An Optimized Least Squares Monte Carlo Approach to Calculate Credit Exposures for Asian and Barrier Options

by

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A thesis

presented to the University of Waterloo

in fulfillment of the

thesis requirement for the degree of

Master of Quantitative Finance

in

Quantitative Finance

Waterloo, Ontario, Canada, 2015

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Counterparty credit risk management has become an important issue for financial institutions since the Basel III framework was introduced. Expected exposure (EE) is defined as the average (positive) exposure at a future date, it is an essential component in the measurement of counterparty credit risk.

This thesis aims to develop an efficient Monte Carlo method to calculate the expected exposures for Asian and barrier options. These options are path-dependent in that their payoffs depend on the historical prices of the underlying assets. Since analytical solutions are generally not available to path-dependent options, the evaluation of the expected exposures has to rely on numerical methods. Monte Carlo method is considered to be more efficient than other methods in particular for high dimension problems.

We briefly introduce the concepts and terms regarding credit exposures in the Basel III framework. Then, we introduce Asian and barrier options as well as some basic pricing models. Next, we will extend the optimized least squares Monte Carlo (OLSM) method to calculate the credit exposures for Asian and barrier options and present our numerical results.

Acknowledgements

I would like to thank Professor Adam Kolkiewicz and Ms. Mary Flatt for their great patience and help.

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Chapter 1

Introduction

Counterparty credit risk has gained more attention since the financial crisis of 2007-2008. Regulators and banks are working together to manage counterparty credit risk in order to build a stable financial market. This chapter briefly introduces counterparty credit risk, and explains the concepts and terminologies regarding credit exposures. In the next chapter we will introduce Asian and barrier options as well as their pricing models.

In the risk coverage of capital framework in Basel III, there is a great deal of emphasis in the area of counterparty credit risk (CCR). While banks have many financial products in their books, not all of them are subject to a counterparty credit risk treatment. According to Basel III, over-the-counter (OTC) derivatives [1] and securities financing transactions (SFT) [2] are in the realm of counterparty credit risk.

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[1]OTC derivatives are derivative products that are negotiated privately by the parties. For example, forwards, swaps, exotic options etc, are in this group.

[2]SFT are transactions such as repurchase agreements, reverse repurchase agreements (repos and reverse repos), security borrowing and lending and margin lending transactions.

For path-dependent options, analytical solutions are generally not available. Thus the values of these options are usually approximated by numerical methods, i.e., binomial option pricing (BOP) method, finite difference (FD) method and Monte Carlo (MC) method.

BOP and FD methods can be used to value European-style as well as American-style options, but they have difficulties when valuing path-dependent options. For example, when BOP method is used to value arithmetic average Asian options, the binomial tree for the averages will not recombine, therefore the number of price paths grows exponentially with respect to time steps. As a result, its computation cost will increase substantially.

Calculating credit exposure of a derivative product is even more challenging. For example, for a plain vanilla European option expiring in three months, there are readily available tools to price the option. We could simply use the Black-Scholes formula to calculate the value of option and we are done. But for calculating the exposure of this option, we are more concerned about the distribution of values of this option at future times. For instance, what is the value (exposure) of this option in two weeks? What will it be in two months? To answer these questions, we have to generate scenarios of market risk factors at different future times. Then, we use a valuation model to calculate the exposure of the option. BOP and FD method are less practical in this regard since they are evaluated at discrete time periods under the risk-neutral measure, while our simulations are evaluated continuously under the physical measure.

The calculation of credit exposures relies on simulation. When valuation models have multiple factors and the portfolios contain many assets, it will be computationally intensive to calculate the credit exposures. BOP method and FD method are not feasible in this

regard due to the high dimensionality of the problems.

We want to develop an efficient method that is suitable to calculate the credit exposures of path-dependent options, and for this we propose to use a Monte Carlo method since its computation cost is relatively low for high dimension options. Monte Carlo method is also flexible about the parameters used in simulation. It can be conducted under either a risk-neutral measure or a physical measure.

Longstaff and Schwartz (2001) [18] introduced a least squares Monte Carlo (LSM) approach to value American options. A similar method was earlier introduced by Tilley (1993). Kan et al. (2010) [17] proposed an optimized least square Monte Carlo approach to measure credit exposures for American options. Using the ideas from the latter paper, we build a least squares Monte Carlo method to calculate the credit exposures for Asian and barrier options. We define backward pricing dynamics that can be easily extended to value other types of options. We also integrate our method with variance reduction techniques to improve the performance of the credit exposures for Asian and barrier options. Therefore we name our method optimized least squares Monte Carlo (OLSM) method.

We will briefly introduce financial risks referred in Basel Accords in the following section.

1.1   Counterparty Credit Risk (CCR)

Counterparty credit risk is the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flows. The loss is usually defined as

Table 1.1: Asymmetrical loss with respect to exposure

	Economic Value to		Exposure to	
	Party A	Party B	Party A	Party B
Case 1	+10 million	-10 million	+10 million	0
Case 2	-5 million	+5 million	0	+5 million

credit exposure or simply exposure, we will use credit exposure or exposure interchangeably throughout this thesis.

Counterparty credit risk is considered as bilateral risk since either party could be subject to loss depending on the economic value at the time of default. For example, in a simple interest rate swap, one party has a positive economic value at one period, thus it faces the risk of loss due to possible default of other party. While in the next period, the other party may have a positive economic value and would face the counterparty credit risk instead. Both parties should monitor their positions closely. In an extreme case, the position could change several times in a very short period.

Note that bilateral risk does not necessarily mean that the potential loss is also bilateral. In fact, loss due to counterparty's default may be asymmetrical. Table 1.1 illustrates the asymmetry feature of counterparty credit risk.

In Table 1.1, suppose party A and party B have entered one transaction. In case 1, the transaction has a positive economic value to party A (obviously that value to party B is negative), then party A has a potential exposure of 10 million dollars if party B defaults. That is, party A has a risk not being able to receive this amount. While in case 2, the economic value to party A is negative, yet party A has a zero exposure instead of 5 million in terms of the counterparty credit risk. This is because party A is still responsible for the

settlement of the transaction when party B defaults in case 2. Of course, in case 2 it is party B who is subject to counterparty credit risk should party A default.

Clearly party A has a possible exposure (or potential loss) in case 1 while it does not "gain" anything in case 2. This possible asymmetry of loss is one feature of counterparty credit risk.

1.2 Credit Exposures under Basel Accords

Since Basel III has evolved from Basel II, many definitions and terminologies are carried over from Basel II. Here we list some definitions related to the concept of "exposures" described in the Basel Accords [2, 3].

- Current Exposure

  Current exposure (CE) is the larger of zero, or the market value of a transaction or a portfolio of transactions within a netting set with a counterparty that would be lost upon the default of the counterparty, assuming no recovery on the value of those transactions in bankruptcy. Current exposure is also called replacement cost.

  From its definition, we have $CE(t) = \max\big(V(t), 0\big)$ for a contract-level exposure where $V(t)$ is the portfolio value at time t, net of applicable collaterals and margin agreements.

  Thus at counterparty-level, for non-netting transactions, we have

  $$CE(t) = \sum_{i=1}^{n} \max\big(V_i(t), 0\big),$$

and when netting is applicable, we have

$$CE(t) = \max\Big(\sum_{i=1}^{n} V_i(t), 0\Big),$$

where n is the total number of transactions in the portfolio with the counterparty, $V_i(t)$ is the value of the i-th transaction at time t.

- Expected Exposure

  Expected exposure (EE) is the mean (average) of the distribution of exposures at any particular future date before the longest-maturity transaction in the netting set matures.

  From the perspective of counterparty credit risk, the Basel Accords are concerned with the positive parts of the expected exposure. Similar to the current exposure, we have the representative of EE as:

  $$EE(t) = \mathbb{E}\Big[\sum_{i=1}^{n} \max\big(V_i(t), 0\big)\Big],$$

  where n is the number of transactions in the portfolio, $V_i(t)$ is the value of the i-th transaction at time t.

- Effective Expected Exposure

  Effective expected exposure (Effective EE) at a specific date is the maximum expected exposure that occurs at that date or any prior date. Then we have:

  $$\text{Effective } EE_{t_k} = \max(\text{Effective EE}_{t_{k-1}}, EE_{t_k}).$$

Alternatively, it may be defined for a specific date as the greater of the expected exposure at that date, or the effective exposure at the previous date. Thus by its definition, effective expected exposure is non-decreasing.

- Expected Positive Exposure

  Expected positive exposure (EPE) is the weighted average over time of expected exposures where the weights are the proportion that an individual expected exposure represents over the entire time interval. When calculating the minimum capital requirement, the average is taken over the first year or, if all of the contracts in the netting set mature before one year, over the time period of the longest-maturity contract in the netting set. Then we have:

$$EPE = \sum_{k=1}^{\min(\text{maturity, 1 year})} (EE_k \cdot \Delta t_k),$$

  where the time interval $\Delta t_k = t_k - t_{k-1}$ is the weight.

- Effective Expected Positive Exposure

  Effective expected positive exposure (Effective EPE) is a weighted average over time of effective expected exposure over the first year, or, if all of the contracts in the netting set mature before one year, over the time period of the longest-maturity contract in the netting set where the weights are the proportion that an individual expected exposure represents over the entire time interval.

Similarly to the effective expected exposure, the effective expected positive exposure (Effective EPE) is given by:

$$\text{Effective } EPE = \sum_{k=1}^{\min(\text{maturity, 1 year})} (\text{Effective } EE_k \cdot \Delta t_k),$$

where the time interval $\Delta t_k = t_k - t_{k-1}$ is the weight.

For the purpose of illustration, we construct Figure 1.1 to show the connections of EE, Effective EE, EPE and Effective EPE for a swap contract.

Figure 1.1: Illustration of EE, Effective EE, EPE and Effective EPE for a swap contract
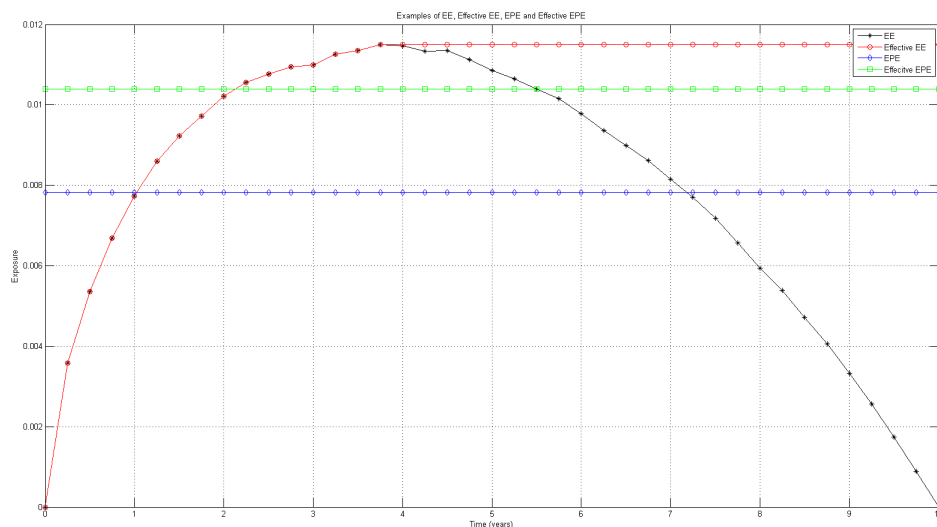
In Figure 1.1, the black line represents the EE; it starts at origin, ascends gradually as time evolves. EE reaches its peak near the mid-term of the contract's life, then it descends and ends when the transaction matures. Effective EE is shown by a red line in the figure. It ascends along with EE at the beginning, then it departs from the EE where EE attains

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its peak. Not like EE which gets smaller thereafter, Effective EE stays at the level of peak through the rest of the duration of the contract.

EPE and Effective EPE are the averages of EE and Effective EE respectively (in Basel Accords, the averages are taken over one year). Since Effective EE is at a higher level of EE, so is the Effective EE over EPE. In Figure 1.1, the green line representing Effective EPE is above the blue line representing EPE.

These concepts of exposures are carried over to Basel III, while Basel III also emphasizes on potential future exposure (PFE). Potential future exposure as a measure of counterparty credit risk, is defined as the maximum expected exposure in future times at a given confidence level. The definition is given by:

$$PFE_\alpha = \inf\{v \in \mathbb{R} : P(EE > v) \leq 1 - \alpha\} = \inf\{v \in \mathbb{R} : F_{EE}(v) \geq \alpha\}. \qquad (1.1)$$

Figure 1.2 shows the expected Mark-to-Market value of a portfolio. The gray area represents the positive exposure, the purple line states the level of EE, and the blue line states PFE at the level of $\alpha = 0.95$.

Notice that the definition of PFE is analogous to Value at Risk (VaR). While VaR is measured with respect to the loss, PFE measures the gain (thus the "exposure" in terms of counterparty credit risk). VaR is usually referring to a relatively short period of time, for example, daily VaR, 10-day VaR. Although Basel III introduces a stressed VaR capital requirement that is based on a continuous 12-month period of significant financial stress. The PFE is typically looking into the future over a longer period of time, in fact it is not uncommon for a PFE to be measured at a time horizon in years.

