Existence Assumptions and Logical Principles: Choice Operators in Intuitionistic Logic

by

Corey Mulvihill

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Philosophy

Waterloo, Ontario, Canada, 2015

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Hilbert's choice operators τ and ε, when added to intuitionistic logic, strengthen it. In the presence of certain extensionality axioms they produce classical logic, while in the presence of weaker decidability conditions for terms they produce various superintuitionistic intermediate logics. In this thesis, I argue that there are important philosophical lessons to be learned from these results. To make the case, I begin with a historical discussion situating the development of Hilbert's operators in relation to his evolving program in the foundations of mathematics and in relation to philosophical motivations leading to the development of intuitionistic logic. This sets the stage for a brief description of the relevant part of Dummett's program to recast debates in metaphysics, and in particular disputes about realism and anti-realism, as closely intertwined with issues in philosophical logic, with the acceptance of classical logic for a domain reflecting a commitment to realism for that domain. Then I review extant results about what is provable and what is not when one adds epsilon to intuitionistic logic, largely due to Bell and DeVidi, and I give several new proofs of intermediate logics from intuitionistic logic+ε without identity. With all this in hand, I turn to a discussion of the philosophical significance of choice operators. Among the conclusions I defend are that these results provide a finer-grained basis for Dummett's contention that commitment to classically valid but intuitionistically invalid principles reflect metaphysical commitments by showing those principles to be derivable from certain existence assumptions; that Dummett's framework is improved by these results as they show that questions of realism and anti-realism are not an "all or nothing" matter, but that there are plausibly metaphysical stances between the poles of anti-realism (corresponding to acceptance just of intutionistic logic) and realism (corresponding to acceptance of classical logic), because different sorts of ontological assumptions yield intermediate rather than classical logic; and that these intermediate positions between classical and intuitionistic logic link up in interesting ways with our intuitions about issues of objectivity and reality, and do so usefully by linking to questions around intriguing everyday concepts such as "is smart," which I suggest involve a number of distinct dimensions which might themselves be objective, but because of their multivalent structure are themselves intermediate between being objective and not. Finally, I discuss the implications of these results for ongoing debates about the status of arbitrary and ideal objects in the foundations of logic, showing among other things that much of the discussion is flawed because it does not recognize the degree to which the claims being made depend on the presumption that one is working with a very strong (*i.e.*, classical) logic.

Acknowledgements

I would first like to thank my advisor David DeVidi, who both encouraged and, when necessary, restrained my efforts, and whose own work inspired so much of the subject matter of this thesis. I would also like to thank my committee, from the Waterloo philosophy department, Tim Kenyon, Doreen Frazer and especially Gerry Callaghan, who valiantly stepped in at the last minute; Steve Furino, from the Waterloo faculty of mathematics; and my external examiner Greg Lavers for his careful reading and helpful comments.

In addition, I must thank all the Waterloo Philosophy Department Faculty and Staff. Especially Debbie Detrich who from my first arrival at UW has advised me, and helped me navigate the UW system, looking out for me as she does all the graduate students in the Philosophy Department, we are very lucky to have her.

I should also thank my graduate colleagues at Waterloo, who created such a wonderful and welcoming environment. Especially my year's Ph.D. cohort: Mike McEwan and Eric Hochstein. They made living in Waterloo tolerable with their friendship, discussions in various uptown establishments, and the Saturday breakfasts that followed.

In a similar vein I would be remiss if I did not thank the Univeristy of Western Ontario philosophers, and Ivey folk, who made living in London bearable.

In addition to those mentioned above, I would also like to thank the people who I encountered when I started my journey in philosophy, at the University of Ottawa—especially since I forgot to include an acknowledgement's section in my master's thesis. Thanks, first, to Mathieu Marion, my master's supervisor, who first introduced me to non-classical logics; and to Sophie Rietti and Patrice Philie, friends and now colleagues at Ottawa, for their friendship and insight. Thanks also are due to my former graduate student colleagues, in particular: Dan McArthur, who encouraged me to return to academia; Mark Young, whose friendship and loyalty can only be described as supererogatory; Shane Ralston, who I am proud to say made it through his Ph.D. at Ottawa without me getting him arrested once; and Dean Lauer whose wonderful companionship kept us out late so many nights, often until closing hours.

Thanks to my wonderful gigantic family, both the Rowes and the Mulvihills, and now, since I made the best decision of my life and married my wonderful wife, the Mazutises and Trences. Please know that I am aware of how lucky I am to have so many wonderful and supportive people in my life. A special thank you is due to my amazing parents, Bob and Helen, who have supported me through the years, both morally and financially, and to my sister, Shauna Mulvihill, whom the fates have punished with me as a sibling.

Finally, to my brilliant and beautiful wife Daina Mazutis and my daughter Emilija Mulvihill; I know I could not have finished this without you Daina. Thank you for your support and the most wonderful daughter in the world. Thank you all.

Contents

Introduction

Chapter 1

Introduction

In 1975 Radu Diaconescu proved that, in a topos, if every epimorphism has a section then every subobject has a complement. If the metaphysical implications of this result are not immediately apparent, the reader can be forgiven. Even mathematically sophisticated readers might not recognize the import of this result, which is why in the same year Goodman and Myhill felt the need to translate the proof into a language more common to mathematical logicians, publishing a short proof of the analogous result in set theory, which showed that in intuitionistic set theory a choice function of a certain type implies the law of excluded middle (*cf.* Diaconescu, 1975; Goodman and Myhill, 1975). The metaphysical implications of this result should be obvious due to the importance the law of excluded middle has played in debates over realism, specifically in the work of Michael Dummett.

It is now well known among specialists that the axiom of choice, added to intuitionistic set theories of various sorts, makes them classical. But even in a logical, rather than set theoretic, framework one can strengthen a logic by adding choice principles. Hilbert's ε and τ operators, for example, are well known examples of logical formulations of the axiom of choice. The τ operator was introduced by Hilbert as the "transfinite axiom" in "*Die logischen Grundlagen der Mathematik*" (Hilbert, 1923, p.156)[1]. Hilbert later replaced τ with the weaker ε axiom (*cf.* Hilbert, 1926, 1927)[2]. The ε and τ axioms may be written as follows:

$$\varphi(x) \to \varphi(\varepsilon_\varphi) \tag{1.1}$$

[1]First presented in a lecture course of the same name over the 1922-1923 winter term (see Grattan-Guinness, 2000, p.473).

[2]That is, ε is weaker that τ when they are added to "weak" systems like intuitionistic logic. They are inter-definable (in fact, dual) operators in classical logic: we can, for instance, define τ_φ to be $\varepsilon_{\neg\varphi}$.

$$(1.2) \qquad \varphi(\tau_\varphi) \rightarrow \varphi(x)$$

Under the standard interpretation of the existential and universal quantifiers (1.1) gives us $\exists x Ax \leftrightarrow A(\varepsilon_A)$ and (1.2) gives us $A(\tau_A) \leftrightarrow \forall x A(x)$. Hence it can be said that the epsilon operator can be thought of as asserting that there is an object ϵ_φ that is most likely to have the property φ, while in sentence (1.2) the tau operator asserts that there is an object τ_φ that is least likely to have the property φ. As Hilbert puts it:

> For us to illustrate its content, if we take the predicate A to mean "to be corrupt," then we would have to understand by τ_A a man of such inviolable sense of justice, that if he should prove to be corrupt, then all human beings are corrupt.[3]

The ϵ and τ term forming operators.[4] are conservative over classical logic; that is their addition to classical logic does not enable one to prove any proposition which one could not prove with classical logic, which doesn't include the operator in question. However, ϵ and τ can strengthen intuitionistic logic, without necessarily making it classical. In certain contexts choice operators strengthen intuitionistic logic by making DeMorgan's intuitionistically invalid law, $\neg(B \wedge C) \vdash \neg B \vee \neg C$, and the linearity axiom, $\vdash (C \rightarrow B) \vee (B \rightarrow C)$, derivable. In other contexts, notably when they include an extensionality

[3] "Um uns seinen Inhalt zu veranschaulichen, nehmen wir etwa für A das Prädikat "bestechlich sein"; dann hätten wir unter τ A einen bestimmten Mann von so unverbrüchlichem Gerechtigkeitssinn zu verstehen, dass, wenn er sich als bestechlich sollte. Tatsächlich alle Menschen überhaupt haupt bestechlich sind" (My translation. Hilbert, 1923, p.156).

[4] Strictly they are variable binding term forming operators which are a category of operators that include Hilbert's ε, τ and η and Russell's $\imath$. They are also referred to by some as subnectors; a term Haskell Curry coined to describe "a functor which converts [a sentence]... into a noun"(Curry, 1966, p.14) Curry's taxonomy of functors was developed from Tarski and Carnap (see Tarski 1935 fn.7 p. 274 or Tarski 1956 fn.2 p.161 and Carnap 1942). However, what Carnap called a functor, Curry names a "nominal functor" (Curry, 1966, p.13 fn. 19). Tarski employs the term functor in this sense due to Kotarbinski and the terms: "'sentence-forming functor' and 'name-forming functor' from Kazimier Ajdukiewicz" (see Ajdukiewicz, 1928). These terms belong to what Curry refered to as the grammatics of communicative languages a "study of the rules for determining the sentences of a language". In Curry's grammatics, the significant class of objects are not expressions, by which he means simply strings of symbols, but phrases of which:

> There are three main classes ... *viz.*, nouns, sentences, and functors. A noun names some object (real or imaginary); a sentence expresses a statement; and a functor is a means of combining phrases to form other phrases (Curry, 1963, p.32).

Nouns and sentences are called "closed phrases" and functors are phrases that combine other phrases (see Curry 1966, p.13, and Curry 1963, pp.32-34). Though Curry notes that his terminology for types of functor, *i.e.* 'junctor' and 'nector' are "suggested by the use of 'junction' and 'nexus' in Jespersen (Jespersen, 1924)" his usage is different. The following table depicts Curry's taxonomy of functors:

axiom, they make the logic classical. Because of the ontological nature of choice operators, *i.e.* they in some manner assert the existence of objects, they provide a particularly interesting example of how logics are strengthened. This is of interest because of the close relations between logical strength and metaphysics.

1.1 Logic and Metaphysics

The goal of this thesis is to extract the metaphysical lessons offered by these results. On their face they suggest that choice principles—which postulate the existence of particular sorts of objects—are intimately related to the validity of principles that are, from the classical point of view, logical. Thus these results seem likely to tell us something important about the relationship between existence and correct reasoning—between metaphysics and logic. We intend to make clear the lessons that results of this sort can teach us. Learning why the results are true will involve becoming clear about how the results are proven. Along the way we prove some new results about the implications of choice principles in logics without identity, which help fend off some misconceptions about what is presupposed when getting the "logic" out of the "metaphysics."

The metaphysical lessons this topic promises are best seen by considering the role of logic in metaphysics. According to Dummett's analysis, a logic is correct for a particular domain of discourse if it reflects the metaphysical status of the entities discussed in that discourse. And so, he argues, metaphysical debates about realism and anti-realism are really about the correct set of logical laws (Dummett, 1978b). Specifically Dummett asserts that one is realist about a domain if one accepts bivalence for that domain, which implies the law of excluded middle, and rejecting the law of excluded middle as a logical law is a prerequisite to any sort of anti-realism or even metaphysically neutral position about a domain.

The most well known example of Dummett's argument, that the law of excluded middle implies realism for a domain, is in reference to the debate between intuitionism and mathematical Platonism (*cf.* Dummett, 1975 and Dummett, 1977). He has also applied this

	Values		
Arguments	Phrase	Noun	Sentence
Phrase	Functor	Nominal Functor	Sentential Functor
All Nominal	Junctor	Adjunctor or Operator	Predicator
Some Sentential	Nector	Subnector	(Pronector)
Mixed	Mixed Nector	(Mixed Nector)	(Ad Nector)
All Sentential	(Pure Nector)	(Pure Subnector)	Connector

In above table of functors, the names in parentheses are not present in Curry's system, and he added them simply to provide completeness (see Curry, 1966, pp.14–15).

argument to anti-realism about the past (*cf.* DUMMETT, 1978b and DUMMETT, 1978a) and critiqued David Lewis's argument that all possible worlds are real (*cf.* LEWIS, 1986 and DUMMETT, 1993c). In all these cases Dummett argues that, to resolve metaphysical debates, we must choose a logic that does not appeal to principles which are not justifiable in that domain (DUMMETT, 1991).

In the *The Logical Basis of Metaphysics* Dummett asserts that most attempts to solve metaphysical debates over realism take a "top down approach" starting at the level of justifying a particular metaphysical position, or principle, and applying it then to the domain in question. Instead he suggests that we should approach such problems from the "bottom up" (DUMMETT, 1991, pp.12-13). By this he means we should look at the logic that underlies the language of the domain of the entities in question.

Realism about a domain involves the entities of that domain being "mind independent" in some suitable sense. That is, if realism is true, the claims we make about those entities are, Dummett suggests, made true or false independently of us. So, at least for well formulated claims in that domain, our claims should be true or false, whether or not we can come to know their truth values. This, broadly stated, is the way Dummett makes the connection between realism and bivalence. Since bivalence implies the correctness of classical logic, we get the connection between realism about a domain and the law of excluded middle holding for that domain. When we think about varieties of anti-realism in the history of philosophy—phenomenalism or nominalism, for instance—we see that the "objects" in question are somehow mind- or language-dependent, and so, Dummett suggests, we have no recourse to mind-independent reality to fix truth values of claims that are in principle unknowable by us. Hence we have no principled reason to accept bivalence in these domains, and so no justification for the adoption of excluded middle.

Thus he simplifies the question about the reality of entities to a more tractable question about the semantic value of statements referring to the entities in question; that is, whether these sentences are or can be true or false, independent of our knowledge, or ability to have knowledge, of them. From there, he makes the connection to debates about the correctness of logical principles in a domain.

In a like manner we can look at how the addition of something like a choice principle would modify that interpretation by strengthening the logic. Not only can we use the formal results about choice principles to investigate the general position that there is a connection between realism and logic, but we can also look at the arguments Dummett has made with reference to particular domains, *e.g.* time. These arguments can be used as examples of investigations of the relation between various domains and different logical systems. By looking at these arguments we gain some insight into the relation between logics that are created by the addition of choice axioms and the objects that choice principles seem to posit.

1.1.1 Problems with Dummett's views

Dummett's argument is if realism is correct then there is a mind independent reality that our language latches onto correctly, that is accurately represents, in certain domains — *i.e.* those for which realism holds. This reality, rather than our knowledge, fixes the truth values of claims in such domains, whether or not we can, even in principle, actually know the truth of a sentence. Since our language hooks up to the reality of those domains, all sentences about such domains are either true or false. This argument thus connects logical principles to realism through truth.

However there are problems with arguments of this sort. There is of course the question of whether or not our language ever totally and completely accurately depicts reality, that is, the problem of radical anti-realism. There is also the problem that one could accept classical logic and not bivalence. While realism perhaps implies classical logic, through the law of excluded middle, classical logic may not necessarily in the same manner imply realism. One could accept some sort of supervaluational semantics or boolean algebra valued semantics. In any boolean algebra, the value of sentences would not necessarily be true or false, but the law of excluded middle would still hold.

1.2 The Axiom of Choice and Hilbert's Programme

The axiom of choice is the claim that for a family of non-empty sets $\mathcal{L}$ there is a function that selects an element of each set which is a member of that family $\mathcal{L}$. Zermelo first formulated the axiom in 1904 as a choice function f on $\mathcal{L}$ such that for each non-empty set $X \in \mathcal{L}$, $f(X) \in X$ (Bell 2009, pp.1-2 *cf.* Zermelo 1904, p.140). Zermelo used the axiom to prove the well ordering theorem, that every set can be well-ordered (Zermelo, 1904, p.140). This answered part of the first question that Hilbert posed in his famous 1900 address where he posed 23 questions that he saw as shaping the future of mathematics.[5]

The existence of a choice function is easily established for *finite* families of sets. Moreover, the "countable" axiom of choice is usually regarded as uncontroversial. For infinities beyond that, however, it encountered criticism almost immediately, with Émile Borel asserting that "any argument where one supposes an arbitrary choice a non-denumerably infinite number of times is outside the domain of mathematics" (quoted in Bell 2009

[5]While the first problem "Cantor's Problem of the Cardinal Number of the Continuum" cannot be said to be solved, or even perhaps solvable, it is in Hilbert's discussion of it that the well-ordering theorem is mentioned. Hilbert asked, "whether the totality of all numbers may not be arranged in another manner so that every partial assemblage may have a first element" and suggests that the solution to this problem may be key to the provability of the entire problem (Hilbert, 1900b, p.446-47).

p.2).[6]

John Bell asserts that the axiom of choice is "the most fertile principle of set theory" and notes that over 200 principles have been proven classically equivalent to the axiom of choice (Bell, 2009, pp.2-4).[7] Foundationally, though, one of the most important developments is Diaconescu's proof showing that the axiom of choice enables one to derive the law of excluded middle in an intuitionistic context.

Errett Bishop argued that the axiom of choice is acceptable in a constructive system as it is implied by what we mean by existence (Bishop, 1967, p.9). Bishop, in fact, criticised the understanding of choice by classical mathematicians, while asserting that choice was constructive:

> When a classical mathematician claims he is a constructivist, he probably means he avoids the axiom of choice. This axiom is unique in its ability to trouble the conscience of the classical mathematician, but in fact it is not a real source of the unconstructivities of classical mathematics. A choice function exists in constructive mathematics, because a choice is implied by the very meaning of existence. Applications of the axiom of choice in classical mathematics either are irrelevant or are combined with a sweeping appeal to the principle of omniscience. The axiom of choice is used to extract elements from equivalence classes where they should never have been put in the first place. For instance, a real number should not be defined as an equivalence class of Cauchy sequences of rational numbers; there is no need to drag in the equivalence classes. The proof that the real numbers can be well ordered is an instance of a proof in which a sweeping use of the principle of omniscience is combined with an appeal to the axiom of choice. Such proofs offer little hope of constructivization. It is not likely that the theorem "the real numbers can be well ordered" will be given a constructive version consonant with the intuitive interpretation of the classical result (Bishop, 1967, pp.9-10).

