $(2, j)$ of σ, we add a corresponding vertex v in row $(n + 1, j)$ of β''. Then, for each pair $\{x_u, x_v\}$ in σ, we pair their corresponding vertices $u \in \mathcal{U} \cup \mathcal{V}$ and v in row $n + 1$. We call the resulting array β'''. Finally, to recover α, we unmark the cells containing the vertices of $\mathcal{U}$.

To show that $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n+1,K;\mathbf{R}}$, we need to show that α satisfies the parameters $\mathbf{R}$ and $\mathbf{s}$, and that it satisfies the balance and forest condition. Since β satisfies the balance condition, by Lemma 3.6, $s'_{i,k,j} = s'_{k,i,j}$ for all $1 \leq i < k \leq n$ and $1 \leq j \leq K$. Note that for each vertex in cell $(1, j)$ of σ, we have added a vertex in cell (n, j) of β''' and have paired it with a vertex in row $n + 1$. Similarly, for each vertex in cell $(2, j)$ of σ, we have added a vertex to cell $(n + 1, j)$ of σ. As σ satisfies the balance condition, we obtain $s_{n,n+1,j} = s_{n+1,n,j}$. Next, note that $s_{i,k,j} = s'_{i,k,j}$ for $1 \leq i, k \leq n$, as the only pairs we have added are between rows n and $n+1$. Furthermore, $s_{n+1,k,j} = s_{k,n+1,j} = 0$ for $k < n$, as all vertices in row $n+1$ are paired with vertices of row n by construction. This means that the $s_{i,k,j}$ satisfies Lemma 3.6, so β''' and α satisfies the balance condition. Now, as the vertices of $\mathcal{U} \cup \mathcal{V}$ are in row n and $|\mathcal{U} \cup \mathcal{V}| = s_{n+1}$, α has s_{n+1} mixed pairs between rows n and $n+1$. Then, as the vertices of row $n+1$ are only paired with vertices of row n, we have $s_{n,n+1} = s_{n+1}$ and $s_{n+1,k} = 0$ for $k < n$. Furthermore, the set of marked cells is the same between α and β for the first $n-1$ rows, so $R'_k = R_k$ for $1 \leq k \leq n-1$. Also note that row n of α has $R'_n - |\mathcal{U}| = R_n$ marked cells in row n, as $R'_n = P$ and $|\mathcal{U}| = P - R_n$. Finally, row $n+1$ of α has R_{n+1} marked cells in row $n+1$, as the set of marked cells in row $n+1$ of α is the same as the set of marked cells in row 2 of σ. Therefore, α satisfies $\mathbf{R}$ and $\mathbf{s}$ as desired.

What remains to be shown is that α satisfies the forest condition. As the other rows are unchanged, we only have to show that the forest condition holds for rows n and $n+1$. To this end, we will show that $\psi_n = \psi'_1$, $\psi_{n+1} = \psi'_2$, $\mathcal{R}_n = \mathcal{R}'_1$, and $\mathcal{R}_{n+1} = \mathcal{R}'_2$, where $\mathcal{R}_n$ and $\mathcal{R}_{n+1}$ are the set of marked cells for rows n and $n+1$ of α, while $\mathcal{R}'_1$ and $\mathcal{R}'_2$ are the marked cells of rows 1 and 2 of σ, respectively. Similarly, ψ_n and ψ_{n+1} are the forest condition functions for rows n and $n+1$ of α, while ψ'_1 and ψ'_2 are the forest condition functions for rows 1 and 2 of (σ, ϕ), respectively. Note that $(\sigma, \phi) \in \Lambda_{\beta,n,\mathcal{W}}$ implies that (σ, ϕ) is Λ-compatible with row n of β, so ϕ is the forest condition for row n of β. Furthermore, if $\mathcal{R}'_n$ is the set of marked cells in row n of β, then by Definition 5.11, $\mathcal{R}'_n$ is also the set of columns that contains the marked cells and critical vertices in row 1 of σ.

Now, consider cell (n, j) of β. Suppose that the cell is empty, then $\phi(j)$ is undefined. As j is not in $\mathcal{R}'_n$, cell $(1, j)$ of σ is also empty, so $\psi'_1(j)$ is undefined. Then, since the vertices of $\mathcal{U}$ and $\mathcal{V}$ are unpaired in β'', they are added only to non-empty cells of β. Therefore, both cell (n, j) of β'' and cell (n, j) of α remains empty, so $\psi_n(j)$ is also undefined. Next, suppose cell (n, j) of β contains a critical vertex u, paired with some other vertex v. In this case, $\phi(j)$ is defined, and $\psi'_1(j) = \phi(j)$. Again, since the vertices of $\mathcal{U}$ and $\mathcal{V}$ are unpaired in β'', they are not the rightmost objects of their cells, so the set of critical vertices of β and β'' are the same. Furthermore, since cell (n, j) of β'' is already unmarked, unmarking marked cells has no effect on u. In particular, u remains a critical vertex in α, and is paired with the same vertex v. Therefore, $\psi'_1(j) = \phi(j) = \psi_n(j)$. On the other hand, if cell (n, j) of β and cell $(1, j)$ of σ are both marked, then cell $(1, j)$ of σ only contain the vertices of $\mathcal{V}$, as we only add vertices of $\mathcal{U}$ to a cell (n, j) of β'' if cell $(1, j)$ of σ contains a critical vertex. Hence, cell (n, j) of α remains marked when we unmark the columns containing vertices of $\mathcal{U}$. In this case, neither $\psi'_1(j)$ nor $\psi_n(j)$ are defined. Finally, if cell (n, j) of β is marked, but cell $(1, j)$ of σ is not, then cell $(1, j)$ must contain a critical vertex x_u paired with some vertex x_v in a column j'. In our construction of β'', we have added vertices

$u \in \mathcal{U}$ to cell (n, j) corresponding to x_u. Furthermore, we have added a vertex v to cell $(n+1, j)$ in β''' corresponding to x_v. When cells containing the vertices of $\mathcal{U}$ are unmarked, u becomes a critical vertex. Since u is paired with v and x_u is paired with x_v, we have $\psi_1'(j) = j' = \psi_n(j)$. Combining these results, we have $\psi_n = \psi_1'$ as desired. Finally, as cell (n, j) of α is marked if and only if cell $(1, j)$ of σ is marked, we have $\mathcal{R}_n = \mathcal{R}_1'$ as well.

The proof that $\psi_{n+1} = \psi_2'$ is similar. Note that cell $(n+1, j)$ is marked in α if and only if cell $(2, j)$ is marked in σ by construction. Therefore, we have $\mathcal{R}_{n+1} = \mathcal{R}_2'$. Next, suppose that cell $(n+1, j)$ of α contains a critical vertex v, paired with some vertex u in cell (n, j') for some j'. Then, their corresponding vertices x_u and x_v must be in cell $(2, j)$ and cell $(1, j')$ of σ respectively. Furthermore, cell $(2, j)$ of σ must be unmarked by construction, so we have $\psi_{n+1}(j) = \psi_2'(j)$. Together, we have $\psi_{n+1} = \psi_2'$, as desired. This shows that α satisfies the forest condition, so $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n+1,K;\mathbf{R}}$.

Finally, we have to show that the two operations presented are inverses of each other. By Proposition 3.12, the extraction and insertion procedures are inverses. Furthermore, if we extract $\mathcal{V}$ and reinsert it, the vertices inserted acquire the same labels as before the extraction. Therefore, we can correspond the non-critical vertices in row 1 of (σ, ϕ) with the vertices of $\mathcal{V}$. Then, the columns which contain the critical vertices $\mathcal{U}$ are exactly the columns of (σ, ϕ) that contain critical vertices in row 1. This allows us to recover the columns of $\mathcal{U}$, so that we can add critical vertices and unmark cells. Similarly, the vertices in row 2 of (σ, ϕ) correspond to the vertices of row $n+1$ of α. As we have a correspondence between the vertices of $\mathcal{U} \cup \mathcal{V}$ and vertices of row $n+1$ with the vertices in row 1 and 2 of (σ, ϕ), respectively, we can recover the pairing of the removed vertices via the pairing of vertices in (σ, ϕ). Therefore, ζ as described, is a bijection. □

Note that in the proof of Theorem 6.7, α' and β''' corresponds to each other, so does α'' and β'', as well as α''' and β'. Also, note that the decomposition works with any row that is a leaf vertex of the support graph. The assumption that the leaf vertex is $n+1$ and is adjacent to vertex n is only for the convenience of proving the theorem. With this decomposition, we can iteratively pick a row where the support graph of $\mathbf{s}$ is a leaf, and remove that row. This leaves arrowed arrays with support graph $\mathbf{s}'$, which is a tree with n rows, so we can repeat the induction. Therefore, we can reduce the problem to 1 row vertical arrays, for which the answer is simply $\binom{K}{R_1}$.

As an example, we will decompose the tree-shaped vertical array in Figure 6.5. As the edges are between rows 1 and 2, rows 2 and 3, and rows 2 and 4, the support graph of $\mathbf{s}$ is a tree. By following the decomposition described in Theorem 6.7, we can decompose row 4 and arrive at the partially-paired array α'' and α''', as depicted in Figure 6.6. For clarity, we

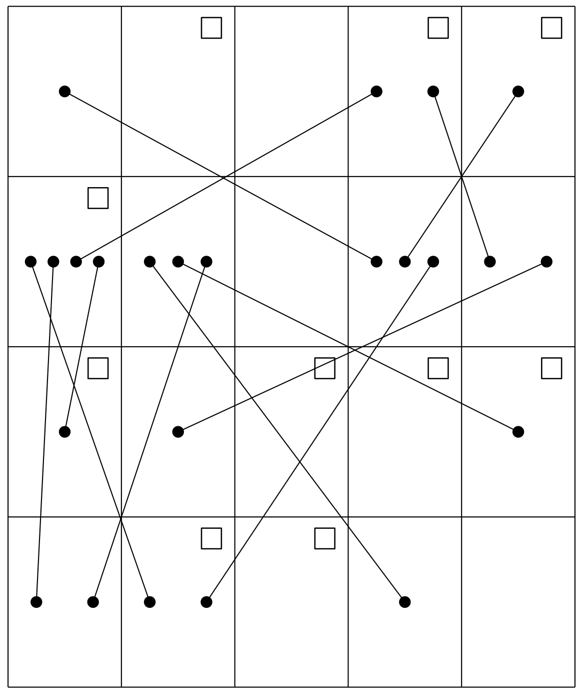


Figure 6.5: A tree-shaped, 4-row vertical array

have marked the vertices of $\mathcal{U}$ and $\mathcal{V}$ in α'', and labelled the objects in row 2 of α'''. After the decomposition, we obtain the minimal array β and the arrowed array (σ, ϕ), depicted in Figure 6.7. We also obtain the subset $\mathcal{W}_4 = \{1, 2, 6\} \in [12; 3]$ and $P_4 = 3$. We can then continue the decomposition with row 3. This gives us the subset $\mathcal{W}_3 = \{2, 4\} \in [9; 2]$, $P_3 = 4$, and the arrowed array in the left figure of Figure 6.8. Subsequently decomposing row 2 gives us the subset $\mathcal{W}_2 = \{3, 4, 6\} \in [6; 3]$, $P_2 = 4$, and the arrowed arrays in the right figure of Figure 6.8. The final resulting vertical array is a 1-row array with no vertices, and cells 1, 2, 4, and 5 marked, as depicted in Figure 6.9.

Now that we have a decomposition of tree-shaped vertical arrays, we can provide an explicit formula for $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ via induction. We start with the following corollary.

Corollary 6.8. *Let $n, K \geq 1$, $\mathbf{s} = (s_{1,2}, s_{1,3}, \ldots, s_{n,n+1}) \geq \mathbf{0}$, and $\mathbf{R} = (R_1, \ldots, R_{n+1}) \in [K]^{n+1}$. Suppose the support graph of $\mathbf{s}$ is a tree with the vertex $n+1$ as a leaf adjacent to*

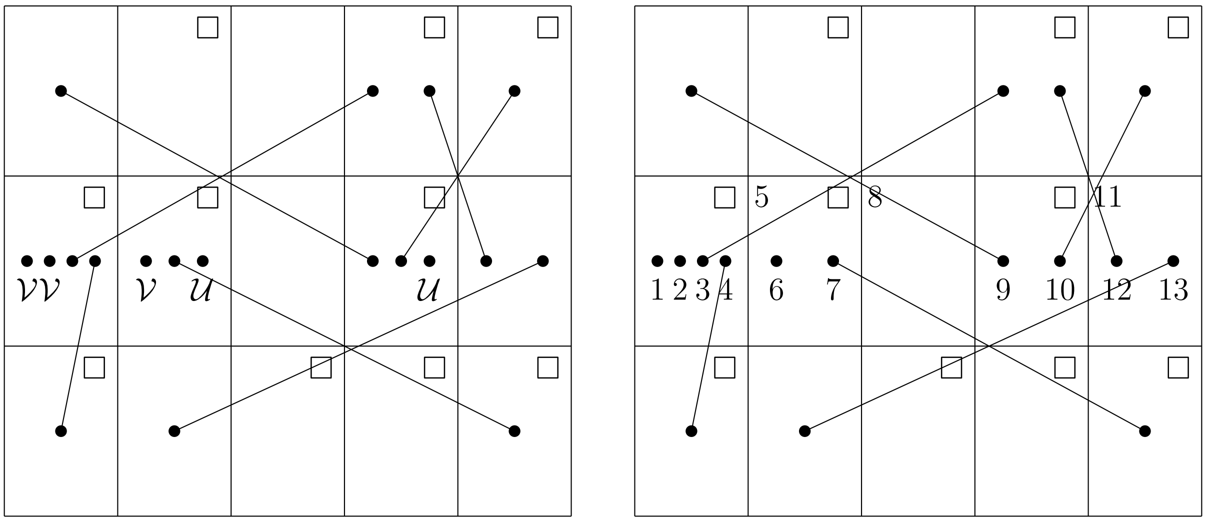


Figure 6.6: Partially-paired array α'' and α''' corresponding to the decomposition of row 4 of Figure 6.5

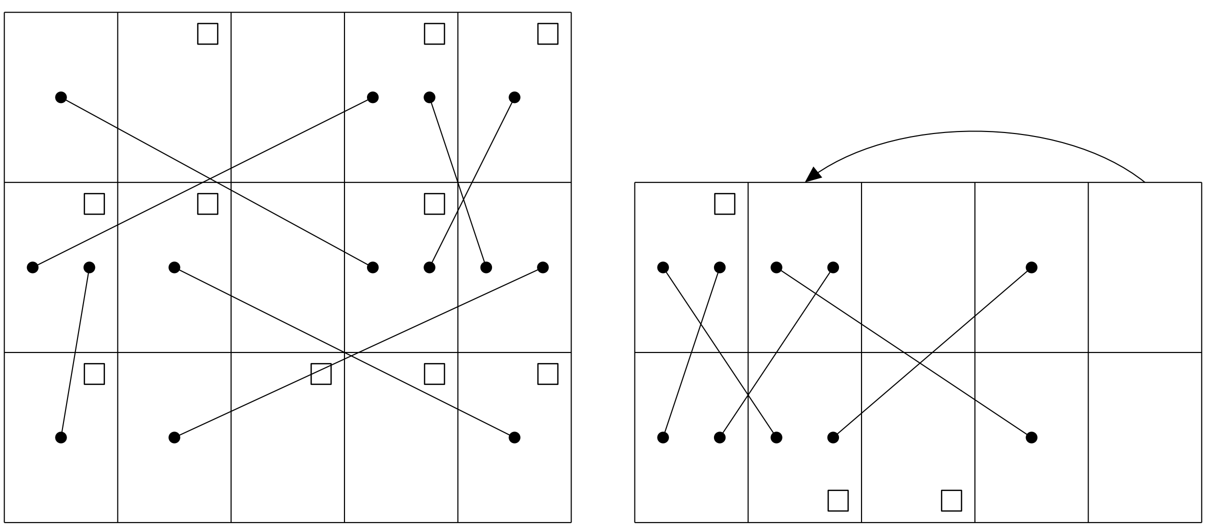


Figure 6.7: Minimal array β and arrowed array (σ, ϕ) corresponding to the decomposition of row 4 of Figure 6.5

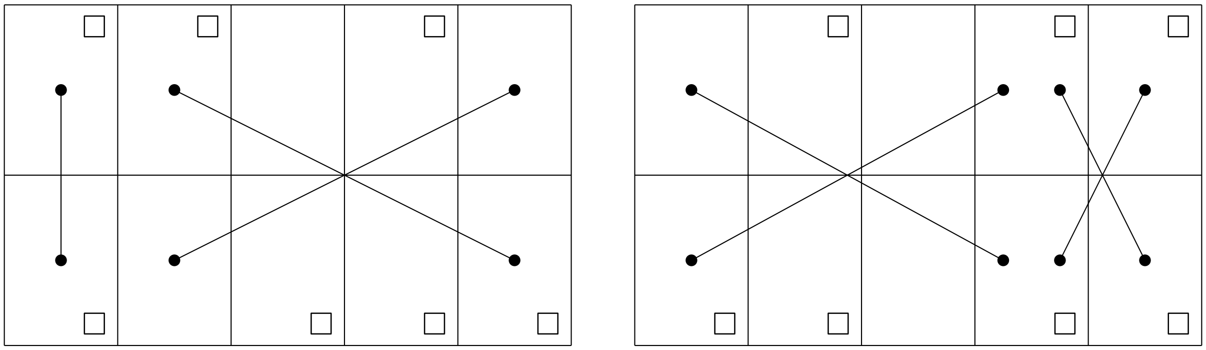


Figure 6.8: Arrowed arrays from the decomposition of row 3, then row 2, of Figure 6.7

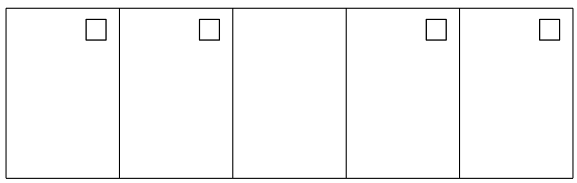


Figure 6.9: 1-row vertical array from the complete decomposition the vertical array in Figure 6.5

the vertex n. Then,

$$v^{(\mathbf{s})}_{n+1,K;\mathbf{R}} = \sum_{P=R_n}^{\min(s_{n+1}+R_n,K)} \sum_{A_{n+1}=0}^{\min(s_{n+1},K)-1} \binom{s_n+R_n-1}{s_{n+1}-P+R_n} v^{(\mathbf{s}')}_{n,K;\mathbf{R}'} \times \frac{(s_{n+1}-P+R_n)\,(K-A_{n+1}-1)!\,(s_{n+1}-A_{n+1}-1)!\,(P-1)!}{(P-R_n-A_{n+1})!\,(K-R_{n+1}-A_{n+1})!\,(R_n-1)!\,(R_{n+1}-1)!}$$

where $\mathbf{s}'$ is $\mathbf{s}$ restricted to an $n \times n$ matrix by removing the last row and column, $s_i = \sum_{k \neq i} s_{i,k}$ for $1 \leq i \leq n+1$, and $\mathbf{R}'$ is a vector of length n given by

$$R'_k = \begin{cases} R_k & k < n \\ P & k = n \end{cases}$$

Proof. Let P be such that $R_n \leq P \leq \min(s_{n+1}+R_n, K)$, $\beta \in \mathcal{PVA}^{(\mathbf{s}')}_{n,K;\mathbf{R}'}$ be an n-row vertical array with parameters as defined in Theorem 6.7, $\mathcal{W}$ be a $(s_{n+1}-P+R_n)$-subset of $[s_n+R_n-1]$, and $\Lambda_{\beta,n,\mathcal{W}}$ be a substructure of $\mathcal{AR}^{(s_{n+1})}_{K;R_n,R_{n+1}}$. As β is a proper vertical array, the forest condition function θ_n for row n satisfies the forest condition, so it is a forest with root vertices $\mathcal{P} = \mathcal{R}'_n$. Therefore, by applying Corollary 5.20, we have

$$T\left(\Lambda_{\beta,n,\mathcal{W}}\right) = \sum_{A_{n+1}=0}^{\min(s_{n+1},K)-1} \frac{(s_{n+1}-P+R_n)\,(K-A_{n+1}-1)!\,(s_{n+1}-A_{n+1}-1)!\,(P-1)!}{(P-R_n-A_{n+1})!\,(K-R_{n+1}-A_{n+1})!\,(R_n-1)!\,(R_{n+1}-1)!}$$

Note that this formula is independent on β and $\mathcal{W}$, and only depends on P. Furthermore, the constraint $R_n \leq P \leq \min(s_{n+1}+R_n, K)$ matches with the definition of substructure Λ. Then, for a given P, there are $\binom{s_n+R_n-1}{s_{n+1}-P+R_n}$ distinct $(s_{n+1}-P+R_n)$-subset of $[s_n+R_n-1]$. Finally, for a given $R'_n = P$, there are $v^{(\mathbf{s}')}_{n,K;\mathbf{R}'}$ proper vertical arrays. Combining these gives the formula of our corollary as desired. □

As we have assumed that the support graph G of $\mathbf{s}$ is a tree, we can repeatedly select a row that corresponds to a leaf vertex in G, and iterate the decomposition in Theorem 6.7. Then, by taking the cardinality of both sides, we obtain the following theorem.

Theorem 6.9. *Let $n, K \geq 1$, $\mathbf{s} \geq \mathbf{0}$, and $\mathbf{R} \geq \mathbf{1}$. Suppose the support graph G of $\mathbf{s}$ is a tree. Then,*

$$v^{(\mathbf{s})}_{n,K;\mathbf{R}} = \sum_{A_{e_1}=0}^{\min\left(s_{e_1},K\right)-1} \cdots \sum_{A_{e_{n-1}}=0}^{\min\left(s_{e_{n-1}},K\right)-1} \left[\prod_{j=1}^{n-1} \frac{\left(K-A_{e_j}-1\right)!}{\left(K+s_{e_j}-A_{e_j}-1\right)!} \times \prod_{i=1}^{n} \frac{\left(K+\sum_{k\sim i}(s_{i,k}-A_{i,k}-1)\right)!\,\left(R_i-1+\sum_{k\sim i}s_{i,k}\right)!}{(R_i-1)!\,\left(K-R_i-\sum_{k\sim i}A_{i,k}\right)!\,\left(R_i+\sum_{k\sim i}(s_{i,k}-1)\right)!}\right]$$

where $e_1, \ldots, e_{n-1}$ are the edges of G. Furthermore, for each edge $e_j = \{i, k\}$ in G, the summation variable A_{e_j} can be equivalently written as $A_{i,k}$ and $A_{k,i}$. Finally, the sum $\sum_{k\sim i}$ is over all indices k that are adjacent to i in the support graph of $\mathbf{s}$.

For example, if $n = 3$ and $s_{2,3} = 0$, the formula reduces to

$$v^{(\mathbf{s})}_{n,K;\mathbf{R}} = \sum_{A_{1,2}=0}^{\min(s_{1,2},K)-1} \sum_{A_{1,3}=0}^{\min(s_{1,3},K)-1} \left[\frac{(K-A_{1,2}-1)!\,(K-A_{1,3}-1)!}{(R_1-1)!\,(R_2-1)!\,(R_3-1)!\,(K-R_2-A_{1,2})!} \times \frac{(K+s_{1,2}+s_{1,3}-A_{1,2}-A_{1,3}-2)!\,(R_1+s_{1,2}+s_{1,3}-1)!}{(K-R_3-A_{1,3})!\,(K-R_1-A_{1,2}-A_{1,3})!\,(R_1+s_{1,2}+s_{1,3}-2)!}\right]$$

Notice that the bounds on A_{e_j} ensure that the factorials in the numerators are non-negative. As we shall later see, we can remove the upper bounds of $K-1$, but upper bounds of $s_{e_j}-1$ are necessary and cannot be removed. For a list of expressions of $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ for small values of n, see A.

Proof. Note that if $R_i > K$ for some i, we have $\frac{1}{(K-R_i-\sum_{k\sim i} A_{i,k})!}$ in the denominator of our summation term. This causes the entire sum to be zero, consistent with our convention that requires $v^{(\mathbf{s})}_{n,K;\mathbf{R}} = 0$ in such cases. Otherwise, we prove this theorem via induction on the number of rows.

Base case:

Suppose $n = 1$, then there are exactly $\binom{K}{R_1}$ vertical arrays in $\mathcal{PVA}^{(\mathbf{s})}_{1,K;R_1}$, as 1-row vertical arrays cannot contain mixed pairs. Hence, arrays in $\mathcal{PVA}^{(\mathbf{s})}_{1,K;R_1}$ have K columns, R_1 marked cells, and no vertices. On the other hand, this also means that the variables s_{e_j} and summations A_{e_j} do not appear in $v^{(\mathbf{s})}_{1,K;R_1}$, so our inductive formula reduces to

$$\begin{aligned} v^{(\mathbf{s})}_{1,K;R_1} &= \frac{K!\,(R_1-1)!}{(R_1-1)!\,(K-R_1)!R_1!} \\ &= \binom{K}{R_1} \end{aligned}$$

which agrees with our base case as desired.