Figure 1.2: Illustration of EE and PFE for a swap contract
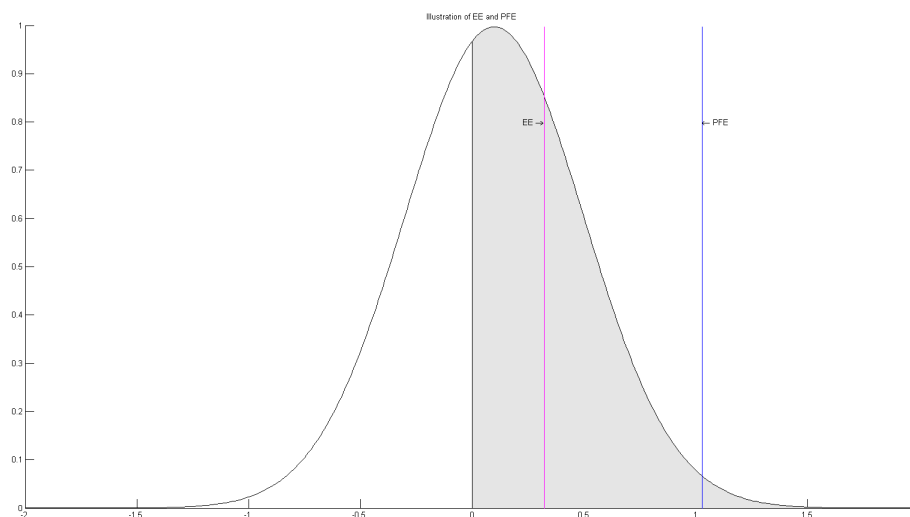
1.3 Credit Valuation Adjustment (CVA) and Expected Exposure (EE)

CVA is a new capital charge introduced in Basel III, and it is required in the calculation of the regulatory capital. In this section we will show that EE is a key element in the calculation of CVA.

CVA is the difference between the CCR-free portfolio value and the CCR-risky portfolio value that takes into account the possibility of a counterparty's default. We can write

$$CVA = V_{risk-free} - V_{risky}. \qquad (1.2)$$

Here the risk refers to the counterparty credit risk, and thus CVA only measures the

value of CCR but not other types of risks.

Another similar concept to CVA is debt valuation adjustment (DVA). For two parties A and B, the CVA measurement from A's point of view is regarding B's credit quality, while DVA measurement from A's point of view is regarding A's own credit quality (and it is actually the CVA from B's point). There is a divergence between the Basel and accounting rules in some regions. For example, Financial Accounting Standards Board (FASB) in U.S. permits firms to recognize CVA and DVA on financial reports, yet Basel III states firms must recognize CVA charges but not DVA charges.

For banks having an approval to apply the Internal Model Method (IMM) for applicable transactions, they can rely on their internal models to calculate CVA charges. Jon Gregory in his book [14] described the way to derive CVA formula, and we will present his idea here.

Denote the value of the risk-free asset at the counterparty level at time t as

$$V(t,T) = \sum_{i=1}^{n} V_i(t,T),$$

where T is the maturity of the asset and $t < T$. Define also the default time of counterparty as τ. Let $I_{default}$ be the indicator function where $I = 1$ when condition is true and $I = 0$ otherwise, we need to find the expression of risky asset $\widetilde{V}(t,T)$, and we assume time $t < \tau < T$. We can consider two cases:

- Case 1: counterparty doesn't default before T.

  If counterparty doesn't default before T, then the risky asset should be considered

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risk-free. Thus the payoff at time t is $I_{(\tau > T)} \cdot V(t, T)$.

- Case 2: counterparty does default before T.

  In this case, we have two parts of the payoff: one is the value we have already received before default time τ, which is $I_{(\tau \leq T)} \cdot V(t, \tau)$; another is the recovery value upon the default.

  When the counterparty defaults, the MtM value of the trade could be positive or negative from a bank's perspective. If it is positive, then the bank will receive a portion of the trade value; if it is negative, the bank still has to pay the amount to counterparty. Thus we have the MtM value as $I_{(\tau \leq T)} \cdot \Big(R \cdot \max \big(V(\tau, T), 0 \big) + \min \big(V(\tau, T), 0 \big) \Big)$ where R is the recovery rate.

The total payoff is the sum of the values in the two cases. Therefore, under a risk-neutral

assumption we have the formula for the risky asset:

$$\widetilde{V}(t,T) = \mathbb{E}\Big[I_{(\tau>T)} \cdot V(t,T) + I_{(\tau \leq T)} \cdot V(t,\tau) +$$

$$+ I_{(\tau \leq T)} \cdot \Big(R \cdot \max\big(V(\tau,T),0\big) + \min\big(V(\tau,T),0\big) \Big) \Big]$$

$$= \mathbb{E}\Big[I_{(\tau>T)} \cdot V(t,T) + I_{(\tau \leq T)} \cdot V(t,\tau) +$$

$$+ I_{(\tau \leq T)} \cdot \Big(R \cdot \max\big(V(\tau,T),0\big) + V(\tau,T) - \max\big(V(\tau,T),0\big) \Big) \Big]$$

$$= \mathbb{E}\Big[I_{(\tau>T)} \cdot V(t,T) + I_{(\tau \leq T)} \cdot V(t,\tau) +$$

$$+ I_{(\tau \leq T)} \cdot \Big((R-1) \cdot \max\big(V(\tau,T),0\big) \Big) + I_{(\tau \leq T)} \cdot V(\tau,T) \Big]$$

$$= \mathbb{E}\Big[I_{(\tau>T)} \cdot V(t,T) + I_{(\tau \leq T)} \cdot V(t,\tau) + I_{(\tau \leq T)} \cdot V(\tau,T)$$

$$+ I_{(\tau \leq T)} \cdot \Big((R-1) \cdot \max\big(V(\tau,T),0\big) \Big) \Big]$$

$$= \mathbb{E}\Big[V(t,T) + I_{(\tau \leq T)} \cdot \Big((R-1) \cdot \max\big(V(\tau,T),0\big) \Big) \Big]$$

$$= V(t,T) + \mathbb{E}\Big[I_{(\tau \leq T)} \cdot (R-1) \cdot \max\big(V(\tau,T),0\big) \Big]$$

(1.3)

Thus we have $\mathbb{E}\Big[I_{(\tau \leq T)} \cdot (1-R) \cdot \max\big(V(\tau,T),0\big) \Big] = V(t,T) - \widetilde{V}(t,T)$. From Equation (1.2), we have $CVA = V_{risk-free} - V_{risky}$. Let $D(\tau) = \frac{B_0}{B_\tau}$ be the discount factor. Then, with the equation above, we have:

$$CVA = \mathbb{E}^{\mathbb{Q}}\Big[D(\tau) \cdot (1-R) \cdot I_{\tau \leq T} \cdot \max\big(V(\tau,T),0\big) \Big]. \tag{1.4}$$

Equation (1.4) gives the definition of CVA.

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With the same definitions as before, in the event a counterparty defaults at time $\tau \leq T$, the bank will experience a loss defined as $L = D(\tau) \cdot (1 - R) \cdot I_{\tau \leq T} \cdot \sum_{i=1}^{n} \max\left(V_i(\tau), 0\right)$, where τ is the time of default. Taking expectation of L, we have $\mathbb{E}[L] = \mathbb{E}^{\mathbb{Q}}\Big[D(\tau) \cdot (1 - R) \cdot I_{\tau \leq T} \cdot \sum_{i=1}^{n} \max\left(V_i(\tau), 0\right)\Big]$.

Since $EE(\tau) = \mathbb{E}\Big[\sum_{i=1}^{n} \max\left(V_i(\tau), 0\right)\Big]$, we have:

$$
\begin{aligned}
\mathbb{E}[L] &= \mathbb{E}^{\mathbb{Q}}\Big[D(\tau) \cdot (1 - R) \cdot I_{\tau \leq T} \cdot \sum_{i=1}^{n} \max\left(V_i(\tau), 0\right)\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\big[D(\tau) \cdot (1 - R) \cdot I_{\tau \leq T} \cdot EE(\tau)\big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[D(\tau) \cdot (1 - R) \cdot I_{\tau \leq T} \cdot \max\left(V(\tau), 0\right)\Big] \\
&= \mathbb{E}^{\mathbb{Q}}\Big[D(\tau) \cdot (1 - R) \cdot I_{\tau \leq T} \cdot \max\left(V(\tau, T), 0\right)\Big] \\
&= CVA
\end{aligned} \tag{1.5}
$$

Note that this definition ignores the possibility of bank defaulting before the counterparty, i.e., we assume that the bank is default-free.

Let $EE^*(t) = \mathbb{E}^{\mathbb{Q}}\big[EE(t) \cdot D(t)\big]$ and $\mathbb{E}[I_{\tau \leq T}] = 1 \cdot PD + 0 \cdot (1 - PD) = PD$ where PD is the probability of default. Since the expectation is over all time until default time τ, assuming exposures and default events are independent, we integrate Equation (1.5) and obtain:

$$
CVA = (1 - R) \cdot \int_0^{\tau} EE^*(t) dPD(t). \tag{1.6}
$$

The integration would be approximated by summing the corresponding values over

piece-wise intervals. Thus we have an approximation of CVA as:

$$CVA \approx (1-R) \cdot \sum_{i=1}^{n} EE^*(t_i) \cdot PD(\Delta t_i) = (1-R) \cdot \sum_{i=1}^{n} EE^*(t_i) \cdot \big(PD(t_i) - PD(t_{i-1})\big). \quad (1.7)$$

It is challenging to quantify the credit exposures for products analytically. For example, EE of an exotic option might have to be obtained by simulation.

Chapter 2

Asian Option, Barrier Option and Basic Asset Pricing Models

In this chapter, we will briefly introduce Asian options and barrier options. Then, we present asset pricing models and analytical solutions for specific Asian and barrier options.

2.1 Options

An option gives the buyer (long) the right but not the obligation to buy or sell the underlying asset. An option giving the holder the right to acquire an asset is referred to as a call option, an option giving the holder the right to sell an asset is referred to as a put option. Depending on the way in which the options are exercised, options can be a European-style which can only be exercised on the expiration date, or an American-style which can be exercised at any time up to the expiration date.

The value of an option contract contains two parts: an intrinsic value and a time value. An intrinsic value is the difference between the current underlying asset price and strike price, and it is zero if current price is lower (higher) than the strike price for a call (put) option. Time value is always positive before the expiration date.

The pricing models for options could be complex. For the European-style options, there are several pricing models available, notably the Black-Scholes formula. For the American-style options, the values of the options are often obtained by approximation as there are usually no closed form solutions for these values.

2.1.1 Asian option

An Asian option is an option with a payoff depending on the average of the prices of the underlying asset over a certain period of time. Asian options are a type of exotic options in that they have a more complicated structure than vanilla options. However, the process of averaging usually results in low volatility and therefore the premium of an Asian option is lower than a plain vanilla option.

The averaging scheme in the Asian option can be either geometric or arithmetic, and

it can be structured as discrete or continuous:

arithmetic average discrete:

$$A(T) = \frac{1}{N} \sum_{i=1}^{N} S(t_i),$$

arithmetic average continuous:

$$A(T) = \frac{1}{T} \int_{0}^{T} S(t)dt,$$

(2.1)

geometric average discrete:

$$A(T) = \sqrt[N]{\prod_{i=1}^{N} S(t_i)},$$

geometric average continuous:

$$A(T) = \exp\left(\frac{1}{T} \int_{0}^{T} \ln\left(S(t)\right)dt\right),$$

where $S(t_i)$ or $S(t)$ are the prices of the underlying asset at time t_i or t respectively.

In terms of strike price, Asian options come in two forms: the fixed strike (known as an average rate) or the floating strike (known as a float rate). The values of corresponding

European Asian call and put options are presented as:

European Asian option with fixed strike:

$$C(T) = e^{-rT}\mathbb{E}\Big[\max\big(A(0,T) - K, 0\big)\Big],$$

$$P(T) = e^{-rT}\mathbb{E}\Big[\max\big(K - A(0,T), 0\big)\Big],$$

$$(2.2)$$

European Asian option with floating strike:

$$C(T) = e^{-rT}\mathbb{E}\Big[\max\big(S(T) - A(0,T), 0\big)\Big],$$

$$P(T) = e^{-rT}\mathbb{E}\Big[\max\big(A(0,T) - S(T), 0\big)\Big],$$

where K is the strike price, $S(T)$ is the price of the underlying asset at time T, $A(0,T)$ is the average of prices of the underlying asset from time 0 to T.

Asian options have no closed form solutions in general, although the geometric Asian option has a closed-formed solution. Therefore we would have to price Asian options by Monte Carlo simulation, PDE approach or tree-based models.

2.1.2 Barrier option

Barrier option is also an exotic option which existence depends on the price path of the underlying asset with respect to a predetermined level.

Barrier options can be classified as knock-out options or knock-in options. For a "knock out" type barrier option, the option ceases to exist when the price of the underlying asset reaches the barrier level (either down to the barrier or up to the barrier), while for a "knock in" type barrier option, the option comes into existence when the price of the underlying

asset reaches the barrier level in a similar way.

Once an option is "knocked out" or "knocked in", its status will not change. For example, an up-and-out call option with barrier level of $100, current price of the underlying asset is $90. The option will be living as long as the price is below $100. If the price goes beyond $100, the option is nil even when price drops below $100 sometime later.

There are in general four types of barrier options:

- up-and-out: initial price is below the barrier level and would move up to the barrier for the option being voided.

- up-and-in: initial price is below the barrier level and would move up to the barrier for the option being activated.

- down-and-out: initial price is above the barrier level and would move down to the barrier for the option being voided.

