

Standard model physics and beyond from non-commutative geometry.

by

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Abstract

Non-commutative differential geometry (NCG) [Con96] extends Riemannian geometry and yields a striking reinterpretation of the standard model of particle physics (SM) as gravity on a ‘non-commutative’ manifold [CCM07]. The basic idea behind NCG is to shift focus away from topological spaces and manifolds, to instead focus on the algebra of functions defined over them (“the algebra of coordinates”). This simple idea allows one to explore geometries where one has only the algebra and there is no classical notion of the underlying space whatsoever [Mar]. In particular, this idea extends to the case in which the input algebra is non-commutative. In this thesis we go a step further: we propose a simple reformulation of the input data corresponding to NCG, which naturally extends to describe geometries which may also be non-associative.

The content of our reformulation is as follows: In the traditional approach to NCG, one replaces the usual geometric data of manifolds and metrics $\{M, g\}$ with ‘spectral’ data held in so called ‘spectral triples’ $T = \{\mathcal{A}, \mathcal{H}, D\}$, consisting of an input algebra \mathcal{A} , and a Dirac operator D represented on a Hilbert space \mathcal{H} . We show that the data held in a spectral triple may be ‘fused’ together into a larger ‘fused’ algebra ΩB [FB15b]. In this way the various elements of NCG are unified together into a single more fundamental object, while their seemingly unrelated geometric axioms and conditions are re-expressed simply and naturally as the intrinsic properties of ΩB . This approach naturally extends to describe non-associative spaces in the sense that ΩB need not be associative. When ΩB has more general associativity properties, then appropriate generalizations of the associative NCG axioms derive readily from the intrinsic properties of ΩB , allowing for the construction of a wide range of non-associative geometries which we showcase in this work.

While our formulation naturally extends to describe non-associative NCG, it also elucidates many aspects of the associative formalism. We show that asking for ΩB to be associative imposes new constraints beyond those traditionally imposed by the NCG axioms. These new constraints resolve a long-standing problem plaguing the NCG construction of the SM, by precisely eliminating from the action a collection of 7 unwanted terms that previously had to be removed by an extra, non-geometric, assumption [FB15b, BBB15]. We also explain how this same reformulation yields a new perspective on the symmetries of a NCG, which arise simply and naturally as the automorphisms of ΩB . Applying this perspective to the NCG traditionally used to describe the SM we find, instead, an extension of the SM by an extra $U(1)$ $B - L$ gauge symmetry, and a single extra complex scalar field σ , which is a singlet under the SM gauge group, but has $B - L = 2$ [FB15b]. The σ field has cosmological implications [BFFS11], and offers a similar solution to the discrepancy between the observed Higgs mass and the NCG prediction as that proposed in [CC12].

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Dedication

To Ian¹, Carolyn², Joshua³, and Benjamin⁴.

¹My fantastic father.

²My marvelous mother.

³The best of big brothers, and his wonderful wife Angela.

⁴The largest of little brothers, and his delightful daughters Annabelle and Eloise.

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Chapter 1

Introduction

The standard model of particle physics (SM)¹, in its current formulation as a quantum field theory (QFT) leaves many foundational questions unanswered. Why for example are there three particle generations, and what mechanism if any singles out the observed SM gauge symmetries? When trying to address such questions in QFT there are unfortunately very few theoretical constraints aside from renormalisability, gauge invariance, and anomaly cancellation. As a result, an abundance of models describing beyond the SM (BSM) physics have emerged. In this work I explore the idea that geometry might provide helpful guidance in singling out those models of greatest relevance; and in particular I will focus on Non-commutative differential geometry (NCG), which is an extension of Riemannian geometry that is of particular physical interest since it offers a novel geometric explanation for certain otherwise-unexplained features of the SM. I will describe a reformulation of NCG initiated by myself and Latham Boyle, which on the one hand is simpler and more generally applicable to answering questions in particle theory, while on the other hand it unifies many features of NCG, and readily generalizes to non-associative differential geometry (NAG).

To motivate the use of NCG in particle physics it will be necessary to explain at the outset what geometry is useful for, starting in the most basic setting with three dimensional Euclidean geometry. Euclidean geometry provides a useful framework for describing the world on familiar human scales: that is, at the low speeds we travel at, and over the distances we are accustomed to. Most importantly, within their range of applicability Euclid's axioms may be used to infer information without ever having to make a direct

¹Here and everywhere else in the text I refer to the ν MSM simply as the standard model, i.e. the base standard model with one extra right handed neutrino per generation [AS05].

measurement, a fact that surveyors take full advantage of every day around the world. That is, to be explicit: the rules of Euclidean geometry allow one to determine all the angles and side lengths of a geometric problem given some suitable subset as input. This usefulness is of course in no way restricted to Euclidean geometry. Breakthroughs in physics often follow from a new framework for describing the geometry of spacetime, and perhaps the best example is seen in the work of Einstein, who taught us to replace Euclid's flat 3D space, first with Minkowski's flat 4D spacetime, and then with Riemann's curved 4D spacetime. Astronomers and cosmologists are now quite comfortable making use of Riemannian geometry to infer information about the curvature of the intervening space between the earth and distant lensed galaxies. One might also argue that black-holes, gravitational waves, and even the big bang are predictions which have arisen from our improved understanding of geometry². Can the same be said for particle physics? Is there a geometric framework appropriate for describing SM physics, and if so can we use it to infer information about the SM and its possible extensions?

Connes' NCG is a generalization of Riemannian geometry that has been developed extensively by pure mathematicians (see *e.g.* [Con94, Lan97, CM08]). It is of physical interest because it provides an elegant framework for describing Yang-Mills theories, coupled to Einstein-Hilbert gravity. When applied to the SM of particle physics, it provides a tighter and more elegant framework for formulating the SM particle content and Lagrangian, including its coupling to Einstein gravity [CL91, Con95a, Con96, CC96, CC97, Bar07, Con06b, CCM07, CC08, CC07] (for a pedagogical introduction, see [vdDvS12, vS15]). In particular, the NCG axioms place heavy restrictions on the allowed particle content, along with its representation under the various gauge symmetries. It is natural to ask whether the axioms of NCG together with the known particle content might also be used to uncover information about particles which as yet remain undiscovered. On initial inspection the outlook is promising: a quick review of the literature [Con96] reveals that indeed NCG did 'predict' the SM Higgs prior to the 2013 detection at CERN [A+12, C+12b], and it also 'postdicts' Majorana neutrinos [CCM07] [Bar07]³. Unfortunately, the formalism also runs almost immediately into problems. For example, an incorrect Higgs mass at 170GeV

²Although technically these ideas all rely on additional information: the equations of motion which describe the dynamics of a geometry.

³Although this could very easily have been a 'prediction' as right handed neutrinos arose in the formalism rather unexpectedly and only a few years after the discovery of neutrino-oscillations when attempting to solve fermion doubling in the NCG SM! In conversations with Alain Connes he claims that Jaques Dixmier had been encouraging him ever since the 98 measurements by Super Kamiokande and SNOLab, to incorporate right handed neutrinos and that he was trying to do so in a model with 16 fermions per generation instead of 15. But the 'Majorana' mass terms in the model were a surprise to him, as he had never heard of the 'see-saw mechanism'.

was initially predicted by the NCG formalism [CIS99, KS06, CCM07], and to obtain the SM scalar content an extra ad-hoc and non-geometric constraint known as the ‘massless photon condition’ is imposed [CC08, CCM07]⁴. Can the formalism be salvaged and made useful for particle physics model building?

Solutions to the above mentioned problems are found via a rather unexpected route: in developing a reformulation of NCG which is general enough to describe non-associative geometries. This work will focus on the generalization from NCG to NAG, its application to the (associative) standard model, and to developing SM extensions. The organization is as follows: in the present section I explain how the work of my thesis fits into the literature. I introduce NCG, focusing on its successes, and on its failures, and explain the motivation for generalizing to non-associative differential geometries. I point to previous work, and summarize the salient features of the non-associative generalization. In Chapter 2 I provide the relevant background for understanding the details of my work, including a review of the necessary mathematical preliminaries, and the NCG formalism in the traditional approach. In Chapter 3 I introduce our reformulation of the NCG input data which generalizes naturally to describe non-associative geometries. The main idea is that the usual NCG input data consisting of five objects $\{\mathcal{A}, \mathcal{H}, D, J, \gamma\}$ is ‘fused’ together into an algebra ΩB , while at the same time many of the NCG axioms derive from the properties of ΩB (such as its associativity). In Chapter 4 I apply this reformulation to the associative NCG SM, and show how it provides natural solutions to a number of outstanding problems. In Chapter 5 I provide example non-associative geometries which correspond to regular associative Yang-Mills theories, before finally closing in Chapter 6.

1.1 What is NCG?

The key idea behind NCG is to shift attention away from topological spaces and manifolds, to instead focus on the algebra of functions defined over them (“the algebra of coordinates”). This simple idea allows one to explore geometries where one has only the algebra and there is no classical notion of the underlying space whatsoever. Alain Connes established NCG in the 80’s and 90’s [Con85, Con94] as an extension of Riemannian differential geometry, much as Riemannian geometry generalizes Euclidean geometry. Just as Riemann extended the framework of geometry to include spaces that are curved, NCG further extends it to include spaces that are “noncommutative” in a sense I will explain

⁴It should be stated at the outset that while the so called ‘massless photon condition’ is ad hoc and lacks geometric motivation, it *is* exceptionally elegant in the sense that it may be stated simply and concisely using only the input data of the NCG SM.

below. In Riemann’s approach, a geometry is specified by providing the following data: the manifold \mathcal{M} and its metric $g_{\mu\nu}$. In the spectral approach, one instead specifies a geometry by providing a so-called “spectral triple” $\{\mathcal{A}, \mathcal{H}, D\}$: here \mathcal{A} is an algebra, which is represented by linear operators acting on the Hilbert space \mathcal{H} , and D is an additional Hermitian operator on \mathcal{H} [Con96]. In fact, to *fully* specify a spectral geometry, one often needs to give two additional operators, called γ and J [Con95a], so that the full spectral data is $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$. Notice that the data $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$ is still called a “spectral triple” in the literature, even though it contains 5 elements! (The various elements of the spectral triple, and their meaning, are explained below.)

Before describing general non-commutative geometries let’s first focus on the canonical example: that of Riemannian geometries. Just as Riemannian geometry contains Euclidean geometry, and reduces to Euclidean geometry for a special class of Riemannian data⁵, NCG contains Riemannian geometry⁶, and reduces to Riemannian geometry for a special class of spectral data: namely, when the spectral data $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$ is given by the so-called “canonical spectral triple” $M = \{\mathcal{A}_c, \mathcal{H}_c, D_c, \gamma_c, J_c\}$. In the canonical triple, the input algebra $\mathcal{A}_c = C_\infty(\mathcal{M})$ is the algebra of smooth complex-valued functions over the manifold \mathcal{M} ; $\mathcal{H}_c = L^2(\mathcal{M}, S)$ is the Hilbert space of (square integrable) Dirac spinors on $\{\mathcal{M}, g_{\mu\nu}\}$; $D_c = -i\gamma^\mu \nabla_\mu^S$ is the ordinary curved-space Dirac operator on $\{\mathcal{M}, g_{\mu\nu}\}$; γ_c is the chirality operator on \mathcal{H}_c (*i.e.* what physicists usually call γ_5 in 4 dimensions); and J_c is the charge conjugation operator on \mathcal{H}_c . As for the representation of \mathcal{A}_c on \mathcal{H}_c , the functions $f \in \mathcal{A}_c$ act on the spinor fields $\psi \in \mathcal{H}_c$ by pointwise multiplication: $\psi(x) \rightarrow f(x)\psi(x)$. The idea is that the Riemannian data $\{\mathcal{M}, g_{\mu\nu}\}$ and the canonical triple M provide dual descriptions of the same geometry, so that the canonical spectral triple may be obtained from the Riemannian data, or vice versa. Following attempts by Rennie and Varilly [RV06] Connes gave a formal proof of this correspondence in [Con13] (see also [GB01] for work on the reconstruction of manifolds and also [Ć12a, Ć14] for more recent work on reconstruction theorems).

When reconstructing a Riemannian geometry from canonical spectral data, specifying the algebra \mathcal{A}_c amounts to specifying the manifold \mathcal{M} , while specifying the operator D_c amounts to specifying the metric $g_{\mu\nu}$. Let’s unpack these statements a bit. How does the algebra \mathcal{A}_c encode the manifold \mathcal{M} ? On the one hand every Riemannian manifold can be seen to give rise to an algebra of smooth functions (*i.e.* maps from the manifold to the complex numbers). Proving the reverse statement is more difficult. According to the commutative Gelfand-Naimark theorem, any commutative algebra (or, more correctly, any

⁵Namely, when the manifold \mathcal{M} is given by \mathbb{R}^n and the metric $g_{\mu\nu}$ is given by the flat Euclidean metric $\delta_{\mu\nu}$.

⁶Technically Riemannian spin geometry *i.e.* Riemannian manifolds which admit spinors.

commutative C^* -algebra) is equivalent to the algebra of complex-valued functions over a certain topological space $X_{\mathcal{A}}$ encoded by \mathcal{A} , where $X_{\mathcal{A}}$ is the space of “characters” of \mathcal{A} – *i.e.* $*$ -homomorphisms $\phi : \mathcal{A} \rightarrow \mathbb{C}$. In particular, for the algebra $\mathcal{A}_c = C^\infty(\mathcal{M})$, the characters (which are in one-to-one correspondence with the points $p \in \mathcal{M}$) are precisely the maps $\phi_p : \mathcal{A} \rightarrow \mathbb{C}$ given by $\phi_p(f) = f(p)$; and $\text{Diff}(\mathcal{M})$ (the group of diffeomorphisms of the manifold \mathcal{M}) is nothing but $\text{Aut}(\mathcal{A}_c)$ (the group of automorphism of the algebra \mathcal{A}_c). Next let us see how D_c (a differential operator) interacts with \mathcal{A}_c (the algebra of functions on \mathcal{M}) to encode the metric $g_{\mu\nu}$ ⁷. To see how differentiation of functions on a manifold may be translated into distances, it is enough to consider a simple 1-dimensional example: we can re-express the distance $|x_a - x_b|$ between two points x_a and x_b on the real line in a dual fashion, as the maximum possible excursion $|f(x_a) - f(x_b)|$ that any function $f(x)$ can make between those two points, subject to the constraint that its derivative cannot be too large ($|df/dx| \leq 1$). But to re-express distances in this way notice that we need a derivative operator which interacts with the functions. This example is just meant to convey the key idea: I provide more details of the reconstruction of Riemannian geometries from spectral data in subsection 2.2.1, and the reader may also consult [Con94, RV06, Č12a, Con13, Con95a, Con96, GB01, Č12a, Č14] and references therein for further information.

Finally, the other elements of the canonical spectral triple also have geometric meaning: the operator γ_c , which provides a notion of chirality (left or right handedness) of spinors, encodes the volume form of the underlying Riemannian geometry; and the anti-unitary operator J_c , which maps spinors to their anti-spinors, provides a “real structure” to the geometry, much as the operation of complex conjugation selects a preferred line (the real line) inside the complex plane.

In order to describe the dynamics of a NCG there is a natural action functional which assigns a real number to each spectral triple. It is given by the simple formula⁸

$$S = \text{Tr}[f(D/\Lambda)] + \langle \psi | D | \psi \rangle \quad (1.1)$$

where $f(x)$ is a real, even function of a single variable, which vanishes rapidly for $|x| > 1$; and ψ is an element of the Hilbert space \mathcal{H} . The first term in the action $S_b = \text{Tr}[f(D/\Lambda)]$ is known as the ‘spectral action’ and when describing physical theories it determines the dynamics of the bosonic degrees of freedom. The second term $S_f = \langle \psi | D | \psi \rangle$ is known as the ‘fermionic action’, and describes the dynamics of the fermionic degrees of freedom. This

⁷Intuitively, the Dirac operator \mathcal{D} knows about the metric since its square is the Laplacian, whose principle symbol is $g^{\mu\nu}$: $\mathcal{D}^2 = g^{\mu\nu} \partial_\mu \partial_\nu + \dots$

⁸Note that actually the fermionic part of the action will depend on the ‘KO-dimension’ of the model being constructed. For Euclidean models in KO-dimension 2 (the dimension of physical interest) the fermionic part of the action is given by $\langle J\psi | D | \psi \rangle$.

general expression may be derived from the “spectral action principle” (*i.e.* the requirement that the action should only depend on the spectrum of the operator D), together with the requirement that the action of the union of two geometries is the sum of their respective actions, as usual [CC96, CC97].

When applied to a canonical spectral triple M , the spectral action term $\text{Tr}[f(D_c/\Lambda)]$ may be expanded in powers of Λ , using the standard heat kernel expansion [Vas03, Gil84]; the leading terms in this expansion are given by

$$\text{Tr}[f(D_c/\Lambda)] = \int d^4x \sqrt{g} \left(\frac{f_4 \Lambda^4}{2\pi^2} - \frac{f_2 \Lambda^2}{24\pi^2} R + \dots \right) \quad (1.2)$$

where $f_n \equiv \int_0^\infty f(x) x^{n-1} dx$. In other words, this term reduces to the ordinary action for Einstein gravity (cosmological constant term plus Einstein-Hilbert term), so that the term $\text{Tr}[f(D/\Lambda)]$ may be regarded as the natural generalization of Einstein gravity in the context of spectral geometry. Meanwhile, the fermionic term in (1.1) becomes:

$$\langle \psi | D_c | \psi \rangle = \int d^4x \sqrt{g} \psi^\dagger(x) D_c \psi(x) = -i \int d^4x \sqrt{g} \psi^\dagger \gamma^\mu \nabla_\mu^S \psi(x). \quad (1.3)$$

In other words, this term reduces to the Euclidean action for a single, massless Dirac fermion.

So much for the correspondence between canonical spectral triples and Riemannian manifolds, what about the more general case? Once geometry has been reformulated in spectral terms, one finds that it naturally extends to a class of spectral triples that is far broader than the subclass of canonical spectral triples. Thus one obtains a natural extension of Riemannian geometry, capable of handling many mathematical spaces that lie beyond the boundaries of ordinary Riemannian geometry. In particular, although the algebra \mathcal{A}_c in the canonical triple is commutative, the algebra \mathcal{A} in the general triple need not be; this is the reason behind the name ‘non-commutative geometry’.

To give a warm-up (but very important) example, consider the simple $SU(N)$ constructions outlined in [CC97, §2], and also in Subsection 3.1.5 of this work: These constructions take as input the algebra of complex $N \times N$ matrices $A_F = M_N(\mathbb{C})$, where A_F is represented on itself, *ie.* $H_F = M_N(\mathbb{C})$. The real structure operator is given by the adjoint on matrices, $J_F h = (h)^*$ for $h \in \mathcal{H}_F$, while the grading on \mathcal{H} is given by the identity operator $\gamma_F = \mathbb{I}_N$. Compatibility with the chirality (*i.e.* $\{\gamma, D\} = 0$ (see section 2.2.2)) then forces $D_F = 0$. This simple finite non-commutative spectral triple $\{A_F, H_F, D_F, J_F, \gamma_F\} = \{M_N(\mathbb{C}), M_N(\mathbb{C}), 0, *, \mathbb{I}\}$ describes a zero dimensional geometry with a symmetry group given by the automorphisms of the input algebra: $SU(N)$. I will discuss at length the symmetries of a NCG in chapter 3.

While ‘non-commutative’ geometries are exciting to explore from a purely mathematical standpoint, their real interest (at least for physicists) lies in their applications to physics. In particular, just as Riemannian geometry provides an elegant framework for describing gravitational theories, NCG provides an economical framework within which to describe more complicated gauge theories as ‘gravity’ on non-commutative space-times. So far the NCGs of interest in such applications seem to be the so called ‘almost-commutative’ geometries: that is, geometries which are formed as the product between a canonical triple M and a finite dimensional non-commutative spectral triple F based on finite dimensional matrix algebras:

$$T = M \times F. \tag{1.4}$$

The basic idea behind almost-commutative geometries is very similar to that of Kaluza-Klein theories: the ‘internal’ gauge degrees of freedom of a physical theory are described at each point on the manifold by the finite space F , with the gauge symmetries arising roughly speaking as the ‘inner automorphisms’ of the finite input algebra. The continuous ‘external’ degrees of freedom are described by the canonical space M , with the diffeomorphisms of the manifold arising as the ‘outer automorphisms’ of the canonical input algebra. This construction is actually quite restrictive, and the vast majority of gauge theories cannot be constructed as a NCG. Rather remarkably however, the standard model of particle physics can be constructed in just this way as an almost-commutative geometry.

When applying the spectral action (1.1) to almost-commutative spectral triples, the key result is that, for a certain (rather simple and natural) class of almost-commutative spectral triples $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$, the spectral action (1.1) reduces precisely to the action for Einstein gravity coupled to the full $SU_c(3) \times SU_w(2) \times U_y(1)$ standard model of particle physics, in all its detail. In particular, $\text{Tr}[f(D/\Lambda)]$ produces the bosonic terms in the action (the gravitational terms, the kinetic terms for the gauge bosons, and the kinetic and potential terms for the Higgs doublet), while $\langle \psi | D | \psi \rangle$ produces the fermionic terms (the kinetic terms for the leptons and quarks, their Yukawa interactions with the Higgs doublet, and the neutrino mass terms). For the detailed derivation of this result, see [CC97, CC96, CCM07, Con06b, vdDvS12, CC08]. I will also discuss the construction the NCG SM in Ch. 4 and Appendix A.

It is worth pausing to emphasize the difference between this perspective on noncommutativity, and the one more commonly encountered in the physics literature. Often, noncommutative geometry is taken to mean the noncommutativity of the 4-dimensional spacetime coordinates themselves; it is regarded as a property of quantum gravity that presumably becomes manifest at the Planck energy scale (*i.e.* the exceedingly high energy scale of 10^{19} GeV). By contrast, from the perspective of the spectral reformulation of

the standard model, all of the non-gravitational fields in nature at low energies are reinterpreted as the direct manifestations of noncommutative geometry, right in front of our nose, staring us in the face!

1.2 A brief historical overview

Although not crucial for understanding the rest of the work, some readers might appreciate a short historical interlude outlining the development of NCG, and in particular the construction of the SM as a NCG. I will provide a rapid-fire and incomplete history here: Attempts towards the construction of a NCG SM date back to the work of Dubois-Violette, Madore, and Kerner in [MDV89b, MDV89a, MDV90], and to that of Connes and Lott in [CL91, Con95b], followed by the work of Iochum and Schücker in [IS96a, Ste09a, IS94, IS96b] and Krajewski in [Kra98]. The earliest models displayed many of the interesting features of the current construction, but did not make use of a real structure operator J , relying instead on so called ‘bi-vector’ potentials. The earliest models also did not feature gravity. Real structure operators were introduced by Connes in [Con95a]. Likewise, it was the later work of Chamseddine and Connes which introduced the current spectral action, and showed how to unify the NCG SM with gravity [Con96, CC96, CC97]. The early models also suffered from a ‘fermion quadrupling problem’, as was pointed out by Lizzi *et al.* [LMMS97]. This problem was solved independently by Connes and Barrett [Bar07, Con06b] by changing the so called ‘KO-signature’ of the NCG SM internal space from zero to 6. This shift in KO-dimension also allowed for a description of right handed neutrinos and neutrino mixing as shown in the work of Chamseddine, Connes, and Marcolli [CCM07], and Barrett [Bar07].

The NCG SM is in very good agreement with phenomenology, however based on a minimal set of assumptions (such as the big desert hypothesis) it made an early prediction for the SM Higgs mass at approximately 170GeV [CIS99, KS06, CIS99]. This prediction was disfavored by the Tevatron, and later ruled out by the LHC. The incorrect Higgs mass prediction prompted the development of models beyond the standard model such as those developed by Stephan [Ste09c, Ste06, Ste08, Ste09b, Ste09a] as well as that constructed by Chamseddine, Connes and van Suijlekom [CCvS13]. A number of minimal solutions to the ‘Higgs mass problem’ were also put forward, notably by Estrada and Marcolli [EM13a] and by Chamseddine and Connes [CC12]. Along with the development of the NCG SM, models of cosmology based on NCG have also been developed, notably by Marcolli and collaborators [Mar11, MP10, MPT11, MPT12, EM13b]

My own work with Latham Boyle [FB13, BF14, FB15b] enters the field quite late and

focuses on cleaning up various aspects of the NCG SM construction *and uncovering their true meaning(!)*, using NCG to explore beyond the standard model physics, and on generalizing the NCG framework to describe non-associative geometries. My work is contemporary with that of Beenakker *et al.* [[BvSB14c](#), [BvSB14b](#), [BvSB14a](#)], and Ishihara *et al.* [[IKM⁺14a](#), [IKM⁺14b](#), [IKM⁺15](#)] which focuses on supersymmetric extensions of the standard model, as well as the work of Chamseddine *et al.* which focuses on Pati-Salam type extensions of the NCG SM [[CCvS13](#), [CCv13](#)], the ‘grand symmetry’ approach of Devastato *et al.* [[DLM14a](#)] which focus on a unification with gravity, and the work of Dungen [[van15](#)] which focuses on Lorentzian spectral triples. For previous work on non-associative geometry in a somewhat different context see [[Wul99b](#), [Wul96](#), [Wul99a](#), [Wul97](#), [AM04](#)].

1.3 What does NCG get right?

What is the physical motivation for reformulating the familiar action for the standard model (coupled to gravity) in the unfamiliar language of spectral triples and the spectral action? In this section I would like to stress that there are two key motivations for the NCG approach: unification and constraint.

Unification

In generalizing from Riemannian geometry to NCG, the algebra \mathcal{A} generalizes the manifold \mathcal{M} ; and the group $\text{Aut}(\mathcal{A})$ of automorphisms of \mathcal{A} generalizes the group $\text{Diff}(\mathcal{M})$ of diffeomorphisms of \mathcal{M} . In particular, in the canonical spectral triple, where $\mathcal{A} = C^\infty(\mathcal{M}, \mathbb{C})$ is the commutative $*$ -algebra of smooth functions $f : \mathcal{M} \rightarrow \mathbb{C}$, we have $\text{Aut}(\mathcal{A}) = \text{Diff}(\mathcal{M})$. Now consider the next simplest case, in which $\mathcal{A} = C^\infty(\mathcal{M}, M_n(\mathbb{C}))$ is the noncommutative $*$ -algebra of smooth functions $f : \mathcal{M} \rightarrow M_n(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the set of $n \times n$ complex matrices (i.e an ‘almost-commutative’ input algebra). In this case, $\text{Aut}(\mathcal{A})$ is the semi-direct product of two groups

$$\text{Aut}(\mathcal{A}) = \text{Map}(\mathcal{M}, SU(N)) \ltimes \text{Diff}(\mathcal{M}) \tag{1.5}$$

where $\text{Map}(\mathcal{M}, SU(N))$ is the group of maps from \mathcal{M} to the group $SU(N)$. Notice that the group on the right-hand side of Eq. (1.5) also has another interpretation: it is the full symmetry group of $SU(N)$ gauge theory coupled to Einstein gravity – namely, the semi-direct product of the group $\text{Map}(\mathcal{M}, SU(N))$ of gauge transformations and the group $\text{Diff}(\mathcal{M})$ of gravitational symmetries. Indeed, if one evaluates the spectral action (1.1) for

a spectral triple based on this algebra $C^\infty(\mathcal{M}, M_n(\mathbb{C}))$, one finds that it reduces to $SU(N)$ gauge theory coupled to Einstein gravity [CC97]. In this example, we see that an elegant and conceptually satisfying picture emerges: the full symmetry group of a gauge theory coupled to gravity is reinterpreted in a unified way as simply $\text{Aut}(\mathcal{A})$, the automorphism group of an underlying algebra; and this, in turn, is interpreted as the group of “purely gravitational” transformations of a corresponding non-commutative space. In essence, this basic picture is also behind the spectral reformulation of the standard model coupled to gravity.

To see what is compelling about this picture, it is interesting to contrast it with Kaluza-Klein (KK) theory. To see the contrast clearly, it is enough to consider the original and simplest KK model. In this model, one starts with the 5D Einstein-Hilbert action $S_{KK} = (16\pi G_5)^{-1} \int d^5X \sqrt{-g_5} R_5$, where X , G_5 , g_5 and R_5 denote the 5D spacetime coordinates, Newton constant, metric determinant and Ricci scalar, respectively. Next, one supposes that the 5D manifold is the product of a 4D manifold and a circle ($\mathcal{M}_5 = \mathcal{M}_4 \times S_1$), where x^μ are the 4D coordinates on \mathcal{M}_4 and z is the coordinate on S_1 . Finally, one writes the general 5D line element in the form

$$ds_5^2 = g_{mn}^{(5)} dX^m dX^n = e^{\varphi/\sqrt{3}} g_{\mu\nu}^{(4)} dx^\mu dx^\nu + e^{-2\varphi/\sqrt{3}} (A_\mu dx^\mu + dz)^2 \quad (1.6)$$

and observes that, if the 5D metric $g_{mn}^{(5)}$ only depends on the 4D coordinates x^μ , then the 5D action S_{KK} reduces to a 4D action of the form

$$S_{KK} = \int d^4x \sqrt{-g_4} \left[\frac{R_4}{16\pi G_4} - \frac{1}{2} (\partial\varphi)^2 - \frac{1}{4} e^{-\sqrt{3}\varphi} F^2 \right] \quad (1.7)$$

where G_4 , g_4 and R_4 are the 4D Newton constant, metric determinant and Ricci scalar, respectively, while $F_{\mu\nu}$ is the 4D Maxwell field strength derived from the 4D gauge potential A_μ . This simple example captures what is appealing about KK theory – that one starts from the simple and purely gravitational action for Einstein gravity in 5D and obtains something tantalizingly close to 4D Einstein gravity plus 4D gauge theory – but it also captures what is unappealing about KK theory. For one thing, one typically obtains extra, unwanted fields with unwanted couplings (in this example, the massless scalar field φ , with its experimentally untenable $e^{-\sqrt{3}\varphi} F^2$ coupling to electromagnetism), and one must explain why these extra fields and couplings are not observed in nature. For another thing, the reduction from the initial 5D action, which has the huge symmetry group $\text{Diff}(\mathcal{M}_5)$, to the final 4D action, which has the much smaller symmetry group $\text{Map}(\mathcal{M}_4, U(1)) \times \text{Diff}(\mathcal{M}_4)$, fundamentally relies on the assumption that the 5D metric $g_{mn}^{(5)}$ only depends on the 4D coordinates x^μ . This assumption is supposed to be justified, in turn, by the fact that the

compactified direction is so small; but this justification assumes that one has stabilized the extra dimension – *i.e.* found a way to make it small and keep it small, without letting it shrink down to a singularity or blow up to macroscopic size. The problem of stabilizing extra dimensions in KK theory is a famously thorny one and, furthermore, is ultimately at the root of the so-called landscape problem in string theory.

Thus the spectral and KK approaches share a similar spirit: in both cases the goal is to reinterpret the action describing ordinary 4-dimensional physics as arising from a simpler action formulated on an “extension” of 4-dimensional spacetime. But the spectral action seems to achieve this goal more elegantly and directly. In KK theory, the starting point is an action with too many fields and too much symmetry, and one must then jump through many hoops to explain why these extra fields and symmetries are unobserved in nature. By contrast, in the NCG approach, the field content and symmetries of the standard model are obtained directly.

In the 5D KK model, three different objects are all packaged together as different parts of the metric on the extended (5D) geometry: (i) the 4D metric, (ii) the 4D gauge field and (iii) the 4D dilaton field. Analogously, in the spectral action, four different objects are unified, in the sense that they are all packaged together as different parts of the Dirac operator D on the extended (spectral) geometry: (i) the 4D Levi-Civita connection (which is related to the 4D metric), (ii) the 4D gauge fields, (iii) the Higgs field and (iv) the matrix of Yukawa couplings. Note that in the spectral case, the 4 objects which are unified in this way are all crucial and experimentally verified components of the standard model of particle physics. In the spectral framework, the remaining fields (*i.e.* the fermions) are nothing but the basis vectors on the Hilbert space \mathcal{H} .

Constraints

The spectral action (1.1) packages all of the complexity of gravity and the standard model of particle physics into two simple and elegant terms which, in turn, follow from a simple principle (the spectral action principle described above). As we will describe, it also ends up explaining aspects of the structure of the SM which are otherwise unexplained. The compactness and tautness of this formulation suggest that it may be a step in the right direction. To give a provocative analogy: much as Minkowski “discovered” that the rather cumbersome Lorentz transformations (which formed the basis of Einstein’s original formulation of special relativity) could be elegantly re-interpreted as the geometrical statement that we live in a 4-dimensional Minkowski spacetime, Chamseddine and Connes seem to have discovered that the rather cumbersome action and particle content for the standard model coupled to gravity can be elegantly re-interpreted as the geometric statement that we

live in a certain type of noncommutative geometry. (Einstein initially rejected Minkowski’s unfamiliar formulation of special relativity as worse than useless, calling it “superfluous learnedness,” and quipping that “since the mathematicians have tackled the theory of relativity, I myself no longer understand it anymore” [Isa07]. Ultimately, of course, it proved to be a crucial step on the road to general relativity.)

The tautness of the NCG formalism is not restricted to the spectral action. The amazing fact is that the set of input data required for constructing the SM as a NCG is greatly reduced when compared with the input required in the more familiar QFT description. To see exactly why this is true, I will provide a side by side comparison of the minimal input data required in each construction. I begin with the familiar QFT construction, where the minimal input data may be provided compactly in three steps, each corresponding roughly to one of the three kinds of particles present in the SM: gauge fields, fermions, and scalars.

	QFT				NCG
Step 1.	$G = U_1 \times SU_2 \times SU_3$				$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$
Step 2.		$SU_c(3)$	$2U_L(2)$	$U_Y(1)$	$\mathcal{H}_F = \mathbb{C}^{96}, \pi$
	q_L	3	3	+1/6	
	u_R	3	1	+2/3	
	d_R	3	1	-1/3	
	l_L	1	2	0	
	ν_R	1	1	-1	
	e_R	1	1	+1/2	
Step 3.	h	1	2	1/2	D_F

Table 1.1: QFT vs NCG: A comparison between SM constructions.

(1) In the first step the symmetries of the theory must be specified, and in particular the gauge symmetries of the model are found experimentally to be $G = U_y(1) \times SU_w(2) \times SU_c(3)$. Once the SM gauge group is provided as input, Yang-Mills theory then says that there exists a gauge field corresponding to each generator of the gauge symmetry, and that these gauge fields must transform into one another in the adjoint representation. The gauge sector of the model is therefore completely specified by the choice of gauge group.

(2) Having specified the gauge sector of the SM, the next step is to outline the fermionic content. There is currently no good understanding for the number of particle species that we observe, and aside from anomaly cancellation there is also little theoretical justification for the various charges and representations under the SM gauge group. All of this information must be provided as input.

(3) Once the fermionic and gauge sectors of the model are specified, the final input required relates to the scalar sector. Just as with the fermionic sector, the number of scalars present, along with their charges and representations is known observationally, but remains undetermined theoretically, and so must be provided as input.

Given the above three pieces of input data, the SM may then be constructed as the most general Lagrangian consistent with the symmetries and particle content. The free parameters in the model are then fit by experiment. This input data is indeed exceptionally minimal, but now let me compare it to that required in the NCG description of the same model. Again the input data may be thought of in three input steps, each corresponding roughly to the three elements of a spectral triple $\{A, H, D\}$:

(1) The first step in constructing the SM as a NCG is the specification of an input algebra, and in particular a finite dimensional algebra. In the case of the SM the finite dimensional algebra is given by $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. This first step is analogous to that taken in the more familiar QFT construction, as the symmetries of the theory correspond, roughly speaking, to the automorphisms of the input algebra (I will make this statement more precise in Section 3.3). However, it has also been argued that under certain very minimal assumptions this input algebra is in some sense the simplest algebra consistent with the NCG formalism [CC08].

(2) The next step in the NCG approach is to specify the representation of the input algebra on a Hilbert space. In practice this means specifying the representation π of the finite algebra A_F , which for the standard model is given on the Hilbert space $H_F = \mathbb{C}^{96}$ (the ‘96’ corresponds to the number of unique fermions in the model including three particle generations and a right handed neutrino per generation). Notice that in contrast to the QFT approach, the main piece of input data being represented here is an algebra and not a group. The representation theory for finite associative $*$ -algebras is much more restrictive than the representation theory for groups, and so in the NCG approach it is no longer a mystery why the particle content always appears in fundamental, adjoint, and singlet representations - this is obtained as an output of the formalism rather than being provided as input. In addition, in the NCG approach it is an explained output that each fermionic species is charged under at most two non-abelian gauge symmetries (a fact which arises from the bi-module structure of the input Hilbert space, ‘bi’ being the operative prefix).

(3) Step three is where the NCG approach really pulls ahead of the competition. *There is in fact no third piece of input data required in the NCG approach!* Just as when specifying the properties of a triangle in Euclidean geometry one need not give all three angles as input, once two pieces of input data are supplied in NCG (an algebra and Hilbert space in this case), the third piece of data, a finite Dirac operator D_F , is then derivable as an output from the NCG axioms alone. The Dirac operator in turn specifies the scalar sector of the model, with the Higgs fields appearing as connections on the finite space in exactly the same way that the gauge fields appear as connections on the continuous space (I will discuss this point in detail in Subsection 2.2.3). In other words the Higgs sector is not an independent input once the gauge and fermion sectors are specified. In effect the Higgs boson is an extra “component” of the gauge field, reflecting the fact that the spacetime is non-commutative.

Finally, once the input data of the NCG SM has been specified one can construct the spectral and fermionic actions. On performing the Heat kernel expansion one then obtains the SM coupled to Einstein-Hilbert gravity in all of its detail.

The key benefit of the NCG approach to physics is that it requires a reduced set of input data as compared with the traditional QFT approach to constructing gauge theories. In this sense, certain aspects of the theory may be explained or understood or predicted which otherwise seem arbitrary. Indeed, from knowledge of the SM fermionic and gauge sectors alone, NCG did ‘predict’ the existence of an SU_2 Higgs field prior to the discovery of the SM Higgs field at the LHC (NCG also makes a ‘post-diction’ for the existence of right handed neutrinos and Majorana mass terms). One would hope to go further, and to predict particle content beyond the standard model. This predictive power of NCG is perhaps its most beautiful feature, and unfortunately, does not seem widely understood or appreciated.

1.4 Where does NCG fail?

NCG gives a remarkably complete account of the SM in all of its detail, including the fermions, gauge bosons, and scalar fields. Bearing this success in mind, one might hope to use the constraints in the NCG formalism to explore beyond the SM, and indeed some have tried [Ste09c, Ste06, Ste08, Ste09b, Ste09a, PS97, SZ01]. Unfortunately the traditional formalism has run into a string of problems which have limited its effectiveness as a model building tool. These include:

1. **Unification** As described in Section 1.3, one of the most beautiful features of the

NCG approach to physics is that it provides a unified description of symmetries: both the manifold diffeomorphisms, and internal gauge symmetries of a physical model arise in the same way as the automorphisms of an input algebra. In addition the gauge and Higgs bosons share a unified description, with scalar fields appearing as connections on the ‘internal’ part of a geometry, and gauge bosons appearing as connections on the ‘continuous’ part of a geometry. While this unification is impressive, the formalism still relies on five separate input elements (ie. a spectral triple $\{\mathcal{A}, \mathcal{H}, D, J, \gamma\}$) for its description, each satisfying its own unique set of rules and conditions. It would be conceptually much more satisfying if there were a more coherent description of the NCG input data and its axioms. As I will describe in Section 3.2, this is precisely what our reformulation does: it packages together all of the data held in a spectral triple into a single involutive, differential, graded algebra which we label ΩB . In our formulation, many of the otherwise separate axioms and conditions satisfied by the elements of a NCG arise simply as the properties of the algebra ΩB .

2. **Euclidean signature:** The current NCG formulation is only well defined for geometries with Euclidean signature. In practice, when constructing a physical theory as a NCG one performs all calculations in Euclidean signature, before computing the spectral action, and then finally Wick rotating at the end. Generalizing NCG to the Lorentzian setting is an active area of research, which is outside the scope of this thesis. The interested reader can find some recent work in [van15, Bar07, BBB15].
3. **Quantization:** While NCG is constructed in the language of quantum mechanics: that is as operators and Hilbert spaces, the spectral action remains a tool for producing a classical action. Recent work on using NCG to construct quantum mechanical models can be found for example in [CCM14, Bar15]. I will mention this topic briefly when I conclude in Ch. 6.
4. **Weinberg angle problem:** The spectral action given in Eq. (1.1) is defined at a scale Λ . When calculating the heat kernel expansion for the SM one finds the same constraint on the gauge couplings that holds at the unification scale Λ_{unif} for $SU(5)$ grand unified theories, i.e. $g_c^2 = g_w = 5/3g_y^2$. This constraint leads to the interpretation that the spectral action must be defined at the unification scale: $\Lambda = \Lambda_{unif}$. Unfortunately it is known experimentally that the gauge couplings do not unify exactly, and so the NCG SM suffers from the well known ‘Weinberg angle problem’. For work on solving the unification problem see for example [DLFV15]. I will discuss our own attempted solution to this problem at the end of Chapter 4.

5. **Three generations:** While NCG explains a number of the otherwise ad-hoc features of the standard model (fermion representations, each fermion being charged only under at most two non-abelian symmetry groups, the Higgs field appearing as an internal connection, etc), it has no explanation for the appearance of three particle generations. This is still taken as unexplained input in the current model. I will motivate a possible solution to this problem in Subsection 5.3.3.
6. **Unwanted Higgs fields:** As discussed in section 1.3, one of the key advantages of the NCG formalism is that it provides a more constrained description of gauge theories. In particular, given the fermionic and gauge bosonic input data of a gauge theory, the NCG axioms may then be used to determine the Higgs sector as output. Unfortunately this is not quite what has traditionally happened. In practice, when constructing the SM, the traditional NCG axioms place very heavy restrictions on the SM Higgs sector but do not uniquely restrict to the SM Higgs. As a solution to this problem an additional ad-hoc non-geometric condition is imposed, which essentially removes the unwanted fields by hand [CCM07, CC08]. Our reformulation of the NCG input data offers a rather natural solution to this problem which I discuss in Subsection 4.1.3.
7. **Higgs Mass:** The NCG SM spectral action is slightly more constrained than the usual QFT description of the standard model. As a result it is possible to obtain, under some mild assumptions, a predicted value for the Higgs mass. Unfortunately the naive value is calculated to be approximately 170GeV [CIS99, KS06], which is ruled out by experiment. As a solution to this problem Chamseddine and Connes add a real scalar field into the model by hand effectively increasing the parameter space of the model and making it compatible with the 125GeV detected Higgs mass value [CC12]. Unfortunately this solution conflicts with one of the most beautiful features of the NCG: the scalar sector is no longer obtained as an output⁹. As it turns out, our reformulation of the NCG input data yields a strikingly simple and natural solution to this problem, which I will discuss in Subsection. 4.2.2
8. **Cosmological constant:** NCG does not provide a solution to the cosmological constant problem. As shown in the most recent version of the NCG SM [CCM07] a cosmological constant term does appear in the expansion of the NCG SM bosonic action, but the formalism does not provide any natural way of matching this constant to its physically measured value.

⁹Another potential solution to the Higgs mass problem has been proposed by Estrada and Marcolli, in which they assume the asymptotic safety of gravity [EM13a], in the sense of Shaposhnikov and Wetterich [SW10], I will however not discuss this solution.

Before making full use of the NCG formalism to explore beyond SM physics, the above mentioned problems indicate one should first carefully understand and distinguish between those NCG axioms which are fundamental and natural, and those which are less well motivated, or are more arbitrary. As I will explain below in Section 1.5, there are good reasons, both mathematical and physical, for removing the axiomatic restriction to associative input algebras taken in the traditional formulation of NCG. As it turns out, in the process of removing this restriction, many otherwise confusing elements of the NCG construction are elucidated, even when considering purely associative geometries such as the NCG SM. In generalizing to the non-associative setting, the solutions to a number of problems in the associative formalism become in some sense ‘obvious’. I discuss the unexpected benefits of extending to a non-associative formalism in chapter 4. For the remainder of this chapter I focus instead on the original motivations for generalizing to a non-associative geometry: there are two.

1.5 Why generalize to non-associative differential geometry?

Over the past few centuries, physicists and mathematicians have become well accustomed to the fact that noncommutative structures are of central and ubiquitous importance. But, for most physicists, nonassociativity still carries a whiff of disreputability¹⁰. It is important then to start by explaining the two key motivations for studying nonassociativity in this paper: first a general motivation, followed by a more specific one.

General mathematical motivation

Let’s start with the more general mathematical motivation. The fundamental point is that, in the ordinary approach to physics, the basic input is a *symmetry group*. By contrast, in the NCG approach, the fundamental input is an *algebra*, and the symmetry group then emerges as the automorphism group of that algebra. Symmetry groups are associative by nature, but algebras are not. Just as some of the most beautiful and important groups are non-commutative, some of the most beautiful and important algebras (including Lie

¹⁰Even though Lie algebras are at the heart of much of physics, interest usually lies in the associative groups that they generate, and so the non-associativity of Lie algebras is conveniently hidden away and forgotten about. For an example of where non-associativity arises explicitly in a physical system see for example the addition of velocities in special relativity [Sbi01].

algebras, Jordan algebras and the Octonions) are non-associative. Just as it would be unnatural to restrict attention to commutative groups (as physicists originally did in studying gauge theory, prior to Yang-Mills), it is unnatural to restrict attention to associative algebras. In either case, imposing such an unnatural restriction amounts to blinding ourselves of something essential that the formalism is trying to say. From this standpoint, it is a natural task to reformulate the spectral approach to physics in such a way that the extension to non-associative algebras becomes obvious and natural.

Physical motivation

Next the physical motivation. Although I would like to use the framework of NCG to explore beyond the standard model of particle physics, many of the most interesting extensions are out of reach of the associative formalism. As a specific example, in order to reformulate the most successful Grand Unified Theories (GUTs) – *e.g.* those based on $SU(5)$, $SO(10)$ and E_6 – in terms of the spectral action, we are forced to use non-associative input algebras. To appreciate this point, first note that the representation theory of associative $*$ -algebras is much more restricted than the representation theory of Lie groups [Kra98]: Lie groups (like $SU(5)$) have an infinite number of irreps, but associative algebras (like the corresponding $*$ -algebra $M_5(\mathbb{C})$ of 5×5 complex matrices, whose automorphism group is $SU(5)$) only have a finite number. In particular, if we ask whether key fermionic representations needed in GUT model building – such as the **10** of $SU(5)$, the **16** of $SO(10)$, or the **27** of E_6 – are available as the irreps of algebras with the correct corresponding automorphism groups, the answer is “no” for associative algebras, and “yes” for non-associative algebras. Furthermore, if we ask whether the exceptional groups (including E_6 , which is of particular interest for GUT model building, and E_8 , which is of particular interest in connection with string theory) appear as the automorphism groups of corresponding algebras, again the answer is “no” for associative algebras and “yes” for non-associative algebras.

1.6 Non-associative geometry: what is it?

Connes generalized Riemannian geometry to NCG by shifting his focus away from geometric spaces to the algebras of functions defined over them. His approach was to first reformulate as much as possible of ordinary Riemannian geometry on a manifold M in terms of the algebra $C^\infty(M)$, including differential forms, bundles, connections, and cohomology [Mar]. The idea is that the data $\{M, g\}$ describing a Riemannian geometry can

be replaced by a dual set of data $\{\mathcal{A}, \mathcal{H}, D\}$ consisting of a very special choice of input algebra \mathcal{A} , Hilbert space \mathcal{H} and Dirac operator D satisfying certain conditions:

$$\{M, g\} \leftrightarrow \{\mathcal{A}, \mathcal{H}, D\} \tag{1.8}$$

The benefit of the ‘spectral’ description of geometry is that the spectral notions of differential forms, bundles, connections, and cohomology all continue to make sense when the input algebra is taken to be non-commutative. The spectral approach therefore readily allows for the description of ‘non-commutative’ geometries.

The spectral data corresponding to a Riemannian geometry satisfy a very special set of conditions [Con96]. What conditions then should a spectral triple corresponding to a more general NCG satisfy when its underlying space is not necessarily known? Connes generalized the conditions satisfied by Riemannian geometries to a set of axioms satisfied by more general NCGs which have non-commutative input algebras (as I will discuss in Ch.2). Our main contribution is the realization that one can go a step further: the input data of a spectral triple can be ‘fused’ together to form an involutive, differential graded $*$ -algebra ΩB . Then many of the traditional NCG axioms as well as a number of new constraints arise from the intrinsic algebraic properties on ΩB if ΩB is taken to be associative.

$$\{M, g\} \leftrightarrow \{\mathcal{A}, \mathcal{H}, D\} \leftrightarrow \Omega B \tag{1.9}$$

A key realization of ours is the fact that ΩB need not be associative. In particular, if ΩB satisfies more general associativity properties (ie. if it is a Jordan algebra for example) then generalized NCG axioms are derived readily from the intrinsic properties of ΩB . In short, I will argue that a non-associative differential geometry is an NCG whose corresponding fused algebra ΩB is non-associative.

Finally, it is important to stress at the outset that the physical theories that are constructible as non-associative NCGs are still normal *associative* gauge theories. Just as (non-associative) Lie algebras generate the associative groups of everyday use in physics, the symmetries of non-associative NCGs are associative. There is no funny non-associativity of space-time which arises, or anything non-associativity of the sort that should worry the reader from a physics stand point.

1.7 Summary of key results

I will close this introductory chapter with a brief listing of our key results:

1. We reformulate the NCG input data in terms of ‘fused algebras’ ΩB , which are constructed as square zero extensions [3.2](#).
2. We show that many of the associative NCG conditions which were previously taken as axiom may be derived from the more fundamental condition that ΩB is an associative, involutive, differential graded algebra [3.2.4](#).
3. We show that the associativity of ΩB imposes new constraints on NCGs beyond those imposed by the traditional NCG axioms, and show how these new constraints may be used to place phenomenologically accurate restrictions on the NCG SM [4.1.3](#).
4. We show that ΩB need not be associative, and as such the fused algebra formulation allows for a natural description of non-associative NCGs [3.2](#).
5. We describe how the symmetries of a NCG are given new meaning as the automorphisms of ΩB [3.3](#).
6. We show how the fused algebra formulation of NCG leads to a natural and phenomenologically viable extension of the SM [4.2.1](#), and describe how this extension provides a possible solution to the NCG Higgs mass problem [4.2.2](#).
7. We construct the first non-associative geometries in [3.1.5](#) and [5](#).

Chapter 2

Preliminaries

NCG grows out of the simple idea that one should shift attention away from geometric spaces, to instead focus on the algebras of functions defined on them. As a result, a good understanding of algebras and their representations will be necessary for understanding the bulk of this work. This chapter has two goals: The first goal is to introduce/review the basic definitions and tools needed when working with (non-associative) algebras. In particular I will review involutive algebras, graded algebras, differential graded algebras (DGAs), opposite algebras, algebra automorphisms and algebra representations. I will also lay out most of the notation that will be used throughout this work. The second goal is to give a pedagogical introduction to NCG in the traditional approach. In particular I will give a brief overview of the NCG axioms, and will describe explicitly how one constructs so called ‘almost-commutative’ geometries. For information beyond that which is provided in this work the reader is encouraged to consult [[vdDvS12](#), [vS15](#), [Lan97](#), [Sch05](#), [CM](#), [GB01](#), [CM08](#)].

2.1 Mathematical preliminaries

2.1.1 Algebras

An algebra \mathcal{A} is a vector space (over a field \mathbb{F}), which is equipped with a product: $ab \in \mathcal{A} \forall a, b \in \mathcal{A}$, which is distributive over addition:

$$a_0(a_1 + a_2) = a_0a_1 + a_0a_2, \quad (a_0 + a_1)a_2 = a_0a_2 + a_1a_2, \quad (2.1)$$

$\forall a_i \in \mathcal{A}$. I will only ever consider algebras which are constructed over the field of real \mathbb{R} or complex \mathbb{C} numbers, and to reduce notation the product between any two algebra elements

will always be notated by juxtaposition unless explicitly stated otherwise. An algebra is said to be unital if it is equipped with a multiplicative unit \mathbb{I} which satisfies

$$a\mathbb{I} = \mathbb{I}a = a, \quad \forall a \in \mathcal{A}. \quad (2.2)$$

An algebra is said to be ‘non-commutative’ if its product is non-commutative: $ab \neq ba$, for some $a, b \in \mathcal{A}$. Similarly, an algebra is said to be ‘non-associative’ if its product is non-associative: $(ab)c \neq a(bc)$, $a, b, c \in \mathcal{A}$. Just as the ‘commutator’ $[a, b]$ is introduced to characterize the failure of commutativity, the ‘associator’ $[a, b, c]$ is introduced to characterize the failure of associativity:

$$[a, b] \equiv ab - ba, \quad [a, b, c] \equiv (ab)c - a(bc) \quad a, b, c \in \mathcal{A}. \quad (2.3)$$

While there are many interesting classes of non-associative algebras, there are three which are most widely known and studied. These are also the only three which I will explore in this work:

1. **Lie algebras:** A Lie algebra \mathcal{A} is a vector space over a field \mathbb{F} equipped with the ‘Lie product’, which satisfies:

$$[a_0, a_1] = 0, \quad (\text{anti-symmetry}) \quad (2.4a)$$

$$a_0(a_1a_2) + a_2(a_0a_1) + a_1(a_2a_0) = 0, \quad (\text{Jacobi identity}) \quad (2.4b)$$

$$\forall a_i \in \mathcal{A}.$$

2. **Jordan algebras:** A Jordan algebra \mathcal{A} is a vector space over a field \mathbb{F} equipped with the ‘Jordan product’, which satisfies:

$$\{a_0, a_1\} = 0, \quad (\text{symmetry}) \quad (2.5a)$$

$$[a_0, a_1, a_0^2] = 0, \quad (\text{Jordan identity}) \quad (2.5b)$$

$\forall a_i \in \mathcal{A}$. Notice that all commutative, associative algebras satisfy Eq. (2.5), and so all commutative, associative algebras are examples of Jordan algebras.

As an interesting historical aside, Jordan algebras were first discovered by physicists: One of the major advances of the last century was the discovery of quantum mechanics in which one passes from the commutative algebra of classical observables to the non-commutative algebra of quantum mechanical observables [Kha04]. In the usual approach to quantum mechanics, the observables of a system are given by Hermitian operators, and two operators can only be measured simultaneously if they commute.

In other words, if X , and Y are two hermitian operators, than XY is not in general observable unless $[X, Y] = 0$. Jordan algebras were first introduced by Pascual Jordan, John von Neumann and Wigner [PJW34], in an attempt to formalize quantum theory in terms of its essential ingredients: observables, states, expectation values and their time evolution [McC78]. They noticed that while observables (hermitian operators) are not in general closed under matrix multiplication (ie. the product of two self-adjoint operators is in general not self-adjoint), they are closed under the ‘Jordan’ product

$$x \circ y = xy + yx, \quad (2.6)$$

and they form a Jordan algebra satisfying the properties outlined in Eq. (2.5). Indeed, there is a Jordan algebra approach to quantum mechanics which is equivalent to the usual C^* algebra formulation [Bis93].

3. **Alternative algebras:** An alternative algebra \mathcal{A} is a vector space over a field \mathbb{F} equipped with an ‘alternative product’, for which the associator is ‘alternating’:

$$[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}] = \text{sgn}(\sigma)[a_1, a_2, a_3], \quad (2.7)$$

for any permutation σ , and $\forall a_i \in \mathcal{A}$. Notice that all associative algebras trivially satisfy the ‘alternative’ identity given in Eq. (2.7), and so all associative algebras are also examples of alternative algebras.

An algebra \mathcal{A} is said to be ‘involutive’¹ if it is equipped with an involution operator $*$: $\mathcal{A} \rightarrow \mathcal{A}$ which satisfies:

$$(a^*)^* = a, \quad (2.8a)$$

$$(a_0 a_1)^* = a_1^* a_0^*, \quad (2.8b)$$

$$(\alpha_0 a_0 + \alpha_1 a_1)^* = \bar{\alpha}_0 a_0^* + \bar{\alpha}_1 a_1^* \quad (2.8c)$$

$\forall a_i \in \mathcal{A}$, and where $\forall \alpha_i \in \mathbb{C}$ the overline denotes the usual complex conjugation. For example: the complex numbers \mathbb{C} are a commutative $*$ -algebra over \mathbb{R} , where the $*$ operation is complex conjugation $z \rightarrow \bar{z}$; and the $n \times n$ complex matrices $M_n(\mathbb{C})$ form a non-commutative $*$ -algebra (over \mathbb{C}), where the $*$ operation is the conjugate transpose (\dagger) operation $m \rightarrow m^\dagger$. I will sometimes indicate explicitly that an algebra \mathcal{A} is equipped with an involution $*$ by writing $\{\mathcal{A}, *\}$.