While the axiom of choice in intuitionistic set theory implies the law of excluded middle, it does so only if, as is usual in most set theories, the sets or functions are extensional. Hence it has been noted that in Martin Löf's constructive type theory the axiom of choice is derivable, but not the law of excluded middle, because there is no extensionality principle. As well, John Bell has presented what he calls a 'weak set theory' in which the law of excluded middle does not follow from the axiom of choice (Bell, 2009, pp 120-131).

[6]Quite a few seemingly paradoxical results can be obtained by use of the axiom of choice, or its equivalents, which is why it is so often criticised, including the fact that, in topology, any solid sphere can be decomposed into a finite number of subsets and recomposed into two spheres of the same size.

[7]These include: Zermelo's well-ordering theorem, the trichotomy principle, König's theorem, Tarski's theorem, Tychonov's theorem, the model existence theorem for first order logic, the Hamel basis theorem, Tukey's lemma, Hausdorff maximal principle, the antichain principle, and Zorn's lemma to name a few.

1.3 Structure of Thesis

This dissertation will be divided into three parts. The first part is a historical contextualization of the inquiry. The metaphysical debates and the positions various philosophers, logicians and mathematicians engaged with were constrained by the historical circumstances. Describing the historical circumstances should help clarify how the various streams come together and why. This story has been told in bits and pieces by various authors but the big picture, as it relates specifically to the topics we want to address, has not been sorted out.

We will start by looking at Hilbert's attempts to fight a rearguard action against constructivists and find a finitist foundation for mathematics that would prevent us from being evicted "from the paradise Cantor has made for us". Then we will look at Dummett's appropriation of Brouwer's mathematical anti-realism, through which he suggests that the link between "mind- or language-dependence" of a domain of discourse and the correctness of intuitionistic logic for it should serve as a model for anti-realism more generally. This will set the stage for part two, where we will introduce certain mathematical results that have only begun to be properly investigated post-Dummett.

In part two we will briefly set out the main results that show the relationship between choice principles and logical principles, and prove a several new ones. We will also survey two semantics for intuitionistic predicate logics + ε.

In part three we will develop the argument that there are important philosophical lessons to be learned from the results in part two. For the main results, the lessons have not been properly drawn because the mathematical results have been presented by mathematicians who have been at times content to pronounce on what they take to be the metaphysical import, but who have not provided the necessary philosophical arguments. In this part of the thesis we will attempt to provide these arguments and show how these results provide two important philosophical insights. The first is a point about the intersection of logic and metaphysics. Dummett's argument about the interconnection of realism about a domain and acceptance of the law of excluded middle, and anti-realism about a domain and anti-realism paints a rather black and white picture.

What we show is that the ontological assumptions inherent in the ε-operator produce logics that describe middle grounds between classical and intuitionistic logics, and hence, between full blown realism and anti-realism about a domain. To do so, we need not tackle a large intractable philosophical problem like "what would the correct logic for ethical language be." We will find that many common properties, that seem at first quite simple and to some degree objective, may have structures that call for a logic weaker than classical logic, but more metaphysically rich than intuitionistic logic. For example, consider the metaphysical implications of a domain where Dummett's scheme $(A \to B) \vee (B \to A)$

holds, but the law of excluded middle doesn't. In such domains we can model examples of properties composed of 'objective' but non-comparable sub-properties. Take, as an example, the concept of intelligence, which is itself composed of various dimensions (or sub-concepts), each of which seems in itself objective but when combined you get something that is less objective—since it might not do to compare mathematical with musical or literary genius, though we might feel quite at ease measuring such aspects of intelligence separately.

The second contribution that our investigations of choice operators will provide is with regard to some long standing discussions of the inter-related issues of abstraction and ideal (or generic or arbitrary) objects. We will show, among other things, that much of the discussion is flawed because it does not recognize the degree to which the claims being made depend on the presumption that one is working with a very strong (*i.e.*, classical) logic.

Part I

Historical Context

Chapter 2

The Origins and Development of Hilbert's Programme

2.1 Introduction

The development of the logical systems we will consider in this thesis was motivated by the debates over the foundations of mathematics in the late nineteenth and early twentieth century. There were several catalysts that brought foundational issues to the foreground at the end of the nineteenth and beginning of the twentieth century. The development in the nineteenth century of non-Euclidean geometry which undermined Kant's position that geometry was the *a priori* intuition of space,[1] and then Dedekind's the construction of the

[1]Efforts were made to recover the Kantian view, for example Bertrand Russell's fellowship dissertation, *An Essay in the Foundations of Geometry* (Russell, 1897), made an attempt at resolving the Kantian position by generalizing on this point to allow for certain non-Euclidean geometries, allowing for geometries that preserved "constant measure of curvature". However, he ended up accidentally asserting the impossibility of Reinmann geometry of which he had not heard (Russell, 1959, p.31). Brouwer notes this in his early paper on the "Nature of Geometry" (Brouwer, 1909A) where he states that Russell's early view that only the projective axioms in physics are *a priori* would work, except that for the recent discovery that "this standpoint becomes untenable in light of modern mechanics, because space and time are no longer considered as independent, and therefore not unambiguously defined". However in his thesis (*cf.* Brouwer, 1907, pp.65-71), before becoming aware of Einstein's theory of general relativity, Brouwer discussed Russell's attempt at reconstructing a Kantian view, admitting that it was not contradictory, but in the end rejected it. Brouwer argued that Euclidean geometry was not *a priori*:

> The three-dimensional Euclidean geometry is a six-parameter group, in which the motions of empirical rigid bodies in our immediate neighbourhood can be represented with a high degree of approximation... but it can be very well imagined that with the same organization of human intellect another mathematical construction would have become as popular (*cf.* Brouwer, 1907, pp.69-71).

real numbers and the work of Cantor on transfinite numbers, which introduced the concept of a completed infinite, both of which were rejected by early constructivists like Leopold Kronecker.[2]

Another catalyst was the "antinomies of set theory", that is the set theoretic paradoxes (Hilbert, 1905) discovered by Russell and Zermelo. These were a profound motivating factor in the development of several foundational programmes including intuitionism and Hilbert's metamathematical theory, as Jean van Heijenoort summarizes:

> The paradoxes were extremely disconcerting to those then engaged in investigating the logical foundations of mathematics, and, together with other factors, they led to four major new developments, which occurred within a very short span of time:
>
> (1) Hilbert's metamathematics ;
>
> (2) Brouwer's intuitionism ;
>
> (3) Zermelo's axiomatic set theory ;
>
> (4) Russell's theory of types .
>
> Hilbert's metamathematics was, at that time, a rudimentary and vague program, which was to be developed only 20 years later. Brouwer's intuitionism was a profound new conception of mathematics that demanded much time to be developed and understood (van Heijenoort, 2012, p.328).

Hilbert knew of Russell's paradox quite early on, not only because Frege himself had written to him about it in 1903 describing Russell's letter that showed the paradox in basic law five of Frege's *Grundgesetze* (Frege, 1893, 1903), but also because his protégée Ernst Zermelo had discovered the same set theoretic paradox "three or four years" before Russell (Grattan-Guinness, 2000, p.216).[3] However, van Heijenoort is correct that both Hilbert's and Brouwer's programmes took 20 or so years to develop, so until then the foundational programme of importance was logicism and the development of type theory. But while Brouwer and Hilbert both gave lip service to the fact that the set theoretic paradoxes were motivators for a new development of mathematical foundations, neither approached them directly. Hilbert's axiomatics became model theory as we know it today, and while Brouwer did produce an intuitionistic set theory, it was not the basis of his foundational arguments, but a consequence.

[2]Who once famously asserted that,"God made integers; all else is the work of man." Original: "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk" (Von Weber, 1891, p.19).

[3]If it should seem an injustice that the paradox is known as Russell's, Grattan-Guinness notes we should not worry as Russell developed a version of the axiom of choice the summer before Zermelo (Grattan-Guinness, 2000, p.340).

Hilbert's work is often divided into several periods. Hermann Weyl's obituary, "David Hilbert and his Mathematical Work" divides it neatly into five periods:

i. Theory of invariants (1885-1893).

ii. Theory of algebraic number fields (1893-1898).

iii. Foundations:

   (a) Foundations of geometry (1898-1902),
   (b) Foundations of mathematics in general (1922-1930).

iv. Integral equations (1902-1912).

v. Physics (1910-1922) (Weyl, 1944, p.617)

However, several people have pointed out this division does not take into account much of the work Hilbert was doing in lecture courses, nor the overlap where Hilbert was working or publishing in multiple areas. Hence we can extend his foundational periods: while it is true that he published on foundations of geometry from 1898 to about 1902 and foundations of arithmetic from 1900 to 1905; he also gave lectures on foundational topics in 1908, 1910, 1913, 1914/15, 1917, and 1918. In addition, much of the work he did on axiomatizing physics is tightly connected with his programme for mathematics. In the 1920s onward he continues to lecture and publish on foundations of mathematics and was also adding to his *Foundations of Geometry* (see Moore 1997, p.68, Sieg 1999, p.2-3, 8, and Ewald 1996, p.1088 for details.)

In this chapter, first we will discuss Hilbert's reaction to early constructive views, which were restrictive with regard both to reasoning in mathematics, and the subject matter of mathematics. Then we will briefly discuss Hilbert's invariant theory papers and how the success of such non-constructive efforts may have affected his views toward the importance of non-constructive proofs, which will lead to a discussion of his early foundational work first on geometry then on analysis. At first Hilbert wanted to justify what he took to be standard mathematical practice by developing a consistent axiom system for arithmetic. Hilbert's second major foundational period produced what is often referred to as Hilbert's programme, this comes out of his reaction to what he saw as the work left to be done after logicism. Hilbert's programme was his attempt to provide a 'finitist' foundational system for mathematics. We will continue to see the development of Hibert's foundational work throughout the period from 1910-1918, including Hilbert's response to Russell and Frege's logicism and how this response informed the logical programme that led Hilbert, Ackermann, and Bernays to introduce first the τ and then the ε axioms in the early 1920s.

The goal of the this chapter then is to tell the story of the development of Hilbert's choice operators in the context of the development of his views on mathematics, logic and philosophy of mathematics. Hence the choice operators will be treated as an end point and we will look at the various influences that brought about the development of Hilbert's ε-calculus through that lens. The development of ε and τ were integral to Hilbert's strategy for developing his 'finitist' foundational system, especially as it pertains to his reaction to the various forms of constructivism from the early constructivism of Kronecker, which Hilbert rightly saw as a form of conservatism in mathematical methods, through to the more philosophically grounded positions held by Poincaré, the French semi-intuitionists, and Brouwer.

Various aspects of Hilbert's metamathematical development influenced or were prerequisites for development of choice operators. As noted above we start with the early desire to protect indirect proof from Kroneckarian type criticism. Hilbert's rejection of the unknowable in mathematics served as an early motivation for the development of Hilbert's programme. Hilbert's development of the axiomatic method applied successfully to geometry and then to arithmetic and his approach to developing manageable systems rather than universal logics enabled him to separate out foundations from ordinary mathematical practice. The development of the views about metamathematics as different from ordinary mathematics was to some extent worked out as a reaction to Poincaré's accusations of circularity, and his interest in Frege and Russell's logicist programmes, which was tempered by the failures of Frege and the limitations of *Principia Mathematica*. Dealing with the criticism of Brouwer and Weyl inspired Hilbert's own finitism with regard to metamathematical reasoning.

Hilbert's desire to protect what he referred to as "transfinite" reasoning from the criticisms of constructivists led him to the strategy of providing a foundation that would be 'finitist' at the level of methamathematics to provide for classical logic at the level of ordinary mathematics. This strategy ended of course with Gödel's incompleteness proof, but the logical machinery that Hilbert invented for this purpose remains.

Hilbert saw the questions of foundations of mathematics not as obstacles to overcome, but as questions that will have a final answer:

> I should like to rid the world of the question of the foundations of mathematics once and for all by making every mathematical statement into a formula that can be concretely exhibited and rigorously derived, and thereby bring mathematical concept formations and inferences into such a form that they are irrefutable and yet furnish a model [*Bild*] of the entire science (Hilbert, 1931, p.1152).

The development of Hilbert's τ and ε-operators must be seen in this light, as part of the machinery to provide methods for the foundation of basic number theory, despite the fact that these operators smuggle in less than constructive assumptions.

2.2 Early Constructivism

Before presenting the development of Hilbert's foundational programme, we should briefly give a short account of Kronecker's position, as it was the dominant view to which Hilbert developed in opposition to, especially in the first phase of his foundational work.

For Kronecker analysis and specifically the then new method of treating the notion of infinity as a number, as Cantor did, was unacceptable. Kronecker asserted that eventually it would be shown that such methods were inexact and lacking rigour.[4]

Unfortunately a detailed account of Kronecker's position in his own words is not available, as William Ewald points out: "Despite his deeply held philosophical convictions, Kronecker's writings on the philosophy of mathematics are scanty and contain little more than a sketch of his position" (EWALD, 1996, p.942). Hilbert surveyed Kronecker's philosophy of mathematics in his 1920 lectures at Göttingen, listing the following aspects of Kronecker's views:

> ...he rejected set theory as a mere game of fantasy containing nothing but illegitimate combinations that are no longer mathematical concepts. In number theory all truths are indubitable, the proofs incontestable and immediately comprehensible to common sense. This rests on their enduring checkability...
>
> On the basis of his way of looking at things, Kronecker forbids already the simplest irrational number $\sqrt{2}$; he introduces the concept of the modulus $x^2 - 2$ in place of this 'inadmissible' concept...

[4]Weierstrass recounts Kronecker's assault on mathematical analysis in the wake of Cantor's work, quoting him as saying:

> If I have time and strength, I myself will show the mathematical world that not only geometry, but also arithmetic can lead the way for analysis and are certainly more rigorous. If I can not do it, those who come after me will ... and they will recognize the inexactitude of all of these findings which you term so-called analysis.

Original:

> Wenn mir noch Jahre und Kräfte genug bleiben, werde ich selber der mathematischen Welt zeigen, dass nicht nur die Geometrie, sondern auch die Arithmetik der Analysis die Wege weisen kann, und sicher die strengeren. Kann ich es nicht mehr thun, so werden's die thun, die nach mir kommen... und sie werden auch die Unrichtigkeit aller jener Schlüsse erkennen, mit denen jetzt die sogenannte Analysis arbeitet. (KRONECKER quoted by WEIERSTRASS in MITTAG-LEFFLER, 1900, p.151. My translation.)

> Kronecker told me personally that the statement that there are infinitely many prime numbers makes no sense until one has shown that after every prime number there is another prime number within a determinable numerical interval...
>
> And Kronecker restricts logic as well. Just as he forbids arbitrary operation with the concepts 'reducible', 'irreducible', etc., so he stands towards the purely logical propositions like the *tertium non datur*, whose applicability he admits only under the condition that there is the possibility of deciding the existential question by a finite procedure (HILBERT, 1920, p.944).

Besides accounts by other people, often those opposed to his position *e.g.* WEIERSTRASS (1874-1888), VON WEBER (1891) and HILBERT (1920), we have a statement of Kronecker's philosophy of mathematics only in his paper "On the Concept of Number" (KRONECKER, 1887) and the introduction to his *Lectures on the theory of numbers* (KRONECKER, 1901) which covers much the same material (EWALD, 1996, p.947 fn.a).

In KRONECKER (1887) he writes that finite arithmetical methods were the only ones that we were able to treat *a priori*, and hence these were the only methods that were on a firm basis:

> The difference in principle between geometry and mechanics on the one hand and the remaining mathematical disciplines (here gathered together under the term 'arithmetic') on the other is, according to Gauss, that the object of the latter, number, is merely our mind's product, while space as well as time also have outside of our mind a reality, whose laws we cannot completely prescribe *a priori* (KRONECKER, 1887, Intro.)

Kronecker presented a view that 'arithmetic' grounds all of mathematics, by arthmetic he meant, "all mathematical disciplines with the exception of geometry and mechanics—especially, therefore, algebra and analysis" (KRONECKER, 1887, Intro.). But Kronecker was sure that soon:

> we shall one day succeed in 'arithmetizing' the entire content of all these mathematical disciplines—that is, in grounding them solely on the number-concept taken in its narrowest sense, and thus in casting off the modifications and extensions of this concept... (KRONECKER, 1887, Intro.)

Kronecker proceeded to define the integers and from there the laws of addition and multiplication of numbers (KRONECKER, 1887, §1-4), then he asserted that the "introduction *in principle* of 'indeterminates'" allows us to disregard "all the concepts that, properly speaking, are foreign to arithmetic — for instance, that of irrational algebraic numbers," and even, should we wish, negative numbers (KRONECKER, 1887, §5).

Kronecker was thus very much opposed to any sort of mathematical practice that was not specifically reducible to the natural numbers. In contrast to this position Hilbert's first large success in mathematics, his invariant theory papers, was based on an infinitely ranging non-constructive existence proof, which was exactly the sort of infinitiary reasoning to which Kronecker objected. Hilbert's programme, the development of his foundational view, is often seen as a rebellion first against Kronecker's constructivism and then only later Brouwer's (*cf.* GRAY, 2000, 1998, 1999; POSY, 1974; REID, 1986). Gray asserts that his opposition to Kronecker "dates from Hilbert's activities in foundations of mathematics, has been read back into the earlier period" (GRAY, 1998, p.24). However the story is more complicated, Hilbert was aware of Kronecker's objection to his work on invariant analysis and clearly would have known that his claim, in "On the Concept of Number" (HILBERT, 1900a, §16), that his method will provide a "proof that the system of real numbers is a consistent (complete) set", would be rejected by Kronecker for whom even negative integer's were suspect (KRONECKER, 1887, §5). Hilbert certainly criticised Kronecker as early as 1904 (see HILBERT 1905, p.130) and Hilbert generally does not mention Brouwer's foundational position until (HILBERT, 1923), after the point that his former Ph.D. student Weyl converted to intuitionsim. Up until this period Hilbert generally uses Kronecker and Poincaré as his foil when discussing constructive mathematics (see HILBERT 1920, pp.944–945). However we should note that Hilbert's desire to provide a finitist foundation for mathematics and his attempts to provide axiomatizations of arithmetic show that he took seriously the challenges and criticisms that views like Kronecker and other early constructivists like Poincaré made. In fact Hilbert's criticism of Kronecker's "dogmatist" views about integers was actually made from a more conservative position regarding their existence and a was part of his attempt to provide a logico-mathematical foundation for the concept of number, rather than taking even the concept of integer as a given (HILBERT, 1905, p.130).