- down-and-in: initial price is above the barrier level and would move down to the barrier for the option being activated.

As illustration, the payoff of an up-and-out call option can be represented as:

$$
C(T) = \begin{cases} \max\left((S_t - K), 0\right), & \text{if } S_t < H \text{ for all } t \leq T, \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}
$$

where S_t is the price of the underlying asset at time t, K is the strike price, H is the barrier level, and T is the time of maturity. Payoffs of other types of barrier options can be defined similarly; so we will not repeat it here.

2.2 Basic Asset Pricing Models

The value of an option is dependent (derived) on the prices of the underlying asset. Here we will introduce some basic models used in this thesis.

2.2.1 Geometric Brownian Motion (GBM)

In this thesis, we assume a simple Geometric Brownian Motion (GBM) model for pricing the underlying asset. It is given by:

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma(t)dW_t, \tag{2.4}$$

where S_t is the value of the asset at time t; $\mu(t)$ is the drift term; $\sigma(t)$ is the volatility term; and W_t is the standard Brownian motion.

As we have seen in Equation (2.4), $\mu(t)$ and $\sigma(t)$ might be dependent on time t. The changing of drift and volatility with respect to time makes the estimation and simulation more complicated. We usually assume that drift and volatility are constant which will simplify the above equation to:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t. \tag{2.5}$$

Then let $f = \ln(S)$ and substitute it into the left side of Equation (2.5), we have

$$\frac{\partial f}{\partial S} = \frac{1}{S}, \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2}, \frac{\partial f}{\partial t} = 0.$$

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By Itō's formula, we have a solution to Equation (2.5) given by

$$S_t = S_0 \cdot \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t\right), \tag{2.6}$$

where S_0 is the initial value of the asset.

Figure 2.1: Simulated sample price paths of an asset
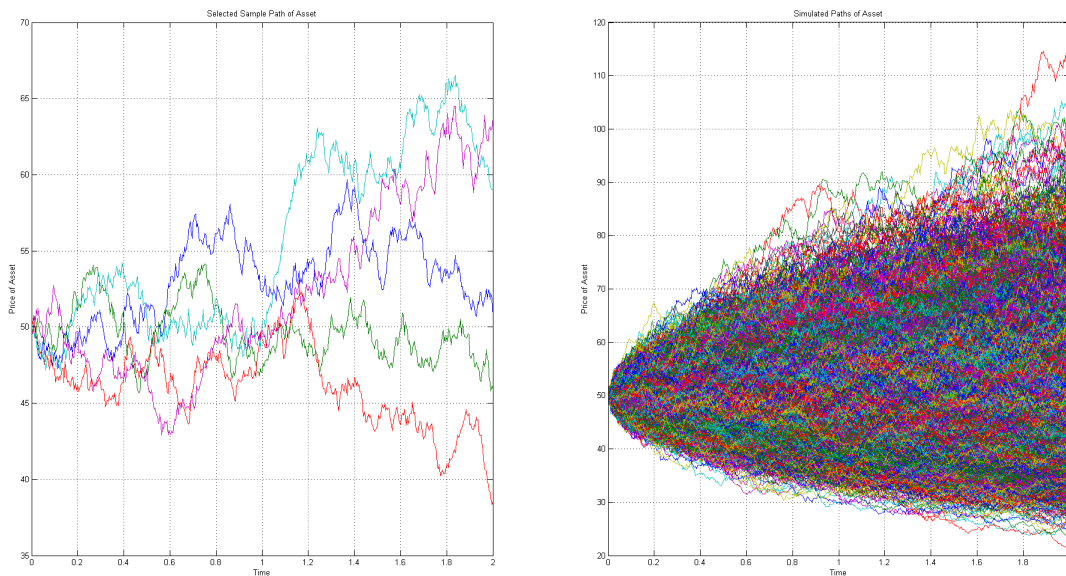
Figure 2.1 represents the sample price paths of an asset under the GBM model. For example, the red line on the left figure shows one possible sample path that the price in 2 years would be at about 38.

2.2.2 Black-Scholes Model

The Black-Scholes or Black-Scholes-Merton model is a popular model for evaluating derivatives products, especially European-style options. It has been one of the most important

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models in finance since it was first introduced by Fischer Black and Myron Scholes and later expanded by Robert C. Merton.

The Black-Scholes formula [1] is deduced from Black-Scholes model. It can be used to value European call and put options. It is given by:

$$C = S \cdot N(d_1) - K \cdot e^{-rT} \cdot N(d_2),$$

$$P = K \cdot e^{-rT} \cdot N(-d_2) - S \cdot N(-d_1),$$

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2}) \cdot T\right),$$

$$d_2 = \frac{1}{\sigma\sqrt{T}}\left(\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2}) \cdot T\right),$$

$$d_2 = d_1 - \sigma\sqrt{T},$$

(2.7)

where

- C is the value of the call option, P is the value of the put option

- S and K are the underlying asset price and strike price respectively

- $N(\cdot)$ is the cumulative distribution function of the standard normal distribution

- T is the maturity

- r is the risk-free rate

- σ is the volatility of returns of the underlying asset

---

[1]In this formula, we simply assume that the asset grows at risk-free rate of r and it does not pay out dividends.

Though widely adopted in financial industry, the Black-Scholes model has its limitations. Originally the Black-Scholes model assumed that the underlying asset will not pay coupon or dividend. This limitation has been addressed by allowing the model to price the options when the underlying asset pays dividends. Another limitation of Black-Scholes model is that it can not price the American options due to a possibly early exercise of the options.

2.3 Analytical Solutions for Certain Asian and Barrier Options

Asian and barrier options are path-dependent options in that the payoffs of the options depend on the prices of underlying asset as well as the paths of the price's evolution. In general, the values of path-dependent options can be approximated by numerical methods, while there are analytical solutions for certain types of options. We refer to Hull [15] for the analytical solutions in this section.

2.3.1 Analytical Solutions for Geometric Average Asian Options

Assuming that the price of the underlying asset follows a GBM, then the price is lognormally distributed. Because the geometric average of a set of lognormally distributed variables is still lognormal, the pricing formula of a geometric average Asian option can be derived in the Black-Scholes framework.

The payoff of a geometric average Asian call option in the discrete case is represented by:

$$C(n) = S \cdot e^{-\left(r + \frac{n+2}{6(n+1)}\sigma^2\right)\frac{T}{2}} \cdot N\left(\frac{\ln\frac{S}{K} + \left(r + \frac{n-1}{6(n+1)}\sigma^2\right)\frac{T}{2}}{\sqrt{\frac{2n+1}{6(n+1)}}\sigma\sqrt{T}}\right) - K \cdot e^{-rT} \cdot N\left(\frac{\ln\frac{S}{K} + \left(r - \frac{\sigma^2}{2}\right)\frac{T}{2}}{\sqrt{\frac{2n+1}{6(n+1)}}\sigma\sqrt{T}}\right),$$

(2.8)

where n is the average frequency.

When in the continuous case, we have $n \to \infty$, then the payoff of a geometric average Asian call option is given by:

$$C = S \cdot e^{-(r + \frac{\sigma^2}{6})\frac{T}{2}} \cdot N\left(\frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{6})\frac{T}{2}}{\sigma\sqrt{T/3}}\right) - K \cdot e^{-rT} \cdot N\left(\frac{\ln\frac{S}{K} + (r - \frac{\sigma^2}{2})\frac{T}{2}}{\sigma\sqrt{T/3}}\right).$$

(2.9)

The above formulas are for Asian call options, the formulas for Asian put options are analogous. However, the payoffs of an arithmetic average Asian options have no analytical solution since the arithmetic average as the sum of lognormal prices will not be lognormal. The values of an arithmetic average Asian option are usually approximated by numerical methods, for example Monte Carlo simulation.

2.3.2 Analytical Solutions for Barrier Options

Assume that the price of the underlying asset is lognormally distributed, a down-and-in call option can be represented analytically as:

$$C_{di} = S \cdot \left(\frac{H}{S}\right)^{2\lambda} \cdot N(y) - K \cdot e^{-rT} \cdot \left(\frac{H}{S}\right)^{2\lambda-2} \cdot N(y - \sigma\sqrt{T}),$$

where

$$\lambda = \frac{r + \sigma^2}{\sigma^2},$$

$$y = \frac{\ln[H^2/(SK)]}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T},$$

(2.10)

and H is the barrier level which is less than or equal to the strike price K.

By the "in-out" parity, the value of a plain vanilla call option is the sum of the value of a down-and-in call option and the value of a down-and-out call option. We have that the value of a down-and-out call option is given by $C_{do} = C - C_{di}$ where C can be obtained by Black-Scholes formula.

Similarly, when H is greater than K, the value of an up-and-in call option can be represented as:

$$C_{ui} = S \cdot N(x_1) - K \cdot e^{-rT} \cdot N(x_1 - \sigma\sqrt{T}) - S \cdot \left(\frac{H}{S}\right)^{2\lambda} \cdot \left(N(-y) - N(-y_1)\right)$$
$$+ K \cdot e^{-rT} \cdot \left(\frac{H}{S}\right)^{2\lambda-2} \cdot \left(N(-y + \sigma\sqrt{T}) - N(-y_1 + \sigma\sqrt{T})\right),$$

where

$$x_1 = \frac{\ln(S/H)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T},$$

$$y_1 = \frac{\ln(H/S)}{\sigma\sqrt{T}} + \lambda\sigma\sqrt{T},$$

(2.11)

and $C_{uo} = C - C_{ui}$.

The barrier put options are defined analogously, and we will not repeat its derivation here.

2.4   Summary

In this chapter, we introduced Asian and barrier options. Then, we presented some basic asset pricing models. These models are to be used to generate the price paths of the underlying asset in our tests. The analytical solutions of certain Asian and barrier options will be used in our study for comparison purpose.

Chapter 3

An Optimized Least Squares Monte Carlo (OLSM) Approach

In the last chapter we have introduced analytical solutions for particular Asian and barrier options. However, these solutions are for European-style options only, and here we consider applicable methods for American options.

Longstaff and Schwartz (2001) [18] introduced a least squares Monte Carlo (LSM) approach to valuing American options. A similar method was earlier introduced by Tilley (1993). Kan et al. (2010) [17] proposed an optimized least squares Monte Carlo approach to measuring exposures for American options.

LSM is a powerful method to calculate derivatives that are both path-dependent and American-style. But calculating the exposure of a derivative product is different than pricing it. For this we propose a new version of least squares Monte Carlo method to cal-

culate the credit exposures of American Asian and American barrier options. We introduce backward pricing dynamics for Asian and barrier options in our method. The backward pricing dynamics can be easily extended to price other exotic options. We also integrate our method with variance reduction techniques to improve the performance. Therefore we name our method the "optimized least squares Monte Carlo" (OLSM) method.

3.1   The LSM Framework

Longstaff and Schwartz (2001) [18] presented an approach to approximating the values of American options by simulation. In their approach, they used cross-sectional information in the simulated paths to find the conditional expectation functions (CEFs). Once the conditional expectation functions were identified, they were used to estimate the conditional expectation of values from continuation. At every step, they determined whether it was optimal to exercise the option on the spot by comparing the values from immediate exercise with the values from continuation. When the former was positive and was greater than the latter, then it was optimal to exercise the option at this point. The procedure was repeated until the complete optimal exercise strategy was discovered. Then the American option can be valued accordingly. Longstaff and Schwartz referred to this technique as the least squares Monte Carlo (LSM) approach.

We now describe the general LSM framework in Longstaff and Schwartz's work. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space, where the state space Ω is the set of all possible realizations of the stochastic economy on the finite time horizon $[0, T]$, element ω be a sample path, $\mathcal{F}$ be the sigma field of distinguishable events at time T, and $\mathcal{P}$ be a

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probability measure defined on the elements of $\mathcal{F}$. Further we define $F = \{\mathcal{F}_t; t \in [0, T]\}$ be the augmented filtration generated by the relevant price processes for the assets in the economy, and assume that $\mathcal{F}_T = \mathcal{F}$.

The objective is to obtain the optimal stopping rule that maximizes the value of the American option. Let $C(\omega, s, t, T)$ be the path of cash flows generated by the option, conditional on that the option has not been exercised at or prior to time t and the option holder has followed the optimal stopping rule for all s where $t < s < T$.

At time t_k, assuming the option has not been exercised, the conditional expectation function $CF(\omega, t_k)$ can be expressed as:

$$CF(\omega, t_k) = \mathbb{E}^Q \left[\sum_{j=k+1}^{K} \exp\left(-\int_{t_k}^{t_j} r(\omega, s) ds \right) C(\omega, t_j, t_k, T) | \mathcal{F}_{t_k} \right], \qquad (3.1)$$

where $r(\omega, s)$ is the discount rate and the expectation is taken conditionally on the information set $\mathcal{F}_{t_k}$ at time $t_k < T$.

LSM assumes that under some assumptions the conditional expectation function is an element of $L^2(\Omega, \mathcal{F}, \mathcal{Q})$. Since L^2 is a Hilbert space, it has a countable orthonormal basis and the conditional expectation function can be represented as a linear function of the elements of the basis. Therefore the functional form of $CF(\omega, t_k)$ in Equation (3.1) can be represented as a linear combination of a countable set of $\mathcal{F}_{t_k}$-measurable basis functions.