¹Involutive algebras are also known as star algebras or $*$ -algebras.

An algebra \mathcal{A} is said to be ‘normed’ if it is equipped with a norm $\|\cdot\| : \mathcal{A} \rightarrow \mathbb{R}$ which has the properties [Lan97]:

$$\|a\| \geq 0, \quad (2.9a)$$

$$\|a\| = 0 \implies a = 0, \quad (2.9b)$$

$$\|\alpha a\| = |\alpha| \|a\|, \quad (2.9c)$$

$$\|a_0 + a_1\| \leq \|a_0\| + \|a_1\|, \quad (2.9d)$$

$$\|a_0 a_1\| \leq \|a_0\| \|a_1\|, \quad (2.9e)$$

$\forall a_i \in \mathcal{A}, \alpha \in \mathbb{F}$. The ‘distance’ between any two elements of a normed algebra can be defined using the norm: $d(a_0, a_1) \equiv \|a_0 - a_1\|$. A ‘Cauchy’ sequence of elements in a normed algebra \mathcal{A} is a sequence of elements a_i who’s ‘distance’ between each other becomes arbitrarily close as the sequence progresses. Explicitly, a sequence is defined to be Cauchy if for every finite positive number $\mu \in \mathbb{R}$, there is an N such that $d(a_n, a_m) \leq \mu$ for $n, m \geq N$. A normed algebra \mathcal{A} is said to be ‘complete’ if every Cauchy sequence of elements $a_i \in \mathcal{A}$ has a limit that is also in \mathcal{A} .

A normed algebra is said to be ‘Banach’ if it is complete with respect to the norm. A C^* -algebra is an associative, involution ‘Banach’ algebra satisfying:

$$\|a^* a\| = \|a\|^2, \quad (2.10)$$

$\forall A \in \mathcal{A}$. Equation (2.9e) together with equation (2.10) can be used to show that C^* -algebras also satisfy the condition:

$$\|a^*\| = \|a\|, \quad (2.11)$$

$\forall a \in \mathcal{A}$.

Given an algebra \mathcal{A} , a left(right) ideal $\mathcal{I}_{L(R)}$ is a subset of \mathcal{A} which is closed under left(right) multiplication by elements \mathcal{A} : $av \in \mathcal{I}_L \forall a \in \mathcal{A}, v \in \mathcal{I}_L$ ($va \in \mathcal{I}_R \forall a \in \mathcal{A}, v \in \mathcal{I}_R$). An ideal which is closed under both left and right multiplication is called ‘two sided’. Given a two-sided ideal $\mathcal{I}_{L,R} \subset \mathcal{A}$, an equivalence relation \sim can be defined such that:

$$a_0 \sim a_1 \text{ iff } a_0 - a_1 \in \mathcal{I} \quad (2.12)$$

It is easy to check that this relation is reflexive, transitive, and symmetric. The equivalence class given in (2.12) defines a new ‘quotient’ algebra \mathcal{A}/\mathcal{I} with elements $[a]$ given by:

$$[a] = \{a + r | r \in \mathcal{I}_{L,R}\}. \quad (2.13)$$

An ideal \mathcal{I} is said to be ‘maximal’ in \mathcal{A} if there is no ideal \mathcal{J} such that $\mathcal{I} \subseteq \mathcal{J} \subseteq \mathcal{A}$.

Finally, in order to describe algebras (especially nonassociative algebras) it is convenient to introduce the standard notation [Sch66] in which L_a denotes the left-action of a , and R_a denotes the right-action of a :

$$L_{a_0}a_1 \equiv a_0a_1, \quad R_{a_0}a_1 \equiv a_1a_0. \quad (2.14)$$

$\forall a_i \in \mathcal{A}$. In other words, L_a and R_a are two different linear operators on the vector space \mathcal{A} . As an illustration of this notation one may write $a_0((a_2v)a_1) = L_{a_0}R_{a_1}L_{a_2}v$ for $a_i, v \in \mathcal{A}$. In particular, note that when \mathcal{A} is nonassociative, the left-hand side of this equation requires parentheses, but the right-hand side does not.

Examples

1. The set of continuous, smooth, complex functions $\mathcal{A} = C^\infty(M, \mathbb{C})$ over a manifold M form a commutative, associative, involutive algebra with multiplication given by the point-wise product $(f.g)(x) := f(x)g(x) \forall f, g \in \mathcal{A}$, and with the involution given by complex conjugation. While this algebra is an example of an associative algebra, it is also an example of a Jordan algebra, and an alternative algebra because it satisfies the properties given in Eq. (2.5) and Eq. (2.7) respectively.
2. The set of finite, complex $n \times n$ matrices $M_n(\mathbb{C})$ equipped with the matrix product, and hermitian conjugation form a complex, non-commutative, associative star-algebra.
3. The set of finite, complex, anti-hermitian $n \times n$ matrices $M_n(\mathbb{C})^-$ equipped with the anti-symmetrised matrix product

$$a_0 \times a_1 = a_0a_1 - a_1a_0, \quad (2.15)$$

form a real Lie algebra (and not a complex algebra because in general an anti-hermitian matrix multiplied by a complex scalar will not be anti-hermitian).

4. The set of finite, complex, hermitian $n \times n$ matrices $M_n(\mathbb{C})^+$ equipped with the symmetrised matrix product

$$a_0 \times a_1 = a_0a_1 + a_1a_0, \quad (2.16)$$

form a real Jordan algebra (and not a complex algebra because in general a hermitian matrix multiplied by a complex scalar will not be hermitian).

5. The octonions are an example of an alternative algebra, and occupy a special place in mathematics. They are one of only four normed division² algebras: the real numbers \mathbb{R} , the complex numbers \mathbb{C} , the quaternions \mathbb{H} and the octonions \mathbb{O} . The algebras \mathbb{R} , \mathbb{C} , \mathbb{H} , and \mathbb{O} are respectively 1,2,4, and 8 dimensional with 0,1,3, and 7 imaginary elements which square to negative one. The octonions are the largest and most general algebra in the natural sequence $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$, and they are intimately connected to some of the most beautiful structures in mathematics, including the exceptional Lie algebras and the exceptional Jordan algebra. For a nice expository introduction to the octonions, and their connections to other areas of mathematics, see [Bae02].

The octonionic product is neatly summarized by the Fano plane given in Figure 2.1. The Fano plane is a diagram with seven points and seven oriented lines. The seven points correspond to the seven imaginary basis elements e_i , $1 \leq i \leq 7$ of the octonions. Each pair of distinct points lies on a unique line and each line runs through exactly three points (and wraps back around again). The multiplication of any two imaginary elements is determined by following the ordered lines in the Fano plane. For example in Figure 2.1, $e_4e_5 = e_1$, while $e_6e_4 = -e_2$. The Fano plane together with the following rules completely defines the octonionic multiplication:

- e_0 is the multiplicative identity,
- $e_i^2 = -1$ for $1 \leq i \leq 7$.

From a NCG model building perspective the interest in the octonions is as follows: In each particle generation of the SM, there are 8 different ‘types’ of particles. That is, each generation has two quarks which act as a triplet under strong $SU(3)$, and each generation also has two leptons which act as singlets under strong $SU(3)$. As a matter of simple minded numerology, the octonions are interesting to consider from the model building perspective because they form an 8 dimensional algebra. In addition their automorphism group is G_2 which has both $SU(3)$, and $SU(2)$ as a subgroup. In particular, one obtains $SU(3)$ as the subgroup which leaves a single imaginary basis element of the octonions fixed such that on symmetry ‘breaking’ from G_2 to $SU(3)$ one obtains a ‘quark’ triplet (the six imaginary elements which transform as a complex triplet under the $SU(3)$ subgroup), and a ‘lepton’ singlet (the real element, and the single imaginary element form a complex ‘singlet’ under the $SU(3)$ sub-group).

²An algebra \mathcal{A} is a division algebra if for any element $a_0 \in \mathcal{A}$ and any non-zero element $a_1 \in \mathcal{A}$ there exists precisely one element $x \in \mathcal{A}$ for which $a_0 = a_1x$ and precisely one element $y \in \mathcal{A}$ such that $a_0 = ya_1$.

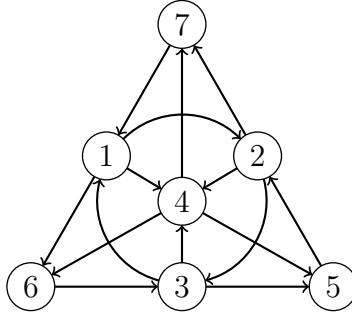


Figure 2.1: The Fano plane: multiplication between any two of the 7 imaginary elements is determined by ‘following’ the direction of the arrows. Notice that any three elements (together with the unit) which lie on the same line (for example e_1, e_2 and e_3 , or e_3, e_4 and e_7) describe a quaternionic sub algebra, and any single imaginary element (together with the unit) describes a complex sub algebra. Notice that unlike the complex numbers, the quaternionic and octonionic products are not unique. The product on the quaternions can be defined in two possible ways corresponding to the sign choice: $e_1e_2 = \pm e_3$, while there are 480 different possibilities for the octonions corresponding the different ways that the arrows may be oriented in the Fano plane [SM96]

2.1.2 Involutive differential graded algebras (*-DGAs)

In Ch. 3 I will explain how to reformulate the input data of a NCG in terms of involutive differential graded algebras (*-DGAs), and so it is necessary that I introduce them here. An algebra \mathcal{A} is said to be ‘graded’ if it decomposes into subspaces $\mathcal{A} = \bigoplus_m \mathcal{A}_m$, and the product respects the decomposition: $\omega_m \in \mathcal{A}_m, \omega_n \in \mathcal{A}_n \Rightarrow \omega_m \omega_n \in \mathcal{A}_{m+n}$. A differential graded algebra is then a graded algebra \mathcal{A} that is also equipped with a differential d : a linear map from \mathcal{A}_m to \mathcal{A}_{m+1} that is nilpotent ($d^2 = 0$) and satisfies the graded Liebniz condition:

$$d(\omega_m \omega_n) = d(\omega_m) \omega_n + (-1)^m \omega_m d(\omega_n), \quad (2.17a)$$

for any $\omega_m \in \mathcal{A}_m$ and $\omega_n \in \mathcal{A}_n$.

A *-DGA is a DGA which is additionally equipped with an involution ω_m^* which satisfies equations (2.8) and the condition:

$$d(\omega_n^*) = \kappa(-1)^n d(\omega_n)^* \quad (2.17b)$$

$\forall \omega_n \in \mathcal{A}^n$, and where κ is a choice of sign.

Examples

1. The canonical example of a DGA is the exterior algebra of differential forms.
2. The universal differential graded algebra $\Omega\mathcal{A} = \Omega^0\mathcal{A} \oplus \Omega^1\mathcal{A} \oplus \dots$ associated to any unital algebra \mathcal{A} over a field \mathbb{F} is an example of a DGA, and is constructed as follows: In degree zero the universal differential graded algebra is given by \mathcal{A} itself, i.e. $\Omega^0\mathcal{A} = \mathcal{A}$. An order one differential operator d is then equipped, which satisfies:

$$d^2 = 0 \tag{2.18a}$$

$$d[a_i a_j] = d[a_i]a_j + a_i d[a_j] \tag{2.18b}$$

$$d[\alpha a_i + \beta a_j] = \alpha d[a_i] + \beta d[a_j] \tag{2.18c}$$

for $a_i, a_j \in \mathcal{A}$, $\alpha, \beta \in \mathbb{F}$. Higher order forms are freely generated by juxtaposition of the elements $a \in \mathcal{A}$ and the formal symbols $d(a) \in \Omega^1\mathcal{A}$. For example $(d[a_2]a_1)d[a_0] \in \Omega^2\mathcal{A}$, $(d[a_2]d[a_0])(a_1d[a_3]) \in \Omega^3\mathcal{A}$. Notice that the product is completely free modulo the conditions given in 2.18, and there is no sort of graded commutativity (usually in the literature associativity is imposed [Lan97], I will drop this assumption however). If $\Omega\mathcal{A}$ is an involutive algebra, then $\Omega\mathcal{A}$ is said to be a $*$ -DGA if it satisfies condition 2.17b. Any differential graded algebra which in degree zero is equal to \mathcal{A} can be constructed from a projection of the universal differential algebra $\Omega\mathcal{A}$ [Lan97]. I discuss universal differential graded algebras again in 2.2.1, a pedagogical introduction in the associative case is given in [Lan97]. Note that I will often use the notation $\Omega\mathcal{A}$ to indicate that an algebra is a DGA although in the math literature this notation is usually reserved for universal differential graded algebras and their projections.

2.1.3 Opposite algebras \mathcal{A}_{op}

Given an (involutive) algebra $\{\mathcal{A}, *\}$, one can always define what is known as the opposite algebra $\{\mathcal{A}_{op}, \times_{op}, *\}$, where \mathcal{A}_{op} has the same vector space as \mathcal{A} , and the same $*$ operation, while the opposite product is defined by:

$$a \times_{op} b \equiv ba, \tag{2.19}$$

for $a, b \in \mathcal{A}_{op}$. Note that the involution $*$ is compatible with \mathcal{A} and \mathcal{A}_{op} :

$$\begin{aligned} (a \times_{op} b)^{*op} &\equiv (ba)^* \\ &= a^* b^* \\ &= b^{*op} \times_{op} a^{*op}. \end{aligned} \tag{2.20}$$

When constructing the opposite algebra $\Omega\mathcal{A}_{op} = \{\mathcal{A}_{op}, \times_{op}, d, *\}$ associated to a $*$ -DGA $\Omega\mathcal{A} = \{\mathcal{A}, d, *\}$ one has to be careful to take account of the grading when defining the product:

$$\omega_m \times_{op} \omega'_n := (-1)^{mn} \omega'_n \omega_m, \tag{2.21}$$

for $\omega_m \in \Omega^m \mathcal{A}$, $\omega'_n \in \Omega^n \mathcal{A}$. The sign on the right hand side of Eq. (2.21) is necessary in order to ensure that the opposite algebra satisfies the graded Liebniz rule:

$$\begin{aligned} d[\omega_m \times_{op} \omega'_n] &\equiv (-1)^{mn} d[\omega'_n \omega_m] \\ &= (-1)^{mn} d[\omega'_n] \omega_m + (-1)^n (-1)^{mn} \omega'_n d[\omega_m] \\ &= (-1)^{mn} (-1)^{m(n+1)} \omega_m \times_{op} d[\omega'_n] + (-1)^n (-1)^{mn} (-1)^{n(m+1)} d[\omega_m] \times_{op} \omega'_n \\ &= d[\omega_m] \times_{op} \omega'_n + (-1)^m \omega_m \times_{op} d[\omega'_n]. \end{aligned} \tag{2.22}$$

2.1.4 Automorphisms and derivations

If \mathcal{A} is a $*$ -algebra, then an *automorphism* of \mathcal{A} is an invertible linear map $\alpha : \mathcal{A} \rightarrow \mathcal{A}$ which respects the product and involution operations in \mathcal{A} :

$$\alpha(a_0 a_1) = \alpha(a_0) \alpha(a_1), \tag{2.23a}$$

$$\alpha(a^*) = (\alpha(a))^*, \tag{2.23b}$$

and a *derivation* of \mathcal{A} is a linear map $\delta : \mathcal{A} \rightarrow \mathcal{A}$ which satisfies

$$\delta(a_0 a_1) = \delta(a_0) a_1 + a_0 \delta(a_1), \tag{2.24a}$$

$$\delta(a^*) = (\delta(a))^*, \tag{2.24b}$$

$\forall a_i \in \mathcal{A}$. The automorphisms of \mathcal{A} form a group which is denoted $Aut(\mathcal{A})$. Note that, when the automorphism α is infinitesimally close to the identity map “ \mathbb{I} ,” it can be written as $\alpha = \mathbb{I} + \delta$ where δ is a derivation, i.e. the derivations of \mathcal{A} are the infinitesimal generators of the automorphisms of \mathcal{A} : $\alpha a = e^\delta a$. They form a Lie algebra which is denoted $Der(\mathcal{A})$, with Lie product given by $[\delta_1, \delta_2] = \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$ (where \circ denotes composition of operators).

Examples

1. Consider the $*$ -algebra of smooth complex functions over a manifold $\mathcal{A} = C^\infty(M, \mathbb{C})$. In this case the automorphisms $\alpha_\varphi : \mathcal{A} \rightarrow \mathcal{A}$ are nothing but the maps $\alpha_\varphi(f) = f \circ \varphi^{-1}$, where $f : \mathcal{M} \rightarrow \mathbb{C}$ is a smooth function and $\varphi : \mathcal{M} \rightarrow \mathcal{M}$ is a diffeomorphism; and if we consider the automorphisms infinitesimally close to the identity, we see that the corresponding derivations have the form $\delta_v(f) = v^\mu \partial_\mu f$, where $v^\mu(x)$ is a real contravariant vector field on \mathcal{M} . To see that the maps α_ϕ act as automorphisms on \mathcal{A} it is sufficient to check that they satisfy conditions (2.23a) and (2.23b):

$$\begin{aligned} \alpha_\phi(f.g)(x) &= (f.g)(\phi^{-1}x) \\ &= f(\phi^{-1}x)g(\phi^{-1}x) \\ &= \alpha_\phi f(x)\alpha_\phi g(x) = (\alpha_\phi f.\alpha_\phi g)(x) \end{aligned} \tag{2.25}$$

$$\begin{aligned} \alpha_\phi \overline{f(x)} &= \overline{\alpha_\phi f(x)} \\ &= \overline{f(\phi^{-1}x)} \\ &= \overline{f(\phi^{-1}x)} = \overline{\alpha_\phi f(x)}. \end{aligned} \tag{2.26}$$

2. Consider the finite, non-commutative, associative, involutive algebra of $n \times n$ complex matrices $\mathcal{A}_F = M_n(\mathbb{C})$. The algebra \mathcal{A}_F has automorphisms $\alpha_u : \mathcal{A}_F \rightarrow \mathcal{A}_F$ which take the form $\alpha_u a = u a u^*$, where $u \in \mathcal{A}_F$ is unitary. Again, to check that the maps α_u act as automorphisms on \mathcal{A}_F it is sufficient to check that they satisfy conditions (2.23a) and (2.23b):

$$\begin{aligned} \alpha_u(a_0 a_1) &= u a_0 a_1 u^* \\ &= u a_0 u^* u a_1 u^* = \alpha_u(a_0)\alpha_u(a_1), \end{aligned} \tag{2.27a}$$

$$\begin{aligned} \alpha_u(a^*) &= u a^* u^* \\ &= (u a u^*)^* = \alpha_u(a)^*, \end{aligned} \tag{2.27b}$$

$\forall a_i \in \mathcal{A}_F$ and unitary $u \in \mathcal{A}_F$. The maps α_u are known as ‘inner’ automorphisms (notated $\text{Inn}(\mathcal{A})$), because they are constructible from elements of the algebra itself³. Noting that the unitary elements u are generated by anti-hermitian elements $x \in \mathcal{A}_F$ ($u = e^x$) and studying the inner automorphisms infinitesimally close to the identity

³For a formal definition of inner automorphisms and their derivations see the work of Schafer and Jacobson [Jac49, Sch49]. Every derivation of a semisimple algebra (that is, direct sum of simple algebras) with a unit over a field of characteristic zero is inner.

map, it is seen that the corresponding ‘inner’ derivations (*i.e.* the generators of the inner automorphisms) are $\delta_x(a) = [x, a]$ or equivalently:

$$\delta_x^{Ass} = L_x - R_x. \quad (2.28a)$$

where x is an anti-hermitian element of \mathcal{A} . The inner automorphisms on $\mathcal{A}_F = M_n(\mathbb{C})$ may therefore be written as $\alpha_u a = e^{\delta_x} a$. It can be shown that every derivation on semi-simple finite associative algebras (like $M_n(\mathbb{C})$) is of the form given in (2.28a) [Jac37].

3. Just as finite non-commutative, associative algebras have inner automorphisms, so too in general do finite non-associative algebras. This thesis focuses on the Lie, Jordan, and alternative algebras, which were introduced in Section 2.1.2. Their respective algebras of inner derivations are generated by elements of the form [Jac49, Sch49]:

$$\delta_x^{Lie} = L_x = -R_x, \quad (2.28b)$$

$$\delta_{w,z}^{Jor} = [L_w, L_z] = [L_w, R_z] = [R_w, R_z], \quad (2.28c)$$

$$\delta_{x,y}^{Alt} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y], \quad (2.28d)$$

for anti-hermitian $x, y \in \mathcal{A}_F$, and hermitian $w, z \in \mathcal{A}_F$. The inner automorphisms on finite (non-associative) algebras may therefore be written as $\alpha_u a = e^{\delta^{Inn}} a$, where the form that δ^{Inn} takes depends on the associativity class that the finite algebra \mathcal{A}_F belongs to. Notice, that Eq. (2.28b) gives new meaning to the Jacobi identity given in Eq. (2.4b): it is nothing but the Leibniz rule given in Eq. (2.24a). The reader is encouraged to check for themselves that the derivations given in Eq. (2.28a), Eq. (2.28c) and Eq. (2.28d) similarly satisfy the Leibniz rule when acting on elements of associative, Jordan, or alternative algebras respectively.

4. As a final example consider the algebra of smooth \mathcal{A}_F valued functions over a manifold M : $\mathcal{A} = C^\infty(M, \mathcal{A}_F)$, where \mathcal{A}_F is a finite algebra. In this case the automorphisms are generated by derivations of the general form $\delta = v^\mu \partial_\mu + \delta^{Inn}$, where the form that the inner derivations δ_{Inn} take depends on the class of algebras to which \mathcal{A}_F belongs. The group of automorphisms $Inn(\mathcal{A})$ generated at each point on M by $Der(\mathcal{A}_F)$ form a normal subgroup of $Aut(\mathcal{A})$, which can be shown as follows. For $\beta \in Aut(\mathcal{A})$

and $\alpha^{Inn} \in Inn(\mathcal{A})$, we find that

$$\begin{aligned}
\beta \circ \alpha^{Inn} \circ \beta^{-1}(a) &= e^\delta e^{\delta^{Inn}} e^{-\delta} a, \\
&= e^{Ad_\delta} e^{\delta^{Inn}} a, \\
&= exp(\delta^{Inn} + [Ad_\delta, \delta^{Inn}] + 1/2[Ad_\delta, [Ad_\delta, \delta^{Inn}]] \dots)(a) \\
&= \tilde{\alpha}^{Inn}(a),
\end{aligned} \tag{2.29}$$

where in the third equality I have used the Baker-Campbell-Hausdorff formula and in the last equality I have used the fact that derivations of inner derivations are inner, which follows directly from (2.24a) and the fact that inner derivations are constructed from elements of the algebra itself. This means that the ‘outer automorphisms’ may be defined by the quotient group:

$$Out(\mathcal{A}) \equiv Aut(\mathcal{A})/Inn(\mathcal{A}). \tag{2.30}$$

2.1.5 Direct products and sums

Throughout the text I will often form direct sums and products of vector spaces and algebras. These are defined as follows:

1. **Direct product:** Given two vector spaces \mathcal{H}_1 and \mathcal{H}_2 over a field \mathbb{F} , one is able to form the direct product, which is denoted:

$$\mathcal{H}_{12} = \mathcal{H}_1 \otimes_{\mathbb{F}} \mathcal{H}_2. \tag{2.31}$$

Notice that when it is obvious which field I am tensoring over I will often drop the subscript \mathbb{F} . The tensor product space \mathcal{H}_{12} is a new vector space over \mathbb{F} of ordered pairs (a, b) , for $a \in \mathcal{H}_1$ and $b \in \mathcal{H}_2$, satisfying the following properties:

$$(q_1, p_1) + (q_1, p_2) = (q_1, p_1 + p_2) \tag{2.32a}$$

$$(q_1, p_1) + (q_2, p_1) = (q_1 + q_2, p_1) \tag{2.32b}$$

$$\lambda(q_1, p_1) = (\lambda q_1, p_1) = (q_1, \lambda p_1) \tag{2.32c}$$

for $q_i \in \mathcal{H}_1$, $p_i \in \mathcal{H}_2$, and $\lambda \in \mathbb{F}$. Ordered pairs (q_i, p_i) satisfying equations (2.36) will be denoted $p_i \otimes q_i$. The tensor product of two algebras \mathcal{A}_1 and \mathcal{A}_2 over a field \mathbb{F} forms a new algebra \mathcal{A}_{12} which in addition to the conditions given in equations (2.36), satisfies:

$$(q_1 \otimes p_1)(q_2 \otimes p_2) = (q_1 q_2 \otimes p_1 p_2) \tag{2.33}$$

Given two linear maps $O_1 : \mathcal{H}_1 \rightarrow \mathcal{V}_1$ and $O_2 : \mathcal{H}_2 \rightarrow \mathcal{V}_2$, their tensor product maps $O_1 \otimes O_2 : \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{V}_1 \otimes \mathcal{V}_2$, and is defined by:

$$(O_1 \otimes O_2)(q_i \otimes p_i) = O_1(q_i) \otimes O_2(p_i) \quad (2.34)$$

Notice that it does not make sense to form the tensor product between two vector spaces or algebras defined over different fields. Whenever I form a tensor product between two real (complex) vector space I am implicitly forming the tensor product over the real (complex) numbers. Whenever I form the tensor product between a complex vector space and a real vector space I am implicitly viewing the complex space as a real space of twice the dimension, i.e. $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. The complex numbers may be viewed as a real vector space equipped with a ‘complex element’ $i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which acts on pairs as $i : (a, b) \rightarrow (-b, a)$.

2. **Direct sum:** Given two vector spaces H_1 and H_2 over a field \mathbb{F} , one is able to form the Direct sum, which is denoted

$$H_{1+2} = H_1 \oplus_{\mathbb{F}} H_2 \quad (2.35)$$

Notice that when it is obvious which field I am forming the sum over I will often drop the subscript \mathbb{F} . The summed space \mathcal{H}_{1+2} is a new vector space over \mathbb{F} of ordered pairs (a, b) , for $a \in \mathcal{H}_1$ and $b \in \mathcal{H}_2$, satisfying the following properties:

$$(q_1, p_1) + (q_2, p_2) = (q_1 + q_2, p_1 + p_2) \quad (2.36a)$$

$$\lambda(q_1, p_1) = (\lambda q_1, \lambda p_1) \quad (2.36b)$$

for $q_i \in \mathcal{H}_1$, $p_i \in \mathcal{H}_1$, and $\lambda \in \mathbb{F}$. Ordered pairs (q_i, p_i) satisfying equations (2.36) will be denoted $p_i \oplus q_i$.

Direct sums only make sense if both vector spaces are taken over the same field. Whenever I form a direct sum between two real (complex) vector space I am implicitly forming the direct sum over the real (complex) numbers. Occasionally we will want to form the direct sum or product between a real vector space $H_{\mathbb{R}}$ and a complex vector space $H_{\mathbb{C}}$ (for example in the NCG SM). In this special case there are two ways that one is able to proceed in principle: (i) One might first complexify the real vector space, and then form the direct sum over the complex numbers: $\mathbb{C} \otimes_{\mathbb{R}} H_{\mathbb{R}} \oplus_{\mathbb{C}} H_{\mathbb{C}}$. (ii) one might instead view the complex vector space $H_{\mathbb{C}}$ as a real vector space of twice the dimension, and then form the direct sum over the real numbers $H_{\mathbb{R}} \oplus_{\mathbb{R}} H_{\mathbb{C}}$. In practice we will only ever consider the second case.

2.1.6 Associative representations

In this subsection I will review representations as they are traditionally defined. The reader should be aware however that the traditional definition and notation are both obstructive when generalizing to the non-associative setting and originally prevented us from making progress. In Ch. 3 I will introduce a definition and corresponding notation which are much more useful for our purposes.

If \mathcal{A} is an associative algebra over \mathbb{F} , and \mathcal{H} is a vector space over \mathbb{F} , then an associative representation π of \mathcal{A} on \mathcal{H} is an algebra homomorphism from \mathcal{A} to the endomorphisms of \mathcal{H} :

$$\pi : \mathcal{A} \rightarrow \text{End}(\mathcal{H}) \quad (2.37)$$

$$\pi(a) : \mathcal{H} \rightarrow \mathcal{H}, \quad (2.38)$$

where the product between elements in $\pi(\mathcal{A})$ is given by composition:

$$\pi(ab) = \pi(a) \circ \pi(b) \quad (2.39)$$

for $a, b \in \mathcal{A}$. If the operators $\pi(a)$ act from the left (right), the representation is known as a left (right) representation.

Equivalently, a left representation is given by a bi-linear map:

$$\phi_L : \mathcal{A} \times \mathcal{H} \rightarrow \mathcal{H}, \quad (2.40)$$

$$\phi_L(a, h) \equiv \pi(a)h \in \mathcal{H}, \quad (2.41)$$

which satisfies:

$$\phi_L(ab, h) = \phi_L(a, \phi_L(b, h)), \quad (2.42)$$

$a, b \in \mathcal{A}$, $h \in \mathcal{H}$. Similarly a right representation is given by a bi-linear map:

$$\pi_R : \mathcal{H} \times \mathcal{A} \rightarrow \mathcal{H} \quad (2.43)$$

$$\phi_R(h, a) \equiv h\pi(a) \in \mathcal{H}, \quad (2.44)$$

which satisfies:

$$\phi_R(h, ab) = \phi_R(\phi_R(h, a), b), \quad (2.45)$$

$a, b \in \mathcal{A}$, $h \in \mathcal{H}$. An algebra \mathcal{A} is said to be ‘bi-represented’ on \mathcal{H} (or equivalently \mathcal{H} acts as a bi-module over \mathcal{A}), if it has both a left and a right representation and satisfies the compatibility condition:

$$\phi_R(\phi_L(a, h), b) = \phi_L(a, \phi_R(h, b)) \quad (2.46)$$

In practice, the notation π , and ϕ is often overly cumbersome, and especially so when dealing with non-associative representations where we also have to carry around parentheses. When there is no possibility of confusion (almost always), I will drop the π , and ϕ notation, and indicate the right or left action of algebra elements on vector space elements simply by juxtaposition. In other words, I will indicate the left action of an algebra element $a \in \mathcal{A}$ on $h \in \mathcal{H}$ simply by $ah \in \mathcal{H}$. For instance it is much more illuminating to write Eq. (2.46) simply as $[a, h, b] = 0$, which shows that it is actually an associator identity!

Involutive representations:

If \mathcal{A} is a $*$ -algebra over \mathbb{F} , and \mathcal{H} is a vector space over \mathbb{F} , then we define a $*$ -representation π of \mathcal{A} on \mathcal{H} to be a bi-representation which is additionally equipped with a unitary anti-linear operator $*$: $\mathcal{H} \rightarrow \mathcal{H}$, which satisfies the following properties:

$$(h^*)^* = \epsilon h \quad (2.47a)$$

$$(\pi(a)h)^* = h^* \pi(a)^* \quad (2.47b)$$

$$(h\pi(a))^* = \pi(a)^* h^* \quad (2.47c)$$

where for ordinary matrix representations the involution $\pi(a)^* = \pi(a^*)$ is given simply by the conjugate transpose, and where $\epsilon = \pm 1$ is a choice of sign. Notice that in comparison to Eq. (2.8) I have not imposed the condition ‘ $*^2 = \mathbb{I}$ ’, but instead the slightly weaker condition ‘ $*^4 = \mathbb{I}$ ’.

Graded representations:

If \mathcal{A} is a graded algebra over \mathbb{F} with grading $\mathcal{A} = \bigoplus_m \mathcal{A}_m$, and \mathcal{H} is a vector space (over \mathbb{F}) with grading $H = \bigoplus_m H_m$, then an associative graded left (right) representation is a bi-linear map: $\mathcal{A}_m \times \mathcal{H}_n \rightarrow \mathcal{H}_{m+n}$ ($\mathcal{H}_n \times \mathcal{A}_m \rightarrow \mathcal{H}_{m+n}$), which satisfies:

$$(\omega\omega')h = \omega(\omega'h) \in \mathcal{H}, \quad (h(\omega\omega')) = (h\omega)\omega' \in \mathcal{H}, \quad (2.48)$$

$\omega, \omega' \in \mathcal{A}$, $h \in \mathcal{H}$. A graded algebra \mathcal{A} is said to be ‘bi-represented’ on \mathcal{H} (or equivalently \mathcal{H} is a graded bi-module over \mathcal{A}), if it has both a left and a right graded representation and satisfies the compatibility condition:

$$((\omega h)\omega') = (\omega(h\omega')) \in \mathcal{H} \tag{2.49}$$

$\forall \omega, \omega' \in \mathcal{A}$, $h \in \mathcal{H}$. Notice that I have dropped the ϕ and π notation here, but otherwise equations (2.48) and (2.49) are simply graded versions of equations (2.42), (2.45), and (2.46).

Differential graded representations (DGRs)

Finally, let \mathcal{A} be a DGA, with grading $\mathcal{A} = \oplus_m \mathcal{A}_m$ and differential d ; and let \mathcal{H} be a vector space with grading $\mathcal{H} = \oplus_m \mathcal{H}_m$. Then we say that a graded bi-representation π of \mathcal{A} on \mathcal{H} is a *differential* graded bi-representation π of \mathcal{A} on \mathcal{H} (or, equivalently \mathcal{H} is a *differential* graded bi-module over \mathcal{A}) if \mathcal{H} is also equipped with its own differential $d_{\mathcal{H}}$: *i.e.* a linear operator from \mathcal{H}_m to \mathcal{H}_{m+1} that is nilpotent ($d_{\mathcal{H}}^2 = 0$) and satisfies the graded Leibniz conditions:

$$d_{\mathcal{H}}[\omega_m h_n] = d[\omega_m] h_n + (-1)^m \omega_m d_{\mathcal{H}}[h_n], \tag{2.50a}$$

$$d_{\mathcal{H}}[h_m \omega_n] = d_{\mathcal{H}}[h_m] \omega_n + (-1)^m h_m d[\omega_n], \tag{2.50b}$$

for any $a_m \in \mathcal{A}_m$ and $h_n \in \mathcal{H}_n$. Notice that I will almost always drop the subscript ‘ \mathcal{H} ’, and simply write the differential on \mathcal{H} as d .

2.2 NCG preliminaries

In this section I will outline the basics of NCG model building. The organization is as follows. I will start in the commutative setting. In Subsection 2.2.1 I will give a little bit more detail on a few of the important ideas behind the reconstruction of Riemannian geometries from spectral triples: the Gel’fand-Naimark theorem, reconstructing geodesic distances, and the exterior algebra. In subsection 2.2.2 I will review the geometric axioms which apply more generally to associative (non-commutative) spectral triples. Then finally in subsection 2.2.3 I describe the construction of so called ‘almost-commutative’ geometries which correspond to Yang-Mills theories coupled to Einstein Hilbert gravity. In particular I describe how the internal gauge symmetries arise, and the so called spectral action associated to any NCG.

2.2.1 Reconstruction

Every compact oriented Riemannian spin manifold M gives rise to a canonical spectral triple, and after substantial attempts by Rennie and Varilly [RV06], more recently Connes put forward a proof of the reverse [Con13]: that a spectral triple with commutative algebra and satisfying certain conditions necessarily arises from a compact oriented Riemannian manifold⁴. This equivalence was first conjectured in 1996 [Con96] and motivated spectral triples more generally as the objects which describe ‘non-commutative Riemannian manifolds’: On the one hand, the Gel’fand-Naimark theorem asserts a one-to-one correspondence between topological spaces and commutative C^* algebras, suggesting we can trade one for the other, which is the starting point of NCG. More formally, the commutative Gelfand-Naimark theorem constructs an equivalence between the category of compact Hausdorff spaces with morphisms given by continuous maps, and the category of commutative unital C^* -algebras with $*$ -homomorphisms. This equivalence motivates the identification in non-commutative geometry of the category of C^* -algebras as the category of non-commutative topological spaces. On the other hand, the Atiyah-Singer Dirac operator of compact spin manifolds motivates generalized Dirac operators as the receptacle for metric data [Ć12a]⁵. Before introducing non-commutative geometries, let us start with a few of the key ideas in the reconstruction of commutative geometries.

The input algebra \mathcal{A} : Gel’fand-Naimark theorem

Let us start with the Gel’fand-Naimark theorem. Note however that while this topic is very interesting, and part of the conceptual underpinnings of NCG, it is not central to my own work and I simply include it here for completeness. In practice it will not be used in any of the calculations that I perform and may be skipped over by the casual reader. I follow closely the presentation given in [Lan97]:

The Gel’fand-Naimark theorem states that any commutative C^* -algebra \mathcal{A} is isometrically $*$ -isomorphic to the algebra of continuous functions $C(M, \mathbb{C})$ over a manifold M . The equivalence is seen as follows:

⁴More recently Ćaćić extended the reconstruction theorem to Connes-Landi deformations of commutative spectral triples [Ć14], and also to the so called almost-commutative spectral triples which describe Yang-Mills theories [Ć12a] coupled to Einstein-Hilbert gravity.

⁵Following [vdDvS12], By a generalized Dirac operator D I mean a first order hermitian differential operator on \mathcal{H} which satisfies $JD = \epsilon'DJ$, $\{D, \gamma\} = 0$ (see Subsection 2.2.2), and whose square is a generalized Laplacian in the sense of [NBV92].

Because \mathcal{A} is commutative, any non-trivial irreducible representation ϕ_λ is one dimensional, and is therefore a $*$ -linear functional $\phi_\lambda : \mathcal{A} \rightarrow \mathbb{C}$, which satisfies $\phi(ab) = \phi(a)\phi(b)$ $a, b \in \mathcal{A}$. The space of such representations of \mathcal{A} is denoted $\hat{\mathcal{A}}$, and can be made into a topological space by endowing it with the so called ‘Gel’fand topology’.

The closed sets S of the Gel’fand topology on $\hat{\mathcal{A}}$ are defined as follows: a sequence $\{\phi_\lambda\}_{\lambda \in \Lambda}$ of elements in $\hat{\mathcal{A}}$ is said to converge to $\phi \in \hat{\mathcal{A}}$ iff $\forall a \in \mathcal{A}$ the set of elements $\{\phi_\lambda(a)\}_{\lambda \in \Lambda}$ in \mathbb{C} converges to $\phi(a) \in \mathbb{C}$ (where Λ is a directed set). A set $S \in \hat{\mathcal{A}}$ is then defined to be closed in the Gel’fand topology if all sequences $\{\phi_\lambda\}_{\lambda \in \Lambda}$ for $\phi_\lambda \in S$ converge to some $\phi \in S$.

Given an element $a \in \mathcal{A}$, its Gel’fand transform $\hat{a} : \hat{\mathcal{A}} \rightarrow \mathbb{C}$ is defined as:

$$\hat{a}(\phi_i) := \phi_i(a) \quad (2.51)$$

Any representation $\phi_i \in \hat{\mathcal{A}}$ satisfies $\phi_i(a_0 a_1) = \phi_i(a_0)\phi_i(a_1)$, and so the Gel’fand transforms satisfy:

$$\begin{aligned} \widehat{a_0 a_1}(\phi_i) &= \phi_i(a_0 a_1) \\ &= \phi_i(a_0)\phi_i(a_1) = \hat{a}_0(\phi_i)\hat{a}_1(\phi_i) \end{aligned} \quad (2.52)$$

$\forall \phi_i \in \hat{\mathcal{A}}$. But this is just a point-wise product of Gel’fand transforms over \mathcal{A} , and \hat{a} is continuous for each $a \in \mathcal{A}$. We therefore get the interpretation that the elements of \mathcal{A} act as \mathbb{C} -valued continuous functions over the topological space $\hat{\mathcal{A}}$. The Gel’fand-Naimark theorem states that all continuous functions over $\hat{\mathcal{A}}$ are of the form given in equation (2.51).

Finally the supremum norm on $\mathcal{A}(\hat{\mathcal{A}})$ satisfies $\|\hat{a}\| = \sup_{\phi_i \in \hat{\mathcal{A}}} |\hat{a}(\phi_i)| = \sup_{\phi_i \in \hat{\mathcal{A}}} |\phi_i(a)| = \|a\|$. This tells us that the Gel’fand transform $a \rightarrow \hat{a}$ is an isometric $*$ -isomorphism of \mathcal{A} onto $\mathcal{A}(\hat{\mathcal{A}})$.

The Dirac operator: geodesic distance

The geodesic distance between any two points x and y on a Riemannian manifold is defined by:

$$d_g(x, y) = \inf_{\gamma} \int_{\gamma} ds = \inf_{\gamma} \int_0^1 \sqrt{g_{\mu\nu} \dot{\gamma}^\mu(t) \dot{\gamma}^\nu(t)} dt, \quad (2.53a)$$

where the infimum is over all smooth curves $\gamma(t)$ on the manifold parametrized such that $\gamma(0) = x$, $\gamma(1) = y$. In other words, the distance is given by the shortest connected

path between two points. When reconstructing Riemannian geometries from the canonical spectral data, the Dirac operator D provides metric information for the spectral triple. The geodesic distance formula given in Eq. (2.53a) is replaced with a new formula which depends only on the spectral input data [Con96]:

$$d_D(x, y) = \sup\{|f(x) - f(y)| : f \in \mathcal{A}, \|[D, f]\| \leq 1\} \quad (2.53b)$$

Let us see how Eq. (2.53b) works on a canonical manifold. In the canonical case \not{D} is described locally in terms of the Levi-Civita connection $\not{D} = -i\gamma^\mu \nabla_\mu^S$ on the spinor bundle $S \rightarrow M$. The spin connection satisfies the Leibniz rule:

$$\nabla_\mu^S(f\psi) = f\nabla_\mu^S\psi + (\partial_\mu f)\psi, \quad (2.54)$$

for $f \in C^\infty(M, \mathcal{C})$, $\psi \in L^2(M, S)$. Contracting this expression on both sides with $-i\gamma^\mu$, then yields [vdDvS12]:

$$[\not{D}, f]\psi = -i\gamma^\mu(\partial_\mu f)\psi. \quad (2.55)$$

In other words, the commutator $[\not{D}, f]$ is given by the Clifford multiplication of the gradient $\nabla(f)$ of f . In particular this means that the operator norm of $[\not{D}, f]$ on $\mathcal{H} = L^2(M, S)$ is given by [CM08]:

$$\|[D, f]\| = \sup_{x \in M} \|\nabla f\| \quad (2.56)$$

It then follows by integration along the path from x to y that $|f(x) - f(y)| \leq d_g(x, y)$, provided Eq. (2.56) is bounded by 1. Hence the equivalence between Eq (2.53a) and Eq. (2.53b). For further information on the reconstruction of geodesic distances from spectral input data see [CM08, Con96].

Junk forms and the exterior algebra

The differential graded algebra of forms associated to a (canonical) NCG can be constructed as a projection from an associative universal $*$ -DGA. A review of $*$ -DGAs and universal differential graded algebras is given in Subsection 2.1.2, but in brief: starting with the input algebra \mathcal{A} , the associative universal differential graded algebra $\Omega\mathcal{A} = \bigoplus_n \Omega^n \mathcal{A}$, is constructed by equipping $\Omega^0 \mathcal{A} = \mathcal{A}$ with a nilpotent ($d^2 = 0$) differential operator d , which satisfies the conditions given in Eq. (2.17). Elements in $\Omega^n \mathcal{A}$ are then constructed by juxtaposing elements $a \in \mathcal{A}$, and n formal symbols $d[a] \in \Omega^1 \mathcal{A}$.

A representation π of the differential graded algebra ΩA on the input Hilbert space H is constructed by making use of the input Dirac operator D :

$$\begin{aligned}\pi : \Omega A &\rightarrow B(H) \\ \pi(a_0 d[a_1] d[a_2] \dots d[a_n]) &:= a_0 [D, a_1] [D, a_2] \dots [D, a_n]\end{aligned}\tag{2.57}$$

$a_i \in A$. While π is an algebra homomorphism, as described in [Lan97] it is usually not a homomorphism between differential graded algebras because in general, $\pi(\omega_n) = 0$ does not imply $\pi(d[\omega_n]) = 0$. Such forms $d[\omega_n]$ for which $\pi(\omega_n) = 0$ are known as ‘junk forms’ and must be removed in order to form a true graded differential representation in which the Liebniz rule makes sense. To see what goes wrong consider for example the following equation for the case in which $\omega_n, \omega'_m \in \Omega \mathcal{A}$, and $\pi(\omega_n) = 0$:

$$\begin{aligned}\pi(d[\omega_n \omega'_m]) &= \pi(d[\omega_n]) \omega'_m + (-1)^n \pi(\omega_n d[\omega'_m]) \\ &= \pi(d[\omega_n]) \omega'_m.\end{aligned}\tag{2.58a}$$

We see that when $\pi(d[\omega_n]) \neq 0$ the Liebniz rule is not satisfied. Consider also for example the following equation:

$$\begin{aligned}d[\omega_n h] &= d[\omega_n] h + (-1)^n \omega_n d[h] \\ &= d[\omega_n] h\end{aligned}\tag{2.58b}$$

for $\omega_n \in \Omega^n \mathcal{A}$, $h \in \mathcal{H}$. When $\pi(\omega_n) = 0$ the left hand side of Eq. (2.58b) is equal to zero, while the right hand side is only zero if $\pi(d[\omega_n]) = 0$. Fortunately, in the associative case the set of all such troubling ‘junk’ forms denoted ‘ J ’ form a two sided ideal of ΩA , and so can be removed by forming the new quotient algebra $\Omega_D A = \Omega A / J$ [CL92]. To see this, consider an element $\omega = \omega_p + d[\omega'_{p-1}] \in J^p$, where $J = J_0 + dJ_0$, and $J_0 = \bigoplus_i J_0^i$, where $J_0^i = \{\omega \in \Omega^i A \mid \pi(\omega) = 0\}$. It is clear that J_0 is an ideal, and so if $\eta \in \Omega^r A$, then $\eta \omega = (\eta \omega_p - (-1)^\eta d[\eta] \omega'_{p-1}) + (-1)^\eta d[\eta \omega'_{p-1}]$, which is also in J^{p+r} .

For canonical spectral triples $M = \{A_c, H_c, D_c\}$, the algebra $\Omega_D A$ is nothing but the usual exterior algebra of differential forms over the manifold M . In this case the first ‘junk forms’ appear at order one. They take the form $fd[g] - d[g]f$ for $f, g \in C^\infty(M, \mathbb{C})$:

$$\pi(fd[g] - d[g]f) = -i\gamma^\mu (f(\partial_\mu g) - (\partial_\mu g)f) = 0,\tag{2.59a}$$

while,

$$\begin{aligned}\pi(d(fd[g] - d[g]f)) &= -\gamma^\nu \gamma^\mu (\partial_\nu f)(\partial_\mu g) - \gamma^\mu \gamma^\nu (\partial_\mu g)(\partial_\nu f) \\ &= -\{\gamma^\nu, \gamma^\mu\} (\partial_\nu f)(\partial_\mu g) \neq 0.\end{aligned}\tag{2.59b}$$

The right hand side of equation (2.59b) is just a symmetric ‘two form’ in $\Omega^2 A$. At all higher orders the junk similarly consists of the symmetric elements in ΩA . Modding out by these ‘junk’ elements, the algebra $\Omega_D A = \Omega A/J$ is nothing but the usual algebra of differential forms over the field $C^\infty(M, \mathbb{C})$, which is generated by the anti-symmetric elements $\gamma^\mu, \gamma^{[\mu}\gamma^{\nu]}$, $\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$, $\gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\tau]}$, where the γ ’s are Dirac gamma matrices. A good account of the differential graded algebra $\Omega_D \mathcal{A}$ in the associative setting is given in [Lan97].

Cyclic homology and the exterior algebra

I have reviewed how to construct the differential graded algebra of forms $\Omega_D \mathcal{A}$ from a projection of the universal DGA $\Omega \mathcal{A}$. Let me briefly discuss an alternative algebraic formulation of differential forms in NCG, which is made in terms of Hochschild and Cyclic (Co)homology. Note that while this alternative formulation is central to NCG, in practice I will only ever make use of the universal differential graded algebra construction. Just as with the Gel’fand-Naimark theorem cyclic (co)homology is not central to my own work, and I simply include this subsection for completeness. I follow closely the presentations given in [Con96] and [HSS15]:

The Hochschild homology $H_*(\mathcal{H}, \mathcal{A})$ of an associative algebra \mathcal{A} bi-represented on the vector space \mathcal{H} is the homology of the complex

$$C_*(\mathcal{H}, \mathcal{A}) = \bigoplus_{n \geq 0} C_n(\mathcal{H}, \mathcal{A}), \quad C_n(\mathcal{H}, \mathcal{A}) = \mathcal{H} \otimes \mathcal{A}^n \quad (2.60)$$

with boundary operator

$$\begin{aligned} b(h \otimes a_1 \otimes a_2 \dots a_n) &\equiv h a_1 \otimes a_2 \dots a_n + (-1)^{n+1} a_n h \otimes a_1 \otimes a_2 \dots a_{n-1} \\ &+ \sum (-1)^k h \otimes a_1 \otimes \dots \otimes a_k a_{k+1} \otimes \dots a_n, \end{aligned} \quad (2.61)$$

which satisfies $b^2 = 0$ even when \mathcal{A} is non-commutative, as long as it is associative. Of course, an associative algebra \mathcal{A} can always be naturally bi-represented on itself, and so one can always construct the homology $H_*(\mathcal{A}, \mathcal{A})$. The algebraic formulation of a differential form is a so called Hochschild ‘cycle’ in the Homology $H_*(\mathcal{A}, \mathcal{A})$ [Con96]. An n -cycle is defined to be a ‘closed’ element $c \in C_n(\mathcal{A}, \mathcal{A})$, ie. $bc = 0$.

When \mathcal{A} is commutative it is easy to construct a Hochschild cycle, it suffices to take any elements $a^i \in \mathcal{A}$ and consider the sum over permutations σ

$$c = \sum_{\sigma} \epsilon(\sigma) a^0 \otimes a^{\sigma(1)} \otimes a^{\sigma(2)} \dots \otimes a^{\sigma(n)} \quad (2.62)$$

The representation of the Hochschild cycle given in Eq. (2.62) on a vector space \mathcal{H} is then given by:

$$\pi(c) = \sum \epsilon(\sigma) a^0 [D, a^{\sigma(1)}] [D, a^{\sigma(2)}] \dots [D, a^{\sigma(n)}] \quad (2.63)$$

where D is a Dirac type operator on \mathcal{H} . Notice that in the case of a canonical geometry this construction exactly reproduces the differential graded algebra of forms which is obtained from the universal differential graded algebra by quotienting out by junk forms: $\Omega_D \mathcal{A} = \Omega \mathcal{A} / J$.

Finally, the cyclic homology of an associative algebra \mathcal{A} is defined by the homology of the quotient space $C_*^\lambda(\mathcal{A}) = C_*(\mathcal{A}) / (\mathbb{I} - t)$, where the operator t acting on n -cycles is defined by:

$$t(a_0 \otimes a_1 \otimes \dots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}. \quad (2.64)$$

For further information on cyclic homology, and also cohomology the reader may consult [[Con96](#), [Con94](#), [CM08](#), [HSS15](#), [Con06a](#)].

2.2.2 Spectral triples and the NCG axioms

NCG extends Riemannian geometry just as Riemannian geometry extends Euclidean Geometry. In order to reconstruct Riemannian geometries one must select a very special set of canonical input data $\{\mathcal{A}_c, \mathcal{H}_c, D_c, J_c, \gamma_c\}$, which satisfies a list of geometric conditions. Similarly, when constructing more general non-commutative geometries, the input data $\{\mathcal{A}, \mathcal{H}, D, J, \gamma\}$ may not be selected arbitrarily, but instead must satisfy a list of geometric axioms which were generalized from the conditions satisfied by commutative spectral triples [[Con96](#)]. In this section I outline the NCG axioms and conditions:

1. The input algebra \mathcal{A} is a real, unital, associative, involutive, (non-commutative) algebra represented faithfully by bounded operators on the Hilbert space \mathcal{H} .
2. The Dirac operator D is a (possibly unbounded) self-adjoint operator on H with compact resolvent.
3. A spectral triple is said to be ‘even’ if it is equipped with a hermitian Z_2 grading operator γ on H which satisfies $\gamma^2 = \mathbb{I}$, $\gamma^* = \gamma^{-1}$, $[\gamma, a] = 0$, $\forall a \in \mathcal{A}$.
4. Compatibility of D with γ

$$\{D, \gamma\} = 0 \quad (2.65)$$

5. A spectral triple is said to be ‘real’ if it is equipped with an anti-linear unitary ‘real structure’ operator J on H satisfying

$$J^2 = \epsilon \mathbb{I} \tag{2.66a}$$

$$JD = \epsilon' DJ \tag{2.66b}$$

$$J\gamma = \epsilon'' \gamma J \tag{2.66c}$$

where the \pm signs $\{\epsilon, \epsilon', \epsilon''\}$ depend on the KO-signature of the geometry being constructed⁶. As explained in [DD11], there are two possible sign choices in the case of

	0	2	4	6	0	2	4	6	1	3	5	7
ϵ	+1	-1	-1	+1	+1	+1	-1	-1	+1	-1	-1	+1
ϵ'	+1	+1	+1	+1	-1	-1	-1	-1	-1	+1	-1	+1
ϵ''	+1	-1	+1	-1	+1	-1	+1	-1				
	•	•	•	•					•	•	•	•

Table 2.1: Mod 8 KO-signature table

even signature, as the real structure operator may always be replaced with $J \rightarrow J\gamma$. The choices marked by • in table 2.1 are those taken by Connes.

6. The left and right actions of \mathcal{A} on \mathcal{H} are related by:

$$R_a = JL_a^* J^{-1} \tag{2.67a}$$

$$L_a = JR_a^* J^{-1}, \quad a \in \mathcal{A}. \tag{2.67b}$$

7. The order zero condition

$$[a_0, Ja_1^* J^*] = 0, \quad a_i \in \mathcal{A}, \tag{2.68}$$

where $J^* = J^{-1}$.

⁶For Riemannian geometries there are several equivalent ways of defining the dimension of a manifold [Con95a, CM08]. For non-commutative spaces these various different notions of dimension may no longer agree. One such notion based on KO-theory is the so called KO-dimension (an integer modulo 8), which is determined by the signs of the commutation relations of J , γ , and D . In the NCG literature the KO table given in 2.1 is usually called a ‘KO-dimension’ table because most NCG practitioners are usually only interested in the euclidean case where there is no distinction between dimension and signature. As pointed out in [Bar07], the signs given in 2.1 really depend on the ‘signature’ of the geometry in question. See also appendix B of [Pol05] for a nice review in the commutative setting.

8. The operator $[D, a]$ is a bounded operator on \mathcal{H} , for all $a \in \mathcal{A}$, and satisfies the order one condition

$$[[D, a_0], J a_1^* J^*] = 0, \quad a_i \in \mathcal{A}. \quad (2.69)$$

In addition to the above mentioned ‘axioms’ satisfied by associative NCG, there are also a number of salient facts about associative NCGs that are worth noting. These include:

1. The symmetries of a NCG are given by the automorphisms of the input algebra \mathcal{A} ‘lifted’ to the Hilbert space \mathcal{H} , or rather in terms of the adjoint representation of the unitary group of the input algebra in the sense of [CCM07]. In particular, the action of the inner derivations of \mathcal{A} on \mathcal{H} are given by $\delta_x = x - Jx^*J^*$, where x is an anti-hermitian element in \mathcal{A} .
2. To ensure covariance of the formalism with respect to the inner automorphisms of the input algebra, the ‘flat’ or ‘ground state’ Dirac operator is replaced with a ‘fluctuated’ Dirac operator $D_A = D + A + \epsilon' J A J^*$, where $A = \sum_{a,b} a [D, b]$ is a hermitian operator on \mathcal{H} which transforms ‘covariantly’ as $A \rightarrow A' = u A u^* + u [D, u^*]$ (explained below in subsection 2.2.3).
3. Given two real, even triples, $T_1 = \{A_1, H_1, D_1, \gamma_1, J_1\}$ and $T_2 = \{A_2, H_2, D_2, \gamma_2, J_2\}$, a third spectral triple $T_{12} = \{A_{12}, H_{12}, D_{12}, \gamma_{12}, J_{12}\}$ may be constructed, where $A_{12} = A_1 \otimes A_2$, $H_{12} = H_1 \otimes H_2$, $D_{12} = D_1 \otimes \mathbb{I}_2 + \gamma_1 \otimes D_2$, $\gamma_{12} = \gamma_1 \otimes \gamma_2$, and $J_{12} = J_1 \otimes J_2$. I will only ever need to take the product between even spectral triples. For the even-odd, odd-even, and odd-odd cases see for example [DD11].