2.3 The Road to Hilbert's Programme

Hilbert's programme is generally said to have developed in the period from 1917 to 1930 when the terms "'formalism' and 'metamathematics' became attached to his philosophy and techniques." However Hilbert's first period of interest in foundations of mathematics which ran until 1905 could be characterised as "axiomatics with proof and model theory" (GRATTAN-GUINNESS, 2000, p.208). [5] Before his production of this work on the founda-

[5]Ewald like Weyl divides Hilberts career into several phases. Ewald describes this period as lasting from 1898 to 1903, being when: "Hilbert's career was devoted to the foundations of geometry and to the

tions of geometry, Hilbert produced the work which brought him notoriety and put him into conflict with those early constructivists whose views he would try to answer in his foundational programme.

2.3.1 The Invariant Theory Papers

From 1885 to 1893 Hilbert published a series of papers that solved several long standing problems in invariant analysis (Hilbert, 1885, 1887, 1890, 1893). Unlike Gordan's earlier proof of a basis for binary forms which was computational, that is constructive, Hilbert provided an existence proof, which relied on the use of the law of excluded middle extended to an infinite case.[6] The invariant theory papers, and specifically the proof of Hilbert's basis theorem, were greeted with mixed responses. Since Gordan's proof there had been little progress on what had become known as Gordan's problem,[7] as Constance Reid writes "in 20 years of effort by English, German, French and Italian mathematicians, no one had been able to extend Gordan's proof beyond binary forms" (Reid, 1986, p.30). Hilbert's basis theorem[8] answered that question, but did not provide a construction of each of the specific systems of invariants. Hilbert provided only an existence proof of the basis theorem, a method well used in geometry, but not so accepted in analysis at the time (pp.36-37 Reid, 1986; Gray, 2000, p.31). As Gray puts it, Hilbert's proof was "decidedly non-constructive" and "clear from everything Hilbert wrote that he thought it at most a small step from geometry to algebra and back" (Gray, 1997, p.8).[9]

---

axiomatic method, with some excursions into the foundations of arithmetic" (Ewald, 1996, p.1088). As we have noted 1905 was not the end of Hilbert's interest in foundations and considering his lectures on foundational topics in 1908, 1910, 1913, 1914/15, 1917, and 1918, dividing his interests into set periods must always be understood with a grain of salt. That his views developed eventually into a more defined programme though cannot be denied.

[6]There have more recently been several constructive proofs of the basis theory (Gray, 1999, p.9). Gray notes the result is recent citing (Sturmfels, 1993, p.11). The first proof I can find of it is by student of Errent Bishop (Tennenbaum, 1973).

[7]This problem could be stated as follows: "was there a basis, a finite system of invariants in terms of which all other invariants, although infinite in number, could be expressed rationally and integrally?" (Reid, 1986, p.30).

[8]that states that "If R is a Noetherian ring, then so is any polynomial ring in a finite number of indeterminates over R" (*cf.* Zariski *et al.*, 1958, pp.200-203)

[9]Gray gives a short description of Hilbert's invariant theory papers:

> The first of these four papers is a rich one, broaching the theory of syzygies, and asking for the generalisation of Noether's theorem to arbitrary dimensions... The second paper shows how to use the basis theorem to illuminate the ideas of dimension, genus, order, and rank of an algebraic variety, and so makes explicit contact with Kronecker's work. Setting the third paper on *Gebilde* [varieties] aside, we come to the famous paper in which the basis theorem is proved. ... Only the final section of the paper is specifically addressed to the theory of algebraic invariants (Gray, 1997, p.8) .

The result was therefore not immediately welcomed by all; Gordan objected to Hilbert's non-computational proof method remarking: "Das ist nicht Mathematik. Das ist Theologie" ("This is not Mathematics. This is Theology") (NOETHER, 1914, p.18),[10] in a letter communicating his review of Hilbert's papers to Felix Klein the editor of *Mathematische Annalen*. Others realized the value of Hilbert's result. Hermann Minkowski for example, in a letter to Hilbert, wrote:

> For a long while it has been clear to me that it could be only a question of time until the old invariant question was settled by you — only the dot was lacking on the 'i'; but that it all turned out to be so surprisingly simple has made me very happy, and I congratulate you (quoted in REID, 1986, p.37).

But it was Felix Klein, who would later bring Hilbert to Göttingen from Königsberg,[11] who was Hilbert's biggest advocate. He, as the editor of the *Annalen Mathematische*, disregarded Gordan's negative review and wrote to Hilbert that: "Without doubt this is the most important work on general algebra that the *Annalen* has ever published" (quoted in ROWE, 1989, p.195). And in this manner the use of existence proofs ranging over an infinite class was tied to Hilbert's first great mathematical achievement. In their defence Hilbert was later to write:

> The value of pure existence proofs consists precisely in that the individual Construction is eliminated by them and that many different constructions are subsumed under one fundamental idea, so that only what is essential to the proof stands out clearly; brevity and economy of thought are the *raison d'étre* of existence proofs. In fact, pure existence theorems have been the most important landmarks in the historical development of our science. But such considerations do not trouble the devout intuitionist (HILBERT, 1927, p.475).

Defence of existence proofs and the mathematical achievements that Hilbert saw flowing from them was one of the early motivations for the development of the Hilbertian foundational programme. Motivated to justify and protect his earlier victories, Hilbert saw that pure existence proofs could be justified in analysis if it could be given a solid foundation that answered the criticisms of Kronecker and other early constructivists. His first attempts at this would lean heavily on his work on axiomatization of Geometry, so that is where we turn to next.

---

[10] Note that as Hilbert's method became much more accepted, Gordan relented and was quoted as saying, "I have convinced myself that even theology has its merits." (REID, 1986, p.37).

[11] On Klein's building of the 'Göttingen Empire' in mathematics, an attempt to move the centre of German mathematics from Berlin to Göttingen, see GRAY (2000, pp.23-35).

2.3.2 From the Foundations of Geometry to the Foundations of Mathematics

At the beginning of his career Hilbert, though an algebraist by training, focused on geometry, which he was concerned with throughout the 1890s, giving a course while still a *Privatedozent* at Königsberg on projective geometry in 1891 and one in 1894 on foundational questions.[12] In 1894, he published a short paper in which he derived a novel proof that "the straight line is the shortest connection between two points" (HILBERT, 1894).[13] In 1895, after moving to Göttingen, Hilbert gave special short course over Easter in 1898 "On the concept of the infinite", dealing with geometrical spaces and continuity (GRATTAN-GUINNESS, 2000, p.207). Since as "part of the growing interest in axiomatics, it had become clear that Euclid had not specified all the assumptions that he needed... Hilbert decided to fill all the remaining gaps" (GRATTAN-GUINNESS, 2000, p.207). The result of this effort was published as *The Foundations of Geometry* (HILBERT, 1899).[14] The study of the foundations of geometry stayed an interest of Hilbert, who continued to add to the book over the years. The original book (HILBERT, 1899) grew from 92 pages to over 320 pages by the seventh edition (HILBERT, 1930). The first edition (HILBERT, 1899) consisted of seven chapters, the first listing five groups of axioms, and the second focusing on independence and consistency of the axioms.[15] In writing the *The Foundations of Geometry* Hilbert developed our modern conception of an axiomatized system, setting the stage for his development of metamathematics and what we now know as model theory. The development of separate logical systems for different purposes, rather than a universal logic, with all the problems of such systems, can be in some ways be traced to his attempt to do separately for arithmetic what he had done for geometry. Much of Hilbert's foundational approach can be related to his understanding of what mathematicians use when they reason about certain parts of mathematics, in geometry we do not need all the axioms we use in algebra, and *vice-versa*.

[12]These included: the independence of axioms, and axioms of connection and continuity *i.e.*'Archimedes's axiom' (GRATTAN-GUINNESS, 2000, p.207).

[13]He derives this as an exercise in the foundations of geometry on the "assumption that points, lines, and planes are taken as elements" and the following axioms: the axioms of the elements' mutual relations (every line has two points), the axioms of segments and sequences of points on a line (that between two points of a line there is a third, and that points can be ordered); and the axiom of continuity (that an infinite sequence of points on a line can be extended) (HILBERT and BERNAYS, 1999, pp.108-109).

[14]The short book was one of a two volume special edition honouring Carl Friedrich Gauss and the physicist Wilhelm Weber, conceived of by Klein.

[15]The other chapters were: "The Theory Of Proportion", "The Theory Of Plane Areas", "Desargues's Theorem", "Pascal's Theorem" and "Geometrical Constructions Based Upon The Axioms I-V" (HILBERT, 1899, 1902).

2.3.3 Hilbert's Problems

While Hilbert's 1904 address to the international mathematics conference in Heidelberg, "Über die Grundlagen der Logik und der Arithmetik" (Hilbert, 1905) has been said to mark the beginning of Hilbert's foundational programme,[16] Hilbert had already mentioned the need for such a proof in the presentation of the second of the problems listed in his famous 1900 address (Hilbert, 1900b) to the Third International Congress of Mathematics.[17] In this address, Hilbert outlined a series of 23 'problems' for mathematics in the next century.[18] In addition to these problems he presented a call to arms asserting that:

> this conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is a problem. Seek its solution. You can find it by pure reason, for in mathematics there is not *ignorabimus* (Hilbert, 1900b).

This optimistic call to arms was a response to a view put forth in Emil du Bois-Reymond's 1872 address that there were limits to scientific knowledge (du Bois-Reymond, 1872, 1880) often summarized in the maxim "*ignoramus et ignorabimus*"[19] (see Bartocci *et al.* 2011, p.3, and, Gray 2000, p.57).

Hilbert's epistemic optimism held that for mathematics and physics that there are no unknowable truths, or perhaps more generously, unsolvable problems. This is one philosophical position that needs to be understood when comparing the motivations for his foundational to those of constructivists. In addition the series of problems Hilbert presented in 1900 included at least two that touch issues we will discuss in this thesis: the first problem, which lead his student Zermelo to first formulate the axiom of choice (Zermelo, 1908), and the second problem, which called for a providing of a foundation for mathematics, specifically one which would be resistant to the criticism of constructivists.

Hilbert's opposition to to constructivist viewpoints, both the conservatism of Kronecker and Brouwer's later ideas, has often been connected with this epistemic optimism, that is, the view that all mathematical problems have a solution. [20] Yet he was not deaf to their

[16] It was, Richard Zack writes, "the first time, he sketched his plan to provide a rigorous foundation for mathematics via syntactic consistency proofs" (Zach, 2009).

[17] Regarded even to this day as, "perhaps the most influential speech ever given to mathematicians" (Joyce 1997 quoted in Hardy *et al.* 2009 p.142, interestingly this sentence shows up in several other books without attribution, *e.g.* Saxe 2002, p.26, Chimakonam 2012 p.102).

[18] The oral address presented 10 problems, the written version (translated to French and available to the congress attendees) provided the full 23 problems.

[19] "We do not know and will not know." Note that du Bois-Reymond did not use this exact phrasing, rather he discussed the *ignoramus* and the *ignorabimus* separately.

[20] *i.e.* his rejection of the *ignorabimus* in mathematics.

arguments, which lead him to attempt to develop a consistency proof for mathematics, specifically for arithmetic, using finitary methods. Another important and central aspect of Hilbert's various research programmes was his use of axiomatic systems, first in his work in geometry and then in arithmetic. Although he did not publish on foundational questions between 1905 and 1917, he returned to these questions in his 1921 paper (HILBERT, 1923). His return to such issues has been suggested to have been in response to what he saw as Weyl's defection to the intuitionism (WEYL, 1921).[21]

Throughout his early writings on foundational issues Hilbert also rejected Frege's logicist view that arithmetic could be reduced to logic, arguing rather that concepts belonging both were presupposed in the definition of the other (HILBERT, 1905, p.131), and hence that they needed to be defined together. Hilbert thus rejected the idea that propositions were primary in a logico-arithmetic system and began with developing his system from 'entities' (*cf.* HILBERT, 1905, p.132ff.). It is not surprising then that he introduced the τ and ϵ axioms into his logical system when he returned to foundational investigations (HILBERT, 1923, 1926) linking as they do elements of the domains with a propositional logic.

2.4 The Origin of the *Ignorabimus*

The famous expression of Hilberts that "there is no *ignorabimus* in mathematics" is often mentioned (REID 1986, p.72, GRAY 2000, p.58, MCCARTY 2004, p.530, CORRY 2004, p.102, BOTTAZZINI 2011, p.2) as response to Emil du Bois-Reymond.[22] Emil du Bois-Reymond was a well respected German physiologist, whose address, "*Über die Grenzen des Naturerkennens*" ("On the limits of our understanding of nature")[23] which he gave to

[21] In the words of Per Martin Löf:

> A new phase in this controversy began in 1921 with the publication of Weyls paper *Über die neue Grundlagenkrise der Mathematik* (WEYL 1921), and that is what really fired it and made it so bitter. (And it seems clear that it had to do with the fact that Weyl was after all Hilberts doctoral student: he took his doctors degree with Hilbert, and I do not know, but presumably Hilbert thought of him as the best of his doctoral students over the years) (MARTIN-LÖF, 2008, p.246).

[22] Brother of the mathematician Paul du Bois-Reymond.

[23] Which ended with the following passage:

> With regard to the enigma of the physical world the investigator of Nature has long been wont to utter his "*Ignoramus*" with manly resignation. As he looks back on the victorious career over which he has passed, he is upheld by the quiet consciousness that wherein he now is ignorant, he may at least under certain conditions be enlightened, and that he yet will know. But as regards the enigma what matter and force are, and how they are to be conceived, he

the of 1872 to the Society of German Scientists and Physicians, in Königsberg (du Bois-Reymond, 1872) and elaborated on in his 1880 speech "*Die sieben Welträthsel*" (Seven World Problems) before the Berlin Academy of Sciences (du Bois-Reymond, 1880), began a wide ranging debate in the German speaking world on the nature and possibility of scientific knowledge. The public debate that followed was referred to as the "*Ignorabimusstreit*" (McCarty, 2004, p.523).[24]

The "no *ignorabimus*" assertion is the key to much of the motivation of Hilbert's programme, also know as his "solvability thesis"; it also sets the background to Brouwer's reaction to Hilbert's programme in his thesis, and early expressions of his rejection of the excluded middle (Brouwer, 1907, 1908C). Hence it will be enlightening to look at the origins of the debate.

Emil du Bois-Reymond stated in his 1880 speech that he was surprised at the reaction to his earlier address and its claims of "the impossibility, on one hand, of comprehending the existence of matter and force, and, on the other hand, of explaining consciousness, even in its lowest degree, on a mechanical theory" which to him seemed "a truism" (du Bois-Reymond, 1880, p.433). The original paper was well discussed and opinions were so divided that du Bois-Reymond asserts that by 1880 the term "*ignorabimus*" had become a "philosophical shibboleth".

Du Bois-Reymond's point was that certain foundational questions in the natural sciences were unanswerable. Though the views of du Bois-Reymond were widely thought to be Kantian, he himself rejected this classification. In fact, throughout both texts du Bois-Reymond leans more heavily on Leibniz than any other philosopher, agreeing about consciousness with Leibniz that even if "he could create an homunculus atom by atom ... he might make the creature think, but not comprehend how it thought" (du Bois-Reymond, 1880, p.435). His argument about the origin of matter and force is based on what he considered the basic properties of matter, its divisibility. He describes atoms as "infinitesimals" which are "regardless of its names ideally divisible" and to which "properties or a state of motion are attribute" (du Bois-Reymond, 1872, p.21).

Likely more important than the details of the Emil's du Bois Reymond's argument, for our interests in Hilbert's reaction, is the fact that his brother, "Paul du Bois-Reymond's 1882 monograph *General Function Theory*... [was] devoted to transplanting a similar skepticism into the realm of pure mathematics" (McCarty, 2004, p.524).

Much of Paul du Bois-Reymond's *General Function Theory*, is written as a form of a

must resign himself once for all to the far more difficult confession "*Ignorabimus*!" (du Bois-Reymond, 1872, p.32)

[24]Denis Charles McCarty notes that Emil's address "...unleashed a whirlwind of argument and counter-argument in the press and learned journals over *Ignorabimus* that continued well into the 20th Century " (McCarty, 2004, p.523).

dialogue between an idealist and an empiricist who answer differently to various questions about infinities and infinitesimals. For example the empiricist accepts that the unit segment can be split into an unlimited points while the idealist expands on this asserting that the number of these points is infinitely large. In this way the idealist "believes in the reality of extensions of concepts that go beyond the imagination but are necessitated by our thought processes", while, on the other hand the "empiricist always remains within the limits of the natural domain of imagination. He concedes and acknowledges the arbitrarily exact in geometry but calls the ideally exact an axiom" (Stolz, 1882, p.4).

Paul du Bois-Reymond distinguished between potentially and actually infinite sets, noting that potential, but not actual, infinities "call into question" the law of excluded middle (McCarty, 2004, p.525). He presents this account, possibly the first example of a description of a lawless sequence, in the following manner:

> One can also think of the following means of generation for an infinite and lawless number: every digit is determined by a throw of the die. Since the assumption can surely be made that throws of the die occur throughout eternity, a conception of lawless number is thereby produced. Indeed the contemplation of nature provides us with even better examples (du Bois-Reymond, 1882b, p.91).[25]

Paul du Bois Reymond argues that the lack of laws that encode sequences found in nature give him reason to agree with his brother that we can never possess complete knowledge of physical systems. Indeed the real infinitesimals that his idealist wishes to claim exist are not definable if we accept du Bois-Reymond's empiricist's view of sequences.[26] In addition unlike Hilbert, and later Brouwer, who despite their differences, both saw mathematics as an autonomous subject, du Bois-Reymond saw his empiricism as needing a foundation in the study of "the simplest constituents of our thinking, the representations" *i.e.* the physiological psychology studied by his brother Emil (McCarty, 2004, pp.529-530).