One example of the basis function is the set of weighted Laguerre polynomials:

$$L_0(x) = \exp(-x/2),$$

$$L_1(x) = \exp(-x/2)(1-x),$$

$$L_2(x) = \exp(-x/2)(1-2x+x^2/2),$$

$$...$$

$$L_n(X) = \exp(-x/2)\frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}),$$

where x is the value of the underlying asset.

Therefore we can approximates $CF(\omega, t_k)$ using a finite set of basis functions such that

$$CF(x, t_k) \approx \sum_{i=0}^{H} a_i L_i(x),$$

where a_i are coefficients.

For simplicity, let $CF(t_k, j) = \exp(-r\Delta t)CF(t_{k+1}, j)$ be the discounted continuation cash flow from t_{k+1} to t at path j, $S_{t_k,j}$ is the price of the underlying asset at time t_k for the path j, in order to find the coefficients a_i, we use a least squares approach to minimize ϵ^2 such that:

$$\epsilon^2 = \sum_{j=1}^{N} \left(CF(t_k, j) - \sum_{i=0}^{H} a_i L_i(S_{t_k,j})\right)^2,$$

where $\{L_0 \quad L_1 \quad ,..., \quad L_H\}$ are the set of the basis functions, and N is the number of in-the-money price paths.

Let $CE(t_k, j)$ be the cash flow from an immediate exercise of the option at time t_k for

path j. Then, after finding the coefficients $\hat{a}_i$, we evaluate $\hat{CF}(t_k, j) = \sum_{i=0}^{H} \hat{a}_i L_i(S_{t_k,j})$ and compare $\hat{CF}(t_k, j)$ with $CE(t_k, j)$. If $CE(t_k, j)$ is positive and is greater than $\hat{CF}(t_k, j)$, then it is optimal to exercise the option and we set succeeding cash flows to be zero for that path.

The procedure is repeated recursively until the complete optimal exercise strategy is identified. Then, the American option can be valued at the starting time t_0 by moving forward along each path until the first exercise occurs, then discounting the value back to time t_0, and taking average over all paths.

3.2 The Optimized Least Square Monte Carlo (OLSM) Framework

3.2.1 The OLSM Algorithm

Since the OLSM framework is based on the original LSM framework, the intuition and algorithm can be adopted in the current context. Suppose that we want to calculate the credit exposure for an American option. For a long position in the option, the value of the exposure is defined as the value of the option. The problem can be simplified to calculate the value of the option. Under the OLSM framework, we proceed backwards to estimate the conditional expectation function at each exercise node by means of a cross-sectional regression. The sample paths are simulated under the risk-neutral measure. Next, the estimated conditional expectation function is used to calculate a continuation value at the

exercise node under the physical measure. Then, we compare the estimated continuation value with the immediate exercise value to determine the optimal exercise strategy. If the immediate exercise value is positive and is greater than the estimated continuation value, we will exercise the option and set succeeding exposures to be zero. We proceed recursively until we have obtained the complete optimal exercise strategy. The exposure of the American option is obtained similarly.

3.2.2 Scenario Generation

The OLSM framework consists of two phases. The first phase is to generate scenarios of risk factors that will be used to estimate the conditional expectation functions. There is a distinction between the LSM and OLSM frameworks with respect to the probability measure. Under the LSM framework, the risk factors are simulated under the risk-neutral measure. However, under the OLSM framework, the risk factors are simulated under the physical measure when they are used to calculate the credit exposures ([8, 25]).

Therefore, in the remainder of the thesis we assume that the underlying prices follow the lognormal model under the risk-neutral measure:

$$S(t)^{\mathbb{Q}} = S_0 \cdot \exp\left[(r - \frac{\sigma^2}{2})t + \sigma W(t)\right], \tag{3.2}$$

where $W(t)$ is the Brownian motion, the risk-free rate r is the drift, σ is the volatility, S_0 is the initial price of the asset. This model is consistent with the GBM model introduced in Chapter 2.

When Model (3.2) is used for generating future scenarios to estimate credit exposures, the lognormal model under the physical measure is:

$$S(t)^{\mathbb{P}} = S_0 \cdot \exp\left[(\mu^* - \frac{\sigma^{*2}}{2})t + \sigma^* W^*(t)\right], \tag{3.3}$$

where $W^*(t)$, μ^* and σ^* are corresponding terms under the physical measure. μ^* and σ^* can be estimated from the historical data as given:

$$\mu_h = \frac{1}{T} \sum_{t=1}^{T} \ln\left(\frac{S(t)}{S(t-1)}\right),$$

$$\sigma_h = \sqrt{\frac{1}{T} \sum_{t=1}^{T} \left(\ln(\frac{S(t)}{S(t-1)}) - \mu_h\right)^2}.$$

We adjust the drift $\mu^* = \mu_h + \frac{1}{2}\sigma_h^2$ to compensate the term of $\frac{\sigma_h^2}{2}$ in Model (3.3), and set $\sigma^* = \sigma_h$.

3.2.3 Backward Pricing Dynamics for Options

The second phase is to calculate exposures by valuation model. We first discuss the pricing dynamics of options that will be used to find the optimal exercise strategies for our options.

Let $\mathbf{X} = (\mathbf{x}_t)$, $0 \leq t \leq T$ be defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{P})$, $\mathbf{x}_t$ presenting market information at time t. If an option can be exercised at time $0 \leq t_m \leq T$,

$m = 0, 1, ..., M$, then the value of the option is defined by:

$$V_m(x_m) = CE_m(x_m),$$

where $V_m(x_m)$ is the value of the option with respect to variable x_m at time t_m, $CE_m(x_m)$ is the value of immediate exercise of the option with respect to variable x_m at time t_m (we use m as short notation for t_m).

For an arithmetic average Asian option, CE is defined as:

$$CE_m(x_m) = \max\left(z \cdot \left(\frac{1}{m} \sum_{i=1}^{m} S(x_i) - K\right), 0 \right), \tag{3.4}$$

where $S(x_i)$ is the price of the underlying asset with respect to variable x_i, K is the strike price, $z = 1$ for call option, $z = -1$ for put option.

For an up-and-out barrier option, CE is defined as:

$$CE_m(x_m) = \max\left(z \cdot (S(x_m) - K), 0 \right) \cdot I_{S(x_m) < H} \quad \text{for all m}, \tag{3.5}$$

where $S(x_m)$ is the price of the underlying asset with respect to variable x_m, K is the strike price, I is the indicator function, H is barrier level, $z = 1$ for call option, $z = -1$ for put option.

Define the conditional expectation function (CF) with respect to variable x_m at time

t_m be

$$CF_m(x_m) = \begin{cases} 0, & m = M, \\ \mathbb{E}^{\mathbb{Q}}\left[V_{m+1}(x_{m+1}) \cdot D(t_m, t_{m+1})|x_m\right] & m < M, \end{cases} \qquad (3.6)$$

where $D(t_m, t_{m+1})$ is the discount factor from t_{m+1} to t_m, $m = 0, 1, ..., M - 1$.

We proceed backwards to evaluate the option. At the expiration date $t_M = T$, V_M is deterministic, we have $V_M = CE_M$. At time t_m before expiration, the value of the option is determined by comparing the value of $CE_m(x_m)$ with the continuation value of $CF_m(x_m)$. If $CE_m(x_m)$ is positive and is greater than $CF_m(x_m)$, we will exercise the option at time t_m and set the succeeding values of the option to be zero. Therefore the value of the option at time t_m is determined recursively by:

$$V_m(x_m) = \max\left(CE_m(x_m), CF_m(x_m)\right), \quad \text{for } 0 \le t_m \le T, \, m = 0, 1, ..., M. \qquad (3.7)$$

The value of the exposure is just the value of the option when the option is alive. Then, by definition, the value of the exposure at time t_m on price path j is obtained by:

$$E_m(x_m, j) = \begin{cases} V_m(x_m, j) & 0 \le m < M, \\ 0 & m = M. \end{cases} \qquad (3.8)$$

Note that the credit exposures at future times should be calculated under the physical measure with respect to the counterparty credit risk. Here we calculate the credit exposures under the risk-neutral measure since it is difficult for us to obtain the physical measure. In this thesis we focus on the methodology for calculating credit exposures and

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this assumption will simplify our calculations.

Then, the expected exposure (EE) at time t_m is given by:

$$EE(t_m) = \mathbb{E}^{\mathbb{Q}}[E_m(x_m)] = \mathbb{E}^{\mathbb{Q}}[V_m(x_m)]. \tag{3.9}$$

We need to discount it back to obtain the present value of $EE(t_m)$. Let $D(s,t), s < t$ be the discount factor from time t to s, then the present value of $EE(t_m)$ at time $t = 0$ is given by:

$$EE^*(t_m) = \mathbb{E}^{\mathbb{Q}}[D(0, t_m) \cdot E_m(x_m)]. \tag{3.10}$$

We have assumed that the option can be exercised at discrete times t_m, $m = 0, 1, ..., M$. In practice, American options are continuously exercisable. By letting $M \to \infty$, we can approximate the values of American options in the continuous case.

3.2.4 Approximation of Conditional Expectation Functions and Expected Exposures

The value of the option can be obtained by Equation (3.7). In Equation (3.7), the immediate exercise value $CE_m(x_m)$ is deterministic at time t_m, but evaluating the conditional expectation function $CF_m(x_m)$ is challenging.

Assume that the conditional expectation function $CF(\cdot)$ is L_2-measurable, we can rep-

resent it as a linear combination of a set of basis functions $L_k(\cdot)$, i.e., at time t_m:

$$CF_m(x_m) \approx \sum_{k=0}^{H-1} \beta_k L_k(x_m),$$

where β_k are coefficients, H is the total number of the basis functions.

We use the least squares regression to approximate the conditional expectation function $CF(\cdot)$ at time t_m for in-the-money price path j, j=1,..., J. That is, we find coefficients $\hat{\beta}_k$ such that they minimize ϵ^2:

$$\epsilon^2 = \sum_{j=1}^{J} \left(V_{m+1}(x_{m+1}, j) \cdot D(t_m, t_{m+1}) - \sum_{k=0}^{H-1} \beta_k L_k(x_m, j) \right)^2,$$

where $V_{m+1}(x_{m+1}, j)$ is the value of the option at time t_{m+1} for price path j, $D(t_m, t_{m+1})$ is the discount factor from time t_{m+1} to t_m.

After obtaining $\hat{\beta}_k$, the continuation value at time t_m can be approximated by

$$\hat{CF}_m(x_m, j) = \sum_{k=0}^{H-1} \hat{\beta}_k L_k(x_m, j).$$

The value of the option can be determined by $\hat{V}_m(x_m, j) = \max\left(CE_m(x_m, j), \hat{CF}_m(x_m, j)\right)$.

Once the value of the option at each exercise node is obtained, we will discover the optimal exercise strategy. We determine whether the option should be exercised accordingly. When option is exercised, the succeeding values along the price path are set to zero.

Then $EE(t_m)$ can be approximated by

$$EE(t_m) \approx \frac{1}{N} \sum_{i=1}^{N} \hat{V}_m(x_m, i), \tag{3.11}$$

and $EE^*(t_m)$ can be obtained by:

$$EE^*(t_m) \approx \frac{1}{N} \sum_{i=1}^{N} \left(D(0, t_m) \cdot \hat{V}_m(x_m, i) \right), \tag{3.12}$$

where N is the number of price paths.

3.2.5 Basis Function

The basis functions are used to approximate the conditional expectation functions $CF(\cdot)$ in the OLSM framework. Longstaff and Schwartz suggested that we can increase the number of basis functions to obtain a better approximation. In our work, we select monomials up to order of 3, i.e., $L(x) = \{1, x, x^2, x^3\}$ as the set of basis functions. Monomial is a candidate for the set of basis functions since it is simple and easy to evaluate. But there are also other choices: orthogonal polynomials are more efficient and useful in some cases. Judd [16] shows that orthogonal polynomials can solve the multicollinearity problem when dealing with multidimensional approximation problems. In Table 3.1 we list some orthogonal polynomials for reference.

One issue with respect to the set of basis functions is that we will have a much larger base of the set of basis functions when two or more state variables are presented. For example, when we have a function $f(x, y)$ with two state variables X and Y, the set of

Table 3.1: Sample orthogonal polynomials

Family	Weight $\omega(x)$	Interval	Definition
Legendre	1	[-1; 1]	$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1 - x^2)^n$
Chebychev	$(1 - x^2)^{-1/2}$	[-1; 1]	$T_n(x) = \cos(n \cosh(x))$
Laguerre	$\exp(-x)$	$[0; \infty)$	$L_n(x) = \frac{\exp(x)}{n!} \frac{d^n}{dx^n} (x^n \exp(-x))$
Hermite	$\exp(-x^2)$	$(-\infty; \infty)$	$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2)$

basis functions should include the terms in x and y, as well as the cross-products of these terms. In our work, we have two state variables when we evaluate the exposure for the Asian option. One is the spot price of the underlying asset, one is the average price along the path. Let $L(X) = \{1, x, x^2\}$ and $L(Y) = \{1, y, y^2\}$. Then, the new set of basis functions is given by $L(X, Y) = \{1, x, y, x^2, y^2, xy, x^2y, xy^2, x^2y^2\}$. This implies that the number of basis functions grows exponentially with the dimension of the problem.