Inductive step:

We want to prove that $v^{(\mathbf{s})}_{n+1,K;\mathbf{R}}$ gives the number of vertical arrays in $\mathcal{PVA}^{(\mathbf{s})}_{n+1,K;\mathbf{R}}$, assuming that the formula is true for n-row vertical arrays. Let $\mathbf{s} = (s_{1,2}, s_{1,3}, \dots, s_{n,n+1}) \geq \mathbf{0}$ and $\mathbf{R} = (R_1, \dots, R_{n+1}) \geq \mathbf{1}$. Without loss of generality, assume that the functional digraph of $\mathbf{s}$ has vertex $n+1$ as a leaf, and is adjacent to vertex n. Then, for convenience of notation, let $s_i = \sum_{k\sim i} s_{i,k}$, $A_i = \sum_{k\sim i} A_{i,k}$, and $\delta_i = \sum_{k\sim i} 1$ for $1 \leq i \leq n+1$. Note that since $n+1$ is a leaf adjacent to n, we have $s_{n+1} = s_{n,n+1}$ and $A_{n+1} = A_{n,n+1}$. Furthermore, let the $e_1, \dots, e_n$ be the edges of the support graph of $\mathbf{s}$, with e_n being the edge between vertex n and $n+1$. This means that for $1 \leq i \leq n-1$, A_i does not contain the variable $A_{n,n+1}$.

Now, by applying $s'_n = s_n - s_{n+1}$ to our inductive hypothesis, we have

$$
\begin{aligned}
v^{(\mathbf{s}')}_{n,K;\mathbf{R}'} = & \sum_{A_{e_1}=0}^{\min\left(s_{e_1},K\right)-1} \cdots \sum_{A_{e_{n-1}}=0}^{\min\left(s_{e_{n-1}},K\right)-1} \left[\prod_{j=1}^{n-1} \frac{\left(K - A_{e_j} - 1\right)!}{\left(K + s_{e_j} - A_{e_j} - 1\right)!} \times \right. \\
& \prod_{i=1}^{n-1} \frac{(K + s_i - A_i - \delta_i)!\,(R_i + s_i - 1)!}{(R_i - 1)!\,(K - R_i - A_i)!\,(R_i + s_i - \delta_i)!} \times \\
& \left. \frac{(K + s_n - s_{n+1} - A_n + A_{n+1} - \delta_n + 1)!\,(P + s_n - s_{n+1} - 1)!}{(P-1)!\,(K - P - A_n + A_{n+1})!\,(P + s_n - s_{n+1} - \delta_n + 1)!} \right]
\end{aligned}
$$

Note that A_n and δ_n are substituted with $A_n - A_{n+1}$ and $\delta_n - 1$, respectively, as the support graph of $\mathbf{s}'$ does not contain the edge $e_n = \{n, n+1\}$. To simplify the expression for further manipulation, we let $C\left(A_{e_1}, \ldots, A_{e_{n-1}}\right)$ to be the first two products inside the sum. That is, we rewrite the above expression as

$$
\begin{aligned}
v^{(\mathbf{s}')}_{n,K;\mathbf{R}'} = & \sum_{A_{e_1},\ldots,A_{e_{n-1}}} C\left(A_{e_1}, \ldots, A_{e_{n-1}}\right) \times \\
& \frac{(K + s_n - s_{n+1} - A_n + A_{n+1} - \delta_n + 1)!\,(P + s_n - s_{n+1} - 1)!}{(P-1)!\,(K - P - A_n + A_{n+1})!\,(P + s_n - s_{n+1} - \delta_n + 1)!}
\end{aligned}
$$

Then, we can substitute this expression into Corollary 6.8, which gives

$$
\begin{aligned}
v^{(\mathbf{s})}_{n+1,K;\mathbf{R}} &= \sum_{P=R_n}^{\min(s_{n+1}+R_n,K)} \sum_{A_{e_1},\ldots,A_{e_{n-1}}} C\left(A_{e_1},\ldots,A_{e_{n-1}}\right) \times \\
&\quad \frac{(K+s_n-s_{n+1}-A_n+A_{n+1}-\delta_n+1)!\,(P+s_n-s_{n+1}-1)!}{(P-1)!\,(K-P-A_n+A_{n+1})!\,(P+s_n-s_{n+1}-\delta_n+1)!} \times \\
&\quad \frac{(s_n+R_n-1)!}{(s_{n+1}-P+R_n)!\,(s_n-s_{n+1}-1+P)!} \times \\
&\quad \sum_{A_{n+1}=0}^{\min(s_{n+1},K)-1} \frac{(s_{n+1}-P+R_n)\,(K-A_{n+1}-1)!\,(s_{n+1}-A_{n+1}-1)!\,(P-1)!}{(P-R_n-A_{n+1})!\,(K-R_{n+1}-A_{n+1})!\,(R_n-1)!\,(R_{n+1}-1)!} \\
&= \sum_{A_{e_1},\ldots,A_{e_{n-1}}} \sum_{A_{n+1}=0}^{\min(s_{n+1},K)-1} \sum_{P=0}^{\min(s_{n+1},K-R_n)} C\left(A_{e_1},\ldots,A_{e_{n-1}}\right) \times \\
&\quad \frac{(K+s_n-s_{n+1}-A_n+A_{n+1}-\delta_n+1)!\,(s_n+R_n-1)!}{(K-P-R_n-A_n+A_{n+1})!\,(P+R_n+s_n-s_{n+1}-\delta_n+1)!} \times \\
&\quad \frac{(K-A_{n+1}-1)!\,(s_{n+1}-A_{n+1}-1)!}{(s_{n+1}-P-1)!\,(P-A_{n+1})!\,(K-R_{n+1}-A_{n+1})!\,(R_n-1)!\,(R_{n+1}-1)!}
\end{aligned}
$$

In the second equation, we have shifted the summation index P down by R_n, and have also rearrange the order of summation. This can be done as the summation bounds are independent of other summation variables. Now, recall that as discussed in Section 1.3, we can take $\frac{1}{x!}$ to be zero if x is a negative integer. This means that for $P \geq s_{n+1} - 1$ and $P \geq K - R_n - A_n + A_{n+1} \geq K - R_n$, the entire summation term is zero, as we have $(s_{n+1} - P - 1)!$ and $(K - P - R_n - A_n + A_{n+1})!$ in the denominator. Note that $A_n \geq A_{n+1}$ comes from the vertex n being the only vertex adjacent to the vertex $n+1$ in the support graph. Also note that P is not part of the numerator, so we can safely increase the upper bound of the P summation to infinity, without creating a negative factorial in the numerator. Furthermore, if $P < A_{n+1}$, then the summation term is also zero, as we have $(P - A_{n+1})!$ in the denominator. This allows us to substitute $P = Q + A_{n+1}$, and sum over $Q \geq 0$ instead. By doing these substitutions, we obtain

$$\begin{aligned}
v^{(\mathbf{s})}_{n+1,K;\mathbf{R}} &= \sum_{A_{e_1},\ldots,A_{e_{n-1}}} \sum_{A_{n+1}=0}^{\min(s_{n+1},K)-1} \sum_{Q\geq 0} C\left(A_{e_1},\ldots,A_{e_{n-1}}\right) \times \\
&\quad \frac{(K+s_n-s_{n+1}-A_n+A_{n+1}-\delta_n+1)!\,(s_n+R_n-1)!}{(K-Q-R_n-A_n)!\,(Q+R_n+A_{n+1}+s_n-s_{n+1}-\delta_n+1)!} \times \\
&\quad \frac{(K-A_{n+1}-1)!\,(s_{n+1}-A_{n+1}-1)!}{(s_{n+1}-A_{n+1}-Q-1)!Q!\,(K-R_{n+1}-A_{n+1})!\,(R_n-1)!\,(R_{n+1}-1)!} \\
&= \sum_{A_{e_1},\ldots,A_{e_{n-1}}} \sum_{A_{n+1}=0}^{\min(s_{n+1},K)-1} C\left(A_{e_1},\ldots,A_{e_{n-1}}\right) \times \\
&\quad {}_2F_1\left(\begin{matrix}-s_{n+1}+A_{n+1}+1,-K+R_n+A_n \\ R_n+A_{n+1}+s_n-s_{n+1}-\delta_n+2\end{matrix};1\right) \times \\
&\quad \frac{(K+s_n-s_{n+1}-A_n+A_{n+1}-\delta_n+1)!\,(s_n+R_n-1)!}{(K-R_n-A_n)!\,(R_n+A_{n+1}+s_n-s_{n+1}-\delta_n+1)!} \times \\
&\quad \frac{(K-A_{n+1}-1)!}{(K-R_{n+1}-A_{n+1})!\,(R_n-1)!\,(R_{n+1}-1)!} \\
&= \sum_{A_{e_1},\ldots,A_{e_{n-1}}} \sum_{A_{n+1}=0}^{\min(s_{n+1},K)-1} C\left(A_{e_1},\ldots,A_{e_{n-1}}\right) \times \\
&\quad \frac{(A_{n+1}+s_n-s_{n+1}-\delta_n+K-A_n+2)^{(s_{n+1}-A_{n+1}-1)}}{(R_n+A_{n+1}+s_n-s_{n+1}-\delta_n+2)^{(s_{n+1}-A_{n+1}-1)}} \times \\
&\quad \frac{(K+s_n-s_{n+1}-A_n+A_{n+1}-\delta_n+1)!\,(s_n+R_n-1)!}{(K-R_n-A_n)!\,(R_n+A_{n+1}+s_n-s_{n+1}-\delta_n+1)!} \times \\
&\quad \frac{(K-A_{n+1}-1)!}{(K-R_{n+1}-A_{n+1})!\,(R_n-1)!\,(R_{n+1}-1)!} \\
&= \sum_{A_{e_1},\ldots,A_{e_{n-1}}} \sum_{A_{n+1}=0}^{\min(s_{n+1},K)-1} C\left(A_{e_1},\ldots,A_{e_{n-1}}\right) \times \\
&\quad \frac{(K+s_n-A_n-\delta_n)!\,(R_n+s_n-1)!\,(K-A_{n+1}-1)!}{(R_n-1)!\,(K-R_n-A_n)!\,(R_n+s_n-\delta_n)!\,(K-R_{n+1}-A_{n+1})!\,(R_{n+1}-1)!}
\end{aligned}$$

where we have used the Chu-Vandermonde identity introduced in Proposition 1.3. Finally, note that since $\delta_{n+1} = 1$, we have

$$\frac{(K + s_{n+1} - A_{n+1} - \delta_{n+1})!\,(R_{n+1} + s_{n+1} - 1)!}{(K + s_{n+1} - A_{n+1} - 1)!\,(R_{n+1} + s_{n+1} - \delta_{n+1})!} = 1$$

Multiplying this by the formula we obtained for $v^{(\mathbf{s})}_{n+1,K;\mathbf{R}}$ and expanding $C\left(A_{e_1}, \ldots, A_{e_{n-1}}\right)$, we obtain

$$\begin{aligned} v^{(\mathbf{s})}_{n+1,K;\mathbf{R}} &= \sum_{A_{e_1}=0}^{\min\left(s_{e_1},K\right)-1} \cdots \sum_{A_{e_n}=0}^{\min(s_{e_n},K)-1} \left[\prod_{j=1}^{n} \frac{\left(K - A_{e_j} - 1\right)!}{\left(K + s_{e_j} - A_{e_j} - 1\right)!}\times\right. \\ &\qquad \left.\prod_{i=1}^{n+1} \frac{(K + s_i - A_i - \delta_i)!\,(R_i + s_i - 1)!}{(R_i - 1)!\,(K - R_i - A_i)!\,(R_i + s_i - \delta_i)!}\right] \end{aligned}$$

which proves our induction as desired. □

To remove the upper bounds of $K - 1$ in Theorem 6.9, we will for each edge e of the support graph of $\mathbf{s}$, assign a vertex v that is incident to e. This will allow us to regroup the factorial terms in $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$, which will allow us to rewrite the expression with rising factorials.

Proposition 6.10. *Let $T = (V, E)$ be a tree on n vertices, and x be a fixed vertex in V. Then, there exists a bijection $\rho\colon E \to V \setminus \{x\}$ such that for $e \in E$, $\rho(e)$ is a vertex incident to e.*

Proof. We prove this by induction. The proposition trivially holds for $n = 1$. Suppose that for some $n > 1$, the result holds for trees with $n - 1$ vertices. Since trees with more than one vertex have at least 2 leaves, let v_0 be a leaf of T distinct from x, and e_0 be the edge incident to v_0. Then, by deleting v_0 and e_0, we obtain a tree $T' = (V', E')$ with $n - 1$ vertices, one of which is x. Hence, our inductive hypothesis gives us a bijection $\rho'\colon E' \to V' \setminus \{x\}$ such that for $e \in E'$, $\rho'(e)$ is a vertex incident to e in T'. From this, we can define $\rho\colon E \to V \setminus \{x\}$ such that

$$\rho(e) = \begin{cases} v_0 & \text{if } e = e_0 \\ \rho'(e) & \text{otherwise} \end{cases}$$

For each $e \in E$, $\rho(e)$ is a vertex incident to e. This is because either $e = e_0$, which is incident to v_0 by construction, or $e \in E'$, in which case the result follows from T' being a subtree of T. Furthermore, as the vertices of V' are distinct from v_0, ρ is an injective function. The fact that $|E| = |V \setminus \{x\}|$ shows that ρ is bijective, as desired. $\square$

Note that in Proposition 6.10, we can let $V = [n]$, $E = \{e_1, \ldots, e_{n-1}\}$, and $x = n$. Furthermore, we can label the edges in E such that for $1 \le j \le n-1$, $\rho(e_j) = j$. By pairing off the factorial terms in $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ involving the edge e_j and terms involving its incident vertex j, we have the following corollary.

Corollary 6.11. *Let $n, K \ge 1$, $\mathbf{s} \ge \mathbf{0}$, and $\mathbf{R} \ge \mathbf{1}$. Suppose that the support graph G of $\mathbf{s}$ is a tree with edges $e_1, \ldots, e_{n-1}$, such that e_j is incident with vertex j in G for $1 \le j \le n-1$. Then,*

$$
\begin{aligned}
v^{(\mathbf{s})}_{n,K;\mathbf{R}} \;=\; & \prod_{i=1}^{n} \frac{(R_i - 1 + \sum_{k\sim i} s_{i,k})!}{(R_i-1)!\,(R_i + \sum_{k\sim i}(s_{i,k}-1))!} \times \\
& \sum_{A_{e_1}=0}^{s_{e_1}-1} \cdots \sum_{A_{e_{n-1}}=0}^{s_{e_{n-1}}-1} \left(K - R_n - \sum_{k\sim n} A_{n,k} + 1\right)^{\left(\sum_{k\sim n}(s_{n,k}-1)+R_n\right)} \times \\
& \prod_{j=1}^{n-1} \left[\left(K - R_j - \sum_{k\sim j} A_{j,k} + 1\right)^{\left(R_j + \sum_{k\sim j} A_{j,k} - A_{e_j} - 1\right)} \times \right. \\
& \left. \left(K + s_{e_j} - A_{e_j}\right)^{\left(\sum_{k\sim j}(s_{j,k} - A_{j,k} - 1) - s_{e_j} + A_{e_j} + 1\right)}\right]
\end{aligned}
\tag{6.1}
$$

where for each edge $e_j = \{j, \ell\}$ in G, the summation variable A_{e_j} can be equivalently written as $A_{j,\ell}$ and $A_{\ell,j}$. As in Theorem 6.9, the sum $\sum_{k\sim j}$ is over all indices k that are adjacent to j in the support graph of $\mathbf{s}$. Furthermore, $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ as expressed in this corollary is a polynomial in K.

Note that we have dropped the upper bounds of $K-1$ from each of the sums in the corollary. However, the bounds of $s_{e_j} - 1$ are vital, and cannot be removed.

For example, suppose $n = 3$ and $s_{1,2} = 0$. Then, we can let $e_1 = \{1, 3\}$ and $e_2 = \{2, 3\}$, so the formula for $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ can be written as

$$\begin{aligned} v^{(\mathbf{s})}_{n,K;\mathbf{R}} &= \frac{(R_1+s_{1,2}+s_{1,3}-1)!}{(R_1-1)!\,(R_2-1)!\,(R_3-1)!\,(R_1+s_{1,2}+s_{1,3}-2)!} \times \\ &\sum_{A_{1,3}=0}^{s_{1,3}-1}\sum_{A_{2,3}=0}^{s_{2,3}-1} (K-R_3-A_{1,3}-A_{2,3}+1)^{(R_3+s_{1,3}+s_{2,3}-2)} \times \\ &\Big[(K-R_1-A_{1,3}+1)^{(R_1-1)}\,(K-R_2-A_{2,3}+1)^{(R_2-1)}\Big] \end{aligned}$$

Alternatively, suppose $n=3$ and $s_{1,3}=0$. Then, we can let $e_1=\{1,2\}$ and $e_2=\{2,3\}$, so the formula for $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ can be written as

$$\begin{aligned} v^{(\mathbf{s})}_{n,K;\mathbf{R}} &= \prod_{i=1}^{n} \frac{(R_1+s_{1,2}+s_{1,3}-1)!}{(R_1-1)!\,(R_2-1)!\,(R_3-1)!\,(R_1+s_{1,2}+s_{1,3}-2)!} \times \\ &\sum_{A_{1,2}=0}^{s_{1,2}-1}\sum_{A_{2,3}=0}^{s_{2,3}-1} (K-R_3-A_{2,3}+1)^{(R_3+s_{2,3}-1)} \times \\ &\Big[(K-R_1-A_{1,2}+1)^{(R_1-1)}\,(K-R_2-A_{1,2}-A_{2,3}+1)^{(R_2+A_{1,2}-1)} \times \\ &(K+s_{2,3}-A_{2,3})^{(s_{1,2}-A_{1,2}-1)}\Big] \end{aligned}$$

In both instances, rising factorials of the form $x^{(0)}$ are omitted for clarity. Furthermore, these are effectively equivalent expressions, the only difference being the labelling of the vertices in the support graph of $\mathbf{s}$.

Proof. First, we need to show that the expression in Corollary 6.11 is well defined. For that, we need to show that the factorial terms in the numerator are non-negative, and that the rising factorials each have a non-negative number of terms. That is, for each rising factorial $x^{(y)}$, we need to show that $y \geq 0$. Observe that for $1 \leq i,k \leq n$, we have $R_i \geq 1$ and $s_{i,k} \geq 0$. Together, this gives $R_i - 1 + \sum_{k\sim i} s_{i,k} \geq 0$, so the factorials in the numerator are well defined. Then, for each $s_{i,k}$ that appears in the sum $\sum_{k\sim n} s_{n,k}$, we have $s_{i,k} \geq 1$, as we are only summing over terms $s_{i,k}$ where $\{i,k\}$ is an edge of the support graph of $\mathbf{s}$. This gives $\sum_{k\sim n}(s_{n,k}-1) + R_n \geq 0$. By our convention in labelling the edges, each edge e_j is incident to the vertex j, so can we let $e_j = \{j,\ell\}$. This means that $A_{j,\ell}$ appears in the sum $\sum_{k\sim j} A_{j,k}$, so $\sum_{k\sim j} A_{j,k} - A_{e_j} = \sum_{\substack{k\sim j\\ k\neq \ell}} A_{j,k} \geq 0$, which gives $R_j + \sum_{k\sim j} A_{j,k} - A_{e_j} - 1 \geq 0$. Similarly, $s_{j,\ell}$ appears in the sum $\sum_{k\sim j} s_{j,k}$, so $\sum_{k\sim j}(s_{j,k}-A_{j,k}-1)-s_{e_j}+A_{e_j}+1=\sum_{\substack{k\sim j\\ k\neq\ell}}(s_{j,k}-A_{j,k}-1)\geq 0$, as the summation bounds of the expression gives $A_{e_j}\leq s_{e_j}-1$ for all edges e_j in the support graph. This shows that the rising factorials all have a non-negative number of terms in them, so the entire expression is well defined.

Next, we need to show that the expression in Corollary 6.11 gives the same values as the expression in Theorem 6.9. Recall that we have assumed that each edge e_j is incident to vertex j in the support graph of $\mathbf{s}$. Therefore, we can rearrange the expression in Theorem 6.9 to

$$\begin{aligned} v_{n,K;\mathbf{R}}^{(\mathbf{s})} &= \prod_{i=1}^{n}\frac{(R_i-1+\sum_{k\sim i}s_{i,k})!}{(R_i-1)!\,(R_i+\sum_{k\sim i}(s_{i,k}-1))!}\times \\ &\sum_{A_{e_1}=0}^{\min(s_{e_1},K)-1}\cdots\sum_{A_{e_{n-1}}=0}^{\min(s_{e_{n-1}},K)-1}\frac{(K+\sum_{k\sim n}(s_{n,k}-A_{n,k}-1))!}{(K-R_n-\sum_{k\sim n}A_{n,k})!}\times \\ &\prod_{j=1}^{n-1}\left[\frac{(K-A_{e_j}-1)!}{\left(K-R_j-\sum_{k\sim j}A_{j,k}\right)!}\cdot\frac{\left(K+\sum_{k\sim j}(s_{j,k}-A_{j,k}-1)\right)!}{(K+s_{e_j}-A_{e_j}-1)!}\right] \end{aligned} \tag{6.2}$$

Furthermore, recall from Chapter 3 that for integers x and y such that $x\geq 0$, we have $\frac{x!}{(x-y)!}=(x-y+1)^{(y)}$. Then, observe that the summation bounds implies that $0\leq A_{e_j}\leq K-1$ and $0\leq A_{e_j}\leq s_{e_j}-1$, so in rows 2 and 3 of (6.2), the factorials in the numerator are non-negative. Therefore, we can use this fact to convert the ratios of factorials into rising factorials. Doing so shows that the summation terms in (6.1) and (6.2) are equal if $0\leq A_{e_j}\leq\min(s_{e_1},K)-1$ holds for all edges e_j. As both expressions contain the bounds $0\leq A_{e_j}\leq s_{e_j}-1$, it remains to show that the summation term in (6.1) is equal to zero if $A_{e_j}\geq K$ for some edge e_j.

Suppose $0\leq A_{e_\ell}\leq s_{e_\ell}-1$ holds for all edges e_ℓ, but there exists some edge e_j such that $A_{e_j}\geq K$. Let $G'=(V',E')$ be the graph on n vertices, such that $\{i,k\}\in E'$ if $A_{i,k}$ is defined and $A_{i,k}\geq K$. As $A_{i,k}$ is defined if and only if $s_{i,k}>0$, G' is a subgraph of G, so G' is a forest. Therefore, each component of G' must have one more vertex than the number of edges in the component. Let $\overline{V}$ be the set of vertices in G' with degree at least one, and $\widehat{V}$ be the set of vertices such that for $v\in\widehat{V}$, we have $e_v\in E'$. As we have assumed that $A_{e_j}\geq K$, $\widehat{V}$ is non-empty, so $\left|\overline{V}\right|>\left|\widehat{V}\right|$. Let $\ell\in\overline{V}\backslash\widehat{V}$. Then, ℓ is incident to some edge

$\{j,\ell\} \in G'$, but $e_\ell \notin G'$. If $\ell = n$, then

$$\begin{aligned}\left(K - R_n - \sum_{k\sim n} A_{n,k} + 1\right)^{\left(\sum_{k\sim n}\left(s_{n,k}-1\right)+R_n\right)} &= \frac{\left(K + \sum_{k\sim n}\left(s_{n,k} - A_{n,k} - 1\right)\right)!}{\left(K - R_n - \sum_{k\sim n} A_{n,k}\right)!} \\ &= 0\end{aligned}$$

as $K + \sum_{k\sim n}\left(s_{n,k} - A_{n,k} - 1\right) \geq K > 0$ implies that the factorial in the numerator is non-negative, while $K - \sum_{k\sim n} A_{n,k} \leq K - A_{n,\ell} \leq 0$ and $R_n \geq 1$ implies that the denominator is the factorial of a negative integer. Otherwise, we have $\ell \neq n$, which yields

$$\begin{aligned}\left(K - R_\ell - \sum_{k\sim \ell} A_{\ell,k} + 1\right)^{\left(R_\ell + \sum_{k\sim\ell} A_{\ell,k} - A_{e_\ell} - 1\right)} &= \frac{\left(K - A_{e_\ell} - 1\right)!}{\left(K - R_\ell - \sum_{k\sim\ell} A_{\ell,k}\right)!} \\ &= 0\end{aligned}$$

as $e_\ell \notin G'$ implies that $0 \leq e_\ell \leq K - 1$, which means that the factorial in the numerator is non-negative. Since $\{j,\ell\} \in G'$, we have $K - \sum_{k\sim\ell} A_{\ell,k} \leq K - A_{\ell,j} \leq 0$ and $R_\ell \geq 1$, so the denominator is the factorial of a negative integer. In both cases, at least one of the rising factorial is zero within the summation term, so the entire term is zero if $A_{e_j} \geq K$, as desired.

Note that the numbers of terms in the rising factorials are independent of K, so each summation term in (6.1) can be written as a polynomial of K. Furthermore, as the number of summation terms is bounded by the s_{e_j}'s, the number of terms is finite and independent of K. Therefore, the entire sum is a polynomial in K. Finally, the factorials outside the sum are independent of K, so the entire expression for $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ as written in this corollary is a polynomial in K, as desired. $\square$

With this, we have obtained an expression for $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ that is a polynomial in K for all $\mathbf{R} \geq \mathbf{1}$, if the support graph of $\mathbf{s}$ is a tree. We can then substitute this into Corollary 6.5 to obtain a polynomial expression for $f_{n,K}^{(\mathbf{q};\mathbf{s})}$ by Theorem 3.7. Then, using Fact 2.4, we can substitute $K = x$ into the expression for $f_{n,K}^{(\mathbf{q};\mathbf{s})}$ to obtain $A_n^{(\mathbf{q};\mathbf{s})}(x)$. This solves the problem we have laid out in Section 2.1, for the cases when the support graph of $\mathbf{s}$ is a tree. Explicit computations of $A_n^{(\mathbf{q};\mathbf{s})}(x)$ for small values of n, $\mathbf{q}$, and $\mathbf{s}$ can be found in B.