Hence what became known in German academic circles as the "*Ignorabimusstreit*" would have been clearly in Hilbert's mind in 1900 when giving his address to Paris. But not only would he be reacting in general to the concept that there were unsolvable problems

[25] Original:

> Man könnte auch an folgende Entstehungsweise einer endlosen und gesetzlosen Zahl denken: Jede Stelle wird einfach ausgewürfelt. Da doch die Annahme gemacht werden darf, dass dies Würfeln von Ewigkeit her oder in alle Ewigkeit stattfindet, so wäre hiermit eine gesetzlose Zahl in der Idee hergestellt. Indessen die Naturbetrachtung liefert uns bessere Beispiele

My translation, extended and adapted from version quoted in McCarty (2004, p.525).

[26] McCarty points out there are many "structural similarities between this argument of du Bois-Reymond and Brouwer's weak counterexamples" (McCarty, 2004, p.525), which we will discuss in the next chapter.

in the sciences, he would be defending mathematics from this claim, and defending its autonomy.

The assertion of an *ignorabimus* fits well within the constructivism of Kronecker and other 19th century mathematicians. The preservation of existence proofs were part of Hilbert's defence of mathematics; but the methods, of infinitely ranging existence proofs, which Hilbert had adapted from his work on geometry could not answer the demands of constructivism. Hence Hilbert's attempts at a what he called a "finitary" foundational project. Hilbert was attempting to preserve mathematical practice from ideological restrictions. And practice was key to why Hilbert introduced his τ and ε-operators, which formalized the manner in which mathematicians proved facts about arbitrary objects of a type.

2.5 Axiomatizing Arithmetic

What we know as Hilbert's programme was his attempt to provide formalization, specifically a axiomatization of all of mathematics and a proof of the the consistency of mathematics. In response to constructive criticism of Hilbert's proof methods in his invariant theory papers, and to the *Ignorabimusstreit* Hilbert wanted this consistency proof to be created using only "finitary" methods. Though Hilbert's programme took this form only in the 1920s, the origins of it can be seen as nascent even in the second problem in his famous 1900 Paris address. We will trace the development of his programme through his early attempts to axiomatize arithmetic and his development of his logical methods.

As mentioned above it was the second problem of Hilbert's famous 1900 Paris address "The Compatibility of the Arithmetical Axioms" (Hilbert, 1900b, pp.447ff.) that actually transformed into what became the focus of what is called Hilbert's programme. Axiom systems had existed before Hilbert, of course, however what was new with Hilbert in the use of the axiomatic method was its model theoretic nature. As Michael Hallett notes, the difference between Hilbert's axiom systems and those that preceded him is that Hilbert separates a "certain body of facts" that are given a special status in relation to the system as a whole (Hallett, 1996, pp.136–137).

> The service of axiomatics is to have stressed a separation into the things of thought [*die gedanklichen Dinge*] of the [axiomatic] framework and the real things of the actual world, and then to have carried this out (Hilbert 1922-1923 lecture quoted in Hallett, 1996, p.137).

In his 1922 paper "Die Bedeutung Hilberts für die Philosophie der Mathematische" ("On the Meaning of Hilbert's Philosophy of Mathematics"), Bernays discusses the difference between Hilbert's and pre-Hilbert axiom systems. Before Hilbert, he writes that an axiom system was created by starting "with a few basic principles, of whose truth one is convinced, [one] places these at the beginning as axioms" developing theorems from these using logical deductive procedures. In contrast Hilbert's axioms:

> ...are not judgements of which it can be said that they are true or false. Only in connection with the axiom system as a whole do they have a sense. And even the entire axiom system does not constitute the expression of a truth. Rather the logical structure ... is in Hilbert's sense purely hypothetical... The axiom system itself does not express a state of affairs but rather represents a possible form of a system of connections, a system which is to be investigated according to its *internal properties* (Bernays, 1922b, pp.95-96)

Hilbert's axiomatic method focused on establishing the consistency of the axioms. All the atomic objects are thus introduced in axioms rather than being added in an *ad hoc* manner. Hence Hilbert's axiomatising of numbers (Hilbert, 1900a, 1905) begins with the primitives (*e.g* 1, =) but does not try to reduce them further, say to equivalence sets. Rather their nature is expressed in the axioms, *i.e.* what can be deduced from them. As Michael Hallett has pointed out, Hilbert was not interested in a universal logic, a *logica magnus* in van Heijenoort's terminology,[27] like Russell and Frege, so "there is no necessity to a say anything about the primitives *prior* to the development of the theory. Thus, in particular, there is not necessity (as regards the primitives) for a strong ambient logic" (Hallett, 1996, p.135–141).

2.5.1 Hilbert's "On the Concept of Number"

Hilbert's Paris address is more famous but it was in an earlier paper published the same year, "Über den Zahlbegriff" ("On the Concept of Number") (Hilbert, 1900a), which is

[27] van Heijenoort described the difference between the two types of logical programmes, *logica magnus* and *logica utens*:

> . . . over what domain are the quantifiers supposed to range?" At this point the opposition between absolutism and relativism in logic strikes us with full force. For an absolutist, there is just one domain, a fixed and all-embracing universe (either on one level or hierarchized in several levels) which comprehends everything about which there can be any discourse. Such was the conception of Frege, such was also the conception of Russell, though for him this universe was stratified according to the theory of types. Under the name of *logica magna*, such a universal system has been the constant dream among logicians. Logicism is a modern form of logica magna. The well known diculties with logicism have led contemporary logicians, for the most part, away from that dream. Rather than being a *logica magna*, present-day logic is a *logica utens*; systems are introduced, here and there, according to needs. Different domains are successively considered for interpretations. In that sense, relativism has at present the upper hand (van Heijenoort, 1985, pp. 79–80)

where Hilbert first tackled the axiomatizing of arithmetic and which can be seen as his first move in what would become his wider foundational programme. This paper can be seen as a rhetorical response to Kronecker's paper of thirteen years earlier of the same name (*cf.* KRONECKER, 1887). In his 1887 paper, Kronecker introduced the integers in a manner Hilbert describes as "*genetic*". Kronecker further argued that the use of algebraic numbers is unnecessary in that one can reduce all higher algebraic terms to natural numbers.

Hilbert begins by describing what he terms the "genetic method". The genetic method of constructing number starts with imagining the: "further positive integers 2, 3, 4 ... as arising through the process of counting" after which "one develops their laws of calculation" including universally applicable "subtraction" through which "one attains the negative numbers."[28] Next, fractions are defined as a pair of numbers such "that every linear function possesses a zero," then finally the real numbers are defined "as a cut or a fundamental sequence" which means that "every entire rational indefinite (and indeed every continuous indefinite) function possesses a zero" (HILBERT, 1900a, p.109–211).

The genetic method, Hilbert states, can be contrasted to the axiomatic method used in geometry which postulates elements and then "brings these elements into relationship" by means of the axioms "of linking, of ordering, of congruence and of continuity." He writes that, these axioms must be shown to be consistent and complete (HILBERT, 1900a, §3) and then states the following opinion:

> My opinion is this: *Despite the high pedagogic and heuristic value of the genetic method, for the final presentation and the complete logical grounding* [Sicherung] *of our knowledge the axiomatic method deserves the first rank.*

Hilbert then describes the form that the axiomatic method should take with reference to "the theory of the concept number":

> We think a system of things [denken ein System von Dingen]; we call these things numbers and designate [bezeichnen] them by $a, b, c, ...$ We think these numbers in certain reciprocal relationships [Beziehungen] whose exact and complete description occurs through the following axioms (HILBERT, 1900a, §7).

Hilbert presented his axiom system for arithmetic, which describe the real numbers axiomatically as "an ordered Archimedean field that cannot be embedded in any larger such field" (EWALD, 1996, p.1090), as follows:

- axioms I 1–6, entitled "Axioms of connection," defined addition and multiplication;

[28]A step we can contrast to Kronecker's paper of the same name where he suggest one can dispense with negative integers (KRONECKER, 1887, §5)

- axioms II 1–6, entitled "Axioms of calculation," covered the use of the equality relation;
- axioms III 1–4, entitled "Axioms of ordering," provided ordering by inequalities; and finally
- axioms IV 1–2 entitled "Axioms of continuity" were the "Archimedian axiom" which stated:

  > If $a > 0$ and $b > 0$ are two arbitrary numbers, then it is always possible to add a to itself so often that the resulting sum has the property that $a + a + ... + a > b$.

  i.e. the real numbers formed an Archimedian field, and the "Axiom of completeness": which stated:

  > It is not possible to add to the system of numbers another system of things so that the axioms I, II, III, and IV-1 are also all satisfied in the combined system; in short, the numbers form a system of things which is incapable of being extended while continuing to satisfy all the axioms (HILBERT, 1900a).

The completeness axiom stood out in that it was not directly about the real numbers. Ewald notes that the axiom "was criticized at the time both for its logical complexity... and for not obviously being the statement of a continuity condition for the real line" (EWALD, 1996, fn. b, pp.1090-1091).[29] After presenting these axioms Hilbert then confidently remarked that: "To prove the consistency of the above axioms, one needs only a suitable modification of familiar methods of inference" (HILBERT, 1900a, §16). Such a "suitable modification" was obviously not easily found and Hilbert continued in his quest to prove the consistency and completeness of arithmetic up until Gödel gave his famous incompleteness proofs.

However in his next paper on the subject (HILBERT, 1905) Hilbert changes his method and provides a syntactic consistency proof.

2.5.2 Hilbert "On the Foundations of Logic and Arithmetic"

In 1904 Hilbert addressed the "Third International Congress of Mathematicians" and sketched his plan to provide a rigorous foundation for mathematics, published as 'On

[29] Specifically by Poincaré for being impredicative (POINCARÉ, 1906)

the Foundations of Logic and Arithmetic' (Hilbert, 1905)[30], specifically by showing the consistency of arithmetic.

Hilbert discussed the positions and labelled the views of others who had investigated the foundations of number dividing them into two groups. In the first group, whom he asserted did not investigate "deeply into the essence of the integer" Hilbert put Kronecker, whom he labelled a "*dogmatist*", and whose restrictive philosophy of mathematics he rejected asserting that his philosophy of mathematics accepted the integer as dogma without providing a foundation; Hermann von Helmholtz whom he labeled an "*empiricist*", and whose views he rejected because they were to his mind not just finitist, but unable to deal with large finites; and Elwin Bruno Christoffel, whom he labelled an "*opportunist*", whom, though an opponent of Kronecker's views *qua* the value of irrational numbers, did not succeed in giving "a pertinent refutation of Kronecker's conception" (Hilbert, 1905, p.130). Then Hilbert turned to those whom he felt had probed "more deeply into the essence of the integer", these included Gottlob Frege, Richard Dedekind and Georg Cantor. However Hilbert realised that Frege's logicism was vulnerable to set theoretic paradoxes, and that: "*from the very beginning a major goal of the investigations into the notion of number should be to avoid such contradictions and to clarify these paradoxes*" (Hilbert, 1905, p.130, emphasis in original). Dedekind's method Hilbert labelled as "*transcedental*" damning it as using a similar method of "philosophers", because it uses the concept of the "totality of all objects" which he asserted would lead to a contradiction. Finally he assessed Cantor's work as "leaving room for *subjective* judgement" in distinguishing between consistent and inconsistent sets, praising him for noticing this distinction, while criticising him for not providing clear criteria with which to distinguish the two (Hilbert, 1905, p.131). Here we begin to see a foreshadowing of Hilbert's approbation of Whitehead's and Russell's *Principia Mathemtica* which would lead Hilbert in 1917 to proclaim their "enterprise of axiomatizing logic" as the "crowning achievement of the work of axiomatization" (Hilbert, 1917, p.9).

After rejecting the above methods of grounding the concept of integer, Hilbert presents his own. He writes that the method should be axiomatic, but at this time, he rejects the logicism of Frege:

> Arithmetic is often considered to be a part of logic, and the traditional fundamental logical notions are usually presupposed when it is a question of establishing a foundation for arithmetic. If we observe attentively, however, we realize that in the traditional exposition of the laws of logic certain fundamental arithmetic notions are already used, for example, the notion of set and, to some extent, also that of number (Hilbert, 1905, p.131)

[30]Grattan-Guinness points out that the original title was simply 'Über die Grundlagen der Arithmetik' but was changed for publication to 'Über die Grundlagen der Logik und der Arithmetik' perhaps to avoid confusion with Frege's publication of the same name (Frege, 1884)

Hilbert first introduces an object, or rather a "thought-object... denoted by a sign... 1 (one)". This (simple) object he then concatenates in combinations *e.g.*: "11, 111, 1111," and combinations of combinations *e.g.*: "(1)(11), (11)(11)(11), ((11)(11))(11), ..." then he adds a second simple object "=" and forms combinations with it:

$$1 =, 11 =, ..., (1)(= 1)(===), ((11)(1)(=))(==), 1 = 1, (11) = (1)(1)$$

Next he introduces the notion of differing combinations: "combinations deviate in any way from each other with regard to the mode and order of succession in the combinations", and divides the combinations into two classes: of entities and of nonentities. A combination is a "true proposition" if it belongs to the class of entities, and its negation true if it belongs to the class of non-entities. He then introduces notation for implication, 'and' and 'or'.

Hilbert now introduces his axioms (1) and (2)[31]

$$\begin{aligned}&(1)\ x = x\\ &(2)\{x = y \wedge w(x) \rightarrow w(y)\}\end{aligned}$$

These two axioms define the notion represented by '=' according to Hilbert, and the consequences of the two axioms are particular sequences of 1s and =s. Note though that axioms (1) and (2) do not provide any sentences of the form $\neg a$ (Hilbert, 1905, pp.131-132).

Hilbert adds the symbols for belonging to a infinite set the 'following' operation and an 'accompanying' operation and uses this notation to introduce the following axioms[32]:

$$\begin{aligned}&(3) f(x_{\in U}) = (f'x)_{\in U}\\ &(4)(f(x_{\in U}) = f(y_{\in U})) \rightarrow (x_{\in U} = y_{\in U})\\ &(5)\neg(f(x_{\in U}) = 1_{\in U})\end{aligned}$$

Now he questions as to whether such axioms could create a contradiction. He notes that only (5) can give rise to axioms of the form $\neg a$, and hence asserts that any axiom that would contradict (5) would be of the form:

$$(6)\exists x f(x_{\in U}) = 1_{\in U}$$

[31] I have used modern notation. Hilbert used: '|' for implication, 'u.'(*und*) for conjunction, and 'o.' (*oder*) for disjunction; $\bar{x}$ for negation; and $A(x^u)$ for universal and $A(x^o)$ for the existential quantifiers.

[32] which he writes, $\mathfrak{u}x$, $\mathfrak{f}$ and $\mathfrak{f}'$ respectively

At this point he shows that all the formulas of the form $a = b$ that can be generated using the axioms have the quality of being "homogeneous" *i.e.* are of the same number of simple objects on either side of the = sign, and notes that (6) is a homogeneous equation, and so provides a syntactic consistency proof of his axiom system (HILBERT, 1905, pp.133-134).

Hilbert's goal in this paper was to "develop logic together with analysis in a common frame, so that proofs can be viewed as *finite mathematical* objects; then show that such formal proofs cannot lead to a contradiction" (SIEG, 1999, p.7). HILBERT (1905) thus shows several features of his later programme. The first is the desire for a consistency proof, specifically one that works on the syntax of the system. Hilbert's later programme focuses on providing a finitist foundation and here we see Hilbert's first real attempt to deal with Kronecker's position, on his own terms. Poincaré's response to Hilbert's 1904 address would further refine his approach to his programme.

2.5.3 Poincaré's Response to Hilbert (1905)

Poincaré contrasts Hilbert's efforts in this paper with Russell's logical system[33], noting that for "Russell any object whatsoever, which he designates by x, is an object absolutely undetermined about which he supposes nothing; for Hilbert it is one of the combinations formed with the symbols 1 and =" (POINCARÉ, 1906, pp.1039-1040). This means that there may not be the introduction of undefined objects only combinations of defined objects.

Poincaré notes that the "contrast with Russell's viewpoint is complete" this is because according to Poincaré:

> Russell is faithful to his point of view, which is that of comprehension. He starts from the general idea of being, and enriches it more and more while restricting it, by adding new qualities. Hilbert on the contrary recognizes as possible beings only combinations of objects already known; so that (looking at only one side of his thought) we might say he takes the viewpoint of extension (POINCARÉ, 1906, p.1040).

Poincaré asserts that this difference can be seen in Hilbert's criticism of Fregean logicism, specifically in näive set theory's susceptibility to paradoxes. Noting that: "in Hilbert's eyes, to take, in an intransigent fashion, the point of view of comprehension (as Russell does) is to be lacking in precision and rigour, and to expose oneself to contradiction" (POINCARÉ, 1906, p.1039-140). Though Poincaré does not dwell on the question, and criticises Hilbert use of complete inductions the principle of induction, he asserts that

[33]That is the system presented in RUSSELL (1903).

"the example of Burali-Forti ... inclines me to say that Hilbert is right" (Poincaré, 1906, p.1039-140).

2.5.4 Hilbert's 1905 Lecture Course 'Logical Principles of Mathematical Thought'

In the lecture course on the 'Logical principles of mathematical thought' in 1905 Hilbert had more time to expand on the paper he had given in 1904 (Hilbert, 1905). The course began by contrasting differing methods of presenting arithmetic: the geometrical, by appeal to diagrams; the genetic, where rationals were ordered pairs of integers, and irrationals treated as decimal expansions; and the axiomatic. Again his preference as in (Hilbert, 1905) was the axiomatic. He followed along the lines of his paper, but with much more attention to consistency and independence. The lectures were not only focused on providing axiom systems for arithmetic and geometry but also physical systems and probability (see Grattan-Guinness 2000, p.215 and Zach 1999, p.333). Hilbert was developing his logical methods, which he would take up again in 1917. He starts the course off by presenting first set theory and introducing the paradoxes discovered by Russell and Zermelo, later noting:

> "The paradoxes we have just introduced show sufficiently that an examination and redevelopment of the foundations of mathematics and logic is urgently necessary" (Hilbert 1905 Lectures quoted in Zach, 1999, p.333).