2.2.3 Almost-commutative geometries

The NCGs that are of most interest in this work, and have so far proven to be the ones of interest for physics are the so called ‘almost-commutative’ geometries discussed briefly in Ch. 1. Almost-commutative geometries are constructed as the product between a commutative canonical spectral triple $M = \{C^\infty(M, \mathbb{C}), L^2(M, S), \nabla^S, J_c, \gamma_c\}$, and a finite non-commutative geometry $F = \{\mathcal{A}_F, \mathcal{H}_F, D_F, J_F, \gamma_F\}$:

$$M \times F = \{C^\infty(M, \mathcal{A}_F), L^2(M, S) \otimes \mathcal{H}_F, \nabla^S \otimes \mathbb{I}_F + \gamma_c \otimes D_F, J_c \otimes J_F, \gamma_c \otimes \gamma_F\}. \quad (2.70)$$

Note that when \mathcal{A}_F is a real algebra, the continuous algebra is replaced with the real algebra $C^\infty(M, \mathbb{R})$, such that $C^\infty(M, \mathbb{R}) \otimes_{\mathbb{R}} \mathcal{A}_F = C^\infty(M, \mathcal{A}_F)$ [C13]. The key idea behind

almost-commutative geometries is that the ‘external’ gravitational degrees of freedom come from the continuous part of the geometry M , while the ‘internal’ gauge/higgs degrees of freedom come from the finite part of the geometry F . As described in example four of Subsection (2.1.4) the automorphisms of the input algebra are given by the semi-direct product of two groups

$$Aut(\mathcal{A}) = Inn(\mathcal{A}) \rtimes Out(\mathcal{A}), \quad (2.71)$$

where $Inn(\mathcal{A}) = \text{Map}(M, Aut(\mathcal{A}_F))$ is the group of maps from M to the group $Aut(\mathcal{A}_F)$, and $Out(\mathcal{A}) = Diff(\mathcal{M})$. Notice that the group on the right-hand side of Eq. (2.71) also has another interpretation: it is the full symmetry group of $Aut(\mathcal{A}_F)$ gauge theory coupled to Einstein gravity. One striking feature of almost-commutative geometries is that, while the metric dimension of F is zero, its KO dimension may be any n modulo 8 [CM08] (for ordinary Riemannian geometries the two notions of dimension are always degenerate). In particular, when constructing the standard particle model one must select the KO-dimension of the finite space F to be 6 modulo 8 in order to obtain the correct fermionic content [Bar07, Con06b]. It is impossible not to wonder about what deep connection might hide behind the 4+6 modulo 8 dimensional NCG SM, and 10 dimensional super-symmetric string theories.

The elegant and conceptually satisfying picture emerges in almost-commutative geometry that the full symmetry group of a gauge theory coupled to gravity is reinterpreted in a unified way as the automorphism group of an underlying algebra \mathcal{A} ; and this, in turn, is interpreted as the group of purely gravitational transformations of a corresponding non-commutative space. The unification which occurs for almost-commutative geometry goes further however: Notice that the Dirac operator of an almost-commutative geometry has both a finite part, and a continuous part. The Higgs degrees of freedom appear as connections on the internal space in exactly the same way that gauge degrees of freedom appear as connections on the continuous space, through a ‘fluctuation’ procedure of the Dirac operator D (explained below in Subsection 2.2.4). Meanwhile all interactions between the input algebra \mathcal{A} and the Dirac operator D are mediated by the Hilbert space \mathcal{H} , which is precisely where the fermionic degrees of freedom live. Indeed, if one evaluates the spectral action (1.1) for an almost-commutative spectral triple, one finds that it reduces to gauge theory coupled to Einstein gravity [CC97].

2.2.4 The spectral action

The dynamics of a NCG are traditionally described by the ‘spectral action’, first introduced by Chamseddine and Connes in [CC97]. In this Subsection I review the construction of

spectral actions from input data satisfying the axioms and conditions outlined in subsection 2.2.2.

Fluctuating the Dirac operator

In ordinary gauge theory, the principle of gauge covariance leads one to replace the partial derivative ∂_μ by the gauge covariant derivative $D_\mu = \partial_\mu + A_\mu$, which is ultimately the object from which a gauge-invariant action is built. In a closely analogous way, in spectral geometry the principle of *-automorphism covariance leads one to replace the ‘flat’ or ‘ground state’ Dirac operator D with the “fluctuated” or “*-automorphism covariant” Dirac operator D_A , which is ultimately the object from which the *-automorphism-invariant spectral action is built.

It is helpful, then, to warm up by reviewing the story in ordinary gauge theory. We can write a general gauge transformation in the form $u(x) = \exp[\alpha^a(x)T_a]$, where T_a are the generators of the gauge group. Now consider a multiplet of matter fields ψ that transforms covariantly under a gauge transformation: $\psi \rightarrow \psi' = u\psi$. We would like to introduce a gauge-covariant derivative operator D_μ with the property that $D_\mu\psi$ also transforms covariantly: $D_\mu\psi \rightarrow D'_\mu\psi' = uD_\mu\psi$. In other words, we want D_μ to transform as

$$D_\mu \rightarrow D'_\mu = uD_\mu u^{-1}. \quad (2.72)$$

Start with the special case where $D_\mu = \partial_\mu$, and perform an infinitesimal gauge transformation to obtain $D'_\mu = \partial_\mu - [\partial_\mu, \alpha^a(x)]T_a$. By inspection of this formula, we see that in the general case we can take

$$D_\mu = \partial_\mu + B_\mu \quad \text{where} \quad B_\mu = B_\mu^a T_a. \quad (2.73)$$

Here B_μ^a are arbitrary gauge fields (one for each linearly independent generator T_a). To make D_μ transform as in Eq. (2.72), we should take B_μ to transform as

$$B_\mu \rightarrow B'_\mu = uB_\mu u^{-1} + u[\partial_\mu, u^{-1}]. \quad (2.74)$$

Let us now present the analogous story in NCG. The symmetries of any geometry are given by maps on the input data which leave the dynamics invariant. The bulk of the differential-topological data of a NCG is held in the input algebra \mathcal{A} , and so its symmetries are given naturally by automorphisms on \mathcal{A} , which are structure preserving, invertible maps. Similarly the symmetries act as invertible maps on the Hilbert space \mathcal{H} which leave its real structure, grading, and the spectrum of the Dirac operator invariant. In other

words the symmetries are given by unitary maps on the Hilbert space which commute with J and γ , and which induce automorphisms on the representation of the input algebra by conjugation. In the cases of interested in this work the group $Inn(A)$ corresponds to the ‘internal’ gauge symmetries, while $Out(A) = Aut(A)/Inn(A)$ corresponds to ‘external’ diffeomorphisms. Here I will focus on the symmetries of a geometry corresponding to the inner automorphisms of an associative input algebra.

The two spectral triples:

$$\{\mathcal{A}, \mathcal{H}, D, J, \gamma\} \simeq \{\mathcal{A}, \mathcal{H}, UDU^{-1}, J, \gamma\}, \quad (2.75)$$

are ‘gauge equivalent’, where $U = uJuJ^{-1} \in B(H)$, and u is the representation of a unitary element of \mathcal{A} as a bounded operator on \mathcal{H} . Let us see what action the elements $U \in B(H)$ have on the elements of a spectral triple. Using the order zero condition given in Eq. (2.68) the operators $U = uJuJ^{-1}$ may be written out in terms of their generating elements δ :

$$U = uJuJ^{-1} = e^x e^{JxJ^*} = e^\delta, \quad (2.76)$$

where

$$\delta \equiv x - Jx^*J^{-1}, \quad (2.77)$$

and $x^* = -x \in \mathcal{A}$. Under the action of U the representation of the input algebra elements transforms as follows:

$$\begin{aligned} UaU^{-1} &= uJuJ^{-1}aJu^{-1}J^{-1}u^{-1} \\ &= uau^* \end{aligned} \quad (2.78)$$

$\forall a \in \pi(A)$, and where the second equality is obtained using the order zero condition. As required, conjugation of input algebra elements by the unitary operators U induces inner automorphisms on the elements of the input algebra (see Subsection 2.1.4). Under conjugation by U the real structure operator and grading operators are invariant:

$$\begin{aligned} UJU^{-1} &= uJuJ^{-1}JJu^{-1}J^{-1}u^{-1} \\ &= J, \end{aligned} \quad (2.79)$$

$$\begin{aligned} U\gamma U^{-1} &= uJuJ^{-1}\gamma Ju^{-1}J^{-1}u^{-1} \\ &= \gamma, \end{aligned} \quad (2.80)$$

where the order zero condition, the condition $[\gamma, a] = 0 \forall a \in A$, the real structure conditions $J^2 = \epsilon \mathbb{1}$, and $J\gamma = \epsilon''\gamma J$, and the unitarity of J have all been used.

Finally, under the action of U the Dirac operator transforms as:

$$\begin{aligned}
UDU^{-1} &= uJuJ^{-1}DJu^{-1}J^{-1}u^{-1} \\
&= D + uJuJ^{-1}Ju^{-1}J^{-1}[D, u^{-1}] + uJuJ^{-1}[D, Ju^{-1}J^{-1}]u^{-1} \\
&= D + u[D, u^{-1}] + \epsilon' uJuJ^{-1}J[D, u^{-1}]J^{-1}u^{-1} \\
&= D + u[D, u^{-1}] + \epsilon' Ju[D, u^{-1}]J^{-1}
\end{aligned} \tag{2.81}$$

where the real structure condition $DJ = \epsilon'JD$ is used in the third equality, and the final equality both the order zero and order one conditions have been used. A gauge transformation of the spectral input data is therefore given by:

$$\begin{aligned}
h &\rightarrow uJuJ^{-1}h \\
a &\rightarrow uau^{-1} \\
D &\rightarrow D + u[D, u^{-1}] + \epsilon' Ju[D, u^{-1}]J^{-1}.
\end{aligned}$$

In order to ensure that the Dirac operator transforms in the appropriate way, the ‘ground state’ or ‘flat’ Dirac operator is replaced by a ‘fluctuated’ Dirac operator:

$$D \rightarrow D_A = D + A + \epsilon'JAJ^*, \tag{2.82}$$

where the ‘gauge potential’ A must gauge transform as:

$$A \rightarrow UAU^{-1} + u[D, u^{-1}]. \tag{2.83}$$

Under the transformation given in equation (2.83), the fluctuated Dirac operator transforms as:

$$\begin{aligned}
D'_A &= D + UAU^{-1} + u[D, u^{-1}] + \epsilon'J(UAU^{-1} + u[D, u^{-1}])J^{-1} \\
&= UD_AU^{-1},
\end{aligned} \tag{2.84}$$

which is the required transformation. The form that the fluctuation terms take is determined using Morita equivalence or semi-group methods to be a bounded hermitian operator on \mathcal{H} of the form [CM, CCv13, CCvS13]:

$$A = \sum_{a,b} a[D, b], \tag{2.85}$$

where $a, b \in \mathcal{A}$. Using the order one condition then, one immediately sees that $UAU^{-1} = uAu^{-1}$, and so finally an internal gauge transformation of the spectral input data is given

by:

$$h \rightarrow uJuJ^{-1}h, \quad (2.86a)$$

$$a \rightarrow uau^{-1}, \quad (2.86b)$$

$$A \rightarrow uAu^{-1} + u[D, u^{-1}] \quad (2.86c)$$

which should be compared for example with Eq. (2.74).

For the special case of an almost-commutative geometry, fluctuating the Dirac operator with respect to the inner automorphisms on \mathcal{A} yields:

$$D_A = -i\gamma^\mu(\partial_\mu + B_\mu) + \gamma_5\Phi, \quad (2.87)$$

where $B_\mu = \sum_{ab} a(\partial_\mu b) + Ja(\partial_\mu b)J$ is the gauge potential, and $\Phi = D_F + \sum_{ab} a[D_F, b] + \epsilon' J_F a[D_F, b] J_F^{-1}$ is a scalar ‘Higgs’ field.

The spectral action

The dynamics of a NCG are described by the action given in Eq. (1.1), and which I reviewed briefly in Section 1.1 for Riemannian geometries. This action functional has two terms: the first term is the so called bosonic or ‘spectral’ action, while the second term is fermionic. In this subsection I will discuss the spectral action or ‘bosonic’ action for almost-commutative geometries:

$$S_b = Tr[f(D_A/\Lambda)], \quad (2.88)$$

where D_A is the fluctuated Dirac operator, f is a positive even function which interpolates between one and zero, and Λ is a cut-off scale. This action depends only on the discrete spectrum of the fluctuated Dirac operator D_A , and so is invariant under the unitary ‘gauge transformations’ shown in Eq. (2.84) [CC97, CC96]. As outlined in [vdDvS12], to obtain a formula which is more recognizable as a local action formula, the spectral action can be expanded in powers of Λ . The first step in this so called ‘heat kernel expansion’ is to square the fluctuated Dirac operator. For an almost-commutative geometry with a fluctuated Dirac operator as in Eq. (2.87), the square is given by (see Appendix A):

$$D_A^2 = \Delta^E - \frac{1}{4}R - \frac{1}{2}\gamma^\mu\gamma^\nu F_{\mu\nu} - \gamma_5[\nabla^E, \Phi] + \Phi^2, \quad (2.89a)$$

where

$$\Delta^E = -g^{\mu\nu}\nabla_\mu^E\nabla_\nu^E, \quad (2.89b)$$

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu], \quad (2.89c)$$

where $\nabla_\mu^E = \nabla_\mu^S \otimes \mathbb{I} + B_\mu^i \otimes \delta_i$, and R is the Ricci scalar. The square of the Dirac operator for an almost-commutative geometry is therefore of the form of a generalized Laplacian in the sense of [NBV92], which allows us to use the ‘heat kernel expansion’ formula for the spectral action reviewed in [vdDvS12]:

$$S_b = \text{Tr}[f(D_A/\Lambda)] \sim 2f_4\Lambda^4 a_0(D_A^2) + 2f_2\Lambda^2 a_2(D_A^2) + f(0)a_4(D_A^2) + O(\Lambda^{-1}) \quad (2.90)$$

where the f_j are the moments of the function f , and the ‘Seeley-DeWitt coefficients’ are:

$$a_k(D_A^2) = \int_M a_k(x, D_A^2) \sqrt{|g|} d^4x, \quad (2.91)$$

and ignoring boundary terms the first three coefficients are given by:

$$a_0(x, D_A^2) = (4\pi)^{-\frac{n}{2}} \text{Tr}(Id) \quad (2.92)$$

$$a_2(x, D_A^2) = (4\pi)^{-\frac{n}{2}} \text{Tr}(\frac{1}{12}R - \Phi^2) \quad (2.93)$$

$$a_4(x, D_A^2) = (4\pi)^{-\frac{n}{2}} \frac{1}{360} \text{Tr}(\frac{5}{4}R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau} + 30\Omega_{\mu\nu}^E\Omega^{E\mu\nu} + 60R\Phi^2 + 180(\frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau F_{\mu\nu}F_{\rho\tau} - \frac{1}{2}R\Phi^2 - [\nabla^E, \Phi]^2 + \Phi^4)). \quad (2.94)$$

The terms $R_{\mu\nu}$, and $R_{\mu\nu\rho\tau}$ are the Ricci tensor and Riemann tensor respectively, while $\Omega_{\mu\nu}^E = \Omega_{\mu\nu}^S \otimes \mathbb{I} + \mathbb{I}_4 \otimes F_{\mu\nu}$, and $\text{Tr}(\Omega_{\mu\nu}^S\Omega^{S\mu\nu}) = -\frac{1}{2}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, where $\Omega_{\mu\nu}^E$ and $\Omega_{\mu\nu}^S$ are the curvatures of the connections ∇_μ^E and ∇_μ^S respectively.

We therefore see that the heat kernel expansion for the spectral action corresponding to an almost-commutative geometry yields the action for a Yang-Mills theory coupled to gravity. For more information on the spectral action and the heat kernel expansion see [Gil84, Vas03, vdDvS12, vS15]. I also compute the heat kernel expansion explicitly for a minimal NCG SM extension in Appendix A (as well as the corresponding fermionic action).

Chapter 3

Non-associative geometry: foundations

When Connes originally proposed spectral triples as the objects which describe ‘non-commutative Riemannian manifolds’, he generalized the conditions satisfied by canonical commutative spectral triples to produce a set of axioms satisfied by more general NCGs with non-commutative input algebras [Con96]. In this chapter I will go a step further, and generalize the axioms of NCG so as to allow for non-associative input algebras. The goal is to develop the foundations of non-associative, non-commutative differential geometry. Ultimately I will describe a reformulation in which the input data of a NCG is ‘fused’ together to form a so called ‘fused \ast -DGA’, denoted ΩB :

$$\{\mathcal{A}, \mathcal{H}, D, J, \gamma\} \longleftrightarrow \Omega B \tag{3.1}$$

This reformulation or ‘fusion’ has four key benefits over the traditional approach:

1. The five otherwise separate input objects $\{\mathcal{A}, \mathcal{H}, D, J, \gamma\}$ of a NCG are unified into a single object ΩB . At the same time many of the otherwise seemingly unrelated axioms of associative NCG are re-expressed, and given new meaning as the intrinsic algebraic properties of ΩB (which for an associative NCG is an associative \ast -DGA).
2. In addition to the traditional NCG constraints, new ‘axioms’ on the NCG input data naturally arise from the algebraic properties of ΩB . These new ‘axioms’ place additional constraints on the allowable gauge theories which may be constructed as NCGs. In particular new ‘higher order’ associativity constraints place phenomenologically accurate restrictions on the NCG SM scalar sector - constraints which were previously put by hand into the construction.

3. While the repackaging of the NCG input data in terms of ΩB elucidates many aspects of the associative formalism, it also allows ready generalization to the non-associative setting. This is because ΩB need not be associative. When ΩB is taken to have more general associative properties (for example if it is taken to be Jordan, or alternative), generalized NCG axioms are ‘derived’ from its intrinsic algebraic properties.
4. The symmetries of a NCG are given new meaning, and are expressed simply and succinctly as the automorphisms of ΩB . In particular on analyzing the symmetries of the algebra ΩB corresponding to the NCG SM, one finds that the formalism forces a minimal SM extension with gauged baryon-lepton number ($B - L$) symmetry, and an additional scalar field which is responsible for Majorana masses, and for breaking the extra $B - L$ symmetry. This extension does not suffer from the various problems which plague the Higgs sector in the traditional approach to the NCG SM.

Before introducing ΩB I will first start in Section 3.1 by reviewing my early attempts at generalizing the axioms of associative NCG. These first steps into the non-associative setting were made in [FB13], and relied heavily on simple finite examples, and I approached the generalization of the spectral data and its axioms in a piecewise fashion. This naive approach gets surprisingly good mileage, and as I will show it is possible to construct very simple non-associative geometries with very little modification of Connes original axioms. Ultimately, however one is led to introduce the fused algebra formulation of NCG as we did in [BF14]. Section 3.2 follows this work. In it I will introduce the fused algebra ΩB by building it up in three steps: to start with in Subsection 3.2.1 I will introduce the algebra $B_0 \subset \Omega B$ which unifies together the elements $\{\mathcal{A}, \mathcal{H}, J\}$ of a spectral triple. In Subsection 3.2.2 I will introduce the algebra B , which provides an incomplete unification of the elements $\{\mathcal{A}, \mathcal{H}, D, J\}$. In Subsections 3.2.3 and 3.2.3 I will introduce the full algebra ΩB , which unifies all elements of the spectral triple $\{\mathcal{A}, \mathcal{H}, D, J, \gamma\}$. Finally, In section 3.3 I will describe the work introduced in [FB15b], in which we described the gauge symmetries associated to finite non-associative, and almost-associative geometries as the automorphisms of ΩB .

3.1 Non-associative geometry: a first attempt.

In subsection 2.2.2 I reviewed the axioms which are imposed on an associative NCG. When attempting to extend to a non-associative geometry a list of ‘obvious’ questions arises relating to these axioms. These include:

1. In associative NCG, the input algebra \mathcal{A} is represented by bounded linear operators on the input Hilbert space \mathcal{H} (see Subsection 2.1.6 for a review of associative representations). The product between elements $\pi(ab) := \pi(a) \circ \pi(b)$ is given by composition, which is associative, so what does it even mean to represent a non-associative input algebra?
2. In associative NCG, a right action of algebra elements on the input Hilbert space is constructed by making use of a so called ‘real structure’ operator J . However the real structure J had its origins as a modular conjugation operator in the Tomita-Takesaki theory of associative Von-Neumann algebras (and also as the charge conjugation operator on spinors) [Con95a, BR87]. An obvious question then is to ask: ‘is there some analogue of the real structure operator in non-associative geometry and can it be used to construct a right action of input algebra elements?’
3. How do the NCG axioms generalize? In particular, the ‘order zero’ and ‘order one’ conditions are both already generalized in going from commutative geometries to associative NCGs [Con96]. Do these axioms need further generalization in the non-associative setting, and if so what are their generalizations?
4. Are the symmetries of a non-associative geometry still related to the automorphisms of the input algebra? If so how do they act on the input Hilbert space?
5. Can non-trivial Dirac operators be constructed in the non-associative setting? What form do they take, and how are their ‘fluctuations’ determined?

This list of questions is of course by no means exhaustive. The goal of this Section however is to outline my first attempts at answering them. In particular I will review my first attempts at generalizing the NCG axioms to allow for non-associative input algebras, and will outline my first attempt at constructing a simple non-associative NCG based on the well known algebra of octonions represented on themselves [FB13]. The organization is as follows: In subsection 3.1.1, I introduce a notion of representation for non-associative $*$ -algebras \mathcal{A} on a Hilbert space \mathcal{H} . In Subsection 3.1.2, I re-introduce the grading and real structure operators γ and J in the non-associative setting, and explain how the usual ‘order zero’ condition generalizes. In subsection 3.1.3, I articulate the principle of $*$ -automorphism covariance, which ties together the transformations of the input algebra \mathcal{A} with those of the Hilbert space \mathcal{H} , and all of the operators that act on it. The principle of $*$ -automorphism covariance subsumes and replaces the traditional covariance principles of physics: diffeomorphism covariance (in Einstein gravity) and gauge covariance (in gauge theory). Then in Subsection 3.1.4, I explain how to obtain a ‘fluctuated’ Dirac operator D_A

from a flat ‘un-fluctuated’ Dirac operator D . In subsection 3.1.5 I introduce the simplest almost-associative geometry based on the algebra of octonions, and explicitly construct its spectral action. Then finally in Subsection 3.1.6 I show how to obtain this same octonionic geometry by ‘twisting’ from an associative geometry.

3.1.1 Representing a non-associative $*$ -algebra

The starting point when constructing a physical theory as a NCG is a $*$ -algebra \mathcal{A} that is represented (or, more correctly, ‘bi-represented’ – *i.e.* represented from both the left and the right) as bounded operators on a vector space \mathcal{H} . However there seems to be an immediate problem in attempting to extend the definition of a bi-representation of \mathcal{A} on \mathcal{H} to the case where \mathcal{A} is non-associative: as described in Subsection 2.1.6 a representation of \mathcal{A} on \mathcal{H} is usually taken to be a linear map from each element $a \in \mathcal{A}$ to a linear operator $\pi(a)$ in $End(\mathcal{H})$, such that the composition of such operators represents the product on \mathcal{A} : $\pi(a) \circ \pi(b) = \pi(ab)$. Yet the composition of linear operators is associative, so it seems one cannot possibly represent the non-associativity of \mathcal{A} in this way. A shift in perspective is required.

Fortunately there is a prototype for what is meant by a non-associative bi-representation: any (non-associative) algebra has a natural bi-representation over itself. That is, take the bi-module \mathcal{H} over an input algebra \mathcal{A} to be the vector space of the input algebra $\mathcal{H} \equiv \mathcal{A}$ itself, and take the action between algebra elements and vector space elements to be inherited from the product on the algebra itself. This natural representation motivates the following definition for a non-associative representation, which although we developed it separately, was first introduced by Samuel Eilenberg [Eil48], and which is nicely explained in Ch. II.4 of [Sch66]:

Definition: Let $ah \in \mathcal{H}$ denote the left-action of $a \in \mathcal{A}$ on $h \in \mathcal{H}$ (a bilinear map from $\mathcal{A} \times \mathcal{H} \rightarrow \mathcal{H}$); and similarly ha denotes the right-action of $a \in \mathcal{A}$ on $h \in \mathcal{H}$ (a bilinear map from $\mathcal{H} \times \mathcal{A} \rightarrow \mathcal{H}$). Given a class \mathcal{C} of (non-associative) algebras defined by a set of multi-linear identities $I_i(a_1, \dots, a_{n_i}) = 0$, and an algebra \mathcal{A} in \mathcal{C} , then \mathcal{A} is said to be bi-represented on \mathcal{H} in \mathcal{C} (or, equivalently, that \mathcal{H} is a bimodule over \mathcal{A} in \mathcal{C}) if all of the identities obtained by replacing any single $a_j \in \mathcal{A}$ by any $h \in \mathcal{H}$ are satisfied.

As an example, consider the class \mathcal{C}_{ass} of associative algebras; that is the class of algebras satisfying the multilinear identity:

$$[a_1, a_2, a_3] = 0 \quad (\forall a_i \in \mathcal{A}). \quad (3.2a)$$

If $\mathcal{A} \in \mathcal{C}_{ass}$ is bi-represented on a vector space \mathcal{H} , then following Eilenberg's definition, replacing any one algebra element in (3.2a) with a vector space element $h \in \mathcal{H}$, results in the following conditions

$$[a_i, a_j, h] = L_{a_i a_j} h - L_{a_i} L_{a_j} h = 0, \quad (3.2b)$$

$$[h, a_i, a_j] = R_{a_j} R_{a_i} h - R_{a_i a_j} h = 0, \quad (3.2c)$$

$$[a_i, h, a_j] = R_j L_i h - L_i R_j h = 0, \quad (3.2d)$$

for all $a_k \in A$, and $h \in H$. But Eq. (3.2b) is just an unfamiliar way of phrasing the familiar fact that \mathcal{A} has an associative left representation on \mathcal{H} : $\pi(ab) = \pi(a) \circ \pi(b)$, i.e. the condition given in Eq. (2.42). Similarly, Eq. (3.2c) says that \mathcal{A} has an associative right representation on \mathcal{H} , i.e. the condition given in Eq. (2.45). Finally, Eq. (3.2d) says that the left and right representations are compatible with one another in the sense that they commute, i.e the condition given in Eq. (2.46). In other words Eilenberg's definition reproduces the usual definition of an associative bi-representation for the class of associative algebras.

As shown in Eq. (3.2), the products between elements in an associative representation $\pi(a), \pi(b) \in \pi(\mathcal{A})$ are given by composition $\pi(a)\pi(b) = \pi(a) \circ \pi(b)$. Composition is associative, and so expressions like $\pi(a)\pi(b)\pi(c)$ and $\pi(a)\pi(b)h$ are unambiguous, and do not require any additional parentheses. By contrast, in the case where \mathcal{A} is non-associative, the operator $\pi(a)$ has two different roles that should be carefully distinguished: on the one hand it can operate on a vector $h \in \mathcal{H}$, mapping it to a new vector $\pi(a)h \in \mathcal{H}$; on the other hand, it can multiply another operator $\pi(b)$ to form a third operator $(\pi(a)\pi(b))$. It is important to note that, since the operators $\pi(a)$ and $\pi(b)$ represent elements in an underlying non-associative algebra \mathcal{A} , their product $(\pi(a)\pi(b))$ will *not* be given by the composition of the operators $\pi(a)$ and $\pi(b)$ on \mathcal{H} (which is associative); instead, it will be given by some other product that reflects the non-associativity of \mathcal{A} : $\pi(a)\pi(b) \equiv \pi(ab)$. In left-right notation, this is again the statement that $L_{\pi(a)}L_{\pi(b)} \neq L_{\pi(ab)}$, and $R_{\pi(a)}R_{\pi(b)} \neq R_{\pi(ba)}$ for non-associative input algebras \mathcal{A} .

Example: The ultimate goal of this Section is to construct a simple example non-associative NCG based on the octonion algebra. Let's therefore consider as an example the algebra of octonions bi-represented on itself $\mathcal{A} = \mathcal{H} = \mathbb{O}$, with the action of algebra elements on \mathcal{H} given by the octonionic product. The octonions do not belong to the class of associative algebras and as a result this bi-representation does not satisfy the conditions given in eq. (3.2). The octonions do however belong to the class of alternative algebras which satisfy the multilinear conditions given in eq. (2.7). This bi-representation will clearly satisfy the conditions of an alternative algebra.

Hilbert spaces

The input vector space \mathcal{H} of a NCG is always taken to be a Hilbert space. A Hilbert space \mathcal{H} is a real or complex vector space equipped with an inner product $\langle \cdot | \cdot \rangle : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{F}$, and where \mathcal{H} is complete with respect to the distance function induced by the inner product:

$$d(x, y) = \sqrt{\langle x - y | x - y \rangle} \quad (3.3)$$

for $x, y \in \mathcal{H}$. For a complex vector space the inner product is skew-linear in its first argument, linear in its second argument, skew-symmetric ($\langle a | c \rangle = \langle c | a \rangle^*$), and positive definite ($\langle a | a \rangle \geq 0$). For a real vector space the inner product is bi-linear, symmetric, and positive definite.

As a simple illustration, consider once again the case where the algebra of octonions is bi-represented on itself $\mathcal{A} = \mathcal{H} = \mathbb{O}$; the algebra homomorphism π is taken to be the identity map ($\tilde{a} = a$); and the product of two operators \tilde{a} and \tilde{b} , and the action of an operator \tilde{a} on a Hilbert space element h , is taken to be given by the underlying product in A : $\tilde{a}\tilde{b} = ab$, $\tilde{a}h = ah$. As a normed algebra the octonions are equipped with a natural inner product $\langle a | b \rangle = (1/2)(a^*b + b^*a) = \text{Re}(a^*b)$ where a^* is the octonionic conjugate of a .

Almost-associative representations

In practice, all of the non-associative geometries that I will describe in this work will be either finite non-associative, or almost-associative in the sense that they will be constructed as the product between a finite non-associative geometry and a canonical geometry¹. In this way the non-associativity is always relegated to the finite part of the geometry. Focusing on finite non-associative geometries has the benefit that all linear operators will be bounded, and so many of the difficulties which arise in the continuous case are avoided. What is not immediately clear however is if the product is taken between a finite dimensional non-associative algebra (represented on a finite Hilbert space) and an associative and commutative algebra of functions over a manifold (which is represented as bounded operators on an infinite dimensional Hilbert space), then will the resulting algebra be represented on the tensor product Hilbert space by bounded operators? One would hope that since the non-associativity is confined to a finite dimensional representation, even if the

¹Almost-associative geometries are the non-associative NCGs which are of most immediate physical interest, as they correspond to Yang-Mills theories non-minimally coupled to Einstein-Hilbert gravity on a normal associative space-time.

total resulting Hilbert space is infinite dimensional the total algebra would still have a representation as bounded operators. Let's check if this is indeed true.

Suppose that there are two Hilbert spaces H_ϕ and H_ψ , then their tensor product is defined such that the inner product on the tensor space is given by:

$$\langle \phi_1 \otimes \psi_1, \phi_2 \otimes \psi_2 \rangle = \langle \phi_1, \phi_2 \rangle \langle \psi_1, \psi_2 \rangle, \quad (3.4)$$

for $\phi_1, \phi_2 \in H_\phi$, and $\psi_1, \psi_2 \in H_\psi$.

Next, let H_ϕ be a finite dimensional Hilbert space on which a finite, non-associative, unital, involutive algebra \mathcal{A}_ϕ is represented, and let $H_\psi = L^2(M, S)$ be the canonical Hilbert space on which $A_\psi = C^*(M, \mathbb{C})$ is represented as bounded operators.

Consider an arbitrary element $v = \sum_i \phi_i \otimes \psi_i \in H_\phi \otimes H_\psi$. One can always choose a basis on H_ϕ and H_ψ which is orthonormal with respect to their inner products. By bilinearity, the ψ_i 's can be replaced by an orthonormal basis of their span, and so it is safe to assume that the ψ_i 's are orthonormal, and also therefore given the definition of the inner product that the $\sum_i \phi_i \otimes \psi_i$'s are. Next consider the norm squared of the algebra element $a_\phi \otimes \mathbb{I}_\psi$ acting on v . Once again the $\sum_i a_\phi \phi_i \otimes \psi_i$'s are orthonormal by the same argument, and so we can write:

$$\begin{aligned} \|(a_\phi \otimes \mathbb{I}_\psi)v\|^2 &= \left\langle \sum_i a_\phi \phi_i \otimes \psi_i, \sum_i a_\phi \phi_i \otimes \psi_i \right\rangle \\ &= \sum_i \langle a_\phi \phi_i, a_\phi \phi_i \rangle \\ &= \sum_i \|a_\phi \phi_i\|^2 \leq \sum_i \|a_\phi\|^2 \|\phi_i\|^2 = \|a_\phi\|^2 \|v\|^2, \end{aligned} \quad (3.5)$$

where the inequality derives from the definition of the operator norm. We therefore have:

$$\|(a_\phi \otimes \mathbb{I}_\psi)\| \leq \|a_\phi\| \quad (3.6a)$$

and by symmetry:

$$\|(\mathbb{I}_\phi \otimes a_\psi)\| \leq \|a_\psi\| \quad (3.6b)$$

By the triangle identity (see Eq. (2.9e)) we therefore have:

$$\|(a_\phi \otimes a_\psi)\| \leq \|a_\phi \otimes \mathbb{I}_\psi\| \cdot \|\mathbb{I}_\phi \otimes a_\psi\| \leq \|a_\phi\| \cdot \|a_\psi\| \quad (3.7)$$

It then follows that every operator in the algebraic tensor product space can be represented by bounded operators on $\mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B}$. Actually, this point should be intuitive from the form of the inner product given in Eq. (3.4) alone, and we really need not have gone through the above proof at all. Notice also that non-associativity never entered into the above proof, as we were never dealing with the product of more than two elements.

3.1.2 The real structure J , and the \mathbb{Z}_2 grading γ

So much for the input algebra \mathcal{A} , and its representation on the Hilbert space \mathcal{H} . Let us next have a look at the operators J and γ . A spectral triple is said to be “real” if it is equipped with a real structure operator J and “even” if it is equipped with a \mathbb{Z}_2 grading operator γ . One uses the real structure operator in associative NCG to construct the right action of associative algebra elements on the input Hilbert space. However the real structure operator has its origins as a modular conjugation operator in the Tomita-Takesaki theory of associative Von-Neumann algebras [Con95a]. Meanwhile the grading operator arises in associative NCG as the representation of a Hochschild cycle $\gamma = \pi(c)$, defined for associative algebras [Con96]. In this section I will briefly discuss the generalization of both operators to the non-associative case. For a complete exposition in the associative case see references [Con95a, Con96, vdDvS12].

The real structure J

First consider the real structure J . The basic observation, which remains perfectly valid when \mathcal{A} is non-associative, is that J can be thought of as extending the $*$ operation from the $*$ -algebra \mathcal{A} to the bimodule \mathcal{H} over \mathcal{A} : The real structure $J : \mathcal{H} \rightarrow \mathcal{H}$ is introduced as a unitary anti-linear operator, which parallels the anti-linear operation $(a \in \mathcal{A}) \rightarrow (a^* \in \mathcal{A})$, satisfying $(ab)^* = b^*a^*$ (see Eq. (2.8)). Viewed in this way, it must therefore have a compatible action on any product of algebra elements $a \in \mathcal{A}$ and Hilbert space elements $h \in \mathcal{H}$: in particular following Eq. (2.47), $J(ah) = (Jh)a^*$ and $J(ha) = a^*(Jh)$. In other words, Connes’ familiar relation between the left and right acting algebra elements on \mathcal{H} as given in Eq. (2.67) is recovered:

$$R_a = JL_{a^*}J^*, \quad (3.8a)$$

$$L_a = JR_{a^*}J^*. \quad (3.8b)$$

The real structure J plays an important role in Connes’ so-called order-zero and order-one conditions (2.68) and (2.69). Let us see what the meaning is behind the order zero condition (I will discuss the order one condition once we have the appropriate tools in Section 3.2):

$$\begin{aligned} [JL_{a_2}^*J^*, L_{a_1}]h &= [R_{a_2}, L_{a_1}]h \\ &= (a_1h)a_2 - a_1(ha_2) \\ &= [a_1, h, a_2] = 0 \end{aligned} \quad (3.9)$$

From the perspective presented here, the order zero condition is really just an assumption about the associativity of the bi-representation of \mathcal{A} on \mathcal{H} (i.e it tells us that the left and right actions are compatible in the sense of Eq. (2.46)). This assumption must be appropriately modified in the case where \mathcal{A} is non-associative. To clarify this point, lets consider two simple examples:

1. Consider first the simple finite $SU(N)$ construction outlined in Section 1.1, which has as input the algebra of complex $n \times n$ matrices $\mathcal{A}_F = M_n(\mathbb{C})$, bi-represented on itself, i.e. $\mathcal{H}_F = M_n(\mathbb{C})$. The real structure operator is given by the adjoint on matrices, $J_F h = (h)^\dagger$ for $h \in \mathcal{H}_F$, while the grading on \mathcal{H}_F is given by the identity operator $\gamma_F = \mathbb{I}_N$. Compatibility with the chirality (i.e. $\{\gamma, D\} = 0$ (see Eq. (2.65))) then forces $D_F = 0$. The order zero condition in this example is given by:

$$\begin{aligned} [Jb^*J^{-1}, a]h &= (b^\dagger(ah)^\dagger)^\dagger - a(b^\dagger h^\dagger)^\dagger \\ &= (ah)b - a(hb) = 0, \quad a, b \in \mathcal{A}_F; h \in \mathcal{H}_F, \end{aligned} \quad (3.10)$$

where the final line is equal to zero because in this example \mathcal{A}_F is an associative algebra with an associative bi-representation on \mathcal{H}_F .²

2. Next, consider a mild variation on the first example: consider the prototype example which is of most interest to us, $\mathcal{A}_F = \mathcal{H}_F = \mathbb{O}$, where the input algebra is taken to be the octonion algebra, bi-represented on itself in the natural way. With the octonions acting on themselves, it is natural to take J to be given by octonionic conjugation. With this input data, the order zero condition yields:

$$\begin{aligned} [JL_b^*J^*, L_a]h &= (b^*(ah)^*)^* - a(b^*h^*)^* \\ &= (ah)b - a(hb) \neq 0, \quad a, b \in \mathcal{A}_F; h \in \mathcal{H}_F. \end{aligned} \quad (3.11)$$

As the octonions are non-associative, the associator is typically non-zero, so we see that the traditional order-zero condition (2.68) is incompatible with the representation of the octonions on themselves, which is the most natural representation.

As discussed in Subsection 3.1.1, rather than satisfying associative order conditions (ie. associative multi-linear identities), a non-associative algebra represented on a Hilbert space should instead satisfy a set of conditions appropriate to the associativity class to which it

²Notice that the associative order 1 condition will be trivially satisfied in this finite geometry because $D = 0$. In addition, on forming the corresponding almost-commutative geometry one would find that the order one condition would follow automatically due to the order zero condition on the finite space.

belongs. The non-associative bimodule given above, $A = H = \mathbb{O}$, will for example satisfy alternative order zero conditions, because the octonions are an alternative algebra³:

$$[R_{\bar{b}}, L_{\bar{a}}] = [L_{\bar{b}}, R_{\bar{a}}], \quad (3.12a)$$

$$[R_{\bar{b}}, L_{\bar{a}}] = L_{\bar{b}\bar{a}} - L_{\bar{b}}L_{\bar{a}} = R_{\bar{a}}R_{\bar{b}} - R_{\bar{a}\bar{b}}. \quad (3.12b)$$

So what is the purpose behind the order zero condition, and does it matter that the associative order zero condition isn't satisfied in general when the input algebra is non-associative? The main purpose of the associative order zero condition is to ensure that the bi-representation is associative when the input algebra is associative. This ensures automorphism covariance, which I will describe shortly in Subsection 3.1.3. In brief, the associative order zero condition ensures that the inner-symmetries of a NCG are generated by inner derivations which take the associative form given in Eq. (2.28a). More generally, the order zero condition, together with the operator J defines the bi-module structure of the Hilbert space \mathcal{H} , and ensures covariance under the automorphisms of the input algebra regardless of its associativity properties. For example, if we wanted to represent a Jordan algebra, then 'Jordan' order zero conditions should be imposed such that the inner symmetries are generated by derivation elements of Jordan form given in (2.28c). All of these statements will be made precise below in Subsection 3.1.3, and in Section 3.3. In this section I will only consider finite examples with zero Dirac operators so as to avoid the need to discuss the order one condition given in Eq. (2.69). I discuss the higher order conditions in Section 3.2.

The grading γ

Now let's consider the \mathbb{Z}_2 grading γ . It is a linear operator on \mathcal{H} that commutes with the action of \mathcal{A} on \mathcal{H} . It is both hermitian ($\gamma^* = \gamma$) and unitary ($\gamma^* = \gamma^{-1}$): hence it satisfies $\gamma^2 = 1$, so its eigenvalues are ± 1 , and it correspondingly decomposes \mathcal{H} into two subspaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Note that all of these defining properties continue to make perfect sense when \mathcal{A} is non-associative, and require no modification.

For physicists, the familiar example is Dirac's helicity operator γ_5 which has the above properties and decomposes the space of Dirac spinors into positive and negative (helicity) subspaces: $L^2(\mathcal{M}, S) = L^2_+(\mathcal{M}, S) \oplus L^2_-(\mathcal{M}, S)$. Another nice way to think of γ_5 is as a volume form. Recall that on a spin manifold the Dirac operator is given by $\not{D} = -i\gamma^\mu \nabla_\mu^S$, where the γ^μ are the Dirac Gamma matrices, and ∇_μ^S is the Levi-Civita connection on

³These order zero conditions simply restate the alternative conditions given in Eq. (2.7) as commutator expressions. For example (3.12a) can be seen as follows: $[R_{\bar{b}}, L_{\bar{a}}]h = [a, h, b] = -[b, h, a] = [L_{\bar{b}}, R_{\bar{a}}]h$.

the spinor bundle. Although this Dirac operator may be unbounded, its commutator with elements of the algebra of functions over the manifold $df = [\not{D}, f] = -i\gamma^\mu(\partial_\mu f)$ is bounded. In fact this bounded operator gives the Clifford representation of the 1-form $df = dx^\mu(\partial_\mu f)$ [vdDvS12]. Similarly, we see that the γ_5 grading operator in the canonical case can be considered as the Clifford representation of a volume form.

$$\frac{1}{4!}\epsilon_{\mu\nu\tau\rho}\gamma^\mu\gamma^\nu\gamma^\tau\gamma^\rho = \gamma^1\gamma^2\gamma^3\gamma^4 := \gamma_5. \quad (3.13)$$

As a volume form, the grading operator may be viewed on an n -dimensional even space as the representation of a Hochschild n -cycle: $\gamma = \pi(c)$ for $c \in C_n(\mathcal{A}, \mathcal{A})$ [Con96]. Unfortunately Hochschild homology is only well defined for associative algebras. To see what goes wrong consider the action of the homology operator b described in Eq. (2.61), on elements of $C_2(\mathcal{A}, \mathcal{A})$:

$$\begin{aligned} b^2(a_0 \otimes a_1 \otimes a_2) &= b(a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 + a_2 a_0 \otimes a_1) \\ &= [a_0, a_1, a_2] + [a_2, a_0, a_1] + [a_1, a_2, a_0]. \end{aligned} \quad (3.14)$$

As can be seen in Eq. (3.14), when acting with b^2 on elements of $C_2(\mathcal{A}, \mathcal{A})$ one obtains a sum of associators. Likewise when acting with b^2 on elements of $C_n(\mathcal{A}, \mathcal{A})$ for $n \geq 3$ one also obtains expressions with associators in them. The condition $b^2 = 0$ is therefore clearly only satisfied for associative algebras. Hochschild (co)homology was generalized for the non-commutative setting to cyclic (co)homology, with major contributions coming from Tsygan, Connes, Loday, Kassel and Quillen [Con94]. When extending to the non-associative case further generalization is necessary. Fortunately, much work has already been done in this direction. The authors Kustermans, Murphy, and Tuset generalize to ‘twisted cyclic cohomology’ in order to describe differential calculi over quantum groups [KMT03], while the authors Akrami, Majid, and Beggs generalize to twisted braided cyclic cohomology to describe the cohomology associated to non-associative algebras obtained by Drinfeld-type cochain twists [AM04, BM10]. More recently Hassanzadeh, Shapiro, and Sütlü [HSS15] generalized to describe the cyclic homology associated to Hom-associative algebras⁴.

I will not describe the non-associative generalization of Hochschild (co)homology any further in this work (it belongs in a separate thesis). Instead I simply take for now the defining properties of the grading operator which continue to make sense in the non-associative setting: $\gamma^2 = \mathbb{I}$, $\gamma^* = \gamma^{-1}$, $\{\gamma, D\} = 0$, $\gamma J = \epsilon'' J \gamma$, and $[\gamma, a] = 0$, for $a \in \mathcal{A}$.

⁴M. Hassanzadeh has told me he is currently also working on generalizations to describe the (co)homology associated to general Jordan and alternative algebras, with some progress already made for Jordan algebras.

3.1.3 The principle of automorphism covariance

I have so far discussed the elements $\{\mathcal{A}, \mathcal{H}, J, \gamma\}$ of a spectral triple. Let us next discuss the symmetries of a non-associative NCG. Consider an automorphism α of the input $*$ -algebra \mathcal{A} , which maps each element $a \in \mathcal{A}$ to a new element $a' \in \mathcal{A}$. Corresponding to each automorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$, one can find corresponding transformations $\tilde{\alpha}$ which map each operator \tilde{a} to a new operator \tilde{a}' , as well as transformations $\hat{\alpha} : \mathcal{H} \rightarrow \mathcal{H}$:

$$a \rightarrow a' = \alpha(a), \quad (3.15a)$$

$$\tilde{a} \rightarrow \tilde{a}' = \tilde{\alpha}(\tilde{a}), \quad (3.15b)$$

$$h \rightarrow h' = \hat{\alpha}(h). \quad (3.15c)$$

To tie the transformations α , $\tilde{\alpha}$ and $\hat{\alpha}$ together, one demands that they satisfy the *principle of automorphism covariance*, which states that the whole formalism should “commute” with automorphisms of the underlying $*$ -algebra. In other words, any sensible expression should have the property that if one first transforms its components and then evaluates the expression, this should be the same as first evaluating the expression and then transforming the result. Note that, given the automorphism $\alpha : \mathcal{A} \rightarrow \mathcal{A}$, the compatible maps $\tilde{\alpha}$ and $\hat{\alpha}$ may be non-unique. Indeed, there may be whole families of maps $\tilde{\alpha}$ and $\hat{\alpha}$ that are compatible with a given α . For example, one may often ‘centrally extend’ the operators $\hat{\alpha}$ by unitary operators on \mathcal{H} which commute with the algebra representation. This point will turn out to be phenomenologically important in our analysis of the standard model later in this thesis in section 3.3.

Let’s see how the principle of automorphism covariance works in practice. For starters, let’s apply the principle to the expression $\tilde{a} = \pi(a)$: it requires that $\pi(\alpha(a)) = \tilde{\alpha}(\pi(a))$, $\forall a \in \mathcal{A}$; or, in other words:

$$\pi \circ \alpha = \tilde{\alpha} \circ \pi \quad (3.16)$$

where \circ denotes composition of functions. Next, apply the principle to the expression $\tilde{a}h$: it requires that $\hat{\alpha}(\tilde{a}h) = \tilde{\alpha}(\tilde{a})\hat{\alpha}(h)$; or, in other words:

$$\tilde{a}' = \tilde{\alpha}(\tilde{a}) = \hat{\alpha} \circ \tilde{a} \circ \hat{\alpha}^{-1} \quad \forall a \in \mathcal{A}. \quad (3.17)$$

Example: For illustration, consider the simple example of an algebra represented on itself: $\mathcal{H} = \mathcal{A}$. In this case we can always find maps $\tilde{\alpha}$ and $\hat{\alpha}$ that are compatible with α in the sense of Eq. (3.16) and Eq. (3.17) by simply choosing $\hat{\alpha} = \alpha$, and $\tilde{\alpha}\tilde{a} = \hat{\alpha}\tilde{a}\hat{\alpha}^*$. This example shows as proof of principle that there exists choices for $\tilde{\alpha}$ and $\hat{\alpha}$ that are compatible with α even in the case where \mathcal{A} is a non-associative algebra.

When constructing a NCG, there are three key linear operators which act on H : namely, D , γ , and J . Applying the principle to the expressions Dh , γh and Jh we see that, under an automorphism α , these operators must transform as

$$D \rightarrow D' = \hat{\alpha} \circ D \circ \hat{\alpha}^{-1}, \quad (3.18a)$$

$$\gamma \rightarrow \gamma' = \hat{\alpha} \circ \gamma \circ \hat{\alpha}^{-1}, \quad (3.18b)$$

$$J \rightarrow J' = \hat{\alpha} \circ J \circ \hat{\alpha}^{-1}. \quad (3.18c)$$

Both the real structure J and the \mathbb{Z}_2 grading γ should be compatible with the automorphisms of the underlying $*$ -algebra, in the sense that automorphisms should not affect the split between positive and negative helicity states, or between particles and anti-particles (the Hilbert space will eventually describe the fermionic degrees of freedom in physical models). We can express this requirement in terms of automorphisms:

$$[\hat{\alpha}, \gamma] = 0, \quad (3.19a)$$

$$[\hat{\alpha}, J] = 0, \quad (3.19b)$$

or in terms of the derivations that generate them:

$$[\tilde{\delta}, \gamma] = 0, \quad (3.20a)$$

$$[\tilde{\delta}, J] = 0, \quad (3.20b)$$

Notice that the condition given in Eq. (3.19b) is really just a rephrasing of the defining property of $*$ -automorphisms given in Eq. (2.23b). The condition given in (3.19a) is really just the statement that the automorphisms on \mathcal{H} should respect the grading on \mathcal{H} .

Example: As described in Eq. (2.77), in the associative setting inner derivations on \mathcal{H} are of the form $\tilde{\delta}_a = a + JaJ^*$, for anti-hermitian $a \in \mathcal{A}$ and so readily satisfy the conditions given in Eq. (3.20):

$$[\tilde{\delta}_a, \gamma] = (a + JaJ^*)\gamma - \gamma(a + \epsilon''\epsilon''JaJ^*) = 0, \quad (3.21a)$$

$$[\tilde{\delta}_a, J] = aJ + Ja - Ja - J^2aJ^* = 0, \quad (3.21b)$$

where in the last line I have used the real structure condition $J^2 = \epsilon\mathbb{I}$ given in Eq. (2.66a), and the fact that $JJ^* = \mathbb{I}$, which together imply $J = \epsilon J^*$.

Example: Consider once again the prototype non-associative geometry where $\mathcal{A} = \mathcal{H} = \mathbb{O}$. The inner derivations on \mathcal{H} are of the form $\tilde{\delta}_{x,y} = [L_x, L_y] + [L_x, JL_y^*J^{-1}] +$

$J[L_x, L_y]J^{-1}$ (see Eq. (2.28d)), and it can be shown that they satisfy the conditions given in Eq. (3.20):

$$\begin{aligned} [\tilde{\delta}_{x,y}, \gamma] &= ([L_x, L_y] + [L_x, JL_y^*J^{-1}] + J[L_x, L_y]J^{-1})\gamma \\ &\quad - ([L_x, L_y] + \epsilon''\epsilon''[L_x, JL_y^*J^{-1}] + \epsilon''\epsilon''J[L_x, L_y]J^{-1})\gamma = 0, \end{aligned} \quad (3.22a)$$

$$\begin{aligned} [\tilde{\delta}_{x,y}, J] &= [L_x, L_y]J + [L_x, JL_y^*J^{-1}]J + J[L_x, L_y] \\ &\quad - J[L_x, L_y] - J[L_x, JL_y^*J^{-1}] - JJ[L_x, L_y]J^{-1} \\ &= J([JL_xJ^{-1}, L_y^*] - [L_x, JL_y^*J^{-1}]) = 0, \end{aligned} \quad (3.22b)$$

where in the last equality of Eq. (3.22b) I have used the alternating condition given in Eq. (2.7), which is satisfied because the octonions are an alternative algebra.

We propose that it is natural to take the compatibility conditions given in Eq. (3.20) to be true more generally; *i.e.* to take them as axiomatic in non-associative geometry. We do not impose the same requirement on D : instead, the automorphisms of the underlying $*$ -algebra \mathcal{A} induce a transformation or “fluctuation” of D , from which the bosonic fields arise.

The principle of automorphism covariance is a fundamental principle lying at the base of the spectral reformulation of physics: it replaces (or subsumes or implies) the more familiar principles of covariance under coordinate transformations and gauge transformations, which are usually taken as the starting points for Einstein gravity and gauge theory. This principle will give us all the guidance necessary in this section, for formulating the spectral action principle unambiguously, even when \mathcal{A} is non-associative. The reader should be aware however that in section 3.2, I will be reformulating spectral triples in terms of $*$ -DGAs denoted ΩB . When I do so, the notion of ‘automorphism covariance’ will itself be subsumed by the compact statement that the symmetries of a NCG are simply the $*$ -automorphisms of ΩB .

3.1.4 Inner fluctuations and covariance

Almost-associative geometries

In Subsection 2.2.3 I reviewed the construction of so called ‘fluctuated’ Dirac operators, which transform covariantly with respect to the automorphisms of the input algebra in a NCG. The usual fluctuation terms given in eq. (2.85) are derived using Morita equivalence [CM] or semi-group methods [CCvS13, CCv13]. Both of these methods rely on the associativity of the input algebra, and so before continuing it is necessary to first develop

a fluctuation procedure which also works in the more general case. Fortunately, following Eq. (3.18a) we already know how covariant Dirac operators must transform, and this tells us what form their fluctuations must take. Following eq. (2.81) a *-automorphism covariant Dirac operator must transform as:

$$D \rightarrow D' = \hat{\alpha}D\hat{\alpha}^{-1} = D - \underbrace{\hat{\alpha}[D, \hat{\alpha}^{-1}]}_{\text{fluctuation}} \quad (3.23)$$

As in regular gauge theory, the form of the covariant Dirac operator must remain stable under continued fluctuation. By inspecting the fluctuation terms in Eq. (3.23) we are therefore able to determine the more general form of the fluctuated Dirac operator. Let's consider the associative case first:

Following eq. (2.28a), the input algebra of an associative NCG geometry has inner derivations which take the form $\delta = L_x - R_x$, for $x = -x^* \in \mathcal{A}$. For an associative NCG eq. (3.23) may therefore be written to first order in the form:

$$\begin{aligned} D' &= e^{\tilde{\delta}_{bc}} D e^{-\tilde{\delta}_{bc}} \simeq D - [D, \hat{\delta}] \\ &= D - [D, x + JxJ^{-1}] = D - \underbrace{[D, x] - \epsilon' J[D, x]J^{-1}}_{\text{fluctuation term}}, \end{aligned} \quad (3.24)$$

The fluctuation term in eq. (3.24) is constructed from hermitian exact one forms. By inspection, general fluctuation terms are therefore constructed from general hermitian one forms as:

$$D_A = D + A_{(1)} + \epsilon' JA_{(1)}J^{-1}, \quad (3.25)$$

where the generalized hermitian forms are given by $A_{(1)} = \sum \tilde{a}[D, \tilde{b}]$, with the sum taken over elements $a, b \in A$. But this approach recovers exactly the traditional fluctuation formula described for the associative case in Eq. (2.85). From this perspective, the structure of the inner fluctuations ultimately comes from the underlying structure of the inner derivations, and is analogous to the connection term that appears in Eq. (2.73), for regular gauge theory.