Chapter 7

Applications and Conclusion

7.1 Reduction of Main Result to $n = 1, 2$

In this section, we will show how our results generalize the formulas of Harer and Zagier [19] for $n = 1$, and Goulden and Slofstra [18] for $n = 2$, introduced in Section 2.3 and Section 2.4 respectively. To achieve this, we combine the results of Corollary 6.5 and Theorem 6.9 to obtain an expression for $f_{n,K}^{(\mathbf{q};\mathbf{s})}$, and do the necessary algebraic manipulations to transform the expression into a form we desire. By Fact 2.4, we can substitute $K = x$ into the expression for $f_{n,K}^{(\mathbf{q};\mathbf{s})}$ to obtain $A_n^{(\mathbf{q};\mathbf{s})}(x)$, and compare it with the formulas presented in those two papers to see that they are equal.

In the case $n = 1$ of our main problem, there are no mixed pairs. Thus, vertical arrays in $\mathcal{VA}_{1,K;R_1}^{(\mathbf{s})}$ contain K columns, zero vertices, and R_1 marked cells, arbitrary placed. By direct computation, we see that $v_{1,K;R_1}^{(\mathbf{s})} = \binom{K}{R_1}$. For simplicity, we let $q = q_1$ and $t = t_1$. Substituting this into Corollary 6.5, and using the fact that $(2q-1)!! = \frac{(2q)!}{2^q q!}$, we have

$$
\begin{aligned}
f_{n,K}^{(\mathbf{q};\mathbf{s})} &= \sum_{t=0}^{q} \frac{(2q)!}{2^t t!\,(q-t)!} \cdot \binom{K}{q-t+1} \\
&= (2q-1)!! \sum_{t=0}^{q} \frac{q!}{2^{t-q} t!\,(q-t)!} \cdot \binom{K}{q-t+1} \\
&= (2q-1)!! \sum_{k=0}^{q} 2^k \frac{q!}{k!\,(q-k)!} \cdot \binom{K}{k+1} \\
&= (2q-1)!! \sum_{k\geq 1} 2^{k-1} \binom{q}{k-1}\binom{K}{k}
\end{aligned}
$$

where we substitute $t = q - k$ to reverse the sum in line 3. To obtain the final result, we shift the summation index by 1, and note that for $k \geq q + 1$, the summand is zero. This allows us to remove the upper bound on the sum. By substituting in $K = x$, we see that $A_n^{(\mathbf{q};\mathbf{s})}(x)$ is the same as the Harer-Zagier formula given in Theorem 2.10, as desired.

In the case $n = 2$, we show that the generating series $A_n^{(\mathbf{q};\mathbf{s})}(x)$ we have computed is equivalent to that of Goulden and Slofstra by using the ${}_3F_2$ identity described in Theorem 1.5. The technique for proving this is motivated by comparing the definitions of paired arrays between the two papers. In particular, they have included in their paired arrays a non-empty condition. As the matrix $\mathbf{s}$ contains only one entry, we will simply denote it as s for convenience.

Definition 7.1. A paired array α satisfies the *non-empty condition* if each column of α contains at least 1 object. We denote the set of proper vertical arrays that satisfy the non-empty condition as $\mathcal{NVA}_{n,K;\mathbf{R}}^{(\mathbf{s})}$, and we let $h_{n,K;\mathbf{R}}^{(\mathbf{s})} = \left|\mathcal{NVA}_{n,K;\mathbf{R}}^{(\mathbf{s})}\right|$. To mirror our definitions for $m_{n,K;\mathbf{R}}^{(\mathbf{q};\mathbf{s})}$ and $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$, we extend our definition of $h_{n,K;\mathbf{R}}^{(\mathbf{s})}$ to all $\mathbf{R} \geq \mathbf{1}$ by letting $h_{n,K;\mathbf{R}}^{(\mathbf{s})} = 0$ if $R_i > K$ for some $1 \leq i \leq n$.

Note that vertical arrays are paired arrays with $\mathbf{q} = \mathbf{0}$, so the definition of $\mathcal{NVA}_{n,K;\mathbf{R}}^{(\mathbf{s})}$ is consistent with the non-empty condition. Now, there is a simple relation between the number of proper vertical arrays satisfying the non-empty condition, and the number of proper vertical arrays in general, given by the following lemma.

Lemma 7.2. *Let $n, K \geq 1$, $\mathbf{s} \geq \mathbf{0}$, and $\mathbf{R} \in [K]^n$. We have*

$$v^{(\mathbf{s})}_{n,K;\mathbf{R}} = \sum_{F=1}^{K} \binom{K}{F} h^{(\mathbf{s})}_{n,F;\mathbf{R}}$$

Proof. To prove this result, we will provide a mapping

$$\zeta\colon \mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}} \quad\rightarrow \bigcup_{F=\max(R_1,\ldots,R_n)}^{K} [K;F] \times \mathcal{NVA}^{(\mathbf{s})}_{n,F;\mathbf{R}}$$

and show that this mapping is a bijection. The idea behind this proof is to remove all columns without vertices, while keeping track of the position of those columns.

Let $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$ be a proper vertical array, $\mathcal{F}$ be the set of columns of α that contain at least one object, and $|\mathcal{F}| = F$. As the boxes used to mark cells are considered objects, we see that $\mathcal{F} \subseteq [K;F]$, where $R_i \leq F \leq K$ holds for $1 \leq i \leq n$. We can permute the columns of $\mathcal{F}$ so that they are the first F columns of the array, while keeping their relative order with each other. This gives us a vertical array, which we denote α'. As discussed back in Section 3.1, permuting the columns preserves the balance and forest conditions, so α' is a proper vertical array. Then, we can simply delete the empty columns. As these columns do not contain vertices or marked cells, they do not affect the balance or the forest conditions. This results in a vertical array β with F columns and R_i marked cells in row i, so $\beta \in \mathcal{NVA}^{(\mathbf{s})}_{n,F;\mathbf{R}}$, as desired.

Conversely, given $\max(R_1, \ldots, R_n) \leq F \leq K$, $\mathcal{F} \subseteq [K;F]$, and $\beta \in \mathcal{NVA}^{(\mathbf{s})}_{n,F;\mathbf{R}}$, we can reconstruct $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$ by reversing the decomposition. First, we add $K - F$ empty columns to the right of the existing columns, giving us a vertical array α' with K columns and R_i marked cells in row i. As the added columns do not contain any vertices or marked cells, both the forest and balance conditions are preserved. Therefore α' is a proper vertical array. To recover α, we permute the columns of α' so that the F non-empty columns of α' are placed in the columns $\mathcal{F}$, doing it in a way that preserves their relative order. The remaining $K - F$ columns, occupying the columns $\mathcal{K} \backslash \mathcal{F}$, are completely empty, so this procedure is unambiguous. As permuting the columns of a paired array preserves the forest and balance conditions, this gives us a proper vertical array, which we denote α. As α has K columns and R_i marked cells in row i, we have $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$, as desired.

Note that each step of the converse simply reverses the step done in the forward direction. Therefore, the function ζ as described is a bijection. To obtain the formula in the lemma, we take the cardinality of both sides of ζ. We can then change the lower bound of the summation to $F = 1$, as $h^{(\mathbf{s})}_{n,F;\mathbf{R}} = 0$ if $R_i > F$ by convention. Doing so gives us the formula we desire. □

Using this lemma, we can derive the values of $v^{(s)}_{2,K;\mathbf{R}}$ given the values of $h^{(s)}_{2,F;\mathbf{R}}$. In Theorem 5.2 of Goulden and Slofstra, they have computed a formula for $h^{(s)}_{2,F;\mathbf{R}}$. Using our notation, their formula can be stated as

$$\begin{aligned} h^{(s)}_{2,F;\mathbf{R}} &= \frac{(s+R_1-1)!\,(s+R_2-1)!}{(s+R_1+R_2-2)!}\binom{s+R_1+R_2-2}{F-1} \times \\ &\quad \left[\binom{F-1}{R_1-1}\binom{F-1}{R_2-1} - \binom{F-1}{s+R_1-1}\binom{F-1}{s+R_2-1}\right] \\ &= \frac{(s+R_1-1)!\,(s+R_2-1)!\,(F-1)!}{(s+R_1+R_2-F-1)!\,(R_1-1)!\,(R_2-1)!\,(F-R_1)!\,(F-R_2)!} - \\ &\quad \frac{(F-1)!}{(s+R_1+R_2-F-1)!\,(F-s-R_1)!\,(F-s-R_2)!} \\ &= h_1 - h_2 \end{aligned} \tag{7.1}$$

where the variables i and j in their paper are given by $R_1 - 1$ and $R_2 - 1$, respectively. Note that this formula holds for all positive integers s, F, R_1, and R_2, since the binomial coefficients inside the brackets are zero if $R_1 > F$ or $R_2 > F$. Equivalently, this follows from having $(F - R_i)!$ and $(F - s - R_i)!$ in the denominator of the two terms, for $i = 1, 2$. For convenience, we let the two terms of the formula be called h_1 and h_2. Combining this formula with Lemma 7.2, we obtain

$$v^{(s)}_{2,K;\mathbf{R}} = \sum_{F=1}^{K}\binom{K}{F}h_1 - \sum_{F=1}^{K}\binom{K}{F}h_2$$

Now, if $R_i > K$ for some $i = 1, 2$, then $R_i > F$ for $1 \le F \le K$. Hence, we have $v^{(s)}_{2,K;\mathbf{R}} = 0$, which matches our formula for $v^{(s)}_{2,K;\mathbf{R}}$ in Theorem 6.9. Otherwise, we have $1 \le R_i \le K$, so we can use the ${}_3F_2$ identity in Theorem 1.5 on both sums separately. By substituting in

$A = K - F$ to reverse the first sum, we have

$$\begin{aligned}\sum_{F=1}^{K}\binom{K}{F}h_1 &= \sum_{A=0}^{K-R_1}\frac{(s+R_1-1)!\,(s+R_2-1)!}{(R_1-1)!\,(R_2-1)!\,(s+R_1+R_2-K+A-1)!}\times\\ &\quad\frac{(K-A-1)!K!}{(K-R_1-A)!\,(K-R_2-A)!\,(K-A)!A!}\\ &= \frac{1}{(R_1-1)!\,(R_2-1)!}\cdot {}_3F_2\left({-K+R_1,-K,-K+R_2 \atop s+R_1+R_2-K,-K+1};1\right)\times\\ &\quad\frac{(s+R_1-1)!\,(s+R_2-1)!\,(K-1)!}{(s+R_1+R_2-K-1)!\,(K-R_1)!\,(K-R_2)!}\end{aligned}$$

Note that for $A > K - R_1$, the summand is zero because of the term $(K - R_1 - A)!$ in the denominator. This allows us to lower the upper bound of the sum to $K - R_1$, which we can write as a terminating ${}_3F_2$, with one of the parameters matching the upper bound. Using the ${}_3F_2$ identity in Theorem 1.5, we have

$$\begin{aligned}\sum_{F=1}^{K}\binom{K}{F}h_1 &= \frac{(s+R_1)^{(K-R_1)}}{(s+R_1+R_2-K)^{(K-R_1)}}\cdot {}_3F_2\left({-K+R_1,1,-K+R_2 \atop 1-K-s,-K+1};1\right)\times\\ &\quad\frac{(s+R_1-1)!\,(s+R_2-1)!\,(K-1)!}{(s+R_1+R_2-K-1)!\,(K-R_1)!\,(K-R_2)!\,(R_1-1)!\,(R_2-1)!}\\ &= \sum_{A=0}^{K-R_1}\frac{(s+K-A-1)!\,(K-A-1)!}{(K-R_1-A)!\,(K-R_2-A)!\,(R_1-1)!\,(R_2-1)!}\end{aligned} \tag{7.2}$$

By applying the substitution $A = K - F$ to the second sum, we have

$$\begin{aligned}\sum_{F=1}^{K}\binom{K}{F}h_2 &= \sum_{A=0}^{K-1}\frac{(K-A-1)!K!}{(s+R_1+R_2-K+A-1)!\,(K-A)!A!}\times\\ &\quad\frac{1}{(K-s-R_1-A)!\,(K-s-R_2-A)!}\end{aligned}$$

To evaluate this sum, we separate it into two cases. If $s > K - R_1$, then $(K - s - R_1 - A)!$ in the denominator forces each summation term to be zero. Hence, the entire sum is zero. Otherwise, $(K - s - R_1 - A)!$ allows us to lower the upper bound of the sum to $K - s - R_1$. This means that we can write the sum as a terminating ${}_3F_2$, with one of the parameters matching the upper bound. Using the same ${}_3F_2$ identity on the sum, we obtain

$$\begin{aligned}
\sum_{F=1}^{K}\binom{K}{F}h_2 &= {}_3F_2\left(\begin{matrix}-K+s+R_1,-K,-K+s+R_2\\ s+R_1+R_2-K,-K+1\end{matrix};1\right)\times \\
&\quad \frac{(K-1)!}{(s+R_1+R_2-K-1)!\,(K-s-R_1)!\,(K-s-R_2)!} \\
&= \frac{R_1^{(K-R_1-s)}}{(s+R_1+R_2-K)^{(K-R_1-s)}}\cdot {}_3F_2\left(\begin{matrix}-K+s+R_1,1,-K+s+R_2\\ 1-K+s,-K+1\end{matrix};1\right)\times \\
&\quad \frac{(K-1)!}{(s+R_1+R_2-K-1)!\,(K-s-R_1)!\,(K-s-R_2)!} \\
&= \sum_{B=0}^{K-R_1-s}\frac{(K-s-B-1)!\,(K-B-1)!}{(R_1-1)!\,(R_2-1)!\,(K-s-R_1-B)!\,(K-s-R_2-B)!} \\
&= \sum_{A=s}^{K-R_1}\frac{(K-A-1)!\,(K+s-A-1)!}{(R_1-1)!\,(R_2-1)!\,(K-R_1-A)!\,(K-R_2-A)!} \qquad (7.3)
\end{aligned}$$

where in the last line we shift the summation index up by letting $A=B+s$. Note that the summands in (7.2) and (7.3) are identical. Furthermore, the summation range of (7.3) is a subset of that of (7.2). In particular, we are summing over $0\le A\le s-1$ for $s\le K-R_1$, and summing over $0\le A\le K-R_1$ otherwise. Therefore, we can combine the two sums to obtain

$$\begin{aligned}
v_{2,K;\mathbf{R}}^{(s)} &= \sum_{F=1}^{K}\binom{K}{F}h_1-\sum_{F=1}^{K}\binom{K}{F}h_2 \\
&= \sum_{A=0}^{\min(s-1,K-R_1)}\frac{(K-A-1)!\,(K+s-A-1)!}{(R_1-1)!\,(R_2-1)!\,(K-R_1-A)!\,(K-R_2-A)!} \\
&= \sum_{A=0}^{\min(s,K)-1}\frac{(K-A-1)!\,(K+s-A-1)!}{(R_1-1)!\,(R_2-1)!\,(K-R_1-A)!\,(K-R_2-A)!}
\end{aligned}$$

where we have raised the upper bound from $K-R_1$ to $K-1$, as the term $(K-R_1-A)!$ in the denominator would make the summand zero for $A>K-R_1$. This matches with our formula for $v_{2,K;\mathbf{R}}^{(s)}$ given in Theorem 6.9, as desired.

This shows how to derive the formula for $v_{2,K;\mathbf{R}}^{(s)}$ given the formula of $h_{2,F;\mathbf{R}}^{(s)}$. To derive the formula for $h^{(s)}_{2,F;\mathbf{R}}$ from that of $v^{(s)}_{2,K;\mathbf{R}}$, we reverse the sequence of computations above to show that $v^{(s)}_{2,K;\mathbf{R}}$ can be written as $\sum_{F=1}^{K}\binom{K}{F}(h_1-h_2)$, then use induction on F with Lemma 7.2 to show that $h^{(s)}_{2,F;\mathbf{R}} = h_1 - h_2$ for all $F \geq 1$. Furthermore, this derivation can be seen as an algebraic proof that shows $v^{(s)}_{2,K;\mathbf{R}} = \sum_{F=1}^{K}\binom{K}{F} h^{(s)}_{2,K;\mathbf{R}}$, given the formulas of $v^{(s)}_{2,K;\mathbf{R}}$ and $h^{(s)}_{2,K;\mathbf{R}}$ as defined in Theorem 6.9 of this thesis and Theorem 5.2 of Goulden and Slofstra, respectively. With this proof, we can algebraically show that the generating series for the number of pairings in $\mathcal{A}^{(\mathbf{q};\mathbf{s})}_{n,L}$ in the paper of Goulden and Slofstra is equivalent to the generating series computed in this thesis. For clarity, we let the series in Goulden and Slofstra be denoted $B^{(\mathbf{q};s)}_2(x)$. By combining Proposition 3.2, Theorem 4.1, and Theorem 4.2 of their paper, we have

$$\begin{aligned}
B^{(\mathbf{q};s)}_2(x) &= \sum_{k\geq 1}\sum_{i\geq 0}\sum_{j\geq 0}\binom{x}{k}\binom{2q_1+s}{2i}(2i-1)!!\binom{2q_2+s}{2j}(2j-1)!! \times \\
&\quad \frac{(2q_1+s-2i)!}{(q_1+s-i)!}\cdot\frac{(2q_2+s-2j)!}{(q_2+s-j)!}\cdot h^{(s)}_{2,k;q_1-i+1,q_2-j+1} \\
&= \sum_{k\geq 1}\sum_{t_1=0}^{q_1}\sum_{t_2=0}^{q_2}\binom{x}{k}\frac{(2q_1+s)!\,(2q_2+s)!}{2^{t_1+t_2}t_1!t_2!\,(q_1+s-t_1)!\,(q_2+s-t_2)!}\cdot h^{(s)}_{2,k;q_1-t_1+1,q_2-t_2+1}
\end{aligned}$$

Again, we have written the generating series using the notation we have developed, where p and q in their paper are given by $2q_1+s$ and $2q_2+s$, respectively. Furthermore, we have lowered the summation bounds to q_1 and q_2, as their convention implies that $h^{(s)}_{2,k;R_1,R_2} = 0$ if $R_1 \leq 0$ or $R_2 \leq 0$. Substituting in $x = K$, we have

$$\begin{aligned}
B^{(\mathbf{q};s)}_2(K) &= \sum_{t_1=0}^{q_1}\sum_{t_2=0}^{q_2}\frac{(2q_1+s)!\,(2q_2+s)!}{2^{t_1+t_2}t_1!t_2!\,(q_1+s-t_1)!\,(q_2+s-t_2)!}\sum_{k=1}^{K}\binom{K}{k}h^{(s)}_{2,k;q_1-t_1+1,q_2-t_2+1} \\
&= \sum_{t_1=0}^{q_1}\sum_{t_2=0}^{q_2}\frac{(2q_1+s)!\,(2q_2+s)!}{2^{t_1+t_2}t_1!t_2!\,(q_1+s-t_1)!\,(q_2+s-t_2)!}\cdot v^{(s)}_{2,K;q_1-t_1+1,q_2-t_2+1} \\
&= f^{(q_1,q_2;s)}_{2,K}
\end{aligned}$$

by Corollary 6.5. Note that the conversion from $\sum_{F=1}^{K}\binom{K}{F} h^{(s)}_{2,K;\mathbf{R}}$ to $v^{(s)}_{2,K;\mathbf{R}}$ is done without using Lemma 7.2, and only using the ${}_3F_2$ identity in Theorem 1.5. By (2.2), we know that $A^{(\mathbf{q};s)}_2(K) = f^{(q_1,q_2;s)}_{2,K}$ for all $K \geq 1$ as well. Therefore, by Fact 2.4, we have that $A^{(\mathbf{q};s)}_2(x) = B^{(\mathbf{q};s)}_2(x)$ for all $q_1 \geq 0$, $q_2 \geq 0$, and $s \geq 1$, as desired.

Remark 7.3. This technique of transforming the sum with a ${}_3F_2$ identity does not appear to be generalizable when applied to Theorem 6.9. It can only be used on summation variables A_{e_j} such that e_j is an edge incident to a leaf vertex in the support graph of $\mathbf{s}$. If e_j is not incident to a leaf vertex in the support graph, then the hypergeometric series with respect to A_{e_j} is a ${}_{p+1}F_p$, where $p \geq 3$. In those cases, there are no known hypergeometric transformations that can be applied.

7.2 Further Reduction to the Goulden-Slofstra Formula

In this section, we will show a method of reducing the number of sums in the formula of Goulden and Slofstra using Pfaff's identity. We start by rewriting Theorem 2.12 using our notation, which gives the formula for $A_2^{(\mathbf{q};s)}(x)$ as

$$\begin{aligned}
A_2^{(\mathbf{q};s)}(x) &= \sum_{k=1}^{d+1}\sum_{t_1\geq 0}\sum_{t_2\geq 0}\frac{(2q_1+s)!\,(2q_2+s)!}{2^{t_1+t_2}t_1!t_2!\,(d-t_1-t_2)!}\cdot\binom{x}{k}\binom{d-t_1-t_2}{k-1}\times\\
&\quad\left[\binom{k-1}{q_1-t_1}\binom{k-1}{q_2-t_2}-\binom{k-1}{s+q_1-t_1}\binom{k-1}{s+q_2-t_2}\right]\\
&= \sum_{k=1}^{d+1}\sum_{t_1\geq 0}\sum_{t_2\geq 0}\frac{(2q_1+s)!\,(2q_2+s)!\,(k-1)!}{2^{t_1+t_2}t_1!t_2!\,(d-t_1-t_2-k+1)!}\cdot\binom{x}{k}\times\\
&\quad\frac{1}{(q_1-t_1)!\,(k-q_1+t_1-1)!\,(q_2-t_2)!\,(k-q_2+t_2-1)!}-\\
&\quad\sum_{k=1}^{d+1}\sum_{t_1\geq 0}\sum_{t_2\geq 0}\frac{(2q_1+s)!\,(2q_2+s)!\,(k-1)!}{2^{t_1+t_2}t_1!t_2!\,(d-t_1-t_2-k+1)!}\cdot\binom{x}{k}\times\\
&\quad\frac{1}{(s+q_1-t_1)!\,(k-s-q_1+t_1-1)!\,(s+q_2-t_2)!\,(k-s-q_2+t_2-1)!}\\
&= g_1-g_2
\end{aligned}$$

where we have $d = q_1 + q_2 + s$ as in the original theorem. Similar to (7.1), we denote the two terms of the formula by g_1 and g_2 for convenience. Note that we have removed the upper bounds for t_1 and t_2. We can justify this by showing that the summation terms can only be non-zero if both $t_1 \leq q_1$ and $t_2 \leq q_2$ hold. For g_1, the term $(q_1 - t_1)!$ in the denominator means that for the summation term to be non-zero, we have $t_1 \leq q_1$. Similarly, the terms in the denominator of g_2 imply that for the summation term to be non-zero, we have $d - t_1 - t_2 - k + 1 \geq 0$ and $k - s - q_2 + t_2 - 1 \geq 0$. Together, this also yields the bound $t_1 \leq q_1$. By changing the indices, the same arguments show that $t_2 \leq q_2$ as well. As q_1 and q_2 are both smaller than the bounds in the original paper, we can safely remove the upper bounds without changing the sum.

To reduce the number of sums in g_1 and g_2, we manipulate them separately with the same transforms. We first use Pfaff's identity to transform the sum involving t_1, then use the Chu-Vandermonde identity to eliminate t_2. Afterwards, we make the summation variables symmetric by making a substitution for k, before combining the results together. For reference, the identities used for this procedure can be found in Proposition 1.3 and Theorem 1.6.