Hilbert presents, in these lectures, an algebraic presentation of propositional logic much like that used in the Heidelberg lecture, Hilbert (1905)[34]

> It would now have to be investigated in how far the axioms are dependent and independent of one another [. . .] What would be most important here, however, is the proof that the 12 axioms do not contradict each other, *i.e.*, that using the process defined one cannot obtain a proposition which contradicts the axioms, say, $X + \bar{X} = 0$. These are only hints which have not been carried out completely as of yet, and one still has free reign in the details; generally speaking this whole section supplies material for the ultimate solution of the interesting questions, rather than give the ultimate solution (Hilbert quoted in Zach, 1999, p.334)

[34] In these lectures Hilbert uses "'≡' for identity, '| ' for implication, and in a reverse from normal ' · ' for disjunction and ' +' for conjunction" and, 0 for truth and 1 for falsehood and $\bar{X}$ for the negation of X (Grattan-Guinness, 2000, p.215).

As seen above much of what he needs for this has been presented in 1905. Zach notes also that Hilbert presents a "nonderivability proof using an arithmetical interpretation of the axioms" (Zach, 1999, p.334). In addition Hilbert shows that "every propositional formula can be brought into one of two normal forms" by using de Morgan's laws to show that all propositions can be turned into sums and products of atomic propositions and their negations, then "using the distributive law, this can be rewritten as a sum of products" (Hilbert 1905 lectures quoted in Zach, 1999, pp.334-335). Hilbert also discussed consequence, interpreting implication using the standard classical definition of the material conditional "Y follows from X if $X \cdot Y = 0$" (Hilbert 1905 lectures quoted in Zach, 1999, p.335). Hilbert proves this about consequence:

> A proposition Y follows from another proposition X if and only if it is of the form $A \cdot X$, where A is some proposition. To deduce is to multiply correct propositions with arbitrary propositions. (Hilbert 1905 lectures quoted in Zach, 1999, p.335).

Hilbert asserts that A is to be defined as a proof. He then uses his normal form theorem as the first attempted proof of decidability for the propositional calculus (Zach, 1999, p.335). This is an example of what Hilbert was looking for in his Paris address when he declared that there is no *Ignorabimus*. But as Zach notes, there are several problems with his presentation, including, "Hilbert's earlier error of claiming that the normal form for a given formula is unique" (Zach, 1999, p.336). Hilbert's previous method could not work because:

> For Hilbert's procedure to work, we would not only have to be able to enumerate all possible proofs A, but also be able to check if $A \cdot (a + b + \cdots) = Y$. This would presumably have to be done by comparing normal forms, since no other method—e.g., truth tables—is available. But normal forms are not unique, so there is no guarantee that the left and right side will result in the same one (Zach, 1999, p.336).

Still this course presents many of the aspects of Hilbert's programme that he would not come back to until 1917. As Zach puts it:

> Here, in 1905, one of Hilbert's aims in the foundations of mathematics is made almost explicit, namely the aim to provide decision procedures for logic on the one hand, and particular systems of mathematics and science, *e.g.*, arithmetic, on the other (Zach, 1999, p.336).

2.6 The Middle Period

Hilbert, though he did not publish on the foundations of mathematics during the period from 1905 to 1917, did however give several lecture courses on set theory, logic and the foundations of mathematics and physics:

> Hilbert lectured on *Zahlbegriff und Prinzipienfragen der Mathematik* [The Concept of Number and Principle Issues of Mathematics] (Summer 1908), on *Elemente und Prinzipienfragen der Mathematik* [Elements and Principle Issues of Mathematics](Summer 1910), on *Grundlagen der Mathematik und Physik* [Foundations of Mathematics and Physics](Summer 1913), *Prinzipien der Mathematik* [Principles of Mathematics] (Summer 1913), *Probleme and Prinzipien der Mathematik* [Problems and Principles of Mathematics](Winter 1914/15), and on *Mengenlehre* [Set Theory] (Summer 1917) (Sieg, 1999, p.8).[35].

Wilfried Sieg notes that none of the courses "broke new ground" and in none but the ones given in the summer of 1905 does he take up the proof theoretic approach of his 1904 paper (Hilbert, 1905) (Sieg, 1999, p.9). In the lecture notes of 1910 Hilbert does provide an extended discussion of the "set theoretic antinomies". As Sieg writes: "This time the fundamental problem is seen as related to what Hilbert calls *genetische Definitionen*," the genetic definitions that Hilbert discussed in his papers *Über den Zahlbegriff* (Hilbert, 1900a). This discussion linkes his previous discussion of Kronecker "to the future, *i.e.*, to a fully developed finitist standpoint" (Sieg, 1999, p.9).

> There is no need to consider irrational numbers; the geometric series $1+1/2+1/4+1/8+$ "and so on" is already an example. Not even formulas in which finite, but only indeterminate whole numbers n occur are immune to our critique. To be able to apply them one sets $n :: 1, 2, 3, 4, 5,$ "and so on". Kronecker who intended to reduce all of mathematics to the whole numbers was consequently not radical enough, for n' does occur in his formula. He should have restricted himself to the specific numbers 7, 15, 24. Thus, one sees what kind of difficulties have to be faced when calculating with letters. Already the simple formula $a + b = b + a$ can be attacked.
>
> ...Despite the high pedagogic and heuristic value of the genetic method, for the final presentation and the complete logical grounding of our knowledge the axiomatic method deserves to be preferred (Hilbert 1910 lecture quoted in Sieg, 1999, p.10).

[35]For more information see also Vito Michele Abrusci's (Abrusci, 1989) and Volker Peckhaus' surveys of Hilbert's lectures (Peckhaus, 1990, 1994)

He then presents an axiom system for natural numbers and notes that it is the "first step in the foundational investigation":

> ... if we set up the axioms of arithmetic, but forego their further reduction and take over uncritically the usual laws of logic, then we have to realize that we have not overcome the difficulties for a first philosophical-epistemological foundation; rather, we have just cut them off in this way (HILBERT 1910 lecture quoted in SIEG, 1999, pp.10-11).

At this point Hilbert suggests that we can further reduce the axioms given to the laws of logic. He does not do this though, and this part of his project will have to wait until 1917 (SIEG, 1999, p.11).

2.6.1 Hilbert and Logicism

As noted above Hilbert's foundational periods are often times divided into two periods from 1900 to 1905 and from 1922 to 1931, however these dates are based on the publications of the early papers on arithmetical foundations (HILBERT, 1900a,b, 1905) and the papers on mathematical logic HILBERT (1923, 1926, 1927) but fail to note his address on "Axiomatic Thinking" HILBERT (1917, 1918) (see SIEG 1999, p.2-3, MOORE 1997, p.68) and of course his courses from 1917/18, 1920 "The problem of mathematical logic" (HILBERT, 1920), and 1921/22. At first Hilbert's goal seemed to be only to derive a consistent system, but by 1913 the logicist programme seemed to have achieved this much success, Hilbert refers to Russell and Whitehead's *Principia Mathematica* (WHITEHEAD and RUSSELL, 1910, 1912, 1913) in glowing terms in his 1917 address[36] "Axiomatische Denken" (Axiomatic Thinking) stating that, "In the completion of this extensive enterprise by Russell for the axiomatization of logic one can behold the crowning of the work of axiomatization in general" (HILBERT, 1918, p.8). And in fact it would be one of Russell's innovations in the *Principia*, specifically the ι-operator, that would lead Hilbert to develop the τ and ε-operators.

Frege and Hilbert's early exchange about the nature of axiomatic systems and the discovery of Russell's paradox or rather the discovery of how it applied to Frege's basic law V of his *Grundgesetze der Arithmetik* (FREGE, 1893, 1903) had originally made Hilbert critical of the logicist programme. In Hilbert's address to the international congress of mathematician in Heidelberg we have him noting that the distinction between mathematics and logic is not clear:

> Arithmetic is often considered to be part of logic and the traditional fundamental, logical notions are usually presupposed when it is a question of establishing a foundation of arithmetic. If we observe attentively, however, we

[36]Which he gave in neutral Zurich, where he invited Paul Bernays to work with him at Göttingen.

> realize that in the traditional exposition of the laws of logic certain fundamental arithmetic notions are already used, for example, the notion of set and, to some extent, also that of number. Thus we find ourselves turning in a circle, and that is why a partly simultaneous development of the laws of logic and arithmetic is required if paradoxes are to be avoided (Hilbert 1905, p.131 translated in Zach 2003, p.211).

Hilbert returned to writing on foundational issues, in his 1917/1918 paper "*Axiomatisches Denken*" (Hilbert, 1917, 1918) where he seems to have actually embraced some aspects of logicism. By this time Hilbert's concerns with set theoretic paradoxes have been dealt with by the development of Russell's type theory.

In his 1917 address Hilbert details the nature of what he sees as the place for the axiomatic method in math and the neighbouring sciences. First "the facts of a specific field of more or less comprehensive knowledge" are collected and "set in order". This ordering of facts is done with the aid of a "*framework of concepts*" which Hilbert notes becomes the "theory" of the field of knowledge. The "*framework of concepts*" is defined by him as the logical relation between concepts that corresponds to first the relation between the concepts and objects of the field of knowledge, and second the relation of the facts of the field of knowledge and relations of concepts to one another (Hilbert, 1918, p.1).

He asks what criteria must such a successful framework satisfy:

> If the theory of a field of knowledge, that is, the framework of concepts that represents the theory, is to serve its purpose, namely the orientation and order, it must then satisfy chiefly two fixed demands: it must offer, first, a general view of the dependence or independence of the propositions of the theory and, second, a guarantee of consistency of all propositions of the theory. In particular, the axioms of each theory have to be proved in accordance with these two viewpoints (Hilbert, 1918, p.3).

With regard to independence results, Hilbert gives several examples: "parallel axiom in geometry offered the classic example for the examination of independence of an axiom"; "arbitrary forces," and "arbitrary secondary conditions" in classical mechanics; and in the analysis of "real numbers" he gives the example of the Archimedian axiom which is "independent of all other axioms of arithmetic" again drawing comparisons with physics (Hilbert, 1918, pp.4-5).

After presenting these examples Hilbert now turns to consistency, with which he is much more concerned. He notes that consistency is "manifestly of greater importance, since the presence of a contradiction in a theory manifestly imperils the stability of the entire theory" (Hilbert, 1918, p.6). Not only is it important, Hilbert notes that it is often contentious as "[t]he understanding of the internal consistency is linked to difficulty even in the long accepted and flourishing theories" (HILBERT, 1918, p.6). This happen Hilbert says because: "It often happens that the internal consistency of a theory is considered self-explanatory while, in truth, deep mathematical developments are necessary for proofs " (HILBERT, 1918, p.6).

Moving from examples in physics to mathematics, Hilbert states that in physics the changes in axiom systems are based on observation of the physical world, but notes that, "the situation changes, however, if contradictions appear in purely theoretical fields of knowledge". He gives the example of the "paradox of the set of all sets" noting that "distinguished mathematicians as Kronecker and Poincaré for instance felt induced to deny set theory ...any justification of existence" (HILBERT, 1918, p.7). Hilbert credits the axiomatic method for resolving the paradox:

> As he [Zermelo] set up suitable axioms to restrict, on the one hand, the arbitrariness in the definitions of sets themselves and, on the other, the admissibility of statements on their elements in a specific way, Zermelo succeeded to develop set theory in such a manner that the paradoxes under discussion fall away and, for all restrictions, the purport and applicability of set theory remains the same (HILBERT, 1918, pp.7-8).

Hilbert asserts that for set theory as well as in physical cases, the contradictions were "brought out in the process of developing a theory" and were eliminated as the definition of the system was revised. Hilbert thus felt that in a properly developed axiomatic systems, "contradictions are always altogether impossible in a field of knowledge founded on the erected system of axioms" (HILBERT, 1918, p.8).

Like he found in geometry, Hilbert notes that the consistency of any axiom system that depends upon the consistency of arithmetic can then be reduced to that problem, pointing out that "no doubt for the fields of physical knowledge, too, it is always sufficient to reduce the question of inner consistency to the consistency of arithmetic axioms" (HILBERT, 1918, p.8). Likewise he continues "the consistency of the axiomatic system for real numbers is reduced, through the use of set theoretic concepts, to the same question for integers" (HILBERT, 1918, p.9). At this point there is nowhere else to go, Hilbert writes:

> Only in two cases, namely if it is a question of the axioms of integers themselves, and if it is a question of the foundation of set theory, this mode of reduction to another specific field of knowledge is manifestly impracticable, since beyond logic there is no more discipline to which an appeal could be lodged (HILBERT, 1918, p.9).

Hence the necessity to "axiomatize logic itself and then to establish that number theory as well as set theory is only a part of logic," which at this point Hilbert writes has finally been accomplished by Russell and Whitehead (Hilbert, 1918, p.9). However Hilbert has not completely bought into the logicist programme, believing that there are still "difficult epistemological questions of specific mathematical coloration" to be answered before the axiomatization of logic could be said to be finished. These include:

1. "the problem of principal *solvability of every mathematical question*"
2. "the problem of supplementary *controllability* of the results of a mathematical investigation"
3. "the question of a *criterion for the simplicity* of mathematical proofs"
4. "the question of relations between *contentualness* (Inhaltlichkeit) and formalism in mathematics and logic", and
5. "the problem of decidability of a mathematical question by a finite number of operations" (Hilbert, 1918, p.9).

These problems, made perhaps more clear by the study of *Principia Mathematica* by Hilbert and his students, started Hilbert on a series of investigations that developed into the programme of the 1920s and on. As Zach puts it:

> These unresolved problems of axiomatics led Hilbert to devote significant effort to work on logic in the following years. In 1917, Paul Bernays joined him as his assistant in Göttingen... The course from 1917, in particular, contains a sophisticated development of first-order logic, and forms the basis of Hilbert and Ackermann's textbook *Principles of Theoretical Logic* ... In 1918, Bernays submitted a treatise on the propositional calculus of *Principia mathematica* as a *Habilitationsschrift*; it contains the first completeness proof of the propositional calculus for truth-functional semantics (Zach, 2006, 415).

The fifth of the above problems is the famous *Entscheidungsproblem* (decision problem) one of key parts of Hilbert's programme. The decision problem was presented by Hilbert and Ackermann in their 1928 book on mathematical logic (Hilbert and Ackermann, 1928), but its roots go back as far as Hilbert's reaction to the *Ignorabimusstreit* in his Paris address where he asserted that all mathematical questions can be answered. As is well known Gödel's incompleteness theorem answers the question of whether their can be such a method to answer all such questions in a finite number of steps in the negative (see Gödel, 1931).

2.7 Hilbert's Programme

As we have seen above, by 1918 Hilbert had devised the questions that would make up the core of his foundational programme. By the summer of 1920 Hilbert had returned to foundational issues giving a lecture course entitled "*Probleme der mathematischen Logik*"(Problems of Mathematical Logic) (HILBERT, 1920). And so by the summer of 1920, Hilbert's foundational programme was now taking shape and was focused around two key principles, as Avigad and Zach summarize:

> First, modern mathematical methods were to be represented in formal deductive systems. Second, these formal systems were to be proved syntactically consistent, not by exhibiting a model or reducing their consistency to another system, but by a direct metamathematical argument of an explicit, "finitary" character (AVIGAD and ZACH, 2013).

Wilfried Sieg points out (SIEG, 1999) Hilbert's notes from the period of 1917 to 1922 "reveal a dialectic progression from a critical logicism through a radical constructivism toward finitism." So by the time of the papers of the early 1920s (HILBERT, 1922, 1923, 1926) Hilbert had abandoned any of his logicist leanings, though he had taken several lessons from his close study of the *Principia Mathematica* including an interest in Russell's use of the term forming operators *(ı)* for definite descriptions, which he subsequently developed into the more useful ε-operator (HILBERT and BERNAYS, 1934, 1939, pp.393-466 and pp.1-209 respectively).

Hilbert and Bernays give the following definition of how their finitary methods were supposed to work and are limited to objects and processes that can be conceived of and completed in principle. Using as an example their method of introducing basic number theory they write:[37]

> Our treatment of the basics of number theory and algebra was meant to demonstrate how to apply and implement direct contentual inference that takes place in thought experiments on intuitively conceived objects and that is free of axiomatic assumptions. Let us call this kind of inference "finitistic" inference for short, and likewise the methodological attitude underlying this kind of inference as the "finitistic attitude or the "finitistic" standpoint. In the same sense, we will speak of finitistic concept formations and assertions: With each use of the word "finitistic", we convey the idea that the relevant consideration, assertion, or definition is confined to

[37]In this quotation Hilbert and Bernays are referring to the material presented in the first chapter of HILBERT and BERNAYS (1934), which resembles very much the methods developed in Hilbert's earlier papers especially HILBERT (1900a, 1905), discussed earlier in this chapter.

- objects that are conceivable in principle, and
- processes that can be effectively executed in principle,

> and thus remains within the scope of a concrete treatment (HILBERT and BERNAYS, 1934, 1939, p.34).

Sieg's survey of Hilbert's lecture notes fills in much historical detail about the development of Hilbert's views. These notes explain the contrasting styles of Hilberts writings in the teens and the twenties, for example, Sieg notes that: "In sharp contrast, [to the 1917 and 1918 papers] the 1922 papers by Hilbert and Bernays seem to set out the philosophical and mathematical-logical goals of the Hilbert Program" (SIEG, 1999, p.3). This did not happen overnight, rather in the winter term of 1920 Hilbert revisits the material in his 1917 and 1918 lectures, and in final third of his 1920 lecture notes Hilbert argues that Dedekind and Frege's set theoretic and logical developments, "did not succeed in establishing the consistency of ordinary number theory," asserting that:

> To solve these problems I don't see any other possibility, but to rebuild number theory from the beginning and to shape concepts and inferences in such a way that paradoxes are excluded from the outset and that proof procedures become completely surveyable (HILBERT 1920 Lectures in SIEG, 1999, p.23).