Before proceeding with the non-associative case let's first consider again the special case of an associative almost-commutative geometry, the input algebra $C^\infty(\mathcal{M}, \mathcal{A}_F)$ is the algebra of \mathcal{A}_F valued functions over a manifold \mathcal{M} . The inner derivations $\tilde{\delta}$ acting on \mathcal{H} , may be written as $\tilde{\delta} = c^i(x) \otimes \tilde{\delta}_i$, where the set $\{\tilde{\delta}_i = L_{e_i} - R_{e_i} | e_i = -e_i^* \in \mathcal{A}\}$ form a basis of anti-hermitian derivations acting on \mathcal{H}_F , while $c^i(x)$ are spatially-varying real coefficient functions (*i.e.* functions from \mathcal{M} to \mathbb{R}). The Dirac operator of an associative

almost-commutative geometry is given by $D = \not{D} \otimes \mathbb{I}_F + \gamma_c \otimes D_F$ (see subsection 2.2.2). Following eq. (3.25) the fluctuated Dirac operator of an almost-commutative geometry is therefore given by:

$$\begin{aligned}
D_A &= -i\gamma^\mu \nabla_\mu^S \otimes \mathbb{I}_F + \sum (i\gamma^\mu a_c [\partial_\mu, b_c] \otimes a_F b_F) + \epsilon' J (i\gamma^\mu a_c [\partial_\mu, b_c] \otimes a_F b_F) J^{-1} \\
&\quad + \gamma_c \otimes D_F + \sum a_c b_c \gamma_c \otimes a_F [D_F, b_F] + \epsilon' J a_c b_c \gamma_c \otimes a_F [D_F, b_F] J^{-1} \\
&= -i\gamma^\mu \nabla_\mu^S \otimes \mathbb{I}_F + \sum i\gamma^\mu a_c (\partial_\mu b_c) \otimes (a_F b_F) + i\gamma^\mu \overline{a_c (\partial_\mu b_c)} \otimes J_F (a_F b_F) J_F^{-1} \\
&\quad + \gamma_c \otimes D_F + \sum a_c b_c \gamma_c \otimes a_F [D_F, b_F] + \epsilon'_F \overline{a_c b_c} \gamma_c \otimes J_F a_F [D_F, b_F] J_F^{-1} \\
&= -i\gamma^\mu \nabla_\mu^S \otimes \mathbb{I}_F - \underbrace{i\gamma^\mu A_\mu^i(x) \otimes \hat{\delta}_i}_{\text{Gauge}} + \underbrace{\gamma_c \phi^{ij} \otimes (e_i [D_F, e_j] + J_F \epsilon'_F e_i [D_F, e_j] J_F^{-1})}_{\text{Higgs}}. \quad (3.26)
\end{aligned}$$

where $a_c \otimes a_F, b_c \otimes b_F \in C^\infty(M) \otimes A_F$. The gauge fields appear in the adjoint representation of the automorphism group of the input algebra, while the representation of the scalar fields depends on the form of the finite Dirac operator.

So much for the associative case, lets consider a non-associative geometry. For a non-associative geometry the fluctuated Dirac operator should transform under inner $*$ -automorphisms of the input algebra as shown in equation (3.23). The only difference is that now the automorphisms will be generated by elements of the algebra of derivations $D(A)$ for the non-associative algebra in question, rather than by associative derivations of the form $\delta_c = L_c - R_c$. Lets consider an input algebra satisfying the Jordan conditions given in eq. (2.5)⁵. Following eq. (2.28c), the inner derivations of a Jordan algebra are given by $\delta_{xy} = [L_x, L_y]$. To first order the Dirac operator must transform as

$$\begin{aligned}
D' &= e^{\tilde{\delta}_{bc}} D e^{-\tilde{\delta}_{bc}} \simeq D - [D, \tilde{\delta}_{bc}] \\
&= D - \underbrace{[[D, L_{\tilde{b}}], L_{\tilde{c}}] + [[D, L_{\tilde{c}}], L_{\tilde{b}}]}_{\text{fluctuation terms}}, \quad (3.27)
\end{aligned}$$

where comparison between equations (3.24) and (3.27) should be stressed. Once again the form of the fluctuated Dirac operator is determined by inspection, and is given by:

$$D_A = D + \sum \delta_{A_{(1)}, A_{(0)}} := \sum [A_{(1)}, A_{(0)}], \quad (3.28)$$

where the sum is taken over generalized hermitian ‘one forms’ $A_{(1)}$, and generalized ‘zero forms’ $A_{(0)}$. The ‘zero forms’ $A_{(0)}$ will simply be given by left acting elements of the Jordan

⁵I choose a Jordan algebra here as an example rather than our prototype octonions because the inner derivations of a Jordan algebra take a simpler form than the derivations of an alternative algebra.

algebra. The generalized ‘one forms’ will depend on the representation of the algebra π and the form of the un-fluctuated Dirac operator D . I will give an explicit example below in Subsection 3.1.5.

Although the fluctuation of D_A involves algebra elements $a \in A$ drawn from the non-associative algebra \mathcal{A} , D_A is simply a linear operator on \mathcal{H} , and is not in any sense non-associative. In particular, note that the fluctuations of D are built not from the elements $a \in A$ themselves, but from L_a and R_a , *i.e.* the (associative) operators which represent the left-action and right-action of a on \mathcal{H} . Furthermore, these operators L_a and R_a are grouped together in a particular way, structured by the derivations of \mathcal{A} . Even when \mathcal{A} is non-associative, its automorphisms still form an ordinary (associative) group, and its derivations (from which the fluctuations of D_A are built) still form an ordinary Lie algebra. This means that, when we take an almost-associative geometry, and plug D_A into the spectral action, the spectral action will yield an ordinary Yang-Mills theory, just as it does in the almost-commutative case. Let us now look at a concrete example.

3.1.5 The octonion example

$SU(N)$ Gauge theory

As a warm up before considering an example non-associative geometry consider first the archetypal almost-commutative geometry $M \times F$ with finite space given by: $F = \{A_F, H_F, D_F, J_F, \gamma_F\}$, where $A_F = M_n(\mathbb{C})$ is the algebra of $n \times n$ complex matrices⁶ represented on itself: $H_F = A_F$. The real structure is given by hermitian conjugation $J_F = \dagger$, and the grading is given by the identity $\gamma_F = \mathbb{I}_F$. Compatibility with the chirality $\{D_F, \gamma_F\} = 0$ then sets $D_F = 0$. This geometry satisfies $J_F^2 = \mathbb{I}_F$, $[D_F, J_F] = 0$, $[\gamma_F, J_F] = 0$ and so is of KO-dimension 0 or 7 (see table 2.1). Only in KO-dimension 0 however does it make sense to impose the condition $\{D_F, \gamma_F\} = 0$, thereby setting $D_F = 0$.

Tensoring the finite space with a four dimensional canonical triple yields a KO-dimension 4 almost-commutative spectral triple given by:

$$M \times F = \{C^\infty(M, M_n(\mathbb{C})), L^2(M, S) \otimes M_n(\mathbb{C}), \not{D} \otimes \mathbb{I}_F, J_c \otimes \dagger, \gamma_c \otimes \mathbb{I}_F\}, \quad (3.29)$$

where all tensor products are taken over the complex numbers. The fluctuated Dirac

⁶Here $n = N$.

operator is given by:

$$\begin{aligned}
D_A &= \not{D} \otimes \mathbb{I}_F + A_{(1)} + \epsilon' J A_{(1)} J^{-1} \\
&= \not{D} \otimes \mathbb{I}_F + i\gamma^\mu A_\mu^i(x) \otimes e_i + J(i\gamma^\mu A_\mu^i(x) \otimes e_i) J^{-1} \\
&= \not{D} \otimes \mathbb{I}_F + i\gamma^\mu A_\mu^i(x) \otimes (e_i - J_F e_i^\dagger J_F^{-1}) \\
&= \not{D} \otimes \mathbb{I}_F + i\gamma^\mu A_\mu^i(x) \otimes \hat{\delta}_i
\end{aligned} \tag{3.30}$$

for anti-hermitian basis elements $e_i \in \mathcal{A}_F$ and with real valued coefficients $A_\mu^i(x) \in \mathcal{A}_c$.

Having constructed the fluctuated Dirac operator, the spectral action is then given by [Gil84, Vas03, vdDvS12, vS15]⁷:

$$\begin{aligned}
S_b &= Tr \left(f \left(\frac{D_A}{\Lambda} \right) \right) \\
&\simeq 2f_4 \Lambda^4 a_0(D_A^2) + 2f_2 \Lambda^2 a_2(D_A^2) + f(0) a_4(D_A^2) + O(\Lambda^{-2})
\end{aligned} \tag{3.31}$$

where the $f_n = \int_0^\infty f(x) x^{n-1} dx$ ($n > 0$) and $a_k(D_*^2)$ are the Seeley-deWitt coefficients, and the square of the fluctuated dirac operator is given by

$$\begin{aligned}
D_A^2 &= (-i\gamma^\mu \nabla_\mu^S \otimes \mathbb{I} - i\gamma^\mu A_\mu^i \otimes \tilde{\delta}_i)^2 \\
&= \Delta_A^E - \frac{1}{2} \gamma^\mu \gamma^\nu \otimes F_{\mu\nu} - \frac{1}{4} R \otimes \mathbb{I},
\end{aligned} \tag{3.32}$$

where R is the Ricci scalar, and

$$\Delta_A^E = -g^{\mu\nu} \nabla_\mu^E \nabla_\nu^E, \tag{3.33}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \tag{3.34}$$

where $\nabla_\mu^E = \nabla_\mu^S \otimes \mathbb{I} + B_\mu^i \otimes \tilde{\delta}_i$. For a compact Euclidean manifold without boundary eq. (3.42) can then be expanded:

$$a_0(D_A^2) = \int_M d^4x \sqrt{g} \frac{1}{4\pi^2} \tag{3.35}$$

$$a_2(D_A^2) = \int_M d^4x \sqrt{g} \frac{1}{48\pi^2} R \tag{3.36}$$

$$\begin{aligned}
a_4(D_A^2) &= \int_M d^4x \sqrt{g} \frac{1}{16\pi^2} \frac{1}{360} \text{Tr} \left[\left(\frac{5}{4} R^2 - 2R_{\mu\nu} R^{\mu\nu} + 2R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right. \right. \\
&\quad \left. \left. + 45\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma F_{\mu\nu} F_{\rho\sigma} + 30\Omega_{\mu\nu}^E (\Omega^E)^{\mu\nu} \right],
\end{aligned} \tag{3.37}$$

⁷See Subsection 2.2.4 for a review of the spectral action.

where $\Omega_{\mu\nu}^E = \Omega_{\mu\nu}^S \otimes \mathbb{I} + \mathbb{I}_4 \otimes F_{\mu\nu}$, and $\text{Tr}(\Omega_{\mu\nu}^S \Omega^{S\mu\nu}) = -\frac{1}{2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$, where $R_{\mu\nu}$, and $R_{\mu\nu\rho\sigma}$ are the Ricci tensor and Riemann tensor respectively and $\Omega_{\mu\nu}^E$ and $\Omega_{\mu\nu}^S$ are the curvatures of the connections ∇_{μ}^E and ∇_{μ}^S respectively. The full bosonic action is then

$$S_b \simeq \int_M d^4x \sqrt{g} \frac{N^2}{(4\pi)^2} \left[8f_4 \Lambda^4 + \frac{2}{3} R f_2 \Lambda^2 + \frac{f(0)}{360} (5R^2 - 8R_{\mu\nu} R^{\mu\nu} - 7R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{240}{N^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu})) \right] \quad (3.38)$$

where I have used the fact that the dimension of the finite Hilbert space is N^2 .

The action given in Eq. (3.38) describes an $SU(N)$ Yang-Mills theory minimally coupled to Einstein-Hilbert gravity (minimally coupled in the sense that there is no term like $\Phi^2 R$ in the action). For this example I have only constructed the bosonic part of the action given in Eq. (1.1). I will construct the fermionic action for more interesting models in Ch. 4.

G_2 Gauge theory

Next, consider our prototype almost-associative geometry $M \times F$ with finite space given by: $F = \{A_F, H_F, D_F, J_F, \gamma_F\}$, where instead of a matrix algebra we take $A_F = \mathbb{O}$, the algebra of octonions represented on itself: $H_F = A_F$. The real structure is given by octonionic conjugation $J_F = *$, and the grading is given by the identity $\gamma_F = \mathbb{I}_F$. Compatibility with the chirality $\{D, \gamma\} = 0$ then sets $D_F = 0$. This geometry satisfies $J_F^2 = \mathbb{I}_F$, $[D_F, J_F] = 0$, $[\gamma_F, J_F] = 0$ and so is of KO-dimension 0 (see table 2.1).

This finite non-associative geometry has a number of properties which make it interesting as a first example:

1. It takes as input a unital, finite, simple algebra, and so all of its automorphisms are well characterized, and inner [Jac49, Sch49].
2. Because the finite Dirac operator is taken to be zero, the first order condition (whatever form it happens to take in the non-associative case) is automatically satisfied, and so can be safely ignored (for now, but I will come back to it later in this thesis).
3. The octonions are normed and so have a canonical inner product, which means they are also naturally a Hilbert space.
4. As will be discussed shortly in Subsection 3.1.6 this non-associative geometry can be obtained by a ‘twisting’ procedure from an associative geometry.

Let us see what the corresponding almost-commutative geometry looks like. Tensoring the finite space F together with a four dimensional canonical triple M yields a KO-dimension 4 almost-commutative spectral triple given by:

$$M \times F = \{C^\infty(M, \mathbb{O}), L^2(M, S) \otimes \mathbb{O}, -i\gamma^\mu \nabla_\mu^S \otimes \mathbb{I}_F, J_c \otimes *, \gamma_c \otimes \mathbb{I}_F\}, \quad (3.39)$$

where each tensor product is taken over \mathbb{R} . Under inner automorphisms of the total input algebra the Dirac operator must transform to first order as (see Eq. (2.28d)):

$$\begin{aligned} D' &= \hat{\alpha} D \hat{\alpha}^{-1} \simeq D - [D, \tilde{\delta}_{b,c}] \\ &= D - [[D, L_{\tilde{b}}], L_{\tilde{c}}] + [[D, L_{\tilde{b}}], J L_{\tilde{c}} J^*] - \epsilon' J [[D, L_{\tilde{b}}], L_{\tilde{c}}] J^* \\ &\quad + [[D, L_{\tilde{c}}], L_{\tilde{b}}] - \epsilon' [J [D, L_{\tilde{c}}] J^*, L_{\tilde{b}}] + \epsilon' J [[D, L_{\tilde{c}}], L_{\tilde{b}}] J^*, \end{aligned} \quad (3.40)$$

By inspection the fluctuated Dirac operator is therefore given by:

$$\begin{aligned} D_A &= \not{D} \otimes \mathbb{I}_F + \sum [A_{(1)}, A_{(0)}] - [A_{(1)}, J A_{(0)} J^*] + \epsilon' J [A_{(1)}, A_{(0)}] J^*, \\ &= \not{D} \otimes \mathbb{I}_F + \sum i\gamma^\mu A_{(1)\mu}^i A_{(0)}^j ([L_{e_i}, L_{e_j}] + [L_{e_i}, J_F L_{e_j}^* J^*] + [J_F L_{e_i}^* J_F^{-1}, J_F L_{e_j}^* J_F^{-1}]), \\ &= \not{D} \otimes \mathbb{I}_F + i\gamma^\mu \mathbb{A}_\mu^i(x) \otimes \hat{\delta}_i \end{aligned} \quad (3.41)$$

where the $e_i \in \mathcal{A}_F$ are the anti-hermitian (imaginary) octonionic basis elements, the $\hat{\delta}_i$ are a basis of derivations of the octonions (these derivations form the g_2 Lie algebra) and the $\mathbb{A}^i(x)$ are real valued coefficient functions (the components of the G_2 gauge field). As expected we obtain a gauge potential in the adjoint representation of g_2 , while the Higgs sector is empty because the finite part of the Dirac operator was set to zero. As there is no scalar field to break the symmetries of the theory, the theory will remain massless. From this fluctuated Dirac operator we are now able to construct the spectral action, which is given by [Gil84, Vas03, vdDvS12, vS15]:

$$\begin{aligned} S_b &= Tr \left(f \left(\frac{D_A}{\Lambda} \right) \right) \\ &\simeq \int_M d^4x \sqrt{g} \frac{8}{(4\pi)^2} [8f_4 \Lambda^4 + \frac{2}{3} R f_2 \Lambda^2 \\ &\quad + \frac{f(0)}{360} (5R^2 - 8R_{\mu\nu} R^{\mu\nu} - 7R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - \frac{240}{8} Tr(F_{\mu\nu} F^{\mu\nu}))] \end{aligned} \quad (3.42)$$

where I have used the fact that the finite Hilbert space has 8 real dimensions. Notice, that once the fluctuated Dirac operator is constructed, the construction of the spectral action goes ahead as usual. This simple almost-associative geometry based on the finite algebra of octonions describes a G_2 Yang-Mills theory minimally coupled to Einstein-Hilbert gravity.

3.1.6 Cochain twisted Geometry

In the previous Subsection I showed how to construct what is in some sense the simplest finite non-associative geometry F and used it to form an almost-associative geometry corresponding to a G_2 gauge theory coupled to gravity. This was in fact the first non-associative geometry constructed [FB13], and which I used as a working example when developing the more general non-associative formalism based on the fused $*$ -DGAs ΩB . The simplicity of the octonion example was not the only reason that I chose to construct it however. It turns out that the finite example octonion geometry may also be obtained through a ‘cochain twist’ of an appropriate *associative* finite spectral triple. One can therefore arrive at our example nonassociative spectral triple F and check that it makes sense, in two different ways. On the one hand, F satisfies all of the required axioms for a spectral triple (including the appropriate nonassociative generalization of the order zero condition presented in Subsection 3.1.2), and is compatible with the principle of automorphism covariance, as explained in Subsection 3.1.3. On the other hand, one can start with an appropriate associative spectral triple that satisfies the standard axioms of associative NCG, and then perform a so called ‘cochain twist’ into the nonassociative triple F . In this subsection I explain this twisting procedure, which relies heavily on the quasi-Hopf algebra description of the octonions developed by Albuquerque and Majid [AM99]. I have tried to make this subsection as self-contained as possible, but for those readers needing more information I highly recommend the references [Maj95, Maj02]. In addition, while this Subsection does present an interesting approach to non-associative geometry (cochain twisting from an associative geometry) the content may be completely skipped over without influencing the reader’s understanding of the remainder of the text in any way.

I begin by introducing a few pieces of mathematical background. The octonions have a so called ‘quasialgebra’ structure. For our present purposes a quasialgebra can be thought of as an algebra that is, in some well defined way, related to certain other algebras. Specifically, starting with an associative algebra (\mathcal{A}, \cdot) , we can perform what is known as a ‘twist’ to obtain a new quasialgebra (\mathcal{A}_F, \times) . The new algebra \mathcal{A}_F shares the same underlying vector space as \mathcal{A} but has a new product (“ \times ” instead of “ \cdot ”). It is possible in this way to describe the non-associativity of a quasialgebra (\mathcal{A}_F, \times) as resulting from a ‘twist’ from an associative quasialgebra (\mathcal{A}, \cdot) .

The authors Albuquerque and Majid [AM99] have already described in full detail the octonions as a quasialgebra resulting from a ‘twist’ on a particular associative group algebra. A group algebra is defined by taking a group G and a field \mathbb{K} ⁸ together in a natural

⁸Here I use the notation \mathbb{K} to denote a field rather than \mathbb{F} , because it is the notation commonly seen

way: namely, arbitrary linear combinations of the form $\sum_i k_i g_i$, where $k_i \in \mathbb{K}$ and $g_i \in G$. These elements may be added and multiplied in the obvious way, and thus form an algebra over the field \mathbb{K} ; the dimension of the algebra $\mathbb{K}G$ is just the order of the group G . $\mathbb{K}G$ is naturally a $*$ -algebra, with the $*$ operation given by $(\sum_i k_i g_i)^* = \sum_i k_i^* g_i^{-1}$; and it is also naturally a Hilbert space, with the inner product of two vectors $v^{(1)} = \sum_i k_i^{(1)} g_i$ and $v^{(2)} = \sum_i k_i^{(2)} g_i$ given by $\langle v^{(1)} | v^{(2)} \rangle = \sum_i (k_i^{(1)*}) k_i^{(2)}$.

In the case of the octonions, the corresponding associative group algebra of interest is $\mathbb{K}G$, where $\mathbb{K} = \mathbb{R}$, and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, so that $\mathbb{K}G$ is an 8-dimensional algebra over the real numbers [AM99, AM04, Maj95, Maj02]. Each basis element of $\mathbb{K}G$ may be written in the form $g_i = (i_1, i_2, i_3)$, where $i_j \in \{0, 1\}$; and then $\mathbb{K}G$ simply inherits the group multiplication law: $j \cdot k$ means adding the two vectors (j and k), mod 2. From here, we can obtain the octonions by performing a ‘twist’ – i.e. by replacing the multiplication law $x \cdot y$ with the new multiplication law:

$$g_i \times g_j = g_i \cdot g_j F(g_i, g_j), \quad \forall g_i, g_j \in G, \quad (3.43)$$

where F is known as a ‘2-cochain twist’ taking values in the field \mathbb{K} over which the algebra \mathcal{A}_F is defined. The 2-cochain F is given in the octonion case as [AM99]:

$$F(g_i, g_j) = (-1)^f, \\ f = i_1(j_1 + j_2 + j_3) + i_2(j_2 + j_3) + i_3 j_3 + j_1 i_2 i_3 + i_1 j_2 i_3 + i_1 i_2 j_3. \quad (3.44)$$

In discussing the twist from $\mathcal{A} = \mathbb{K}G$ to $\mathcal{A}_F = \mathbb{O}$ the authors Albuquerque and Majid [AM99] give a ‘natural involution’ ($*$ operation) on the twisted algebra basis:

$$J e_i = F(e_i, e_i) e_i. \quad (3.45)$$

From equation (3.44) it can be seen that this involution is simply octonionic conjugation. Prior to twisting however the involution is simply given by $F(e_i, e_i) = 1, \forall e_i \in \mathbb{K}G$. Notice that in $\mathbb{K}G$ each basis element is its own inverse. For this reason the ‘natural’ $*$ operation coincides in the untwisted case with what is known as the ‘antipode’ operator S on $\mathbb{K}G$:

$$J e_i = S e_i = e_i^{-1}. \quad (3.46)$$

The triple $\{\mathcal{A}, \mathcal{H}, J\} = \{\mathbb{O}, \mathbb{O}, J_F\}$ may be considered as being ‘twisted’ from the data $\{\mathbb{K}G, \mathbb{K}G, S\}$. It is therefore natural to consider a spectral triple $\{\mathcal{A}, \mathcal{H}, D, \gamma, J\}$ where \mathcal{A} and \mathcal{H} are both given by $\mathbb{K}G$, and \mathcal{A} is represented in the obvious way: *i.e.* π is the

in the relevant literature.

identity map (so $\tilde{a} = a$), and the action of the operator \tilde{a} on an element of H is given by the ordinary product in $\mathbb{K}G$. Furthermore, we can take $\gamma = 1$; the condition $\{\gamma, D\} = 0$ then implies $D = 0$. Finally, the action of J on \mathcal{H} is naturally given by the $*$ -operation in $\mathbb{K}G$: $Jh = J(\sum_i k_i g_i) = (\sum_i k_i g_i)^* = \sum_i k_i^* g_i^{-1}$. The twist given in equation (3.43) then maps between the associative finite spectral triple corresponding to the group algebra $\mathbb{K}G$ and the non-associative finite spectral triple corresponding to the octonion algebra $A_F = \mathbb{O}$.

We are now in a position to analyze how the order zero condition behaves under a ‘twist’ from the associative $\mathcal{A} = \mathcal{H} = \mathbb{K}G$ to the non-associative $\mathcal{A}_F = \mathcal{H}_F = \mathbb{O}$. As \mathcal{A} is associative it will satisfy the order zero condition given in Eq.(2.68):

$$\begin{aligned}
[\pi_{g_j}^0, \pi_{g_i}] \tilde{g}_k &= (\tilde{g}_i \cdot \tilde{g}_k) \cdot \tilde{g}_j - \tilde{g}_i \cdot (\tilde{g}_k \cdot \tilde{g}_j) = 0, & g_i, g_j, g_k \in G, \\
\text{‘twist’} \rightarrow 0 &= F^{-1}(g_i, g_k) F^{-1}(g_i \cdot g_k, g_j) (\tilde{g}_i \times \tilde{g}_k) \times \tilde{g}_j \\
&\quad - F^{-1}(g_i, g_k \cdot g_j) F^{-1}(g_k, g_j) \tilde{g}_i \times (\tilde{g}_k \times \tilde{g}_j) \\
&= \frac{1}{F(g_i, g_k \cdot g_j) F(g_k, g_j)} (\Phi_{\tilde{g}_i, \tilde{g}_k, \tilde{g}_j}^{-1} (\tilde{g}_i \times \tilde{g}_k) \times \tilde{g}_j - \tilde{g}_i \times (\tilde{g}_k \times \tilde{g}_j)) \\
&\equiv \frac{1}{F(g_i, g_k \cdot g_j) F(g_k, g_j)} [R_{\tilde{g}_j}, L_{\tilde{g}_i}] \Phi \tilde{g}_k
\end{aligned} \tag{3.47}$$

where the ‘associator’ is defined as $\Phi_{\tilde{g}_i, \tilde{g}_k, \tilde{g}_j} := \frac{F(g_i, g_k) F(g_i \cdot g_k, g_j)}{F(g_i, g_k \cdot g_j) F(g_k, g_j)}$. In other words, an augmented order zero condition is obtained, and is given by:

$$[R_{\tilde{b}}, L_{\tilde{a}}]_{\Phi} = 0 \quad \forall a, b \in \mathcal{A}_F, \tag{3.48}$$

where the subscript Φ can be seen as telling us when to ‘flip’ the brackets on one side of the commutator when acting on a hilbert space element. Note that for an associative algebra, the ‘associator’ Φ will be trivial and our augmented order zero condition will collapse back to that given in the associative case (2.68). Note also, that in the octonion example, when $a = b$, the ‘associator’ Φ will be trivial, as would be expected following the alternative order conditions given in Eq. (3.12). Indeed the twisted order zero condition in this case contains the same information as the alternative conditions given in Eq. (3.12), it is simply hidden away in the ‘associator’ Φ .

It should be stressed that one can arrive at our prototype nonassociative spectral triple F , and check that it makes sense in two different ways. On the one hand, F satisfies all of the required axioms for a spectral triple (including the appropriate alternative generalization of the order zero condition given in Eq. (3.12)), and is compatible with the principle of automorphism covariance, as explained in Subsection 3.1.3. On the other hand, one can start with the associative spectral triple: $F_0 = \{\mathbb{K}G, \mathbb{K}G, 0, \mathbb{I}, J_{\mathbb{K}G}\}$, where $\mathbb{K}G$ is the group algebra based on $\mathbb{K} = \mathbb{R}$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and $J_{\mathbb{K}G}$ denotes the natural $*$ operation in

$\mathbb{K}G$. This spectral triple satisfies the standard axioms for an associative spectral triple of KO dimension zero. But then, when one twists $\mathbb{K}G$ into \mathbb{O} , the associative spectral triple F_0 is correspondingly twisted into the finite octonion nonassociative triple F .

3.2 The fused algebra approach and non-associative geometry.

In Section 3.1 I discussed the first steps towards a reformulation of the NCG input data, which would naturally describe non-associative geometries. There were four key ideas explored in the approach:

1. All algebras, including non-associative algebras have a natural representation on themselves. This ‘natural’ representation acts as a prototype for what is meant by a non-associative representation.
2. The idea of automorphism covariance: all elements of a spectral triple should transform covariantly with respect to the automorphisms of the input algebra.
3. The fluctuated Dirac operator is nothing but a ‘covariant’ Dirac operator, which acts ‘covariantly’ with respect to the automorphisms of the input algebra.
4. The traditional order zero condition is an ‘associativity’ condition, which ensures that the symmetries of a NCG are generated by the derivations of the associative form given in eq. (2.28a). More generally however, the order zero condition should be seen as an associator, which describes the associativity properties of the bi-representation of \mathcal{A} on \mathcal{H} .

These four ideas ultimately allow one under very minimal assumptions to construct the simplest almost-associative models as in Subsection 3.1.5. Unfortunately, while the elements of a spectral triple do seem at least naively to generalize quite naturally, up to this point their generalization has been made in a piecewise fashion. The ultimate goal of this thesis is to produce a reformulation of the NCG input data which not only extends to describe the most general non-associative geometries, but also gives a more unified description of the five elements of a spectral triple, along with their axioms. In this section I introduce just such an approach: I will show how to fuse the five elements of a spectral triple together into a so called ‘fused’ *-DGA denoted ΩB . The organisation is as follows: In Subsection 3.2.1 I will introduce the algebra B_0 which unifies together the

elements $\{\mathcal{A}, \mathcal{H}, J\}$ of a spectral triple. In Subsection 3.2.2 I will introduce the algebra B , which provides an incomplete unification of the elements $\{\mathcal{A}, \mathcal{H}, D, J\}$. Finally, in Subsections 3.2.3 and 3.2.3 I will introduce the full algebra ΩB , which unifies all elements of the spectral triple $\{\mathcal{A}, \mathcal{H}, D, J, \gamma\}$.

3.2.1 The algebra $B_0 = \mathcal{A} \oplus \mathcal{H}$

As outlined in Section 3.1, developing a clean notion of non-associative algebra representations is essential if one wishes to generalize the NCG formalism to accept non-associative input data. In Subsection 3.1.1 I introduced bi-representations in the sense of Eilenberg. As it turns out, Eilenberg's definition can be constructed in a way which draws together the various elements of a triple in a cohesive, and unified fashion. The general idea is actually very simple, and it involves replacing the spectral data $\{\mathcal{A}, \mathcal{H}, D, J, \gamma\}$ with a larger algebra ΩB , which we call a 'fused algebra'. I will begin in this Subsection by constructing the subalgebra $B_0 \subset \Omega B$, which unifies together the elements $\{\mathcal{A}, \mathcal{H}, J\}$.

Definition: Given an algebra \mathcal{A} over a field \mathbb{F} and a vector space \mathcal{H} over the same field \mathbb{F} , a bi-representation π of \mathcal{A} on \mathcal{H} (or equivalently a bi-module \mathcal{H} over \mathcal{A}) is nothing but a pair of \mathbb{F} -bilinear products $ah \in \mathcal{H}$ and $ha \in \mathcal{H}$ ($a \in \mathcal{A}$, $h \in \mathcal{H}$), where $ah \in \mathcal{H}$ denotes the left-action of $a \in \mathcal{A}$ on $h \in \mathcal{H}$ (a bilinear map from $\mathcal{A} \times \mathcal{H} \rightarrow \mathcal{H}$); and similarly ha denotes the right-action of $a \in \mathcal{A}$ on $h \in \mathcal{H}$ (a bilinear map from $\mathcal{H} \times \mathcal{A} \rightarrow \mathcal{H}$) [Eil48].

This definition is equivalent to the definition of a new fused algebra B_0 ⁹, with vector space:

$$B_0 = \mathcal{A} \oplus \mathcal{H}, \tag{3.49}$$

which is equipped with the following bi-linear product between two elements of B_0 ($b_0 = a + h$ and $b'_0 = a' + h'$):

$$b_0 b'_0 = aa' + ah' + ha' \tag{3.50}$$

where $aa' \in \mathcal{A}$ is the product inherited from \mathcal{A} , while $ah' \in \mathcal{H}$ and $ha' \in \mathcal{H}$ are the products inherited from π , and $hh' = 0$. In addition, the algebra B_0 defined this way is automatically a superalgebra – *i.e.* a \mathbb{Z}_2 -graded algebra, with “even” and “odd” subspaces

⁹In general one can only construct the direct sum between two vector spaces formed over the same field \mathbb{F} . Notice however in the finite NCG SM the input algebra is real, while the vector space is complex. The representation is made by embedding the real input algebra in a complex matrix algebra such that $\phi(a, \lambda v) = \lambda \phi(a, v)$ for $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$, $v \in \mathcal{H}$. It appears as though the correct interpretation may be to view the finite NCG SM Hilbert space as a real vector space, although this is a subtle point we have not yet completely dealt with.

\mathcal{A} and \mathcal{H} , respectively. Because $hh' = 0$ for all $h, h' \in \mathcal{H}$, B_0 is referred to in the math literature as a ‘square zero extension’ of \mathcal{A} .

So far nothing has been assumed about the associativity or any other properties of \mathcal{A} or B_0 (in other words nothing has been assumed about the associativity of the representation $\pi(ab) = \pi(a)\pi(b)$). On the one hand, if B_0 is assumed to be associative, then (as explained below) we precisely recover the traditional associative definition of the bi-representation of \mathcal{A} on \mathcal{H} . But, on the other hand, one need not necessarily assume that B_0 is associative: for example, if \mathcal{A} is a Jordan algebra (see Subsection 2.1.2 for the Jordan algebra definition), then it is natural to define its representation on \mathcal{H} by taking B_0 to also be a Jordan algebra [Eil48, Sch66, Jac73]. We adopted fused algebras in [BF14, FB15b] as a way of defining the representation of \mathcal{A} on \mathcal{H} that naturally generalizes from non-commutative geometry (where \mathcal{A} , the algebra of coordinates, may be non-commutative) to non-associative geometry (where \mathcal{A} may also be non-associative). The fused algebra, or algebra extensions definition of bi-modules was first introduced by Eilenberg [Eil48], and is equivalent to that defined in Subsection 3.1.1 for the special case in which B_0 is taken to have the same associativity properties as \mathcal{A} . The key benefit of this later definition is that it provides a perspective in which two otherwise separate elements of a spectral triple \mathcal{A} , and \mathcal{H} are ‘fused’ or ‘unified’ into a larger object B_0 , which will be key in going further as we shall see.

Let us now explain our assertion (from the previous paragraph) that if we assume B_0 is associative, then we precisely recover the *traditional* definition of an associative representation of \mathcal{A} on \mathcal{H} . If B_0 is associative, all the associators $[b_0, b'_0, b''_0]$ must vanish. This implies four non-trivial constraints:

$$[a, a', a''] = 0, \tag{3.51a}$$

$$[a, a', h''] = 0, \tag{3.51b}$$

$$[h, a', a''] = 0, \tag{3.51c}$$

$$[a, h', a''] = 0, \tag{3.51d}$$

while the remaining associators (in which two or three arguments are from \mathcal{H}) vanish trivially because $hh' = 0$. Comparing with Eq. (3.2), Eq. (3.51a) is simply the requirement that \mathcal{A} itself is associative; Eq. (3.51b) says that ah is a traditional associative left-representation of \mathcal{A} on \mathcal{H} ; Eq. (3.51c) says that ha is a traditional associative right-representation of \mathcal{A} on \mathcal{H} ; and Eq. (3.51d) says that the left- and right-representations commute with each other. In other words, we recover the traditional associative definition of a left-right bi-representation of \mathcal{A} on \mathcal{H} (or, equivalently, the traditional associative definition of a left-right bi-module \mathcal{H} over \mathcal{A}); and the special cases of a left-representation

(left-module) or right-representation (right-module) are recovered, respectively, when either the right action ha or the left action ah vanishes identically.

Representing $*$ -algebras

When constructing NCGs, the input algebras of interest are always involutive in the sense that they are always equipped with an involution $*$ satisfying the properties outlined in Eq. (2.8). Just as one constructs a fused algebra by extending the product of \mathcal{A} to a product on $\mathcal{A} \oplus \mathcal{H}$, one is able to construct a fused $*$ -algebra by extending the involution on the input algebra to $\mathcal{A} \oplus \mathcal{H}$. Indeed many of the traditional axioms/assumptions of NCG follow from the requirement that B_0 is equipped with an involution.

To extend Eilenberg's construction from algebras to $*$ -algebras, we equip an anti-linear involution $*$: $B_0 \rightarrow B_0$, which satisfies $(a_0 a_1)^* = a_1^* a_0^*$ (see Eq. (2.8)). Rather than assume that $*$ has period two as in Eq. (2.8a) (*i.e.* $a^{**} = a$), we only assume the slightly weaker condition that it has period four $(a^{**})^{**} = a$. The involution on B_0 must also have two more properties in addition to those outlined in Eq. (2.8): (i) compatibility with the $*$ operation on the sub-algebra $\mathcal{A} \subset B_0$; and (ii) compatibility with the natural inner product on \mathcal{H} (see below). The first requirement together with the fact that the involution is of period four further implies compatibility with the intrinsic \mathbb{Z}_2 grading on B_0 ¹⁰; the first requirement therefore fixes the $*$ operation to be of the form

$$b_0^* = a^* + Jh \tag{3.52}$$

where a^* is the $*$ -operation on \mathcal{A} , while J is an invertible anti-linear operator on \mathcal{H} . The second requirement forces

$$L_{a^*} = L_a^\dagger. \tag{3.53}$$

The fact that B_0 is a $*$ -algebra then implies and unifies six traditionally-assumed facts about NCG, including: (i) that \mathcal{A} is a $*$ -algebra; and (ii) that \mathcal{H} is equipped with an invertible anti-linear operator J . In addition, (iii) the anti-homomorphism property implies $(ah)^* = h^* a^*$ and $(ha)^* = a^* h^*$, which implies that \mathcal{A} is not just left-represented or right-represented on \mathcal{H} , but left-right bi-represented on \mathcal{H} , with the left and right representations related by $R_a = JL_{a^*} J^{-1}$ and $L_a = JR_{a^*} J^{-1}$. Consistency of these two equations then

¹⁰To be completely explicit: Compatibility with the involution on the sub-algebra means that $b_0^* \in \mathcal{A}$ when $b_0 \in \mathcal{A}$, and then the period four condition forces $b_0^* \in \mathcal{H}$ when $b_0 \in \mathcal{H}$ because if the involution mapped elements out of \mathcal{H} into \mathcal{A} the closure of \mathcal{A} under its own involution would violate the period four requirement.

implies (iv) that $[J^2, L_a] = [J^2, R_a] = 0$ for any $a \in \mathcal{A}$ (in NCG, one assumes J^2 is proportional to the identity). Finally, (v) if J^2 is proportional to the identity, then the fact that the $*$ -operation has period four implies $J^2 = \epsilon$, with $\epsilon = \pm 1$ ¹¹.

Example $*$ -representations

Any (non-associative) $*$ -algebra has a natural representation on itself. However, let's consider two example non-associative representations which are slightly more interesting:

- **Lie algebra representations** Consider the real Lie algebra $\mathcal{A} = M_n(\mathbb{C})^-$ of $n \times n$ anti-hermitian complex matrices bi-represented on $\mathcal{H} = \mathbb{C}^{2n}$ (viewed as a real vector space) by the matrix elements:

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & -a^T \end{pmatrix}. \quad (3.54)$$

To be explicit, the product between elements in the algebra \mathcal{A} is simply the lie product (the commutator), while the product between elements in \mathcal{A} and those in \mathcal{H} is given by the matrix product: $\pi(ab)h = (\pi(a)\pi(b) - \pi(b)\pi(a))h$. The involution is extended from the input algebra to the algebra $B_0 = \mathcal{A} \oplus \mathcal{H}$ by introducing a real structure operator $(a + h)^* = a^\dagger + Jh$, where:

$$J = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} \circ cc, \quad (3.55)$$

where again the action of J on \mathcal{H} is simply given by ordinary matrix multiplication. The right action is readily determined to be $R_{\pi(a)} = JL_{\pi(a)}^\dagger J^{-1} = -L_{\pi(a)}$. The algebra B_0 is clearly not an associative algebra, because for example $\pi(ab)h \neq \pi(a)(\pi(b)h)$. It is however a Lie algebra, as the sub algebra \mathcal{A} is a Lie algebra, while the product involving an element $h \in \mathcal{H}$ is also anti-symmetric and satisfies the Jacobi identity:

$$\begin{aligned} \pi(a)(\pi(b)h) + h\pi(ab) + \pi(b)(h\pi(a)) &= \pi(a)\pi(b)h - [\pi(a), \pi(b)]h - \pi(b)\pi(a)h \\ &= 0, \quad a, b \in \mathcal{A}; h \in \mathcal{H}, \end{aligned} \quad (3.56)$$

where on the right hand side, all products are given by composition, and so brackets are not required.

¹¹Although it is worth stressing that the spectral triple corresponding to the Lorentzian NCG SM is of KO-dimension 0 [Bar07], for which $\epsilon = 1$, and so its fused algebra has a true involution with period 2 (this is possibly an important hint about why spacetime has the dimension it does).

- **Jordan algebra representations** Consider the real Jordan algebra $\mathcal{A} = M_n(\mathbb{C})^+$ of $n \times n$ complex hermitian matrices bi-represented on $\mathcal{H} = \mathbb{C}^{2n}$ (viewed as a real vector space) by the matrix elements:

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a^T \end{pmatrix}. \quad (3.57)$$

To be explicit, the product between elements in the algebra \mathcal{A} is simply the Jordan product (i.e. the symmetric product), while the product between elements in \mathcal{A} and \mathcal{H} is given by the matrix product: $\pi(ab)h = \frac{1}{2}(\pi(a)\pi(b)h + \pi(b)\pi(a)h)$ ¹². The involution is extended from the input algebra to the algebra $B_0 = \mathcal{A} \oplus \mathcal{H}$ by introducing a real structure operator $(a + h)^* = a^\dagger + Jh$, where:

$$J = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} \circ cc, \quad (3.58)$$

and again the action of J on \mathcal{H} is simply given by matrix multiplication. The right action is readily determined to be $R_{\pi(a)} = JL_{\pi(a)}^\dagger J^{-1} = L_{\pi(a)}$. The algebra B_0 is clearly not an associative algebra, because for example $\pi(ab)h \neq \pi(a)(\pi(b)h)$. It is however a Jordan algebra, as the sub algebra \mathcal{A} is a Jordan algebra, while the product involving an element $h \in \mathcal{H}$ is also anti-symmetric and satisfies the Jordan identity:

$$\begin{aligned} \pi(a)(h\pi(a^2)) - (\pi(a)h)\pi(a^2) &= \frac{1}{2}\pi(a)\{\pi(a), \pi(a)\}h - \frac{1}{2}\{\pi(a), \pi(a)\}\pi(a)h \\ &= 0, \quad a \in \mathcal{A}; h \in \mathcal{H}, \end{aligned} \quad (3.59)$$

where on the right hand side all products are given by composition and so brackets are not required.

3.2.2 The algebra $B = \Omega\mathcal{A} \oplus \mathcal{H}$

So far I have introduced the subalgebra $B_0 \subset \Omega B$, which seems to successfully ‘unify’ the data $\{\mathcal{A}, \mathcal{H}, J\}$ into one object B_0 . From the single requirement that B_0 is an associative $*$ -algebra, six of the traditional NCG axioms and assumptions are recovered, namely: (i) that \mathcal{A} is an associative $*$ -algebra; (ii) that \mathcal{A} is (associatively) left-represented on \mathcal{H} ; (iii) that \mathcal{A} is (associatively) right-represented on \mathcal{H} ; (iv) that the left and right representations

¹²Where again I am being a little bit lazy with notation: all products on the right of the expression are given by composition.

of \mathcal{A} on \mathcal{H} commute with each other (the so-called “order-zero condition”); (v) that \mathcal{H} is equipped with an anti-unitary operator J ; and (vi) that the left and right actions of $a \in \mathcal{A}$ on \mathcal{H} must be related via the formula $R_a = JL_a^*J^*$. On the other-hand, this new perspective naturally generalizes from non-commutative to non-associative geometry (in the sense that neither \mathcal{A} nor B_0 needs to be associative for the formalism to make sense).

In order for B_0 to completely describe a geometry, it must additionally be equipped with a Dirac operator D . It is the interaction between D and \mathcal{A} on \mathcal{H} that provides metric information. The operator D will also provide a means of explicitly constructing gauge potentials, higher order forms, and eventually an action. The next step is therefore to work out how to incorporate a Dirac operator D into the algebra B_0 . The answer is that when constructing a NCG it is not only the algebra \mathcal{A} which is represented on \mathcal{H} , but instead the full differential graded algebra $\Omega\mathcal{A}$ constructed from \mathcal{A} and D which is represented on \mathcal{H} . In this Subsection I therefore discuss the construction of a differential graded $*$ -algebra $B = \Omega\mathcal{A} \oplus \mathcal{H}$ ¹³.

Representing $\Omega\mathcal{A}$ on \mathcal{H}

Any associative differential graded $*$ -algebra which in degree zero is equal to \mathcal{A} can be constructed as a projection from the associative universal differential graded algebra $\Omega\mathcal{A}$ [Lan97]. As reviewed in Ch. 2, the associative universal differential graded $*$ -algebra $\Omega\mathcal{A} = \Omega^0\mathcal{A} \oplus \Omega^1\mathcal{A} \oplus \dots$, with $\Omega^0\mathcal{A} = \mathcal{A}$, is constructed by equipping \mathcal{A} with a differential operator d which satisfies:

$$d^2 = 0, \tag{3.60a}$$

$$d[\alpha a + \beta b] = \alpha d[a] + \beta d[b], \tag{3.60b}$$

$$d[ab] = d[a]b + ad[b], \tag{3.60c}$$

$$d[a^*] = \kappa d[a]^* \tag{3.60d}$$

for $\alpha, \beta \in \mathbb{F}$, $a, b \in \mathcal{A}$, and $\kappa = \pm 1$. $\Omega\mathcal{A}$ is generated by elements $a \in \Omega^0\mathcal{A}$, and the formal symbols $d[a] \in \Omega^1\mathcal{A}$ by juxtaposition, such that the subset $\Omega^n\mathcal{A}$ is populated by elements in which ‘ d ’ appears n times. For example $\omega_2 = d[a]d[b]c \in \Omega^2\mathcal{A}$. The associative algebra $\Omega\mathcal{A}$ is free in the sense that aside from the conditions given in Eq. (3.60), and those inherited from the product on \mathcal{A} , this algebra is free. Notice however that none of the defining properties of $\Omega\mathcal{A}$, including the properties d given in Eq. (3.60), relies on associativity, and so universal DGAs continue to make sense even when the base algebra

¹³Although as we will see B does not really satisfy all the properties of a $*$ -DGA.

$\Omega^0\mathcal{A} = \mathcal{A}$ is non-associative. One simply has to be careful to take account of bracketing, and $\Omega\mathcal{A}$ must be taken to be completely free in the sense that not even associativity is assumed.

To obtain a representation of the elements $d[a] \in \Omega^1\mathcal{A}$ on \mathcal{H} , an operator D is introduced, which is regarded as the representation of the map $d : \Omega^0\mathcal{A} \rightarrow \Omega^1\mathcal{A}$ on \mathcal{H} . The operator D 's interaction with \mathcal{A} and \mathcal{H} satisfies a Leibniz condition $Dah = d[a]h + aDh$, for $a \in \mathcal{A}$, $h \in \Omega\mathcal{H}$. In this way the left-action of the exact one-form $d[a]$ is represented by

$$L_{\pi(d[a])} = [D, L_a]. \quad (3.61)$$

In the associative setting the product between elements of $\pi(\Omega\mathcal{A})$ is simply given by composition, and so the representation of exact forms given in Eq. (3.61) naturally extends to higher order forms, $\pi : \Omega\mathcal{A} \rightarrow B(\mathcal{H})$:

$$\pi(a_0d[a_1]\dots d[a_n]) = a_0[D, a_1]\dots[D, a_n] \quad (3.62)$$

for $a_i \in \mathcal{A}$. In the non-associative case the situation is not so simple. To see why, consider for example the case in which a non-associative input algebra \mathcal{A} is represented on $2n$ copies of itself $\mathcal{H} = \mathcal{A}^{2n}$ (i.e a column vector of $2n$ copies of \mathcal{A}). Just as in the associative case, the universal DGA $\Omega\mathcal{A}$ constructed from \mathcal{A} can be represented on \mathcal{H} by introducing an operator D on \mathcal{H} . However, the Leibniz condition given in Eq. (3.60) together with the left action of exact forms given in Eq. (3.61) implies:

$$\begin{aligned} [D, L_{ab}] &= L_{\pi(d[ab])} \\ &= L_{\pi(d[a]b)} + L_{\pi(ad[b])}, \quad a, b \in \mathcal{A}. \end{aligned} \quad (3.63)$$

Or in other words the operator $[D, _]$ must act as a derivation on $\pi(\mathcal{A})$. In the associative case this places no restriction on the operator D , as the product for an associative representation is simply given by composition, which is associative, and commutators act like derivations on associative products.

In the non-associative setting it will be interesting to consider the large class of Dirac operators of the form:

$$D = \gamma^i \delta_i, \quad (3.64)$$

where γ^i is a hermitian matrix operator with values in \mathbb{F} , and which commutes with both the left and right representation of the input algebra; and δ_i is an anti-hermitian derivation element on \mathcal{A} . Notice that the canonical Dirac operator $i\gamma^\mu \partial_\mu$ is already in this form. For

the class of Dirac operators given in Eq. (3.64), the representation of exact one forms naturally extends to the representation of higher order forms as:

$$\pi(a_0(d[a_1]\dots(d[a_{n-1}]d[a_n]))) \equiv \gamma^i \dots \gamma^j \gamma^k (a_0(\delta_i(a_1)\dots(\delta_j(a_{n-1})\delta_k(a_n)))), \quad (3.65)$$

for $a_i \in \mathcal{A}$. In Eq. (3.65) all brackets are shifted right, but this is just an example, all other bracketings can also occur. I provide explicit examples later in this Section.

The representation of $\Omega\mathcal{A}$ on \mathcal{H} is equivalent to a fused algebra $B = \Omega\mathcal{A} \oplus \mathcal{H}$ (with several problems as I will discuss shortly). Just as B_0 was constructed as a $*$ -algebra in Subsection 3.2.1, we ultimately want to use the representation π to construct B as a ‘fused’ differential graded $*$ -algebra. In constructing $\Omega\mathcal{A}$ no modifications have been made to the Hilbert space \mathcal{H} , and so it is quite natural to take the involution on the whole of B to be given by the extension of the one given on B_0 :

$$(\omega_n + h)^* = \omega_n^* + Jh \quad (3.66)$$

for $\omega_n \in \Omega^n\mathcal{A}$. The right action of $\Omega^0\mathcal{A}$ is therefore naturally extended to the whole of $\Omega\mathcal{A}$, and is given by:

$$R_\omega = JL_\omega^*J^{-1}, \quad (3.67a)$$

$$L_\omega = JR_\omega^*J^{-1}, \quad (3.67b)$$

Order conditions and the algebra $\Omega\mathcal{A} \oplus \mathcal{H}$

So far in the construction of B , nothing has been assumed about its associativity, or about the associativity of $\Omega\mathcal{A}$. If however we assume that B is an associative differential graded $*$ -algebra, then many of Connes’ axioms and assumptions are recovered in addition to the six which were recovered when constructing B_0 . Let’s explore what happens when we take B to be an associative $*$ -algebra: Demanding that B is associative, is equivalent to demanding that all associators vanish, ie. $[b, b', b''] = 0 \forall b \in B$. This demand imposes four non-trivial conditions on B :

$$[\omega, \omega', \omega''] = 0, \quad (3.68a)$$

$$[\omega, \omega', h] = 0 \quad (3.68b)$$

$$[h, \omega, \omega'] = 0 \quad (3.68c)$$

$$[\omega, h, \omega'] = 0, \quad (3.68d)$$

for $\omega, \omega' \in \Omega\mathcal{A}$, $h \in \mathcal{H}$, while the remaining associators (in which two or three arguments are from \mathcal{H}) vanish trivially because $hh' = 0$. As expected the conditions given in Eq. (3.68) coincide with those of Eq. (3.51) when $\omega, \omega' \in \Omega^0\mathcal{A}$. Note that (3.68a) is simply the requirement that $\Omega\mathcal{A}$ itself is associative; (3.68b) says that ωh is a traditional associative left-representation of $\Omega\mathcal{A}$ on \mathcal{H} ; (3.68c) says that $h\omega$ is a traditional associative right-representation of $\Omega\mathcal{A}$ on \mathcal{H} ; and (3.68d) says that the left- and right-representations commute with each other. In other words, we recover the traditional associative definition of a left-right bi-representation of $\Omega\mathcal{A}$ on \mathcal{H} (or, equivalently, the traditional associative definition of a left-right bi-module \mathcal{H} over $\Omega\mathcal{A}$); and the special cases of a left-representation (left-module) or right-representation (right-module) are recovered, respectively, when either the right action $h\omega$ or the left action ωh vanishes identically.

Now let us have a closer look at the associativity condition given in Eq. 3.68d. As expected, in the special case where $\omega, \omega' \in \Omega^0\mathcal{A}$, this associativity condition reproduces Connes order zero condition outlined in Eq. (2.68). In the special case where $\omega = d[a] \in \Omega^1\mathcal{A}$, and $\omega' = b \in \Omega^0\mathcal{A}$, we find:

$$\begin{aligned} [\omega, h, \omega'] &= (d[a]h)b - d[a](hb) \\ &= [Jb^*J^*, [D, a]]h = 0, \end{aligned} \tag{3.69}$$

which is just the order one condition outlined in Eq. (2.69), which was first imposed by Connes [Con96]. In other words, from the fused algebra perspective, Connes' order conditions are simply replaced by associativity conditions on B . Notice however that there are now also new higher order conditions. For the case where $\omega = d[a], \omega' = d[b] \in \Omega^1\mathcal{A}$, we find:

$$\begin{aligned} [\omega, h, \omega'] &= (d[a]h)d[b] - d[a](hd[b]) \\ &= [J[D, b]^*J^*, [D, a]]h = 0, \end{aligned} \tag{3.70}$$

This higher order associativity condition places constraints on the input Dirac operator beyond those already imposed in the traditional approach to NCG. As it turns out this condition exactly removes unwanted terms from the finite Dirac operator of the NCG SM, which were previously removed by an additional non-geometric assumption known as the ‘massless-photon condition’ [CC08, CCM07]. I will discuss this situation in detail in Subsection 4.1.3.

Removing the junk

While $B = \Omega\mathcal{A} \oplus \mathcal{H}$ might at first sight appear to be a well defined differential graded $*$ -algebra, it is in general not. The problem, as described in 2.2.1 for the associative case,

is that while π is an algebra homomorphism it is usually not a homomorphism between differential graded algebras because in general $\pi(\omega) = 0$ does not imply $\pi(d[\omega]) = 0$ [Lan97]. Such ‘junk forms’ must be removed in order to form a true graded differential representation in which the Leibniz rule makes sense. Fortunately, as outlined in Section 2.2.1, in the associative case such troublesome elements form an ideal of ΩA , and so may be modded out to form a new quotient algebra $\Omega_D A = \Omega A / J$, where $J = \oplus_i J_0^i + dJ_0^i$ and $J_0^i = \{\omega \in \Omega^i A | \pi(\omega) = 0\}$. Once the junk forms have been modded out we arrive at a new fused algebra which we denote $B' = \Omega_D \mathcal{A} \oplus \mathcal{H}$ (but again this algebra has several interrelated problems which I will discuss shortly).

In the associative case junk forms appear because the algebra $\Omega \mathcal{A}$ is free in the sense that the product is simply defined by juxtaposition, and there is nothing like graded commutativity. When constructing a representation of $\Omega \mathcal{A}$, introducing the operator D introduces an explicit product on $\pi(\Omega \mathcal{A})$ and in particular the commutator of certain elements may be zero. The situation is even more interesting and must be worked out in full detail for the non-associative case, where not just commutators, but also associators of elements must be taken into consideration. There is also the added complication when constructing a non-associative representation of the universal differential graded $*$ -algebra $\Omega \mathcal{A}$, that the elements $J_0 = \oplus_i J_0^i = \{\omega \in \Omega^i A | \pi(\omega) = 0\}$ do not in general form an ideal of $\Omega \mathcal{A}$. In turn $J = \oplus_i J_0^i + dJ_0^i$, will not in general form an ideal of $\Omega \mathcal{A}$, which prevents one from constructing the well defined quotient algebra $\Omega_D A = A / J$. We therefore introduce the following condition which all sensible non-associative NCGs must satisfy:

Axiom: Given the input data $\{\mathcal{A}, \mathcal{H}, D\}$, the set $J_0 = \oplus_i J_0^i = \{\omega \in \Omega^i A | \pi(\omega) = 0\}$ forms a two sided ideal of $\Omega \mathcal{A}$.

All of the finite associative, and almost-associative NCGs that I will consider naturally satisfy this condition. Just as in the associative case, when J_0 forms an ideal of $\Omega \mathcal{A}$, the elements $J = J_0 + dJ_0$ also form an ideal and so just as in the associative case we can mod out by the junk to form a new differential graded algebra $\Omega_D A = A / J$ (associativity does not enter at any point into the proof given in 2.2.1 that J is a two sided ideal).

- **Octonion example** Consider the finite non-associative geometry given by:

$$\{A_F, H_F, D_F\} = \{\mathbb{O}, \mathbb{O}^2, i\sigma_2\delta\}, \quad (3.71)$$

where the representation of the input algebra is given by $\pi(a) = a\mathbb{I}_2$, δ is an arbitrary element in $Der(\mathbb{O})$, and σ_2 is the second Pauli matrix. We also equip the real structure and grading operators $J_F = \mathbb{I}_2 \circ *$, $\gamma_F = diag\{1, -1\}$. As required, this

geometry satisfies $\{D_F, \gamma_F\} = 0$. It also satisfies $J^2 = \mathbb{I}_2$, $[D_F, J_F] = 0$, $[\gamma_F, J_F] = 0$ and so is of KO-dimension 0 (see Table 2.1).