In order to address this issue, we could use complete polynomials up to the total degree of n to build the set of basis functions. In two-dimension, a complete polynomial of degree of n is given by:

$$P_n(x, y) = \sum_{k=0}^{n} a_k x^i y^j \quad i + j \leq k.$$

For example, the set of complete polynomials of order 2 is given by: $L(X, Y)_{n=2} = \{1, x, y, x^2, y^2, xy\}$. Judd [16] shows that the k-th degree convergence is obtained asymptotically while the size of basis functions grows only polynomially with the dimension of the problem. In our work, we evaluate the set of basis functions containing complete polynomials up to degree of 2, and the full set of basis functions. They both provide a good approximation to the exposures. The results are shown in next chapter.

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3.3   Optimization of Monte Carlo Simulation

Under the OLSM framework, we first estimate CFs under the risk-neutral measure. Then, we make use of the estimated CFs to calculate credit exposures under the physical measure. By the no-arbitrage paradigm, we assume that two measures are equivalent. However, when the real-world values of the underlying asset are outside the range of the risk-neutral values used in the least squares regression, it may result in estimated CFs producing a suboptimal exercise strategy. We optimize the procedure to improve the efficiency and precision in simulation to help deal with this problem.

3.3.1   Variance Reduction

Crude Monte Carlo simulation is relatively straightforward to implement, yet the process may not be very efficient. The rate of convergence of the estimation error is $O(1/\sqrt{n})$ where n is the number of simulation trials. In our simulation study, we use the following two methods to improve the efficiency of Monte Carlo simulation by reducing the variance.

Antithetic Variates

Antithetic variates technique is to generate pairs of negative correlated sample paths to reduce the variance in simulation. For example, for random variable X, we want to estimate $\theta = \mathbb{E}[X]$. Suppose that we have two identically distributed random variables X_1 and X_2

with mean θ, then an unbiased estimate of θ is:

$$\hat{\theta} = \frac{X_1 + X_2}{2},$$

and

$$Var(\hat{\theta}) = \frac{Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)}{4}.$$

Since X_1 and X_2 are identically distributed, then we have $Var(\hat{\theta}) = \frac{Var(X_1)}{2} + \frac{Cov(X_1,X_2)}{2}$. If X_1 and X_2 are negative correlated, the variance $Var(\hat{\theta})$ becomes smaller since $Cov(X_1, X_2) < 0$.

In our simulation, instead of generating the whole set of the sample paths, we generate only half of them using dw, then we use $-dw$ as antithetic variates to generate the remaining paths in the simulation.

Control Variates

Another way to reduce variance in simulation is to use control variates to estimate the variable. Suppose we want to estimate $\theta = E[X]$, there is another random variable Y such that $\mu_Y = E[Y]$ is known. Then for any number a, the random variable $Z = X + a(Y - E[Y])$ is an unbiased estimator of θ.

We have $E[Z] = E[X + a(Y - E[Y])] = E[X] + a(E[Y] - \mu_Y) = \theta$, and $Var(Z) = Var(X + aY) = Var(X) + a^2 Var(Y) + 2aCov(X, Y)$. To minimize $Var(Z)$ with respect to a, we have: $a^* = -Cov(X, Y)/Var(Y)$, then $Var(Z) = Var(X + a^*Y) = Var(X) - Cov(X, Y)^2/Var(Y)$.

We can apply control variates in the cross-sectional regression to reduce the variance of estimation under the OLSM framework. We write $CF_{P,i}(\cdot)$ as a weighted average of $CF_{Q,j}(\cdot)$ such that:

$$CF_{P,i} = \sum_{j=1}^{N} b_{j,j} CF_{Q,j},$$

where $b_{i,j}$ are the weights connecting path i to path j, $CF_{P,i}$ and $CF_{Q,j}$ are the conditional expectation functions on paths i and j under the physical measure and the risk-neutral measure respectively, N is the number of the price paths.

Let Y_j be some control variates , we have:

$$\begin{aligned}
CF_{P,i}(a) &= \sum_{j=1}^{N} b_{i,j} CF_{Q,j} - a\Big(\sum_{j=1}^{N} b_{i,j} Y_j - \sum_{j=1}^{N} b_{i,j}\mathbb{E}[Y_j]\Big) \\
&= \sum_{j=1}^{N} b_{i,j}\big(CF_{Q,j} - a(Y_j - \mathbb{E}[Y_j])\big).
\end{aligned}$$

We obtain $a^* = Cov(Y_j, CF_{Q,j})/Var(Y_j)$ to minimize the variance.

The choice of control variates is arbitrary. In general, we should choose the one that is highly correlated with the conditional expectation functions being estimated.

In our implementation tests, we evaluate Asian and barrier options under the OLSM framework, and we use the European options as control variates accordingly.

3.3.2   Approximation Improvement

Variance reduction techniques can help to reduce the simulation error. To further improve the efficiency of the OLSM method, we can employ some additional techniques. We describe

them briefly below.

Initial State Dispersion

In [17], Kan et al. proposed to disperse the initial state so as to improve the stability of the regression. In a normal simulation, we generate scenarios with one initial value. For example, when using GBM model to generate the price paths of the underlying asset, we use single price at the initial point. When we calculate the credit exposures, we should have a forward-looking view. The prices in the real world will be different than the prices under the risk-neutral measure. The method of initial state dispersion will use a range of prices at the beginning to generate a wider support for regression. This will help to reduce extrapolation error when calculating the credit exposures. The following figures illustrate the simulated price paths with single initial price and initial state dispersion.

In our testing, we divide the range of the initial state into 3 regions by using $\pm$ 2 standard deviations because this will cover 95% of price variation under the normal distribution. Further, we draw 70% of sample paths uniformly from the $\pm$ 1 standard deviation since this region covers nearly 68% of the price variation. We draw 15% of sample paths in each region between $\pm$ 1 and ± 2 standard deviations respectively .

Multiple Bucketing

While initial state dispersion deals with data at initiation in simulation, multiple bucketing is aimed to improve the fit of regression models in the later period. One drawback of the

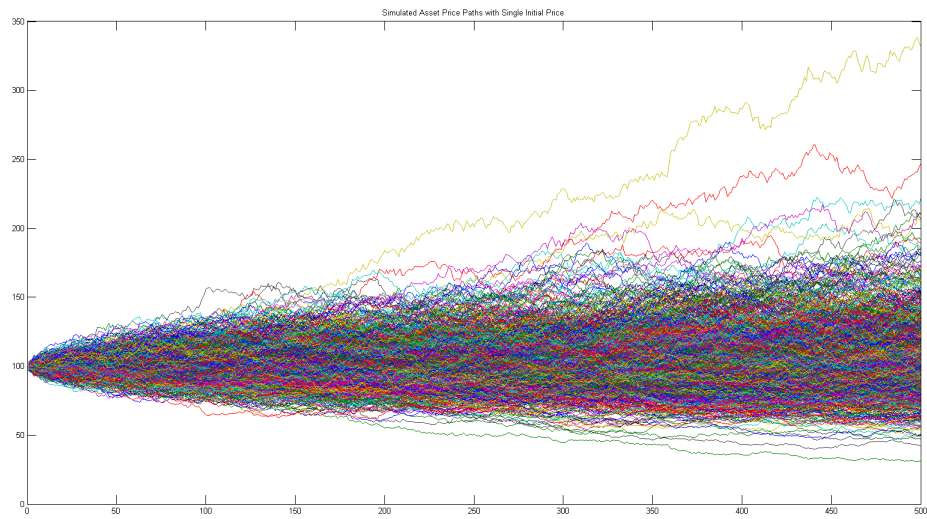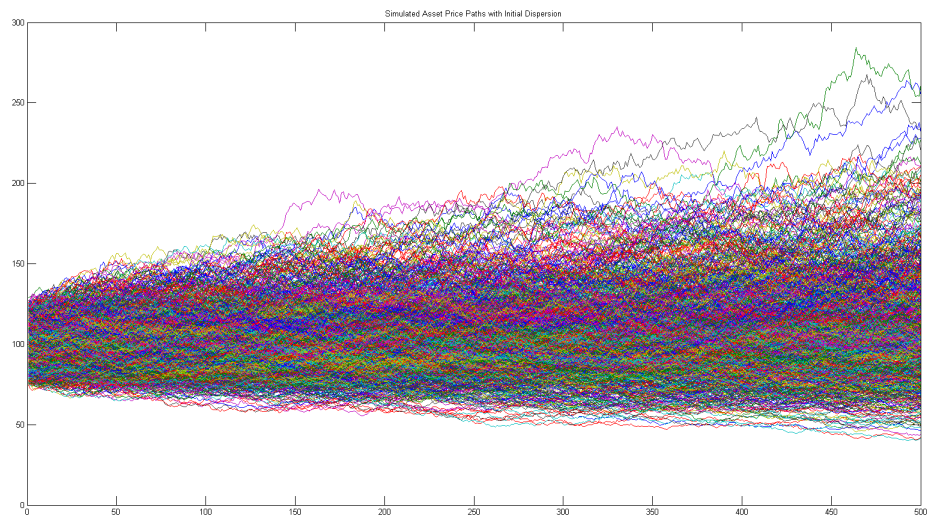Figure 3.1: Simulated sample price paths, single initial price

Simulated Asset Price Paths with Single Initial Price

Figure 3.2: Simulated sample price paths, initial state dispersion

Simulated Asset Price Paths with Initial Dispersion

least squares method is that it is sensitive to outliers. Outliers will distort the functionality of least squares method, leading to possible biased estimates.

In Kan et al. [17], the authors divided the simulation paths into buckets. Then they ran a single bucket regression in the first quarter with bucket boundary for the underlying stock prices be 100, then after the first quarter till expiration date, they ran two regressions for in-the-money (ITM) and out-of-the-money (OTM) buckets, the boundary is the strike price. Kan et al. pointed out that the price of the financial instrument being evaluated is sensitive to the choices of bucket boundaries, but the authors didn't give a general approach to address this issue.

In our test, we calculate the exposures for an Asian call option using two buckets, where we arbitrarily divide our simulation process into two parts at the mid-term (i.e. T/2). We run single least squares regression in the first part, then we run two regressions with boundary being the expected price of the underlying asset in the second part. Since we are concerned about the in-the-money paths, the buckets (or intervals) are [strike price, expected asset price] and (expected asset price, +). Our result suggests that this issue is not very important for Asian options, since the averaging effectively smooths the stochastic paths of prices.

3.4   Summary

In this chapter, we have introduced our optimized least squares Monte Carlo method to calculate exposures for some path-dependent options. Because calculating the exposure of

an option is more difficult than pricing the option, numerical methods like the binomial option pricing method and the finite difference method are less practical in this regards.

We present the backward pricing dynamics for American options, the explicit expressions for Asian and barrier options are given in this study. The backward pricing dynamics can be easily extended to other types of options. We also discuss the set of basis functions that are used in the approximation of conditional expectation functions. We integrate some techniques to improve the performance in Monte Carlo simulations. In next chapter we will present our numerical results.

Chapter 4

Numerical Results

In previous chapter, we have introduced the optimized least squares Monte Carlo approach which aims to calculate EEs for options that are early exercisable and path-dependent. In this chapter, we will calculate EEs for Asian and barrier options under the OLSM framework. All options studied in this chapter are American-style.

4.1 Asian and Barrier options

Before we can calculate EEs for Asian and barrier options on shares of stock, we want to verify that our models will price the options correctly under the OLSM framework. In this section, we first calculate the values of the American put option in Longstaff and Schwartz's work [18], and compare our results with theirs. Then, we will focus on pricing an arithmetic average Asian call option, and an "up-and-out" barrier call option.

4.1.1   OLSM versus LSM for American Put Option

The OLSM method is an extension of the original LSM method. However, there are several distinctions between these two methods. The LSM method is used to price the American options, thus it is conducted under the risk-neutral measure. The OLSM method can be used to price the American options and other types of exotic options. In this case, it is conducted under the risk-neutral measure. This is consistent with the original LSM method. The OLSM method can also be used to calculate the credit exposures of the options with respect to counterparty credit risk. Then, it is carried out under the physical measure in this case.

There is no variance reduction technique (except the antithetic variates) integrated into the original LSM method. Our OLSM method incorporates variance reduction techniques, as well as initial state dispersion and multiple bucketing, which improves its performance. For comparison, below we use the OLSM method to calculate prices of the American put options presented in the Longstaff and Schwartz's paper.

The underlying stock price follows a lognormal model. The option is exercisable 50 times per year, the strike price is 40, the interest rate is 0.06. The underlying stock price S, the volatility σ and the number of years until the final expiration of the option T are as indicated in Table 4.1.

Longstaff and Schwartz used a constant and the first three Laguerre polynomials as the set of basis functions. In our test, we use monomials up to order of 3 as the set of basis functions.

The simulation is based on 100000 (50000 and 50000 antithetic) paths for the stock-

price process. We repeat the simulation 100 times, and the averages of the results are listed in Table 4.1.