2.7.1 Hilbert's "New Grounding of Mathematics"

Given his views presented above, when describing Hilbert's programme the term's finitist and formalist are both often used, but both of these terms are misnomers. The goal of the programme was not to limit mathematics to a finite fragment nor was the goal of the programme to reduce mathematics to simply the marks on the page. Sieg notes that in "On the Infinite" (HILBERT, 1926) Hilbert's goal was to defend two claims:

> (i) proof theory can secure the foundations of classical mathematics "once and for all", and
>
> (ii) proof theory can answer "pre-existent questions that the theory was not specifically created to answer" (SIEG, 1988, p.341)

Hilbert thus continued to believed in the autonomy of mathematics, something he had defended first in his 1900 Paris address. Hence Michael Hallett writes, that a key feature of Hilbert's particular philosophy of mathematics is: "the desire to show that mathematics is autonomous, not dependant on 'foreign elements' or on appeals to intuition, as least in its deductive development" (HALLETT, 1996, p.138). This meant that mathematics would never be dependant on a physical theory. What then are the objects of number theory? Hilbert answers that

> ...the objects (*Gegenstäinde*) of number theory are for me—in direct contrast to Dedekind and Frege—the signs themselves, whose shape (*Gestalt*) can be generally and certainly recognized by us—independently of space and time, of the special conditions of the production of the sign, and of insignificant differences in the finished product (Hilbert, 1922, p.202).

Hence, the subject of mathematics is not the physical marks or any physical phenomena, but the signs themselves which are intuited (see section 3.2.1 on intuition). Hilbert's stated goal remained the same, a consistency proof for the axioms of analysis. Hilbert is thus found stating in a lecture in 1922 that:

> a satisfactory conclusion to the research into these [mathematical] foundations can only be attained by the solution of the problem of the consistency of the axioms of analysis. If we can produce this proof, then we can say that mathematical statements are in fact incontestable and ultimate truths—a piece of knowledge that (also because of its general philosophical character) is of the greatest signicance for us (Hilbert, 1922, p.202).

Thus asserted that a proper foundation would be developed by building on the work of the logicists and Zermelo's set theory, and rejecting what he saw as Weyl's criticisms that the "modern critique of analysis" had ended in "chaos and senselessness" (Hilbert, 1922, p.202). His lecture begins by referring to the problems encountered, as he saw it by the logicist project:

> ...abstract operation with general concept-scopes and contents has proved to be inadequate and uncertain. Instead, as a precondition for the application of logical inferences and for the activation of logical operations, something must already be given in representation (in der Vorstellung): certain extra-logical discrete objects, which exist intuitively as immediate experience before all thought (Hilbert, 1922, p.202).

Hilbert continues:

> If logical inference is to be certain, then these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something that cannot be reduced to something else (Hilbert, 1922, p.202).

And thus it is in his 1922 paper "The New Grounding of Mathematics" that Hilbert first describes a method of constructing a fragment of number theory by the manipulation of signs, though this is still much along the lines of his 1900 paper "On the Concept of Number" (HILBERT, 1900a):

> When we develop number theory in this way, there are no axioms, and no contradictions of any sort are possible. We simply have concrete signs as objects, we operate with them, and we make contentual [inhaltliche] statements about them. And in particular, regarding the proof just given that $a + b = b + a$, I should like to stress that this proof is merely a procedure that rests on the construction and deconstruction of number signs and that it is essentially different from the principle that plays such a prominent role in higher arithmetic, namely, the principle of complete induction or of inference from n to $n + 1$ (HILBERT, 1922, p.203).[38]

He continues that while we can "make considerable further progress in number theory" by using such a method we cannot develop all of mathematics in this manner, rather such procedures "break down" when "we cross over into higher arithmetic and algebra" and want to discuss "assertions about infinitely many numbers or functions" (HILBERT, 1922, pp.203-204). Rather, he notes that to understand analysis, "we need real, actual formulas for its construction." However, we can move up a level if we consider "axioms, formulae, and proofs of the mathematical theory" themselves to be "the objects of a contentual investigation," hence Hilbert asserts that:

> we need to have a strict formalization of the entire mathematical theory, inclusive of its proofs, so that-... mathematical inferences and definitions become a formal part of the edice of mathematics. The axioms, formulae, and proofs that make up this formal edice are precisely what the number-signs were in the construction of elementary number theory... and with them alone, as with the number-signs in number theory, contentual thought takes place—i.e. only with them is actual thought practiced. (HILBERT, 1922, p.203).

Contentual thoughts he then writes, are "removed elsewhere—to a higher plane, as it were" but even as these thoughts are elevated we can still "draw a sharp and systematic distinction in mathematics between the formulae and formal proofs on the one hand, and the contentual ideas on the other" (HILBERT, 1922, p.203).

After sketching his proof theory Hilbert notes that his system is consistent, but does not give a proof, which is likely because to make a proof later will actually take considerable

[38]Compare to (HILBERT, 1900a) discussed above.

modifications *i.e.* the removal of the all sign and the replacement of the axiom of complete induction with an induction *schema* (Hilbert, 1922, p.214, fn. d (by Bernays)).

Hilbert then describes what he sees as the "general tendency and direction in which the new grounding of mathematics ought to proceed" (Hilbert, 1922, p.211). He focuses on two points:

> First: everything that hitherto made up mathematics proper is now to be strictly formalized, so that mathematics proper, or mathematics in the strict sense, becomes a stock of provable formulae. The formulae of this stock are distinguished from the usual formulae of mathematics only by the fact that, besides the mathematical signs, they also contain the $\rightarrow$ sign, the all-sign, and the sign for statements. This circumstance corresponds to a conviction have long maintained? namely, that a simultaneous construction of arithmetic and formal logic is necessary because of the close connection and inseparability of arithmetical and logical truths.
>
> Secondly: in addition to this proper mathematics, there appears a mathematics that is to some extent new, a metamathematics which serves to safeguard it by protecting it from the terror of unnecessary prohibitions as well as from the difficulty of paradoxes. In this metamathematics—in contrast to the purely formal modes of inference in mathematics proper—we apply contentual inference, in particular, to the proof of the consistency of the axioms (Hilbert, 1922, pp.211-212)

Around this point Hilbert's also provides a description of his philosophy of mathematics, which as Zach points out, appears in several of his publications almost word for word:

> as a condition for the use of logical inferences and the performance of logical operations, something must already be given to our faculty of representation, certain extra-logical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that can neither be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication (Hilbert, 1926, p.376)[39].

[39]Zach (Zach, 2006, p.206) notes that this passage appears more or less the same in 1922b; 1926; 1928; 1931b

Hilbert's goal was to provide a foundation for arithmetic through "finitist"methods, which he hoped could rescue it, and logic, from constructivist criticisms. This involved dividing mathematics into two parts: a set of "meaningful propositions, as well as finitarily admissible constructions and methods of proof" and the "rest, which includes classical infinitary mathematics (full first-order arithmetic, analysis, and set theory, in particular)" which were to be understood as simply instrumentally meaningful (Zach, 2006, pp.419–420).

That the objects of mathematics were "intuitively present as immediate experience prior to all thought" means that Hilbert understood them as being apprehended by intuition, in a Kantian manner. As Bernays describes:

> the objects of intuitive number theory, the number signs, are, according to Hilbert, also not 'created by thought'. But this does not mean that they exist independently of their intuitive construction, to use the Kantian term that is quite appropriate here (Bernays, 1923, p.226)

As noted above, the objects of investigation were not physical nor the physical symbols but the intuitively given mathematical objects. Mancosu (Mancosu, 1998b, pp.145ff.) has argued that between the early papers (*cf.* Hilbert, 1900a,b, 1902, 1905) and his later writings (/cf. Bernays, 1928) one can detect a shift, from a form of perceptual intuition, to something more like form of pure intuition in the Kantian sense. In Hilbert (1931), he asserts that although:

> we can no longer agree with Kant in the details, nevertheless the most general fundamental idea of Kantian epistemology retains its significance: to acertain the *a priori* in intuitive mode of thought (Hilbert, 1931, pp.1149–1150).

Hence Hilbert writes, in what we may take to be a Kantian tone, that "besides experience and thought, there is yet a third source of knowledge" which is *a priori*. He refers to it as the "fundamental mode of thought" and the "finite mode of thought". This mode of thought is how we apprehend that which is "already given to us in advance in our faculty of representation: certain extra-logical concrete objects that exist intuitively as an immediate experience before all thought" (Hilbert, 1931, p.1150). There are limitations to what the Kantian inspired *a priori* can achieve. As Hilbert stated the details of Kant's theory could not be accepted, including Kant's belief in geomentry's status as *a priori*, but Hilbert also includes as examples of propositions that are generally held to be *a priori*, "but cannot be achieved within the frame of the finite mode of thought," both the "*tertium non datur* as well as the so-called transfinite statements generally" (Hilbert, 1931, p.1150). Hilbert however notes that number theory demands the use of transfinite axioms, and writes that:

> These difficulties must be overcome; for how can knowledge be possible at all if not even number theory can be given a firm foundation, and if full unity and absolute correctness cannot be demanded even here! (HILBERT, 1931, p.1150)

According to Hilbert the problems with the transfinite axioms and law of excluded middle occur only in the infinite case—though this is of course the only case about which constructivists like Brouwer would worry—and hence the "operation with the infinite must be secured in the finite; and precisely this occurs in my proof theory" (HILBERT, 1931, p.1151). He then gives the following description of what he terms the "fundamental idea" of his proof theory:

> Everything that makes up mathematics in the traditional sense is rigorously formalized, so that mathematics proper (or mathematics in the narrow sense) becomes a stock of formulae. These are distinguished from the usual formulae of mathematics only in the following way: that besides the usual signs, the logical signs appear as wellin particular, the signs for 'implies' ($\rightarrow$) and for 'not' ($^{-}$). Certain formulae that serve as a foundation for the formal edifice of mathematics are called axioms. A proof is a figure, which must be intuitively presented to us as such; it consists of inferences, where each of the premisses is either an axiom, or agrees with the end-formula of an inference that comes earlier in the proof, or results from such a formula by substitution. Instead of contentual inference, in proof theory we have an external action according to rules, namely, the use of the inference schemata and of substitution. A formula shall be called provable if it is either an axiom or the end-formula of a proof (HILBERT, 1931, p.1152).

Hilbert then writes that this "proper, formalized mathematics" is secured by a new form of mathematics. This he calls "a metamathematics that is necessary to secure formalized mathematics". It is in this metamathematics, which he contrasts to "the purely formal modes of inference of mathematics proper," where "contentual inference" occurs, but, Hilbert continues, the only point of metamathematics is to "prove the consistency of the axioms" (HILBERT, 1931, pp.1152-1153). After making these distinctions, Hilbert defended his proof theory asserting that the critics objections have been unjustified:

> My theory has been subjected to the reproach that, although the theorems are indeed consistent, they are not for that reason proved. To be sure, they are provable, as I have shown here in simple cases. More generally, it turns out (as I was convinced from the outset) that the attainment of consistency is the essential thing in proof theory, and the question of provability (possibly with a suitable extension of the conditions that preserves the finite character)

> is settled at the same time. However, it cannot be demanded of a theory that all the relevant questions which it poses be fully solved at the outset; it suffices if the path to that goal has been indicated (Hilbert, 1931, pp.1155-1156).

Hilbert was certain at this point that his theory stood on its own and needed no outside justification:

> And for justification I need neither God, like Kronecker, nor the assumption of a special capacity of our understanding directed towards the principle of complete induction, like Poincaré, nor some *ur*-intuition like Brouwer, nor, like Whitehead and Russell, the axioms of infinity and reducibility, which are real, contentual presuppositions, not compensated for by proofs of consistency, and of which the latter is not even plausible (Hilbert, 1931, p.1156).

Hilbert's criticism of Brouwer's ur-intuitionism was well placed as Brouwer's philosophical justifications obscured his important investigations; not until intuitionistic logic was separated from Brouwer's naïve philosophy was its importance made clear, and more sophisticated arguments for mathematical and logical intuitionism given.

Hilbert's optimism was finally answered by Gödel's incompleteness theorems. Gödel had already proved the completeness theorem for first-order logic (Gödel, 1930), before proving his incompleteness proofs (Gödel, 1931) which showed that part of Hilbert's programme was not actually possible. Specifically the first incompleteness proof shows that in a logical theory capable of expressing elementary arithmetic there is no decision procedure for provability, *i.e.* it is not complete [40]. The second incompleteness theorem proves that the consistency of a logical theory strong enough to describe elementary arithmetic cannot be proven in that same system [41]

[40] Kleene puts it in the following manner:

> Theorem IV. There is no decision procedure for provability in the formal system N of §38 [predicate calculus with the addition of Peano's third, forth and fifth axioms (providing that zero, its successors, and any successor of a number is a numbers); the axioms of equality; and axioms that provide recursive definitions of addition and multiplication (*cf.* Kleene, 1967, pp.206–210); or briefly, N is undecidable (Kleene, 1967, p.248).

Kleene continues:

> More generally, this applies not just to the formal system N of §38, but to any formal system N in which, to each a, there can be found effectively a closed formula C_a such that (a) and (b) hold.

where a and b are the Church-Turning Thesis and its inverse, which Kleene symbolizes as:

$$\{\vdash C_a \text{in } \mathbf{N}\} \rightarrow (\exists x)T(a,a,x) \text{ and, } (\exists x)T(a,a,x) \rightarrow \{\vdash C_a \text{in } \mathbf{N}\}$$

(Kleene, 1967, pp.247–248)

[41] In Gödel's terms:

2.8 Choice Operators

As noted above the τ-operator was first introduced by Hilbert in his 1922 paper "*Die logischen Grundlagen der Mathematik*" (The Logical Foundation of Mathematics) as the transfinite axiom. Starting in 1925 in the paper "On the Infinite", Hilbert began to work instead with the more familiar ε operator (Hilbert, 1926). The value of ε, Hilbert argued, was that it allows us to prove the "transfinite axioms":

$$\forall x A(x) \to A(y)$$

$$\neg\forall x A(x) \to \exists x \neg A(x)$$

$$\neg\exists x A(x) \to \forall x \neg A(x)$$

Hilbert notes that all of these, "transfinite axioms are derived from a single axiom, one that also contains the core of one of the most attacked axioms in the literature of mathematics, namely the axiom of choice":

$$A(x) \to A\varepsilon_A$$

Hilbert refers to this as a "transfinite logical choice function" (Hilbert, 1926, p.382). These axioms while introduced in these papers, and Ackermann's dissertation, were presented in a much more developed manner, with reference to Russell's ι-operator from which they were developed, in Hilbert and Bernay's text on mathematical foundations the *Grundlagen der Arithmetik*; and hence it is to this book we now turn our attention.

2.8.1 Hilbert and Bernay's *Grundlagen der Arithmetik*

Hilbert and Bernay's two volume *Grundlagen der Arithmetik* was, in the words of Claus-Peter Wirth, "the central and most involved presentation of Hilberts program and Hilberts proof theory" (Hilbert and Bernays, 2013, p. vi). The second volume in particular

> Theorem XI. Let κ be any recursive consistent class of formulas; then the sentential formula stating that κ is consistent is not κ-provable; in particular, the consistency of P is not provable in P, provided P is consistent (in the opposite case, of course, every proposition is provable [in P]) (Gödel, 1932, p.614).

where κ is a theory strong enough to describe elementary arithmetic.

interests us, as it is here where Hilbert presents the ε-operator and derives the various ε theorems.

Hilbert does not introduce the ε operator directly as he did in his (Hilbert, 1922, 1926) but rather by adapting the definite description operator *($\imath$)* introduced by Russell: "We are thus led to the following definition, in which ' $(\imath x)(\varphi x)$' is to be read 'the term x which satisfies φx'." (Russell, 1908)[42].

Hilbert writes that using the ε-operator one can introduce a function symbol $f(a, ..., k, x)$ from what Hilbert calls the "symbolic resolution"[43] of the existential fromula:

$$(\exists x), A(a, ..., k, x)$$

in a manner that differs from the introduction of the same function symbol *via* the ι-rule only in that there is no longer the condition that one must also be able to derive the "uniqueness formula"[44]:

$$(\forall x)(\forall y)(A(a, ..., k, x) \wedge A(a, ..., k, y) \rightarrow x = y)$$

(Hilbert and Bernays, 1939, p.9).

As an intermediate step Hilbert introduces the ν-operator which differs from the ι-operator only in that in comparison with the ι-rule the second uniqueness formula is no longer required as a premise (Hilbert and Bernays, 1939, p.10 esp. fn.1).