This is the first non-associative geometry that we have so far seen with non-trivial Dirac operator. We can use D_F to construct a representation of $\Omega\mathcal{A}$ on \mathcal{H} :

$$L_{\pi(d[a])}h = [D_F, L_{\pi(a)}]h = i\sigma_2[\delta, L_a]h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} L_{\delta(a)}h, \quad (3.72)$$

where in the last equality I have used the fact that inner derivations satisfy the Leibniz rule: $\delta(ah) = \delta(a)h + a(\delta h) \implies [\delta, L_a] = \delta(a)h$. This representation naturally extends to higher order forms, $\pi : \Omega\mathcal{A} \rightarrow B(\mathcal{H})$. For example:

$$\pi(bd[a]) \equiv \begin{pmatrix} 0 & b\delta[a] \\ -b\delta[a] & 0 \end{pmatrix}, \quad \pi(c(d[b]d[a])) \equiv \begin{pmatrix} -c(\delta[b]\delta[a]) & 0 \\ 0 & -c(\delta[b]\delta[a]) \end{pmatrix}, \quad (3.73)$$

for $a, b, c \in \mathcal{A}$. Higher order forms of odd order are similarly off-diagonal, while higher order forms of even order are diagonal. We see that there is no distinction between the representation of the element $c(d[b]d[a]) \in \Omega^2\mathcal{A}$, and the element $-c(\delta[b]\delta[a]) \in \Omega^0\mathcal{A}$. This is identical to the situation which occurs for the canonical spectral triple prior to quotienting out by the junk forms. For canonical spectral triples the representation of the element $\{d[a], d[b]\} \in \Omega^2\mathcal{A}$ is indistinguishable from the representation of the element $2\partial_\mu(a)\partial^\mu(b) \in \Omega^0\mathcal{A}$ (see Subsection 2.2.1).

The octonions are a division algebra, and so it is clear from the representations given in Eq. (3.73) that the elements $J_0 = \oplus_i J_0^i = \{\omega \in \Omega^i\mathcal{A} | \pi(\omega) = 0\}$ form an ideal of the differential graded algebra $\Omega\mathcal{A}$ when $\mathcal{A} = \mathbb{O}$. Just as in the associative case we can mod out by the junk, to form a new differential graded algebra $\Omega_D\mathcal{A} = \Omega\mathcal{A}/J$. Let's have a look at the form that the junk takes.

Consider elements of the form $\omega_1 = c[a, d[b], a] = c((ad[b])a - a(d[b]a))$. The differential graded $*$ -algebra $\Omega\mathcal{A}$ has a priori no graded commutativity or associativity properties and so $\omega_1 \neq 0$. When we represent ω_1 on $\mathcal{H} = \mathbb{O}^2$ however we find:

$$\pi(\omega_1) = \begin{pmatrix} 0 & c((a\delta[b])a - a(\delta[b]a)) \\ -c((a\delta[b])a - a(\delta[b]a)) & 0 \end{pmatrix} = 0, \quad (3.74)$$

where the second equality holds due to alternativity (see Eq. (2.7)). For the representation of $d[\omega_1]$ we find:

$$\begin{aligned} \pi(d[\omega_1]) &= -c((\delta[a]\delta[b])a - (a\delta[b])\delta[a] + \delta[a](\delta[b]a) - a(\delta[b]\delta[a]))\mathbb{I}_2 \\ &= -c([\delta[a], \delta[b], a] - [a, \delta[b], \delta[a]])\mathbb{I}_2 \\ &= 2c[a, \delta[b], \delta[a]]\mathbb{I}_2 \neq 0 \end{aligned} \quad (3.75)$$

where the last equality holds due to alternativity (see Eq. (2.7)). By appropriate choice of $a, b, c \in \mathbb{O}$ one can find $\pi(d[\omega_1]) = a'\mathbb{I}_2$ for any $a' \in \mathbb{O}$. We therefore see that by quotienting out by the junk we impose alternativity conditions, and remove all forms of order two and higher. In other words:

$$\Omega_D \mathcal{A} \simeq \pi(\mathcal{A} \oplus \Omega^1 \mathcal{A}), \quad (3.76)$$

where $\Omega_D^1 \mathcal{A}$ is an alternative bi-module over \mathcal{A} .

- **Jordan example** Consider the finite non-associative geometry given by:

$$\{A_F, H_F, D_F\} = \{M_3(\mathbb{C})^+, \mathbb{C}^{4n}, \text{diag}\{i\sigma_2\delta, i\sigma_2\bar{\delta}\}\}, \quad (3.77)$$

where the representation of the input algebra is given by $\pi(a) = \text{diag}\{a\mathbb{I}_2, a^T\mathbb{I}_2\}$, δ is an arbitrary element in $\text{Der}(M_3(\mathbb{C})^+)$, and σ_2 is the second Pauli matrix. The real structure and grading operators are taken to be:

$$J_F = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \circ cc, \quad \gamma_F = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \quad (3.78)$$

where σ_3 is the third Pauli matrix. As required, this geometry satisfies $\{D_F, \gamma_F\} = 0$. It also satisfies $J_F^2 = 0$, $[J_F, D_F] = 0$, $[J_F, \gamma_F] = 0$, and so is once again of KO-dimension 0. The Dirac operator D_F can be used to construct a representation of $\Omega \mathcal{A}$ on \mathcal{H} . The action of exact one-forms on \mathcal{H} is given by:

$$\begin{aligned} L_{\pi(d[a])}h &= [D_F, L_{\pi(a)}]h = \text{diag}\{i\sigma_2 L_{\delta a}, i\sigma_2 L_{\bar{\delta} a}\}h, \\ \rightarrow \pi(d[a]) &= \begin{pmatrix} 0 & \delta(a) & 0 & 0 \\ -\delta(a) & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{\delta(a)} \\ 0 & 0 & -\overline{\delta(a)} & 0 \end{pmatrix} \end{aligned} \quad (3.79)$$

Higher order forms are then similarly constructed. As an example, an element of $\Omega^2 \mathcal{A}$ may be represented as:

$$\pi(c(d[b]d[a])) = \begin{pmatrix} -c(\delta[b]\delta[a]) & 0 & 0 & 0 \\ 0 & -c(\delta[b]\delta[a]) & 0 & 0 \\ 0 & 0 & -\overline{c(\delta[b]\delta[a])} & 0 \\ 0 & 0 & 0 & -\overline{c(\delta[b]\delta[a])} \end{pmatrix}, \quad (3.80)$$

As in the octonion example above, we see that there is no distinction between the representation of the element $c(d[b]d[a]) \in \Omega^2 \mathcal{A}$, and the element $-c(\delta[b]\delta[a]) \in \Omega^0 \mathcal{A}$.

Once again the elements $J_0 = \oplus_i J_0^i = \{\omega \in \Omega^i A \mid \pi(\omega) = 0\}$ form an ideal of the differential graded algebra ΩA . Just as in the associative case we can mod out by the junk to form a new differential graded algebra $\Omega_D A = A/J$. Let's have a look at the form that the junk takes.

Consider elements of the form $\omega_1 = c(ad[b] - d[b]a)$. The differential graded $*$ -algebra $\Omega \mathcal{A}$ has a priori no graded commutativity or associativity properties and so $\omega_1 \neq 0$. When we represent ω_1 on $\mathcal{H} = \mathbb{C}^{4n}$ however we find:

$$\begin{aligned} \pi(\omega_1) &= \begin{pmatrix} 0 & c(a\delta(b) - \delta(b)a) & 0 & 0 \\ -c(a\delta(b) - \delta(b)a) & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{c(a\delta(b) - \delta(b)a)} \\ 0 & 0 & -\overline{c(a\delta(b) - \delta(b)a)} & 0 \end{pmatrix} \\ &= 0, \end{aligned} \tag{3.81}$$

where the second equality holds because Jordan algebras are commutative (see Eq. (2.5)). For the representation of $d[\omega_1]$ we find:

$$\pi(d[\omega_1]) = -2 \text{diag}\{c(\delta(a)\delta(b))\mathbb{I}_2, \overline{c(\delta(a)\delta(b))}\mathbb{I}_2\} \neq 0 \tag{3.82}$$

One can always find an appropriate form $\omega = \sum_{a,b,c} c[b, d[a]] \in \Omega^1 \mathcal{A}$ such that $\pi(d[\omega]) = \text{diag}\{a'\mathbb{I}_2, a'^T\mathbb{I}_2\}$ for any $a' \in M_3(\mathbb{C})^+$. We therefore see that by quotienting out by the junk we impose Jordan conditions, and remove all forms of order two and higher. In other words:

$$\Omega_D \mathcal{A} \simeq \pi(\mathcal{A} \oplus \Omega_D^1 \mathcal{A}), \tag{3.83}$$

where $\Omega_D^1 \mathcal{A}$ acts as a Jordan bi-module over \mathcal{A} .

Problems with the algebra $B' = \Omega_D \mathcal{A} \oplus \mathcal{H}$

So far the fused algebra $B' = \Omega_D \mathcal{A} \oplus \mathcal{H}$ seems to successfully bring together the elements $\{\mathcal{A}, \mathcal{H}, D, J\}$ of a spectral triple. In addition many of the traditional axioms and conditions imposed in associative NCG are obtained from the simple statement that B' is an associative, differential, graded $*$ -algebra. Unfortunately B' suffers from four major interrelated problems which until now I have avoided discussing. These are:

- **Associativity:** In order for B to be associative the condition $[b, b', b''] = 0$ must be satisfied for all $b, b', b'' \in B$, as described in Eq. (3.68). In particular, asking for compatibility between the left and right actions in the sense of Eq. (3.68d) seems to not

only reproduce Connes' order conditions, but also to provide additional higher order constraints on D . Unfortunately, while this condition imposes phenomenologically accurate constraints on the *finite* part of the NCG SM (see Subsection 4.1.3), it is not in general satisfied by the *canonical* spectral triples corresponding to Riemannian geometries where for $\omega = d[a], \omega' = d[b] \in \Omega_D^1 \mathcal{A}$, we find:

$$\begin{aligned} [\omega, h, \omega'] &= [J[D, b]^* J^*, [D, a]]h \\ &= -[\gamma^\mu, \gamma^\nu](\partial_\mu b)(\partial_\nu a) \neq 0, \end{aligned} \tag{3.84}$$

for $a, b \in C^\infty(M, \mathbb{C})$, $D = -i\gamma^\mu \nabla_\mu^S$, $J = \gamma^0 \gamma^2 \circ cc$. Either canonical spectral triples are examples of non-associative geometries, or else some incorrect assumption has been made in the construction of B' .

- **The algebra B' is in general not well defined:** As it turns out B' will in general suffer from a much more serious problem than non-associativity, which hints that we have made some incorrect assumption in the construction of B' . It turns out that in general B' is not well defined. On first inspection this seems odd. If one starts with an associative algebra and mods out by a two-sided ideal, then the resulting algebra is well defined and always associative. The differential graded algebra $\Omega_D \mathcal{A}$ is constructed from $\Omega \mathcal{A}$ by modding out by the junk J , which forms a two-sided ideal, so what could go wrong? The trouble is that the junk J is only an ideal of the sub algebra $\Omega \mathcal{A}$, and not of the entire algebra B . As a result B' will in general be poorly defined. To avoid this problem, one would need to mod out by an appropriate ideal of B . Unfortunately in the cases of physical interest this would necessitate modding out by the whole of \mathcal{H} .
- **The grading on \mathcal{H} :** The operator D was incorporated into the fused algebra formulation of NCG by making the extension:

$$B_0 = \mathcal{A} \oplus \mathcal{H} \longrightarrow \Omega_D \mathcal{A} \oplus \mathcal{H} = B.$$

Notice that there is something strange about this extension. While \mathcal{A} is extended to the differential graded algebra $\Omega_D \mathcal{A}$, the Hilbert space \mathcal{H} is left unaffected and so it does not provide a true differential graded representation in the sense of Subsection 2.1.6. In addition for even spectral triples the Hilbert space is equipped with its own \mathbb{Z}_2 grading γ . But this grading on \mathcal{H} seems completely unrelated to the grading on $\Omega_D \mathcal{A}$. What would be preferable, would be a formulation which naturally draws together all five elements of a spectral triple along with their defining axioms. The algebra B' only seems to give a partial picture, and while it does reproduce some of

the properties of J , and the order conditions, it says nothing about the real structure table 2.1, or the properties of γ .

A possible remedy to this undesirable situation is hinted at if one pays closer attention to the ‘derivation’ of the right action given in Eq. (3.67). The form of the right action was determined using the anti-automorphism property of the involution given in Eq. (2.8): $R_\omega = JL_\omega^*J^{-1}$. In comparison, the left action was determined in Eq. (3.61) by making use of the Leibniz rule $Dah = d[a]h + aDh$, for $a \in \mathcal{A}$, and $h \in \mathcal{H}$. One could repeat the ‘derivation’ of the right action of forms in exactly the same way by making use of the Leibniz rule: $D(ha) = (Dh)a + hd[a]$, for $a \in \mathcal{A}$, and $h \in \mathcal{H}$. If one does so, one finds that the right action of exact forms is given by:

$$\begin{aligned} R_{d[a]} &= [D, R_a] \\ &= -\epsilon' J[D, L_a]^* J^{-1} \end{aligned} \quad (3.85)$$

which is clearly in disagreement up to sign with the right action of forms given previously in Eq. (3.67). To derive the ‘correct’ right action of forms it appears that one should instead use the ‘graded’ Leibniz rule

$$D(ha) = (Dh)a - \epsilon' hd[a], \quad (3.86)$$

where the ‘order’ of the Hilbert space element h is determined by the real structure sign ϵ' . If this is the correct interpretation, then one notices immediately that if \mathcal{H} is to be viewed as ‘graded’, then its order is fixed in two ways:

1. The elements of $\Omega_D \mathcal{A}$ are represented as bounded operators on \mathcal{H} , and so the action of any n -form will map \mathcal{H} to \mathcal{H} . In other words the ‘order’ of \mathcal{H} seems unaffected by the action of n -forms.
2. The sign ϵ' is fixed by the geometry.

It is tempting for these reasons to consider the space \mathcal{H} to be assigned a grading equal to ∞ , ie. \mathcal{H}^∞ . As I will explain shortly in Subsection 3.2.3 however, this is not quite the correct approach to take. The correct interpretation is that the Hilbert space should simply be graded $\mathcal{H} = \bigoplus_n \mathcal{H}^n$, such that the representation becomes a true differential, graded, involutive representation.

- **Nilpotency $d^2 = 0$:** For B' to truly be a differential graded $*$ -algebra then it must be equipped with a differential operator d which squares to zero. We introduced the Dirac operator D on \mathcal{H} , which we regarded as the representation of the map $d : \Omega_D^0 \mathcal{A} \rightarrow \Omega_D^1 \mathcal{A}$, and demanded that it satisfy the Leibniz property $Dah = d[a]h + aDh$. Unfortunately this Dirac operator does not square to zero, and so it is not a differential operator in the sense of Eq. (3.60).

3.2.3 The work of Brouder *et al.*

In subsection 3.2.1 I constructed the fused algebra $B_0 = \mathcal{A} \oplus \mathcal{H}$ as a square zero extension of \mathcal{A} . The key idea was that B_0 should extend all of the properties of \mathcal{A} . By extending the associative product on \mathcal{A} and its involution, many aspects of the NCG formalism were recovered. While this initial construction is promising, in practice when constructing a NCG one is not interested in just the algebra \mathcal{A} , but instead the full differential graded $*$ -algebra $\Omega_D \mathcal{A}$ constructed from \mathcal{A} and D , which comes with a lot more structure. In Subsection 3.2.2 the algebra $B = \Omega \mathcal{A} \oplus \mathcal{H}$ was constructed in an attempt to extend the involutive, differential, graded structure of $\Omega \mathcal{A}$. Again many aspects of the NCG formalism were captured by this construction including the order one condition. Unfortunately B is not a differential graded $*$ -algebra, and for the NCGs which are of physical interest, B is certainly not associative (even when considering physical models with associative input algebra \mathcal{A} such as the NCG SM). A solution to these problems was pointed out by Brouder *et al.* in [BBB15], who suggested that as a DGA, $\Omega \mathcal{A}$ should have a true differential graded representation, and made a proposal for how the fused algebra could extend the differential properties of $\Omega \mathcal{A}$. In this subsection I provide a brief review of the Brouder *et al.* proposal.

As outlined in Subsection 2.1.6, a graded algebra $\mathcal{A} = \bigoplus_n \mathcal{A}^n$ is a graded vector space equipped with a product which is compatible with the grading: $\mathcal{A}^n \times \mathcal{A}^m \rightarrow \mathcal{A}^{n+m}$. In other words, for any two elements $\omega_n \in \mathcal{A}^n$, and $\omega'_m \in \mathcal{A}^m$, their product is a new element $(\omega_n \omega'_m) \in \mathcal{A}^{n+m}$. A differential graded algebra is a graded algebra which is additionally equipped with an order one differential operator $d : \mathcal{A}^n \rightarrow \mathcal{A}^{n+1}$, which satisfies the properties outlined in Eq. (2.17), ie. $d^2 = 0$, $d[\omega_n \omega'_m] = d[\omega_n] \omega'_m + (-1)^n \omega_n d[\omega'_m]$.

A graded bi-representation of an algebra $\mathcal{A} = \bigoplus_n \mathcal{A}^n$, is a graded vector space $\mathcal{H} = \bigoplus_n \mathcal{H}^n$, which is equipped with a bilinear left action $\mathcal{A}^n \times \mathcal{H}^m \rightarrow \mathcal{H}^{n+m}$, and a bilinear right action $\mathcal{H}^n \times \mathcal{A}^m \rightarrow \mathcal{H}^{n+m}$. In other words, for $\omega_n \in \mathcal{A}^n$, and $h_m \in \mathcal{H}^m$, the left action is given by $(\omega_n h_m) \in \mathcal{H}^{n+m}$, and the right action is given by $(h_m \omega_n) \in \mathcal{H}^{n+m}$. A differential graded representation of a differential graded algebra is a graded representation, which is additionally equipped with an order one differential operator $d : \mathcal{H}^n \rightarrow \mathcal{H}^{n+1}$ satisfying the compatibility conditions $d^2 = 0$ ¹⁴, $d[\omega_n h_m] = d[\omega_n] h_m + \omega_n d[h_m]$ (again see Subsection 2.1.6 for further details).

A differential graded bi-representation of \mathcal{A} on \mathcal{H} is equivalent to the differential graded

¹⁴Actually the condition $d^2 = 0$ is not usually imposed, but from our perspective it should be, such that our differential graded representations are equivalent to true fused DGAs.

fused algebra

$$B = \bigoplus_n (\mathcal{A}^n \oplus \mathcal{H}^n). \quad (3.87)$$

Note that while I have used the notation ‘ B ’ here in Eq. (3.87) to denote a general differential graded fused algebra, it should not be confused with the fused algebra $B = \Omega\mathcal{A} \oplus \mathcal{H}$ introduced in Subsection 3.2.2. The fused algebra B given in (3.87) is graded in two different ways or, more precisely, it is graded over the ring $\mathbb{Z} \times \mathbb{Z}_2$. In other words, it has its new grading $\bigoplus_m (\mathcal{A}_m + \mathcal{H}_m)$ over \mathbb{Z} ; but, in addition, it still has another independent \mathbb{Z}_2 grading that splits it into an even part \mathcal{A} and an odd part \mathcal{H} , thereby making it a super-algebra (or, in this case, a super-DGA).

Brouder *et al.* in [BBB15], proposed that the differential graded algebra $\Omega_D\mathcal{A}$ should be bi-represented on a differential graded vector space which I denote $\Omega\mathcal{H} = \bigoplus \Omega^n\mathcal{H}$. This graded vector space could be copies of the bounded operators $B(\mathcal{H})$ on \mathcal{H} , or copies of \mathcal{H} , or something else (for now I leave it general). I will use the notation ΩB to denote the super-DGA associated to the differential graded representation of $\Omega_D\mathcal{A}$ on $\Omega\mathcal{H}$:

$$\Omega B = \bigoplus_n (\Omega_D^n\mathcal{A} \oplus \Omega^n\mathcal{H}). \quad (3.88)$$

One of the main problems with the fused algebra $B = \Omega\mathcal{A} \oplus \mathcal{H}$ introduced in Subsection 3.2.2, is that it does not in general have the same associativity properties as the sub-algebra \mathcal{A} (in particular the canonical NCGs have a corresponding algebra B which is non-associative). Let us see what effect the grading has on the associativity properties of ΩB . The condition which is of most immediate interest is that given in Eq. (3.68d), which describes the compatibility between the left and right action of forms. This is the condition which places constraints on the Dirac operator when constructing the NCG SM. For a graded representation the associativity condition given in Eq. (3.68d) becomes:

$$[\omega_m, h_n, \omega'_p] = 0, \quad \forall \omega_m, \omega'_p \in \Omega_D\mathcal{A}, h_n \in \Omega\mathcal{H}. \quad (3.89)$$

In order to make use of Eq. (3.89) it is first necessary to work out how to construct the right action of forms on $\Omega\mathcal{H}$. In Subsection 3.2.1 I constructed a right action of algebra elements by extending the involution on \mathcal{A} to B_0 . In going from the fused algebra B_0 to B , no changes were made to the representation space, and so in Subsection 3.2.2 I once again used J to extend the involution from $\Omega\mathcal{A}$ to the whole of B , and to construct the right action of forms given in Eq. (3.67). It is no longer obvious how one should extend the involution to the whole of $\Omega\mathcal{H}$ when constructing the algebra ΩB . Rather than assuming

the form of the involution, Brouder *et al.* instead use the opposite algebra ΩB_{op} to derive the right action of forms:

$$h_m \times \omega_n = (-1)^{mn} \omega_n \times_{op} h_m, \quad \omega_n \in \Omega^n \mathcal{A}, h_m \in \Omega^m \mathcal{H}. \quad (3.90)$$

The compatibility conditions for the opposite product defined in Subsection 2.1.3 uniquely determine the form that the right action takes. The right action of zero forms is already known, which allows one to determine:

$$\begin{aligned} a \times_{op} h_m &= h_m a \\ &= JaJ^{-1}h_m \end{aligned} \quad (3.91)$$

for $a \in \Omega_D^0 \mathcal{A}$, and $h_m \in \Omega \mathcal{H}$. Knowing the right action of zero forms, the right action of exact one forms is then readily derived:

$$\begin{aligned} d[a] \times_{op} h_m &= (-1)^m h_m d[a] \\ &= d[h_m a] - (dh_m)a \\ &= [D, JaJ^{-1}]h_m, \end{aligned} \quad (3.92)$$

for $a \in \Omega_D^0 \mathcal{A}$, and $h_m \in \Omega \mathcal{H}$. In the first equality I used of the definition for the graded opposite product given in Eq. (2.21), and in the second equality I have used the graded Liebniz rule given in Eq. (2.17). Restricting to the associative case, higher order forms are constructed by composition of zero and one forms and so Eq. (3.91) and Eq. (3.92) are all that is needed to determine the right action of general forms. In particular:

$$\begin{aligned} \omega_m \times_{op} (\omega'_n \times_{op} h_p) &= (-1)^{pn+m(p+n)} (h_p \omega'_n) \omega_m \\ &= (-1)^{pn+m(p+n)} h_p (\omega'_n \omega_m) \\ &= (-1)^{mn} (\omega'_n \omega_m) \times_{op} h_p, \end{aligned} \quad (3.93)$$

for $\omega_m, \omega'_n \in \Omega_D \mathcal{A}$, and $h_p \in \Omega \mathcal{H}$, which tells us that the right action of higher order forms $\omega_m = a_0 d[a_1] \dots d[a_m] \in \Omega_D^m \mathcal{A}$ can be broken down into the much more manageable task of representing m one forms:

$$\begin{aligned} h_n \omega_m &= (-1)^{mn} \omega_m \times_{op} h_n \\ &= (-1)^{mn+(m-1)+(m-2)\dots+1} d[a_m] \times_{op} \dots d[a_1] \times_{op} a_0 \times_{op} h_n \\ &= (-1)^{mn+(m-1)+(m-2)\dots+1} (\epsilon')^m (-1)^m J[D, a_m^*] \dots a_0^* J^{-1} h_n \\ &= (\epsilon')^m (-1)^{mn+m(m+1)/2} J \omega_m^\dagger J^{-1} h_n \end{aligned} \quad (3.94)$$

Eq. (3.94) gives the right action of forms on a graded vector space. Importantly, beyond order zero the right action clearly differs up to signs from the right action determined in Eq. 3.67.

Having determined the form of the right action, it is interesting to once again check the associativity conditions given in Eq. (3.68). In particular, Eq. (3.68d) yields:

$$\begin{aligned} [a, h_m, b] &= (\epsilon')^0 (-1)^0 Jb^* J(ah_m) - (\epsilon')^0 (-1)^0 a(Jb^* Jh_m) \\ &= [Jb^* J^*, a]h_m = 0 \end{aligned} \quad (3.95a)$$

$$\begin{aligned} [d[a], h_m, b] &= (\epsilon')^0 (-1)^0 Jb^* J^*(d[a]h) - (\epsilon')^0 (-1)^0 d[a](Jb^* J^{-1}h_m) \\ &= [Jb^* J^*, [D, a]]h_m = 0 \end{aligned} \quad (3.95b)$$

$$\begin{aligned} [b, h_m, d[a]] &= (\epsilon')^1 (-1)^{m+1} Jd[a]^\dagger J^*(bh) - (\epsilon')^1 (-1)^{m+1} b(Jd[a]^\dagger J^{-1}h_m) \\ &= -\epsilon'(-1)^m [J[D, a]^\dagger J^*, b]h_m = 0 \end{aligned} \quad (3.95c)$$

$$\begin{aligned} [d[a], h_m, d[b]] &= (\epsilon')^1 (-1)^{m+2} Jd[b]^\dagger J^*(d[a]h_m) - (\epsilon')^1 (-1)^{m+1} d[a](Jd[b]^\dagger J^*h_m) \\ &= \epsilon'(-1)^m \{[J[D, b]^\dagger J^*, [D, a]]\}h_m = 0 \end{aligned} \quad (3.95d)$$

for $a, b \in \Omega^0 \mathcal{A}$.

As can be seen, both the order zero and order one conditions remain unchanged because the right action of zero forms is unchanged by the grading on the Hilbert space, and because zero forms do not change the grading when acting on elements of $\Omega \mathcal{H}$. Comparing with Eq. (3.70) however, the second order order associativity condition in Eq. (3.95d) differs by a sign, ie. we obtain an anti-commutator condition rather than a commutator condition. As I will describe in Ch. 4, rather surprisingly Eq. (3.70) and Eq. (3.95d) place exactly the same constraints on the finite part of the NCG SM. The big difference is that Eq. (3.95d) is satisfied modulo ‘junk’ by the continuous part of the geometry¹⁵, while Eq. (3.70) is violated. In other words Riemannian geometries, and the NCG SM are associative so long as one takes full account of their grading, and the junk.

Junk forms and $\Omega \mathcal{H}$

The bi-representation of $\Omega \mathcal{A}$ as bounded operators on \mathcal{H} is not a graded representation because $B(\mathcal{H})$ is not graded. To obtain a true graded representation Brouder et al. replace the representation $\pi : \Omega \mathcal{A} \rightarrow B(\mathcal{H})$ with a new representation $\tilde{\pi} : \Omega \mathcal{A} \rightarrow B^\infty(\mathcal{H}) = \bigoplus_n B^n(\mathcal{H})$, where the $B^n(\mathcal{H})$ are copies of $B(\mathcal{H})$. In practice, so long as one keeps track

¹⁵As shown in Eq. (2.59b) for canonical geometries the second order junk consist of symmetric elements of the form $\{[D, a], [D, b]\}$.

of the degree there is no need to distinguish between π and $\tilde{\pi}$ and so I will drop all tildes from this point on.

So far I have not described the graded space $\Omega\mathcal{H}$ on which $\Omega\mathcal{A}$ is represented. The obvious choice is to take multiple copies of \mathcal{H} :

$$\Omega\mathcal{H} = \bigoplus_m \mathcal{H}^m, \quad (3.96)$$

But this choice runs into similar problems as when representing on a single copy of \mathcal{H} : To ensure that the representation π (ie. $\tilde{\pi}$) is a well defined differential graded representation, one must remove the junk forms: $\Omega_D\mathcal{A} = \Omega\mathcal{A}/J$. However, in such a picture one would have to represent over the graded vector space $\Omega\mathcal{H}$, where $\Omega^n\mathcal{H} = \pi(\omega^n\mathcal{A})\mathcal{H}/\pi(\delta[J_0^{n-1}\mathcal{A}])\mathcal{H}$. Often this results in the removal of all elements $\Omega^n\mathcal{H}$ for $n \geq 2$, leaving a Hilbert space with only two levels. This is a priori not an issue, but it does result in the higher order associativity conditions given in Eq. (3.95d) becoming trivial.

3.2.4 The *-DGA $\Omega B = \Omega\mathcal{A} \oplus \Omega\mathcal{H}$

In subsection 3.2.3 the algebra ΩB was constructed as a graded extension of $\Omega_D\mathcal{A}$, following the work of Brouder *et al.* [BBB15]. So far however ΩB has not been equipped with an involution which naturally extends from $\Omega_D\mathcal{A}$, and so it is not a true involutive graded algebra. If one could find a natural extension of the involution on $\Omega_D\mathcal{A}$ compatible with the grading, then presumably one would be able to use it to define the right action of forms, replacing the derivation given in Eq. (3.94), which relies heavily on the associativity of ΩB . In addition if one regards the Dirac operator as the representation of the differential d acting on ΩH , then one still has the problem that D is not a nilpotent operator, ie. $D^2 \neq 0$. The algebra ΩB is therefore not truly a *-DGA. In this Subsection I address both of these problems. What we will find is that once ΩB is made into a true *-DGA, almost *all* of Connes axioms outlined in Subsection 2.2.2 derive from the algebraic properties of ΩB .

The involution on ΩB

Let's consider the involution on ΩB first. We would like to see if we can construct an involution which is compatible with the right action derived in Eq. (3.94). The anti-linear involution on ΩB must satisfy the conditions outlined in Eq. (2.8): $(a_0 a_1)^* = a_1^* a_0^*$. Following Subsection 3.2.1, rather than assume that $*$ has period two as in Eq. (2.8a) (*i.e.*

$a^{**} = a$), we only assume the slightly weaker condition that it has period four $(a^{**})^{**} = a$. The involution on ΩB must be compatible with the $*$ operation on the sub-algebra $\Omega_D \mathcal{A} \subset B$, and also compatible with the $\mathbb{Z}_2 \times \mathbb{Z}$ grading. This fixes the $*$ operation to be

$$(\omega_m + h_n)^* = \omega_m^* + J_n h_n, \quad (3.97)$$

for $\omega_m \in \Omega_D^m \mathcal{A}$, $h_n \in \Omega^n H$, and where $J_n : \Omega^n \mathcal{H} \rightarrow \Omega^n \mathcal{H}$ ¹⁶. Following Eq. (2.8), the J_n must be anti-linear, and following Eq. (2.11), the J_n must also be unitary to ensure that the involution acts as an isometry. This means that without loss of generality we may write:

$$J_n = J_0 \chi_n \quad (3.98)$$

where χ_n is unitary. Compatibility with the derivation d on ΩB (see Eq. (3.60)) then implies further that the involution must satisfy $d(b_m)^* = \kappa(-1)^m d(b_m^*)$, or when acting on $\Omega \mathcal{H}$:

$$J_{n+1} d(h_n) = \kappa(-1)^n d(J_n h_n). \quad (3.99)$$

for $h_n \in \Omega^n \mathcal{H}$. If we adopt the convention:

$$\chi_{n+1} = \pm(-1)^n \chi_n \implies J_n = (-1)^{n(n\mp 1)/2} J_0 \quad (3.100)$$

Then Eq. (3.99) reduces to:

$$d[J_0 h_n] = \pm \kappa J_0 d[h_n], \quad (3.101)$$

from which we infer $\kappa = \pm \epsilon'$. There seem to be two sensible choices for ϵ' . Notice for now that this is exactly the situation which occurs for even K0-signature, as shown in table 2.1 [DD11].

So much for the involution on elements of $\Omega \mathcal{H}$, let us next determine the form that the involution should take on elements of $\Omega_D \mathcal{A}$. Once again, following Eq. (3.60), compatibility with the differential d implies:

$$\begin{aligned} d[a]^* &= \kappa d[a^*] \\ &= \pm \epsilon' [D, L_a^\dagger] = \mp \epsilon' [D^\dagger, L_a]^\dagger, \end{aligned} \quad (3.102)$$

for $a \in \mathcal{A}$, and where \dagger will depend on the algebra representation in question. For associative matrix algebras it will indicate hermitian conjugation. For matrices of octonions it

¹⁶Later on we will consider the interpretation in which the grading on $\Omega \mathcal{H}$ is given by γ , in which case we would need to relax this condition when $\epsilon'' = -1$.

will indicate transpose octonionic conjugation, etc. If additionally we take the condition $D = D^\dagger$, then the involution on higher order forms may be written compactly as:

$$\omega_n^* = (\mp\epsilon')^n \omega_n^\dagger \quad (3.103)$$

Having constructed an involution on both $\Omega_D \mathcal{A}$, and on $\Omega \mathcal{H}$, we are now in a position to construct a right action of forms by making use of the involution property $h_m \omega_n = (\omega_n^* h_m^*)^*$ (see Eq. (2.8)). The right action is then given by:

$$\begin{aligned} h_m \omega_n &= J_{n+m} \omega_n^* J_m h_m \\ &= (\mp\epsilon')^n (-1)^{(m+n)(m+n\mp 1)/2 + m(m\mp 1)/2} J_0 \omega_n^\dagger J_0^{-1} h_m \\ &= (\epsilon')^n (-1)^{n(n+1)/2} J_0 \omega_n^\dagger J_0^{-1} h_m \end{aligned} \quad (3.104)$$

which is exactly the same expression for the right action of forms as was determined in Eq. (3.94).

Junk forms

Having introduced the algebra ΩB , let us next take the ‘obvious’ choice for the bi-representation, which is to take multiple copies of \mathcal{H} : $\Omega \mathcal{H} = \bigoplus_m \mathcal{H}^m$. Let us introduce some matrix notation to describe the representation of the input algebra $\Omega^0 \mathcal{A}$ on $\Omega \mathcal{H}$:

$$\pi_{\mathcal{H}}(a) = \begin{pmatrix} \ddots & & & & \\ & \pi_3(a) & 0 & 0 & 0 \\ & 0 & \pi_2(a) & 0 & 0 \\ & 0 & 0 & \pi_1(a) & 0 \\ & 0 & 0 & 0 & \pi_0(a) \end{pmatrix}, \quad (3.105)$$

where $a \in \mathcal{A}$ and for now I will take $\pi = \pi_n$, for all n . The differential operator and real structure on $\Omega \mathcal{H}$ are given by $d[\omega_m + h_n] = d[\omega_m] + d_{\mathcal{H}} h_n$, and $(\omega_m + h_n)^* = \omega_m^* + J h_n$, where

$$d_{\mathcal{H}} = \begin{pmatrix} \ddots & & & & \\ & 0 & d_2 & 0 & 0 \\ & 0 & 0 & d_1 & 0 \\ & 0 & 0 & 0 & d_0 \\ & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} \ddots & & & & \\ & J_3 & 0 & 0 & 0 \\ & 0 & J_2 & 0 & 0 \\ & 0 & 0 & J_1 & 0 \\ & 0 & 0 & 0 & J_0 \end{pmatrix}, \quad (3.106)$$

and $J_n = (-1)^{n(n+1)/2} J_0$. For now the interpretation is that $D = d_0 = d_1 = d_2 \dots$ is a Dirac operator which satisfies the properties outlined in Subsection 2.2.2, such that the representation of exact one forms on $\Omega\mathcal{H}$ is given by:

$$\pi_{\Omega\mathcal{H}}(d[a]) \equiv [d_{\mathcal{H}}, L_{\pi_{\Omega\mathcal{H}}(a)}] = \begin{pmatrix} \ddots & & & & \\ & 0 & [D, L_{\pi(a)}] & 0 & 0 \\ & 0 & 0 & [D, L_{\pi(a)}] & 0 \\ & 0 & 0 & 0 & [D, L_{\pi(a)}] \\ & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.107)$$

$a \in \mathcal{A}$. We are also in the midst of analyzing an alternative interpretation in which each d_i is actually ‘half’ of a Dirac operator such that $D = d_{\mathcal{H}} + d_{\mathcal{H}}^*$, which I will discuss shortly in this section. Notice that while zero forms are represented along the diagonal, one forms are adjacent to the diagonal, two forms will likewise be two from diagonal, etc. This is nothing but the statement that the representation is graded, $\pi : \Omega\mathcal{A} \rightarrow B^\infty(\mathcal{H})$.

To explore the junk, let’s consider as an example the canonical spectral data corresponding to Riemannian geometries: $M = \{C^\infty(M), L^2(M, S), \not{D}, \gamma^0 \gamma^2 \circ cc, \gamma_5\}$. For a graded representation the one-form elements $\{\omega \in \Omega^1\mathcal{A} | \pi(\omega) = 0\}$ will be of the form $\omega = fd[g] - d[g]f$, for $f, g \in \mathcal{A}$ and so the representation of the junk is given by:

$$\pi(\omega) = \begin{pmatrix} \ddots & & & & \\ & 0 & [\pi(f), [D, \pi(g)]] & 0 & 0 \\ & 0 & 0 & [\pi(f), [D, \pi(g)]] & 0 \\ & 0 & 0 & 0 & [\pi(f), [D, \pi(g)]] \\ & 0 & 0 & 0 & 0 \end{pmatrix} = 0, \quad (3.108)$$

$$\pi(d[\omega]) = \begin{pmatrix} \ddots & & & & \\ & 0 & 0 & \{[D, \pi(f)], [D, \pi(g)]\} & 0 \\ & 0 & 0 & 0 & \{[D, \pi(f)], [D, \pi(g)]\} \\ & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.109)$$

We cannot simply represent $\Omega_D\mathcal{A} = \Omega\mathcal{A}/J$ on $\Omega\mathcal{H}$ because the representation would not in general be well defined. Instead we must represent on $\pi(\Omega^n)\mathcal{H}/\pi(dJ_0^{n-1})\mathcal{H}$. For canonical NCGs this quotient is trivial for $n \geq 2$.

Re-interpreting D

Having introduced the graded vector space ΩH , one immediately starts to wonder what the physical significance of the grading is. Is the grading ‘invisible’ aside from the higher order conditions as suggested by Brouder *et al.* [BBB15]? or could it be related to the number of particle generations? Or, if the grading really does only have two levels, is this the same grading as that singled out by the γ operator? Let us explore this last option: Brouder *et al.* avoid representing $\Omega_D \mathcal{A}$ on copies of \mathcal{H} because at least in the cases of physical interest (Riemannian geometries and the NCG SM) the quotient $\mathcal{H}/\pi(dJ_0^{n-1})\mathcal{H}$ is trivial for $n \geq 2$. This situation is not necessarily undesirable however; far from being a problem we think this might be a feature. It means the differential operator is nil-potent up to junk: $d_{\mathcal{H}}^2 h_m = 0 +$ ‘junk’ for $h_m \in \Omega \mathcal{H}$, which means that ΩB is truly a $*$ -DGA. Let us therefore take seriously the two level grading, and assign to it a \mathbb{Z}_2 grading operator $\gamma = \text{diag}(-\mathbb{I}, \mathbb{I})$. Note that this automatically implies two more standard NCG assumptions $\{d_{\mathcal{H}}, \gamma\} = 0$, and $[a, \gamma] = [Ja^*J^{-1}, \gamma] = 0$.

With this interpretation let us consider once again the canonical NCG input data $\{C^\infty(M), \not{D}, L^2(M, S), J, \gamma_5\}$, but this time in a different guise. Take the representation

$$\pi_{\mathcal{H}}(f) = \begin{pmatrix} \ddots & & & & \\ & f\mathbb{I}_2 & 0 & 0 & 0 \\ & 0 & f\mathbb{I}_2 & 0 & 0 \\ & 0 & 0 & f\mathbb{I}_2 & 0 \\ & 0 & 0 & 0 & f\mathbb{I}_2 \end{pmatrix}, \quad (3.110)$$

on the vector space $\Omega \mathcal{H} = \bigoplus_m \mathcal{H}^m$, where $\mathcal{H} = \mathbb{C}^2$ is the space of 2-component Weyl spinors on M . Once again take the differential and involution operations on $\Omega \mathcal{H}$ to be given by

$$d_{\mathcal{H}} = \begin{pmatrix} \ddots & & & & \\ & 0 & \partial & 0 & 0 \\ & 0 & 0 & \partial & 0 \\ & 0 & 0 & 0 & \partial \\ & 0 & 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} \ddots & & & & \\ & J_3 & 0 & 0 & 0 \\ & 0 & J_2 & 0 & 0 \\ & 0 & 0 & J_1 & 0 \\ & 0 & 0 & 0 & J_0 \end{pmatrix}, \quad (3.111)$$

where $J_0 = i\sigma_2 \circ cc$, $J_n = (-1)^{n(n\mp 1)/2} J_0$; and $\partial = \mathbb{I}_2 \partial_0 - i\sigma^i \partial_i$. After modding out by junk we find once again that $\Omega^n \mathcal{H} = \mathcal{H}/\pi(dJ_0^{n-1})\mathcal{H}$ is trivial for $n \geq 2$. We therefore obtain a

differential graded algebra where $\Omega\mathcal{H} \simeq L^2(M, S)$, and:

$$\begin{aligned} \pi_{\mathcal{H}}(f) &\simeq \begin{pmatrix} f\mathbb{I}_2 & 0 \\ 0 & f\mathbb{I}_2 \end{pmatrix}, & d_{\mathcal{H}} &\simeq \begin{pmatrix} 0 & \partial \\ 0 & 0 \end{pmatrix}, \\ J_{\pm} &\simeq \begin{pmatrix} \pm J_0 & 0 \\ 0 & J_0 \end{pmatrix}, & \gamma &\simeq \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}. \end{aligned} \quad (3.112)$$

The choice of sign for the real structure corresponds to the convention selected in Eq. (3.100). The two choices relate via the map $J_+ = \gamma J_-$, and following Table 2.1 both correspond to a geometry with KO-signature 4. The Dirac operator is then constructed as:

$$\mathcal{D} = d_{\mathcal{H}} + d_{\mathcal{H}}^{\dagger} \quad (3.113)$$

As a final remark, notice that if we wished to tensor together two fused $*$ -DGAs, to form a new fused $*$ -DGA (ie. a product space), then the naive order one differential operator $d_T = d_1 \otimes \mathbb{I}_2 + \mathbb{I}_1 \otimes d$ would not satisfy the nilpotency condition $d_T^2 = 0$:

$$\begin{aligned} d_T^2[\omega_n \otimes \omega'_m] &= d_T[d_1[\omega_n] \otimes \omega'_m + \omega_n \otimes d_2[\omega'_m]] \\ &= d_1^2[\omega_n] \otimes \omega'_m + d_1[\omega_n] \otimes d_2[\omega'_m] + d_1[\omega_n] \otimes d_2[\omega'_m] + \omega_n \otimes d_2^2[\omega'_m] \\ &= d_1[\omega_n] \otimes d_2[\omega'_m] + d_1[\omega_n] \otimes d_2[\omega'_m] \end{aligned} \quad (3.114)$$

for $\omega_n \otimes \omega'_m \in \Omega B_T = \Omega B_1 \otimes \Omega B_2$. Instead, one must define the graded differential operator: $d_T = d_1 \otimes \mathbb{I}_2 + (-1)^n \mathbb{I}_1 \otimes d_2$ acting on an element $\omega_n \otimes \omega'_m \in \Omega B_T = \Omega B_1 \otimes \Omega B_2$. With this definition in place the total Dirac operator is given by:

$$D_T = d_T + d_T^{\dagger} = D_1 \otimes \mathbb{I}_2 + \gamma_1 \otimes D_2 \quad (3.115)$$

Which recovers the standard form for the Dirac operator on the tensor product of two spectral triples, giving further evidence for the consistency of this interpretation. With this interpretation, the simple statement that ΩB is an associative $*$ -DGA reproduces the following NCG axioms and assumptions as output:

1. The associativity of the algebra $\Omega_D \mathcal{A}$, which results from the associativity of ΩB .
2. The associativity of the left and right actions of $\Omega_D \mathcal{A}$ on $\Omega \mathcal{H}$, which results from the associativity of ΩB .
3. The extension of the involution on \mathcal{A} to the whole of ΩB gives a raison d'être for J and the bi-module structure of ΩH on $\Omega_D A$.

4. The right action of forms on ΩH , arising from the grading compatible involution on ΩB .
5. The order zero condition $[a, Jb^*J^{-1}] = 0$, which results from the associativity of ΩB .
6. The order one condition $[[D, a], Jb^*J^{-1}] = 0$, which results from the associativity of ΩB .
7. New higher order conditions which replace the ‘massless photon condition’ introduced in [CCM07] (in the case where the Hilbert space elements of degree 2 are non-trivial after removing the junk).
8. The differential graded structure, and the removal of junk gives a raison d’etre for the introduction of a \mathbb{Z}_2 grading operator.
9. The conditions $\gamma^2 = 1$, $\gamma^{-1} = \gamma^\dagger$, $\{D, \gamma\} = 0$.
10. The hermiticity of D , is built in: $D \equiv d_{\mathcal{H}} + d_{\mathcal{H}}^* \Rightarrow D = D^*$
11. The sign in the real structure condition $J^2 = \epsilon \mathbb{I}$, ($\epsilon = \pm 1$) is linked to the period of the involution. The condition $JD = \epsilon' DJ$ is linked to the choice of sign κ in the compatibility condition $d[\omega_n^*] = \kappa(-1)d[\omega_n]^2$ given in Eq. (2.17).
12. The formula for the Dirac operator on a tensored space $D_T = D_1 \otimes \mathbb{I}_2 + \gamma_1 \otimes D_2$ derives from the requirement that the differential operator d_T square to zero.
13. The symmetries of a NCG arise cleanly as the automorphisms of ΩB (discussed below in Sec. 3.3).

Finally, if this interpretation is correct, then one notices two additional interesting points:

1. It is natural to consider the ‘right action’ of all operators acting on the graded space $\Omega \mathcal{H}$. If one does so, one finds:

$$R_J = JJ^*J^{-1} = \epsilon J, \tag{3.116a}$$

$$R_D = JD^*J^{-1} = \epsilon' D, \tag{3.116b}$$

$$R_\gamma = J\gamma^*J^{-1} = \epsilon'' \gamma. \tag{3.116c}$$

where the epsilons correspond to those in the KO-signature table 2.1. It is not clear what meaning is behind this observation however.

2. This construction seems to single out geometries with even KO-signature. In addition, a true $*$ -DGA should really be equipped with an involution of period 2. If we want D to be of order 1 and J to be of order zero this further requires that the involution should respect the grading in the sense that $[J, \gamma] = 0$. In other words the construction seems to single out the KO-signature 0. This happens to be exactly the KO-signature of the Lorentzian NCG SM [Bar07]. This may be an intriguing hint as to why the universe has the dimension that it does!

3.3 Symmetries in non-associative NCG.

The ‘fused algebra’ or ‘square zero extension’ formulation of NCG outlined in Section 3.2 replaces the algebra \mathcal{A} as the main piece of input data with a full $*$ -DGA ΩB which naturally incorporates all five elements of the spectral triple $\{\mathcal{A}, \mathcal{H}, D, J, \gamma\}$. From this perspective the symmetries of a NCG are not described by the automorphisms of \mathcal{A} , but by the automorphisms of the full differential graded algebra ΩB . This should not come as a surprise to the reader, as even in the traditional approach the symmetries of a NCG are not given exactly by the automorphisms of \mathcal{A} . To elucidate this point consider for example the symmetries of the finite NCG SM input algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. While the $*$ -automorphisms of $M_3(\mathbb{C})$ are $SU(3)$, the $*$ -automorphisms of \mathbb{C} are not $U(1)$ but merely \mathbb{Z}_2 , and the $*$ -automorphisms of the quaternion algebra \mathbb{H} are $SO(3)$. For this reason non-commutative geometers have had to identify the gauge group with the unitary group of the input algebra despite the fact that the automorphisms gave the original motivation [CCM07]. One of the motivating ideas behind NCG was that the symmetries of a geometry should arise as the automorphisms of the input algebra. In this Section I will introduce the idea that the symmetries of a NCG are not the automorphisms of \mathcal{A} , but instead they arise simply and naturally as the automorphisms of the full $*$ -DGA ΩB .

3.3.1 The automorphisms of B_0 .

Let’s start by first considering the automorphisms of the sub-algebra $B_0 = \mathcal{A} \oplus \Omega \mathcal{H} \subset \Omega B$ ¹⁷, i.e. the invertible linear transformations $\alpha : B_0 \rightarrow B_0$ that also preserve the grading,

¹⁷Notice that I now include the whole graded Hilbert space in B_0 , because I am working with the interpretation introduced in Subsection 3.2.4, that the grading on $\Omega \mathcal{H}$ corresponds to the grading γ . However, even if one does not take this interpretation, the content of this Subsection remains unchanged because the zero forms do not change the degree when acting on an element of $\Omega \mathcal{H}$.

product and $*$ -operation on B_0 :

$$\alpha = \alpha_{\mathcal{A}} \oplus \alpha_{\mathcal{H}}, \quad (3.117a)$$

$$\alpha(bb') = \alpha(b)\alpha(b'), \quad (3.117b)$$

$$\alpha(b^*) = \alpha(b)^*, \quad (3.117c)$$

where $\alpha_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$, $\alpha_{\mathcal{H}} : \Omega\mathcal{H} \rightarrow \Omega\mathcal{H}$, and $b, b' \in B_0$. The direct sum notation in Eq. (3.117a), means that $\alpha a = \alpha_{\mathcal{A}} a$ when $a \in \mathcal{A}$, and $\alpha h = \alpha_{\mathcal{H}} h$ when $h \in \Omega\mathcal{H}$. For an even geometry, the fact that diffeomorphisms are either orientation-preserving or orientation-reversing translates to the NCG condition that $\alpha_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{H}$ either commutes or anti-commutes with the orientation γ . We are interested here in the orientation or grading preserving automorphisms, which translates to the condition (see also Eq. (3.19a)):

$$[\alpha_{\mathcal{H}}, \gamma] = 0. \quad (3.117d)$$

The automorphisms $\alpha_{\mathcal{H}}$ must leave the action given in Eq. (1.1) invariant. As outlined in Eq. (2.84), the fluctuated Dirac operator of a NCG transforms under the action of $\alpha_{\mathcal{H}}$ by conjugation $D_A \rightarrow D'_A = \alpha_{\mathcal{H}} D_A \alpha_{\mathcal{H}}^{-1}$. As the spectral action depends only on the spectrum of the Dirac operator D_A , this tells us that the maps $\alpha_{\mathcal{H}}$ on \mathcal{H} must be unitary such that the spectrum of D_A is fixed:

$$\alpha_{\mathcal{H}}^\dagger = \alpha_{\mathcal{H}}^{-1}. \quad (3.117e)$$

The conditions (3.117a, 3.117b, 3.117c, 3.117d, 3.117e) on the automorphism $\alpha = e^\delta$ are readily translated into conditions on its infinitesimal generator, the derivation δ :

$$\delta = \delta_{\mathcal{A}} \oplus \delta_{\mathcal{H}}, \quad (3.118a)$$

$$\delta(bb') = \delta(b)b' + b\delta(b'), \quad (3.118b)$$

$$\delta(b^*) = \delta(b)^*, \quad (3.118c)$$

$$[\delta_{\mathcal{H}}, \gamma] = 0, \quad (3.118d)$$

$$\delta_{\mathcal{H}}^\dagger = -\delta_{\mathcal{H}}. \quad (3.118e)$$

where $\delta_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$, $\delta_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{H}$, and $b, b' \in B_0$. By the direct sum notation in Eq. (3.118a), I mean that $\delta a = \delta_{\mathcal{A}} a$ when $a \in \mathcal{A}$, and $\delta h = \delta_{\mathcal{H}} h$ when $h \in \mathcal{H}$.

It is often more elucidating to work with the derivations δ rather than with the automorphisms they generate. To see what constraints the conditions given in Eq. (3.118) place on the form that the derivations take, consider the case where a derivation element

δ acts on an element $ah \in \mathcal{H}$, where $a \in \mathcal{A}$ and $h \in \mathcal{H}$:

$$\begin{aligned}\delta_{\mathcal{H}}(ah) &= \delta_{\mathcal{A}}(a)h + a\delta_{\mathcal{H}}h \\ \Rightarrow [\delta_{\mathcal{H}}, L_a] &= L_{\delta_{\mathcal{A}}a}\end{aligned}\tag{3.119a}$$

$$\begin{aligned}\delta_{\mathcal{H}}(ha) &= \delta_{\mathcal{H}}(h)a + h(\delta_{\mathcal{H}}a) \\ \Rightarrow [\delta_{\mathcal{H}}, R_a] &= R_{\delta_{\mathcal{A}}a}\end{aligned}\tag{3.119b}$$

In other words, for a graded derivation $\delta = \delta_{\mathcal{A}} \oplus \delta_{\mathcal{H}}$, the elements $\delta_{\mathcal{A}}$ and $\delta_{\mathcal{H}}$ are not arbitrary with respect to one another, but are related by Eq. (3.119). This is extremely useful in practice as the form that the derivation elements take on \mathcal{A} is usually well known. Further constraint can also be placed on $\delta_{\mathcal{H}}$ by use of Eq. (3.118c), which implies:

$$[\delta_{\mathcal{H}}, J] = 0.\tag{3.120}$$

Examples

1. So far the discussion of symmetries has not relied in any way on the associativity of either \mathcal{A} or of B_0 . It is instructive however to consider as an example the derivations of the sub-algebra B_0 corresponding to a finite associative NCG. All the derivations of a finite semi-simple algebra are inner [Jac49, Sch49], and as the input algebra \mathcal{A} is associative all of its derivations are well known, and of the form given in Eq. (2.28a). The derivations on B_0 may therefore all be written in the form:

$$\begin{aligned}\delta &= \delta_{\mathcal{A}} \oplus \delta_{\mathcal{H}} \\ &= (L_x - R_x) \oplus \delta_{\mathcal{H}},\end{aligned}\tag{3.121}$$

where $x^* = -x \in \mathcal{A}$. The conditions given in Eq. (3.119) together with the associative order zero condition (2.68) then further restricts the derivations on B_0 to be of the form:

$$\delta = (L_x - R_x) \oplus (x - Jx^*J^{-1} + T),\tag{3.122}$$

where $x^* = -x \in \mathcal{A}$, and T is an operator on \mathcal{H} which satisfies:

$$[T, L_a] = [T, R_a] = [T, J] = [T, \gamma] = 0, \quad T^* = -T.\tag{3.123}$$

If the conditions given in Eq. (3.123) are satisfied, then $\delta_{\mathcal{H}}$ automatically satisfies conditions Eq. (3.118).

Comparing Eq. (3.125) with Eq. (2.76), it is clear that the inner derivations of B_0 reproduce the infinitesimal generators of Connes, but there are also additional generators of the form T . These derivations generate ‘central extensions’ in the sense of Schücker [Sch00, LS01], although in this case they may be non-abelian. The reason that B_0 is able to have derivations which are not inner is because even when \mathcal{A} is semi-simple, B_0 is not (because \mathcal{H} is a non-trivial nilpotent ideal).

2. Consider as a second example the derivations of the sub-algebra B_0 corresponding to a finite alternative NCG (that is where B_0 is an alternative algebra). As a finite semi-simple alternative algebra, all derivations of the input algebra \mathcal{A} are well known, and of the form given in Eq. (2.28d) [Jac49, Sch49]. The derivations on B_0 may therefore all be written in the form:

$$\begin{aligned}\delta &= \delta_{\mathcal{A}} \oplus \delta_{\mathcal{H}} \\ &= ([L_x, L_y] + [L_x, R_y] + [R_x, R_y]) \oplus \delta_{\mathcal{H}},\end{aligned}\tag{3.124}$$

where $x^* = -x, y^* = -y \in \mathcal{A}$. Conditions Eq. (3.119) together with the alternative order zero conditions (3.12) then further restricts the derivations on B_0 to be of the form:

$$\delta = ([L_x, L_y] + [L_x, R_y] + [R_x, R_y]) \oplus ([L_x, L_y] - [L_x, JL_yJ^{-1}] + J[L_x, L_y]J^{-1} + T),\tag{3.125}$$

where $x^* = -x, y^* = -y \in \mathcal{A}$, and T is once again an operator on \mathcal{H} which satisfies the conditions given in Eq. (3.123). If the conditions given in Eq. (3.123) are satisfied, then $\delta_{\mathcal{H}}$ automatically satisfies conditions Eq. (3.118).