Table 4.1: Comparison of FD, OLSM and LSM methods for American put option

S	σ	T	Finite Difference	OLSM	(s.e.)	LSM	(s.e.)	Difference	Difference
A	B	C	D	E	F	G	H	D-E	D-G
36	0.2	1	4.478	4.476	(0.013)	4.472	(0.010)	0.002	0.006
36	0.2	2	4.840	4.837	(0.019)	4.821	(0.012)	0.003	0.019
36	0.4	1	7.101	7.101	(0.005)	7.091	(0.020)	0	0.010
36	0.4	2	8.508	8.504	(0.011)	8.488	(0.024)	0.004	0.020
40	0.2	1	2.314	2.313	(0.004)	2.313	(0.009)	0.001	0.001
40	0.2	2	2.885	2.882	(0.008)	2.879	(0.010)	0.003	0.006
40	0.4	1	5.312	5.312	(0.004)	5.308	(0.018)	0	0.004
40	0.4	2	6.920	6.915	(0.011)	6.921	(0.022)	0.005	-0.001
44	0.2	1	1.110	1.110	(0.001)	1.118	(0.007)	0	-0.008
44	0.2	2	1.690	1.689	(0.004)	1.675	(0.009)	0.001	0.015
44	0.4	1	3.948	3.949	(0.004)	3.957	(0.017)	-0.001	-0.009
44	0.4	2	5.647	5.641	(0.010)	5.622	(0.021)	0.006	0.025

In Table 4.1, values in columns D, G and H are excerpted from Longstaff and Schwartz's example. To be consistent with their work, we use the results obtained by the finite difference (FD) method (column D in Table 4.1) as the benchmarks.

Columns E and F in Table 4.1 are obtained by our OLSM method, the differences between OLSM and FD methods are listed. The results imply that our OLSM method is very accurate with respect to the valuation of American-style options.

4.1.2 Evaluation of Asian option

Here we price an arithmetic average Asian call option on a share of stock. We assume that the stock does not pay dividends, and the options are cash-settled. The initial price of the stock is 100, the strike price is 100, the interest rate is 5%, the volatility of the price of the underlying asset is 20%, and the life of option is 2 years. We also assume that the option is exercisable 50 times per year.

We assume that the price of the underlying asset follows GBM model:

$$S(t)_Q = S_0 \cdot \exp\left[(\mu - \frac{\sigma^2}{2})t + \sigma W(t)\right]$$

For valuing the Asian option, we have two state variables, one is the spot price, and another is the average stock price. We use the complete polynomials up to order of 2 as the basis functions as well as monomials up to order of 3.

The summary of parameters in the tests is listed below:

Table 4.2: Parameters for arithmetic average Asian option

Parameter	Asian option
initial price	100
strike price	100
risk-free rate	5%
time to expiration	2
volatility	20%
number of simulations	10000

We first price the Asian option using single initial price point without variance reduction. We then price the option by adopting the antithetic variates and control variates

techniques respectively. The simulations are based on 10000 (5000 and 5000 antithetic) paths. We use the corresponding European option as a control variate. We repeat the simulations 100 times and the averages of the results are listed in Table 4.3.

Table 4.3: Pricing of arithmetic average Asian call option under OLSM

No.	Scheme	European Asian call option	American Asian call option	(s.e.)
1	single price	8.7150	9.4683	(0.0093)
2	antithetic variate	8.6732	9.4067	(0.0088)
3	antithetic, control variates	8.6732	9.3347	(0.0090)
4	single price, two buckets	8.6732	9.3347	(0.0094)
5	state dispersion	8.7916	9.5731	(0.0098)
6	antithetic variate	8.7500	9.5397	(0.0095)
7	antithetic, control variates	8.7500	9.4648	(0.0102)
8	state dispersion, two buckets	8.7500	9.4648	(0.0107)

In Table 4.3, columns 3 and 4 list the prices of European and American Asian call options respectively. The last column shows the standard errors of the simulation. In this test, we first use a single initial price without incorporating the variance reduction into the model. Then we price the options with variance reduction techniques. When the antithetic variate technique is introduced, the prices of the European and American Asian options are improved. The price of American Asian option is further improved by introducing the control variates technique. The European Asian call option is used as control variates. Finally, we use the two buckets technique to price the European and American Asian options. Table 4.3 shows that the price of the American Asian option is the same as in the case of control variates.

We then price the option using initial states dispersion. For this, we divide the initial

state into 3 regions by using ± 1 and ± 2 standard deviations, we draw 70% of sample paths within ± 1 standard deviation from the initial price, and 15% of sample paths from each region between ± 1 and ± 2 standard deviations respectively. When the initial price is a range of values, this means that more in-the-money paths should be included in the estimation. Then, the price of the option should be higher than when we use a single initial price in simulation. Both the European and American Asian options have higher prices than in the previous test. Antithetic variates and control variates technique work similarly for the initial state dispersion in that they reduce the bias in original LSM approach.

In our test, we repeat the simulation 100 times with the sample size of 10000 and time step of 100. It takes about 30 seconds to run the program on a quad-core CPU at 3.0GHz when we utilize all the optimization techniques. The computational cost is justifiable. Thus we should utilize these techniques in our simulation to improve the approximation.

4.1.3   Evaluation of Barrier option

In this section, we want to price an up-and-out barrier call option with single barrier. We assume that the price of the underlying asset follows GBM model as in the case of the Asian option. The asset does not pay out dividends and there is no rebate. The initial price is 100, strike price is 100, interest rate is 5%, the volatility of the price of the underlying asset is 20%, the life of the option is 2 years and the option is exercisable 250 time per year.

The summary of parameters in the tests is listed in Table 4.4:

We use 4 different barrier levels: 120, 150, 200 and 500, we will focus on the levels of 120 and 150. We include the barrier levels of 200 and 500 for comparison purposes. For an

Table 4.4: Parameters for barrier (up-and-out) call option

Parameter	Barrier option
initial price	100
strike price	100
risk-free rate	5%
time to expiration	2
volatility	20%
number of simulations	10000
barrier level	120, 150, 200, 500

"up-and-out" barrier call option, the higher the barrier level, the higher the option price (option is less likely being "knocked out" in this case), and the higher the exposure would be.

The lowest barrier level is 120. In this case, many price paths are excluded as the options on these paths are quickly knocked out. Thus the option is most valuable at the beginning, consequently EE is highest at the beginning as well. The second lowest level is 150. Since this barrier level is higher, it allows more price paths being included in the calculation, and hence the price of the option is higher. For barrier levels of 200 and 500, the barrier options are like a regular call option in that the barriers are less likely to be reached.

Table 4.5: Pricing of barrier (up-and-out) call option under OLSM

Barrier level	European up-and-out option	American up-and-out option
120	0.5718	2.1166
150	6.1818	8.4480
200	14.0458	14.5696
500	16.1097	16.1493

We now focus on the barrier levels of 120 and 150. Each simulation has 10000 paths,

and we repeat the process for 100 times and take the average of the results. We also test different variance reduction techniques.

Table 4.6: Pricing of barrier (up-and-out) call option under OLSM

No.	Scheme	barrier level 120	barrier level 150
1	single price	2.1194	8.4286
2	antithetic variate	2.1221	8.4244
3	state dispersion	2.1177	8.4241
4	antithetic variate	2.1224	8.4217

Table 4.6 reports the prices of barrier options with two barrier levels. For a barrier option, it is not as sensitive as an Asian option with respect to the initial state dispersion. This is because the effective paths are bounded by the strike and barrier. If paths generated by initial state dispersion are outside of this region, they are excluded from the calculation of the value of the option.

4.2   Plots of Expected Exposures

Once we have the complete values of the option at every exercise node, the expected exposure is calculated by Equation (3.12).

We will calculate the expected exposures for the Asian and barrier options that we have studied. Since EE is not a single number, we will plot EE for illustration purpose.

4.2.1 Plots of Exposures for Asian Option

Here we show the plots of the expected exposures (EE) for an arithmetic average Asian option. The initial price of the stock is 100, the strike price is 100, interest rate is 5%, the volatility of the price of the underlying asset is 20%, and the life of the option is 2 years. We also assume that the option is exercisable 50 times per year.

The thin lines in the plots represent exposures for selected sample paths, while the thick line is the exposure that is the average of the results based on 100 trials.

The plots show that the expected exposure of the Asian option reaches its peak level near the mid-term of its life. As the price of the underlying stock increases, the average of the price also increases, thus the exposure increases as well. When the option is exercised before expiration, the succeeding exposures vanish. We can see that the exposure gradually decreases toward expiration. The exposure profile exhibits a humped curve because of the averaging feature of the Asian option.

Figure 4.1: Exposure for an Asian option, single initial price
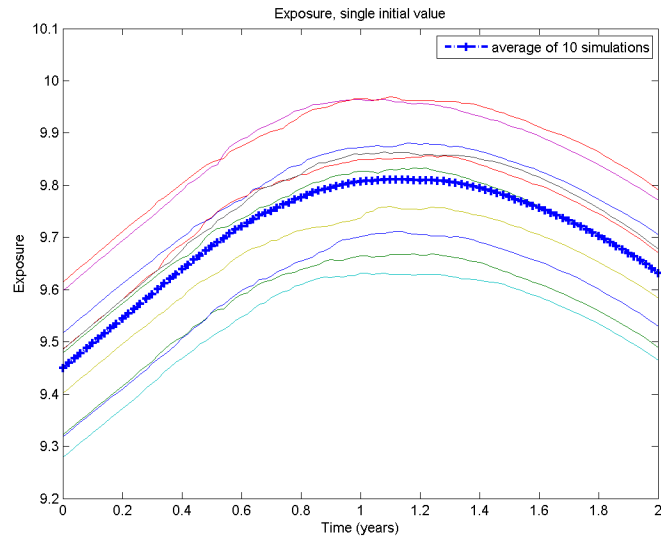
Figure 4.2: Exposure for an Asian option, single initial price with antithetic variates
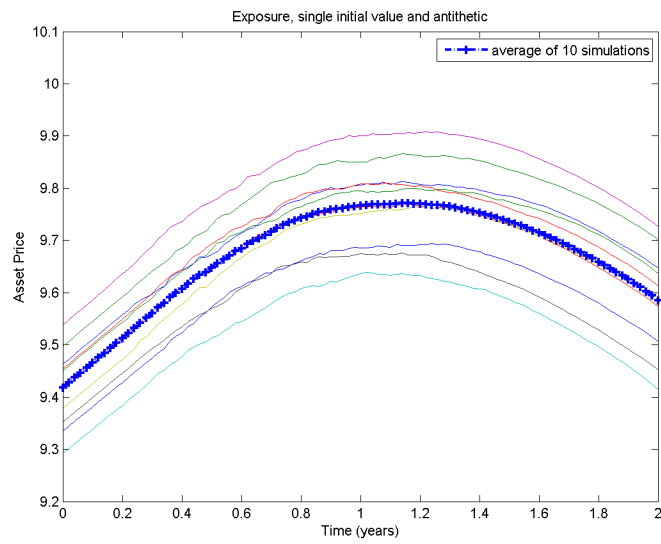
Figure 4.3: Exposure for an Asian option, single initial price with antithetic variates and control variates
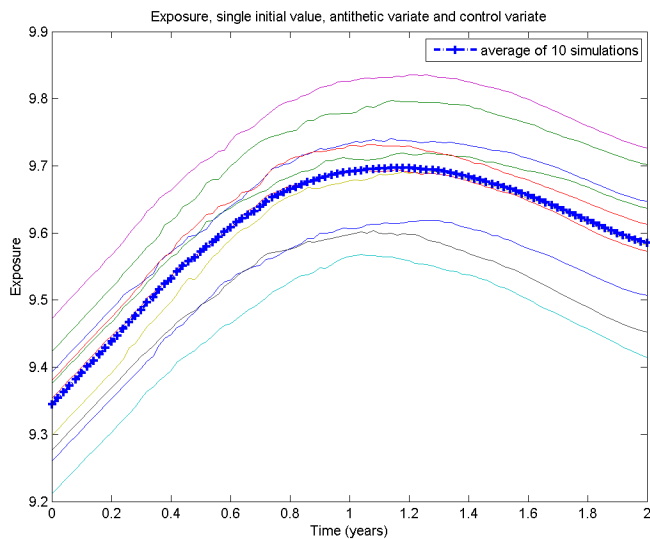
Figure 4.4: Exposure for an Asian option, single initial price with antithetic variates, control variates and two buckets
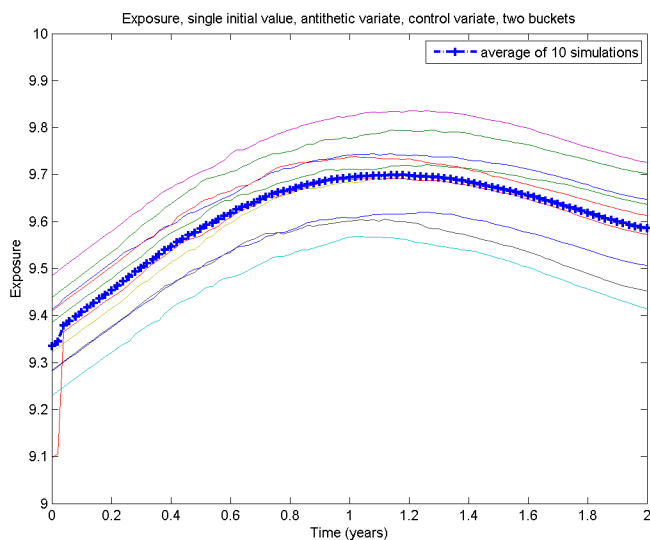
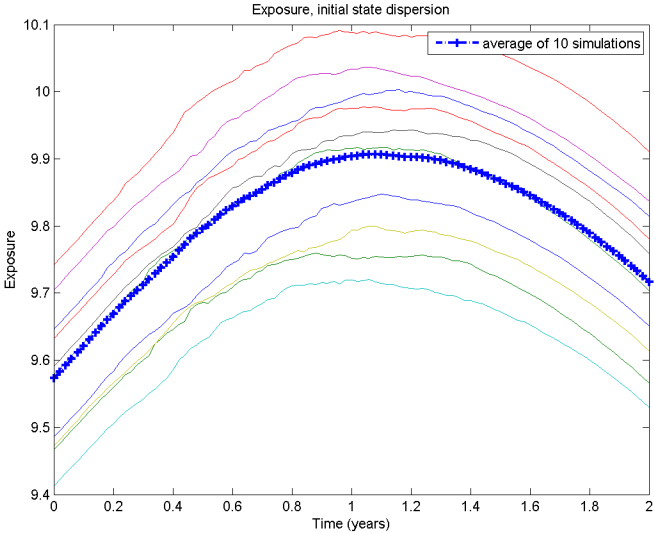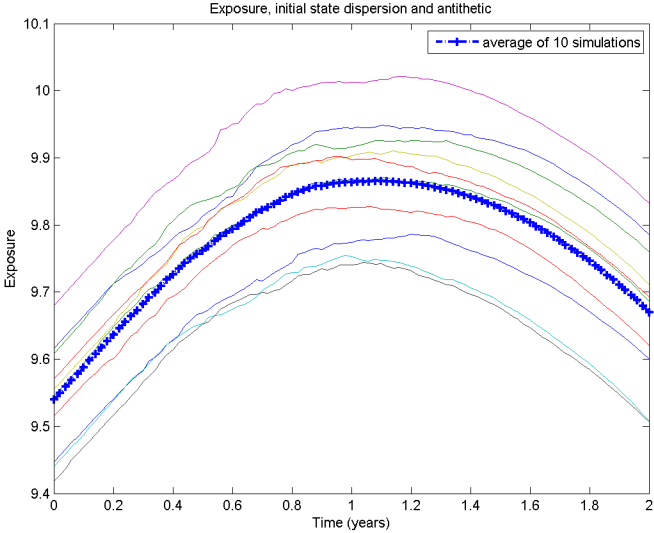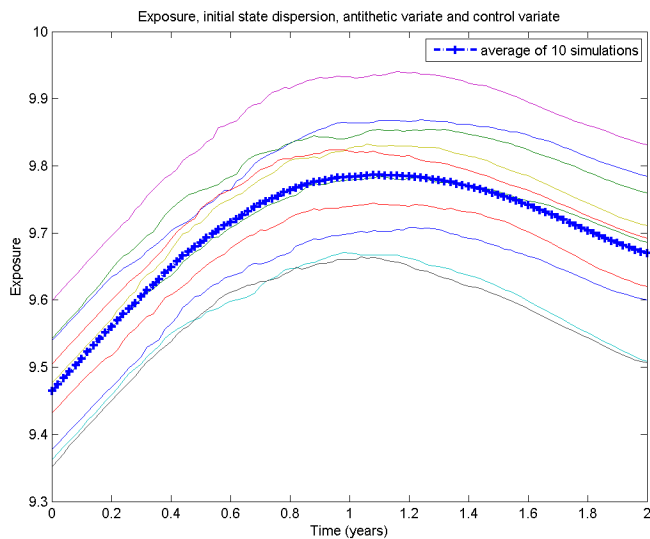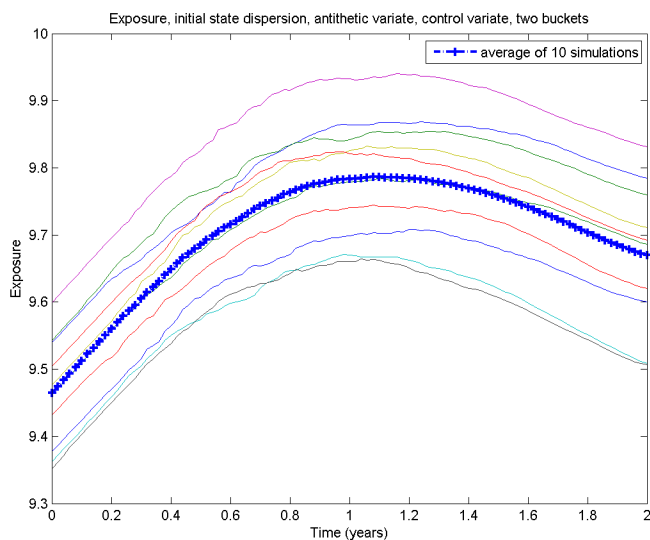Figure 4.5: Exposure for an Asian option, initial state dispersion