2.8.2 Russell's $\imath$-Operator and Hilbert's ε-Operator

At the end of book one of Hilbert and Bernays *Grundlagen* "we find a rival theory of definite descriptions to Russells", and in book 2 the epsilon terms are introduced in relation to this theory (Slater, 2009, p.387). Recall that Russells theory of definite descriptions an expression like: "The King of France is bald" is broken into three parts:

1. 'there is a king of France'
2. 'there is only one king of France'
3. 'he is bald'

[42] Hilbert writes the operator simply as ι.
[43] "der symbolischen Auflösung"
[44] "Unitätsformel"

Russell used the $\imath$-operator to symbolise the definite description. In Russell's system it is an incomplete symbol *i.e.* an abbreviations for formulas not containing them. By incomplete symbol, Russell meant that: "Every use of '$(\imath x)(\varphi x)$,' where it apparently occurs as a constituent of a proposition in the place of an object, is defined in terms of the primitive ideas already on hand" (Whitehead and Russell, 1910, p.31). Hence Russell notes that both his term defining operators are incomplete:

> Both symbols [$\imath$ and $\hat{x}$] are incomplete symbols defined only in use... We now proceed to define $E!(\imath x)(\varphi x)$ so that it can be read 'the x satisfying φx exists.' ... Its definition is:
>
> $$E!(\imath x)(\varphi x). =: (\exists c) : \varphi x. \equiv_x .x = c \quad Df,$$
>
> *i.e.* "the x satisfying $\varphi \hat{x}$ exists" is to mean "there is an object c such that φx is true when x is c but not otherwise" (Whitehead and Russell, 1910, pp.31-32).[45]

This means that $\imath$-terms are nothing more than abbreviations for longer terms and are not part of the language proper. This is because Russell thought that normal proper names are nothing more than definite descriptions in disguise, while the only names that were strictly name were denoted by indexicals like 'this' and 'that' which drew upon immediate experience.[46]

[45]Whitehead and Russell then note the equivalent forms of iota-terms:

> The following are equivalent forms:
>
> $$E!(\imath x)(\varphi x). =: (\exists c) : \varphi c : \varphi x. \supset_x .x = c,$$
> $$E!(\imath x)(\varphi x). =: (\exists c).\varphi c : \varphi x.\varphi y. \supset_{x,y} .x = y,$$
> $$E!(\imath x)(\varphi x). =: (\exists c) : \varphi c : x \neq c. \supset_x . \sim \varphi x.$$
>
> (Whitehead and Russell, 1910, p.32)

[46]Russell makes a distinction between "ordinary" proper names and "logically" proper names *e.g.*:

> While logically proper names (words such as "this" or "that" which refer to sensations of which an agent is immediately aware) do have referents associated with them, descriptive phrases (such as "the smallest number less than pi") should be viewed merely as collections of quantifiers (such as "all" and "some") and propositional functions (such as "x is a number"). As such, they are not to be viewed as referring terms but, rather, as "incomplete symbols" (Irvine, 2014).

Note that Russell, in the papers generally taken to describe this view ("On Denoting" (1905), "Knowledge by Acquaintance and Knowledge by Description" (1910), "Descriptions" (1919) (Chapter 16 of *Introduction to Mathematical Philosophy*) and "The Philosophy of Logical Atomism" (1918, 1919)), never used the

Russell writes in "The Philosophy of Logical Atomism" that the nature of names is not simply apparent. Rather it is:

> ... very difficult to get any instance of a name at all in the proper strict logical sense of the word. The only words one does use as names in the logical sense are words like 'this' or 'that.' One can use 'this' as a name to stand for a particular with which one is acquainted at the moment. We say 'This is white.' If you agree that 'This is white,' meaning the 'this' that you see, you are using 'this' as a proper name. But if you try to apprehend the proposition that I am expressing when I say 'This is white,' you cannot do it. " (RUSSELL, 1918, p.524)

In comparison to the $\imath$-terms, ε-terms are not "incomplete" symbols, that is they are not abbreviations for predicates, rather they are to be understood to be themselves terms.

In §8 of book one of the *Grundlagen*[47] on definite descriptions and their elimination Hilbert and Bernays describe what they call "an important logical notion, which is often used in common reasoning and is especially often used in mathematics".[48] Specifically this notation formalizes expressions in natural language where on might use the definite article. They give several examples such as "the greatest common divisor of 63 and 84" or the "highest mountaion in the alps". They note that all these examples are unique, *i.e.* they hold for one and only one object.[49] For this sort of definite description they follow Russell and Whitehead and represent as $\iota_x \mathfrak{A}(x)$ (HILBERT and BERNAYS, 1934, p.393).[50] But they say for a predicate $\mathfrak{A}(a)$ where the a represents just a place holder, the ι-term can be introduced if the predicate hold for one and only one object which they express with two formulae:[51]

$$(\exists x)\mathfrak{A}x$$

expression "logically proper name" rather he uses it only in a quotation of Strawson's critique of his view (RUSSELL, 1957, p.386).

[47]Entitled "Der Begiff 'derjenige, wlecher' und seine Eliminierbarkeit" (HILBERT and BERNAYS, 1934, p392), Claus-Peter Wirth trans: "The Notion 'that which' and its Eliminability" (HILBERT and BERNAYS, 2013), Elsenbroich trans: "Definite Descriptions and the Possibility of their Elimination" (HILBERT and BERNAYS, 2004), Gaillard and Guillaume trans: "La notion 'le, qui' et son éliminabilité" (HILBERT and BERNAYS, 2001).

[48]"Dennoch fehlt darin die Darstellung einer gewissen logischen Begriffsbildung, welche sowohl im alltäglichen Denken wie insbesondere in der Mathematik viel gebraucht wird" (HILBERT and BERNAYS, 1934) translation from (HILBERT and BERNAYS, 2004).

[49]"Hier wird jedesmal ein Gegenstand dadurch charakterisiert, daßein bestimmtes Prädikat auf ihn und auf ihn allein zutrifft. Im Bereich der von uns betrachteten Aussagen stellt sich ein solches" (HILBERT and BERNAYS, 1934, p.392).

[50]Hilbert and Bernays do not rotate the iota as Russell does.

[51]I have used modern symbols for the quantifier below to improve readability.

$$(\forall x)(\forall y)(\mathfrak{A}x \wedge \mathfrak{A}y \rightarrow x = y)$$

These Hilbert and Bernays refer to as the "*Unitäformen*" (*i.e.* uniqueness formulae) belonging to the predicate $\mathfrak{A}$. The ι-term then $\iota_x \mathfrak{A}(x)$ represents "the object x for which $\mathfrak{A}$ holds"[52] (Hilbert and Bernays, 2004). Hilbert and Bernays insist that the uniqueness formulae for $\iota_x \mathfrak{A}(x)$ must already be derived to introduce the ι-term, because they note that Russell and Whitehead parse the formulae $\mathfrak{B}(\iota_x \mathfrak{A})(x)$ as asserting that "There exists a single object for which $\mathfrak{A}(a)$ holds and for which $\mathfrak{B}(a)$ holds as well"[53] (Hilbert and Bernays, 2004).

They then define what they refer to as the ι-rule: For every formula $A(x)$ the expression $\iota x A(x)$ is a term if the following derivation can be made:

$$\frac{\begin{array}{c}(\exists x)A(x)\\ (\forall x)(\forall y)(A(x) \wedge A(y) \rightarrow x = y)\end{array}}{A(\iota x A(x))}.$$

The rules for *renaming variables* holds for term forming operators like the ι-symbol in the same manner as for variables bound by quantifiers (Hilbert and Bernays, 1934, pp.393-394).

The ε symbol as defined by Hilbert subsumes the ι-symbol when they both are in the language. In fact one can expect it to "absorb the weaker ι-symbol whenever that symbol can be used; and such is indeed the case" (Kneebone, 1963, p.102)

> If, for some formula $\mathfrak{A}$, the uniqueness formulae
>
> (2.1) $$(\exists x)\mathfrak{A}(x)$$
>
> and
>
> (2.2) $$(\forall x)(\forall y)[\mathfrak{A}x \ \& \ \mathfrak{A}y \rightarrow x = y]$$
>
> are both derivable, we are able by the ι-schema to derive the formula
>
> (2.3) $$\mathfrak{A}(\iota_x \mathfrak{A}(x)).$$
>
> But since, by substitution in the ε-formula, we have
>
> $$\mathfrak{A}(\iota_x \mathfrak{A}(x)) \rightarrow \mathfrak{A}(\varepsilon_x \mathfrak{A}(x)),$$

[52] "dasjenige Ding x, für welches $\mathfrak{A}(x)$ besteht" (Hilbert and Bernays, 1934, 393).

[53] "Es gibt ein einziges Ding, auf welches $\mathfrak{A}(a)$ zutrifft, und auf dieses trifft auch $\mathfrak{B}(a)$ zu" (Hilbert and Bernays, 1934, p.393).

the derived formula (2.3) at once yields the formula

$$\mathfrak{A}(\varepsilon_x \mathfrak{A}(x)). \tag{2.4}$$

From (2.2), (2.3), and (2.4) we now easily derive the formula

$$\varepsilon_x \mathfrak{A}(x) = \iota_x \mathfrak{A}(x)$$

and thus the ε-term is demonstrably identical with the ι-term whenever the latter may be introduced (Kneebone, 1963, p.101-102).

This does not mean that the epsilon term is equivalent, despite Kneebone's confusing phrasing. When then ι-term can be introduced, that is when both of what Hilbert refers to as the "uniqueness formulae" (*Unitätsformeln*) are premises.

Hilbert and Bernays do not proceed directly to the ε-term from ι-terms. They develop an intermediate term η-term:

> The modified ι-rule we use the letter η to distinguish it from ι reads as follows: Let a formula $(\exists \mathfrak{v})\mathfrak{A}(\mathfrak{v})$ be either derivable or an axiom. Then the term $\eta\mathfrak{v}\mathfrak{A}(\mathfrak{v})$ can be introduced and the formula
>
> $$\mathfrak{A}(\eta\mathfrak{v}\mathfrak{A}(\mathfrak{v}))$$
>
> can be used as an axiom. (Hilbert and Bernays, 1939, p.10)

> According to the η-rule, the term $\eta\mathfrak{v}\mathfrak{A}(\mathfrak{v})$, in correspondence with the term $\iota\mathfrak{v}\mathfrak{A}(\mathfrak{v})$, is still dependent on the derivability of one formula, i.e. the formula $(\exists \mathfrak{v})\mathfrak{A}(\mathfrak{v})$. We can now lift this restriction as follows. (Hilbert and Bernays, 1939, p.11)

> Let us assume the derivable formula
>
> $$\neg((\exists y)A(y)) \vee (\exists x)A(x),$$
>
> from which we obtain the formula
>
> $$(\exists x((\exists y A(y) \rightarrow A(x)),$$
>
> by transformations of the predicate calculus. By the derivability of this formula and according to the η -rule we can introduce the term
>
> $$\eta x((\exists y)A(y) \rightarrow A(x)),$$

and take the formula

$$(\exists y)A(y) \to \eta x((\exists y)A(y) \to A(x)),$$

as an axiom. Let us now define the ε-symbol explicitly by the equality

$$\varepsilon x A(x) = \eta x((\exists y)A(y) \to A(x)).$$

Then we obtain the formula

$$(\exists x)A(x) \to A(\varepsilon x A(x)),$$

(Hilbert and Bernays, 1939, p.11)

They continue by noting that if the we rename the y variable in the antecedant of $(\exists y)A(y) \to \eta x((\exists y)A(y) \to A(x))$,

> if the variable y is renamed into x in the antecedent. The application of a η-symbol becomes superfluous by this formula, in which we can rename the bound variable in the antecedent as well as in the consequent into any other variable, as, if the formula
>
> $$(\exists \mathfrak{v})\mathfrak{A}(\mathfrak{v})$$
>
> is derivable, we can immediately obtain
>
> $$A(\varepsilon \mathfrak{v} \mathfrak{A}(\mathfrak{v}))$$
>
> by substitution and the inference schema.
>
> It is now natural to eliminate the η-rule completely and to introduce the symbol $\varepsilon_{\mathfrak{v}} A(v)$ as a basic symbol together with the formula
>
> $$(\varepsilon_0)(\exists x)A(x) \to A(\varepsilon x A(x))$$
>
> into the formalism (Hilbert and Bernays, 1939, pp.11-12)

2.8.3 Hilbert's Epsilon Theorems

Hilbert's use of ε operators was not focused on the use of ideal elements themselves but rather the derivation of finitist consistency proofs. The epsilon terms themselves are to be removed in the final results. The first and second epsilon theorems were described by Hilbert and Bernays in the following manner:

These theorems refer to a formalism F, obtained from the predicate calculus on addition of the ε-symbol and some additional object, predicate and function symbols and on extension of the axioms by the ε-formula plus some basic axioms not containing ε-symbols $\mathfrak{P}, ..., \mathfrak{P}\mathfrak{k}$. Now the following two theorems mean the following for this formalism F :

> 1. Let $\mathfrak{E}$ be a formula derivable from F , not containing bound variables. Let the axioms $\mathfrak{P}, ..., \mathfrak{P}\mathfrak{k}$ not contain bound variables either. Then the formula E can be derived from the axioms $\mathfrak{P}, ..., \mathfrak{P}\mathfrak{k}$ not using bound variables, i.e. simply using the elementary calculus with free variables ('first ε-theorem').
>
> 2. Let E be a formula derivable from F , not containing the ε-symbol. Then this formula can be derived from the axioms $\mathfrak{P}, ..., \mathfrak{P}\mathfrak{k}$ only using the predicate calculus ('second ε-theorem').

(Hilbert and Bernays, 1939, p.18).[54]

Hilbert's classical epsilon theorems form the basis for the proof of Herbrand's theorem which can be stated in the following manner:

If an existential formula

$$\exists x_1 ... \exists x_k A(x_1, ..., x_k)$$

is derivable in first-order predicate logic (without equality), where A is quantifier-free, then there are sequences of terms $t_1^1, ..., t_k^1, ..., t_1^n, ..., t_k^n$ such that:

$$A(t_1^1, ..., t_k^1) \vee ... \vee A(t_1^n, ..., t_k^n)$$

is a tautology (Avigad and Zach, 2013).

[54] Avigad and Zack describe the first epsilon theorem thusly:

> First epsilon theorem: Suppose $\Gamma \cup \{A\}$ is a set of quantifier-free formulae not involving the epsilon symbol. If A is derivable from Γ in the epsilon calculus, then A is derivable from Γ in quantifier-free predicate logic

and the second in the following manner:

> Second epsilon theorem: Suppose $\Gamma \cup \{A\}$ is a set of formulae not involving the epsilon symbol. If A is derivable from Γ in the epsilon calculus, then A is derivable from Γ in predicate logic (Avigad and Zach, 2013).

Hilbert and Bernays used the ideal elements in his proof of Herbrand's theorem, but it should be noted that the ideal elements and quantified statements do not appear in the conclusion. In fact Hilbert's first two epsilon theorems actually prove that if something can be proved in the epsilon calculus then they are derivable in a quantifier free predicate logic (Avigad and Zach, 2013).

Classical epsilon calculus is conservative over classical logic as these theorems attest. However, as we will demonstrate in Chapter 4, in the intuitionistic case the addition of the epsilon operator enables one to derive results one cannot in intuitionistic logic without epsilon. The logics one can derive which are weaker than classical logic but stronger than intuitionistic logics are often termed superintuitionistic or intermediate logics.

Chapter 3

Intuitionistism, Intuitionistic Logic and Anti-Realism

3.1 Introduction

The proofs in the next chapter involve adding Hilbert's epsilon operator to first order intuitionisitic logic. We encountered epsilon in the last chapter as part of our investigation of Hilbert's efforts to put mathematics on firm finitist foundations. In addition, we have also encountered intuitionism with relation to and as a movement critical of Hilbert's programme. What we are trying to do in this chapter then is to draw metaphysical lessons from what may merely seem to be results having to do with formal mathematical systems. The claim rests on there being at least something persuasive about the Dummettian view of the relationship between logic and metaphysics—roughly, that intuitionistic logic is logic *per se*, and the acceptance of classical logic is legitimate only in the case one accepts realism. The goals of this chapter are, essentially, three: to make clear what that claim amounts to; and to explain Dummett's grounds for saying it, so that the claim has at least a *prima facie* claim to plausibility; and to present some of the material that will be important to set up the eventual case for the philosophical importance of the formal results to follow.

Dummett has argued that the great insight of intuitionism is the connection between logic and metaphysics, noting that while many views rejected realism about a diverse set of subject-matters,[1] it was Brouwer who first understood that this had implications for

[1]Dummett notes that there are:

> ...several philosophical doctrines that reject realist views of various subject-matters: phenomenalism rejects realism about the material universe, behaviourism rejects realism about the mental, instrumentalism rejects realism about scientific theories. But, historically, these have all concentrated on what constitute the objects of the sphere within which they oppose

reasoning. As Dummett puts it:

> ...[Brouwer's] genius enabled him to perceive, what the opponents of other varieties of realism had failed to realise, that the improved conception must be applied, first of all, to our understanding of propositions about the subject-matter in question, in Brouwer's case to our understanding of mathematical propositions. Only when this is done does it become apparent that the new non-realist conception demands a change in the logic with which we operate (DUMMETT, 2008, pp.341-342).

Though both Hilbert's programme and intuitionism to some degree attempt to adhere to constructivist principles, differently understood, the ontological cleavage between these two major twentieth century developments can be understood with regard to their differing goals. Hilbert is attempting to derive, from principles he deems constructive, classical principles, which are needed to allow for the infinitely ranging existence proofs that he felt were necessary to mathematical practice. In fact, the epsilon operator too is justified in terms of the mathematical practice of choosing an ideal or arbitrary object to reason about, thereby enabling one to reason about an entire class of objects (see HILBERT, 1926). Intuitionistic logic, and mathematics, on the other hand, is developed in almost the opposite manner. It is defined not by an end but rather by its philosophical starting point. Intuitionsitic logic hence is an attempt to define a logic that does not offend its philosophical principles.

The contrast is thus between a movement that attempts to recover "seemingly" non-constructive principles from arguably constructive principles, while the other is delimiting the class of logically correct principles by not admitting those principles from which non-constructive principles can be proven. From a strictly intuitionisitic viewpoint, if the addition of a principle P allowed the deduction of a non-constructive principle Q, then that principle P would be considered non-constructive. Hibert's method, on the other hand, is to attempt to assert that a "seemingly" non-constructive principle Q is constructive because it can be proven by the addition of some principle R which we have convinced ourselves is constructive.

Michael Dummett argues that the intuitionist's attention to logical hygiene give intuitionistic logic a special status in that it is metaphysically neutral. Specifically, he argues that the introduction and elimination rules of intuitionistic logic make clear that it is truth preserving without smuggling in realist metaphysical assumptions. The view that there is a metaphysically neutral logic is, of course, disputed by many. For example, Timothy Williamson, in his 2013 book *Modal Logic as Metaphysics*, writes that "all major logical

realism (DUMMETT, 2008, p.341).

principles have been rejected on metaphysical grounds." For example, he notes how arguments about the unreality of time, including those of Dummett, have been used to reject the law of excluded middle, how Graham Priest's arguments about true contradictions have been used to reject explosion, and how it has been argued that the sorites paradox is made true simply by the "structural principle that chaining together valid arguments yields another valid argument". Williamson then asserts simply that each "deviant metaphysics corresponds to the deviant logic", and since this is the case, it follows simply that:

> Any logical principle has persuasive force in some dialectical contexts and not in others. Logic has no metaphysically neutral core. Like any other science, its findings are open to legitimate challenge, even when the challenges are in fact mistaken (WILLIAMSON, 2013, p.146).