Now that we have discussed the automorphisms of B_0 we can once again ask the question: ‘so what is the purpose behind the order zero condition, and does it matter that the associative order zero condition isn’t satisfied in general when the input algebra is non-associative?’ As described in Sec 3.1 the main purpose of the order zero condition is to ensure automorphism covariance. From the perspective of our reformulation the associative order zero condition simply ensures that B_0 is an associative algebra so that its inner automorphisms are generated by inner derivations which take the associative form given in Eq. (2.28a). More generally, the order zero conditions determine the properties of B_0 , and in turn the form that its inner derivations take.

3.3.2 The automorphisms of ΩB

In practice the symmetries of a NCG will be given not just by the automorphisms of the sub-algebra B_0 , but of the full $*$ -DGA ΩB . The $*$ -automorphisms of ΩB will once again satisfy the conditions given in Eq. (3.117) however they now act on graded spaces in the sense:

$$\alpha_{\mathcal{A}} = \alpha_{\mathcal{A}}^0 \oplus \alpha_{\mathcal{A}}^1 \oplus \dots \quad (3.126a)$$

$$\alpha_{\mathcal{H}} = \alpha_{\mathcal{H}}^0 \oplus \alpha_{\mathcal{H}}^1 \oplus \dots \quad (3.126b)$$

Similarly the automorphism generating derivations still satisfy the conditions given in Eq. (3.118), but now act on graded spaces in the sense:

$$\delta_{\mathcal{A}} = \delta_{\mathcal{A}}^0 \oplus \delta_{\mathcal{A}}^1 \oplus \dots \quad (3.126c)$$

$$\delta_{\mathcal{H}} = \delta_{\mathcal{H}}^0 \oplus \delta_{\mathcal{H}}^1 \oplus \dots \quad (3.126d)$$

In general, the derivations of ΩB will be more more restricted than the derivations of B_0 , because in addition to the conditions given in Eq. (3.119), the elements $\delta_{\mathcal{H}}$ on ΩB will have to also satisfy conditions of the form:

$$[\delta_H, L_{\omega_n}] = L_{\delta_{\mathcal{A}}^n \omega_n} \in L_{\pi(\Omega_D^n \mathcal{A})} \quad (3.127a)$$

$$[\delta_H, R_{\omega_n}] = R_{\delta_{\mathcal{A}}^n \omega_n} \in R_{\pi(\Omega_D^n \mathcal{A})} \quad (3.127b)$$

where $\omega_n \in \Omega_D^n \mathcal{A}$, and no sum is implied.

So far, we have characterized the *classical* symmetries associated to a given NCG; these will generate the symmetries of the corresponding classical gauge theory obtained from the spectral action. In order for these gauge symmetries to remain consistent at the quantum level, they must also be anomaly free. If $\{\delta_{\mathcal{H}}^{\alpha}\}$ denotes a basis for the space of all operators $\delta_{\mathcal{H}}$ obtained by satisfying the restrictions (3.118), then anomaly freedom corresponds to the additional constraint

$$\text{Tr}[\gamma \delta_{\mathcal{H}}^{\alpha} \{\delta_{\mathcal{H}}^{\beta}, \delta_{\mathcal{H}}^{\gamma}\}] = 0 \quad (3.128)$$

for any basis elements $\delta_{\mathcal{H}}^{\alpha}$, $\delta_{\mathcal{H}}^{\beta}$ and $\delta_{\mathcal{H}}^{\gamma}$ – see Eq. (20.81) in Ref. [PS95]. In contrast to the classical constraints (3.118), we do not know if the quantum constraint (3.128) has a more fundamental geometric reinterpretation in our formalism.

Chapter 4

Non-associative geometry and particle physics applications

In this Chapter I will explain in detail how the SM is constructed as an almost-commutative geometry, and discuss NCG SM phenomenology. In particular I will focus on problems in the NCG construction, and the solutions that our reformulation proposes. The organization is as follows: in Section 4.1 I introduce the finite dimensional input data for the NCG SM, focusing on the constraints in the Higgs sector which arise from the traditional NCG geometric axioms. I will also review the ‘massless photon’ constraint which must be imposed in the traditional approach to NCG in order to single out the SM higgs sector uniquely [CCM07]. In Subsection 4.1.2 I will show explicitly how to fluctuate the NCG SM Dirac operator to obtain the $SU_c(3) \times SU_w(2) \times U_y(1)$ gauge fields, and the SM electroweak Higgs field. In Subsection 4.1.3 I will describe the higher order associativity conditions which we first introduced in [BF14], and I will show explicitly how they constrain the NCG SM Higgs sector, replacing the ad hoc ‘massless photon condition’. In Subsection 4.1.4 I will explain how the incorrect 170GeV Higgs mass prediction was made within the framework of NCG. Having discussed the NCG SM in Section 4.1, in Section 4.2 I will describe a viable extension of the NCG SM which we first introduced in [FB15b], and which arises naturally in the fused algebra formulation of NCG. This extension is similar to that which we discussed in [BFFS11] outside of the context of NCG, and features the SM symmetry group extended by a gauged $U(1)$ baryon-lepton number symmetry (B-L), which also fluctuates an extended Higgs sector. In Section 4.2.2 I will review how the gauged B-L extension of the NCG SM does not suffer from the same Higgs mass problem as the NCG SM. In Section 4.2.3 I will discuss an attempted solution to the Weinberg angle problem.

As a final note, many of the calculations discussed in this section are currently underway. This is particularly true of Subsections 4.1.4 and 4.2.2 where I calculate the Higgs boson mass for the standard model, and for a standard model extension, and also in Subsection 4.2.3 where I discuss the Weinberg angle problem. These sections are included to provide a snapshot of my current work.

4.1 The NCG SM: input data and particle content

The standard model is described by an almost commutative geometry which is constructed as the product between a finite non-commutative spectral triple with a continuous commutative spectral triple $M \times F$. In the following construction I will deal with a single-generation of standard model fermions; the extension to the full set of three generations is straightforward. For further information the reader may refer to [Con06b, CC08, CC07, BBB15, BF14, CC12, vdDvS12, vS15].

4.1.1 The finite input data

The finite geometry corresponding to the NCG SM is described by a real even spectral triple $F = \{A_F, H_F, D_F, J_F, \gamma_F\}$ of KO-signature 6 [Bar07, Con06b]. The finite input algebra \mathcal{A}_F is given by the real $*$ -algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, where \mathbb{C} is the algebra of complex numbers, \mathbb{H} is the algebra of quaternions, and $M_3(\mathbb{C})$ is the algebra of 3×3 complex matrices. The finite Hilbert space \mathcal{H}_F is a 32-dimensional complex Hilbert space, where the dimensionality relates to the number of fermionic degrees of freedom in one standard model generation, including the right-handed neutrino (six quarks and two leptons, each with a choice of chirality and a corresponding anti-particle, $(6 + 2) \times 2 \times 2 = 32$). To describe the action of the grading γ_F and real structure operator J_F on the Hilbert space \mathcal{H}_F , it is convenient to split H_F into four 8-dimensional subspaces $\mathcal{H}_F = \mathcal{H}_R \oplus \mathcal{H}_L \oplus \bar{\mathcal{H}}_R \oplus \bar{\mathcal{H}}_L$. The spaces \mathcal{H}_R and \mathcal{H}_L contain the right-handed and left-handed particles respectively, while $\bar{\mathcal{H}}_R$ and $\bar{\mathcal{H}}_L$ contain their corresponding anti-particles. If $h_R \in \mathcal{H}_R$ is a right-handed particle (with $\bar{h}_R \in \bar{\mathcal{H}}_R$ the corresponding anti-particle) and $h_L \in \mathcal{H}_L$ is a left-handed particle (with $\bar{h}_L \in \bar{\mathcal{H}}_L$ the corresponding anti-particle), then the finite helicity operator γ_F and the anti-linear charge conjugation operator J_F act as follows:

$$\begin{aligned} \gamma_F h_R &= -h_R, & \gamma_F h_L &= h_L, & \gamma_F \bar{h}_R &= \bar{h}_R, & \gamma_F \bar{h}_L &= -\bar{h}_L, \\ J_F h_R &= \bar{h}_R, & J_F h_L &= \bar{h}_L, & J_F \bar{h}_R &= h_R, & J_F \bar{h}_L &= h_L. \end{aligned} \tag{4.1}$$

To describe the action of \mathcal{A}_F on \mathcal{H}_F , it is convenient to further split each of the four spaces $(\mathcal{H}_R, \mathcal{H}_L, \bar{\mathcal{H}}_R, \bar{\mathcal{H}}_L)$ into a lepton and quark subspace: $\mathcal{H}_R = L_R \oplus Q_R$, $\mathcal{H}_L = L_L \oplus Q_L$, $\bar{\mathcal{H}}_R = \bar{L}_R \oplus \bar{Q}_R$, and $\bar{\mathcal{H}}_L = \bar{L}_L \oplus \bar{Q}_L$. Each of the four lepton spaces $\{L_R, L_L, \bar{L}_R, \bar{L}_L\}$ is a copy of \mathbb{C}^2 ; an element of any of these four spaces correspondingly carries a doublet (neutrino vs. electron) index. Each of the four quark spaces $\{Q_R, Q_L, \bar{Q}_R, \bar{Q}_L\}$ is a copy of $\mathbb{C}^2 \otimes \mathbb{C}^3$; an element of any one of these four spaces correspondingly carries two indices: a doublet (up quark vs. down quark) index and a triplet (color) index. Now consider an element $a = \{\lambda, q, m\} \in \mathcal{A}_F$, where $\lambda \in \mathbb{C}$ is a complex number, $q \in \mathbb{H}$ is a quaternion, and $m \in M_3(\mathbb{C})$ is a 3×3 complex matrix, and write

$$q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}, \quad (4.2)$$

where α and β are complex numbers. Here q is the standard 2×2 complex matrix representation of a quaternion, and q_λ is the corresponding diagonal embedding of \mathbb{C} in \mathbb{H} . Then L_a (the left action of a on \mathcal{H}) is given by

$$\begin{aligned} L_a L_R &= q_\lambda L_R, & L_a L_L &= q L_L, \\ L_a Q_R &= q_\lambda Q_R, & L_a Q_L &= q Q_L, \\ L_a \bar{L}_R &= \lambda \mathbb{I}_{2 \times 2} \bar{L}_R, & L_a \bar{L}_L &= \lambda \mathbb{I}_{2 \times 2} \bar{L}_L, \\ L_a \bar{Q}_R &= m \bar{Q}_R, & L_a \bar{Q}_L &= m \bar{Q}_L. \end{aligned} \quad (4.3)$$

where q , q_λ and $\lambda \mathbb{I}_{2 \times 2}$ act on the doublet index, while m acts on the color index.

As outlined in Subsection 2.2.2, the finite Dirac operator D_F obeys the following four geometric constraints in KO-dimension 6: $D_F^\dagger = D_F$, $\{D_F, \gamma_F\} = 0$, $[D_F, J_F] = 0$ and $[[D_F, L_a], R_b] = 0$. In the basis $\{L_R, Q_R, L_L, Q_L, \bar{L}_R, \bar{Q}_R, \bar{L}_L, \bar{Q}_L\}$, these imply

$$D_F = \begin{pmatrix} 0 & 0 & y_l^\dagger & 0 & m^\dagger & n^\dagger & 0 & 0 \\ 0 & 0 & 0 & y_q^\dagger & \bar{n} & 0 & 0 & 0 \\ \hline y_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & y_q & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline m & n^T & 0 & 0 & 0 & 0 & y_l^T & 0 \\ n & 0 & 0 & 0 & 0 & 0 & 0 & y_q^T \\ \hline 0 & 0 & 0 & 0 & \bar{y}_l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{y}_q & 0 & 0 \end{pmatrix} \quad (4.4)$$

where

$$y_l = \begin{pmatrix} y_{l,11} & y_{l,12} \\ y_{l,21} & y_{l,22} \end{pmatrix} \quad \text{and} \quad y_q = \begin{pmatrix} y_{q,11} & y_{q,12} \\ y_{q,21} & y_{q,22} \end{pmatrix} \quad (4.5)$$

are arbitrary 2×2 matrices that act on the doublet indices in the lepton and quark sectors, respectively, while

$$m = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix} \quad \text{and} \quad n = \begin{pmatrix} \mathbf{c} & \mathbf{d} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (4.6)$$

are 2×2 and 6×2 matrices, respectively; and in n I have used bold notation to emphasize that \mathbf{c} , \mathbf{d} and $\mathbf{0}$ are 3×1 columns. Of the 8 complex parameters $\{a, b, \mathbf{c}, \mathbf{d}\}$, only a is present in the standard model (where it corresponds to the right-handed neutrino's majorana self-coupling). The remaining 7 parameters $\{b, \mathbf{c}, \mathbf{d}\}$ present a puzzle – they are an unwanted blemish that must be removed in order to match observations. If they are not removed, they result in Higgs $SU_c(3)$ triplets when the Dirac operator is fluctuated, which are ruled out experimentally. Further, on fluctuating D the terms y_l and y_q will only result in the correct SM $SU_2(2)$ Higgs doublet if they satisfy the additional conditions:

$$y_{l,11}\overline{y_{l,12}} = -y_{l,21}\overline{y_{l,22}}, \quad y_{q,11}\overline{y_{q,12}} = -y_{q,21}\overline{y_{q,22}} \quad (4.7)$$

Traditionally, the seven unwanted terms in D_F are removed by introducing an extra non-geometric assumption (non-geometric in the sense that no similar constraint is satisfied by the continuous geometry) known as the ‘massless photon condition’ [CCM07]:

$$[L_a, D_F] = 0, \quad a = \{\lambda, q_\lambda, 0\} \in A \quad (4.8)$$

This constraint also diagonalizes y_l and y_q , ensuring that they satisfy Eq. (4.7) trivially; but as emphasized by Chamseddine and Connes (see *e.g.* Sec. 5 of Ref. [CC08]), this solution is *ad hoc* and unsatisfying, and cries out for a better understanding.

4.1.2 Fluctuating the Dirac operator

While the fermionic degrees of freedom are held in the Hilbert space \mathcal{H} , the gauge bosons appear just as in regular gauge theory as connections on the continuous space. As reviewed in Subsection 2.2.4, a rather novel feature of NCG is that scalar fields arise in exactly the same way as gauge bosons do: that is, as connections on the internal space. To determine the form that the Gauge and Higgs fields take, Connes provides a prescription for ‘fluctuating’ the Dirac operator of a NCG: one first starts with the ‘ground state’ Dirac operator which is ‘un-fluctuated’ by inner automorphisms. For the full standard model the ‘ground state’ Dirac operator is given by:

$$D = -i\gamma^\mu \nabla_\mu^S \otimes \mathbb{I}_F + \gamma_5 \otimes D_F, \quad (4.9)$$

where D_F is given as in Eq. (4.4) with the ‘massless photon condition’ of Eq. (4.8) applied such that all the unwanted extra terms are removed. Following Eq. (2.85) the fluctuated Dirac operator is then given by:

$$D_A = D + A + \epsilon' JAJ^{-1} \quad (4.10)$$

where $A = \sum_{ab} a[D, b]$ is hermitian, and $a, b \in \mathcal{A}$. Gauge fields appear when one ‘fluctuates’ the continuous part of the Dirac operator, while Higgs fields arise from the finite part. I will review the gauge and Higgs sectors of the NCG SM separately.

The NCG SM gauge sector

The fluctuations of the continuous part of the Dirac operator are given by hermitian operators of the form:

$$\begin{aligned} A &= \sum (a_M \otimes a_F) [-i\gamma^\mu \partial_\mu \otimes \mathbb{I}_F, b_M \otimes b_F] \\ &= \sum -i\gamma^\mu (b_M \partial_\mu b_M) \otimes (a_F b_F) \\ &\equiv -i\gamma^\mu A_\mu^i(x) \otimes e_i \end{aligned} \quad (4.11)$$

where $a_M \otimes a_F, b_M \otimes b_F \in C^\infty(M) \otimes \mathcal{A}_F$, the $A_\mu^i(x) \in C^\infty(M)$ are real continuous functions, and the e_i are anti-hermitian basis elements in $\pi(\mathcal{A}_F)$. The full fluctuated continuous Dirac operator is therefore given by:

$$\begin{aligned} D &= -i\gamma^\mu (\partial_\mu + \omega_\mu) - i\gamma^\mu A_\mu^i(x) \otimes e_i - \epsilon' J_c i\gamma^\mu A_\mu^i(x) J_c^{-1} \otimes J_F e_i J_F^{-1} \\ &= -i\gamma^\mu (\partial_\mu + \omega_\mu) - i\gamma^\mu A_\mu^i(x) \otimes [e_i + \epsilon' \epsilon'_c J_F e_i J_F^{-1}] \\ &= -i\gamma^\mu (\partial_\mu + \omega_\mu) - i\gamma^\mu A_\mu^i(x) \otimes [e_i - J_F e_i^\dagger J_F^{-1}] \\ &= -i\gamma^\mu (\partial_\mu + \omega_\mu) - i\gamma^\mu A_\mu^i(x) \otimes \delta_i \end{aligned} \quad (4.12)$$

where the subscript ‘c’ indicates that the real structure sign ϵ'_c corresponds to the ‘continuous’ part of the geometry. Using the representation of the finite algebra given in Eq. (4.3), and the finite real structure operator given in Eq. (4.1), it is found that the action of the ‘gauge potential’ $A_\mu^i(x) \otimes \delta_i$ on the basis $\{L_R, Q_R, L_L, Q_L, \bar{L}_R, \bar{Q}_R, \bar{L}_L, \bar{Q}_L\}$ is given by:

$$i\gamma^\mu A_\mu^i(x) \otimes \delta_i = i\gamma^\mu B_\mu(x) \otimes \delta^{(1)} + i\gamma^\mu W_\mu^i(x) \otimes \delta_i^{(2)} + i\gamma^\mu G_\mu^i(x) \otimes \delta_i^{(3)} \quad (4.13)$$

where

$$\delta^{(1)} = \{y_R^{(l)}, y_R^{(q)} \otimes \mathbb{I}_3, y_L^{(l)}, y_L^{(q)} \otimes \mathbb{I}_3, \bar{y}_R^{(l)}, \bar{y}_R^{(q)} \otimes \mathbb{I}_3, \bar{y}_L^{(l)}, \bar{y}_L^{(q)} \otimes \mathbb{I}_3\}, \quad (4.14a)$$

$$\delta_i^{(2)} = \{0, 0, i\sigma_i, i\sigma_i \otimes \mathbb{I}_3, 0, 0, i\bar{\sigma}_i, i\bar{\sigma}_i \otimes \mathbb{I}_3\} \quad (4.14b)$$

$$\delta_i^{(3)} = \{0, \mathbb{I}_2 \otimes i\lambda_i, 0, \mathbb{I}_2 \otimes i\lambda_i, 0, \mathbb{I}_2 \otimes i\bar{\lambda}_i, 0, \mathbb{I}_2 \otimes i\bar{\lambda}_i\}, \quad (4.14c)$$

σ^i and λ^i are the Pauli and Gell-Mann matrices respectively, and

$$\begin{aligned} y_L^{(l)} &= 2i \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, & y_R^{(l)} &= 2i \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \\ y_L^{(q)} &= 2i \begin{pmatrix} +\frac{1}{6} & 0 \\ 0 & +\frac{1}{6} \end{pmatrix}, & y_R^{(q)} &= 2i \begin{pmatrix} +\frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}. \end{aligned} \quad (4.15)$$

In other words, the derivations $\delta_i = e_i - J e_i^\dagger J^{-1}$ correspond precisely to the generators $\delta^{(1)}$, $\delta_i^{(2)}$ and $\delta_i^{(3)}$ of the familiar standard model gauge group $U(1)_Y \times SU(2)_L \times SU(3)_C$, with the right- and left-handed leptons and quarks transforming in their familiar representations, including the correct hyper-charges.

The NCG SM Higgs sector

The fluctuations of the finite part of the Dirac operator are given by:

$$\begin{aligned} A &= \sum (a_M \otimes a_F) [\gamma_5 \otimes D_F, b_M \otimes b_F] \\ &= \sum \gamma_5 a_M b_M \otimes a_F [D_F, b_F] \end{aligned} \quad (4.16)$$

where $a_M \otimes a_F, b_M \otimes b_F \in C^\infty(M) \otimes \mathcal{A}_F$, and where a_M, b_M are real coefficient functions. The full fluctuated finite Dirac operator is therefore given by:

$$\begin{aligned} D &= \gamma_5 \otimes D_F + \sum \gamma_5 a_M b_M \otimes a_F [D_F, b_F] + \epsilon' J_c \gamma_5 a_M b_M J_c^{-1} \otimes J_F a_F [D_F, b_F] J_F^{-1} \\ &= \gamma_5 \otimes D_F + \sum \gamma_5 a_M b_M \otimes (a_F [D_F, b_F] + J_F a_F [D_F, b_F] J_F^{-1}) \\ &= \gamma_5 \Phi \end{aligned} \quad (4.17)$$

where by an abuse of notation I have dropped the tensor notation in the last line. The Dirac operator corresponds to the Dirac operator D_F given in Eq. (4.4) taken after applying the so called ‘massless photon condition’, such that the terms b, \vec{c}, \vec{d} in Eq. (4.6) are set to zero,

and the terms y_q and y_l in Eq. (4.4) are diagonalized [CCM07]. Using the representation of the finite algebra given in Eq. (4.3), and the finite real structure operator given in Eq. (4.1), the operator Φ on the basis $\{L_R, Q_R, L_L, Q_L, \bar{L}_R, \bar{Q}_R, \bar{L}_L, \bar{Q}_L\}$ is then given by

$$\Phi = \left(\begin{array}{cc|cc|cc|cc} 0 & 0 & Y_l^\dagger & 0 & m^\dagger & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_q^\dagger & 0 & 0 & 0 & 0 \\ \hline Y_l & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_q & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline m & 0 & 0 & 0 & 0 & 0 & Y_l^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & Y_q^T \\ \hline 0 & 0 & 0 & 0 & \bar{Y}_l & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{Y}_q & 0 & 0 \end{array} \right) \quad (4.18)$$

where

$$Y_l = \begin{pmatrix} Y_\nu \varphi_1 & -Y_e \bar{\varphi}_2 \\ Y_\nu \varphi_2 & +Y_e \bar{\varphi}_1 \end{pmatrix}, \quad Y_q = \begin{pmatrix} Y_u \varphi_1 & -Y_d \bar{\varphi}_2 \\ Y_u \varphi_2 & +Y_d \bar{\varphi}_1 \end{pmatrix}, \quad (4.19)$$

with $\{Y_\nu, Y_e, Y_u, Y_d, \varphi_1, \varphi_2\} \in \mathbb{C}$. The fields ϕ_i form a complex Higgs doublet. Notice that the Majorana term m remains un-fluctuated under the action of the SM inner automorphisms, and so is given just as in Eq. (4.6) with $b = 0$.

The full Dirac operator is simply the sum of the fluctuated continuous and finite parts. This is precisely the almost-commutative data that generates the standard model of particle physics [CCM07, CC08, CC07, vdDvS12]! The standard model action is generated by substituting the full fluctuated Dirac operator into the action 1.1:

$$S_{SM} = \langle J\psi | D_A \psi \rangle + Tr[f(D_A/\Lambda)] \quad (4.20)$$

where the fermionic part is restricted to the set of anti-commuting Grassmann variables ψ' for even vectors $\psi \in H^+$, with $H^+ = \{\psi \in H | \gamma\psi = \psi\}$. I give a complete description of (a minimal extension to) the standard model action in Appendix A.

4.1.3 Higgs sector constraints revisited

In Subsections 4.1.1 and 4.1.2 I reviewed the construction of the SM as a NCG. In the traditional construction the geometric constraints $D = D^*$, $\{\gamma, D\} = 0$, $[D, J] = 0$, $[[D, a], JbJ^*] = 0$ are not enough to fully constrain the SM Higgs sector. To obtain

the correct phenomenology, an additional non-geometric condition known as the ‘massless photon’ condition must be imposed in order to constrain the Higgs sector [CCM07]. But this additional condition amounts to removing the extra unwanted scalar fields by hand. Our reformulation yields a simple and satisfying alternative solution. As we saw in Subsections 3.2.2 and 3.2.3, when constructing the fused algebras $B = \Omega_D \mathcal{A} \oplus \mathcal{H}$ and $\Omega B = \Omega_D \mathcal{A} \oplus \Omega \mathcal{H}$ associated to a NCG, associativity implies new constraints beyond the traditional constraints usually imposed:

$$[\omega_n, h, \omega'_m] = 0 \quad (4.21)$$

for $\omega_n, \omega'_m \in \Omega_D \mathcal{A}$, $h \in \mathcal{H}$. For $n + m = 0$, Eq. (4.21) implies the traditional order zero condition ($[L_a, R_b] = 0$), while for $n + m = 1$ it implies the order one condition ($[[D, L_a], R_b] = 0$). For $n + m \geq 2$ it implies new constraints on D . Let us see what these additional constraints imply in the case of the NCG SM.

Associativity constraints on the algebra B

We first introduced the associativity constraints $[\omega_n, h, \omega'_m] = 0$ given in Eq. (3.68), in [BF14]. At that time we were still working with the incomplete description of spectral triples provided by the fused algebra $B = \Omega \mathcal{A} \oplus \mathcal{H}$. For this algebra the right action of forms on \mathcal{H} is given in Eq. (3.67) as $R_\omega = J\omega^*J^{-1}$, and so the associativity conditions yield:

$$\begin{aligned} & [\omega_n, h, \omega'_m] = 0 \\ \Rightarrow & [J\omega'_m{}^*J^{-1}, \omega_n] = 0 \end{aligned} \quad (4.22)$$

for $\omega_n, \omega'_m \in \Omega \mathcal{A}$, $h \in \mathcal{H}$. In particular the ‘order two’ condition:

$$[[D, a], J[D, b]^*J^*] = 0, \quad (4.23)$$

must hold $\forall a, b \in \mathcal{A}$. For the Dirac operator given in Eq. (4.4) this constraint may be satisfied in four different ways: by setting (i) $b = \mathbf{c} = \mathbf{d} = 0$; (ii) $Y_{q,11} = Y_{q,21} = b = 0$; (iii) $Y_{l,11} = Y_{l,21} = \mathbf{c} = \mathbf{d} = 0$; or (iv) $Y_{l,11} = Y_{l,21} = Y_{q,11} = Y_{q,21} = \mathbf{c} = 0$. Note, in particular, that solution (i) precisely corresponds to setting the 7 unwanted parameters ($b, \mathbf{c}, \mathbf{d}$) to zero, without the additional *ad hoc* assumption (the massless photon condition) described above!

We can go further by noting that the general embedding of \mathbb{C} in \mathbb{H} is given by $q_\lambda(\hat{n}) = \text{Re}(\lambda)\mathbb{1}_{2 \times 2} + \text{Im}(\lambda)\hat{n} \cdot \vec{\sigma}$, where $\vec{\sigma}$ are the three Pauli matrices, and \hat{n} is a unit 3-vector specifying the embedding direction. Since all of these embeddings are equivalent, the

diagonal embedding $q_\lambda = q_\lambda(\hat{z})$ in Eq. (4.2) was arbitrary, and may be replaced by the more general possibility $L_a L_R = q_\lambda(\hat{n}_l) L_R$ and $L_a Q_R = q_\lambda(\hat{n}_q) Q_R$. If we redo the preceding analysis with this modification, the four solutions for D are modified accordingly: in particular, in solution (i), D is given by Eq. (4.4), where the 2×2 matrices y_l and y_q are arbitrary, the 6×2 matrix n vanishes, and the 2×2 matrix m is given by $m = P^T M P$, with M an arbitrary 2×2 symmetric matrix and $P = (\mathbb{I}_{2 \times 2} + \hat{n}_l \cdot \vec{\sigma})/\sqrt{2}$ a projection operator. Then, one can check the following result: given the arbitrary 2×2 matrices y_l , y_q and M , there is a preferred choice for the embedding directions \hat{n}_l and \hat{n}_q such that, after a change of basis on \mathcal{H} , L_a is given by Eq. (4.2) [with the diagonal embedding $q_\lambda = q_\lambda(\hat{z})$], D is given by Eq. (4.4), m and n are given by Eq. (4.6) with $b = \mathbf{c} = \mathbf{d} = 0$, while Y_l and Y_q are given by:

$$Y_l = \begin{pmatrix} Y_\nu \varphi_1 & -Y_e \bar{\varphi}_2 \\ Y_\nu \varphi_2 & +Y_e \bar{\varphi}_1 \end{pmatrix}, \quad Y_q = \begin{pmatrix} Y_u \varphi_1 & -Y_d \bar{\varphi}_2 \\ Y_u \varphi_2 & +Y_d \bar{\varphi}_1 \end{pmatrix}, \quad (4.24)$$

with $\{Y_\nu, Y_e, Y_u, Y_d, \varphi_1, \varphi_2\} \in \mathbb{C}$.

Associativity constraints on the algebra ΩB

While the fused algebra $B = \Omega_D \mathcal{A} \oplus \mathcal{H}$ seems to give a very good account of the NCG SM, as I discussed in Subsection 3.2.3 it suffers from the serious drawback that B can not be made into an associative $*$ -DGA for the full NCG SM. In particular, while the associativity condition given in Eq. (4.22) imposes phenomenologically accurate constraints on the finite part of the NCG SM, it is not satisfied by the continuous part.

In [BBB15] Brouder *et al.* introduced differential graded representations of $\Omega_D \mathcal{A}$, ie. the fused algebra $\Omega B = \Omega_D \mathcal{A} \oplus \Omega \mathcal{H}$. The right action of forms on a graded vector space must take account of the grading and is given by: $h_m \omega_n = (\epsilon')^n (-1)^{n(n+1)/2} J_0 \omega_n^\dagger J_0^{-1} h_m$ (see Subsections 3.2.3 and 3.2.4). Because the right action of zero forms remains unchanged, and because zero forms do not affect the degree when acting on elements in $\Omega \mathcal{H}$, both the order zero and order one conditions remain unaffected by the graded representation. The higher order associativity condition given in Eq. (4.22) however now results in the anti-commutator constraint on D (see Eq. (3.95d)):

$$\{[D, a], J[D, b]^* J^{-1}\} = 0, \quad a, b \in \mathcal{A}. \quad (4.25)$$

Unlike the commutator relation which arose when imposing associativity constraints on the algebra B , this anti-commutator is satisfied by the continuous part of the standard

model up to junk¹. Let us see what constraints this anti-commutator condition places on the finite part of the NCG SM.

For the NCG SM the associativity condition given in Eq. (4.25) must hold up to junk in the finite part for the corresponding fused algebra ΩB to be associative. What Brouder *et al.* found was that for the finite Dirac operator given in (4.4) (prior to imposing the massless photon condition of Eq. (4.8)) the order two condition for ΩB does not completely overlap with the junk of order two. In other words, if one calculates the 32×32 dimensional matrix corresponding to a general element of the finite second order junk (for one particle generation), and then one calculates the 32×32 dimensional matrix $\{[D_F, a], J[D_F, b]^* J^{-1}\}$ for general $a, b \in \mathcal{A}_F$, one finds that the two do not completely overlap (as they would for the continuous geometry). The non-zero second order terms which do not overlap with the junk must be set to zero, and this places constraints on the finite Dirac operator. In their words:

“Among the pairs of indices where the junk is zero, 68 of them correspond to matrix elements of $\{[D_F, a], J[D_F, b]^* J^{-1}\}$ that are not generically zero. This situation is shown in figure 4.1. Since these 68 elements cannot be compensated by the junk, the condition of order two implies that these 68 matrix elements must be equal to zero... we recover exactly the four solutions ... (i) $b = \mathbf{c} = \mathbf{d} = 0$; (ii) $y_{q,11} = y_{q,21} = b = 0$; (iii) $y_{l,11} = y_{l,21} = \mathbf{c} = \mathbf{d} = 0$; or (iv) $y_{l,11} = y_{l,21} = y_{q,11} = y_{q,21} = \mathbf{c} = 0$. Three of these four solutions are not physically acceptable... while the remaining solution (i) is precisely the result of the condition of zero photon mass.”

Notice that even though imposing associativity on B yields a commutator condition, and imposing associativity on ΩB yields an anti-commutator condition, both place the same constraint on D_F . This is because the non-zero elements of $[D_F, a]J[D_F, b]^* J^{-1}$ and $J[D_F, b]^* J^{-1}[D_F, a]$ do not overlap and so to cancel the commutator or the anti-commutator requires $[D_F, a]J[D_F, b]^* J^{-1} = J[D_F, b]^* J^{-1}[D_F, a] = 0$.

Associativity constraints, junk, and the \mathbb{Z}_2 grading on \mathcal{H}

So far in the discussion I have avoided describing the graded vector space on which the NCG SM finite algebra $\Omega_D \mathcal{A}$ is bi-represented. Recall, that for canonical spectral triples if the graded vector space is taken to be $\Omega \mathcal{H} = \bigoplus_n \mathcal{H}^n$ where each \mathcal{H}^n is a copy of $L^2(M, S)$, then on modding out by the junk, all elements of $\Omega \mathcal{H}$ for $n \geq 2$ are removed along with the junk (see Subsections 3.2.4 and 3.2.3). If a similar effect were to occur for the finite part of

¹For canonical spectral triples the second order junk consists of symmetric elements of the form $\{\gamma^\mu, \gamma^\nu\}(\partial_\mu a)(\partial_\nu b)$, see Eq. (2.59b) for a full description

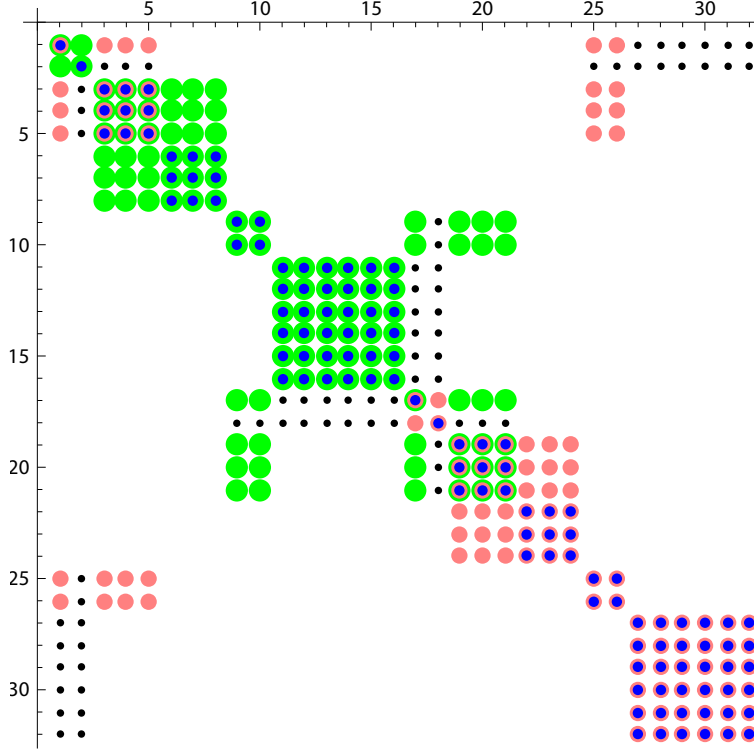


Figure 4.1: Each dot at position (i, j) corresponds to a generally non-zero element at line i and column j . The elements of the condition of order two $\{[D, a], J[D, b^*]J^{-1}\}$ are black dots. The other dots describe the junk. The green dots correspond to $[D, a][D, b]Jc^*J^{-1}$, the pink dots to $[D, Ja^*J^{-1}][D, Jb^*J^{-1}]c$ and the blue dots to aJb^*J^{-1} . Note that the non-zero matrix elements of the junk and of the second-order condition do not overlap (figure taken with permission from [BBB15]).

the NCG SM then the second order condition would become trivial. In addition, while the associativity constraints on the finite algebras B and ΩB may be used to set $b = c = d = 0$ in Eq. (4.4), associativity alone does not seem to ‘diagonalize’ the Yukawa terms Y as is done by the massless photon condition given in Eq. (4.8). Diagonalizing the Yukawa terms Y , ensures that they are trivially of the form given in Eq. (4.7), which in turn ensures that only the SM scalar degrees of freedom are fluctuated. So what is the graded representation space for the finite NCG SM, and can the Dirac operator be consistently constrained from the associativity of the representation alone?

Let us take a closer look at the ‘obvious’ proposal: let us see what form the junk takes

when we represent the differential graded algebra $\Omega_D \mathcal{A}$ on copies of \mathbb{C}^{96} . To simplify the discussion notice that the finite Dirac operator does not mix leptons and quarks. I will therefore focus on one generation of leptons only, i.e. an electroweak model represented on the graded vector space $\Omega \mathcal{H} = \bigoplus_n \mathcal{H}^n$, where each \mathcal{H}^n is taken to be a copy of \mathbb{C}^8 with basis $\{L_R, L_L, \bar{L}_R, \bar{L}_L\}$. This restriction greatly simplifies the calculation. The calculation for the full a NCG SM proceeds in exactly the same way, but where the salient features are more easily lost for the long and tedious algebra.

As in Eq. (4.3), the left action of the algebra $\mathbb{C} \oplus \mathbb{H}$ on $\{L_R, L_L, \bar{L}_R, \bar{L}_L\}$ is given by

$$\begin{aligned} L_a L_R &= q_\lambda L_R, & L_a L_L &= q L_L, \\ L_a \bar{L}_R &= \lambda \mathbb{I}_{2 \times 2} \bar{L}_R, & L_a \bar{L}_L &= \lambda \mathbb{I}_{2 \times 2} \bar{L}_L, \end{aligned} \quad (4.26)$$

where q, q_λ are given as in Eq. (4.2), and together with $\lambda \mathbb{I}_{2 \times 2}$ act on the lepton doublet index. The leptons do not carry any $SU_c(3)$ triplet index. The helicity operator γ_F and the anti-linear charge conjugation operator J_F are given just as in Eq. (4.1):

$$\begin{aligned} \gamma_F L_R &= -L_R, & \gamma_F L_L &= L_L, & \gamma_F \bar{L}_R &= \bar{L}_R, & \gamma_F \bar{L}_L &= -\bar{L}_L, \\ J_F L_R &= \bar{L}_R, & J_F L_L &= \bar{L}_L, & J_F \bar{L}_R &= L_R, & J_F \bar{L}_L &= L_L. \end{aligned} \quad (4.27)$$

Meanwhile the finite Dirac operator (prior to imposing the massless photon condition or the second order condition) is given by

$$D_F = \left(\begin{array}{cc|cc} 0 & y_l^\dagger & m^\dagger & 0 \\ y_l & 0 & 0 & 0 \\ \hline m & 0 & 0 & y_l^T \\ 0 & 0 & \bar{y}_l & 0 \end{array} \right) \quad (4.28)$$

where

$$y_l = \begin{pmatrix} y_{l,11} & y_{l,12} \\ y_{l,21} & y_{l,22} \end{pmatrix} \quad \text{and} \quad m = \begin{pmatrix} a & b \\ b & 0 \end{pmatrix} \quad (4.29)$$

are both 2×2 complex matrices which act on the lepton doublet indices.

To work out the second degree junk for this finite electroweak data, we must look for elements $\omega \in \Omega^1 \mathcal{A}$ such that, $\pi(\omega) = 0$, while $\pi(d[\omega]) \neq 0$. To start with, a general element

$\omega = \sum a' d[a] \in \Omega^1 \mathcal{A}$ is left represented by:

$$\pi(\omega) = \sum \left(\begin{array}{cc|cc} 0 & q'_\lambda(Y_l^\dagger q - q_\lambda Y_l^\dagger) & \begin{pmatrix} 0 & 0 \\ \bar{b}\lambda' \delta\lambda & 0 \end{pmatrix} & 0 \\ \hline q'(Y_l q_\lambda - q Y_l) & 0 & 0 & 0 \\ \begin{pmatrix} 0 & b\lambda' \delta\lambda \\ 0 & 0 \end{pmatrix} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \quad (4.30)$$

where the sum is over algebra elements $a' = \{\lambda', q'\}$, $a = \{\lambda, q\} \in \mathcal{A}_F$, and where I have defined $\delta\lambda \equiv \lambda - \bar{\lambda}$. For a generic one form to be equal to zero we therefore require:

$$\sum q'(Y_l q_\lambda - q Y_l) = \begin{pmatrix} A_1 Y_{11} + A_2 Y_{21} & \bar{A}_4 Y_{12} + \bar{A}_3 Y_{22} \\ -A_3 Y_{11} + A_4 Y_{21} & -\bar{A}_2 Y_{12} + \bar{A}_1 Y_{22} \end{pmatrix} = 0, \quad (4.31a)$$

$$\sum q'_\lambda(Y_l^\dagger q - q_\lambda Y_l^\dagger) = \begin{pmatrix} B_1 \bar{Y}_{11} - B_2 \bar{Y}_{21} & B_3 \bar{Y}_{11} + B_4 \bar{Y}_{21} \\ \bar{B}_4 \bar{Y}_{12} - \bar{B}_3 \bar{Y}_{22} & \bar{B}_2 \bar{Y}_{12} + \bar{B}_1 \bar{Y}_{22} \end{pmatrix} = 0, \quad (4.31b)$$

$$\sum \lambda' \lambda = \sum \lambda' \bar{\lambda}, \quad (4.31c)$$

where

$$\begin{aligned} A_1 &= \sum \alpha'(\lambda - \alpha) + \beta' \bar{\beta}, & B_1 &= \sum \lambda'(\alpha - \lambda), \\ A_2 &= \sum \beta'(\lambda - \bar{\alpha}) - \alpha' \beta, & B_2 &= \sum \lambda' \bar{\beta}, \\ A_3 &= \sum \bar{\beta}'(\lambda - \alpha) - \bar{\alpha}' \bar{\beta}, & B_3 &= \sum \lambda' \beta, \\ A_4 &= \sum \bar{\alpha}'(\lambda - \bar{\alpha}) + \bar{\beta}' \beta, & B_4 &= \sum \lambda'(\bar{\alpha} - \lambda). \end{aligned} \quad (4.31d)$$

Let us next look at general second order elements of the form $\pi(d[\omega]) = \sum [D, a'] [D, a]$ subject to the conditions given in Eq. (4.31). To start with, a general two form satisfying condition Eq. (4.31c) is given by:

$$\left(\begin{array}{cc|cc} (Y_l^\dagger q' - q'_\lambda Y_l^\dagger)(Y_l q_\lambda - q Y_l) & 0 & 0 & 0 \\ 0 & (Y_l q'_\lambda - q' Y_l)(Y_l^\dagger q - q_\lambda Y_l^\dagger) & (Y_l q'_\lambda - q' Y_l) \begin{pmatrix} 0 & 0 \\ \bar{b} \delta\lambda & 0 \end{pmatrix} & 0 \\ \hline 0 & \begin{pmatrix} 0 & b\delta\lambda' \\ 0 & 0 \end{pmatrix} (Y_l^\dagger q - q_\lambda Y_l^\dagger) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad (4.32)$$

Next, notice that the conditions given in Eq. (4.31a) and Eq. (4.31b) may be satisfied in two different ways: **Case (i)**: For generic Y_l , the conditions given in Eq. (4.31a) and Eq. (4.31b) are satisfied when each A_i , and B_i vanish separately. Imposing these conditions, the left action of the general second order form given in Eq. (4.32) becomes:

$$\pi(d[\omega]) = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & (Y_l q'_\lambda - q' Y_l)(Y_l^\dagger q - q_\lambda Y^\dagger) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad (4.33)$$

while the corresponding right acting junk element is given by:

$$J\pi(d[\omega])^\dagger J^{-1} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ((Y_l q'_\lambda - q' Y_l)(Y_l^\dagger q - q_\lambda Y^\dagger))^T \end{array} \right). \quad (4.34)$$

Notice that in this first case the junk only acts on the basis elements $\{L_L, \bar{L}_L\}$, and so it is clear that when we form the quotient algebra ΩB we will not need to mod out by the whole of $\Omega^2 \mathcal{H}$. One would hope therefore that the second order condition given in Eq. (4.25) would place non-trivial constraints on D_F . Unfortunately this does not turn out to be the case, the second order condition is indeed trivial on the quotient algebra for this input data.

Case (ii): Notice, that in the special case when $Y_{11}\bar{Y}_{12} = -Y_{12}\bar{Y}_{22}$, then the conditions given in Eq. (4.31a) and Eq. (4.31b) have additional solutions when $A_1 Y_{11} = -A_2 Y_{21}$, $A_3 Y_{11} = A_4 Y_{21}$, $B_1 \bar{Y}_{11} = B_2 \bar{Y}_{21}$, and $B_3 \bar{Y}_{11} = -B_4 \bar{Y}_{21}$. But note that this special case is exactly the case of physical interest outlined in Eq. (4.7)! Notice also that in this case the Junk will be much larger (because in this special case the junk does not necessarily satisfy the conditions $A_i = B_j = 0$ for all i, j , and it is likely that once again this will result in all of $\Omega \mathcal{H}$ being removed for $n \geq 2$, in which case the condition $d^2 = 0$ would automatically be satisfied up to junk. Calculations are currently ongoing. It is not yet clear how one should consistently impose the higher order associativity conditions along with the condition $d^2 = 0$ in order to obtain the correct SM Dirac operator.

4.1.4 Higgs boson mass prediction

NCG made a prediction for the SM Higgs at approximately 170GeV [CIS99, KS06, CCM07], which is now ruled out by experiment [A+12, C+12b]. Since then a number of solutions

to the ‘Higgs mass problem’ have been proposed. Notably by Estrada and Marcolli who include gravitation corrections to obtain a 125 GeV Higgs [EM13a], while Chamseddine and Connes propose an alternative solution in which they add an extra scalar field σ into the model by hand [CC12]. Later papers by Devastato *et al.* [DLM14a] and Chamseddine *et al.* [CCvS13] construct standard model extensions which naturally include the σ field as an output. Our fused algebra formulation of the NCG SM offers a natural solution to the problem which I will outline in full detail in Section 4.2.2: in short a complex variant of the scalar field σ introduced Chamseddine and Connes [CC12] arises naturally as an output in our formalism, *without* the need to alter the NCG SM input data or the axioms of the formalism. In this section I lay the ground work for that discussion by reviewing very briefly how the original 170GeV prediction was made. For further detail the reader should refer to the appropriate literature [CIS99, KS06, CCM07].

A Higgs mass prediction is possible within the NCG formulation of the SM because the spectral action is slightly more constrained than the usual SM bosonic action. This can be seen immediately when calculating the heat kernel expansion of the bosonic action. Following Appendix A, and ignoring boundary terms, gravitational interactions and the cosmological constant term, the expansion of the bosonic action is given by:

$$\begin{aligned}
S_b &= \int_M \frac{d^4x}{8\pi^2} f(0) \text{Tr}[-[\nabla^E, \Phi]^2 - \frac{4f_2}{f(0)} \Lambda^2 \Phi^2 + \Phi^4 - \frac{1}{3} F_{\mu\nu} F^{\mu\nu}] + O(\Lambda^{-1}) \\
&= \int_M \frac{d^4x}{2\pi^2} a f(0) [|\partial_\mu \phi - \frac{1}{2} g_y B_\mu i \phi + \frac{1}{2} g_w i W_\mu^i \sigma_i \phi|^2 \\
&\quad + 2(f(0)e - 2f_2 a \Lambda^2) |\phi|^2 + f(0) b |\phi|^4 \\
&\quad - f(0) (\frac{5}{3} g_y^2 B_{\mu\nu}^i B^{j\mu\nu} + g_w^2 W_{\mu\nu} W^{\mu\nu} + g_c^2 G_{\mu\nu} G^{\mu\nu})] + O(\Lambda^{-1}), \tag{4.35}
\end{aligned}$$

where the terms a, b and c relate to the Yukawa couplings, and are given in Eq. (A.15). To express Eq. (4.35) in a more canonical form, the scalar and gauge kinetic terms are normalized:

$$\begin{aligned}
S_b &= \int_M d^4x \frac{1}{2} |\partial_\mu \phi' - \frac{1}{2} g_y B_\mu i \phi' + \frac{1}{2} g_w i W_\mu^i \sigma_i \phi'|^2 - \frac{1}{2} \mu_\phi |\phi'|^2 + \frac{1}{4} \lambda_\phi |\phi'|^4 \\
&\quad + \frac{1}{4} (B_{\mu\nu}^i B^{j\mu\nu} + W_{\mu\nu} W^{\mu\nu} + G_{\mu\nu} G^{\mu\nu}) + O(\Lambda^{-1}), \tag{4.36}
\end{aligned}$$

where

$$\phi' = \sqrt{\frac{af(0)}{\pi^2}} \phi, \quad \frac{\mu_\phi}{2} = -\left(\frac{ef(0) - 2af_2\Lambda^2}{af(0)}\right), \quad \frac{\lambda_\phi}{4} = \frac{\pi^2 b}{2a^2 f(0)}, \tag{4.37a}$$

and

$$\frac{5f(0)}{6\pi^2} g_y^2 = \frac{f(0)}{2\pi^2} g_w^2 = \frac{f(0)}{2\pi^2} g_c^2 = \frac{1}{4}. \tag{4.37b}$$

On normalizing the gauge kinetic terms as in Eq. (4.37b), one finds that the relationship $5g_y^2/3 = g_w^2 = g_c^2$ holds. This is the familiar relationship found in $SU(5)$ grand unification at the unification scale Λ_{unif} . The interpretation is that the spectral action is defined at the grand unification scale, and so the Higgs mass calculation relies on the so called ‘big desert hypothesis’ [KS06]. The important point to note is that in the heat kernel expansion of the NCG SM spectral action, the gauge kinetic terms and the scalar quartic couplings appear at the same order and so their coefficients are related (note that this relationship, and the one given in Eq. (4.37b) only hold at the scale Λ_{unif} however!). This is where the additional constraint arises in the bosonic action, allowing for a prediction to be made for the Higgs mass.

Having re-expressed the bosonic action in the canonical form of Eq. (4.36), lets now turn to the calculation of the NCG SM tree level bosonic mass spectrum starting first with the gauge bosons and then working out the Higgs mass. For $\mu_\phi > 0$, the minimum of the scalar potential occurs at

$$|\phi|^2 = \frac{\mu_\phi}{\lambda_\phi} = \frac{2a^2 f_2 \Lambda^2 - aef(0)}{b\pi^2} \quad (4.38)$$

The fields that satisfy this relation are called the vacuum states of the Higgs field [vdDvS12]. Without any loss of generality I will choose the vacuum state $diag\{0, v_h\}$, where the vacuum expectation value v_h is a real parameter given by Eq. (4.38). Working in the ‘unitary’ gauge $\phi' = \{0, v_h + H\}$, the relevant terms in the bosonic action for determining the boson masses are expressed as:

$$\begin{aligned} \frac{1}{2}|\frac{1}{2}g_w iW_\mu^i \sigma_i \phi' - \frac{1}{2}g_y B_\mu i\phi'|^2 - \frac{\mu_\phi}{2}|\phi'|^2 + \frac{\lambda_\phi}{4}|\phi'|^4 &= \frac{1}{8}g_w^2 |iW_\mu^1 + W_\mu^2|^2 (v_\phi + H)^2 \\ &+ \frac{1}{8}|g_w iW_\mu^3 + g_y iB_\mu|^2 (v_\phi + H)^2 \\ &- \frac{\mu_\phi}{2}(v_h + H)^2 + \frac{\lambda_\phi}{4}(v_h + H)^4. \end{aligned} \quad (4.39)$$

By inspecting the coefficients of the scalar quadratic terms, the gauge boson mass squared matrix on the basis $\{W_\mu^+ = \frac{1}{\sqrt{2}}(W_\mu^1 + iW_\mu^2), W_\mu^- = \frac{1}{\sqrt{2}}(W_\mu^1 - iW_\mu^2), W_\mu^3, B_\mu\}$, is therefore:

$$M_Z^2 = \begin{pmatrix} \frac{1}{4}g_w^2 v_\phi^2 & 0 & 0 & 0 \\ 0 & \frac{1}{4}g_w^2 v_\phi^2 & 0 & 0 \\ 0 & 0 & \frac{1}{4}g_w^2 v_\phi^2 & \frac{1}{4}g_w g_y v_\phi^2 \\ 0 & 0 & \frac{1}{4}g_w g_y v_\phi^2 & \frac{1}{4}g_y^2 v_\phi^2 \end{pmatrix}. \quad (4.40)$$

The eigenvalues of M_Z^2 are given by: $\{\frac{1}{4}g_w^2 v_\phi^2, \frac{1}{4}g_w^2 v_\phi^2, \frac{1}{4}(g_w^2 + g_y^2)v_\phi^2, 0\}$. The masses of the W_μ^\pm and Z_0 bosons are therefore $m_{W^\pm} = \frac{1}{2}g_w v_\phi$, and $M_Z = \frac{1}{2}\sqrt{g_w^2 + g_y^2}v_\phi$ respectively,

while the photon is massless. Experimentally, physicists have very good measurements of Fermi's coupling constant $\frac{G_{Fermi}}{(\hbar c)^3} = \sqrt{2}g_w^2/8M_W^2$, as well as the W and Z boson masses, which allows both the weak coupling constant, and the Higgs VEV v_ϕ to be determined; both of which are necessary for determining the mass of the Higgs. Next let us determine the Higgs mass. Inspecting the coefficient of the quadratic scalar term in Eq. (4.39), the Higgs boson mass is given by:

$$\begin{aligned} M_H^2 &= -\mu_\phi + 3\lambda_\phi v_h^2 \\ &= 2\lambda_\phi v_h^2 \end{aligned} \tag{4.41}$$

where in the second line the value for the Higgs VEV $\mu_\phi = \lambda_\phi v_\phi^2$ given in Eq. (4.38) has been used. The only remaining step is to determine the value of λ_ϕ at the electroweak scale. Fortunately the value of the quartic coupling is known at the unification scale, and so its value at the electroweak scale can be determined using the renormalization flow. Substituting Eq. (4.37b) into Eq. (4.37a) yields:

$$\lambda_\phi = 4g_w^2 b/a^2 \tag{4.42}$$

at the scale Λ . If further it is assumed that the top quark Yukawa coupling is much larger than the other Yukawa couplings² then Eq. (4.42) simplifies to

$$\lambda_\phi = 4g_w^2/3. \tag{4.43}$$

The first step in determining the scalar quartic coupling is therefore to determine g_w at Λ_{unif} . Following [KM10], the one loop SM beta functions for the gauge couplings are:

$$(4\pi)^2\beta(g_3) = -\frac{42}{6}g_3^3 \tag{4.44a}$$

$$(4\pi)^2\beta(g_2) = -\frac{19}{6}g_2^3 \tag{4.44b}$$

$$(4\pi)^2\beta(g_y) = +\frac{41}{6}g_y^3, \tag{4.44c}$$

The gauge couplings are well known at the electroweak scale, and on examining their running it is clear that there is no scale at which all three unify (see Fig. 4.2). I will return to this point when I discuss the Weinberg angle problem in Eq. 4.2.3.

²This is not necessarily a good assumption, as depending on the mechanism behind the neutrino masses the τ -neutrino Yukawa coupling could for example also be quite large. I will be more careful in Section 4.2.2 when discussing our solution to the Higgs mass problem.

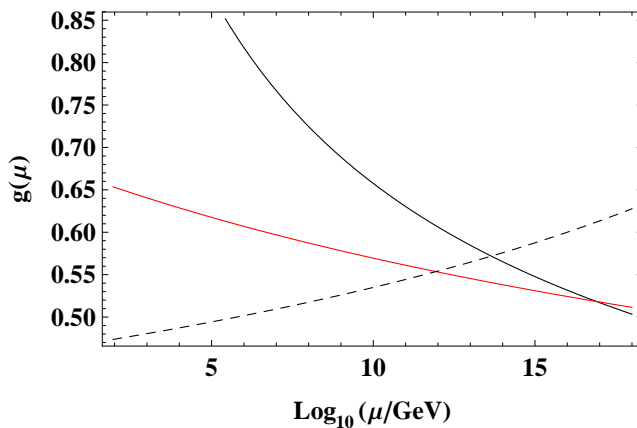


Figure 4.2: Standard model gauge couplings: g_c (black), g_w (red), g_y (black, dashed).

The next step is to determine the running of the top Yukawa coupling, as the quartic coupling beta function depends on it. The one loop beta function for the top Yukawa coupling is [KM10]:

$$(4\pi)^2 Y_t^{-1} \frac{dY_t}{dt} = \frac{9}{2} (Y_t^\dagger Y_t) + -8g_3^2 - \frac{9}{4}g_2^2 - \frac{17}{12}g_y^2 \quad (4.45a)$$

where the g_i are determined as solutions to Eq. 4.44, and I have ignored all Yukawa couplings other than the top Yukawa coupling (see Fig. 4.3).