By rewriting the t_1 sum of g_1 as a hypergeometric series and using Pfaff's identity, we have

$$
\begin{aligned}
g_1 &= \sum_{k=1}^{d+1} \sum_{t_2 \geq 0} \frac{1}{2^{t_2} t_2!} \cdot \binom{x}{k} {}_2F_1\left(\begin{matrix} -d + t_2 + k - 1, -q_1 \\ k - q_1 \end{matrix}; \frac{1}{2}\right) \times \\
&\qquad \frac{(2q_1 + s)!\,(2q_2 + s)!\,(k-1)!}{(d - t_2 - k + 1)! q_1!\,(k - q_1 - 1)!\,(q_2 - t_2)!\,(k - q_2 + t_2 - 1)!} \\
&= \sum_{k=1}^{d+1} \sum_{t_2 \geq 0} \frac{1}{2^{t_2} t_2!} \cdot \binom{x}{k} \left(1 - \frac{1}{2}\right)^{d - t_2 - k + 1} {}_2F_1\left(\begin{matrix} -d + t_2 + k - 1, k \\ k - q_1 \end{matrix}; -1\right) \times \\
&\qquad \frac{(2q_1 + s)!\,(2q_2 + s)!\,(k-1)!}{(d - t_2 - k + 1)! q_1!\,(k - q_1 - 1)!\,(q_2 - t_2)!\,(k - q_2 + t_2 - 1)!} \\
&= \sum_{k=1}^{d+1} \sum_{t_2 \geq 0} \sum_{t_1 \geq 0} \frac{1}{2^{d-k+1} t_1! t_2!\,(d - t_1 - t_2 - k + 1)!} \cdot \binom{x}{k} \times \\
&\qquad \frac{(2q_1 + s)!\,(2q_2 + s)!\,(k + t_1 - 1)!}{q_1!\,(k - q_1 + t_1 - 1)!\,(q_2 - t_2)!\,(k - q_2 + t_2 - 1)!}
\end{aligned}
$$

While there is no upper bound for t_1, the term $(d - t_1 - t_2 - k + 1)!$ in the denominator causes the sum to terminate. Furthermore, for the summation term to be non-zero, we must have $d - t_1 - t_2 - k + 1 \geq 0$ and $k - q_2 + t_2 - 1 \geq 0$ at the same time. Combining these inequalities together gives us $t_1 \leq q_1 + s$, which can be used as an upper bound for t_1. Next, we rewrite the t_2 sum as a hypergeometric series, and note that it satisfies the

Chu-Vandermonde identity. This yields,

$$\begin{aligned}
g_1 &= \sum_{k=1}^{d+1}\sum_{t_1=0}^{q_1+s} \frac{1}{2^{d-k+1}t_1!}\cdot\binom{x}{k}\,{}_2F_1\left(\begin{matrix}-q_2, -d+t_1+k-1\\ k-q_2\end{matrix};1\right)\times \\
&\quad \frac{(2q_1+s)!\,(2q_2+s)!\,(k+t_1-1)!}{(d-t_1-k+1)!q_1!\,(k-q_1+t_1-1)!q_2!\,(k-q_2-1)!} \\
&= \sum_{k=1}^{d+1}\sum_{t_1=0}^{q_1+s} \frac{(s+q_1-t_1+1)^{(q_2)}}{2^{d-k+1}t_1!\,(k-q_2)^{(q_2)}}\cdot\binom{x}{k}\times \\
&\quad \frac{(2q_1+s)!\,(2q_2+s)!\,(k+t_1-1)!}{(d-t_1-k+1)!q_1!\,(k-q_1+t_1-1)!q_2!\,(k-q_2-1)!} \\
&= \sum_{k=1}^{d+1}\sum_{t_1=0}^{q_1+s} \frac{(d-t_1)!}{2^{d-k+1}t_1!\,(d-t_1-k+1)!}\cdot\binom{x}{k}\times \\
&\quad \frac{(2q_1+s)!\,(2q_2+s)!\,(k+t_1-1)!}{q_1!\,(s+q_1-t_1)!\,(k-q_1+t_1-1)!q_2!\,(k-1)!}
\end{aligned}$$

Note that the term $(d-t_1-k+1)!$ in the denominator means that for $k > d-t_1+1$, the summation term is zero. Therefore, we can switch the two sums and lower the upper bound of k to $d-t_1+1$. Next, the terms $(k-q_1+t_1-1)!$ and $(k-1)!$ in the denominator means that for the summand to be non-zero, we have $k \geq \max\{q_1-t_1+1, 1\}$. Hence, we can change the lower bound of k to q_1-t_1+1. As $k+t_1-1 \geq q_1 \geq 0$ with this new lower bound, the factorial term in the numerator remains non-negative. After changing the bounds, we can reverse the sum with the substitution $k = d-t_1-t_2+1$. This gives us the formula

$$\begin{aligned}
g_1 &= \sum_{t_1=0}^{q_1+s}\sum_{t_2=0}^{q_2+s} \frac{(d-t_1)!\,(d-t_2)!\,(2q_1+s)!\,(2q_2+s)!}{2^{t_1+t_2}t_1!t_2!\,(d-t_1-t_2)!}\cdot\binom{x}{d-t_1-t_2+1}\times \\
&\quad \frac{1}{q_1!q_2!\,(s+q_1-t_1)!\,(s+q_2-t_2)!} \qquad (7.4)
\end{aligned}$$

which is symmetric between t_1 and t_2.

We now apply the same transformations to g_2. By rewriting the t_1 sum of g_2 as a hypergeometric series and using Pfaff's identity, we have

$$
\begin{aligned}
g_2 &= \sum_{k=1}^{d+1} \sum_{t_2 \geq 0} \frac{1}{2^{t_2} t_2!} \cdot \binom{x}{k} {}_2F_1\left(\begin{matrix} -d+t_2+k-1, -q_1-s \\ k-q_1-s \end{matrix}; \frac{1}{2}\right) \times \\
&\quad \frac{(2q_1+s)!\,(2q_2+s)!\,(k-1)!}{(d-t_2-k+1)!\,(q_1+s)!\,(k-q_1-s-1)!\,(q_2+s-t_2)!\,(k-q_2-s+t_2-1)!} \\
&= \sum_{k=1}^{d+1} \sum_{t_2 \geq 0} \frac{1}{2^{t_2} t_2!} \cdot \binom{x}{k} \left(1-\frac{1}{2}\right)^{d-t_2-k+1} {}_2F_1\left(\begin{matrix} -d+t_2+k-1, k \\ k-q_1-s \end{matrix}; -1\right) \times \\
&\quad \frac{(2q_1+s)!\,(2q_2+s)!\,(k-1)!}{(d-t_2-k+1)!\,(q_1+s)!\,(k-q_1-s-1)!\,(q_2+s-t_2)!\,(k-q_2-s+t_2-1)!} \\
&= \sum_{k=1}^{d+1} \sum_{t_2 \geq 0} \sum_{t_1 \geq 0} \frac{1}{2^{d-k+1} t_1! t_2!\,(d-t_1-t_2-k+1)!} \cdot \binom{x}{k} \times \\
&\quad \frac{(2q_1+s)!\,(2q_2+s)!\,(k+t_1-1)!}{(q_1+s)!\,(k-q_1-s+t_1-1)!\,(q_2+s-t_2)!\,(k-q_2-s+t_2-1)!}
\end{aligned}
$$

As with the case for g_1, the sum of t_1 terminates because of the term $(d-t_1-t_2-k+1)!$ in the denominator. Also, for the summation term to be non-zero, we must have $d-t_1-t_2-k+1 \geq 0$ and $k-q_2-s+t_2-1 \geq 0$ at the same time. Combining these inequalities together gives us $t_1 \leq q_1$, which can be used as an upper bound for t_1. Next, we rewrite the t_2 sum as a hypergeometric series, and note that it satisfies the Chu-Vandermonde identity. This yields,

$$
\begin{aligned}
g_2 &= \sum_{k=1}^{d+1}\sum_{t_1=0}^{q_1} \frac{1}{2^{d-k+1}t_1!}\cdot\binom{x}{k}{}_2F_1\left(\begin{matrix}-q_2-s,-d+t_1+k-1\\ k-q_2-s\end{matrix};1\right)\times \\
&\quad \frac{(2q_1+s)!\,(2q_2+s)!\,(k+t_1-1)!}{(d-t_1-k+1)!\,(q_1+s)!\,(k-q_1-s+t_1-1)!\,(q_2+s)!\,(k-q_2-s-1)!} \\
&= \sum_{k=1}^{d+1}\sum_{t_1=0}^{q_1} \frac{(q_1-t_1+1)^{(q_2+s)}}{2^{d-k+1}t_1!\,(k-q_2-s)^{(q_2+s)}}\cdot\binom{x}{k}\times \\
&\quad \frac{(2q_1+s)!\,(2q_2+s)!\,(k+t_1-1)!}{(d-t_1-k+1)!\,(q_1+s)!\,(k-q_1-s+t_1-1)!\,(q_2+s)!\,(k-q_2-s-1)!} \\
&= \sum_{k=1}^{d+1}\sum_{t_1=0}^{q_1} \frac{(d-t_1)!}{2^{d-k+1}t_1!\,(d-t_1-k+1)!}\cdot\binom{x}{k}\times \\
&\quad \frac{(2q_1+s)!\,(2q_2+s)!\,(k+t_1-1)!}{(q_1+s)!\,(q_1-t_1)!\,(k-q_1-s+t_1-1)!\,(q_2+s)!\,(k-1)!}
\end{aligned}
$$

Note that the term $(d-t_1-k+1)!$ in the denominator means that for $k > d-t_1+1$, the summation term is zero. Similarly, the term $(k-q_1-s+t_1-1)!$ in the denominator means for $k < q_1+s-t_1+1$, the summation term is also zero. Therefore, we can tighten the bounds of k to $q_1+s-t_1+1 \le k \le d-t_1+1$, as $d-t_1+1 \le d+1$ and $q_1+s-t_1+1 \ge 1$. After doing so, we can reverse the sum with the substitution $k = d-t_1-t_2+1$. This gives us the formula

$$
\begin{aligned}
g_2 &= \sum_{t_1=0}^{q_1}\sum_{t_2=0}^{q_2} \frac{(d-t_1)!\,(d-t_2)!\,(2q_1+s)!\,(2q_2+s)!}{2^{t_1+t_2}t_1!t_2!\,(d-t_1-t_2)!}\cdot\binom{x}{d-t_1-t_2+1}\times \\
&\quad \frac{1}{(q_1+s)!\,(q_2+s)!\,(q_1-t_1)!\,(q_2-t_2)!}
\end{aligned} \tag{7.5}
$$

which is again symmetric in t_1 and t_2.

As we have $(q_1-t_1)!$ and $(q_2-t_2)!$ in the denominator of g_2, we can actually increase the bounds of t_1 and t_2 to q_1+s and q_2+s without changing the sum, matching the bounds of g_1. Finally, we can put (7.4) and (7.5) together and obtain

$$\begin{aligned} A_2^{(\mathbf{q};s)}(x) &= g_1 - g_2 \\ &= \sum_{t_1=0}^{q_1+s}\sum_{t_2=0}^{q_2+s} \frac{(d-t_1)!\,(d-t_2)!\,(2q_1+s)!\,(2q_2+s)!}{2^{t_1+t_2}t_1!t_2!\,(d-t_1-t_2)!} \cdot \binom{x}{d-t_1-t_2+1} \times \\ &\quad \left[\frac{1}{q_1!q_2!\,(s+q_1-t_1)!\,(s+q_2-t_2)!} - \frac{1}{(q_1+s)!\,(q_2+s)!\,(q_1-t_1)!\,(q_2-t_2)!}\right] \end{aligned}$$

where $d = q_1 + q_2 + s$.

Remark 7.4. Note that the transformations used in this section cannot be directly applied to the formula of $A_n^{(\mathbf{q};\mathbf{s})}(x)$ derived from Theorem 6.9, as the variables R_i do not appear together in the same factorial terms. This means that applying a ${}_2F_1$ transform on a summation variable t_i will not have any effect on the type or the parameters of the hypergeometric series of another summation variable t_k. One potential method to remedy this is to use the techniques in Section 7.1. This involves picking an edge $e_j = \{i, k\}$ in the support graph of $\mathbf{s}$, such that i is a leaf vertex. This allows us to break up the sum of A_{e_j} into two parts by raising the upper bound of A_{e_j} from $\min\left(s_{e_j}, K\right) - 1$ to $K - 1$. Then, we can use one of the eighteen ${}_3F_2$ transforms on the summation variable A_{e_j}, in hopes of creating an expression that has the variables t_i and t_k in a common factorial term. However, for $n \geq 3$, the results of these transformations either fail to create a factorial term with both t_i and t_k, or change the type of the hypergeometric series of t_i or t_k into a ${}_3F_2$, with $x = \pm\frac{1}{2}$ as a parameter. This prevents us from using one of Kummer's 24 solutions. Other variations of this technique, such as applying hypergeometric transformations to t_i, t_k, and A_{e_j} in various orders, have also failed to create an expression that can be further simplified.

7.3 Enumeration of Vertical Arrays with s Non-Tree

In this section, we will describe a method of computing $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ for small values of n, K, $\mathbf{R}$, and $\mathbf{s}$, when the support graph of $\mathbf{s}$ is not a tree. This will allow for the computation of $A_n^{(\mathbf{q};\mathbf{s})}(x)$ for small values of n, $\mathbf{q}$, and $\mathbf{s}$. In general, this method of computation is more efficient than doing an exhaustive search over all potential vertical arrays, then counting the ones that satisfy the balance and forest conditions. Furthermore, for very small values of n and $\mathbf{s}$, we can derive a formula for $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ that holds for all $\mathbf{R} \geq \mathbf{1}$, and is a polynomial in K.

Definition 7.5. A paired array α satisfies the *full condition* if each column of α contains at least 1 vertex. We denote the set of proper vertical arrays that satisfy the full condition as $\mathcal{FVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$, and we let $g^{(\mathbf{s})}_{n,K;\mathbf{R}} = \left|\mathcal{FVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}\right|$. Again, to mirror our definitions for $m^{(\mathbf{q};\mathbf{s})}_{n,K;\mathbf{R}}$ and $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$, we extend our definition of $g^{(\mathbf{s})}_{n,K;\mathbf{R}}$ to all $\mathbf{R} \geq \mathbf{1}$ by letting $g^{(\mathbf{s})}_{n,K;\mathbf{R}} = 0$ if $R_i > K$ for some $1 \leq i \leq n$.

In contrast to the full condition for arrowed arrays in Definition 4.1, each cell must contain a vertex instead of an object. Consider a column j of α for $1 \leq j \leq K$. It must contain a mixed vertex in some cell (i, j) of that column, as α is a vertical array that satisfies the full condition. Since α also satisfies the balance condition, by Definition 3.3, α must contain a vertex pair $\{u, v\}$ such that u is in row i and v is in column j, but not row i. This means that every column of α must contain at least 2 vertices, which also implies that $g^{(\mathbf{s})}_{n,K;\mathbf{R}}$ can only be non-zero if $K \leq s$, where $s = \sum_{i<k} s_{i,k}$ is the total number of pairs in the array. Combining this with the fact that $1 \leq R_i \leq K$, we have that for fixed values of n and $\mathbf{s}$, there is a finite number of values for K and $\mathbf{R}$ such that $g^{(\mathbf{s})}_{n,K;\mathbf{R}}$ is non-zero.

Theorem 7.6. *Let $n, K \geq 1$, $\mathbf{s} \geq \mathbf{0}$, and $\mathbf{R} \in [K]^n$. We have*

$$v^{(\mathbf{s})}_{n,K;\mathbf{R}} = \sum_{F=1}^{s} \sum_{r_1=1}^{R_1} \cdots \sum_{r_n=1}^{R_n} \binom{K}{F} \binom{K-F}{R_1 - r_1} \cdots \binom{K-F}{R_n - r_n} g^{(\mathbf{s})}_{n,F;\mathbf{r}}$$

where $\mathbf{r} = (r_1, \ldots, r_n)$. Furthermore, for fixed values of $\mathbf{s}$ and $\mathbf{R}$, $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ is a polynomial in K.

Remark 7.7. Notice that for fixed values of $\mathbf{s}$ and $\mathbf{R}$, $g^{(\mathbf{s})}_{n,K;\mathbf{R}}$ is non-zero only for a finite number of values of K. Therefore, $g^{(\mathbf{s})}_{n,K;\mathbf{R}}$ cannot be a polynomial in K without being identically zero, which is not true in general.

Proof. The proof of this theorem is similar to that of Lemma 7.2. We will provide a mapping

$$\begin{aligned} \zeta\colon \mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}} \;\to\; & \bigcup_{F=1}^{K} \bigcup_{r_1=1}^{\min(R_1,F)} \cdots \bigcup_{r_n=1}^{\min(R_n,F)} \\ & [K;F] \times [K-F; R_1 - r_1] \times \cdots \times [K-F; R_n - r_n] \times \mathcal{FVA}^{(\mathbf{s})}_{n,F;\mathbf{r}} \end{aligned}$$

and show that this mapping is a bijection. Again, we remove all columns without vertices, while keeping track of the position of those columns. In addition, we need to keep track of the number and positions of the marked cells within the columns without vertices.

Let $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$ be a proper vertical array, $\mathcal{F}$ be the set of columns of α that contain at least one vertex, and $|\mathcal{F}| = F$. We see that $1 \leq F \leq K$, and $\mathcal{F} \subseteq [K;F]$. Then, we label the columns of $\mathcal{F}$ from left to right with $[F] = 1,\ldots,F$, and label the columns of $\mathcal{K}\backslash\mathcal{F}$ with $[K-F]' = 1',\ldots,(K-F)'$. We can permute the columns of α so that they are in the order $1,\ldots,F,1',\ldots,(K-F)'$, and call the resulting vertical array α'. As discussed back in Section 3.1, permuting the columns preserves the balance and forest conditions, so α' is a proper vertical array.

Now, let ψ'_i be the forest condition function for row i of α'. As all vertices of α' are in the first F columns, both the domain and range of ψ'_i are in $[F]$, so the functional digraph of ψ'_i has all its edges in $[F]$. This means that a column $j' \in [K-F]'$ is either an isolated root vertex in the functional digraph if cell (i,j') is marked, or does not appear at all if that cell is unmarked. In either case, the columns of $[K-F]'$ can be removed without violating the forest condition of row i. Furthermore, as these columns do not contain any vertices, removing them preserves the balance condition as well. Then, recall our overarching assumption that the support graph of $\mathbf{s}$ is connected. This means that row i must contain at least one vertex, in some column $j \in [F]$. Again using the fact that the edges of the functional digraph are in $[F]$, we have that the root vertex of the component containing j in the functional digraph of ψ'_i must also be in $[F]$. Consequently, there must be some r_i marked cells in the columns of $[F]$ in row i, with $1 \leq r_i \leq \min(R_i, F)$. Therefore, we can remove the columns $[K-F]'$ and be left with a proper vertical array β, where each column of β has at least one vertex. By letting $\mathbf{r} = (r_1,\ldots,r_n)$, we see that $\in \mathcal{FVA}^{(\mathbf{s})}_{n,F;\mathbf{r}}$. Finally, note that for $1 \leq i \leq n$, there are $R_i - r_i$ marked cells in the columns of $[K-F]'$, with no restrictions on how they are placed. Hence, we can represent them with a set $\mathcal{S}_i \subseteq [K-F; R_i - r_i]$, where $j \in \mathcal{S}_i$ if and only if cell (i,j') is marked in α'. This shows that given a vertical array α, we can determine the values of F and $\mathbf{r}$, then decompose α into the objects $\mathcal{F}$, β, and $\mathcal{S}_i$'s, as desired.

Conversely, given $1 \leq F \leq K$, $1 \leq r_i \leq \min(R_i, F)$, $\mathcal{F} \subseteq [K;F]$, $\mathcal{S}_i \subseteq [K-F; R_i - r_i]$, and $\beta \in \mathcal{FVA}^{(\mathbf{s})}_{n,F;\mathbf{r}}$, with $1 \leq i \leq n$ and $\mathbf{r} = (r_1,\ldots,r_n)$, we can reconstruct $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$ by reversing the decomposition. First, we label the columns of β with $[F] = 1,\ldots,F$, then add $K-F$ columns labelled $[K-F]' = 1',\ldots,(K-F)'$ to the right of the existing columns. Next, we mark the cells in the columns of $[K-F]'$ with the sets $\mathcal{S}_i$. For each i, we mark cell (i,j') if and only if $j \in \mathcal{S}_i$, marking $R_i - r_i$ cells in total. This gives us a vertical array α' with K columns and R_i marked cells in row i. As adding empty columns does not add any vertices, the balance condition is preserved. Similarly, the only change to the functional digraph ψ_i' of row i is adding isolated vertices, corresponding to the marked cells in $[K-F]'$. Therefore, the forest condition is also preserved, so α' is proper vertical array. To recover α, we let the set of all columns, denoted $\mathcal{K}$, be labelled with $1, \ldots, K$ from left to right, and note that $\mathcal{F}$ represents a subset of the columns of size F. This means that we can permute the columns of α' so that the columns of $[F]$ are in $\mathcal{F}$, and that the columns of $[K-F]'$ are in $\mathcal{K} \backslash \mathcal{F}$, preserving the relative order of both sets. As permuting the columns of a paired array preserves the forest and balance conditions, this gives us a proper vertical array α. So, we have $\alpha \in \mathcal{PVA}^{(\mathbf{s})}_{n,K;\mathbf{R}}$, as desired.

As with the proof of Lemma 7.2, each step of the converse simply reverses the step done in the forward direction. Therefore, the function ζ as described is a bijection. To obtain the formula in the theorem, we take the cardinality of both sides of ζ, then adjust the summation bounds. As $\binom{F}{K} = 0$ for $K > F$, we can raise the upper bound of F from K to s if $K < s$. Alternatively, we can also decrease the upper bound of F to s if $K > s$, as $g^{(\mathbf{s})}_{n,K;\mathbf{R}} = 0$ for $K > s$. Finally, we can remove the upper bounds of K from the summations of the r_i's, since we take $g^{(\mathbf{s})}_{n,F;\mathbf{r}} = 0$ if $r_i > F$ for some $1 \leq i \leq n$.

To show that $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ is a polynomial in K, recall from Section 1.2 that for integer $k \geq 0$, we have $\binom{n}{k} = \frac{(n-k+1)^{(k)}}{k!}$. As K only appears in the numerator of the binomial coefficients, they are each a polynomial in K. Therefore, each summation term is a polynomial in K, as the $g^{(\mathbf{s})}_{n,F;\mathbf{r}}$ are constants. As the number of summation terms is independent of K, the entire sum is also a polynomial in K, as desired. ☐

Next, we will manipulate the expression in Theorem 7.6, so that we can remove the upper bounds for r_i. This will allow us to expand the sum and write an expression for $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ that does not require special cases for small values of $\mathbf{R}$. Recall from Section 1.2 that for integer $k \geq 0$, we have $\binom{n}{k} = \frac{(n-k+1)^{(k)}}{k!}$. By viewing the binomial coefficient as a function of n and manipulating this expression, we obtain the following identities

$$\begin{aligned}\binom{n}{k} &= \frac{k+1}{n-k}\binom{n}{k+1} \\ &= \frac{n-k+1}{n+1}\binom{n+1}{k}\end{aligned}$$

Effectively, these identities multiply both the numerator and the denominator of the expression by the same term, so they introduce removable singularities for n. Therefore, for certain integer values of n, these expressions are undefined. However, we will generally simplify our expressions before evaluating them in our application, so these singularities are removed, and the expression remains valid. By using these identities to expand $\binom{K-F}{R_i-r_i}$, we have for $r_i \leq R_i$

$$\begin{aligned}\binom{K-F}{R_i-r_i} &= \frac{(R_i-r_i+1)^{(r_i-1)}}{(K-F-R_i+2)^{(r_i-1)}} \cdot \binom{K-F}{R_i-1} \\ &= \frac{(R_i-r_i+1)^{(r_i-1)}\,(K-F-R_i+2)^{(F-1)}}{(K-F-R_i+2)^{(r_i-1)}\,(K-F+1)^{(F-1)}} \cdot \binom{K-1}{R_i-1}\end{aligned} \tag{7.6}$$

If we are to treat K as a variable, then these identities hold for all integers $R_i \geq r_i > 0$, as we can write everything as rising factorials and simplify the expressions as necessary. Furthermore, by writing the binomial coefficients as rising factorials, we see that the second expression simplifies to the first, regardless of whether $R_i \geq r_i$. Now, for fixed integers R_i and r_i such that $r_i > R_i > 0$, the rising factorial $(R_i - r_r + 1)^{(r_i-1)}$ evaluates to zero. This agrees with the upper bound of r_i in the expression of Theorem 7.6, which allows us to remove that upper bound. Then, to make the sums finite, we can decrease the upper bound for each r_i to F, as $g^{(\mathbf{s})}_{n,F;\mathbf{r}} = 0$ if $r_i > F$. Doing so yields

$$\begin{aligned}v^{(\mathbf{s})}_{n,K;\mathbf{R}} &= \sum_{F=1}^{s}\sum_{r_1=1}^{F}\cdots\sum_{r_n=1}^{F} g^{(\mathbf{s})}_{n,F;\mathbf{r}} \cdot \binom{K}{F} \times \\ &\quad \prod_{i=1}^{n} \frac{(R_i-r_r+1)^{(r_i-1)}\,(K-F-R_i+2)^{(F-1)}}{(K-F-R_i+2)^{(r_i-1)}\,(K-F+1)^{(F-1)}} \cdot \binom{K-1}{R_i-1}\end{aligned} \tag{7.7}$$

Note that when evaluating this expression, we should first substitute in the values for $\mathbf{R}$, simplify, then substitute in the value of K. Alternatively, we can leave the simplified expression as a function in K, which by Theorem 7.6 is a polynomial. From there, we can substitute these expressions for $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ into Corollary 6.5 to obtain a generating series for $A^{(\mathbf{q};\mathbf{s})}_n(x)$, via the use of Fact 2.4.

Remark 7.8. Depending on the method of evaluation, the expression for $\binom{K-F}{R_i-r_i}$ in (7.6) can be non-zero for $r_i > R_i$. In particular, if we are to substitute in K before R_i, then we can have $(K-F-R_i+2)^{(r_i-1)} = 0$, which causes the expression to be undefined. However, for fixed values of r_i, K, and F, $\frac{(R_i-r_i+1)^{(r_i-1)}}{(K-F-R_i+2)^{(r_i-1)}}$ is a rational function in R_i that contains the factor R_i in both the numerator and denominator. As at most one term in each rising factorial can be zero, we can cancel out this factor, which causes the expression to evaluate to a finite number. Furthermore, we can also deduce that $K-F-R_i+2 \leq 0$, or $K-F < R_i-1$. However, for $\binom{K-F}{R_i-1}$ to be non-zero, we must have either $K-F \geq R_i-1$ or $K-F<0$, the former of which we have already ruled out. In the latter case, the term $\binom{K}{F}$ in our expression for $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ makes the summation term zero regardless. Therefore, our expression for $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ remains valid for computing the number of vertical arrays, no matter how we choose to evaluate this expression.

As an example, we will compute the formula for $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ in the simplest case where the support graph of $\mathbf{s}$ is not a tree. This computation can be repeated for larger values of $\mathbf{s}$, as long as it remains feasible to compute all non-zero values of $g_{n,F;\mathbf{r}}^{(\mathbf{s})}$.