Figure 4.6: Exposure for an Asian option, initial state dispersion with antithetic variates

Figure 4.7: Exposure for an Asian option, initial state dispersion with antithetic variates and control variates

Exposure, initial state dispersion, antithetic variate and control variate

Figure 4.8: Exposure for an Asian option, initial state dispersion with antithetic variates, control variates and two buckets

Exposure, initial state dispersion, antithetic variate, control variate, two buckets

When the initial state dispersion is utilized in the simulation, the value of the Asian option increases. This is because more in-the-money paths are available.

Figure 4.9 shows the exposures in the cases of single initial price and initial state dispersion. The curve representing exposure in the case of initial state dispersion is on the top.

When we evaluate credit exposure, we should have a forward-looking view. We know the initial price today, but the future prices are unknown to us. We use the estimated conditional expectation functions to calculate the exposures under the physical measure. By using initial state dispersion technique in simulation, the simulated paths will cover a broader range of real prices at future times. Since expected exposure is a measurement of risk, the expected exposure obtained in the case of initial state dispersion is conservative.

Figure 4.9: Exposures for an Asian option, single initial price versus initial state dispersion
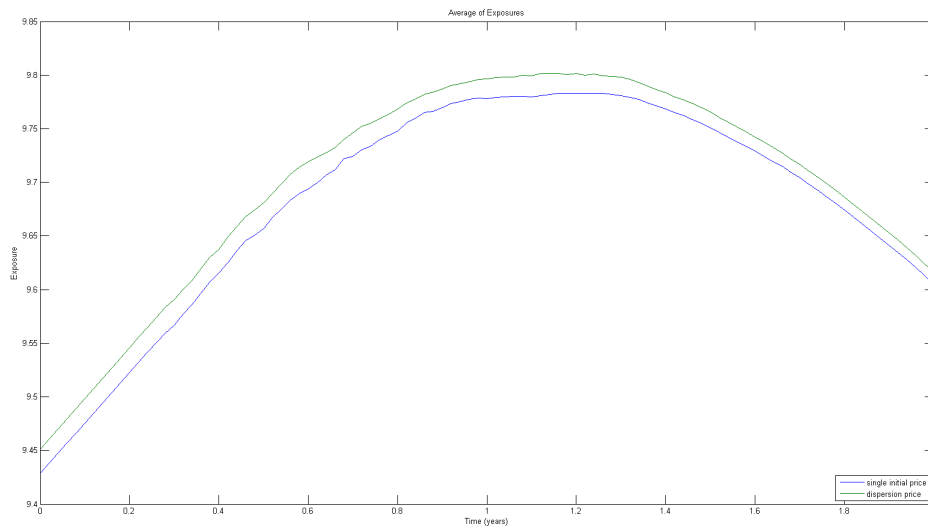
4.2.2 Plots of Exposures for Barrier Options

Here we show the plots of exposures for barrier options.

The first four plots show the exposures of the up-an-out barrier call options with barrier levels of 120, 150, 200 and 500 respectively. As can be seen from the plots, the up-and-out barrier call option behaves like a regular call option when the barrier level is high.

At the barrier level of 120, the exposure is at peak at the beginning; at the barrier level of 150, exposure increases as the price of the underlying asset evolves, then quickly drops when prices hit the barrier. At the barrier levels of 200 and 500, exposures are similar to those of regular call options as the barriers are less likely to be reached.

Figure 4.10: Exposure for a barrier (up-and-out) call option, single initial price, barrier level=120
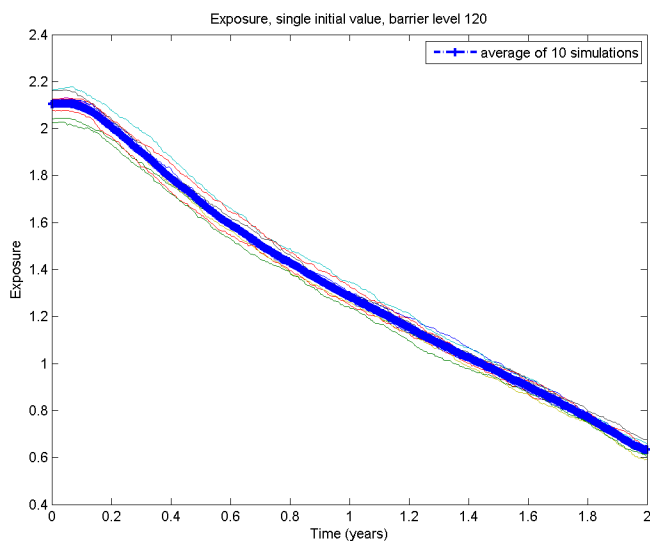
Figure 4.11: Exposure for a barrier (up-and-out) call option, single initial price, barrier level=150
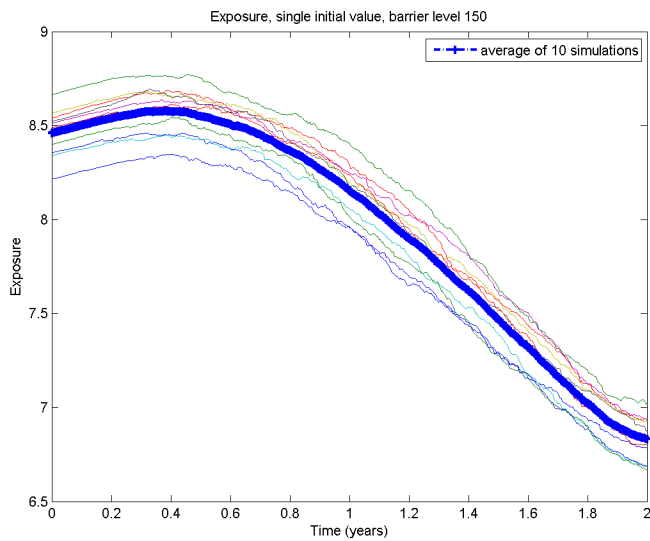

Exposure, single initial value, barrier level 150

Figure 4.12: Exposure for a barrier (up-and-out) call option, single initial price, barrier level=200
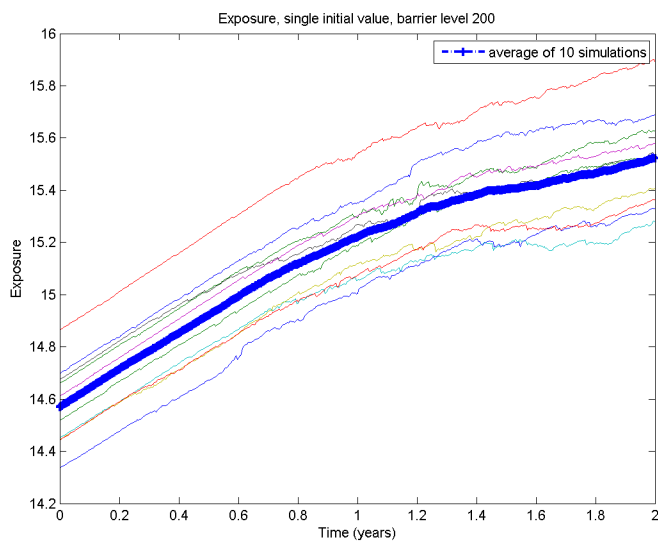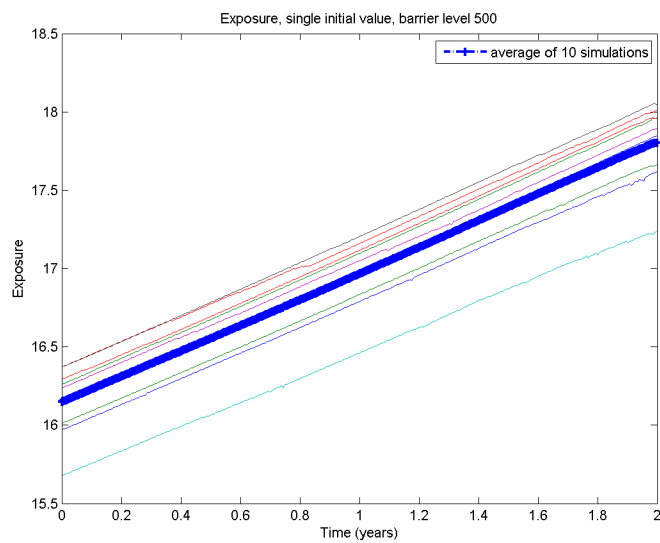

Exposure, single initial value, barrier level 200

Figure 4.13: Exposure for a barrier (up-and-out) call option, single initial price, barrier level=500

The following plots are the exposures of an up-and-out barrier call option when different variance reduction techniques are applied. We have not evaluated the barrier option using two buckets technique since the effective price paths are bounded by the strike price and the barrier. Paths outside of this range are automatically excluded, therefore there is no improvement using two buckets in the evaluation.