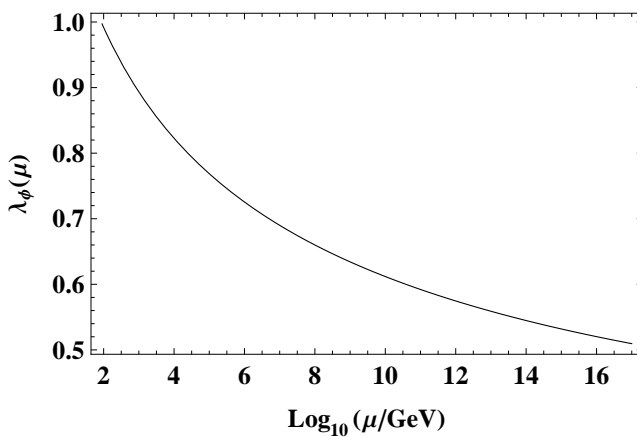


Figure 4.3: Standard model top quark Yukawa coupling.

Finally the one loop beta function for the Higgs quartic coupling is given by [KM10]

(see Fig. 4.4):

$$(4\pi)^2\beta_h = 24\lambda_h^2 - 6(Y_t^\dagger Y_t)^2 + 12Y_t^\dagger Y_t \lambda_h - 3(g_y^2 + 3g_2^2)\lambda_h + \frac{9}{8}g_2^4 + \frac{3}{8}g_y^4 + \frac{3}{4}g_2^2 g_y^2 \quad (4.46)$$

The initial value for the quartic coupling is set at the unification scale, but as shown

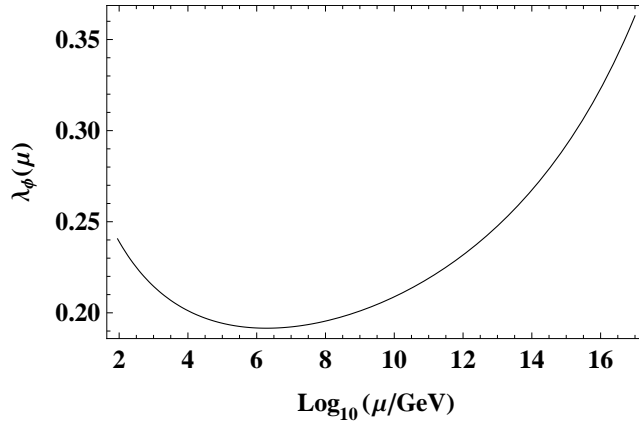


Figure 4.4: Standard model quartic coupling running.

in Fig. 4.2 there is no scale at which all three coupling constants unify. I will therefore determine the Higgs mass at $10^{12}GeV \leq \Lambda_{unif} \leq 10^{17}GeV$. The determined value of the quartic coupling at the electroweak scale can then be substituted into Eq. 4.41 to determine the tree level Higgs mass. The Higgs mass is determined in this way to lie approximately in the range $170GeV \leq m_{Higgs} \leq 178GeV$.

4.2 The NCG SM symmetries revisited (a gauged B-L SM extension).

The ‘fused algebra’ or ‘square zero extensions’ formulation of NCG outlined in Ch. 3 replaces the algebra \mathcal{A} as the main piece of input data with a full *-DGA ΩB which incorporates all five elements of the spectral triple $\{\mathcal{A}, \mathcal{H}, D, J, \gamma\}$. As explained in Section 3.3, from this perspective it is clear that the symmetries of a NCG are not given by the automorphisms of the input algebra \mathcal{A} , but by the automorphisms of the full *-DGA ΩB . In this section I re-examine the symmetries of the NCG SM from the fused algebra perspective.

4.2.1 The symmetries of ΩB

Symmetries of the NCG SM sub-algebra B_0

Often the simplest approach to determining the symmetries of a NCG is to work out the form that the infinitesimal generators take. This is the approach I will take here. To begin with, first consider the NCG SM sub-algebra $B_0 = \mathcal{A} \oplus \Omega\mathcal{H}^3$, where $\mathcal{A} = C^\infty(M, \mathcal{A}_F)$, and $\Omega\mathcal{H} = L^2(M, S) \otimes \mathbb{C}^{32}$. As described in example four of Subsection 2.1.4, general derivations of the sub-algebra \mathcal{A} are of the form $\delta_{\mathcal{A}} = v^\mu \partial_\mu + L_x - R_x$, where the v^μ are real functions on M , and x is any anti-hermitian element of \mathcal{A} . The grading preserving derivations on the full algebra $\delta : B_0 \rightarrow B_0$ are therefore of the form $\delta = (v^\mu \partial_\mu + L_x - R_x) \oplus \delta_{\mathcal{H}}$. The conditions given in Eq. (3.118), can then be used to fix the form of the $\delta_{\mathcal{H}}$. In particular the Leibniz condition given in Eq. (3.118b) restricts the generating derivations on B_0 to be of the form:

$$\delta = (v^\mu \partial_\mu + L_x - R_x) \oplus (v^\mu \partial_\mu + x - Jx^\dagger J^{-1} + T), \quad (4.47)$$

where the term $v^\mu \partial_\mu$ generates manifold translations, and the term $x - Jx^\dagger J^{-1}$ generates inner automorphisms. Note that the term $v^\mu \partial_\mu$ on the right hand side of Eq. (4.47) is not covariant. The covariant term would be of the form $v^\mu \nabla_\mu^S$, and so we should expect to find an appropriate connection term amongst the possible T's. This is in fact exactly what I will show below. To display the inner derivations more explicitly, let us denote an element of the algebra $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ by $a = (\lambda, q, m)$ where $\lambda \in \mathbb{C}$ is a complex number, $q \in \mathbb{H}$ is a quaternion, and $m \in M_3(\mathbb{C})$ is a 3×3 complex matrix. The anti-hermitian elements of A_F can be split into 3 pieces: namely (i) $a_1 = (\lambda, 0, \mu \mathbb{I}_3)$ where $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$ are pure imaginary and \mathbb{I}_3 is the 3×3 identity matrix, (ii) $a_2 = (0, q, 0)$ where q is a general anti-hermitian 2×2 matrix, and (iii) $a_3 = (0, 0, m)$ where m is a general traceless anti-hermitian 3×3 matrix. Demanding that the corresponding symmetry generators $\delta_{\mathcal{H}}^{(i)} = L_{a_i} - R_{a_i}$ are anomaly free (3.128) yields the additional restriction $\mu = -\lambda/3$. The $\delta_{\mathcal{H}}^{(i)}$ are block diagonal; if, as in Ref. [BF14], we label the subspaces of H_F as $\{L_R, Q_R, L_L, Q_L, \bar{L}_R, \bar{Q}_R, \bar{L}_L, \bar{Q}_L\}$, the blocks are

$$\delta_{\mathcal{H}}^{(1)} = \{y_R^{(l)}, y_R^{(q)} \otimes \mathbb{I}_3, y_L^{(l)}, y_L^{(q)} \otimes \mathbb{I}_3, \bar{y}_R^{(l)}, \bar{y}_R^{(q)} \otimes \mathbb{I}_3, \bar{y}_L^{(l)}, \bar{y}_L^{(q)} \otimes \mathbb{I}_3\} \quad (4.48a)$$

$$\delta_{\mathcal{H}}^{(2)} = \{0, 0, q, q \otimes \mathbb{I}_3, 0, 0, \bar{q}, \bar{q} \otimes \mathbb{I}_3\} \quad (4.48b)$$

$$\delta_{\mathcal{H}}^{(3)} = \{0, \mathbb{I}_2 \otimes m, 0, \mathbb{I}_2 \otimes m, 0, \mathbb{I}_2 \otimes \bar{m}, 0, \mathbb{I}_2 \otimes \bar{m}\} \quad (4.48c)$$

³Notice that I am including all of $\Omega\mathcal{H}$ in B_0 here, and not just $\Omega^0\mathcal{H}$.

where

$$\begin{aligned} y_L^{(l)} &= 2\lambda \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, & y_R^{(l)} &= 2\lambda \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \\ y_L^{(q)} &= 2\lambda \begin{pmatrix} +\frac{1}{6} & 0 \\ 0 & +\frac{1}{6} \end{pmatrix}, & y_R^{(q)} &= 2\lambda \begin{pmatrix} +\frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}. \end{aligned} \quad (4.49)$$

In other words, from the derivations $\delta_n = L_a - R_a$ we precisely obtain the generators $\delta_{\mathcal{H}}^{(1)}$, $\delta_{\mathcal{H}}^{(2)}$ and $\delta_{\mathcal{H}}^{(3)}$ of the familiar standard model gauge group $U(1)_Y \times SU(2)_L \times SU(3)_C$, with the right- and left-handed leptons and quarks transforming in their familiar representations.

Without loss of generality the operator T in Eq. (4.47) may be written as $T = \sum T_c \otimes T_F$, where each T_c may be taken to be an arbitrary hermitian matrix operator on $\mathcal{H}_c = L^2(M, S)$, while each T_F is an arbitrary anti-hermitian matrix operator acting on $\mathcal{H}_F = \mathbb{C}^{96}$, which satisfies $[T_F, a_F] = [T_F, J_F a_F^* J^{-1}] = 0$, $\forall a_F \in \mathcal{A}_F$. In order to be a derivation, the operator T must also satisfy the commutator conditions given in Eq. (3.118c, 3.118a), ie. $[T, J] = [T, \gamma] = 0$. It is easy to show that there are only two possible types of non-trivial solutions to these commutator equations. The first is when each $T_c \otimes T_F$ independently satisfies $[T_c, \gamma_c] = [T_F, \gamma_F] = [T_c, J_c] = [T_F, J_F] = 0$. The second is when each $T_c \otimes T_F$ independently satisfies $[T_c, \gamma_c] = [T_F, \gamma_F] = \{T_c, J_c\} = \{T_F, J_F\} = 0$. It turns out that the two solutions $\{T_c, \gamma_c\} = \{T_F, \gamma_F\} = [T_c, J_c] = [T_F, J_F] = 0$ and $\{T_c, \gamma_c\} = \{T_F, \gamma_F\} = [T_c, J_c] = [T_F, J_F] = 0$ to Eq. (3.118c, 3.118a) are trivial because the condition $[T_F, a_F] = [T, J_F a_F^* J^{-1}] = 0$, $\forall a_F \in \mathcal{A}_F$ diagonalizes each T_F (and γ_F is diagonal). Similarly there are no other interesting solutions constructed from sums $\sum T_c \otimes T_F$. The two possible types of solutions for the operator $T = T_c \otimes T_F$ are shown in table 4.1.

	$[T_c, \gamma_c] = [T_F, \gamma_F] = 0$
$[T_c, J_c] = [T_F, J_F] = 0$	$T^- = \sum \begin{pmatrix} f_0 \mathbb{I}_2 & 0 \\ 0 & f'_0 \mathbb{I}_2 \end{pmatrix} \otimes T_F^-, \quad f_0 \in C^\infty(M, \mathbb{R})$
$\{T_c, J_c\} = \{T_F, J_F\} = 0$	$T^+ = \sum \begin{pmatrix} f_i \sigma^i & 0 \\ 0 & f'_j \sigma^j \end{pmatrix} \otimes T_F^+, \quad f_i, f'_j \in C^\infty(M, \mathbb{R})$

Table 4.1: The outer derivations T for the NCG SM fused algebra B_0 .

In table 4.1 the operators T_F^- , and T_F^+ are given by:

$$T_F^- = \{x_{L_R}^-, x_{Q_R}^- \otimes \mathbb{I}_3, ic_{L_L}^- \otimes \mathbb{I}_2, ic_{Q_L}^- \otimes \mathbb{I}_6, \bar{x}_{L_R}^-, \bar{x}_{Q_R}^- \otimes \mathbb{I}_3, \bar{ic}_{L_L}^- \otimes \mathbb{I}_2, \bar{ic}_{Q_L}^- \otimes \mathbb{I}_6\}, \quad (4.50a)$$

$$T_F^+ = \{x_{L_R}^+, x_{Q_R}^+ \otimes \mathbb{I}_3, ic_{L_L}^+ \otimes \mathbb{I}_2, ic_{Q_L}^+ \otimes \mathbb{I}_6, x_{L_R}^+, x_{Q_R}^+ \otimes \mathbb{I}_3, ic_{L_L}^+ \otimes \mathbb{I}_2, ic_{Q_L}^+ \otimes \mathbb{I}_6\}, \quad (4.50b)$$

where

$$x_l^\pm = i \begin{pmatrix} c_{LR}^\pm & 0 \\ 0 & c'_{LR}{}^\pm \end{pmatrix}, \quad x_q^\pm = i \begin{pmatrix} c_{QR}^\pm & 0 \\ 0 & c'_{QR}{}^\pm \end{pmatrix} \quad (4.50c)$$

and $c_i \in \mathbb{R}$. Aside from anomaly cancellation, no further restriction can be placed on the generators of the form T^\pm at the level of the sub-algebra $B_0 \subset \Omega B$.

Symmetries of the NCG SM algebra ΩB

To put further constraint on the generators T , one must consider the constraints imposed on the full $*$ -DGA ΩB . In particular, derivations on ΩB must map n forms to n forms: $[\delta, \omega_n] = \omega'_n$ (See Eq. (3.127)). Consider for example an arbitrary one form:

$$\omega_1 = \sum a_c [\delta_{\mathcal{H}1}, b_c] \otimes c_F + \sum \gamma_5 c_c \otimes a_F [\delta_{\mathcal{H}2}, b_F], \quad (4.51)$$

for $a_c, b_c, c'_c \in \mathcal{A}_c$, $a_F, b_F, c_F \in \mathcal{A}_F$. For one forms given as in Eq. (4.51) to remain ‘closed’ under commutation, we must further restrict the allowed generators T^\pm to be of the form:

$$T^- = \sum f_0 \mathbb{I}_4 \otimes \delta_{B,L} \quad (4.52a)$$

$$T^+ = \sum \begin{pmatrix} f_i i \sigma^i & 0 \\ 0 & f'_j i \sigma^j \end{pmatrix} \otimes \mathbb{I}_F, \quad (4.52b)$$

where the generators $\delta_{B,L}$ are of the form:

$$\delta_{B,L} = \{i l \mathbb{I}_2, i b \mathbb{I}_6, i l \mathbb{I}_2, i b \mathbb{I}_6, -i l \mathbb{I}_2, -i b \mathbb{I}_6, -i l \mathbb{I}_2, -i b \mathbb{I}_6\}, \quad (4.52c)$$

and $b, l \in \mathbb{R}$. Finally, demanding that the symmetry generators T are anomaly free (3.128) yields the additional restriction $l = -3b$. The derivation elements given in Eq. (4.52a) therefore generate a gauged baryon - Lepton number ($B-L$) symmetry, while the derivation elements given in Eq. (4.52b) generate local $SO(4)$ transformations.

4.2.2 Higgs boson mass revisited

The fused algebra formulation of the NCG SM offers a natural solution to the ‘Higgs mass problem’ (See Section 4.1.4) which does not rely on any modification of the NCG SM input data. The idea is very simple: When constructing a NCG the scalar fields appear as connections on the internal space through a ‘fluctuation’ procedure (see Section 4.1).

In the traditional approach only the inner fluctuations of the finite Dirac operator were considered, and so only the scalar fields charged under the SM gauge group $SU_c(3) \times SU_w(2) \times U_y(1)$ were ‘turned on’. As shown in Eq. (4.18), the term in the finite Dirac operator responsible for Neutrino Majorana masses is completely unaffected by the NCG SM inner automorphisms and so remains unfluctuated. The story is different however if we consider the full automorphism group of the fused algebra ΩB corresponding to the NCG SM, which includes a gauged B-L symmetry. Under this extra symmetry the term in the finite Dirac operator responsible for Majorana masses fluctuates or ‘turns on’. In other words, we obtain a minimal extension of the SM with an additional gauged boson associated the gauged B-L symmetry, and an additional complex scalar, which Higgses the gauged B-L symmetry. As shown by Chamseddine and Connes in [CC12], such an extended Higgs sector can be made compatible with the 125GeV detection at the LHC, and stabilizes the electroweak vacuum (previously there was also a concern that the Higgs quartic self-coupling would become negative at high energy [DLM14b, EMEG+12, DDVEM+12, CT12]). In their work however, Chamseddine and Connes introduced their real scalar field into the model by hand. In contrast, we obtain a complex analog of their scalar field which is charged under the gauged B-L symmetry as an output. In this Section I show how this all happens explicitly, and follow the computation given in [CC12], in which the Higgs mass is recalculated taking into consideration the extended scalar sector.

Under an automorphism $\alpha : \Omega B \rightarrow \Omega B$, the Dirac operator $D : \Omega \mathcal{H} \rightarrow \Omega \mathcal{H}$ must transform covariantly: $D \rightarrow D' = \alpha_{\mathcal{H}} D \alpha_{\mathcal{H}}^{-1} \approx D - [D, \delta_{\mathcal{H}}]$. As in ordinary gauge theory, by inspecting the fluctuation term $[D, \delta_{\mathcal{H}}]$, we can read off the ‘connection’ terms which must be added to D in order to make it covariant. The Dirac operator D on the product space is the sum of two terms, $D = D_c \otimes \mathbb{I}_F + \gamma_c \otimes D_F$, where $D_c = \gamma^\mu \nabla_\mu$ is the ordinary curved space Dirac operator, while D_F is a finite dimensional Hermitian matrix (see Ref. [vdDvS12]); thus its fluctuation has two terms as well:

$$[D, \delta_{\mathcal{H}}] = [D_c \otimes \mathbb{I}_F, \delta_{\mathcal{H}}] + [\gamma_c \otimes D_F, \delta_{\mathcal{H}}]. \quad (4.53)$$

Although it may be expressed in unfamiliar notation, the first (D_c) term on the right-hand side of (4.53) is nothing but the familiar term that, in ordinary gauge theory, forces one to introduce a gauge field A_μ^a corresponding to each generator t^a of the gauge group [in this case, $SU(3)_C \times SU(2)_L \times U(1)_Y \times U(1)_{B-L}$] in order to make the derivatives transform covariantly. In an analogous way, the second (D_F) term on the right-hand side of (4.53) forces us to add extra fields; but, whereas the first term involves the regular curved space Dirac operator $D_c = -i\gamma^\mu \nabla_\mu^S$, and thus induces fields $\gamma^\mu A_\mu$ with a spacetime index μ , the second term involves the finite matrix D_F , with no spacetime index, and thus induces fields with no spacetime index – *i.e.* scalars. This is one of the most important advantages of

Connes' approach: the gauge fields and scalar fields and their properties emerge hand in hand, from a single formula, as an inevitable consequence of covariance (in contrast to the standard approach, where the gauge fields and their properties emerge this way, but the scalar fields and their properties do not, and must instead just be added to the theory by hand). Let us now compute fluctuations associated to the D_F term in (4.53) and inspect the result.

As explained in Section 4.1.3 (see also Ref. [BF14, BBB15]), there are only four matrices D_F compatible with the associative algebras B_F and ΩB_F . The one which is relevant to describing the standard model is given by (4.4), where Y_l and Y_q are arbitrary 2×2 matrices that act on the doublet indices in the lepton and quark sectors, respectively, $m = \text{diag}\{a, 0\}$ is 2×2 diagonal, and for brevity we have written Y_q in place of $Y_q \otimes \mathbb{I}_3$. Thus, if we calculate the fluctuation $D \rightarrow D' \approx D - [\gamma_c \otimes D_F, \delta_\infty]$, where $\delta_{\mathcal{H}} = \delta_{\mathcal{H}}^{(3)}(x) + \delta_{\mathcal{H}}^{(2)}(x) + \delta_{\mathcal{H}}^{(1)}(x) + \alpha(x)\delta_{\mathcal{H}}^{(1)'}$, we find Y_l , Y_q and m transform as

$$Y'_l = Y_l - Y_l q_\lambda(x) + q(x)Y_l \quad (4.54a)$$

$$Y'_q = Y_q - Y_q q_\lambda(x) + q(x)Y_q \quad (4.54b)$$

$$m' = m + 2i\alpha(x)m \quad (4.54c)$$

where $q_\lambda(x) = \text{diag}\{\lambda(x), \bar{\lambda}(x)\}$. From this, we read off that, to make D covariant, as before Y_l and Y_q and must be promoted to fields

$$Y_l \rightarrow \begin{pmatrix} Y_\nu \phi_1 & Y_e \psi_1 \\ Y_\nu \phi_2 & Y_e \psi_2 \end{pmatrix}, \quad Y_q \rightarrow \begin{pmatrix} Y_u \phi_1 & Y_d \psi_1 \\ Y_u \phi_2 & Y_d \psi_2 \end{pmatrix}, \quad (4.55)$$

where $\{\varphi_1(x), \varphi_2(x)\}$ and $\{\psi_1(x), \psi_2(x)\}$ are scalar fields that transform as $SU(2)_L$ doublets, with hypercharge $y = +1/2$ and $y = -1/2$, respectively, while now we also find $m \rightarrow \text{diag}\{\sigma, 0\}$, where $\sigma(x)$ transforms with charge $+2$ under $U(1)_{B-L}$, but is a singlet under $SU(3)_C \times SU(2)_L \times U(1)_Y$. Finally, as explained in Ref. [BF14], one can choose the embedding of \mathbb{C} in \mathbb{H} so that $\{\psi_1, \psi_2\} = \{-\bar{\varphi}_2, \bar{\varphi}_1\}$; in this way, instead of obtaining a 2-higgs doublet model, one obtains a single higgs doublet $\{\varphi_1, \varphi_2\}$ (or alternatively one could keep the usual embedding and start with a 'flat' Dirac operator satisfying Eq. (4.7) and obtain the same result).

The scalar field σ has important phenomenological consequences. (i) As noted already, although the traditional NCG construction of the standard model predicted an incorrect Higgs mass ($m_h \approx 170$ GeV), several recent works [CC12, DLM14a, CCv13, CCvS13] have explained that an additional real singlet scalar field σ can resolve this problem, and also restore the stability of the Higgs vacuum. Our σ field, although somewhat different (since

it is complex, and charged under $B - L$), solves these same two problems for exactly the same reasons (as may be readily seen in the $U(1)_{B-L}$ gauge where σ is real). (ii) Second, precisely this field content (the standard model, including a right-handed neutrino in each generation of fermions, plus a $U(1)_{B-L}$ gauge boson C_μ , and a complex scalar field σ that is a singlet under $SU(3)_C \times SU(2)_L \times U(1)_Y$ but carries $B - L = 2$) has been previously considered [IOO09, BFFS11] because it provides a minimal and experimentally viable extension of the standard model that can account for several cosmological phenomena that are otherwise not explained: namely, the cosmological matter-antimatter asymmetry, the nature and abundance of dark matter, and the origin of nearly scale-invariant, gaussian and adiabatic spectrum of primordial curvature perturbations.

Next let us see more explicitly what effect the extended scalar sector has on the Higgs mass prediction. The following calculation follows closely the procedure for calculating the Higgs mass outlined in [CC12].

The bosonic action

Following Appendix A, and ignoring boundary terms, gravitational interactions and the cosmological constant term, the expansion of the bosonic action is given by:

$$\begin{aligned}
S_b &= i \int_M \frac{d^4x}{8\pi^2} f(0) \text{Tr}[-[\nabla^E, \Phi]^2 - \frac{4f_2}{f(0)} \Lambda^2 \Phi^2 + \Phi^4 - \frac{1}{3} F_{\mu\nu} F^{\mu\nu}] + O(\Lambda^{-1}) \\
&= i \int_M \frac{d^4x}{2\pi^2} f(0) [a |\partial_\mu \phi - \frac{1}{2} g_y B_\mu i \phi + \frac{1}{2} g_w i W_\mu^i \sigma_i \phi|^2 + \frac{c}{2} |(\partial_\mu - 2i g_{b-l} C_\mu) \sigma|^2 \\
&\quad - \frac{f_2}{f(0)} \Lambda^2 (4a |\phi|^2 + 2c |\sigma|^2) + b |\phi|^4 + 2e |\sigma|^2 |\phi|^2 + \frac{1}{2} d |\sigma|^4 \\
&\quad - (\frac{5}{3} g_y^2 B_{\mu\nu}^i B^{j\mu\nu} + \frac{4}{3} g_y g_{b-l} B_{\mu\nu}^i C^{j\mu\nu} + \frac{8}{3} g_{b-l}^2 C_{\mu\nu}^i C^{j\mu\nu} \\
&\quad + g_w^2 W_{\mu\nu} W^{\mu\nu} + g_c^2 G_{\mu\nu} G^{\mu\nu})] + O(\Lambda^{-1}), \tag{4.56a}
\end{aligned}$$

where terms a, b, c, d and e relate to the Yukawa couplings, and are given in Eq. (A.15). To express Eq. (4.35) in a more canonical form, the kinetic terms may be normalised:

$$\begin{aligned}
S_b &\sim i \int_M d^4x \frac{1}{2} [|(\partial_\mu - \frac{1}{2} g_y B_\mu i + \frac{1}{2} g_w i W_\mu^i \sigma_i) \phi'|^2 + \frac{1}{2} |(\partial_\mu - 2i g_{b-l} C_\mu) \sigma'|^2 \\
&\quad - \frac{1}{2} (\mu_\phi^2 |\phi'|^2 + \mu_\sigma^2 |\sigma'|^2) + \frac{1}{4} (\lambda_\phi |\phi'|^4 + 2\lambda_m |\sigma'|^2 |\phi'|^2 + \lambda_\sigma |\sigma'|^4) \\
&\quad - \frac{1}{4} (\xi_{ij} A_{\mu\nu}^i A^{j\mu\nu} + W_{\mu\nu} W^{\mu\nu} + G_{\mu\nu} G^{\mu\nu})] \tag{4.56b}
\end{aligned}$$

where

$$\phi' = \sqrt{\frac{af(0)}{\pi^2}}\phi, \quad \sigma' = \sqrt{\frac{cf(0)}{2\pi^2}}\sigma, \quad \frac{\mu_\phi^2}{2} = \frac{2f_2\Lambda^2}{f(0)} = \frac{\mu_\sigma^2}{2}, \quad (4.57a)$$

$$\frac{\lambda_\phi}{4} = \frac{\pi^2 b}{2a^2 f(0)}, \quad \frac{\lambda_m}{2} = \frac{2\pi^2 e}{ca f(0)}, \quad \frac{\lambda_\sigma}{4} = \frac{\pi^2 d}{c^2 f(0)}. \quad (4.57b)$$

As is done in [LX03], I have also re-expressed the $U(1)$ kinetic terms in the action as $-i \int_M d^4x \frac{1}{4} \xi_{ij} A_{\mu\nu}^i A^{j\mu\nu}$, where $A_{\mu\nu}^B = B_{\mu\nu}$, and $A_{\mu\nu}^C = C_{\mu\nu}$, and the coefficients ξ are given by:

$$\xi_{BB} = \frac{2f(0)}{\pi^2} \frac{5}{3} g_y^2, \quad \xi_{BC} = \xi_{CB} = \frac{f(0)}{\pi^2} \frac{4}{3} g_y g_{b-l}, \quad \xi_{CC} = \frac{2f(0)}{\pi^2} \frac{8}{3} g_{b-l}^2, \quad (4.58a)$$

Setting $\xi_{BB} = \xi_{CC} = 1$ fixes $\xi_{BC} = \xi_{CB} = \sqrt{1/10}$, and $\frac{5}{3} g_y^2 = \frac{8}{3} g_{b-l}^2 = g_w = g_s$. This normalization also fixes the value $\frac{10f(0)}{3\pi^2} g_y^2 = 1$.

The scalar mass spectrum

Having constructed the bosonic action, the next task is to determine the tree level mass spectrum of this NCG SM B-L extension. To make the calculation easier we can work in the unitary gauge where three real scalars of the complex Higgs doublet, and also one real scalar of the σ' complex singlet are gauged away:

$$\phi' = \begin{pmatrix} 0 \\ v_\phi + H \end{pmatrix}, \quad \sigma' = v_\sigma + \rho. \quad (4.59)$$

For tree-level symmetry breaking, we find

$$v_\phi^2 = \frac{\lambda_\sigma m_\phi^2 - \lambda_m m_\sigma^2}{\lambda_\phi \lambda_\sigma - \lambda_m^2}, \quad (4.60a)$$

$$v_\sigma^2 = \frac{\lambda_\phi m_\sigma^2 - \lambda_m m_\phi^2}{\lambda_\sigma \lambda_\phi - \lambda_m^2}. \quad (4.60b)$$

We take v_ϕ to be equal to the Higgs VEV at $246 GeV$, and for now we take $v_\sigma = R v_\phi$, where the ratio of the VEVs R is large, but so far undefined (it will need to be large to fit experiment as discussed below in Subsection 4.2.3). If we expand about the vacuum, we find that the potential energy at the minimum is negative and the mass matrix at the minimum is

$$2 \begin{pmatrix} \lambda_\phi v_\phi^2 & \lambda_m v_\phi^2 R \\ \lambda_m v_\phi^2 R & \lambda_\sigma R^2 v_\phi^2 \end{pmatrix}, \quad (4.61)$$

We can diagonalize this matrix to find the mass eigenstates and the corresponding mass-squared eigenvalues:

$$m_{\pm}^2 = v_{\phi}^2 [(\lambda_{\phi} + \lambda_{\sigma} R^2) \pm \sqrt{(\lambda_{\phi} - \lambda_{\sigma} R^2)^2 + 4\lambda_m^2 R^2}]. \quad (4.62)$$

For $R \gg 1$ the two mass eigenstates are given by:

$$m_{+}^2 \simeq 2\lambda_{\phi} v_{\phi}^2 (1 - \frac{\lambda_m^2}{\lambda_{\phi} \lambda_{\sigma}}), \quad m_{-}^2 \simeq 2\lambda_{\sigma} v_{\phi}^2 (R^2 + \frac{\lambda_m^2}{\lambda_{\sigma}^2}) \quad (4.63)$$

We have only to determine the value of the scalar quartic couplings. Fortunately, as was the case in Section 4.1.4 we can relate the quartic couplings in the bosonic action (4.56a) to the gauge couplings and Yukawa couplings at the unification scale Λ . Following [CC12], I will work in the rough approximation where the Yukawa couplings of the top quark Y_t , and tau neutrino (both Dirac and Majorana) Y_{τ} , Y_{σ} are dominant. I define $Y_{\tau} = \sqrt{n} Y_t$, $Y_{\sigma} = \sqrt{m} Y_t$, where n , and m are dimensionless constants that relate Y_t to Y_{ν} and Y_{σ} at the scale Λ . Using this approximation the terms given in Eq. (A.15) may be written as $a = (3 + n) Y_t^{\dagger} Y_t$, $b = (3 + n^2) (Y_t^{\dagger} Y_t)^2$, $c = Y_{\sigma}^{\dagger} Y_{\sigma}$, $d = (Y_{\sigma}^{\dagger} Y_{\sigma})^2$, and $e = n Y_{\sigma}^{\dagger} Y_{\sigma} Y_t^{\dagger} Y_t$. Substituting these relations and the relation $\frac{2f(0)}{\pi^2} g_w^2 = 1$ into Eq. (4.57b) yields:

$$\lambda_{\phi} = 4(3 + n^2) g_w^2 / (3 + n)^2, \quad \lambda_m = 8n g_w^2 / (3 + n), \quad \lambda_{\sigma} = 8g_w^2, \quad (4.64)$$

which hold only at the unification scale Λ . To determine the Higgs mass, we can therefore run the quartic couplings from the scale Λ down to the electroweak scale, and then substitute them into Eq. (4.63).

The quartic coupling beta functions depend on the Yukawa, and gauge couplings, and so their running must first be determined. The beta functions for the B-L extended SM, are slightly more complicated than those outlined in Subsection 4.1.4 because there will be mixing between the two species of scalar, and between the two $U(1)$ kinetic terms. There are two possible ways to deal with the mixing of the two $U(1)$ gauge couplings which occurs when running their gauge couplings. One could either continuously re-diagonalize the gauge kinetic terms, or alternatively one could simply run the off-diagonal mixing term ξ_{ij} introduced in Eq. (4.56). I follow the second approach, which is developed in [dACQ88, dAMPV95, FH91, Hol86, LX03]. In this approach the one loop beta functions for the $SU_c(3) \times SU_w(2) \times U_y(1)$ gauge couplings are unchanged from those given in Eq. (4.44). The beta function for the $U_{bl}(1)$ gauge coupling is given by:

$$(4\pi)^2 \beta(g_{bl}) = +\frac{72}{6} g_y^3, \quad (4.65a)$$

As explained in Subsection 4.1.4, the strong, electroweak, and hypercharge gauge couplings are well known at the electroweak scale, and on examining their running it is clear that there is no scale at which all three unify. To simplify the following calculations I will simply take the unification scale to be defined as the point at which the $SU_c(3)$ and $SU_w(2)$ gauge couplings meet. As shown in [CC12] however a 125GeV Higgs mass solution can be found for unification scale ranging between $10^{12}GeV$ and $10^{17}GeV$. One might hope that the extra $U(1)$ gauge symmetry might help fix the so called ‘Weinberg angle’ problem, and this is something that I will discuss in 4.2.3. The gauge coupling runnings are shown in Figure 4.5.

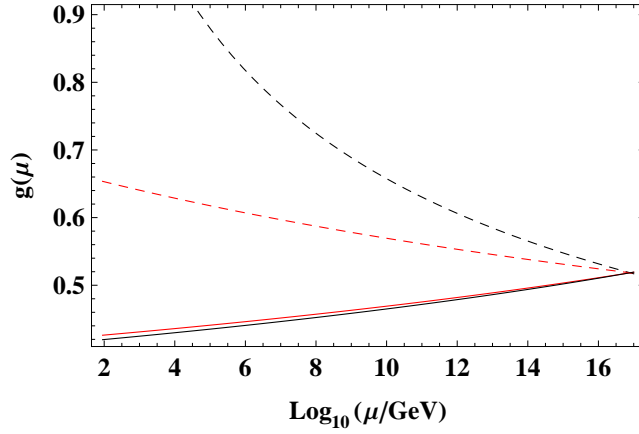


Figure 4.5: Gauge coupling of the NCG SM with gauged B-L extension: g_c (black,dashed), g_w (red,dashed), $\sqrt{5/3}g_y$ (red,solid), $\sqrt{8/3}g_{b-l}$ (black,solid).

Our next goal is to determine the running of the gauge-kinetic mixing coefficients. Their one loop beta functions are given by:

$$(4\pi)^2\beta_{yy} = \frac{41}{3}(\xi_{yy} - 1)g_y^2 \quad (4.66a)$$

$$(4\pi)^2\beta_{xx} = \frac{72}{3}(\xi_{xx} - 1)g_{bl}^2 \quad (4.66b)$$

$$(4\pi)^2\beta_{\kappa} = \left(\frac{41}{6}g_y^2 + 12g_{bl}^2\right)\kappa - \frac{32}{3}g_y g_{bl} \quad (4.66c)$$

where I have defined $\kappa \equiv \xi_{BC} = \xi_{CB}$. The gauge kinetic mixing terms have been canonically normalized in Eq. (4.56) such that $\xi_{BB} = \xi_{CC} = 1$, which fixes $\xi_{BC} = \xi_{CB} = \sqrt{1/10}$ at the unification scale. Note that with this choice of normalization the diagonal terms ξ_{BB} , and

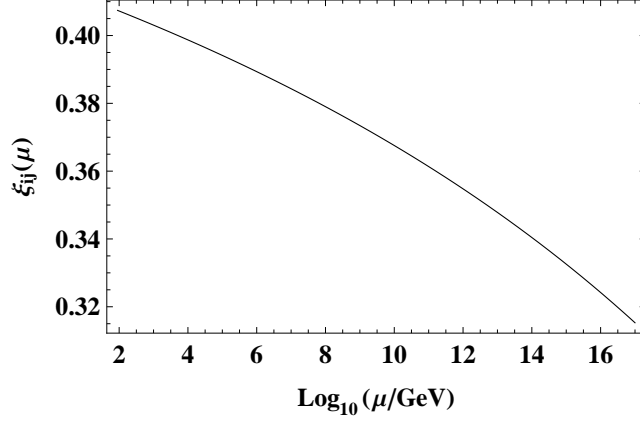


Figure 4.6: Gauge coupling kinetic mixing term ξ_{CB} for the NCG SM with gauged B-L extension

ξ_{CC} remain normalized and do not run. The running of the off diagonal term is shown in Figure 4.6.

The next step is to determine the running of the Yukawa couplings, as the scalar quartic coupling beta function depends on them. The one loop beta functions for the Yukawa couplings are given by:

$$(4\pi)^2 Y_t^{-1} \frac{dY_t}{dt} = \frac{9}{2} Y_t^\dagger Y_t + Y_\tau^\dagger Y_\tau - 8g_3^2 - \frac{9}{4} g_2^2 + \frac{1}{1-\kappa^2} \left[-\frac{17}{12} g_y^2 + \frac{5}{3} \kappa g_y g_{bl} - \frac{2}{3} g_{bl}^2 \right] \quad (4.67a)$$

$$(4\pi)^2 Y_\nu^{-1} \frac{dY_\nu}{dt} = \frac{5}{2} Y_\tau^\dagger Y_\tau + 3Y_t^\dagger Y_t - \frac{9}{4} g_2^2 + \frac{1}{2} Y_\nu^{-1} Y_\sigma Y_\sigma^\dagger Y_\nu + \frac{1}{1-\kappa^2} \left[-\frac{3}{4} g_y^2 + 3\kappa g_y g_{bl} - 6g_{bl}^2 \right] \quad (4.67b)$$

$$(4\pi)^2 \frac{dY_\sigma}{dt} = Y_\nu Y_\nu^\dagger Y_\sigma + (Y_\nu Y_\nu^\dagger Y_\sigma)^T + Y_\sigma Y_\sigma^\dagger Y_\sigma + \frac{1}{2} Y_\sigma \text{Tr}[Y_\sigma^\dagger Y_\sigma] + \frac{1}{1-\kappa^2} [-6g_{bl}^2 Y_\sigma] \quad (4.67c)$$

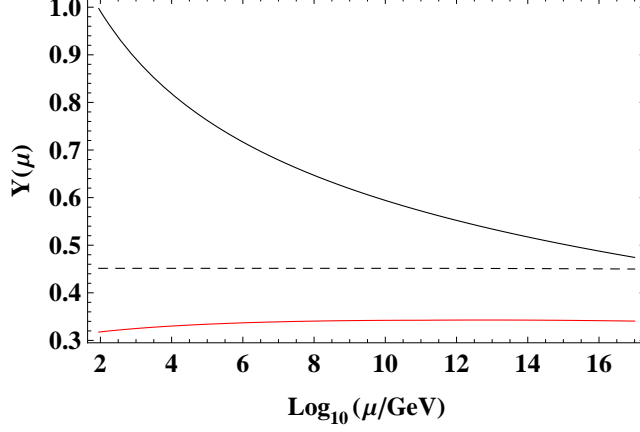


Figure 4.7: Yukawa coupling terms for the NCG SM with gauged B-L extension: Y_t (Black), Y_τ (Red), Y_M (black, dashed).

While the beta functions for the scalar quartic couplings are given by:

$$(4\pi)^2\beta_h = 24\lambda_h^2 + 4\lambda_m^2 - 2b + 4a\lambda_h - 3\left(\frac{g_y^2}{1-\kappa^2} + 3g_2^2\right)\lambda_h + \frac{9}{8}g_2^4 + \frac{3}{8}\frac{g_y^4}{(1-\kappa^2)^2} + \frac{3}{4}\frac{g_2^2g_y^2}{1-\kappa^2} \quad (4.68a)$$

$$(4\pi)^2\beta_\varphi = 20\lambda_\varphi^2 + 8\lambda_m^2 - \text{Tr}[(Y_m^\dagger Y_m)^2] + \text{Tr}[Y_m^\dagger Y_m]\lambda_\varphi - 48\frac{g_{bl}^2}{1-\kappa^2}\lambda_\varphi + 96\frac{g_{bl}^4}{(1-\kappa^2)^2} \quad (4.68b)$$

$$(4\pi)^2\beta_m = [12\lambda_h + 8\lambda_m + 8\lambda_\varphi]\lambda_m - 2\text{Tr}[Y_\nu Y_\nu^\dagger Y_m Y_m^\dagger] + 6\frac{\kappa^2 g_y^2 g_{bl}^2}{(1-\kappa^2)^2} + (2a + \text{Tr}[Y_m^\dagger Y_m])\lambda_m - \frac{3}{2}\lambda_m\left(3g_2^2 + \frac{g_y^2 + 16g_{bl}^2}{1-\kappa^2}\right) \quad (4.68c)$$

Having run the quartic couplings, their value at the electroweak scale can be substituted into (4.63) to determine the value of the Higgs mass which will now be given as a function of the free parameters m and n (and the unification scale). Chamseddine and Connes claim to find 125GeV Higgs mass solutions [CC12], which we are currently in the midst of checking.

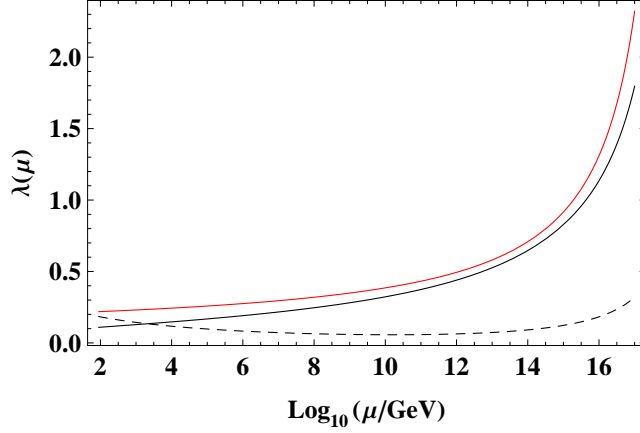


Figure 4.8: Scalar quadratic couplings terms for the NCG SM with gauged B-L extension: λ_ϕ (black, dashed), λ_m (black), λ_σ (red).

4.2.3 The Weinberg angle problem.

Let us have a closer look at the gauge boson mass spectrum in our gauged B-L extended SM, and compare it to the mass spectrum of the standard model. To begin with consider once again the SM. The SM bosonic mass matrix M^2 given in Eq. (4.40) is expressed in the so called ‘kinetic basis’. To determine the physical masses of the system one must first change to the ‘mass basis’ in which the mass matrix is diagonalized. One can diagonalize by conjugating with the ‘rotation’ matrix R:

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{g_w}{\sqrt{g_w^2 + g_y^2}} & \frac{g_y}{\sqrt{g_w^2 + g_y^2}} \\ 0 & 0 & -\frac{g_y}{\sqrt{g_w^2 + g_y^2}} & \frac{g_w}{\sqrt{g_w^2 + g_y^2}} \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos[\theta_W] & \sin[\theta_W] \\ 0 & 0 & -\sin[\theta_W] & \cos[\theta_W] \end{pmatrix} \quad (4.69)$$

where I have defined $\sin[\theta_W] \equiv \frac{g_y}{\sqrt{g_w^2 + g_y^2}}$, $\cos[\theta_W] \equiv \frac{g_w}{\sqrt{g_w^2 + g_y^2}}$, and θ_W is known as the ‘Weinberg’ or ‘electroweak mixing’ angle. The mass eigenvalues are given by: $\{\frac{1}{4}g_w^2 v_\phi^2, \frac{1}{4}g_w^2 v_\phi^2, \frac{1}{4}(g_y^2 + g_w^2)v_\phi^2, 0\}$, and one notices in particular that the ratio of the W bosons to the Z boson is given by:

$$\frac{M_W}{M_Z} = \frac{g_w}{\sqrt{g_w^2 + g_y^2}} = \cos[\theta_W] \quad (4.70)$$

The masses of the W and Z bosons are measured to extremely high precision, which places very strong constraint on the value of the electroweak coupling at the electroweak scale. In

particular, without new physics entering above the electroweak scale this rules out gauge coupling unification. Said another way: if we impose the GUT relation $\frac{5}{3}g_y^2 = g_w^2 = g_c^2$ at the point at which the strong and electroweak couplings are equal, then on running down all three couplings to the electroweak scale, they do not satisfy Eq. (4.70).

Our extended SM with gauged B-L symmetry has additional particle content, and one would hope that this might allow for a consistent picture in which all four gauge couplings do unify. Let's see if this is the case. The first step is to diagonalize the $U(1)$ gauge kinetic terms $\xi_i^j A_{\mu\nu}^i A_j^{\mu\nu}$ given in Eq. (4.56). We do this by making the transformation $A_i^{\mu\nu} \rightarrow (S_1^{-1})_i^k A_k^{\mu\nu}$, and $\xi_i^j \rightarrow (S_1^{-1})_i^l \xi_l^k (S_1)_k^j$ with the matrix (remembering our normalization $\xi_{BB} = \xi_{CC} = 1, \xi_{BC} = \xi_{CB} = \kappa$):

$$S_1^{-1} = \begin{pmatrix} \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/2} & \sqrt{1/2} \end{pmatrix} \quad (4.71)$$

After this transformation $\xi = \text{diag}\{1 - \kappa, 1 + \kappa\}$, and we can absorb these diagonal values into a renormalization of the gauge potentials. We therefore define the diagonalized, renormalized $U(1)$ field strength tensors to be:

$$\begin{pmatrix} A_{\mu\nu}^- \\ A_{\mu\nu}^+ \end{pmatrix} = \begin{pmatrix} \cos[\theta] & -\sin[\theta] \\ \sin[\theta] & \cos[\theta] \end{pmatrix} \begin{pmatrix} \sqrt{1 - \kappa} & 0 \\ 0 & \sqrt{1 + \kappa} \end{pmatrix} \begin{pmatrix} \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/2} & \sqrt{1/2} \end{pmatrix} \begin{pmatrix} B_{\mu\nu} \\ C_{\mu\nu} \end{pmatrix} \quad (4.72)$$

where the final matrix on the left represents the freedom to make an arbitrary $SO(2)$ transformation of the fields once they have been diagonalized and normalized. We will leave the angle θ arbitrary for now. To ensure the rest of the action remains invariant under this field re-definition we similarly define new fermionic and scalar couplings and charges:

$$\begin{pmatrix} q_j^- g_- \\ q_j^+ g_+ \end{pmatrix} = \begin{pmatrix} \cos[\theta] & -\sin[\theta] \\ \sin[\theta] & \cos[\theta] \end{pmatrix} \begin{pmatrix} 1/\sqrt{1 - \kappa} & 0 \\ 0 & 1/\sqrt{1 + \kappa} \end{pmatrix} \begin{pmatrix} \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/2} & \sqrt{1/2} \end{pmatrix} \begin{pmatrix} q_j^B g_y \\ q_j^C g_{b-l} \end{pmatrix} \quad (4.73)$$

where j ranges over all fermions and scalars. Having Diagonalized the gauge kinetic terms we can now determine the mass spectrum of the gauge bosons at tree level by considering the scalar kinetic terms in Eq. (4.56):

$$\begin{aligned} \frac{1}{2}|D\phi'|^2 + \frac{1}{2}|D\sigma'|^2 &= \frac{1}{2}|\partial_\mu + iq_\phi^- g_- A_\mu^- + iq_\phi^+ g_+ A_\mu^+ + ig_w W_\mu^i \sigma_i \phi'|^2 \\ &+ \frac{1}{2}|\partial_\mu + iq_\sigma^- g_- A_\mu^- + iq_\sigma^+ g_+ A_\mu^+ \sigma'|^2 \end{aligned} \quad (4.74)$$

Working in the Unitary gauge (see Eq. (4.59)) the relevant terms in the extended Lagrangian for determining the masses of the gauge bosons on the basis $\{W_\mu^+, W_\mu^-, W_\mu^3, A_\mu^+, A_\mu^-\}$ then become:

$$\frac{v_h^2}{4} \begin{pmatrix} g_w^2 & 0 & 0 & 0 & 0 \\ 0 & g_w^2 & 0 & 0 & 0 \\ 0 & 0 & g_w^2 & -2q_\phi^+ g_w g_+ & -2q_\phi^- g_w g_- \\ 0 & 0 & -2q_\phi^+ g_w g_+ & g_+^2 ((q_\phi^+)^2 + R^2 (q_\sigma^+)^2) & g_- g_+ (q_\phi^- q_\phi^+ + R^2 q_\sigma^- q_\sigma^+) \\ 0 & 0 & -2q_\phi^- g_w g_- & g_- g_+ (q_\phi^- q_\phi^+ + R^2 q_\sigma^- q_\sigma^+) & g_-^2 ((q_\phi^-)^2 + R^2 (q_\sigma^-)^2) \end{pmatrix}, \quad (4.75)$$

which has eigenvalues $\{M_{W^+}^2, M_{W^-}^2, M_Z^2, M_{Z'}^2, 0\}$, where the eigenstate with zero mass is the photon γ_μ ; the two eigenstates with mass-squared $(1/4)g_w^2 v_h^2$ are the familiar W_μ^+ and the W_μ^- bosons; and the remaining two eigenvalues correspond to two additional massive gauge bosons: the Z_μ boson and a new heavy boson Z'_μ , with mass squared values

$$M_Z^2 = \frac{m_1 v_\phi^2}{2} (1 - \sqrt{1 - 4R^2 m_2 / m_1^2}), \quad M_{Z'}^2 = \frac{m_1 v_\phi^2}{2} (1 + \sqrt{1 - 4R^2 m_2 / m_1^2}), \quad (4.76a)$$

$$m_1 = g_-^2 ((q_\phi^-)^2 + R^2 (q_\sigma^-)^2) + g_+^2 ((q_\phi^+)^2 + R^2 (q_\sigma^+)^2) + g_w q_w^2, \quad (4.76b)$$

$$m_2 = -2g_-^2 g_+^2 q_\phi^- q_\sigma^- q_\phi^+ q_\sigma^+ + g_+^2 (q_\sigma^+)^2 (g_-^2 (q_\phi^-)^2 + g_w^2 q_w^2) + g_-^2 (q_\sigma^-)^2 (g_+^2 (q_\phi^+)^2 + g_w^2 q_w^2). \quad (4.76c)$$

The main point to note is that the ratio of the W boson and Z boson masses is no longer given as in Eq. (4.70). The ratio of the masses is determined now as a function of the scalar charges which are known, and the gauge couplings g_w, g_y, g_{b-l} , which are fixed at the electroweak scale by the running: ie we can run the couplings g_w and g_c from the electroweak scale up to the scale at which they ‘unify’ and then impose the relation $\frac{5}{3}g_y^2 = \frac{8}{3}g_{bl}^2 = g_w^2 = g_c^2$ obtained from the expansion of the spectral action. We can then run all four couplings back down to the electroweak scale (see Fig. 4.5). Notice however that the ratio of the W boson and Z boson masses is now also a function of ratio between the two scalar VEVs R , which is a free parameter. It might be possible to select a value of R which gives the correct electroweak phenomenology, and which is also large enough so that the Z' boson is out of range of current detectors. Unfortunately this is not what happens.

As might have been guessed. Rather than ‘fix’ the Weinberg angle ‘problem’ the model simply asymptotes to the same incorrect value of m_Z as would be obtained by assuming unification in the SM. If it had asymptoted from above, then a solution might have been possible.

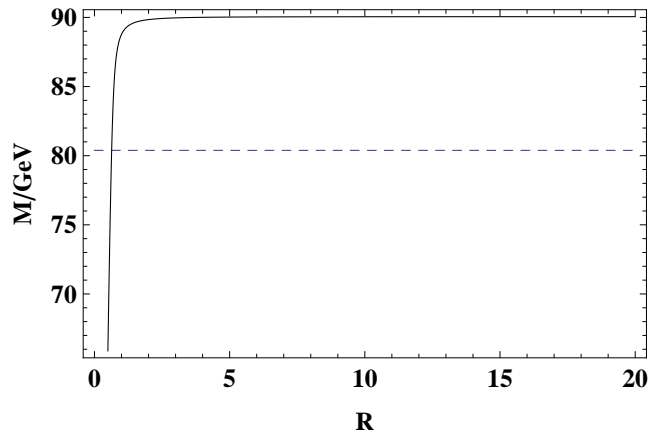


Figure 4.9: W and Z boson masses for the NCG SM with gauged B-L extension as a function of the ratio of the two scalar VEVs: M_W (dashed), M_Z (solid).

Chapter 5

Example non-associative geometries

The fused algebra perspective allows access to a range of new geometries beyond those of the associative formalism. One might imagine therefore that once we relax the associativity requirements of ΩB , that there would be a flood of new example geometries to play with. As it turns out however, the formalism remains quite restrictive, and in particular it is very difficult to construct sensible finite non-associative geometries that display a non-trivial Dirac operator. Indeed, I introduced the first almost-associative geometry in [FB13], which described Einstein gravity coupled to a G_2 gauge theory. This model however had a trivial finite Dirac operator, which resulted in a trivial Higgs sector, and massless fermions (see Subsection 3.1.5 for details). In this chapter I will provide a wide range of examples. The organization is as follows: in Section 5.1 I construct a family of simple non-associative geometries in each KO-dimension. In Subsection 5.2 I provide a pair of representations B_0 , which are ‘exotic’ in the sense that B_0 has different associativity properties to \mathcal{A} . In Section 5.3 I explore models in which non-abelian symmetries arise as outer automorphisms. In Subsection 5.3.2 I show how to construct Pati-Salam-like models from non-associative input data. Finally in Subsection 5.3.3 I discuss more exotic grand unified theories based on the exceptional Jordan algebra.

5.1 A family of fully non-associative spaces

In this Section I construct a family of finite non-associative geometries which have both non-associative input algebras, and non-trivial Higgs sectors. The family is based on the simple non-associative octonionic example which I outlined in Subsection 3.1.5. We start by introducing the representation π of a finite (non-associative) $*$ -algebra \mathcal{A} over the field

\mathbb{F} on a Hilbert space \mathcal{H} . We will be interested in simple models in which the finite Hilbert space \mathcal{H} , is the space of n -vectors with components valued in \mathcal{A} . The inner product on \mathcal{H} is given by $\langle x|y \rangle = \sum_{m=1}^n \langle x_m|y_m \rangle$. Elements $a \in \mathcal{A}$ will be represented on \mathcal{H} as $n \times n$ matrices, proportional to a itself, so we can factor out a and write

$$\pi(a) = p a \quad (5.1)$$

where p is an $n \times n$ matrix with its components valued in \mathbb{F} . The condition $\pi(ab) = \pi(a)\pi(b)$ implies p must be idempotent: $p^2 = p$; and the condition $\pi(a^*) = \pi(a)^*$ implies that either $p = \bar{p}$ or $p = p^\dagger$, depending on whether we choose to define $\pi(a)^*$ as $\bar{p}a^*$ or $p^\dagger a^*$. Eliminating any ambiguity, we assume that both conditions hold, which implies that p is also symmetric ($p = p^T$). Next, we introduce the generalized Dirac operator D , as a $n \times n$ matrix, with components valued in the derivations of \mathcal{A} , $\mathcal{D}(\mathcal{A})$ – that is, each component D_{ij} is an inner derivation acting on the components of elements of \mathcal{H} . Since D is hermitian and each component (*i.e.* each derivation on \mathcal{A}) is anti-hermitian \mathcal{A} we have

$$D_{ij} = -D_{ji}. \quad (5.2)$$

Such a Dirac operator can be written in the form:

$$D = \gamma_F^\alpha \delta_\alpha, \quad (5.3)$$

where the γ_F^α are $n \times n$ anti-hermitian matrices with elements valued in \mathbb{F} , and the δ_α are a basis of inner derivation elements. We will adopt this notation from here on. Next we introduce the anti-linear, unitary operator J :

$$J = j \circ “ * ”. \quad (5.4)$$

In other words, J is the composition of j with $*$, where j is an ordinary $n \times n$ unitary matrix with components valued in \mathbb{F} , and $“ * ”$ is the involution operation in \mathcal{A} . As J is an anti-linear, unitary operator, its adjoint is defined by $\langle J^\dagger x|y \rangle := \overline{\langle x|Jy \rangle}$ ($\forall x, y \in H_F$), so that we have $J^\dagger J = 1$, with $J^\dagger = j^T \circ “ * ”$. Finally, in the case of even K0 dimension, we also introduce the operator γ , an $n \times n$ matrix valued in \mathbb{F} ; it is both hermitian and unitary ($\gamma = \gamma^\dagger = \gamma^{-1}$), and hence has eigenvalues ± 1 . We require that γ commutes with the action of the algebra on \mathcal{H} , and anti-commutes with D :

$$[\gamma, L_a] = 0 \quad (\forall a \in \mathcal{A}), \quad (5.5)$$

$$\{\gamma_F, \gamma_F^\alpha\} = 0, \quad (5.6)$$

for all $a \in \mathcal{A}$, and all α labelling the basis elements of $\mathcal{D}(\mathcal{A})$. Finally, to get the right K0 dimension, we must choose j , D and γ such that

$$J^2 = \epsilon \quad \Leftrightarrow \quad j^T = \epsilon j, \quad (5.7a)$$

$$JD = \epsilon' DJ \quad \Leftrightarrow \quad j\gamma_F^\alpha = \epsilon' \gamma_F^\alpha j, \quad (5.7b)$$

$$J\gamma = \epsilon'' \gamma J \quad \Leftrightarrow \quad j\gamma = \epsilon'' \gamma j. \quad (5.7c)$$

where the signs ϵ , ϵ' and ϵ'' are either plus or minus one [vdDvS12]. We will be interested in constructing simple geometries in which the algebra ΩB has the same associative properties as \mathcal{A} . The simplest way of doing this is to ensure that both the order zero, and order one commutators are proportional to associators. We will not consider the restrictions placed on the Dirac operator by the higher order conditions here. The order zero commutator evaluates to

$$[L_a, JL_b^* J^\dagger]h = (pjppj^\dagger)_j^i a(h_i b) - (jppj^\dagger p)_j^i (ah_i) b, \quad (5.8)$$

where $h \in H$; $a, b \in A$, and the i, j are matrix and vector indices ranging from 1 to n . We therefore demand that the matrices p and j satisfy

$$[p, jppj^\dagger] = 0, \quad (5.9)$$

which results in the commutator evaluating on any Hilbert space element to an n -vector of \mathcal{A} valued associators:

$$[L_a, JL_b^* J^\dagger]h = -(pjppj^\dagger)_j^i [a, h_i, b]. \quad (5.10)$$

Notice that in the special case in which the input algebra is associative, the right hand side of (5.10) is identically zero for any choice of $h \in \mathcal{H}$, and the expression therefore collapses to the familiar order zero condition. More generally, the associator on the right hand side of equation 5.10 will by definition satisfy the required multilinear associator identities of any input algebra \mathcal{A} , because $a, b, h_i \in \mathcal{A}$. Next we should consider the order one commutator, but first however note that

$$[D, L_a]h = \gamma_F^\alpha L_{\delta_\alpha a} h + \delta_\alpha [\gamma_F^\alpha, L_a]h. \quad (5.11)$$

We eliminate the term proportional to δ_α in the above expression by demanding that the matrices D and p commute:

$$[\gamma_F^\alpha, p] = 0. \quad (5.12)$$

With this condition, the order one commutator automatically evaluates on any Hilbert space element to an n -vector of \mathcal{A} valued associators:

$$[JL_b^* J^*, [D, L_a]]_j^i h_i = (pjppj^\dagger \gamma_F^\alpha)_j^i [(\delta_\alpha a), h_i, b], \quad (5.13)$$

where $h \in \mathcal{H}; b, \delta_\alpha a \in \mathcal{A}$, and the i, j are matrix and vector indices ranging from 1 to n . By definition, the right hand side of equation 5.13 satisfies the required multilinear associative identities of \mathcal{A} . In the case where \mathcal{A} is associative, equation 5.13 reduces to the familiar order one condition.