Example 7.9. Let $n=3$ and $\mathbf{s}=(1,1,1)$. As $g_{n,F;\mathbf{r}}^{(\mathbf{s})}$ can only be non-zero for $F \leq s$, we only need to compute the values of $g_{n,F;\mathbf{r}}^{(\mathbf{s})}$ for $1 \leq F \leq 3$ and $1 \leq r_1, r_2, r_3 \leq F$. This gives us the table of values in Table 7.1. Then, by substituting these values into (7.7) and simplifying the result, we can obtain the following formula

$$\begin{aligned} v_{n,K;\mathbf{R}}^{(\mathbf{s})} &= \binom{K-1}{R_1-1}\binom{K-1}{R_2-1}\binom{K-1}{R_3-1} \times \left[(R_1+1)(R_2+1)(R_3+1) + \right. \\ &\left. \frac{(K+1)(2K-R_1-1)(2K-R_2-1)(2K-R_3-1)}{(K-1)^2}\right] \end{aligned}$$

Note that $\binom{K-1}{R_1-1}\binom{K-1}{R_2-1}\binom{K-1}{R_3-1}(R_1+1)(R_2+1)(R_3+1)$ matches the summation term of $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ in Theorem 6.9, if we are to extend the term in a straightforward manner. However, in cases where the support graph is not a tree, this expression comes with a correction term like the one in the second row. Furthermore, the size of the correction term increases dramatically as the values in $\mathbf{s}$ grows large, even in the case $n=3$, which makes producing a compact formula for $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ difficult. This prohibits the creation of a compact formula for $A_n^{(\mathbf{q};\mathbf{s})}(x)$.

Now, for fixed values of $\mathbf{R}$, $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ can be simplified to polynomials in K. In particular, we have $(2K-R_i-1) = 2(K-1)$ if $R_i=1$. Otherwise, $\frac{(K-1)!}{(K-R_i)!}$ is a polynomial in K that contains the factor $K-1$. In both cases, this cancels out the term $(K-1)^2$ in the denominator, leaving us with a polynomial in K. Doing so for small values of $\mathbf{R}$ gives us Table 7.2. By substituting these values into Corollary 6.5 and using Fact 2.4, we can obtain the generating series for $A_n^{(\mathbf{q};\mathbf{s})}(x)$ for $\mathbf{0} \leq \mathbf{q} \leq \mathbf{3}$, where $\mathbf{s}=(1,1,1)$. The values of these series can be found in Table B.10 of B, where we have also computed the series $A_n^{(\mathbf{q};\mathbf{s})}(x)$ for other small values of n, $\mathbf{q}$, and $\mathbf{s}$.

F	(r_1, r_2, r_3)	$g^{(\mathbf{s})}_{n,F;\mathbf{r}}$
1	$(1,1,1)$	8
2	$(1,1,1)$	16
	$(1,1,2)$	8
	$(1,2,2)$	8
	$(2,2,2)$	14
3	$(1,1,1)$	0
	$(1,1,2)$	0
	$(1,1,3)$	0
	$(1,2,2)$	0
	$(1,2,3)$	0
	$(1,3,3)$	0
	$(2,2,2)$	6
	$(2,2,3)$	6
	$(2,3,3)$	6
	$(3,3,3)$	6

Table 7.1: Table of values for $g^{(\mathbf{s})}_{n,F;\mathbf{r}}$ with $\mathbf{s} = (1,1,1)$ and $1 \le F \le 3$

7.4 Future Work

In this thesis, we have devised methods for finding the generating series $A_n^{(\mathbf{q};\mathbf{s})}(x)$, which effectively counts the number of rooted embeddings of a given graph G by genus. In the case where the support graph of $\mathbf{s}$ is a tree, this problem is completely solved by the combined application of Fact 2.4, Corollary 6.5, and Theorem 6.9. Topologically, this corresponds to G being a tree that contains loops and multiple edges. However, the question remains open to find the generating series $A_n^{(\mathbf{q};\mathbf{s})}(x)$ for an arbitrary connected graph G. As Fact 2.4 and Corollary 6.5 remain valid regardless of whether the support graph of $\mathbf{s}$ is a tree, it suffices to find a polynomial expression in K for $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$, which is the number of proper vertical arrays. To find such an expression, there are two main obstacles that need to be overcome, one for each of the two conditions that make a paired array proper.

The first obstacle that needs to be overcome is the forest condition. In the case where the support graph of $\mathbf{s}$ is a tree, we have shown in Section 6.2 that we can decompose vertical arrays row by row. Let row i be a leaf vertex in the support graph of $\mathbf{s}$, and row k be its neighbour. Then, deleting row i is the same as deleting all mixed pairs with one vertex in row i and one vertex in row k. Therefore, one way of decomposing vertical arrays

(R_1, R_2, R_3)	$v^{(\mathbf{s})}_{n,K;\mathbf{R}}$
$(1,1,1)$	$8K^2$
$(1,1,2)$	$4K\,(K-1)\,(2K-1)$
$(1,1,3)$	$4K\,(K-1)^2\,(K-2)$
$(1,1,4)$	$\frac{2}{3}K\,(K-1)\,(K-2)\,(K-3)\,(2K-3)$
$(1,2,2)$	$4K\,(K-1)\,(2K^2-4K+3))$
$(1,2,3)$	$2K\,(K-1)\,(K-2)\,(2K^2-5K+5))$
$(1,2,4)$	$\frac{2}{3}K\,(K-1)\,(K-2)\,(K-3)\,(2K^2-6K+7)$
$(1,3,3)$	$2K\,(K-1)\,(K-2)^2\,(K^2-3K+4)$
$(1,3,4)$	$\frac{1}{3}K\,(K-1)\,(K-2)^2\,(K-3)\,(2K^2-7K+11)$
$(1,4,4)$	$\frac{1}{9}K\,(K-1)\,(K-2)^2\,(K-3)^2\,(2K^2-8K+15)$
$(2,2,2)$	$K\,(K-1)\,(8K^3-28K^2+45K-27)$
$(2,2,3)$	$K\,(K-1)\,(K-2)\,(4K^3-16K^2+31K-21)$
$(2,2,4)$	$\frac{1}{6}K\,(K-1)\,(K-2)\,(K-3)\,(8K^3-36K^2+79K-57)$
$(2,3,3)$	$K\,(K-1)\,(K-2)^2\,(2K^3-9K^2+21K-16)$
$(2,3,4)$	$\frac{1}{6}K\,(K-1)\,(K-2)^2\,(K-3)\,(4K^3-20K^2+53K-43)$
$(2,4,4)$	$\frac{1}{36}K\,(K-1)\,(K-2)^2\,(K-3)^2\,(8K^3-44K^2+133K-115)$
$(3,3,3)$	$K\,(K-1)\,(K-2)^3\,(K^3-5K^2+14K-12)$
$(3,3,4)$	$\frac{1}{6}K\,(K-1)\,(K-2)^3\,(K-3)\,(2K^3-11K^2+35K-32)$
$(3,4,4)$	$\frac{1}{36}K\,(K-1)\,(K-2)^3\,(K-3)^2\,(4K^3-24K^2+87K-85)$
$(4,4,4)$	$\frac{1}{216}K\,(K-1)\,(K-2)^3\,(K-3)^3\,(8K^3-52K^2+215K-225)$

Table 7.2: Table of values for $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$ with $\mathbf{s}=(1,1,1)$ and $1 \le R_i \le 4$

when the support graph of $\mathbf{s}$ is not a tree is to remove all vertex pairs between a given pair of rows. By doing this, we only have to concern ourselves with the forest condition functions of those two rows, instead of the forest condition functions of multiple rows if we are to entirely remove a row from the vertical array.

To facilitate this, we can extend arrowed arrays to allow for arrows in both rows, which we call *double arrowed arrays*. Note that the arrow simplification lemmas and irreducible substructures introduced in Section 4.2 generalize to double arrowed arrays. However, the irreducible structures for double arrowed arrays can be significantly more complex than the ones when we have arrows only on row 1. For example, an arrow in row 1 can point to a column that has an arrow-tail in row 2, but is unmarked in row 1. This arrow in row 2 can then point to another column that has an arrow-tail in row 1, and is unmarked in row 2. This means that we can have chains, trees, or even cycles of arrows alternating between row 1 and row 2, and some examples of that can see be in Figure 7.1. These can then be combined in various ways, and while we have conjectures in limited cases, we have yet to find a formula or a proof in the general case.

For example, if we are to take the top diagram of Figure 7.1 and leave the cell R unmarked, then it is a column of type $\mathcal{A}$, as described in Definition 4.10. The number of double arrowed arrays satisfying this substructure is given by

$$T(\Gamma) = d_{1,1} d_{2,1} (s-2)!$$

However, if the cell R is marked, then it becomes a column of type $\mathcal{C}$, and the number of double arrowed arrays satisfying this substructure is the sum

$$T(\Gamma) = d_{2,1} \left((s-1)! + \sum_{i=1}^{\left\lfloor \frac{K-1}{2} \right\rfloor} (-1)^i (i-1)! \left(a_{1,2i+1} + a_{1,2i+2} + \cdots + a_{1,K}\right)(s-i-1)! \right)$$

Notice that whether the term $a_{1,j}$ appears in a given sum depends on the distance of column j to the column of type $\mathcal{C}$ in the chain of arrows. This fact appears to hold true even if multiple arrows are pointing to the same cell, so that the chain of arrows forms a tree instead.

Another way we can chain the arrows together is to make them form a cycle, like in the bottom diagram of Figure 7.1. The number of double arrowed arrays satisfying this substructure gives another sum

$$T(\Gamma) = d_{1,1}d_{2,1}\sum_{i=1}^{\left\lfloor \frac{K-1}{2}\right\rfloor}(-1)^{i-1}(i-1)!\,(s-i-1)!$$

In all cases, we have included a column of type $\mathcal{D}$ to ensure that there are root vertices for the functional digraphs. Now, the way we have created these conjectures is to fix the number of columns, as well as the positions of the marked cells and arrows, but without fixing the number of vertices in each cell. This way, we can simply enumerate all possible ways of pairing the critical vertices in both rows so that they satisfy the forest condition, and pair all other vertices arbitrarily.

The second obstacle that needs to be overcome is the balance condition. In the case where the support graph of $\mathbf{s}$ is a tree, the balance condition is radically simplified by Lemma 3.6. If row i is a leaf vertex in the support graph of $\mathbf{s}$, and row k is its neighbour, then the number of vertices in cell (i, j) is equal to the number of vertices in cell (k, j) that is paired with a vertex in row i. This means that if we delete row i, the remaining rows satisfy the balance condition, so we can recursively decompose vertical arrays. Furthermore, this gave us a clean method of determining the number of ways to place the vertices into row k during the decomposition. None of these hold true when the support graph of $\mathbf{s}$ is not a tree. As an example, the vertical array in Figure 7.2 satisfies both the balance and forest conditions. However, deleting any of the three rows will leave a vertical array that violates the balance condition, as will deleting all vertex pairs between any two rows. Despite that, by using Proposition 3.4, we can determine whether a given paired array $\alpha \in \mathcal{PA}_{n,K;\mathbf{R}}^{(\mathbf{q};\mathbf{s})}$ satisfies the balance condition by only knowing the rows of the partners of each vertex, as this gives the number of vertices in row i such that its partner is in column j. Therefore, this allows us to separate the balance and the forest condition, even though we currently do not have a formula for the number of ways to place the vertices.

In cases where the support graph of $\mathbf{s}$ contains a bridge, it may be possible to separate the balance and forest conditions in another manner. If $e = \{i, k\}$ is a bridge in the support graph, then by modifying the proof of Lemma 3.6, we see that the balance condition implies that $s_{i,k,j} = s_{k,i,j}$ holds for $1 \le j \le K$. Note that this condition only holds between rows i and k, with $s_{i,k,j}$ being the number of vertices in cell (i, j) that are paired with a vertex in row k, and vice-versa.

Aside from directly generalizing our work for arbitrary graphs, here are some other observations and directions that may be pursued. In Section 2.1, we noted that the generating series $A_n^{(\mathbf{q};\mathbf{s})}(x)$ is either an odd or a even polynomial, depending on the parity of

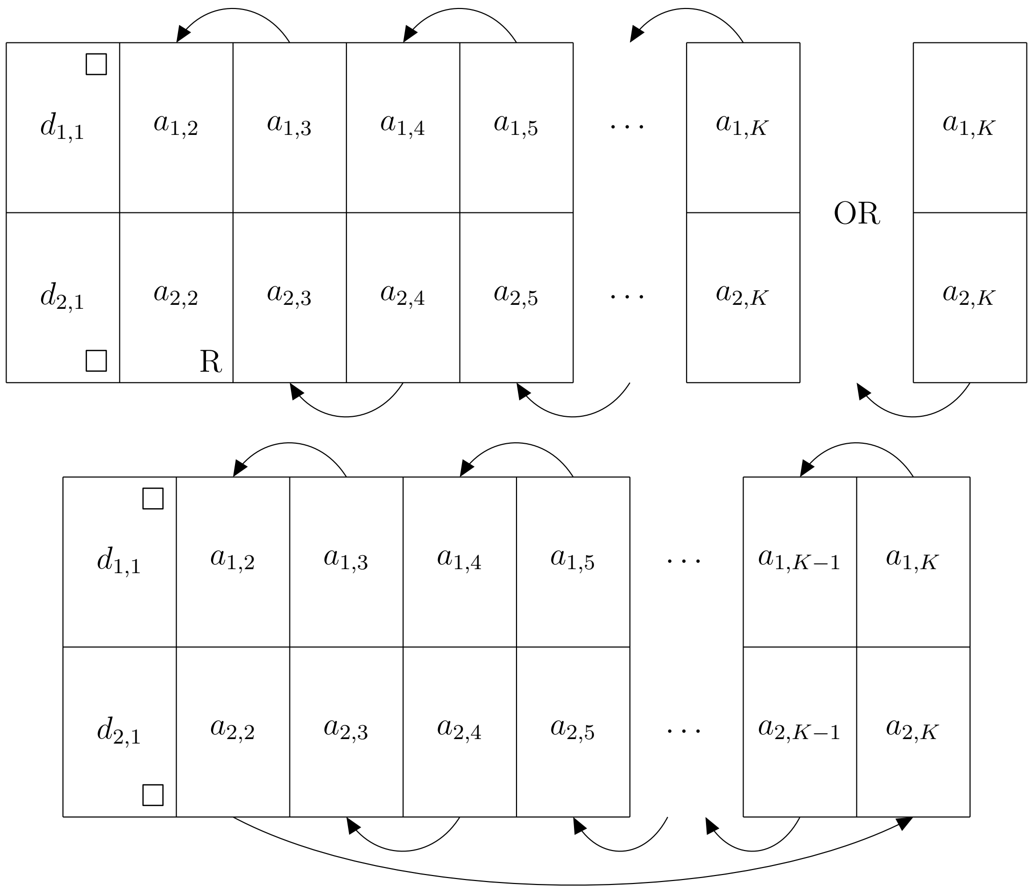


Figure 7.1: Examples of double arrowed arrays

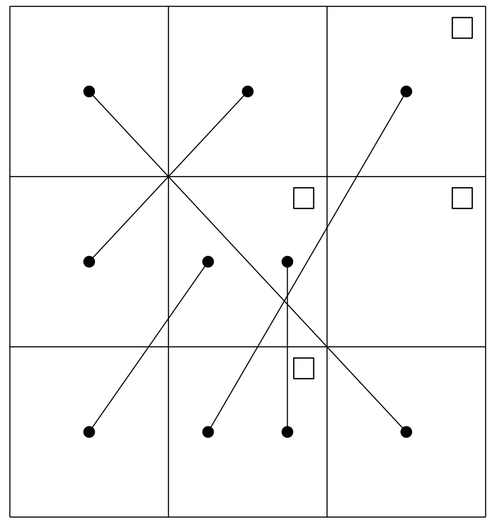


Figure 7.2: Non-tree vertical array that satisfies the forest and balance conditions

$n + d$. However, the formula we have computed by combining Corollary 6.5 and Theorem 6.9 gives no indication on why this should be the case. It would be useful to find a direct proof of this, as it may allow us to determine the properties and relationships of the numbers $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$. On a more practical note, this fact can be used to help compute specific values of $A^{(\mathbf{q};\mathbf{s})}_{n}(x)$ when direct computation is infeasible. By combining Corollary 6.5 and Theorem 7.6, we know that we can write $f^{(\mathbf{q};\mathbf{s})}_{n,K}$ as an expression that is polynomial in K, and has coefficients that are linear combinations of $g^{(\mathbf{s})}_{n,F;\mathbf{r}}$, the number of full vertical arrays. For a fixed $\mathbf{s}$, the number of $g^{(\mathbf{s})}_{n,F;\mathbf{r}}$ that are non-zero is known and finite. Combining this with Fact 2.4, we can equate the coefficients of $f^{(\mathbf{q};\mathbf{s})}_{n,K}$ that are zero, and set up a system of linear equations to solve for $g^{(\mathbf{s})}_{n,F;\mathbf{r}}$.

Next, recall that in Section 2.3 and Section 2.4, we have used the matrix integral method to find the generating series $A^{(\mathbf{q};\mathbf{s})}_{n}(x)$ for $n = 1, 2$. As far as we are aware, the are no generalizations of this method that can be used to derive the results in this thesis. The only generalizations that we have are similar to the ones covered in Section 2.5. Therefore, it may be useful to derive the results of this thesis with algebraic methods, and see whether those methods can be extended to arbitrary graphs.

Finally, in bijective proofs used in Goulden, Nica, and Slofstra [17, 18], they have used partitions to label the cycles of $\mu\gamma^{-1}$, which is effectively a surjective colouring of the cycles. This translates to the non-empty condition that exists in their version of the paired array, as described in Section 7.1. As we saw in Section 2.5, the same technique was used in Schaeffer and Vassilieva [31], and in the follow up paper of Vassilieva [39]. In contrast, in both the algebraic technique for map enumeration, as well as our extension of the combinatorial technique in this thesis, we do not require the colouring function to be a surjection. This eliminates the need of a non-empty condition. Consequently, our proofs have become simpler in many aspects. It may be useful to see whether this approach can be used to simplify other combinatorial proofs in map enumeration.

Appendix A

Table of Formulas for $v^{(\mathbf{s})}_{n,K;\mathbf{R}}$

The following expressions give the number of vertical arrays for $1 \leq n \leq 5$ where the support graph of $\mathbf{s}$ is a tree. This comes from specializing the formula in Theorem 6.9.

- $n = 1$, support graph of $\mathbf{s}$ is empty

$$v^{(\mathbf{s})}_{n,K;\mathbf{R}} = \frac{K!}{R_1!\,(K-R_1)!}$$

- $n = 2$, support graph of $\mathbf{s}$ contains the edge $1-2$

$$\sum_{A_{1,2}=0}^{\min(s_{1,2},K)-1} \frac{(K-A_{1,2}-1)!\,(K+s_{1,2}-A_{1,2}-1)!}{(R_1-1)!\,(R_2-1)!\,(K-R_1-A_{1,2})!\,(K-R_2-A_{1,2})!}$$

- $n = 3$, support graph of $\mathbf{s}$ contains the edges $1-2$ and $1-3$

$$\begin{aligned} v^{(\mathbf{s})}_{n,K;\mathbf{R}} \quad = \quad & \sum_{A_{1,2}=0}^{\min(s_{1,2},K)-1} \sum_{A_{1,3}=0}^{\min(s_{1,3},K)-1} \left[\frac{(K-A_{1,2}-1)!\,(K-A_{1,3}-1)!}{(R_1-1)!\,(R_2-1)!\,(R_3-1)!\,(K-R_2-A_{1,2})!} \times \right. \\ & \left. \frac{(K+s_{1,2}+s_{1,3}-A_{1,2}-A_{1,3}-2)!\,(R_1+s_{1,2}+s_{1,3}-1)!}{(K-R_3-A_{1,3})!\,(K-R_1-A_{1,2}-A_{1,3})!\,(R_1+s_{1,2}+s_{1,3}-2)!} \right] \end{aligned}$$

- $n = 4$, support graph of $\mathbf{s}$ contains the edges $1-2$, $1-3$, and $1-4$

$$v_{n,K;\mathbf{R}}^{(\mathbf{s})} = \sum_{A_{1,2}=0}^{\min(s_{1,2},K)-1} \sum_{A_{1,3}=0}^{\min(s_{1,3},K)-1} \sum_{A_{1,4}=0}^{\min(s_{1,4},K)-1} \left[\frac{(K-A_{1,2}-1)!\,(K-A_{1,3}-1)!}{(R_1-1)!\,(R_2-1)!\,(R_3-1)!} \times \frac{(K-A_{1,4}-1)!\,(K+s_{1,2}+s_{1,3}+s_{1,4}-A_{1,2}-A_{1,3}-A_{1,4}-3)!}{(R_4-1)!\,(K-R_2-A_{1,2})!\,(K-R_3-A_{1,3})!\,(K-R_4-A_{1,4})!} \times \frac{(R_1+s_{1,2}+s_{1,3}+s_{1,4}-1)!}{(K-R_1-A_{1,2}-A_{1,3}-A_{1,4})!\,(R_1+s_{1,2}+s_{1,3}+s_{1,4}-3)!} \right]$$

- $n = 4$, support graph of $\mathbf{s}$ contains the edges $1-2$, $2-3$, and $3-4$

$$v_{n,K;\mathbf{R}}^{(\mathbf{s})} = \sum_{A_{1,2}=0}^{\min(s_{1,2},K)-1} \sum_{A_{2,3}=0}^{\min(s_{2,3},K)-1} \sum_{A_{3,4}=0}^{\min(s_{3,4},K)-1} \left[\frac{(K-A_{1,2}-1)!\,(K-A_{2,3}-1)!}{(R_1-1)!\,(R_2-1)!\,(R_3-1)!} \times \frac{(K-A_{3,4}-1)!\,(R_2+s_{1,2}+s_{2,3}-1)!\,(R_3+s_{2,3}+s_{3,4}-1)!}{(R_4-1)!\,(K-R_1-A_{1,2})!\,(K-R_2-A_{1,2}-A_{2,3})!} \times \frac{(K+s_{1,2}+s_{2,3}-A_{1,2}-A_{2,3}-2)!}{(K-R_3-A_{2,3}-A_{3,4})!\,(K-R_4-A_{3,4})!\,(K+s_{2,3}-A_{2,3}-1)!} \times \frac{(K+s_{2,3}+s_{3,4}-A_{2,3}-A_{3,4}-2)!}{(R_2+s_{1,2}+s_{2,3}-2)!\,(R_3+s_{2,3}+s_{3,4}-2)!} \right]$$

- $n = 5$, support graph of $\mathbf{s}$ contains the edges $1-2$, $1-3$, $1-4$, and $1-5$

$$v_{n,K;\mathbf{R}}^{(\mathbf{s})} = \sum_{A_{1,2}=0}^{\min(s_{1,2},K)-1} \sum_{A_{1,3}=0}^{\min(s_{1,3},K)-1} \sum_{A_{1,4}=0}^{\min(s_{1,4},K)-1} \sum_{A_{1,5}=0}^{\min(s_{1,5},K)-1} \left[\frac{(K-A_{1,2}-1)!}{(R_1-1)!\,(R_2-1)!} \times \frac{(K-A_{1,3}-1)!\,(K-A_{1,4}-1)!\,(K-A_{1,5}-1)!}{(R_3-1)!\,(R_4-1)!\,(R_5-1)!\,(K-R_1-A_{1,2}-A_{1,3}-A_{1,4}-A_{1,5})!} \times \frac{(K+s_{1,2}+s_{1,3}+s_{1,4}+s_{1,5}-A_{1,2}-A_{1,3}-A_{1,4}-A_{1,5}-4)!}{(R_1+s_{1,2}+s_{1,3}+s_{1,4}+s_{1,5}-4)!} \times \frac{(R_1+s_{1,2}+s_{1,3}+s_{1,4}+s_{1,5}-1)!}{(K-R_2-A_{1,2})!\,(K-R_3-A_{1,3})!\,(K-R_4-A_{1,4})!\,(K-R_5-A_{1,5})!} \right]$$

- $n=5$, support graph of $\mathbf{s}$ contains the edges $1-2$, $2-3$, $3-4$, and $4-5$

$$
\begin{aligned}
v_{n,K;\mathbf{R}}^{(\mathbf{s})} \;=\; & \sum_{A_{1,2}=0}^{\min(s_{1,2},K)-1} \sum_{A_{2,3}=0}^{\min(s_{2,3},K)-1} \sum_{A_{3,4}=0}^{\min(s_{3,4},K)-1} \sum_{A_{4,5}=0}^{\min(s_{4,5},K)-1} \left[\frac{(K-A_{1,2}-1)!}{(R_1-1)!\,(R_2-1)!} \times \right. \\
& \frac{(K-A_{2,3}-1)!\,(K-A_{3,4}-1)!\,(K-A_{4,5}-1)!}{(R_3-1)!\,(R_4-1)!\,(R_5-1)!\,(K-R_2-A_{1,2}-A_{2,3})!} \times \\
& \frac{(R_2+s_{1,2}+s_{2,3}-1)!\,(R_3+s_{2,3}+s_{3,4}-1)!}{(K-R_3-A_{2,3}-A_{3,4})!\,(K-R_4-A_{3,4}-A_{4,5})!\,(K-R_1-A_{1,2})!} \times \\
& \frac{(R_4+s_{3,4}+s_{4,5}-1)!\,(K+s_{1,2}+s_{2,3}-A_{1,2}-A_{2,3}-2)!}{(K-R_5-A_{4,5})!\,(R_2+s_{1,2}+s_{2,3}-2)!\,(R_3+s_{2,3}+s_{3,4}-2)!} \times \\
& \left. \frac{(K+s_{2,3}+s_{3,4}-A_{2,3}-A_{3,4}-2)!\,(K+s_{3,4}+s_{4,5}-A_{3,4}-A_{4,5}-2)!}{(R_4+s_{3,4}+s_{4,5}-2)!\,(K+s_{2,3}-A_{2,3}-1)!\,(K+s_{3,4}-A_{3,4}-1)!} \right]
\end{aligned}
$$

- $n=5$, support graph of $\mathbf{s}$ contains the edges $1-2$, $2-3$, $3-4$, and $3-5$

$$
\begin{aligned}
v_{n,K;\mathbf{R}}^{(\mathbf{s})} \;=\; & \sum_{A_{1,2}=0}^{\min(s_{1,2},K)-1} \sum_{A_{2,3}=0}^{\min(s_{2,3},K)-1} \sum_{A_{3,4}=0}^{\min(s_{3,4},K)-1} \sum_{A_{3,5}=0}^{\min(s_{3,5},K)-1} \left[\frac{(K-A_{1,2}-1)!}{(R_1-1)!\,(R_2-1)!} \times \right. \\
& \frac{(K-A_{2,3}-1)!\,(K-A_{3,4}-1)!\,(K-A_{3,5}-1)!}{(R_3-1)!\,(R_4-1)!\,(R_5-1)!\,(K-R_1-A_{1,2})!} \times \\
& \frac{(R_2+s_{1,2}+s_{2,3}-1)!\,(K+s_{1,2}+s_{2,3}-A_{1,2}-A_{2,3}-2)!}{(K-R_2-A_{1,2}-A_{2,3})!\,(K-R_3-A_{2,3}-A_{3,4}-A_{3,5})!} \times \\
& \frac{(R_3+s_{2,3}+s_{3,4}+s_{3,5}-1)!}{(K-R_4-A_{3,4})!\,(K-R_5-A_{3,5})!\,(K+s_{2,3}-A_{2,3}-1)!} \times \\
& \left. \frac{(K+s_{2,3}+s_{3,4}+s_{3,5}-A_{2,3}-A_{3,4}-A_{3,5}-3)!}{(R_2+s_{1,2}+s_{2,3}-2)!\,(R_3+s_{2,3}+s_{3,4}+s_{3,5}-3)!} \right]
\end{aligned}
$$

Appendix B

Table of Formulas for $A_n^{(\mathbf{q};\mathbf{s})}(x)$

The following is a selection of the values for $A_n^{(\mathbf{q};\mathbf{s})}(x)$, where $1 \le n \le 4$. They are computed by the combined use of Corollary 6.5, Theorem 6.9, and Fact 2.4. When the support graph of $\mathbf{s}$ is a tree, the values of $v_{n,K;\mathbf{R}}^{(\mathbf{s})}$ are computed using Theorem 6.9. Otherwise, the technique described in Section 7.3 is used instead.