Figure 4.14: Exposure for a barrier (up-and-out) call option, single initial price with antithetic variates, barrier level=120
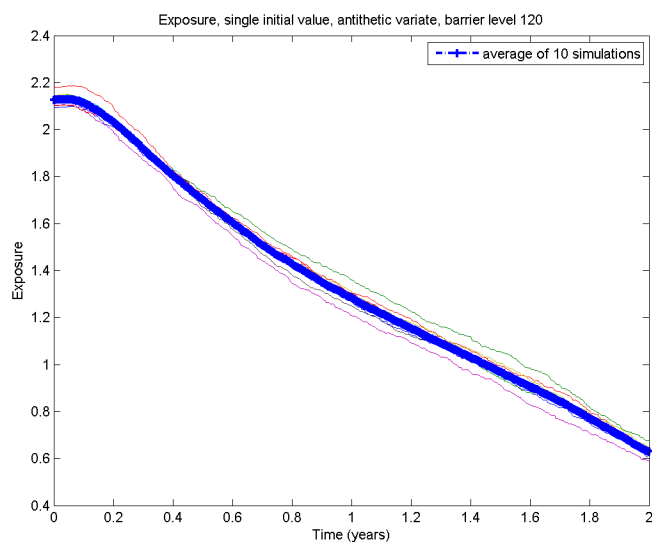
Figure 4.15: Exposure for a barrier (up-and-out) call option, single initial price with antithetic variates and control variates, barrier level=120
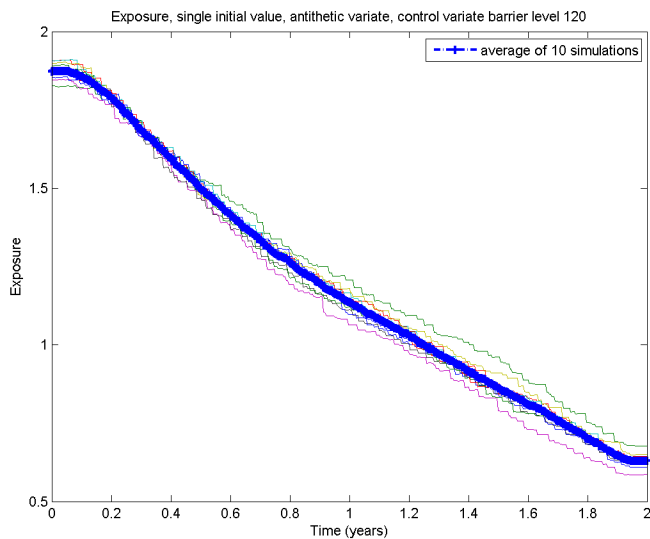

Figure 4.16: Exposure for a barrier (up-and-out) call option, initial state dispersion with antithetic variates, barrier level=120
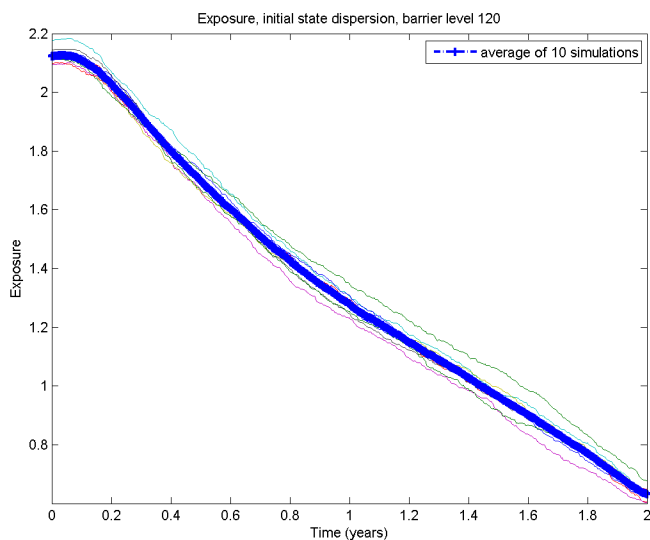
Figure 4.17: Exposure for a barrier (up-and-out) call option, initial state dispersion with antithetic variates and control variates, barrier level=120
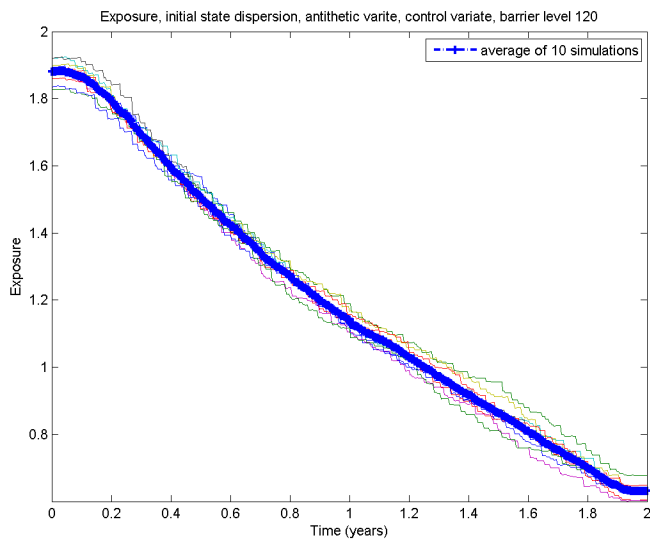

Figure 4.18: Exposure for a barrier (up-and-out) call option, single initial price with antithetic variates, barrier level=150
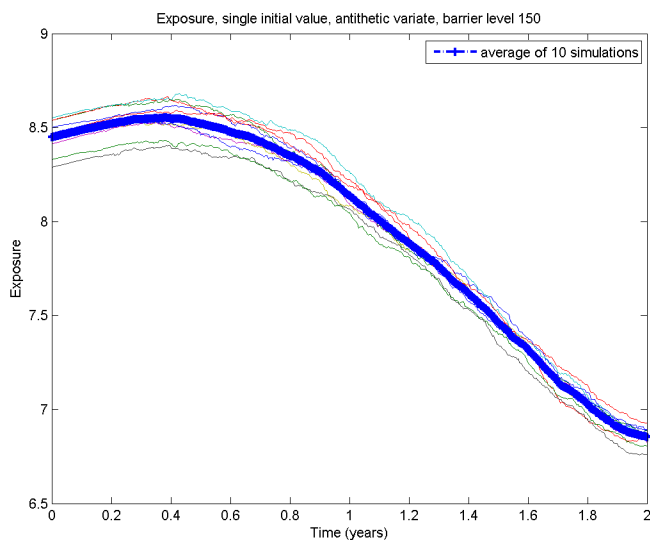
Figure 4.19: Exposure for a barrier (up-and-out) call option, single initial price with antithetic variates and control variates, barrier level=150
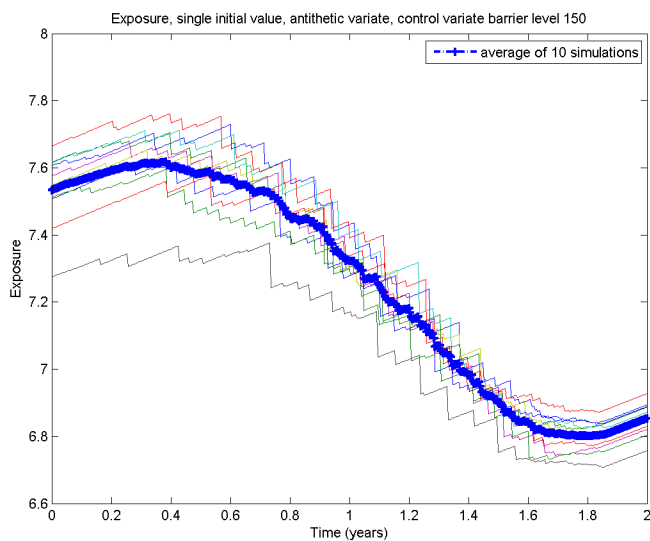

Figure 4.20: Exposure for a barrier (up-and-out) call option, initial state dispersion with antithetic variates, barrier level=150
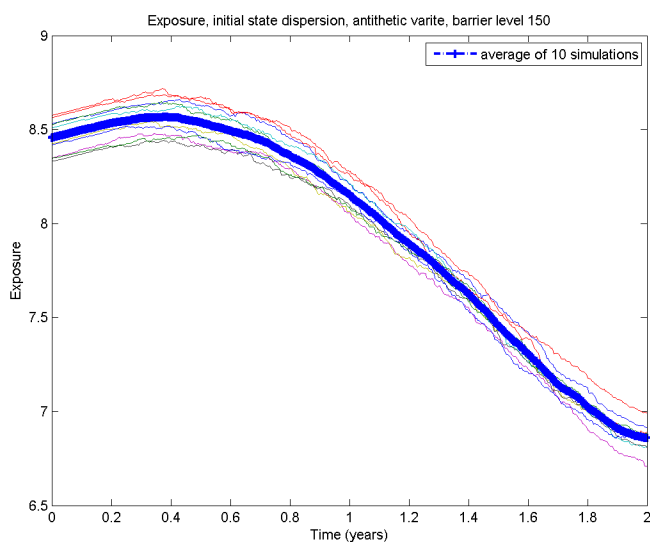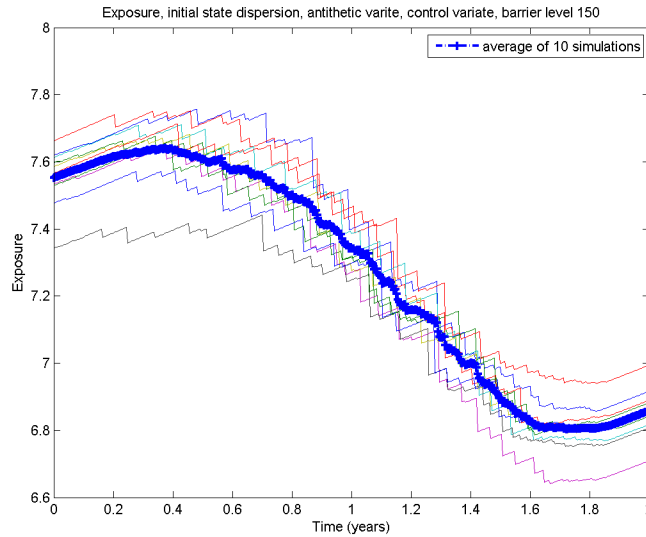
Figure 4.21: Exposure for a barrier (up-and-out) call option, initial state dispersion with antithetic variates and control variates, barrier level=150

For the barrier options we have studied, the numerical results of respective exposures are very close with or without incorporating the variance reduction techniques. This is because the value of the barrier option depends on the strike and the barrier level. Paths outside of this region will not affect the value or exposure of the barrier option.

When we use European call options as control variates, the exposures drop sharply. The plots show that the sample exposures are more dispersive and irregular. The value of an European call option is not affected by the barrier. When we use it as the control variates to estimate the conditional expectation functions, the estimated conditional expectation functions might be flawed. In this case, an European option is not a good control variate for a barrier option.

4.3 Summary

In this chapter, we have calculated the expected exposures for Asian and barrier options under the OLSM framework.

We have implemented variance reduction, initial state dispersion and multiple bucketing techniques in our tests. The numerical results show that OLSM method can calculate the exposures accurately and efficiently. We use monomials up to order of 3 and complete polynomials up to order of 2 as the sets of basis functions. We obtain approximately the same results in our tests. This implies OLSM method is stable with respect to the choice of basis functions.

Chapter 5

Summary

In this thesis, we have extended the optimized least squares Monte Carlo method that is based on the original LSM method. The OLSM method is a simple and yet powerful approach to evaluate credit exposures for path-dependent options.

In our study, we have evaluated the credit exposures for Asian and barrier options under the OLSM framework. The results show that our method is accurate and efficient.

The OLSM method has the advantage that it is not limited to calculate exposures for path-dependent options. We have presented backward pricing dynamics for Asian and barrier options. With properly defined pricing dynamics, the OLSM method can be used to calculate exposures for other types of options.

We have made several assumptions to simplify the calculation in our study. For example, we assume GBM model for the evolution of the prices, and we only evaluate one option at a time. In practice, other more advanced models are available, and a typical

portfolio usually contains many contracts. The OLSM method has the advantage when multiple factors and a high dimension problem are of concern since we have adopted the Monte Carlo method in our approach.

Appendix A

Standard Model for CVA in Basel III

Basel III permits the banks to calculate CVA charges by two methods, one is standardized approach (SA), one is advanced approach. We will briefly introduce SA-CCR under Basel III framework.

For banks without supervisors' approval for internal model method (IMM) for CCR, they must now use SA-CCR method described in Basel III to calculate CVA charges.

The formula in SA method is defined as:

$$
K = 2.33 \cdot h^{1/2} \cdot \left(\left(\sum_i 0.5 \cdot w_i \cdot (M_i \cdot EAD_i^{total} - M_i^{hedge} \cdot B_i) - \sum_{ind} w_{ind} \cdot M_{ind} \cdot B_{ind} \right)^2 + \right.
$$
$$
\left. \sum_i 0.75 \cdot w_i^2 \cdot (M_i \cdot EAD_i^{total} - M_i^{hedge} \cdot B_i)^2 \right)^{1/2}
$$

(A.1)

where

- h is the one-year risk horizon, $h = 1$

- w_i is the weight applicable to the rating of counterparty 'i', w_{ind} is the weight applicable to index hedges.

- EAD_i^{total} is the exposure at default of counterparty 'i', including the effect of collateral as per applicable rules.

- B_i is the notional of purchased single name CDS hedges (summed if more than one position) referencing counterparty 'i', and used to hedge CVA risk. M_i^{hedge} is the maturity of the hedge instrument for B_i

- B_{ind} is the full notional of one or more index CDS of purchased protection, used to hedge CVA risk. M_{ind} is the maturity of the index hedge 'ind'.

- M_i is the effective maturity of the transactions with counterparty 'i'.

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