Now that we have laid out the conditions that must be satisfied, we can present the simplest geometry (with non-vanishing D) which satisfies these conditions in each KO dimension:

$$K0 = 0 : \quad \pi(a) = a \mathbb{I}_2 \quad D = \begin{pmatrix} 0 & +\delta \\ -\delta & 0 \end{pmatrix} \quad J = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \circ \text{“ * ”} \quad (5.14a)$$

$$K0 = 1 : \quad \pi(a) = a \mathbb{I}_2 \quad D = \begin{pmatrix} 0 & +\delta \\ -\delta & 0 \end{pmatrix} \quad J = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \circ \text{“ * ”} \quad (5.14b)$$

$$K0 = 2 : \quad \pi(a) = a \mathbb{I}_2 \quad D = \begin{pmatrix} 0 & +\delta \\ -\delta & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \circ \text{“ * ”} \quad (5.14c)$$

$$K0 = 3 : \quad \pi(a) = a \mathbb{I}_2 \quad D = \begin{pmatrix} 0 & +\delta \\ -\delta & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \circ \text{“ * ”} \quad (5.14d)$$

$$K0 = 4 : \quad \pi(a) = a \mathbb{I}_4 \quad D = \begin{pmatrix} 0 & +\Delta_+ \\ +\Delta_+ & 0 \end{pmatrix} \quad J = \begin{pmatrix} +\sigma & 0 \\ 0 & +\sigma \end{pmatrix} \circ \text{“ * ”} \quad (5.14e)$$

$$K0 = 5 : \quad \pi(a) = a \mathbb{I}_4 \quad D = \begin{pmatrix} 0 & +\Delta_- \\ -\Delta_- & 0 \end{pmatrix} \quad J = \begin{pmatrix} +\sigma & 0 \\ 0 & +\sigma \end{pmatrix} \circ \text{“ * ”} \quad (5.14f)$$

$$K0 = 6 : \quad \pi(a) = a \mathbb{I}_4 \quad D = \begin{pmatrix} 0 & \sigma\delta \\ \sigma\delta & 0 \end{pmatrix} \quad J = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \circ \text{“ * ”} \quad (5.14g)$$

$$K0 = 7 : \quad \pi(a) = a \mathbb{I}_2 \quad D = \begin{pmatrix} 0 & +\delta \\ -\delta & 0 \end{pmatrix} \quad J = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix} \circ \text{“ * ”} \quad (5.14h)$$

where \mathbb{I}_2 and \mathbb{I}_4 are the 2×2 and 4×4 identity matrices, respectively; the 2×2 matrices σ and Δ_\pm are given by

$$\sigma \equiv \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \quad \Delta_\pm = \begin{pmatrix} +\delta_1 & +\delta_2 \\ \mp\delta_2 & \pm\delta_1 \end{pmatrix}, \quad (5.15)$$

and, in the case of even KO dimension, γ is given by

$$\gamma = \begin{pmatrix} +\mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}. \quad (5.16)$$

Notice that in KO-dimensions 4, 5, and 6 the simplest models displaying a non-trivial Dirac operator require a larger vector space representation than the rest ($n = 4$).

5.2 Exotic spaces.

The ‘square zero extensions’ definition of algebra representations outlined in Section (3.2) is equivalent to the ‘multi-linear identities’ definition outlined in 3.1 only in the special case in which the input algebra \mathcal{A} is taken to have the same associativity properties as the algebra B_0 . To this point, I have only explored models which are of this type. While these models are in some sense ‘natural’ there is no reason a priori for this restriction (it is as natural as restricting to associative models for example). In this section I will explicitly construct two non-trivial example fused algebras in which B_0 and \mathcal{A} have different associative properties, and determine the form that inner derivations take on B_0 .

5.2.1 Associative-Jordan representations

As a first example, consider a fused algebra $B_0 = \mathcal{A} \oplus \mathcal{H}$, where the input algebra \mathcal{A} is taken to be a finite, non-commutative associative algebra represented on itself (ie. $\mathcal{H} = \mathcal{A}$) via the Jordan product $L_a = R_a = k\{a, -\}$, for $k \in \mathbb{R}$. As usual, the product on B_0 is defined as in equation (3.50). This extended algebra is clearly not a Jordan algebra, because \mathcal{A} is associative and not commutative, but it is also not an associative algebra because the product between elements of \mathcal{A} on \mathcal{H} does not associate with the product between elements of \mathcal{A} , ie. $[a, b, h] \neq 0$. I will call such an algebra an ‘associative-Jordan’ square zero extension, or an associative-Jordan fused algebra.

Let’s determine the general form that inner derivations take on the full algebra B_0 . For an associative algebra the inner derivations are given by equation (2.28a), while for a Jordan algebra the derivations are given by equation (2.28c). B_0 however is neither purely associative, nor purely Jordan. It is instead some kind of hybrid. Let us therefore make the following guess for the general form of the grading preserving inner derivations on B_0 :

$$\delta_{xy} = [L_x, L_y] + S[L_x, R_y] + [R_x, R_y], \quad (5.17)$$

for $x, y \in B_0$ and $S \in \mathbb{R}$. Actually, for derivations of the form given in Eq. (5.17) to generate unitary automorphisms then $x, y \in \mathcal{A}$. Derivations of this form are automatically compatible with the natural involution on B_0 in the sense of Eq. (2.24b) for anti-hermitian $x, y \in \mathcal{A}$. We do however have to check that these derivations satisfy the leibniz property given in equation (2.24a):

$$\delta_{xy}(ab) = \delta_{xy}(a)b + a\delta_{xy}(b) \quad (5.18a)$$

$$\delta_{xy}(L_a h) = L_{\delta_{xy}a} h + L_a \delta_{xy}(h), \quad (5.18b)$$

Checking (5.18a) is fairly straightforward; we have:

$$\begin{aligned}\delta_{xy}(a)b + a\delta_{xy}(b) &= [[x, y], a]b + a[[x, y]b] \\ &= [[x, y], ab] = \delta_{xy}(ab)\end{aligned}\tag{5.19}$$

for $a, b, x, y \in A$. We see that the required Leibniz property is satisfied for all values of $k, S \in \mathbb{R}$. Checking condition (5.18b) is a little more leg work, but yields:

$$\begin{aligned}L_{\delta_{xy}a}h + L_a\delta_{xy}(h) &= L_{[[x, y], a]}h + (2 + S)L_a[L_x, L_y]h \\ &= \frac{1}{k^2}[L_x, L_y]L_a h + (2 + S - \frac{1}{k^2})L_a[L_x, L_y]h.\end{aligned}\tag{5.20}$$

We see that condition (5.18b) is only satisfied when $S = \frac{1}{k^2} - 2$. We therefore take inner derivations on B_0 to be of the form

$$\delta_{xy} = [L_x, L_y] + (\frac{1}{k^2} - 2)[L_x, R_y] + [R_x, R_y]\tag{5.21a}$$

where x, y are anti-hermitian elements in A . Notice that when acting on elements in \mathcal{A} the derivations given in Eq. (5.21a) collapsed to the usual associative form $\delta_{xy} = L_{[x, y]} - R_{[x, y]}$, while on elements of \mathcal{H} it collapses to the Jordan form $\delta_{xy} = \frac{1}{k^2}[L_x, L_y]$. In other words, we could have constructed these ‘associative-jordan’ inner derivations directly by considering at the outset the general form $\delta = \delta_A \oplus \delta_H$, in which case we would find:

$$\delta = L_{[x, y]} - R_{[x, y]} \oplus \frac{1}{k^2}[L_x, L_y]\tag{5.21b}$$

5.2.2 Associative-alternative representations

In associative NCG many left right symmetric models of interest beyond the standard model are excluded by the associative order one condition. In a recent paper [CCvS13] the associative order one condition was removed in order to build a Pati-Salam type model. Let’s go one step further and look at representations in which the order zero condition is not satisfied. Consider for example a model in which the finite associative algebra $\mathcal{A} = M_n(\mathbb{C})$ is represented as a real algebra on the input Hilbert space $\mathcal{H} = \mathbb{C}^{2n}$:

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ cc,\tag{5.22}$$

for $a \in \mathcal{A}$. While \mathcal{A} is associative, B_0 is not because the order zero condition is not satisfied. Unlike in Subsection 5.2.1, let us use the prescription for determining the inner derivations

of B_0 that we developed in Subsection 3.3.1. To start with we know that general inner derivations will take the form:

$$\delta = \delta_A \oplus \delta_H, \quad (5.23)$$

where because the sub-algebra \mathcal{A} is associative, δ_A will be of the form $L_x - R_x$ where x is an anti-hermitian element of \mathcal{A} . Derivations on B_0 must satisfy the Leibniz conditions given in equation (2.24a), and so we have:

$$\begin{aligned} [\delta_H, L_a] &= L_{[x,a]} = [L_x, L_a] \\ &\Rightarrow \delta_H = L_x + \delta_R \end{aligned} \quad (5.24)$$

$$\begin{aligned} [\delta_H, R_a] &= R_{[x,a]} = -[R_x, R_a] \\ &\Rightarrow \delta_H = -R_x + \delta_L. \end{aligned} \quad (5.25)$$

where $[L_a, \delta_R] = [R_a, \delta_L] = 0$ for all $a \in \mathcal{A}$. For an associative representation satisfying the order zero condition (ie $[R_a, L_x] = 0 \forall a, b \in \mathcal{A}$) one could simply set $\delta_L = L_x$, and $\delta_R = -R_x$, in which case a general derivation would be of the form $\delta = (L_x - R_x) \oplus (L_x - R_x)$. In other words we would recover Connes inner derivations because B_0 would be associative. Unfortunately this example representation does not satisfy the order zero condition, and so at first glance it looks as though B_0 may not have any non-zero inner derivations. The reader may check however, that for anti-hermitian $x \in A$ that indeed $L_x = -R_x$. So finally, we may write general inner derivations on B_0 as:

$$\delta = (L_x - R_x) \oplus L_x = (L_x - R_x) \oplus -R_x \quad (5.26)$$

for any anti-hermitian $x \in A$. Notice that in the case were we may write $x = [y, z]$ for anti-hermitian $y, z \in A$, then (5.26) may be written as:

$$\begin{aligned} \delta &= ([L_y, L_z] - [R_z, R_y]) \oplus ([L_y, L_z] + [L_y, L_z] - [L_y, L_z]) \\ &= ([L_y, L_z] + [R_y, R_z] + [L_y, R_z]) \oplus ([L_y, L_z] + [R_y, R_z] + [L_y, R_z]), \end{aligned} \quad (5.27)$$

where on the associative input algebra, terms of the form $[L_y, R_z]$ (ie. associators) are equal to zero. We may therefore write all such inner derivations on B_0 compactly as $\delta_{xy} = [L_y, L_z] + [L_y, R_z] + [R_y, R_z]$ for $x, y \in \mathcal{A}$, which comparing with Eq. (2.28d) we notice to be of alternative form. Notice also that $[a, h, b] = -[b, h, a]$, $[a, b, h] = -[b, a, h]$, and $[h, a, b] = -[h, b, a] \forall a, b \in \mathcal{A}; h \in \mathcal{H}$, which are all alternative conditions as given in Eq. (2.7)¹. For this reason I will call this kind of representation an associative-alternative representation.

¹Actually these last two conditions are trivially true because the representation is both left associative, and right associative

5.3 Non-abelian outer symmetries

As outlined in Section 3.3, finite NCGs often have automorphism groups which are larger than just the group of inner automorphisms of \mathcal{A} because their corresponding fused algebras B_0 are not semi-simple. These extra symmetries (in the $U(1)$ case) have been explored in the NCG literature, and are known as central extensions [Sch00, LS01]. The gauged B-L symmetry, and local $SO(4)$ symmetry which appears in the extended NCG SM are both examples of such ‘extended’ symmetries, as they are not constructible from input algebra elements (see Section 4.2). In this section I provide example models which display non-abelian ‘extended’ or ‘accidental’ symmetries.

5.3.1 Jordan algebra example

Consider the input data $\{\mathcal{A}, \mathcal{H} = \mathcal{A}^4, J, \gamma\}$, where \mathcal{A} is the finite, unital, Jordan algebra of $n \times n$ complex hermitian matrices with representation $\pi(a) = a\mathbb{I}_4$ on \mathcal{H} , and:

$$J = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \circ (*), \quad \gamma = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix} \equiv \gamma_5. \quad (5.28)$$

The grading preserving automorphisms of the fused algebra $B_0 = \mathcal{A} \oplus \mathcal{H}$ are generated by derivations of the form:

$$\delta_B = \delta_A \oplus \delta_H, \quad (5.29)$$

which must satisfy the Leibniz condition given in Eq. (2.24a), and also preserve the involution on B in the sense of Eq. (2.24b):

$$\begin{aligned} \delta_B[(a+h)(a'+h')] &= \delta_B[a+h](a'+h') + (a+h)\delta_B[a'+h'] \\ &= (\delta_A a + \delta_H h)(a'+h') + (a+h)(\delta_A a' + \delta_H h') \end{aligned} \quad (5.30)$$

$$\begin{aligned} \delta_B(a^* + h^*) &= \delta_A(a^*) + \delta_H Jh \\ &= \delta_A(a)^* + J\delta_H h = \delta_B(a+h)^*. \end{aligned} \quad (5.31)$$

for $a, a', \delta_B a, \delta_B a' \in A$, and $h, h', \delta_B h, \delta_B h' \in H$. The Leibniz and involution conditions together with grading compatibility imply:

$$[\delta_H, L_a] = L_{\delta_A a}, \quad [\delta_H, R_a] = R_{\delta_A a}, \quad [\delta_H, \gamma] = 0, \quad [\delta_H, J] = 0, \quad (5.32)$$

for $a \in \mathcal{A}$ and where the derivations on the subalgebra \mathcal{A} are all inner, and take the form given in Eq. (2.28c): $\delta_{\mathcal{A}} = [L_x, L_y]$. The anti-hermitian derivations $\delta_B = \delta_{\mathcal{A}} \oplus \delta_H$ which satisfy all of these conditions on B are given by:

$$\delta_{xy} = [L_x, L_y] \oplus ([L_x, L_y] + T), \quad x, y \in \mathcal{A} \quad (5.33)$$

where:

$$[T, L_a] = [T, R_a] = [T, \gamma] = [T, J] = 0, \quad T^* = -T. \quad (5.34)$$

The generators T which satisfy these conditions are of the form:

$$T_0 = \begin{pmatrix} i\mathbb{I} & 0 \\ 0 & -i\mathbb{I} \end{pmatrix}, \quad T_i = \begin{pmatrix} i\sigma_i & 0 \\ 0 & i\sigma_i \end{pmatrix} \quad (5.35)$$

But these are the infinitesimal generators of $U(2)$, and so the full symmetry group of this model will be given by $SU(n) \times U(2)$.

5.3.2 Pati-Salam revisited.

One of the unexplained features of the NCG SM is that it takes as part of its finite input the Quaternionic algebra. Chamseddine and Connes are able to derive much of the NCG SM data from very minimal assumptions [CC08], but the quaternions \mathbb{H} remain as a mysterious input. Why for example is the input algebra of the NCG SM given by $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and not the seemingly more natural $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ [SZ01, PS97]? Recent attempts at going beyond the NCG SM also rely heavily on the quaternions. For example, attempts have been made to construct Pati-Salam² type extensions and ‘grand symmetry’ extensions by enlarging the finite input algebra to $\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$ [CCvS13] or $M_4(\mathbb{H}) \oplus M_8(\mathbb{C})$ [DLM14a]. These extensions however do not satisfy the associative order one condition. Rather than enlarge the input algebra, it might be possible to obtain the desired symmetries of a model as ‘extended’ symmetries by reducing the size of the input algebra. In particular Subsection 5.3.1 promotes the idea that we might be able to

²The Pati-Salam model is a grand unified theory in which the quarks and leptons are grouped together into a ‘lepto-quark’ quadruplet charged under an $SU(4)$ gauge symmetry [PS74]. The model has left-right symmetry, in the sense that both left handed and right handed particles are charged under $SU(2)$ gauge symmetries, such that the full gauge group of the model is given by $SU_{lq}(4) \times SU_L(2) \times SU_R(2)$. The model predicts the existence of a high energy right handed weak interaction with heavy W' and Z' bosons. It is a viable alternative to Georgi-Glashow $SU(5)$ unified models, and can be embedded within an $SO(10)$ unification model.

construct Pati-Salam type models in which the group $SU(2) \times SU(2) \times SU(4)$ is obtained in this way.

Consider the finite data $\{\mathcal{A}, D, \mathcal{H} = \mathbb{C}^{32}, J, \gamma\}$, where \mathcal{A} is the associative algebra of 4×4 complex matrices $M_4(\mathbb{C})$, with representation $\pi(a) = \text{diag}\{a\mathbb{I}_4, \bar{a}\mathbb{I}_4\}$ and on the basis of ‘lepto-quarks’ $\{u_L, d_L, u_R, d_R, \bar{u}_L, \bar{d}_L, \bar{u}_R, \bar{d}_R\}$ we have:

$$J = \begin{pmatrix} 0 & \mathbb{I}_4 \\ \mathbb{I}_4 & 0 \end{pmatrix} \circ (*), \quad \gamma = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix}, \quad (5.36)$$

where the $(*)$ operation is complex conjugation. We can again form the fused algebra $B_0 = \mathcal{A} \oplus \mathcal{H}$, and determine its derivations, which are of the form:

$$\delta_x = (L_x - R_x) \oplus (L_x + T) \quad (5.37)$$

for anti-hermitian $x \in \mathcal{A}$. The inner derivations therefore generate the group $SU(4)$. The infinitesimal generators T must satisfy the conditions given in (5.34), and are of the form:

$$T = \text{diag}\{a^i i \sigma_i + \alpha^0 i \mathbb{I}_2, b^i i \sigma_i + \beta^0 i \mathbb{I}_2, a^i \bar{i} \sigma_i - \alpha^0 i \mathbb{I}_2, b^i \bar{i} \sigma_i - \beta^0 i \mathbb{I}_2\} \quad (5.38)$$

for $a, b, \alpha, \beta \in \mathbb{R}$. Eventually if we were to construct an almost-associative NCG out of this model, then anomaly cancellation would impose that $\alpha = \beta$, which means that the extra $U(1)$ ’s are actually degenerate. In other words, the full symmetry group is given by.

$$SU_L(2) \times SU_R(2) \times SU(4) \times U(1), \quad (5.39)$$

Note that one might also consider imposing the so called ‘unimodularity condition’ [AGBM95, LS98], to restrict further to the group $SU_L(2) \times SU_R(2) \times SU(4)$, but I won’t consider this possibility here.

Next consider equipping a finite Dirac operator D

$$D = \begin{pmatrix} D_S & D_T \\ D_T^* & \bar{D}_S \end{pmatrix}. \quad (5.40)$$

where in KO-dimension 6:

$$D_S = \begin{pmatrix} 0 & D_H \\ D_H^* & 0 \end{pmatrix}, \quad D_T = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \quad (5.41)$$

where D_T is symmetric. This is as far as D can be restricted without imposing higher order conditions. Notice that in the construction of the Pati-Salam model given in [CCvS13] the

authors build a NCG which satisfies the associative order zero condition, but which does not satisfy the associative order one condition. In other words B_0 is associative, but ΩB is not. Without an order one condition there is no reason a priori for restricting the finite Dirac operator any further than that shown in Eq. (5.41), and one should expect when fluctuating the Dirac operator to see the full set of Higgs fields outlined in the appendix of [CCvS13] (rather than the restricted set mentioned in their abstract).

Fortunately we do have higher order conditions which we are able to impose. In particular this geometry has been constructed as an associative-alternative representation in the sense of Subsection 5.2.2. It is therefore natural to impose the associative-alternative order one condition:

$$[d[a], h, b] = -[b, h, d[a]] \quad (5.42)$$

A full analysis of the resulting Higgs sector is yet to be completed but one might hope that the order one condition given in Eq. (5.42) might leave only the phenomenologically desirable Higgs fields, while eliminating the excess unwanted fields found in [CCvS13].

5.3.3 Exceptional Jordan algebras and unification.

Apart from a few exotic cases, all of the non-associative NCGs that I have described in this work have been based either on the octonion algebra, or on Jordan matrix algebras. This thesis would be incomplete without a brief discussion of *the* algebra which is most interesting for non-associative model building, and which draws together many of the most interesting properties of both the Jordan algebras and octonions: the exceptional Jordan algebra. The vector space of the exceptional Jordan algebra $J_3(\mathbb{O})$ (sometimes called the ‘Albert algebra’) is the 27 dimensional real vector space of 3×3 hermitian octonionic matrices, ie. elements of the form:

$$\begin{pmatrix} a & X & Y \\ \bar{X} & b & Z \\ \bar{Y} & \bar{Z} & c \end{pmatrix} \quad (5.43)$$

for $a, b, c \in \mathbb{R}$, and $X, Y, Z \in \mathbb{O}$, and where the bar indicates octonionic conjugation. The product is given by the symmetric octonionic matrix product $A \circ B = AB + BA$, for $A, B \in J_3(\mathbb{O})$. This algebra is known as exceptional because according to a famous classification theorem of Jordan, Wigner, and Von Neumann, Jordan algebras fall into one of four infinite families with just this one fascinating exception.

As mentioned in Subsection 2.1.2 the octonion algebra is interesting from the perspective of non-associative model building because it is 8 dimensional (the SM fermions come in groups of 8), and it has both $SU(2)$ and $SU(3)$ as subgroups of its automorphisms. One of the downsides of using the octonions however is that one still has to explain the three generations of fermions. Three ‘generations’ (and a tiny bit of ‘dark’ matter) is a feature of the exceptional Jordan algebra. Unfortunately, the Exceptional Jordan algebra is often discarded as a useful candidate for unified physics because it has F_4 as its automorphism group. The group F_4 is of rank 4, but it does not have $SU(3) \times SU(2) \times U(1)$ as a subgroup [Sla81]. Fortunately this is not a problem in our approach because the symmetry group of an NCG is in general larger than the inner automorphisms of its input algebra.

Let us look a bit closer at the representation outlined in Eq. (5.43). When analyzing the symmetries of the octonion algebra one finds that the subgroup of the automorphisms which leaves both the identity element, and one of the imaginary basis elements fixed, is $SU(3)$. Similarly, the exceptional Jordan algebra is constructed from three off-diagonal octonion elements, and three diagonal real elements, and so it is natural to ask: ‘does F_4 have a subgroup which leaves the three diagonal real elements, as well as the real element and one imaginary element from each off-diagonal octonion invariant?’ The subgroup is once again exactly $SU(3)$. In other words, on breaking down from F_4 to $SU(3)$ one obtains three ‘quark’ triplets, three ‘lepton’ singlets, and three ‘Majorana’ like dark matter candidates. The hope would be to obtain the electroweak symmetry as outer ‘extended’ automorphisms between two copies of the exceptional Jordan algebra, and as is outlined in Section 5.1 if one represents $J_3(\mathbb{O})$ on itself in KO dimension 6 one is forced to take multiple copies in order to obtain a non-trivial Higgs sector.

These are just some preliminary thoughts on this topic, which I hope to flesh out into a more complete model in the future.

Chapter 6

Conclusion and outlook

We have presented an approach to NCG which naturally extends to describe non-associative geometries. The key idea is very simple: we draw together the elements held in a spectral triple into a single ‘fused’ algebra ΩB :

$$\{\mathcal{A}, \mathcal{H}, D, \gamma, J\} \leftrightarrow \Omega B$$

The elements $\{\mathcal{H}, D, J, \gamma\}$ are seen as extending the $*$ -DGA $\Omega\mathcal{A} = \{\mathcal{A}, d, *\}$, where the representation \mathcal{H} is seen as the extension of the underlying vectorspace and product, D is seen as extending the differential, J is seen as extending the involution, and γ is seen as extending the grading¹. At the same time the various axioms and assumptions of NCG are reformulated in terms the intrinsic properties of ΩB : for associative NCG, ΩB is an associative, involutive $*$ -DGA. Their are two key benefits to this approach:

1. It provides a more unified description of the various elements of a NCG, and a natural description of the symmetries of a NCG as the automorphisms of ΩB .
2. It readily generalizes to describe non-associative geometry in the sense that ΩB need not be associative.

The simplicity of our formulation elucidates many aspects of NCG in both the associative and non-associative setting. In particular it has already seen three applications:

1. The axioms of a NCG become in some sense ‘derivable’ from the intrinsic properties of ΩB . What we find are a set of ‘new’ constraints in addition the traditional NCG

¹depending on the perspective taken.

axioms which arise as higher order associativity conditions on ΩB . These new conditions can be used to place phenomenologically accurate restrictions on the NCG SM Higgs sector.

2. If one analyses the automorphisms of the fused algebra ΩB corresponding to the NCG SM, one finds instead a minimal and viable extension by an additional gauged $B - L$ symmetry. This symmetry fluctuates an additional complex scalar field which has two roles: (i) it extends the scalar sector, and allows compatibility of the model with the 125GeV Higgs detection. (ii) The extra scalar ‘Higgses’ the additional gauged $B - L$ symmetry. The perspective that the symmetries of NCG arise as the automorphisms of ΩB also allows one to explore more exotic extensions to the SM in which non-abelian gauge symmetries are obtained as outer automorphisms.
3. If one constructs a non-associative fused algebra ΩB , then the appropriate non-associative generalizations of the NCG axioms become in some sense ‘derivable’ from the intrinsic properties of ΩB . In other words, one need not analyze the generalization of each axiom independently! This fact has allowed us to construct a wide range of example non-associative geometries.

There is now fertile new ground for future work. We must continue exploring the particle phenomenology and cosmology of our gauged $B - L$ extended SM, and there are now a wide range of models to consider building, including Pati-Salam-type GUT models, and GUT models based on the exceptional Jordan algebra. There are also still many subtle issues surrounding the Junk and the higher order conditions that must be ironed out, as well as the appropriate generalizations of cyclic (Co)homology to be made for the classes of non-associative algebras of interest. It is interesting to ask if our formalism has anything to say about the three particle generations, and also to what extent KO-dimension 0 models are singled out. We now have an object ΩB which contains all of the physically relevant fields, and so it is curious to ask if one could quantize it using well-known tools which already exist for quantizing differential graded algebras such as those methods applied in the Batalin-Vilkovisky approach. These are among the exciting new areas of research which are opened up by our fused algebra approach to NCG, and which we are currently considering.

APPENDICES

Appendix A

The B-L extended NCG SM action

In this Section I will explicitly construct the bosonic and fermionic actions of the gauged $B - L$ extended NCG SM which was introduced in Section 4.2.1. The calculation follows exactly the prescription outlined in [vdDvS12] for the base standard model, and the reader is encouraged to consult that reference for further detail.

A.1 Squaring the Dirac operator

The action corresponding to an almost-commutative geometry is constructed from both a bosonic, and a fermionic part. The first step in constructing the bosonic part of the action is squaring the fluctuated Dirac operator. The full fluctuated Dirac operator for an almost commutative geometry is given by $D = -i\gamma^\mu(\partial_\mu + \omega_\mu) \otimes \mathbb{I}_F - i\gamma^\mu F_\mu^i(x) \otimes \delta_i + \gamma_5 \otimes \Phi$ (see

Eq. (2.87)). We start by first working out the square of the curved space Dirac operator:

$$\begin{aligned}
(\not{D})^2 &= -\gamma^\mu(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab})\gamma^\nu(\partial_\nu + \frac{1}{4}\omega_{\nu fg}\gamma^{fg}) \\
&= -\gamma^\mu\gamma^\nu(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab})(\partial_\nu + \frac{1}{4}\omega_{\nu fg}\gamma^{fg}) \\
&\quad - \gamma^\mu(\gamma^e\partial_\mu\epsilon_e^\nu + \frac{1}{8}\epsilon_e^\nu\omega_{\mu ab}[[\gamma^a, \gamma^b], \gamma^e])(\partial_\nu + \frac{1}{4}\omega_{\nu fg}\gamma^{fg}) \\
&= -\gamma^\mu\gamma^\nu(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab})(\partial_\nu + \frac{1}{4}\omega_{\nu fg}\gamma^{fg}) \\
&\quad - \gamma^\mu(\gamma^e\partial_\mu\epsilon_e^\nu + \frac{4}{8}\epsilon_e^\nu\omega_{\mu ab}(\eta^{be}\gamma^a - \eta^{ae}\gamma^b))(\partial_\nu + \frac{1}{4}\omega_{\nu fg}\gamma^{fg}) \\
&= -\gamma^\mu\gamma^\nu(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab})(\partial_\nu + \frac{1}{4}\omega_{\nu fg}\gamma^{fg}) \\
&\quad - \gamma^\mu\gamma^e(\partial_\mu\epsilon_e^\nu + \epsilon_a^\nu\omega_{\mu eb}\eta^{ba})(\partial_\nu + \frac{1}{4}\omega_{\nu fg}\gamma^{fg}) \\
&= -\gamma^\mu\gamma^\nu\{(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab})(\partial_\nu + \frac{1}{4}\omega_{\nu fg}\gamma^{fg}) - \Gamma_{\nu\mu}^\sigma(\partial_\sigma + \frac{1}{4}\omega_{\sigma fg}\gamma^{fg})\} \\
&= -\gamma^\mu\gamma^\nu\{\nabla_\mu^S\nabla_\nu^S - \Gamma_{\nu\mu}^\sigma\nabla_\sigma^S\}. \tag{A.1}
\end{aligned}$$

Having determined the square of the curved space Dirac operator, we next include the gauge connection terms by defining $-i\gamma^\mu\nabla_\mu^E = -i\gamma^\mu\nabla_\mu^S + F$, where $F = -i\gamma^\mu F_\mu^i\delta_i = -i\gamma^\mu(\Lambda_\mu\delta_y + K_\mu\delta_x + Q_\mu^i\delta_i^w + V_\mu^i\delta_i^c)$, with the potentials corresponding to gauged $B - L$, hypercharge, weak, and strong forces respectively. Using this notation, squaring $(-i\gamma^\mu\nabla_\mu^E)^2$ then yields:

$$\begin{aligned}
(-i\gamma^\mu\nabla_\mu^E)^2 &= -\gamma^\mu\gamma^\nu\nabla_\mu^E\nabla_\nu^E - \gamma^\mu[\nabla_\mu^S, \gamma^\nu]\nabla_\nu^E \\
&= -\gamma^\mu\gamma^\nu(\nabla_\mu^E\nabla_\nu^E - \Gamma_{\mu\nu}^\tau\nabla_\tau^E) \\
&= -g^{\mu\nu}(\nabla_\mu^E\nabla_\nu^E - \Gamma_{\mu\nu}^\tau\nabla_\tau^E) - \frac{1}{2}[\gamma^\mu, \gamma^\nu]\nabla_\mu^E\nabla_\nu^E \\
&= -g^{\mu\nu}(\nabla_\mu^E\nabla_\nu^E - \Gamma_{\mu\nu}^\tau\nabla_\tau^E) \\
&\quad - \frac{1}{2}[\gamma^\mu, \gamma^\nu](\partial_\mu\omega_\nu + \omega_\mu\partial_\nu + \partial_\mu F_\nu + F_\mu\partial_\nu + \omega_\mu\omega_\nu + \omega_\mu F_\nu + F_\mu\omega_\nu + F_\mu F_\nu) \\
&= -g^{\mu\nu}(\nabla_\mu^E\nabla_\nu^E - \Gamma_{\mu\nu}^\tau\nabla_\tau^E) \\
&\quad - \frac{1}{4}[\gamma^\mu, \gamma^\nu](\partial_\mu\omega_\nu - \partial_\nu\omega_\mu + [\omega_\mu, \omega_\nu] + [\partial_\mu, Z_\nu] - [\partial_\nu, F_\mu] + [F_\mu, F_\nu]) \\
&= -g^{\mu\nu}(\nabla_\mu^E\nabla_\nu^E - \Gamma_{\mu\nu}^\tau\nabla_\tau^E) - \frac{1}{2}\gamma^\mu\gamma^\nu(\Omega_{\mu\nu}^S + F_{\mu\nu}), \tag{A.2}
\end{aligned}$$

where $\Omega_{\mu\nu}^S \equiv [\nabla_\mu^S, \nabla_\nu^S] \equiv \frac{1}{4}R_{\mu\nu\rho\tau}\gamma^\rho\gamma^\tau$ and $F_{\mu\nu} \equiv \partial_\mu F_\nu - \partial_\nu F_\mu + [F_\mu, F_\nu]$. Finally, the square of the full Dirac operator is given by:

$$\begin{aligned}
D^2 &= (\nabla^S)^2 + \Delta^S F + F\Delta^S + F^2 + \Delta^S\gamma_5\Phi + \gamma_5\Phi\Delta^S + F\Phi + \Phi F + \Phi^2 \\
&= (\nabla^E)^2 + \gamma_5\Phi\Delta^E + \Delta^E\gamma_5\Phi + \Phi^2 \\
&= -g^{\mu\nu}(\nabla_\mu^E\nabla_\nu^E - \Gamma_{\mu\nu}^\tau\nabla_\tau^E) - \frac{1}{2}\gamma^\mu\gamma^\nu(\Omega_{\mu\nu}^S + F_{\mu\nu}) - \gamma_5[\nabla^E, \Phi] + \Phi^2 \\
&:= \Delta^E - \frac{1}{4}R - \frac{1}{2}\gamma^\mu\gamma^\nu Z_{\mu\nu} - \gamma_5[\nabla^E, \Phi] + \Phi^2 \tag{A.3}
\end{aligned}$$

A.2 The heat kernel expansion

Having determined the form of the square of the Dirac operator the next task is to calculate the heat kernel expansion of the bosonic action. Following [vdDvS12] the Heat kernel expansion for an almost commutative spectral triple is given as:

$$S_b = Tr[f(D/\Lambda)] \sim 2f_4\Lambda^4 a_0(D^2) + 2f_2\Lambda^2 a_2(D^2) + f(0)a_4(D^2) + O(\Lambda^{-1}) \quad (\text{A.4})$$

where the Seeley-DeWitt coefficients are:

$$a_k(D^2) = \int_M a_k(x, D^2) \sqrt{|g|} d^4x, \quad (\text{A.5})$$

and ignoring boundary terms the first three coefficients are given by:

$$a_0(x, D^2) = (4\pi)^{-\frac{n}{2}} Tr(Id) \quad (\text{A.6})$$

$$a_2(x, D^2) = (4\pi)^{-\frac{n}{2}} Tr(\frac{1}{12}R - \Phi^2) \quad (\text{A.7})$$

$$a_4(x, D^2) = (4\pi)^{-\frac{n}{2}} \frac{1}{360} Tr(\frac{5}{4}R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\tau}R^{\mu\nu\rho\tau} + 30\Omega_{\mu\nu}^E\Omega^{E\mu\nu} + 60R\Phi^2 + 180(\frac{1}{4}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\tau Z_{\mu\nu}Z_{\rho\tau} - \frac{1}{2}R\Phi^2 - [-i\gamma^\mu\nabla_\mu^E, \Phi]^2 + \Phi^4)) \quad (\text{A.8})$$

I will ignore gravitational interactions from here out, in which case the full bosonic action is given by:

$$S_b \sim \int_M \frac{d^4x}{8\pi^2} \sqrt{|\eta|} f(0) Tr[-[-i\gamma^\mu\nabla_\mu^E, \Phi]^2 - \frac{4f_2}{f(0)}\Lambda^2\Phi^2 + \Phi^4 - \frac{1}{3}F_{\mu\nu}F^{\mu\nu}] + O(\Lambda^{-1}) \quad (\text{A.9})$$

where the trace is over the finite degrees of freedom. We can use the form of the gauge generators to determine more explicitly the form of the gauge kinetic terms:

$$Tr[F_{\mu\nu}F^{\mu\nu}] = Tr[\Lambda_{\mu\nu}\Lambda^{\mu\nu} + K_{\mu\nu}K^{\mu\nu} + Q_{\mu\nu}Q^{\mu\nu} + V_{\mu\nu}V^{\mu\nu}] = 24(\frac{10}{3}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \frac{16}{3}K_{\mu\nu}K^{\mu\nu} + \frac{8}{3}Q_{\mu\nu}Q^{\mu\nu} + Tr[Q_{\mu\nu}Q^{\mu\nu} + V_{\mu\nu}V^{\mu\nu}]) \quad (\text{A.10})$$

where in the last line the trace is only taken over the generators σ_i and λ_i . Finally we are able to introduce gauge coupling constants by defining:

$$\Lambda_\mu = \frac{1}{2}g_y B_\mu, \quad K_\mu = \frac{1}{2}g_{b-l} C_\mu, \quad Q_\mu^i = \frac{1}{2}g_w W_\mu^i, \quad V_\mu^i = \frac{1}{2}g_c G_\mu^i. \quad (\text{A.11})$$

In which case the gauge kinetic terms can then be written as:

$$Tr[F_{\mu\nu}F^{\mu\nu}] = \frac{24}{2}(\frac{5}{3}g_y^2 B_{\mu\nu}B^{\mu\nu} + \frac{8}{3}g_{b-l}^2 C_{\mu\nu}C^{\mu\nu} + \frac{4}{3}g_{b-l}g_y B_{\mu\nu}C^{\mu\nu} + g_w^2 W_{\mu\nu}W^{\mu\nu} + g_c^2 G_{\mu\nu}G^{\mu\nu}) \quad (\text{A.12})$$

Next we calculate the scalar kinetic terms as:

$$\begin{aligned} \text{Tr}[-[-i\gamma^\mu \nabla_\mu^E, \Phi]^2] &= \eta^{\mu\nu} \text{Tr}[(\partial_\mu + Z_\mu, \Phi)(\partial_\nu + Z_\nu, \Phi)] \\ &= 4a|\partial_\mu \phi - \frac{1}{2}B_\mu i\phi + \frac{1}{2}iW_\mu^i \sigma_i \phi|^2 + 2c|(\partial_\mu - 2iC_\mu)\sigma|^2, \end{aligned} \quad (\text{A.13})$$

and the scalar potential terms as:

$$\text{Tr}[-\frac{4f_2}{f(0)}\Lambda^2\Phi^2 + \Phi^4] = -\frac{4f_2}{f(0)}\Lambda^2(4a|\phi|^2 + 2c|\sigma|^2) + 4b|\phi|^4 + 8e|\sigma|^2|\phi|^2 + 2d|\sigma|^4 \quad (\text{A.14})$$

where we have defined:

$$a = \text{Tr}[Y_\nu^* Y_\nu + Y_e^* Y_e + 3Y_u^* Y_u + 3Y_d^* Y_d] \quad (\text{A.15a})$$

$$b = \text{Tr}[(Y_\nu^* Y_\nu)^2 + (Y_e^* Y_e)^2 + 3(Y_u^* Y_u)^2 + 3(Y_d^* Y_d)^2] \quad (\text{A.15b})$$

$$c = \text{Tr}[Y_\sigma^* Y_\sigma] \quad (\text{A.15c})$$

$$d = \text{Tr}[(Y_\sigma^* Y_\sigma)^2] \quad (\text{A.15d})$$

$$e = \text{Tr}[Y_\sigma^* Y_\sigma Y_\nu^* Y_\nu] \quad (\text{A.15e})$$

Putting everything together, the full bosonic action is:

$$\begin{aligned} S_b &\sim i \int_M d^4x [|\partial_\mu \phi - \frac{1}{2}g_y B_\mu i\phi + \frac{1}{2}g_w iW_\mu^i \sigma_i \phi|^2 + |(\partial_\mu - 2ig_{b-l}C_\mu)\sigma|^2 \\ &\quad - m_\phi^2|\phi|^2 - m_\sigma^2|\sigma|^2 + \lambda_\phi|\phi|^4 + 2\lambda_m|\sigma|^2|\phi|^2 + \lambda_\sigma|\sigma|^4 \\ &\quad - \frac{1}{4}(\xi_{ij}A_{\mu\nu}^i A^{j\mu\nu} + W_{\mu\nu}W^{\mu\nu} + G_{\mu\nu}G^{\mu\nu})] \end{aligned} \quad (\text{A.16a})$$

where we have re-expressed the $U(1)$ kinetic terms in the action as $-i \int_M d^4x \frac{1}{4}\xi_{ij}A_{\mu\nu}^i A^{j\mu\nu}$, where $A_{\mu\nu}^B = B_{\mu\nu}$, and $A_{\mu\nu}^C = C_{\mu\nu}$. We have also normalized the scalar fields to absorb the coefficients in their kinetic terms. The various coefficients are then given by:

$$m_\phi^2 = m_\sigma^2 = \frac{f_2}{2\pi^2}\Lambda^2 \quad (\text{A.17a})$$

$$\lambda_\phi = \frac{2\pi^2 b}{f(0)a^2}, \quad \lambda_m = \frac{8\pi^2 e}{f(0)ac}, \quad \lambda_\sigma = \frac{4\pi^2 d}{f(0)c^2} \quad (\text{A.17b})$$

$$\xi_{BB} = \frac{2f(0)}{\pi^2} \frac{5}{3}g_y^2, \quad \xi_{BC} = \xi_{CB} = \frac{f(0)}{\pi^2} \frac{4}{3}g_y g_{b-l}, \quad \xi_{CC} = \frac{2f(0)}{\pi^2} \frac{8}{3}g_{b-l}^2, \quad (\text{A.17c})$$

Normalizing the gauge kinetic terms, and setting $\xi_{BB} = \xi_{CC} = 1$ fixes $\xi_{BC} = \xi_{CB} = \sqrt{1/10}$, and $\frac{5}{3}g_y^2 = \frac{8}{3}g_{b-l}^2 = g_w = g_s$. This normalization also fixes the value $\frac{10f(0)}{3\pi^2}g_y^2 = 1$ at the scale Λ at which the action is defined.

A.3 The NCG SM fermionic action

The total NCG SM action is composed of a bosonic, and a fermionic part. Our next task then is therefore to determine the form of the fermionic action, which Connes and chamseddine give in KO dimension 2 as:

$$S_f = \frac{1}{2} \langle J\psi' | D\psi' \rangle = \frac{i}{2} \int_M d^4x (J\psi' | D\psi') \quad (\text{A.18})$$

restricted to the set of anti-commuting Grassmann variables ψ' for even vectors $\psi \in H^+$, with $H^+ = \{\psi \in H | \gamma\psi = \psi\}$. We can decompose the fermionic action into three parts:

$$\begin{aligned} \frac{1}{2} \langle J\psi' | D\psi' \rangle &= \frac{1}{2} \langle J\psi' | (-i\gamma^\mu \partial_\mu \otimes \mathbb{I}) \psi' \rangle \\ &\quad + \frac{1}{2} \langle J\psi' | (-i\gamma^\mu F_\mu^i \otimes \delta_i) \psi' \rangle \\ &\quad + \frac{1}{2} \langle J\psi' | (\gamma_5 \otimes \Phi) \psi' \rangle \end{aligned} \quad (\text{A.19})$$

A general element of the tensor product of two spaces consists of the sums of tensor products, so an arbitrary element of H^+ can be written as:

$$\psi = \chi_R \otimes \Psi_R + \chi_L \otimes \Psi_L + \xi_R \otimes \bar{\Psi}_L + \xi_L \otimes \bar{\Psi}_R \quad (\text{A.20})$$

By an abuse of notation, let us write $\nu^\lambda, \bar{\nu}^\lambda, e^\lambda, \bar{e}^\lambda, u^\lambda, \bar{u}^\lambda, d^\lambda, \bar{d}^\lambda$ as a set of independent Dirac spinors, where the λ labels family indices, and c labels colour. We can then write a generic anti-commuting Grassman variable for the NCG SM as:

$$\begin{aligned} \psi' &= \nu_R^\lambda \otimes \nu_R^\lambda + \nu_L^\lambda \otimes \nu_L^\lambda + \bar{\nu}_R^\lambda \otimes \bar{\nu}_L^\lambda + \bar{\nu}_L^\lambda \otimes \bar{\nu}_R^\lambda \\ &\quad + e_R^\lambda \otimes e_R^\lambda + e_L^\lambda \otimes e_L^\lambda + \bar{e}_R^\lambda \otimes \bar{e}_L^\lambda + \bar{e}_L^\lambda \otimes \bar{e}_R^\lambda \\ &\quad + u_R^\lambda \otimes u_R^\lambda + u_L^\lambda \otimes u_L^\lambda + \bar{u}_R^\lambda \otimes \bar{u}_L^\lambda + \bar{u}_L^\lambda \otimes \bar{u}_R^\lambda \\ &\quad + d_R^\lambda \otimes d_R^\lambda + d_L^\lambda \otimes d_L^\lambda + \bar{d}_R^\lambda \otimes \bar{d}_L^\lambda + \bar{d}_L^\lambda \otimes \bar{d}_R^\lambda \end{aligned}$$

Next, notice that the weak force generators will mix $\nu_L \in H_F$ and $e_L \in H_F$, and similarly for the left handed quarks. Choosing the normalization $e^a = i\sigma^a$, $Tr[\sigma^a \sigma^b] = 2\delta^{ab}$ for the $SU(2)$ generators, and also $T^a = i\lambda^a$, $Tr[\lambda^a \lambda^b] = 2\delta^{ab}$ for the $SU(3)$, the gauge coupling terms are then given as:

$$\begin{aligned} \frac{i}{2} \langle J\psi' | (-\frac{i}{2} g_{b-l} \gamma^\mu C_\mu \otimes \delta^{b-l}) \psi' \rangle &= \frac{1}{2} g_{b-l} C_\mu [+ \langle J_M \bar{\nu}^\lambda | \gamma^\mu \nu^\lambda \rangle + \langle J_M \bar{e}^\lambda | \gamma^\mu e^\lambda \rangle \\ &\quad - \frac{1}{3} \langle J_M \bar{u}^\lambda | \gamma^\mu u^\lambda \rangle - \frac{1}{3} \langle J_M \bar{d}^\lambda | \gamma^\mu d^\lambda \rangle] \\ &= \frac{1}{2} g_{b-l} C_\mu [\langle J_M \bar{l}^\lambda | \gamma^\mu l^\lambda \rangle - \frac{1}{3} \langle J_M \bar{q}^\lambda | \gamma^\mu q^\lambda \rangle] \end{aligned} \quad (\text{A.21})$$

$$\begin{aligned}
\frac{1}{2}\langle J\psi'|(-\frac{i}{2}g_y\gamma^\mu B_\mu \otimes \delta^y)\psi'\rangle &= -\frac{1}{4}g_y B_\mu[\langle J_M\bar{\nu}^\lambda|\gamma^\mu(1+\gamma_5)\nu^\lambda\rangle \\
&\quad - \langle J_M\bar{e}^\lambda|\gamma^\mu(1+\gamma_5)e^\lambda\rangle + 4\langle J_M\bar{e}^\lambda|\gamma^\mu e^\lambda\rangle \\
&\quad + \langle J_M\bar{u}^\lambda|\gamma^\mu(1+\gamma_5)u^\lambda\rangle - \frac{8}{3}\langle J_M\bar{u}|\gamma^\mu u^\lambda\rangle \\
&\quad - \langle J_M\bar{d}^\lambda|\gamma^\mu(1+\gamma_5)d^\lambda\rangle + \frac{4}{3}\langle J_M\bar{d}^\lambda|\gamma^\mu d^\lambda\rangle] \\
&= -\frac{1}{4}g_y B_\mu[\langle J_M\bar{l}^\lambda|\gamma^\mu(1+\gamma_5)\sigma^3 l^\lambda\rangle \\
&\quad + \langle J_M\bar{l}^\lambda|\gamma^\mu(1+\gamma_5)\sigma_3 l^\lambda\rangle \\
&\quad + 4\langle J_M\bar{e}^\lambda|\gamma^\mu e^\lambda\rangle - \frac{8}{3}\langle J_M\bar{u}|\gamma^\mu u^\lambda\rangle \\
&\quad + \frac{4}{3}\langle J_M\bar{d}^\lambda|\gamma^\mu d^\lambda\rangle] \tag{A.22}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\langle J\psi'|(-\frac{i}{2}g_w\gamma^\mu W_\mu^a \otimes \delta_a^w)\psi'\rangle &= \frac{1}{4}g_w W_\mu^1[\langle J_M\bar{\nu}^\lambda|\gamma^\mu(1+\gamma_5)e^\lambda\rangle \\
&\quad + \langle J_M\bar{e}^\lambda|\gamma^\mu(1+\gamma_5)\nu^\lambda\rangle \\
&\quad + \langle J_M\bar{u}^\lambda|\gamma^\mu(1+\gamma_5)d^\lambda\rangle \\
&\quad + \langle J_M\bar{d}^\lambda|\gamma^\mu(1+\gamma_5)u^\lambda\rangle \\
&\quad + \frac{1}{4}g_w W_\mu^2[-\langle J_M\bar{\nu}^\lambda|i\gamma^\mu(1+\gamma_5)e^\lambda\rangle \\
&\quad + \langle J_M\bar{e}^\lambda|i\gamma^\mu(1+\gamma_5)\nu^\lambda\rangle \\
&\quad - \langle J_M\bar{u}^\lambda|i\gamma^\mu(1+\gamma_5)d^\lambda\rangle \\
&\quad + \langle J_M\bar{d}^\lambda|i\gamma^\mu(1+\gamma_5)u^\lambda\rangle \\
&\quad + \frac{1}{4}g_w W_\mu^3[\langle J_M\bar{\nu}^\lambda|\gamma^\mu(1+\gamma_5)\nu^\lambda\rangle \\
&\quad - \langle J_M\bar{e}^\lambda|\gamma^\mu(1+\gamma_5)e^\lambda\rangle \\
&\quad + \langle J_M\bar{u}^\lambda|\gamma^\mu(1+\gamma_5)u^\lambda\rangle \\
&\quad - \langle J_M\bar{d}^\lambda|\gamma^\mu(1+\gamma_5)d^\lambda\rangle \\
&= \frac{1}{4}g_w W_\mu^i[\langle J_M\bar{l}^\lambda|\gamma^\mu(1+\gamma_5)\sigma_i l^\lambda\rangle \\
&\quad + \langle J_M\bar{q}^\lambda|\gamma^\mu(1+\gamma_5)\sigma_i q^\lambda\rangle] \tag{A.23}
\end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\langle J\psi'|(-\frac{i}{2}g_c\gamma^\mu G_\mu^a \otimes \delta_i^c)\psi'\rangle &= \frac{1}{2}g_c G_\mu^i[\langle J_M\bar{u}^\lambda|\gamma^\mu i\lambda_i u^\lambda\rangle + \langle J_M\bar{d}^\lambda|\gamma^\mu i\lambda_i d^\lambda\rangle] \\
&= \frac{1}{2}g_c G_\mu^i[\langle J_M\bar{q}^\lambda|\gamma^\mu i\lambda_i q^\lambda\rangle] \tag{A.24}
\end{aligned}$$

Next we determine the ‘derivative’ part of the fermion kinetic part of the action:

$$\begin{aligned} \frac{1}{2}\langle J\psi'|(-i\gamma^\mu\partial_\mu\otimes\mathbb{I})\psi'\rangle &= -i[\langle J_M\bar{\nu}^\lambda|\gamma^\mu\partial_\mu\nu^\lambda\rangle + \langle J_M\bar{e}^\lambda|\gamma^\mu\partial_\mu e^\lambda\rangle \\ &\quad + \langle J_M\bar{u}^\lambda|\gamma^\mu\partial_\mu u^\lambda\rangle + \langle J_M\bar{d}^\lambda|\gamma^\mu\partial_\mu d^\lambda\rangle] \end{aligned}$$

Finally the Yukawa terms are given by:

$$\begin{aligned} \frac{1}{2}\langle J\psi'|(\gamma_5\otimes\Phi)\psi'\rangle &= \frac{1}{2}[2\langle J_M\nu_R|Y_\nu(\phi_1\bar{\nu}_R + \phi_2\bar{e}_R)\rangle \\ &\quad - 2\langle J_M\bar{\nu}_L|Y_\nu^*(\bar{\phi}_1\nu_L + \bar{\phi}_2e_L)\rangle \\ &\quad - 2\langle J_M\bar{e}_L|Y_e^*(\phi_1e_L - \phi_2\nu_L)\rangle \\ &\quad + 2\langle J_Me_R|Y_e(\bar{\phi}_1\bar{e}_R - \bar{\phi}_2\bar{\nu}_R)\rangle \\ &\quad + \langle J_M\nu_R|Y_\sigma\sigma\nu_R\rangle - \langle J_M\bar{\nu}_L|Y_\sigma^*\bar{\sigma}\bar{\nu}_L\rangle \\ &\quad + 2\langle J_Mu_R|Y_u(\phi_1\bar{u}_R + \phi_2\bar{d}_R)\rangle \\ &\quad - 2\langle J_M\bar{u}_L|Y_u^*(\bar{\phi}_1u_L + \bar{\phi}_2d_L)\rangle \\ &\quad - 2\langle J_M\bar{d}_L|Y_d^*(\phi_1d_L - \phi_2u_L)\rangle \\ &\quad + 2\langle J_Md_R|Y_d(\bar{\phi}_1\bar{d}_R - \bar{\phi}_2\bar{u}_R)\rangle \end{aligned} \tag{A.25}$$

Which we re-write compactly as:

$$\begin{aligned} \frac{1}{2}\langle J\psi'|(\gamma_5\otimes\Phi)\psi'\rangle &= [\langle J_M\nu_R|Y_\nu\bar{\phi}l_R\rangle - \langle J_M\bar{\nu}_L|Y_\nu^*\bar{\phi}l_L\rangle \\ &\quad + \langle J_Me_R|Y_e\bar{\phi}i\sigma_2l_R\rangle - \langle J_M\bar{e}_L|Y_e^*\bar{\phi}i\sigma_2l_L\rangle \\ &\quad + \langle J_Mu_R|Y_u\bar{\phi}q_R\rangle - \langle J_M\bar{u}_L|Y_u^*\bar{\phi}q_L\rangle \\ &\quad + \langle J_Md_R|Y_d\bar{\phi}i\sigma_2q_R\rangle - \langle J_M\bar{d}_L|Y_d^*\bar{\phi}i\sigma_2q_L\rangle \\ &\quad + \frac{1}{2}\langle J_M\nu_R|Y_\sigma\sigma\nu_R\rangle - \frac{1}{2}\langle J_M\bar{\nu}_L|Y_\sigma^*\bar{\sigma}\bar{\nu}_L\rangle] \end{aligned}$$

where the l and q are lepton and quark doublets.

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