B.1 Formulas for $n = 1$

q_1	$A(x)$
0	x
1	x^2
2	$2x^3 + x$
3	$5x^4 + 10x^2$
4	$14x^5 + 70x^3 + 21x$
5	$42x^6 + 420x^4 + 483x^2$
6	$132x^7 + 2310x^5 + 6468x^3 + 1485x$
7	$429x^8 + 12012x^6 + 66066x^4 + 56628x^2$
8	$1430x^9 + 60060x^7 + 570570x^5 + 1169740x^3 + 225225x$
9	$4862x^{10} + 291720x^8 + 4390386x^6 + 17454580x^4 + 12317877x^2$
10	$16796x^{11} + 1385670x^9 + 31039008x^7 + 211083730x^5 + 351683046x^3 + 59520825x$
11	$58786x^{12} + 6466460x^{10} + 205633428x^8$ $+2198596400x^6 + 7034538511x^4 + 4304016990x^2$
12	$208012x^{13} + 29745716x^{11} + 1293938646x^9$ $+20465052608x^7 + 111159740692x^5 + 158959754226x^3 + 24325703325x$
13	$742900x^{14} + 135207800x^{12} + 7808250450x^{10}$ $+174437377400x^8 + 1480593013900x^6 + 4034735959800x^4 + 2208143028375x^2$
14	$2674440x^{15} + 608435100x^{13}$ $+45510945480x^{11} + 1384928666550x^9 + 17302190625720x^7$ $+79553497760100x^5 + 100940771124360x^3 + 14230536445125x$
15	$9694845x^{16} + 2714556600x^{14}$ $+257611421340x^{12} + 10369994005800x^{10} + 182231849209410x^8$ $+1302772718028600x^6 + 3130208769783780x^4 + 1564439686929000x^2$

Table B.1: Values of $A_n^{(\mathbf{q;s})}$ for $n = 1$

B.2 Formulas for $n = 2$

$(s_{1,2}, q_1, q_2)$	$A(x)$
$(1,0,0)$	x
$(1,0,1)$	$3x^2$
$(1,0,2)$	$10x^3 + 5x$
$(1,0,3)$	$35x^4 + 70x^2$
$(1,0,4)$	$126x^5 + 630x^3 + 189x$
$(1,1,1)$	$9x^3$
$(1,1,2)$	$30x^4 + 15x^2$
$(1,1,3)$	$105x^5 + 210x^3$
$(1,1,4)$	$378x^6 + 1890x^4 + 567x^2$
$(1,2,2)$	$100x^5 + 100x^3 + 25x$
$(1,2,3)$	$350x^6 + 875x^4 + 350x^2$
$(1,2,4)$	$1260x^7 + 6930x^5 + 5040x^3 + 945x$
$(1,3,3)$	$1225x^7 + 4900x^5 + 4900x^3$
$(1,3,4)$	$4410x^8 + 30870x^6 + 50715x^4 + 13230x^2$
$(1,4,4)$	$15876x^9 + 158760x^7 + 444528x^5 + 238140x^3 + 35721x$
$(2,0,0)$	$2x^2$
$(2,0,1)$	$8x^3 + 4x$
$(2,0,2)$	$30x^4 + 60x^2$
$(2,0,3)$	$112x^5 + 560x^3 + 168x$
$(2,0,4)$	$420x^6 + 4200x^4 + 4830x^2$
$(2,1,1)$	$32x^4 + 40x^2$
$(2,1,2)$	$120x^5 + 360x^3 + 60x$
$(2,1,3)$	$448x^6 + 2800x^4 + 1792x^2$
$(2,1,4)$	$1680x^7 + 19320x^5 + 31920x^3 + 3780x$
$(2,2,2)$	$450x^6 + 2250x^4 + 1350x^2$
$(2,2,3)$	$1680x^7 + 14280x^5 + 19320x^3 + 2520x$
$(2,2,4)$	$6300x^8 + 88200x^6 + 229950x^4 + 100800x^2$
$(2,3,3)$	$6272x^8 + 76832x^6 + 189728x^4 + 79968x^2$
$(2,3,4)$	$23520x^9 + 423360x^7 + 1728720x^5 + 1634640x^3 + 158760x$
$(2,4,4)$	$88200x^{10} + 2116800x^8 + 12965400x^6 + 22138200x^4 + 7342650x^2$

Table B.2: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 2$, Part 1

$(s_{1,2}, q_1, q_2)$	$A(x)$
$(3,0,0)$	$3x^3 + 3x$
$(3,0,1)$	$15x^4 + 45x^2$
$(3,0,2)$	$63x^5 + 420x^3 + 147x$
$(3,0,3)$	$252x^6 + 3150x^4 + 4158x^2$
$(3,0,4)$	$990x^7 + 20790x^5 + 65835x^3 + 16335x$
$(3,1,1)$	$75x^5 + 425x^3 + 100x$
$(3,1,2)$	$315x^6 + 3150x^4 + 2835x^2$
$(3,1,3)$	$1260x^7 + 20790x^5 + 45990x^3 + 7560x$
$(3,1,4)$	$4950x^8 + 127050x^6 + 560175x^4 + 347325x^2$
$(3,2,2)$	$1323x^7 + 19845x^5 + 38367x^3 + 6615x$
$(3,2,3)$	$5292x^8 + 117306x^6 + 426888x^4 + 244314x^2$
$(3,2,4)$	$20790x^9 + 665280x^7 + 4147605x^5 + 5363820x^3 + 717255x$
$(3,3,3)$	$21168x^9 + 635040x^7 + 3667356x^5 + 4630500x^3 + 571536x$
$(3,3,4)$	$83160x^{10} + 3367980x^8 + 29688120x^6 + 68461470x^4 + 29376270x^2$
$(3,4,4)$	$326700x^{11} + 16879500x^9 + 208107900x^7$ $+772645500x^5 + 725791275x^3 + 77182875x$
$(4,0,0)$	$4x^4 + 20x^2$
$(4,0,1)$	$24x^5 + 240x^3 + 96x$
$(4,0,2)$	$112x^6 + 1960x^4 + 2968x^2$
$(4,0,3)$	$480x^7 + 13440x^5 + 48720x^3 + 12960x$
$(4,0,4)$	$1980x^8 + 83160x^6 + 582120x^4 + 580140x^2$
$(4,1,1)$	$144x^6 + 2340x^4 + 2916x^2$
$(4,1,2)$	$672x^7 + 16800x^5 + 48048x^3 + 10080x$
$(4,1,3)$	$2880x^8 + 105840x^6 + 582120x^4 + 443160x^2$
$(4,1,4)$	$11880x^9 + 617760x^7 + 5821200x^5 + 10549440x^3 + 1710720x$
$(4,2,2)$	$3136x^8 + 109760x^6 + 549584x^4 + 395920x^2$
$(4,2,3)$	$13440x^9 + 645120x^7 + 5292000x^5 + 8594880x^3 + 1330560x$
$(4,2,4)$	$55440x^{10} + 3575880x^8 + 45405360x^6 + 138489120x^4 + 74428200x^2$
$(4,3,3)$	$57600x^{10} + 3585600x^8 + 42940800x^6 + 125816400x^4 + 65739600x^2$
$(4,3,4)$	$237600x^{11} + 19008000x^9 + 324324000x^7$ $+1551528000x^5 + 1809680400x^3 + 224532000x$
$(4,4,4)$	$980100x^{12} + 97029900x^{10} + 2208165300x^8$ $+15682580100x^6 + 33040151100x^4 + 13804708500x^2$

Table B.3: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 2$, Part 2

B.3 Formulas for $n = 3$, Tree Support Graph

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$2x$
$(0,0,1)$	$6x^2$
$(0,0,2)$	$20x^3 + 10x$
$(0,1,1)$	$18x^3$
$(0,1,2)$	$60x^4 + 30x^2$
$(0,2,2)$	$200x^5 + 200x^3 + 50x$
$(1,0,0)$	$12x^2$
$(1,0,1)$	$36x^3$
$(1,0,2)$	$120x^4 + 60x^2$
$(1,1,1)$	$108x^4$
$(1,1,2)$	$360x^5 + 180x^3$
$(1,2,2)$	$1200x^6 + 1200x^4 + 300x^2$
$(2,0,0)$	$60x^3 + 30x$
$(2,0,1)$	$180x^4 + 90x^2$
$(2,0,2)$	$600x^5 + 600x^3 + 150x$
$(2,1,1)$	$540x^5 + 270x^3$
$(2,1,2)$	$1800x^6 + 1800x^4 + 450x^2$
$(2,2,2)$	$6000x^7 + 9000x^5 + 4500x^3 + 750x$

Table B.4: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1, 1, 0)$

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$6x^2$
$(0,0,1)$	$24x^3 + 12x$
$(0,0,2)$	$90x^4 + 180x^2$
$(0,1,0)$	$18x^3$
$(0,1,1)$	$72x^4 + 36x^2$
$(0,1,2)$	$270x^5 + 540x^3$
$(0,2,0)$	$60x^4 + 30x^2$
$(0,2,1)$	$240x^5 + 240x^3 + 60x$
$(0,2,2)$	$900x^6 + 2250x^4 + 900x^2$
$(1,0,0)$	$40x^3 + 20x$
$(1,0,1)$	$160x^4 + 200x^2$
$(1,0,2)$	$600x^5 + 1800x^3 + 300x$
$(1,1,0)$	$120x^4 + 60x^2$
$(1,1,1)$	$480x^5 + 600x^3$
$(1,1,2)$	$1800x^6 + 5400x^4 + 900x^2$
$(1,2,0)$	$400x^5 + 400x^3 + 100x$
$(1,2,1)$	$1600x^6 + 2800x^4 + 1000x^2$
$(1,2,2)$	$6000x^7 + 21000x^5 + 12000x^3 + 1500x$
$(2,0,0)$	$210x^4 + 420x^2$
$(2,0,1)$	$840x^5 + 2520x^3 + 420x$
$(2,0,2)$	$3150x^6 + 15750x^4 + 9450x^2$
$(2,1,0)$	$630x^5 + 1260x^3$
$(2,1,1)$	$2520x^6 + 7560x^4 + 1260x^2$
$(2,1,2)$	$9450x^7 + 47250x^5 + 28350x^3$
$(2,2,0)$	$2100x^6 + 5250x^4 + 2100x^2$
$(2,2,1)$	$8400x^7 + 29400x^5 + 16800x^3 + 2100x$
$(2,2,2)$	$31500x^8 + 173250x^6 + 173250x^4 + 47250x^2$

Table B.5: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1, 2, 0)$

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$12x^3 + 12x$
$(0,0,1)$	$60x^4 + 180x^2$
$(0,0,2)$	$252x^5 + 1680x^3 + 588x$
$(0,1,0)$	$36x^4 + 36x^2$
$(0,1,1)$	$180x^5 + 540x^3$
$(0,1,2)$	$756x^6 + 5040x^4 + 1764x^2$
$(0,2,0)$	$120x^5 + 180x^3 + 60x$
$(0,2,1)$	$600x^6 + 2100x^4 + 900x^2$
$(0,2,2)$	$2520x^7 + 18060x^5 + 14280x^3 + 2940x$
$(1,0,0)$	$90x^4 + 270x^2$
$(1,0,1)$	$450x^5 + 2550x^3 + 600x$
$(1,0,2)$	$1890x^6 + 18900x^4 + 17010x^2$
$(1,1,0)$	$270x^5 + 810x^3$
$(1,1,1)$	$1350x^6 + 7650x^4 + 1800x^2$
$(1,1,2)$	$5670x^7 + 56700x^5 + 51030x^3$
$(1,2,0)$	$900x^6 + 3150x^4 + 1350x^2$
$(1,2,1)$	$4500x^7 + 27750x^5 + 18750x^3 + 3000x$
$(1,2,2)$	$18900x^8 + 198450x^6 + 264600x^4 + 85050x^2$
$(2,0,0)$	$504x^5 + 3360x^3 + 1176x$
$(2,0,1)$	$2520x^6 + 25200x^4 + 22680x^2$
$(2,0,2)$	$10584x^7 + 158760x^5 + 306936x^3 + 52920x$
$(2,1,0)$	$1512x^6 + 10080x^4 + 3528x^2$
$(2,1,1)$	$7560x^7 + 75600x^5 + 68040x^3$
$(2,1,2)$	$31752x^8 + 476280x^6 + 920808x^4 + 158760x^2$
$(2,2,0)$	$5040x^7 + 36120x^5 + 28560x^3 + 5880x$
$(2,2,1)$	$25200x^8 + 264600x^6 + 352800x^4 + 113400x^2$
$(2,2,2)$	$105840x^9 + 1640520x^7 + 3863160x^5 + 2063880x^3 + 264600x$

Table B.6: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1, 3, 0)$

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$16x^3 + 8x$
$(0,0,1)$	$64x^4 + 80x^2$
$(0,0,2)$	$240x^5 + 720x^3 + 120x$
$(0,1,1)$	$256x^5 + 512x^3 + 96x$
$(0,1,2)$	$960x^6 + 3600x^4 + 1920x^2$
$(0,2,2)$	$3600x^7 + 19800x^5 + 23400x^3 + 1800x$
$(1,0,0)$	$120x^4 + 240x^2$
$(1,0,1)$	$480x^5 + 1440x^3 + 240x$
$(1,0,2)$	$1800x^6 + 9000x^4 + 5400x^2$
$(1,1,1)$	$1920x^6 + 7680x^4 + 3360x^2$
$(1,1,2)$	$7200x^7 + 43200x^5 + 43200x^3 + 3600x$
$(1,2,2)$	$27000x^8 + 216000x^6 + 378000x^4 + 108000x^2$
$(2,0,0)$	$672x^5 + 3360x^3 + 1008x$
$(2,0,1)$	$2688x^6 + 16800x^4 + 10752x^2$
$(2,0,2)$	$10080x^7 + 85680x^5 + 115920x^3 + 15120x$
$(2,1,1)$	$10752x^7 + 80640x^5 + 83328x^3 + 6720x$
$(2,1,2)$	$40320x^8 + 393120x^6 + 715680x^4 + 211680x^2$
$(2,2,2)$	$151200x^9 + 1814400x^7 + 4914000x^5 + 3099600x^3 + 226800x$

Table B.7: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (2,2,0)$

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$30x^4 + 90x^2$
$(0,0,1)$	$150x^5 + 850x^3 + 200x$
$(0,0,2)$	$630x^6 + 6300x^4 + 5670x^2$
$(0,1,0)$	$120x^5 + 480x^3 + 120x$
$(0,1,1)$	$600x^6 + 4000x^4 + 2600x^2$
$(0,1,2)$	$2520x^7 + 27720x^5 + 39480x^3 + 5880x$
$(0,2,0)$	$450x^6 + 2700x^4 + 2250x^2$
$(0,2,1)$	$2250x^7 + 19500x^5 + 29250x^3 + 3000x$
$(0,2,2)$	$9450x^8 + 122850x^6 + 305550x^4 + 129150x^2$
$(1,0,0)$	$252x^5 + 1680x^3 + 588x$
$(1,0,1)$	$1260x^6 + 12600x^4 + 11340x^2$
$(1,0,2)$	$5292x^7 + 79380x^5 + 153468x^3 + 26460x$
$(1,1,0)$	$1008x^6 + 7980x^4 + 6132x^2$
$(1,1,1)$	$5040x^7 + 56700x^5 + 81060x^3 + 8400x$
$(1,1,2)$	$21168x^8 + 343980x^6 + 878472x^4 + 343980x^2$
$(1,2,0)$	$3780x^7 + 38430x^5 + 62370x^3 + 8820x$
$(1,2,1)$	$18900x^8 + 255150x^6 + 626850x^4 + 233100x^2$
$(1,2,2)$	$79380x^9 + 1468530x^7 + 5477220x^5 + 4484970x^3 + 396900x$
$(2,0,0)$	$1512x^6 + 18900x^4 + 24948x^2$
$(2,0,1)$	$7560x^7 + 124740x^5 + 275940x^3 + 45360x$
$(2,0,2)$	$31752x^8 + 703836x^6 + 2561328x^4 + 1465884x^2$
$(2,1,0)$	$6048x^7 + 84672x^5 + 160272x^3 + 21168x$
$(2,1,1)$	$30240x^8 + 544320x^6 + 1557360x^4 + 589680x^2$
$(2,1,2)$	$127008x^9 + 3005856x^7 + 13102992x^5 + 11388384x^3 + 952560x$
$(2,2,0)$	$22680x^8 + 374220x^6 + 1111320x^4 + 532980x^2$
$(2,2,1)$	$113400x^9 + 2324700x^7 + 9412200x^5 + 7881300x^3 + 680400x$
$(2,2,2)$	$476280x^{10} + 12462660x^8 + 70410060x^6 + 101844540x^4 + 29132460x^2$

Table B.8: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (2, 3, 0)$

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$54x^5 + 450x^3 + 216x$
$(0,0,1)$	$270x^6 + 3150x^4 + 3780x^2$
$(0,0,2)$	$1134x^7 + 18900x^5 + 45486x^3 + 10080x$
$(0,1,1)$	$1350x^7 + 20250x^5 + 44400x^3 + 6000x$
$(0,1,2)$	$5670x^8 + 113400x^6 + 416430x^4 + 220500x^2$
$(0,2,2)$	$23814x^9 + 595350x^7 + 3257226x^5 + 3585330x^3 + 476280x$
$(1,0,0)$	$504x^6 + 7560x^4 + 12096x^2$
$(1,0,1)$	$2520x^7 + 47880x^5 + 127680x^3 + 23520x$
$(1,0,2)$	$10584x^8 + 261072x^6 + 1118376x^4 + 726768x^2$
$(1,1,1)$	$12600x^8 + 289800x^6 + 1142400x^4 + 571200x^2$
$(1,1,2)$	$52920x^9 + 1517040x^7 + 8767080x^5 + 9772560x^3 + 1058400x$
$(1,2,2)$	$222264x^{10} + 7631064x^8 + 60035976x^6 + 113379336x^4 + 40995360x^2$
$(2,0,0)$	$3240x^7 + 79380x^5 + 291060x^3 + 79920x$
$(2,0,1)$	$16200x^8 + 472500x^6 + 2400300x^4 + 1647000x^2$
$(2,0,2)$	$68040x^9 + 2415420x^7 + 17146080x^5 + 24460380x^3 + 3538080x$
$(2,1,1)$	$81000x^9 + 2740500x^7 + 18238500x^5 + 22032000x^3 + 2268000x$
$(2,1,2)$	$340200x^{10} + 13664700x^8 + 120922200x^6 + 250368300x^4 + 90984600x^2$
$(2,2,2)$	$1428840x^{11} + 66441060x^9 + 746886420x^7$ $+2289557340x^5 + 1739453940x^3 + 157172400x$

Table B.9: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (3, 3, 0)$

B.4 Formulas for $n = 3$, Non-Tree Support Graph

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$8x^2$
$(0,0,1)$	$32x^3 + 16x$
$(0,0,2)$	$120x^4 + 240x^2$
$(0,0,3)$	$448x^5 + 2240x^3 + 672x$
$(0,1,1)$	$128x^4 + 160x^2$
$(0,1,2)$	$480x^5 + 1440x^3 + 240x$
$(0,1,3)$	$1792x^6 + 11200x^4 + 7168x^2$
$(0,2,2)$	$1800x^6 + 9000x^4 + 5400x^2$
$(0,2,3)$	$6720x^7 + 57120x^5 + 77280x^3 + 10080x$
$(0,3,3)$	$25088x^8 + 307328x^6 + 758912x^4 + 319872x^2$
$(1,1,1)$	$512x^5 + 1216x^3$
$(1,1,2)$	$1920x^6 + 8640x^4 + 2400x^2$
$(1,1,3)$	$7168x^7 + 58240x^5 + 55552x^3$
$(1,2,2)$	$7200x^7 + 50400x^5 + 36000x^3 + 3600x$
$(1,2,3)$	$26880x^8 + 295680x^6 + 477120x^4 + 107520x^2$
$(1,3,3)$	$100352x^9 + 1542912x^7 + 4290048x^5 + 2533888x^3$
$(2,2,2)$	$27000x^8 + 270000x^6 + 324000x^4 + 108000x^2$
$(2,2,3)$	$100800x^9 + 1461600x^7 + 3276000x^5 + 1814400x^3 + 151200x$
$(2,3,3)$	$376320x^{10} + 7338240x^8 + 25401600x^6 + 25589760x^4 + 4798080x^2$
$(3,3,3)$	$1404928x^{11} + 35298816x^9 + 168591360x^7 + 272029184x^5 + 115379712x^3$

Table B.10: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1,1,1)$

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$18x^3 + 18x$
$(0,0,1)$	$90x^4 + 270x^2$
$(0,0,2)$	$378x^5 + 2520x^3 + 882x$
$(0,0,3)$	$1512x^6 + 18900x^4 + 24948x^2$
$(0,1,1)$	$450x^5 + 2550x^3 + 600x$
$(0,1,2)$	$1890x^6 + 18900x^4 + 17010x^2$
$(0,1,3)$	$7560x^7 + 124740x^5 + 275940x^3 + 45360x$
$(0,2,2)$	$7938x^7 + 119070x^5 + 230202x^3 + 39690x$
$(0,2,3)$	$31752x^8 + 703836x^6 + 2561328x^4 + 1465884x^2$
$(0,3,3)$	$127008x^9 + 3810240x^7 + 22004136x^5 + 27783000x^3 + 3429216x$
$(1,0,0)$	$72x^4 + 144x^2$
$(1,0,1)$	$360x^5 + 1560x^3 + 240x$
$(1,0,2)$	$1512x^6 + 12600x^4 + 8568x^2$
$(1,0,3)$	$6048x^7 + 87696x^5 + 160272x^3 + 18144x$
$(1,1,1)$	$1800x^6 + 13400x^4 + 6400x^2$
$(1,1,2)$	$7560x^7 + 92400x^5 + 118440x^3 + 8400x$
$(1,1,3)$	$30240x^8 + 579600x^6 + 1607760x^4 + 504000x^2$
$(1,2,2)$	$31752x^8 + 564480x^6 + 1361808x^4 + 423360x^2$
$(1,2,3)$	$127008x^9 + 3238704x^7 + 13843872x^5 + 10732176x^3 + 635040x$
$(1,3,3)$	$508032x^{10} + 17273088x^8 + 112910112x^6 + 172603872x^4 + 39626496x^2$
$(2,0,0)$	$270x^5 + 1080x^3 + 270x$
$(2,0,1)$	$1350x^6 + 9000x^4 + 5850x^2$
$(2,0,2)$	$5670x^7 + 62370x^5 + 88830x^3 + 13230x$
$(2,0,3)$	$22680x^8 + 396900x^6 + 1111320x^4 + 510300x^2$
$(2,1,1)$	$6750x^7 + 69000x^5 + 77250x^3 + 9000x$
$(2,1,2)$	$28350x^8 + 437850x^6 + 916650x^4 + 318150x^2$
$(2,1,3)$	$113400x^9 + 2589300x^7 + 9790200x^5 + 7238700x^3 + 680400x$
$(2,2,2)$	$119070x^9 + 2566620x^7 + 8546580x^5 + 6032880x^3 + 595350x$
$(2,2,3)$	$476280x^{10} + 14209020x^8 + 75966660x^6 + 96922980x^4 + 26751060x^2$
$(2,3,3)$	$1905120x^{11} + 74299680x^9 + 573917400x^7$ $+1207846080x^5 + 662505480x^3 + 51438240x$

Table B.11: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1,1,2)$, Part 1

(q_1, q_2, q_3)	$A(x)$
$(3,0,0)$	$1008x^6 + 7560x^4 + 6552x^2$
$(3,0,1)$	$5040x^7 + 52920x^5 + 83160x^3 + 10080x$
$(3,0,2)$	$21168x^8 + 321048x^6 + 860832x^4 + 384552x^2$
$(3,0,3)$	$84672x^9 + 1862784x^7 + 8149680x^5 + 8192016x^3 + 762048x$
$(3,1,1)$	$25200x^8 + 365400x^6 + 835800x^4 + 285600x^2$
$(3,1,2)$	$105840x^9 + 2134440x^7 + 7479360x^5 + 5803560x^3 + 352800x$
$(3,1,3)$	$423360x^{10} + 11854080x^8 + 64456560x^6 + 91339920x^4 + 22438080x^2$
$(3,2,2)$	$444528x^{10} + 11928168x^8 + 58455432x^6 + 76977432x^4 + 18892440x^2$
$(3,2,3)$	$1778112x^{11} + 63419328x^9 + 450306864x^7$ $+939732192x^5 + 518467824x^3 + 26671680x$
$(3,3,3)$	$7112448x^{12} + 323616384x^{10} + 3109917888x^8$ $+9496007136x^6 + 9307527264x^4 + 1760330880x^2$

Table B.12: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1,1,2)$, Part 2

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$36x^4 + 180x^2$
$(0,0,1)$	$216x^5 + 2160x^3 + 864x$
$(0,0,2)$	$1008x^6 + 17640x^4 + 26712x^2$
$(0,0,3)$	$4320x^7 + 120960x^5 + 438480x^3 + 116640x$
$(0,1,0)$	$180x^5 + 1500x^3 + 480x$
$(0,1,1)$	$1080x^6 + 15300x^4 + 16020x^2$
$(0,1,2)$	$5040x^7 + 113400x^5 + 284760x^3 + 50400x$
$(0,1,3)$	$21600x^8 + 730800x^6 + 3641400x^4 + 2410200x^2$
$(0,2,0)$	$756x^6 + 10080x^4 + 11844x^2$
$(0,2,1)$	$4536x^7 + 90720x^5 + 207144x^3 + 37800x$
$(0,2,2)$	$21168x^8 + 617400x^6 + 2554272x^4 + 1569960x^2$
$(0,2,3)$	$90720x^9 + 3749760x^7 + 26248320x^5 + 36318240x^3 + 5034960x$
$(0,3,0)$	$3024x^7 + 61992x^5 + 170856x^3 + 36288x$
$(0,3,1)$	$18144x^8 + 508032x^6 + 2181816x^4 + 1374408x^2$
$(0,3,2)$	$84672x^9 + 3217536x^7 + 21189168x^5 + 28851984x^3 + 3810240x$
$(0,3,3)$	$362880x^{10} + 18506880x^8 + 182891520x^6 + 452103120x^4 + 203439600x^2$

Table B.13: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1,2,2)$, Part 1

(q_1, q_2, q_3)	$A(x)$
$(1,1,0)$	$900x^6 + 10900x^4 + 9800x^2$
$(1,1,1)$	$5400x^7 + 102000x^5 + 192600x^3 + 24000x$
$(1,1,2)$	$25200x^8 + 709800x^6 + 2599800x^4 + 1201200x^2$
$(1,1,3)$	$108000x^9 + 4368000x^7 + 28308000x^5 + 32232000x^3 + 3024000x$
$(1,2,0)$	$3780x^7 + 66360x^5 + 134820x^3 + 21840x$
$(1,2,1)$	$22680x^8 + 573300x^6 + 1917720x^4 + 888300x^2$
$(1,2,2)$	$105840x^9 + 3757320x^7 + 20532960x^5 + 21115080x^3 + 2116800x$
$(1,2,3)$	$453600x^{10} + 22100400x^8 + 191129400x^6 + 374043600x^4 + 126693000x^2$
$(1,3,0)$	$15120x^8 + 380520x^6 + 1484280x^4 + 841680x^2$
$(1,3,1)$	$90720x^9 + 3069360x^7 + 17221680x^5 + 18627840x^3 + 1814400x$
$(1,3,2)$	$423360x^{10} + 19051200x^8 + 154843920x^6 + 297939600x^4 + 99277920x^2$
$(1,3,3)$	$1814400x^{11} + 107352000x^9 + 1257379200x^7$ $+3983061600x^5 + 2994818400x^3 + 228614400x$
$(2,2,0)$	$15876x^8 + 372204x^6 + 1289484x^4 + 703836x^2$
$(2,2,1)$	$95256x^9 + 3069360x^7 + 15463224x^5 + 15505560x^3 + 1587600x$
$(2,2,2)$	$444528x^{10} + 19336968x^8 + 144150552x^6 + 252319032x^4 + 83842920x^2$
$(2,2,3)$	$1905120x^{11} + 110073600x^9 + 1207740240x^7$ $+3486898800x^5 + 2485229040x^3 + 209563200x$
$(2,3,0)$	$63504x^9 + 2000376x^7 + 11303712x^5 + 13558104x^3 + 1651104x$
$(2,3,1)$	$381024x^{10} + 15748992x^8 + 119165256x^6 + 219850848x^4 + 73505880x^2$
$(2,3,2)$	$1778112x^{11} + 95425344x^9 + 986111280x^7$ $+2789561376x^5 + 1968221808x^3 + 160030080x$
$(2,3,3)$	$7620480x^{12} + 525813120x^{10} + 7475690880x^8$ $+31671667440x^6 + 39935125440x^4 + 10401002640x^2$
$(3,3,0)$	$254016x^{10} + 10160640x^8 + 83126736x^6 + 178128720x^4 + 71251488x^2$
$(3,3,1)$	$1524096x^{11} + 77728896x^9 + 806627808x^7$ $+2383686144x^5 + 1737088416x^3 + 137168640x$
$(3,3,2)$	$7112448x^{12} + 458752896x^{10} + 6168270528x^8$ $+25468787232x^6 + 31765081824x^4 + 8145531072x^2$
$(3,3,3)$	$30481920x^{13} + 2469035520x^{11} + 43596766080x^9$ $+250896683520x^7 + 494584392960x^5 + 271342431360x^3 + 17283248640x$

Table B.14: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1, 2, 2)$, Part 2

(q_1, q_2, q_3)	$A(x)$
$(0,0,0)$	$64x^5 + 1088x^3 + 576x$
$(0,0,1)$	$384x^6 + 9600x^4 + 15936x^2$
$(0,0,2)$	$1792x^7 + 63616x^5 + 235648x^3 + 61824x$
$(0,0,3)$	$7680x^8 + 376320x^6 + 2593920x^4 + 2465280x^2$
$(0,1,1)$	$2304x^7 + 78480x^5 + 253296x^3 + 54720x$
$(0,1,2)$	$10752x^8 + 490560x^6 + 2804928x^4 + 2136960x^2$
$(0,1,3)$	$46080x^9 + 2773440x^7 + 25976160x^5 + 45646560x^3 + 7205760x$
$(0,2,2)$	$50176x^9 + 2922752x^7 + 25313792x^5 + 41483008x^3 + 6435072x$
$(0,2,3)$	$215040x^{10} + 15886080x^8 + 202930560x^6 + 606318720x^4 + 317721600x^2$
$(0,3,3)$	$921600x^{11} + 83462400x^9 + 1449100800x^7$ $+6845716800x^5 + 7819761600x^3 + 947116800x$
$(1,1,1)$	$13824x^8 + 612576x^6 + 3174336x^4 + 2031264x^2$
$(1,1,2)$	$64512x^9 + 3681216x^7 + 29742048x^5 + 42717024x^3 + 5443200x$
$(1,1,3)$	$276480x^{10} + 20131200x^8 + 244823040x^6 + 663854400x^4 + 295634880x^2$
$(1,2,2)$	$301056x^{10} + 21374976x^8 + 243554304x^6 + 614229504x^4 + 263612160x^2$
$(1,2,3)$	$1290240x^{11} + 113460480x^9 + 1805448960x^7$ $+7429101120x^5 + 7089163200x^3 + 707616000x$
$(1,3,3)$	$5529600x^{12} + 586483200x^{10} + 12244608000x^8$ $+74183472000x^6 + 127789574400x^4 + 42381532800x^2$
$(2,2,2)$	$1404928x^{11} + 121175040x^9 + 1829040640x^7$ $+7038689280x^5 + 6372577792x^3 + 640120320x$
$(2,2,3)$	$6021120x^{12} + 629207040x^{10} + 12596935680x^8$ $+71987758080x^6 + 116939558400x^4 + 37885639680x^2$
$(2,3,3)$	$25804800x^{13} + 3201408000x^{11} + 81210124800x^9$ $+640543276800x^7 + 1645874697600x^5 + 1137918499200x^3 + 91902988800x$
$(3,3,3)$	$110592000x^{14} + 16042752000x^{12}$ $+499074048000x^{10} + 5150333376000x^8$ $+19147577472000x^6 + 23193996192000x^4 + 6003017568000x^2$

Table B.15: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (2,2,2)$

(q_1,q_2,q_3)	$A(x)$
$(0,0,0)$	$60x^5+900x^3+480x$
$(0,0,1)$	$420x^6+10500x^4+19320x^2$
$(0,0,2)$	$2160x^7+83160x^5+355320x^3+103680x$
$(0,0,3)$	$9900x^8+554400x^6+4504500x^4+4910400x^2$
$(0,1,0)$	$360x^6+8100x^4+13140x^2$
$(0,1,1)$	$2520x^7+85680x^5+297360x^3+68040x$
$(0,1,2)$	$12960x^8+635040x^6+4105080x^4+3411720x^2$
$(0,1,3)$	$59400x^9+4039200x^7+43991640x^5+86842800x^3+14754960x$
$(0,2,0)$	$1680x^7+54600x^5+195720x^3+50400x$
$(0,2,1)$	$11760x^8+535080x^6+3216360x^4+2587200x^2$
$(0,2,2)$	$60480x^9+3749760x^7+35964432x^5+63972720x^3+10559808x$
$(0,2,3)$	$277200x^{10}+22869000x^8+334136880x^6+1108661400x^4+629687520x^2$
$(0,3,0)$	$7200x^8+327600x^6+2179800x^4+2021400x^2$
$(0,3,1)$	$50400x^9+3024000x^7+29317680x^5+53928000x^3+8935920x$
$(0,3,2)$	$259200x^{10}+20217600x^8+280869120x^6+906714000x^4+506548080x^2$
$(0,3,3)$	$1188000x^{11}+118800000x^9+2320164000x^7$ $+12088612800x^5+14974740000x^3+1930975200x$
$(1,0,0)$	$300x^6+5900x^4+8200x^2$
$(1,0,1)$	$2100x^7+64260x^5+197400x^3+38640x$
$(1,0,2)$	$10800x^8+486360x^6+2860200x^4+2085840x^2$
$(1,0,3)$	$49500x^9+3141600x^7+31762500x^5+56522400x^3+8316000x$
$(1,1,0)$	$1800x^7+49800x^5+139200x^3+25200x$
$(1,1,1)$	$12600x^8+506520x^6+2520000x^4+1496880x^2$
$(1,1,2)$	$64800x^9+3643920x^7+30315600x^5+42634080x^3+4989600x$
$(1,1,3)$	$297000x^{10}+22651200x^8+297019800x^6+824788800x^4+352123200x^2$

Table B.16: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n=3$, $(s_{1,2},s_{1,3},s_{2,3})=(1,2,3)$, Part 1

(q_1, q_2, q_3)	$A(x)$
$(1,2,0)$	$8400x^8 + 320600x^6 + 1622600x^4 + 1072400x^2$
$(1,2,1)$	$58800x^9 + 3075240x^7 + 23961000x^5 + 32528160x^3 + 3880800x$
$(1,2,2)$	$302400x^{10} + 21147840x^8 + 247489200x^6 + 620166960x^4 + 253965600x^2$
$(1,2,3)$	$1386000x^{11} + 126911400x^9 + 2167426800x^7$ $+9199020600x^5 + 8646607200x^3 + 814968000x$
$(1,3,0)$	$36000x^9 + 1860000x^7 + 15666000x^5 + 24450000x^3 + 3348000x$
$(1,3,1)$	$252000x^{10} + 16984800x^8 + 199684800x^6 + 518011200x^4 + 217627200x^2$
$(1,3,2)$	$1296000x^{11} + 112276800x^9 + 1827100800x^7$ $+7538896800x^5 + 7016047200x^3 + 650462400x$
$(1,3,3)$	$5940000x^{12} + 652608000x^{10} + 14525676000x^8$ $+91212264000x^6 + 157617504000x^4 + 50330808000x^2$
$(2,0,0)$	$1260x^7 + 32760x^5 + 95340x^3 + 21840x$
$(2,0,1)$	$8820x^8 + 333984x^6 + 1672860x^4 + 1159536x^2$
$(2,0,2)$	$45360x^9 + 2414664x^7 + 19855584x^5 + 30320136x^3 + 4517856x$
$(2,0,3)$	$207900x^{10} + 15093540x^8 + 193942980x^6 + 558433260x^4 + 280138320x^2$
$(2,1,0)$	$7560x^8 + 260820x^6 + 1214640x^4 + 784980x^2$
$(2,1,1)$	$52920x^9 + 2543688x^7 + 18319140x^5 + 23801652x^3 + 2910600x$
$(2,1,2)$	$272160x^{10} + 17726688x^8 + 193528440x^6 + 458734752x^4 + 187041960x^2$
$(2,1,3)$	$1247400x^{11} + 107525880x^9 + 1729977480x^7$ $+6932383920x^5 + 6324900120x^3 + 621205200x$
$(2,2,0)$	$35280x^9 + 1605240x^7 + 11642400x^5 + 16211160x^3 + 2257920x$
$(2,2,1)$	$246960x^{10} + 15006936x^8 + 154679280x^6 + 353704344x^4 + 143154480x^2$
$(2,2,2)$	$1270080x^{11} + 101013696x^9 + 1470900816x^7$ $+5364775584x^5 + 4614687504x^3 + 449608320x$
$(2,2,3)$	$5821200x^{12} + 595508760x^{10} + 12087721800x^8$ $+67852489320x^6 + 106100101800x^4 + 33399717120x^2$
$(2,3,0)$	$151200x^{10} + 8996400x^8 + 98343000x^6 + 252453600x^4 + 116335800x^2$
$(2,3,1)$	$1058400x^{11} + 80932320x^9 + 1178634240x^7$ $+4428292680x^5 + 3933525960x^3 + 379436400x$
$(2,3,2)$	$5443200x^{12} + 527627520x^{10} + 10237026240x^8$ $+55889825040x^6 + 86486823360x^4 + 26887094640x^2$
$(2,3,3)$	$24948000x^{13} + 3028687200x^{11} + 77939215200x^9$ $+608197312800x^7 + 1517334865200x^5 + 1013308758000x^3 + 80786613600x$

Table B.17: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1,2,3)$, Part 2

(q_1, q_2, q_3)	$A(x)$
$(3,0,0)$	$5040x^8 + 173880x^6 + 914760x^4 + 720720x^2$
$(3,0,1)$	$35280x^9 + 1668744x^7 + 13078296x^5 + 20398896x^3 + 2921184x$
$(3,0,2)$	$181440x^{10} + 11539584x^8 + 132786864x^6 + 364268016x^4 + 177067296x^2$
$(3,0,3)$	$831600x^{11} + 69771240x^9 + 1156090320x^7$ $+5152676760x^5 + 5565732480x^3 + 628689600x$
$(3,1,0)$	$30240x^9 + 1315440x^7 + 9707040x^5 + 14258160x^3 + 1905120x$
$(3,1,1)$	$211680x^{10} + 12298608x^8 + 127304352x^6 + 306491472x^4 + 125229888x^2$
$(3,1,2)$	$1088640x^{11} + 82954368x^9 + 1201368672x^7$ $+4555196352x^5 + 4069826208x^3 + 377213760x$
$(3,1,3)$	$4989600x^{12} + 490477680x^{10} + 9840489120x^8$ $+56587552560x^6 + 92566061280x^4 + 29117309760x^2$
$(3,2,0)$	$141120x^{10} + 7761600x^8 + 80262000x^6 + 201625200x^4 + 91234080x^2$
$(3,2,1)$	$987840x^{11} + 70531776x^9 + 971392464x^7$ $+3553112304x^5 + 3112091136x^3 + 293388480x$
$(3,2,2)$	$5080320x^{12} + 463833216x^{10} + 8537223744x^8$ $+45120735072x^6 + 68853195936x^4 + 21047003712x^2$
$(3,2,3)$	$23284800x^{13} + 2682408960x^{11} + 65792366640x^9$ $+495314265600x^7 + 1213241697360x^5 + 801830715840x^3 + 61611580800x$
$(3,3,0)$	$604800x^{11} + 42033600x^9 + 604195200x^7$ $+2418897600x^5 + 2396520000x^3 + 253108800x$
$(3,3,1)$	$4233600x^{12} + 371286720x^{10} + 6810168960x^8$ $+36807130080x^6 + 58026356640x^4 + 18003384000x^2$
$(3,3,2)$	$21772800x^{13} + 2381944320x^{11} + 56046453120x^9$ $+410638273920x^7 + 992067834240x^5 + 650074844160x^3 + 49174957440x$
$(3,3,3)$	$99792000x^{14} + 13481899200x^{12}$ $+407490652800x^{10} + 4035279124800x^8$ $+14355847598400x^6 + 16669604952000x^4 + 4125640780800x^2$

Table B.18: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 3$, $(s_{1,2}, s_{1,3}, s_{2,3}) = (1, 2, 3)$, Part 3

B.5 Formulas for $n = 4$, Tree Support Graph

(q_1, q_2, q_3, q_4)	$A(x)$
$(0,0,0,0)$	$6x$
$(0,0,0,1)$	$18x^2$
$(0,0,0,2)$	$60x^3 + 30x$
$(0,0,1,1)$	$54x^3$
$(0,0,1,2)$	$180x^4 + 90x^2$
$(0,0,2,2)$	$600x^5 + 600x^3 + 150x$
$(0,1,1,1)$	$162x^4$
$(0,1,1,2)$	$540x^5 + 270x^3$
$(0,1,2,2)$	$1800x^6 + 1800x^4 + 450x^2$
$(0,2,2,2)$	$6000x^7 + 9000x^5 + 4500x^3 + 750x$
$(1,0,0,0)$	$60x^2$
$(1,0,0,1)$	$180x^3$
$(1,0,0,2)$	$600x^4 + 300x^2$
$(1,0,1,1)$	$540x^4$
$(1,0,1,2)$	$1800x^5 + 900x^3$
$(1,0,2,2)$	$6000x^6 + 6000x^4 + 1500x^2$
$(1,1,1,1)$	$1620x^5$
$(1,1,1,2)$	$5400x^6 + 2700x^4$
$(1,1,2,2)$	$18000x^7 + 18000x^5 + 4500x^3$
$(1,2,2,2)$	$60000x^8 + 90000x^6 + 45000x^4 + 7500x^2$
$(2,0,0,0)$	$420x^3 + 210x$
$(2,0,0,1)$	$1260x^4 + 630x^2$
$(2,0,0,2)$	$4200x^5 + 4200x^3 + 1050x$
$(2,0,1,1)$	$3780x^5 + 1890x^3$
$(2,0,1,2)$	$12600x^6 + 12600x^4 + 3150x^2$
$(2,0,2,2)$	$42000x^7 + 63000x^5 + 31500x^3 + 5250x$
$(2,1,1,1)$	$11340x^6 + 5670x^4$
$(2,1,1,2)$	$37800x^7 + 37800x^5 + 9450x^3$
$(2,1,2,2)$	$126000x^8 + 189000x^6 + 94500x^4 + 15750x^2$
$(2,2,2,2)$	$420000x^9 + 840000x^7 + 630000x^5 + 210000x^3 + 26250x$

Table B.19: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,1,1,0,0,0)$

(q_1, q_2, q_3, q_4)	$A(x)$
$(0,0,0,0)$	$24x^2$
$(0,0,0,1)$	$96x^3 + 48x$
$(0,0,0,2)$	$360x^4 + 720x^2$
$(0,0,1,0)$	$72x^3$
$(0,0,1,1)$	$288x^4 + 144x^2$
$(0,0,1,2)$	$1080x^5 + 2160x^3$
$(0,0,2,0)$	$240x^4 + 120x^2$
$(0,0,2,1)$	$960x^5 + 960x^3 + 240x$
$(0,0,2,2)$	$3600x^6 + 9000x^4 + 3600x^2$
$(0,1,1,0)$	$216x^4$
$(0,1,1,1)$	$864x^5 + 432x^3$
$(0,1,1,2)$	$3240x^6 + 6480x^4$
$(0,1,2,0)$	$720x^5 + 360x^3$
$(0,1,2,1)$	$2880x^6 + 2880x^4 + 720x^2$
$(0,1,2,2)$	$10800x^7 + 27000x^5 + 10800x^3$
$(0,2,2,0)$	$2400x^6 + 2400x^4 + 600x^2$
$(0,2,2,1)$	$9600x^7 + 14400x^5 + 7200x^3 + 1200x$
$(0,2,2,2)$	$36000x^8 + 108000x^6 + 81000x^4 + 18000x^2$
$(1,0,0,0)$	$240x^3 + 120x$
$(1,0,0,1)$	$960x^4 + 1200x^2$
$(1,0,0,2)$	$3600x^5 + 10800x^3 + 1800x$
$(1,0,1,0)$	$720x^4 + 360x^2$
$(1,0,1,1)$	$2880x^5 + 3600x^3$
$(1,0,1,2)$	$10800x^6 + 32400x^4 + 5400x^2$
$(1,0,2,0)$	$2400x^5 + 2400x^3 + 600x$
$(1,0,2,1)$	$9600x^6 + 16800x^4 + 6000x^2$
$(1,0,2,2)$	$36000x^7 + 126000x^5 + 72000x^3 + 9000x$

Table B.20: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,1,2,0,0,0)$, Part 1

(q_1, q_2, q_3, q_4)	$A(x)$
$(1,1,1,0)$	$2160x^5 + 1080x^3$
$(1,1,1,1)$	$8640x^6 + 10800x^4$
$(1,1,1,2)$	$32400x^7 + 97200x^5 + 16200x^3$
$(1,1,2,0)$	$7200x^6 + 7200x^4 + 1800x^2$
$(1,1,2,1)$	$28800x^7 + 50400x^5 + 18000x^3$
$(1,1,2,2)$	$108000x^8 + 378000x^6 + 216000x^4 + 27000x^2$
$(1,2,2,0)$	$24000x^7 + 36000x^5 + 18000x^3 + 3000x$
$(1,2,2,1)$	$96000x^8 + 216000x^6 + 144000x^4 + 30000x^2$
$(1,2,2,2)$	$360000x^9 + 1440000x^7 + 1350000x^5 + 450000x^3 + 45000x$
$(2,0,0,0)$	$1680x^4 + 3360x^2$
$(2,0,0,1)$	$6720x^5 + 20160x^3 + 3360x$
$(2,0,0,2)$	$25200x^6 + 126000x^4 + 75600x^2$
$(2,0,1,0)$	$5040x^5 + 10080x^3$
$(2,0,1,1)$	$20160x^6 + 60480x^4 + 10080x^2$
$(2,0,1,2)$	$75600x^7 + 378000x^5 + 226800x^3$
$(2,0,2,0)$	$16800x^6 + 42000x^4 + 16800x^2$
$(2,0,2,1)$	$67200x^7 + 235200x^5 + 134400x^3 + 16800x$
$(2,0,2,2)$	$252000x^8 + 1386000x^6 + 1386000x^4 + 378000x^2$
$(2,1,1,0)$	$15120x^6 + 30240x^4$
$(2,1,1,1)$	$60480x^7 + 181440x^5 + 30240x^3$
$(2,1,1,2)$	$226800x^8 + 1134000x^6 + 680400x^4$
$(2,1,2,0)$	$50400x^7 + 126000x^5 + 50400x^3$
$(2,1,2,1)$	$201600x^8 + 705600x^6 + 403200x^4 + 50400x^2$
$(2,1,2,2)$	$756000x^9 + 4158000x^7 + 4158000x^5 + 1134000x^3$
$(2,2,2,0)$	$168000x^8 + 504000x^6 + 378000x^4 + 84000x^2$
$(2,2,2,1)$	$672000x^9 + 2688000x^7 + 2520000x^5 + 840000x^3 + 84000x$
$(2,2,2,2)$	$2520000x^{10} + 15120000x^8 + 20790000x^6 + 10710000x^4 + 1890000x^2$

Table B.21: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,1,2,0,0,0)$, Part 2

(q_1, q_2, q_3, q_4)	$A(x)$
$(0,0,0,0)$	$4x$
$(0,0,0,1)$	$12x^2$
$(0,0,0,2)$	$40x^3 + 20x$
$(0,0,1,0)$	$24x^2$
$(0,0,1,1)$	$72x^3$
$(0,0,1,2)$	$240x^4 + 120x^2$
$(0,0,2,0)$	$120x^3 + 60x$
$(0,0,2,1)$	$360x^4 + 180x^2$
$(0,0,2,2)$	$1200x^5 + 1200x^3 + 300x$
$(0,1,0,1)$	$72x^3$
$(0,1,0,2)$	$240x^4 + 120x^2$
$(0,1,1,0)$	$144x^3$
$(0,1,1,1)$	$432x^4$
$(0,1,1,2)$	$1440x^5 + 720x^3$
$(0,1,2,0)$	$720x^4 + 360x^2$
$(0,1,2,1)$	$2160x^5 + 1080x^3$
$(0,1,2,2)$	$7200x^6 + 7200x^4 + 1800x^2$
$(0,2,0,1)$	$360x^4 + 180x^2$
$(0,2,0,2)$	$1200x^5 + 1200x^3 + 300x$
$(0,2,1,1)$	$2160x^5 + 1080x^3$
$(0,2,1,2)$	$7200x^6 + 7200x^4 + 1800x^2$
$(0,2,2,0)$	$3600x^5 + 3600x^3 + 900x$
$(0,2,2,1)$	$10800x^6 + 10800x^4 + 2700x^2$
$(0,2,2,2)$	$36000x^7 + 54000x^5 + 27000x^3 + 4500x$
$(1,0,0,1)$	$36x^3$
$(1,0,0,2)$	$120x^4 + 60x^2$
$(1,0,1,1)$	$216x^4$
$(1,0,1,2)$	$720x^5 + 360x^3$
$(1,0,2,1)$	$1080x^5 + 540x^3$
$(1,0,2,2)$	$3600x^6 + 3600x^4 + 900x^2$

Table B.22: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,0,0,1,0,1)$, Part 1

(q_1, q_2, q_3, q_4)	$A(x)$
$(1,1,0,2)$	$720x^5 + 360x^3$
$(1,1,1,1)$	$1296x^5$
$(1,1,1,2)$	$4320x^6 + 2160x^4$
$(1,1,2,1)$	$6480x^6 + 3240x^4$
$(1,1,2,2)$	$21600x^7 + 21600x^5 + 5400x^3$
$(1,2,0,2)$	$3600x^6 + 3600x^4 + 900x^2$
$(1,2,1,2)$	$21600x^7 + 21600x^5 + 5400x^3$
$(1,2,2,1)$	$32400x^7 + 32400x^5 + 8100x^3$
$(1,2,2,2)$	$108000x^8 + 162000x^6 + 81000x^4 + 13500x^2$
$(2,0,0,2)$	$400x^5 + 400x^3 + 100x$
$(2,0,1,2)$	$2400x^6 + 2400x^4 + 600x^2$
$(2,0,2,2)$	$12000x^7 + 18000x^5 + 9000x^3 + 1500x$
$(2,1,1,2)$	$14400x^7 + 14400x^5 + 3600x^3$
$(2,1,2,2)$	$72000x^8 + 108000x^6 + 54000x^4 + 9000x^2$
$(2,2,2,2)$	$360000x^9 + 720000x^7 + 540000x^5 + 180000x^3 + 22500x$

Table B.23: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1, 0, 0, 1, 0, 1)$, Part 2

(q_1, q_2, q_3, q_4)	$A(x)$
$(0,0,0,0)$	$18x^2$
$(0,0,0,1)$	$54x^3$
$(0,0,0,2)$	$180x^4 + 90x^2$
$(0,0,1,0)$	$120x^3 + 60x$
$(0,0,1,1)$	$360x^4 + 180x^2$
$(0,0,1,2)$	$1200x^5 + 1200x^3 + 300x$
$(0,0,2,0)$	$630x^4 + 1260x^2$
$(0,0,2,1)$	$1890x^5 + 3780x^3$
$(0,0,2,2)$	$6300x^6 + 15750x^4 + 6300x^2$
$(0,1,0,1)$	$360x^4 + 180x^2$
$(0,1,0,2)$	$1200x^5 + 1200x^3 + 300x$
$(0,1,1,0)$	$800x^4 + 1000x^2$
$(0,1,1,1)$	$2400x^5 + 3000x^3$
$(0,1,1,2)$	$8000x^6 + 14000x^4 + 5000x^2$

Table B.24: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1, 0, 0, 2, 0, 1)$, Part 1

(q_1, q_2, q_3, q_4)	$A(x)$
$(0,1,2,0)$	$4200x^5 + 12600x^3 + 2100x$
$(0,1,2,1)$	$12600x^6 + 37800x^4 + 6300x^2$
$(0,1,2,2)$	$42000x^7 + 147000x^5 + 84000x^3 + 10500x$
$(0,2,0,1)$	$1890x^5 + 3780x^3$
$(0,2,0,2)$	$6300x^6 + 15750x^4 + 6300x^2$
$(0,2,1,1)$	$12600x^6 + 37800x^4 + 6300x^2$
$(0,2,1,2)$	$42000x^7 + 147000x^5 + 84000x^3 + 10500x$
$(0,2,2,0)$	$22050x^6 + 110250x^4 + 66150x^2$
$(0,2,2,1)$	$66150x^7 + 330750x^5 + 198450x^3$
$(0,2,2,2)$	$220500x^8 + 1212750x^6 + 1212750x^4 + 330750x^2$
$(1,0,0,1)$	$162x^4$
$(1,0,0,2)$	$540x^5 + 270x^3$
$(1,0,1,1)$	$1080x^5 + 540x^3$
$(1,0,1,2)$	$3600x^6 + 3600x^4 + 900x^2$
$(1,0,2,1)$	$5670x^6 + 11340x^4$
$(1,0,2,2)$	$18900x^7 + 47250x^5 + 18900x^3$
$(1,1,0,2)$	$3600x^6 + 3600x^4 + 900x^2$
$(1,1,1,1)$	$7200x^6 + 9000x^4$
$(1,1,1,2)$	$24000x^7 + 42000x^5 + 15000x^3$
$(1,1,2,1)$	$37800x^7 + 113400x^5 + 18900x^3$
$(1,1,2,2)$	$126000x^8 + 441000x^6 + 252000x^4 + 31500x^2$
$(1,2,0,2)$	$18900x^7 + 47250x^5 + 18900x^3$
$(1,2,1,2)$	$126000x^8 + 441000x^6 + 252000x^4 + 31500x^2$
$(1,2,2,1)$	$198450x^8 + 992250x^6 + 595350x^4$
$(1,2,2,2)$	$661500x^9 + 3638250x^7 + 3638250x^5 + 992250x^3$
$(2,0,0,2)$	$1800x^6 + 1800x^4 + 450x^2$
$(2,0,1,2)$	$12000x^7 + 18000x^5 + 9000x^3 + 1500x$
$(2,0,2,2)$	$63000x^8 + 189000x^6 + 141750x^4 + 31500x^2$
$(2,1,1,2)$	$80000x^8 + 180000x^6 + 120000x^4 + 25000x^2$
$(2,1,2,2)$	$420000x^9 + 1680000x^7 + 1575000x^5 + 525000x^3 + 52500x$
$(2,2,2,2)$	$2205000x^{10} + 13230000x^8 + 18191250x^6 + 9371250x^4 + 1653750x^2$

Table B.25: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,0,0,2,0,1)$, Part 2

(q_1, q_2, q_3, q_4)	$A(x)$
$(0,0,0,0)$	$12x^2$
$(0,0,0,1)$	$48x^3 + 24x$
$(0,0,0,2)$	$180x^4 + 360x^2$
$(0,0,1,0)$	$80x^3 + 40x$
$(0,0,1,1)$	$320x^4 + 400x^2$
$(0,0,1,2)$	$1200x^5 + 3600x^3 + 600x$
$(0,0,2,0)$	$420x^4 + 840x^2$
$(0,0,2,1)$	$1680x^5 + 5040x^3 + 840x$
$(0,0,2,2)$	$6300x^6 + 31500x^4 + 18900x^2$
$(0,1,0,0)$	$72x^3$
$(0,1,0,1)$	$288x^4 + 144x^2$
$(0,1,0,2)$	$1080x^5 + 2160x^3$
$(0,1,1,0)$	$480x^4 + 240x^2$
$(0,1,1,1)$	$1920x^5 + 2400x^3$
$(0,1,1,2)$	$7200x^6 + 21600x^4 + 3600x^2$
$(0,1,2,0)$	$2520x^5 + 5040x^3$
$(0,1,2,1)$	$10080x^6 + 30240x^4 + 5040x^2$
$(0,1,2,2)$	$37800x^7 + 189000x^5 + 113400x^3$
$(0,2,0,0)$	$360x^4 + 180x^2$
$(0,2,0,1)$	$1440x^5 + 1440x^3 + 360x$
$(0,2,0,2)$	$5400x^6 + 13500x^4 + 5400x^2$
$(0,2,1,0)$	$2400x^5 + 2400x^3 + 600x$
$(0,2,1,1)$	$9600x^6 + 16800x^4 + 6000x^2$
$(0,2,1,2)$	$36000x^7 + 126000x^5 + 72000x^3 + 9000x$
$(0,2,2,0)$	$12600x^6 + 31500x^4 + 12600x^2$
$(0,2,2,1)$	$50400x^7 + 176400x^5 + 100800x^3 + 12600x$
$(0,2,2,2)$	$189000x^8 + 1039500x^6 + 1039500x^4 + 283500x^2$

Table B.26: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1, 0, 0, 1, 0, 2)$, Part 1

(q_1, q_2, q_3, q_4)	$A(x)$
$(1,0,0,0)$	$36x^3$
$(1,0,0,1)$	$144x^4 + 72x^2$
$(1,0,0,2)$	$540x^5 + 1080x^3$
$(1,0,1,0)$	$240x^4 + 120x^2$
$(1,0,1,1)$	$960x^5 + 1200x^3$
$(1,0,1,2)$	$3600x^6 + 10800x^4 + 1800x^2$
$(1,0,2,0)$	$1260x^5 + 2520x^3$
$(1,0,2,1)$	$5040x^6 + 15120x^4 + 2520x^2$
$(1,0,2,2)$	$18900x^7 + 94500x^5 + 56700x^3$
$(1,1,0,0)$	$216x^4$
$(1,1,0,1)$	$864x^5 + 432x^3$
$(1,1,0,2)$	$3240x^6 + 6480x^4$
$(1,1,1,0)$	$1440x^5 + 720x^3$
$(1,1,1,1)$	$5760x^6 + 7200x^4$
$(1,1,1,2)$	$21600x^7 + 64800x^5 + 10800x^3$
$(1,1,2,0)$	$7560x^6 + 15120x^4$
$(1,1,2,1)$	$30240x^7 + 90720x^5 + 15120x^3$
$(1,1,2,2)$	$113400x^8 + 567000x^6 + 340200x^4$
$(1,2,0,0)$	$1080x^5 + 540x^3$
$(1,2,0,1)$	$4320x^6 + 4320x^4 + 1080x^2$
$(1,2,0,2)$	$16200x^7 + 40500x^5 + 16200x^3$
$(1,2,1,0)$	$7200x^6 + 7200x^4 + 1800x^2$
$(1,2,1,1)$	$28800x^7 + 50400x^5 + 18000x^3$
$(1,2,1,2)$	$108000x^8 + 378000x^6 + 216000x^4 + 27000x^2$
$(1,2,2,0)$	$37800x^7 + 94500x^5 + 37800x^3$
$(1,2,2,1)$	$151200x^8 + 529200x^6 + 302400x^4 + 37800x^2$
$(1,2,2,2)$	$567000x^9 + 3118500x^7 + 3118500x^5 + 850500x^3$

Table B.27: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,0,0,1,0,2)$, Part 2

(q_1,q_2,q_3,q_4)	$A(x)$
$(2,0,0,0)$	$120x^4+60x^2$
$(2,0,0,1)$	$480x^5+480x^3+120x$
$(2,0,0,2)$	$1800x^6+4500x^4+1800x^2$
$(2,0,1,0)$	$800x^5+800x^3+200x$
$(2,0,1,1)$	$3200x^6+5600x^4+2000x^2$
$(2,0,1,2)$	$12000x^7+42000x^5+24000x^3+3000x$
$(2,0,2,0)$	$4200x^6+10500x^4+4200x^2$
$(2,0,2,1)$	$16800x^7+58800x^5+33600x^3+4200x$
$(2,0,2,2)$	$63000x^8+346500x^6+346500x^4+94500x^2$
$(2,1,0,0)$	$720x^5+360x^3$
$(2,1,0,1)$	$2880x^6+2880x^4+720x^2$
$(2,1,0,2)$	$10800x^7+27000x^5+10800x^3$
$(2,1,1,0)$	$4800x^6+4800x^4+1200x^2$
$(2,1,1,1)$	$19200x^7+33600x^5+12000x^3$
$(2,1,1,2)$	$72000x^8+252000x^6+144000x^4+18000x^2$
$(2,1,2,0)$	$25200x^7+63000x^5+25200x^3$
$(2,1,2,1)$	$100800x^8+352800x^6+201600x^4+25200x^2$
$(2,1,2,2)$	$378000x^9+2079000x^7+2079000x^5+567000x^3$
$(2,2,0,0)$	$3600x^6+3600x^4+900x^2$
$(2,2,0,1)$	$14400x^7+21600x^5+10800x^3+1800x$
$(2,2,0,2)$	$54000x^8+162000x^6+121500x^4+27000x^2$
$(2,2,1,0)$	$24000x^7+36000x^5+18000x^3+3000x$
$(2,2,1,1)$	$96000x^8+216000x^6+144000x^4+30000x^2$
$(2,2,1,2)$	$360000x^9+1440000x^7+1350000x^5+450000x^3+45000x$
$(2,2,2,0)$	$126000x^8+378000x^6+283500x^4+63000x^2$
$(2,2,2,1)$	$504000x^9+2016000x^7+1890000x^5+630000x^3+63000x$
$(2,2,2,2)$	$1890000x^{10}+11340000x^8+15592500x^6+8032500x^4+1417500x^2$

Table B.28: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n=4$, $(s_{1,2},s_{1,3},s_{1,4},s_{2,3},s_{2,4},s_{3,4})=(1,0,0,1,0,2)$, Part 3

B.6 Formulas for $n = 4$, Non-Tree Support Graph

(q_1, q_2, q_3, q_4)	$A(x)$
$(0,0,0,0)$	$24x^2$
$(0,0,0,1)$	$72x^3$
$(0,0,0,2)$	$240x^4 + 120x^2$
$(0,0,1,0)$	$96x^3 + 48x$
$(0,0,1,1)$	$288x^4 + 144x^2$
$(0,0,1,2)$	$960x^5 + 960x^3 + 240x$
$(0,0,2,0)$	$360x^4 + 720x^2$
$(0,0,2,1)$	$1080x^5 + 2160x^3$
$(0,0,2,2)$	$3600x^6 + 9000x^4 + 3600x^2$
$(0,1,1,0)$	$384x^4 + 480x^2$
$(0,1,1,1)$	$1152x^5 + 1440x^3$
$(0,1,1,2)$	$3840x^6 + 6720x^4 + 2400x^2$
$(0,1,2,0)$	$1440x^5 + 4320x^3 + 720x$
$(0,1,2,1)$	$4320x^6 + 12960x^4 + 2160x^2$
$(0,1,2,2)$	$14400x^7 + 50400x^5 + 28800x^3 + 3600x$
$(0,2,2,0)$	$5400x^6 + 27000x^4 + 16200x^2$
$(0,2,2,1)$	$16200x^7 + 81000x^5 + 48600x^3$
$(0,2,2,2)$	$54000x^8 + 297000x^6 + 297000x^4 + 81000x^2$
$(1,0,0,0)$	$160x^3 + 80x$
$(1,0,0,1)$	$480x^4 + 240x^2$
$(1,0,0,2)$	$1600x^5 + 1600x^3 + 400x$
$(1,0,1,0)$	$640x^4 + 800x^2$
$(1,0,1,1)$	$1920x^5 + 2400x^3$
$(1,0,1,2)$	$6400x^6 + 11200x^4 + 4000x^2$
$(1,0,2,0)$	$2400x^5 + 7200x^3 + 1200x$
$(1,0,2,1)$	$7200x^6 + 21600x^4 + 3600x^2$
$(1,0,2,2)$	$24000x^7 + 84000x^5 + 48000x^3 + 6000x$

Table B.29: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,1,1,1,0,0)$, Part 1

(q_1, q_2, q_3, q_4)	$A(x)$
$(1,1,1,0)$	$2560x^5 + 6080x^3$
$(1,1,1,1)$	$7680x^6 + 18240x^4$
$(1,1,1,2)$	$25600x^7 + 73600x^5 + 30400x^3$
$(1,1,2,0)$	$9600x^6 + 43200x^4 + 12000x^2$
$(1,1,2,1)$	$28800x^7 + 129600x^5 + 36000x^3$
$(1,1,2,2)$	$96000x^8 + 480000x^6 + 336000x^4 + 60000x^2$
$(1,2,2,0)$	$36000x^7 + 252000x^5 + 180000x^3 + 18000x$
$(1,2,2,1)$	$108000x^8 + 756000x^6 + 540000x^4 + 54000x^2$
$(1,2,2,2)$	$360000x^9 + 2700000x^7 + 3060000x^5 + 1080000x^3 + 90000x$
$(2,0,0,0)$	$840x^4 + 1680x^2$
$(2,0,0,1)$	$2520x^5 + 5040x^3$
$(2,0,0,2)$	$8400x^6 + 21000x^4 + 8400x^2$
$(2,0,1,0)$	$3360x^5 + 10080x^3 + 1680x$
$(2,0,1,1)$	$10080x^6 + 30240x^4 + 5040x^2$
$(2,0,1,2)$	$33600x^7 + 117600x^5 + 67200x^3 + 8400x$
$(2,0,2,0)$	$12600x^6 + 63000x^4 + 37800x^2$
$(2,0,2,1)$	$37800x^7 + 189000x^5 + 113400x^3$
$(2,0,2,2)$	$126000x^8 + 693000x^6 + 693000x^4 + 189000x^2$
$(2,1,1,0)$	$13440x^6 + 60480x^4 + 16800x^2$
$(2,1,1,1)$	$40320x^7 + 181440x^5 + 50400x^3$
$(2,1,1,2)$	$134400x^8 + 672000x^6 + 470400x^4 + 84000x^2$
$(2,1,2,0)$	$50400x^7 + 352800x^5 + 252000x^3 + 25200x$
$(2,1,2,1)$	$151200x^8 + 1058400x^6 + 756000x^4 + 75600x^2$
$(2,1,2,2)$	$504000x^9 + 3780000x^7 + 4284000x^5 + 1512000x^3 + 126000x$
$(2,2,2,0)$	$189000x^8 + 1890000x^6 + 2268000x^4 + 756000x^2$
$(2,2,2,1)$	$567000x^9 + 5670000x^7 + 6804000x^5 + 2268000x^3$
$(2,2,2,2)$	$1890000x^{10} + 19845000x^8 + 32130000x^6 + 18900000x^4 + 3780000x^2$

Table B.30: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,1,1,1,0,0)$, Part 2

(q_1, q_2, q_3, q_4)	$A(x)$
$(0,0,0,0)$	$16x^2$
$(0,0,0,1)$	$64x^3 + 32x$
$(0,0,0,2)$	$240x^4 + 480x^2$
$(0,0,1,1)$	$256x^4 + 320x^2$
$(0,0,1,2)$	$960x^5 + 2880x^3 + 480x$
$(0,0,2,2)$	$3600x^6 + 18000x^4 + 10800x^2$
$(0,1,0,1)$	$256x^4 + 320x^2$
$(0,1,0,2)$	$960x^5 + 2880x^3 + 480x$
$(0,1,1,1)$	$1024x^5 + 2432x^3$
$(0,1,1,2)$	$3840x^6 + 17280x^4 + 4800x^2$
$(0,1,2,1)$	$3840x^6 + 17280x^4 + 4800x^2$
$(0,1,2,2)$	$14400x^7 + 100800x^5 + 72000x^3 + 7200x$
$(0,2,0,2)$	$3600x^6 + 18000x^4 + 10800x^2$
$(0,2,1,2)$	$14400x^7 + 100800x^5 + 72000x^3 + 7200x$
$(0,2,2,2)$	$54000x^8 + 540000x^6 + 648000x^4 + 216000x^2$
$(1,1,1,1)$	$4096x^6 + 16640x^4$
$(1,1,1,2)$	$15360x^7 + 103680x^5 + 36480x^3$
$(1,1,2,2)$	$57600x^8 + 576000x^6 + 460800x^4 + 72000x^2$
$(1,2,1,2)$	$57600x^8 + 576000x^6 + 460800x^4 + 72000x^2$
$(1,2,2,2)$	$216000x^9 + 3024000x^7 + 3888000x^5 + 1512000x^3 + 108000x$
$(2,2,2,2)$	$810000x^{10} + 15390000x^8 + 27540000x^6 + 17820000x^4 + 4050000x^2$

Table B.31: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,0,1,1,0,1)$

(q_1,q_2,q_3,q_4)	$A(x)$
$(0,0,0,0)$	$72x^3+72x$
$(0,0,0,1)$	$288x^4+576x^2$
$(0,0,0,2)$	$1080x^5+4320x^3+1080x$
$(0,0,1,1)$	$1152x^5+3456x^3+576x$
$(0,0,1,2)$	$4320x^6+21600x^4+12960x^2$
$(0,0,2,2)$	$16200x^7+113400x^5+145800x^3+16200x$
$(0,1,0,0)$	$360x^4+1080x^2$
$(0,1,0,1)$	$1440x^5+6240x^3+960x$
$(0,1,0,2)$	$5400x^6+36000x^4+23400x^2$
$(0,1,1,1)$	$5760x^6+32640x^4+13440x^2$
$(0,1,1,2)$	$21600x^7+172800x^5+180000x^3+14400x$
$(0,1,2,2)$	$81000x^8+837000x^6+1539000x^4+459000x^2$
$(0,2,0,0)$	$1512x^5+10080x^3+3528x$
$(0,2,0,1)$	$6048x^6+50400x^4+34272x^2$
$(0,2,0,2)$	$22680x^7+249480x^5+355320x^3+52920x$
$(0,2,1,1)$	$24192x^7+241920x^5+258048x^3+20160x$
$(0,2,1,2)$	$90720x^8+1149120x^6+2177280x^4+665280x^2$
$(0,2,2,2)$	$340200x^9+5216400x^7+14742000x^5+9525600x^3+793800x$
$(1,1,0,0)$	$1800x^5+10200x^3+2400x$
$(1,1,0,1)$	$7200x^6+53600x^4+25600x^2$
$(1,1,0,2)$	$27000x^7+276000x^5+309000x^3+36000x$
$(1,1,1,1)$	$28800x^7+265600x^5+224000x^3$
$(1,1,1,2)$	$108000x^8+1296000x^6+2100000x^4+384000x^2$
$(1,1,2,2)$	$405000x^9+5985000x^7+15255000x^5+6975000x^3+540000x$
$(1,2,0,0)$	$7560x^6+75600x^4+68040x^2$
$(1,2,0,1)$	$30240x^7+369600x^5+473760x^3+33600x$
$(1,2,0,2)$	$113400x^8+1751400x^6+3666600x^4+1272600x^2$
$(1,2,1,1)$	$120960x^8+1747200x^6+3104640x^4+470400x^2$
$(1,2,1,2)$	$453600x^9+8013600x^7+21722400x^5+10130400x^3+504000x$
$(1,2,2,2)$	$1701000x^{10}+35532000x^8+134190000x^6+111888000x^4+22869000x^2$

Table B.32: Values of $A_n^{(\mathbf{q};\mathbf{s})}$ for $n=4$, $(s_{1,2},s_{1,3},s_{1,4},s_{2,3},s_{2,4},s_{3,4})=(1,1,1,1,1,0)$, Part 1

(q_1, q_2, q_3, q_4)	$A(x)$
$(2,2,0,0)$	$31752x^7 + 476280x^5 + 920808x^3 + 158760x$
$(2,2,0,1)$	$127008x^8 + 2257920x^6 + 5447232x^4 + 1693440x^2$
$(2,2,0,2)$	$476280x^9 + 10266480x^7 + 34186320x^5 + 24131520x^3 + 2381400x$
$(2,2,1,1)$	$508032x^9 + 10442880x^7 + 31667328x^5 + 13829760x^3 + 705600x$
$(2,2,1,2)$	$1905120x^{10} + 46357920x^8 + 189665280x^6 + 160030080x^4 + 30693600x^2$
$(2,2,2,2)$	$7144200x^{11} + 200831400x^9 + 1063692000x^7$ $+1351047600x^5 + 556453800x^3 + 35721000x$

Table B.33: Values of $A_n^{(\mathbf{q;s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,1,1,1,1,0)$, Part 2

(q_1, q_2, q_3, q_4)	$A(x)$
$(0,0,0,0)$	$162x^4 + 1134x^2$
$(0,0,0,1)$	$810x^5 + 8910x^3 + 3240x$
$(0,0,0,2)$	$3402x^6 + 56700x^4 + 75978x^2$
$(0,0,1,1)$	$4050x^6 + 62550x^4 + 63000x^2$
$(0,0,1,2)$	$17010x^7 + 366660x^5 + 833490x^3 + 143640x$
$(0,0,2,2)$	$71442x^8 + 2008314x^6 + 7707798x^4 + 4500846x^2$
$(0,1,1,1)$	$20250x^7 + 411750x^5 + 780000x^3 + 84000x$
$(0,1,1,2)$	$85050x^8 + 2286900x^6 + 7821450x^4 + 3414600x^2$
$(0,1,2,2)$	$357210x^9 + 12105450x^7 + 63146790x^5 + 61188750x^3 + 6085800x$
$(0,2,2,2)$	$1500282x^{10} + 62011656x^8 + 449973468x^6 + 744806664x^4 + 241989930x^2$
$(1,1,1,1)$	$101250x^8 + 2598750x^6 + 7920000x^4 + 2340000x^2$
$(1,1,1,2)$	$425250x^9 + 13891500x^7 + 67247250x^5 + 51576000x^3 + 2940000x$
$(1,1,2,2)$	$1786050x^{10} + 71640450x^8 + 494779950x^6 + 698786550x^4 + 161847000x^2$
$(1,2,2,2)$	$7501410x^{11} + 360067680x^9 + 3327847740x^7$ $+7313596920x^5 + 3743759250x^3 + 250047000x$
$(2,2,2,2)$	$31505922x^{12} + 1774833606x^{10} + 21063848148x^8$ $+65431965564x^6 + 57101566410x^4 + 12125890350x^2$

Table B.34: Values of $A_n^{(\mathbf{q;s})}$ for $n = 4$, $(s_{1,2}, s_{1,3}, s_{1,4}, s_{2,3}, s_{2,4}, s_{3,4}) = (1,1,1,1,1,1)$

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