He anticipates the obvious objection that such a definition of logic leaves it open to such a relativism and asserts that while:

> ...some readers may prefer to use the word 'logic' differently. They can rephrase the conclusions of this chapter by using another word. But whatever advantages may accrue to their way of using 'logic', they will not include isolating some claims that are in principle metaphysically uncontroversial. There are none (WILLIAMSON, 2013, pp.146-147).

Another objection sometimes levelled at intuitionism is one that, for instance, TROELSTRA (2003, p.14) levels at the so-called BHK interpretation,[2], but that applies to constructive mathematics and logic more generally, that is that the "abstract notion of a constructive proof and construction are used as primitives" and are hence unanalysed. The analysis of what a constructive proof is, however, is key to Dummett's arguments as we shall see below.

Rather than trying to respond to all these objections directly, I will simply note that, like most worthwhile things in philosophy, Dummett's view is not without critics. I will instead try to explain why Dummett's views, especially his claim that intuitionistic logic is 'metaphysically neutral' while stronger logics are not, are plausible. To the extent that it has plausibility, the results discussed in later chapters constitute a useful filling in of details about the relationship between logic and metaphysics.

Despite these *caveats*, that the abstract notion of construction is assumed, and that one can always generate a deviant metaphysics that can call into question any logical

[2]Brouwer–Heyting–Kolomogorov

system, Dummett's arguments still merit special attention.[3] Dummett's arguments point to a connection between realism about some entity or phenomena, say time or numbers, and the acceptance of the law of excluded middle with regard to the context in which this realism is accepted.[4] Since this is the direction in which choice terms strengthen intuitionisitic logic, it is by examining these arguments that we should most likely find worthwhile answers.

This chapter will proceed as follows. We begin with a sketch of the history of intuitionism. In section 3.2, we will discuss what we might call intuitionism's pre-history, before it took shape, in the hands of LEJ Brouwer and others in the 20th Century, in the sense the term has that gives rise to the phrase "intuitionistic logic". Moving from there to consider the nearer origins of how this logic was introduced. In section 3.3 we will briefly discuss Brouwer's version of intuitionism, and contrast it to its key rival in the foundations of mathematics in the early 20th century, namely Hilbert's formalism. This discussion will make clear Brouwer's distinctive contribution: a compelling argument that one's metaphysical views are linked in a fundamental way to which principles of reasoning one ought to accept as correct.

At this point, it is a good idea to give some more detail about the outline of the chapter, section 3.4 will be a sort of formal interlude. The main items of business will be presentations of two different formalizations of intuitionistic logic, one axiomatic and one as a system of natural deduction. Each system will be important for subsequent discussion. We will also briefly introduce some basic ideas of intermediate logic, since in later chapters we will be considering proofs that demonstrate how adding choice principles to intuitionistic logic convert it into one or another intermediate logic.

In section 3.5, I consider the use that Dummett makes of the insights he reads out of the work of traditional intuitionists like Brouwer. One might regard Dummett's view as "Brouwer meets Wittgenstein," as Dummett links his insight on the connection between logic and metaphysics with the idea that meaning is use. "Use" of a linguistic item, in Dummett's hands and stated in a rough and ready way, is to be spelled out in terms of what information is needed to legitimately infer statements involving the expression, and in terms of what can be inferred from such statements.

[3] This is not to say that many of these divergent metaphysics could not be re-worked or re-worded in some manner as to make the intuitionist-to-classical logic difference the important one for describing them in relation to some other metaphysical system.

[4] Though Dummett's general line of argument is similar in these various cases, he did not assert that anti-realism is globally the case:

> I saw the matter, rather, as the posing of a question how far, and in what contexts, a certain generic line of argument could be pushed, where the answers 'No distance at all' and 'In no context at all' could not be credibly entertained, and the answers 'To the bitter end' and 'In all conceivable contexts' were almost as unlikely to be right (Dummett, 1993b, p.464).

In section 3.6, I will look at how this general view applies to the specific case of logical vocabulary. This is the lynchpin of Dummett's argument that intuitionistic logic is metaphysically neutral (because the relevant introduction and elimination rules are "harmonious"), while stronger systems are not.

Finally, in section 3.7 I briefly introduce some attempts to discuss "harmonious" versions of the epsilon principle one finds in the literature, mostly as something that it will be useful to contrast the original, "non-harmonious", version of epsilon to be investigated in later chapters.

3.2 Defining Intuitionism

Broadly stated intuitionism traditionally asserts that some faculty of the mind intuits certain knowledge, in the case of mathematical intuitionism, mathematical knowledge. Intuitionism thus is a form of constructivism; mathematical constructivists are those who believe one "determines the legitimacy of mathematical objects by whether they are cognitively graspable by our human faculties" (POSY, 1974, p.126). As Posy points out both as a constructivist and an intuitionist Brouwer shares much with Kant, since for Kant, mathematical objects are constructed in intuition (see KANT, 1724-1804, A735/B763).

However Brouwer's view does differ from Kant's. In fact, he first used the term "neo-intuitionism" to differentiate himself from earlier intuitionist views.[5] Brouwer likely used the term neo-intuitionist because he rejected the Kantian view that time and space were *a priori* (*cf.* BROUWER, 1913C, pp.78-80). Brouwer's position can be seen in relation to both neo-Kantian philosophy, and the positions of the French semi-intuitionists and earlier mathematical constructivists, such as Leopold Kronecker and other critics of Cantor's work on set theory, but intuitionism can be taken to mean several things. The philosophical motivations for Brouwer's intuitionism are not necessarily the only, or the best, motivations for accepting the mathematical and logical consequences of his view. It may be best to survey what is generally meant by philosophical intuitionism and then look at how that relates to Brouwer's views about mathematics and logic.

[5]Brouwer did not use the term intuitionist until 1911 (BROUWER, 1911A) referring to what we now know as the French pre-intuitionists Poincaré and Borel. He used the term neo-intuitionist in 1912 to refer to his own views. As Per Martin Löf writes "... it was not until the twenties that he took the shrewd step of calling his own conception intuitionism *tout court*, qualifying his predecessors instead as pre-intuitionists or the old-intuitionists" (MARTIN-LÖF, 2008, p.245). Heyting later referred to these views as semi-intuitionist (first in a chapter entitled "Die französischen Halbintuitionisten"/"Le Semi-intuitionnisme Français", HEYTING 1934, p.4, expanded and translated (to French) as HEYTING 1955, p.6).

3.2.1 Philosophical Intuitionism

'Intuitionism' is traditionally used, in philosophy, to describe a group of views that assert that there is an aspect of human consciousness as the source of knowledge. Hence an intuitionist view asserts that we do not simply have a faculty of discursive reason, but also an "act of direct apprehension" called intuition, as one of the foundations of knowledge (van Stigt, 1998, p.4).

The philosophical roots of intuitionism can be traced back through Kant and Descartes and some would say as far back as Aristotle (van Stigt, 1998, p.4). Intuitionism broadly considered is a philosophical doctrine, or rather intuitionistic theories are philosophical doctrines, that assert the primacy of individual consciousness or some aspect of individual consciousness as the origin or source of knowledge (see van Stigt 1998, p4, Largeault 1993, p.8).

The origin of the distinction between intuiting and reasoning is not clear. Walter van Stigt asserts that "elements of intuitionism" may be found in Aristotle's concept of *νοῦς* (van Stigt, 1998, p.4), (intellect) which is distinguishes from *λόγος*, (reason).[6] Jean Largeault asserts, on the other hand, that the best place to locate the origin of the distinction between discursive thought and intuition is in Plotinus (Largeault, 1993, p.9).[7]

However it is with Descartes that the modern concept of intuition as the origin of all true knowledge is probably best identified. It is likely this Cartesian understanding of

[6]The passage he is most likely referring to is from the *Nicomachean Ethics* where Aritotle is comparing intelligence and the other intellectual virtues:

> For understanding [νοῦς] is about the (first) terms, (those) have not account of them; but intelligence [λόγος] is about the last thing, the object of perception, not scientific knowledge. This is not the perception of special objects, the sort by which we perceive that the last among mathematical objects is a triangle; for it will stop here too. This is another species (of perception of special objects); but it is still perception rather than intelligence" (Aristotle, 1985, 1142a, p. 161).

Original:

> ὁ μὲν γὰρ νοῦς τὼν ὅρων, ὧν οὐκ ἔστι λόγος, ἣ δὲ τοῦ ἐσξάτου,οὗ οὐκ ἔστιν ἐπιστήμη ἀλλ᾽ αἴσθησις, οὐξ ἡ τῶν ἰδίων, ἀλλ᾽ οἵᾳ αἰσθανόνεθα ὅτι τὸ ἐν τοϊς μαθηματικοϊς ἔσξατον τρίγωον: στήσεται γὰρ κἀκεῖ. ἀλλ῾ αὕτη μᾶλλον αἴσθησις, ἐκείνης δ᾽ ἄλλο εἶδος (Aristotle, 1894, 1142a).

[7]Plotinus made a "distinction between 'universal' Intellect and particular intellects" (Gerson, 1994) "For when it [Intellect] is active in itself, the products of its activity are the other intellects" Φυξῆς δὲ ἐνεργούσης ὡς γένους ἢ εἴδους αἱ ἀλλαι φυξαὶ ὡς εἰδη. (Enneads VI.2.22.267). The universal being that which apprehended the Platonic forms and which is hence shared"among all things". The particular intellects νοῖ or νοἕς which include among others "immediate consciousness" συναὶσθησις (Gerson, 1994, p.45).

intuition is the origin of the modern philosophical intuitionism, giving rise to a philosophical tradition that eventually influenced the development of mathematical intuitionism of both the French semi-intuitionists and Brouwer. Recall that for Descartes intuition is the sole origin of knowledge:

> It is an indubitable conception formed by an unclouded mental mind; one that originates solely from the light of reason, and is more certain even than deduction, because it is simpler ... Thus, anybody can see by mental intuition that he himself exists, that he thinks, that a triangle is bounded by just three lines, and a globe by a single surface, and so on; there are far more of such truths than most people observe, because they disdain to turn their mind to such easy topics.

More complex things, on the other hand:

> are known although not self-evident, so long as they are deduced from principles known to be true by a continuous and uninterrupted movement of thought, with clear intuition of each point (DESCARTES, 1954, pp.10-12)

Brouwer, in his 1909 inaugural address (BROUWER, 1909A) as a *privatedocent* on the foundations of geometry and topology, discussed his view on the intuition of time and *two-in-one-ness* and traced the idea of the *a priori* nature of the intuition of time back to Descartes.

As we noted in the last chapter, Hilbert too believed in the pure intuition of mathematical reasoning, even though he criticised justifications of mathematics based on an "ur-intuition like Brouwer" (HILBERT, 1931, p.1156). In fact, he continued to believed that the foundation of mathematics was to be grounded in a Kantian inspired "*a priori* intuitive mode of thought" (HILBERT, 1931, pp.1149–1150).[8]

3.3 Brouwer's intuitionism

Intuitionism, specifically mathematical intuitionism, is most often connected with Brouwer, but as a philosophical position it did not simply spring into existence in 1907 when Brouwer wrote his Ph.D. dissertation (BROUWER, 1907).[9]

[8]It should also be noted that, as part of his PhD public promotion exercise in a formal debate against two fellow mathematics students, Hilbert defended the proposition: "That the objections to Kant's theory of the *a priori* nature of arithmetical judgement are unfounded" (REID, 1986, p.17).

[9]Nor in 1911 when Brouwer first used the term intuitionism in his review of Manoury's book on elementary mathematics (BROUWER, 1911A).

Mathematical intuitionism, though philosophically motivated, is not just a philosophical position, but has implications for the actual practice of mathematics. The methods to an intuitionist are constructive, but of course while a Brouwerian intuitionist is a constructivist of a sort, not all constructivists are Brouwerian intuitionists. Not only are there other types of constructivists in terms of degree, but also in kind. In addition Michael Dummett and others have argued that one need not accept Brouwer's philosophical intuitionism to agree that intuitionist methods in logic and mathematics are correct. Dummett first pointed out in Dummett (1975) that any philosophical position that opposes a realist metaphysics, not only about philosophy of mathematics, but about any subject matter, requires that we use a different reasoning criteria than classical logic.

In short, it is true, as Carl Posy writes, that "Kant's philosophy of mathematics is [Brouwer's intuitionism's] forerunner" (Posy, 1974, p.132). But Brouwer's intuitionism is an important turning point, because it was Brouwer who gave clearly articulated arguments for the rejection of certain principles of classical logic, the law of excluded middle in particular, because they were incompatible with his metaphysical assumptions.

That is not to say that earlier constructivists did not reject certain proof methods. Some of the methods employed by Georg Cantor in the development of his set theory were criticised by Leopold Kronecker (see Dauben 1979, p.66 and also Troelstra 1991, pp.1-2) and Henri Poincaré (Dauben, 1979, p.266). In fact through the 1870s and the 1880s, Kronecker became a more and more strident opponent of methods of which he disapproved (see Dauben 1979, p.66 and Mittag-Leffler 1900, pp.150-151, and for the reception of these views see Hilbert 1905, and Hilbert 1920). But it was Brouwer who first argued, in print, that a change in our metaphysical views could be directly related to the need to change our logic, and was the first to express why this was in regard to his intuitionism. Brouwer's metaphysical views were for him a starting point and he allowed them to lead him to a decision as to what methods were in line with his philosophy of mathematics.

3.3.1 Intuitionism and Formalism

Brouwer's thesis Brouwer (1907) was the first place we see him confronting Hilbert's programme. At this stage Brouwer focused his criticism on Hilbert's optimistic "solvability conviction" that every mathematical problem has a solution, which is not surprising given "that the only items [written by Hilbert] that were available to him were the Heidelberg lecture of 1904 and the Mathematical Problems paper from 1900" (Martin-Löf, 2008, p.245). Then in 1908 in "The unreliability of the logical principles" Brouwer turns this into the criticism of the law of excluded middle asserting that:

> The question of the validity of the *principium tertii exclusi* is equivalent to the question *whether unsolvable mathematical problems can exist.* There is not a

> shred of proof for this conviction, which has sometimes been put forward that there exists no unsolvable mathematical problem (Brouwer, 1908C, p.109).

Troelstra writes that Brouwer's thesis 'Over de Grondslagen der Wiskunde' (Brouwer, 1907) defended a position "more radically and more consistently than the semi-intuitionists" though Troelstra asserts that the essentials of Brouwer's philosophical position were already to be found in his 1905 paper (Brouwer, 1905A). Brouwer's philosophical views did not change much throughout his life. Troelstra summarizes the main points of his philosophical views based on Brouwer (1949C) as follows:

> Brouwer's main ideas are:
>
> 1. Mathematics is not formal; the objects of mathematics are mental constructions in the mind of the (ideal) mathematician. Only the thought constructions of the (idealized) mathematician are exact.
> 2. Mathematics is independent of experience in the outside world, and mathematics is in principle also independent of language. Communication by language may serve to suggest similar thought constructions to others, but there is no guarantee that these other constructions are the same. (This is a solipsistic element in Brouwer's philosophy.)
> 3. Mathematics does not depend on logic; on the contrary, logic is part of mathematics (Troelstra, 1991, p.8).

Brouwer's programme was based on these principles, though Brouwer did not at first pursue intuitionisitic mathematics, rather he worked on classical topology from the period between 1907 and 1913. Troelstra notes that:

> In these years his view of the continuum and of countable sets is quite similar to Borel's position on these matters. Thus he writes:
>
> > The continuum as a whole was intuitively given to us; a construction of the continuum, an act which would create "all" its parts as individualized by the mathematical intuition is unthinkable and impossible. The mathematical intuition is not capable of creating other than countable quantities in an individualized way (Brouwer, 1907, p.62, cf. also p.10).
>
> (Troelstra, 1991, pp.8-9)

However Brouwer had already started to differentiate his views from the French semi-intuitionists: Brouwer's view was that the natural numbers and the continuum were both aspects of the same "primeval intuition" a view which differed from Borel's "pragmatic intersubjectivism" (Troelstra, 1991, p.9). Troelstra notes that almost immediately after completing his thesis Brouwer had concluded that classical logic did not apply to mathematics and until around 1913 Brouwer "did not publicly dissociate himself from [the French semi-intuitionists]" (Troelstra, 1991, p.9). But it was not until after 1912 when he obtained his professorship at the University of Amsterdam, that he returned to the foundations of mathematics and developed his own views in earnest.

3.3.2 Hilbert and Brouwer

As van Dalen writes, perhaps overstating the case somewhat for rhetorical effect:

> Nowadays the names of Brouwer and Hilbert are automatically associated as the chief antagonists in the most prominent conflict in the mathematical world of this century, the notorious *Grundlagenstreit* (van Dalen, 1990, p.18)

Well, perhaps, but as noted in the previous chapter Hilbert does not mention Brouwer's foundational position until (Hilbert, 1923), after his former student Weyl for a time converted to Brouwer's intuitionism. It was only after seeing his prized former student change sides, that Hilbert criticised Brouwer more directly. Indeed, at first Hilbert and Brouwer were well disposed to one another, with Hilbert writing him a letter of recommendation for the chair at Amsterdam, and even offering Brouwer a chair at Göttingen. However, as van Dalen writes:

> Gradually the scientific differences between the two adversaries turned into a personal animosity. The *Grundlagenstreit* is in part the collision of two strong characters, both convinced that they were under a personal obligation to save mathematics from destruction (van Dalen, 1990, p.19).

The disagreement between Brouwer and Hilbert finally became well known because of a conflict over the journal *Mathematische Annalen* wherein Hilbert, to remove Brouwer from the editorial board, convinced Springer to dissolve the entire board and to reconstitute the journal with Hilbert at its helm (for details see van Dalen, 1990).[10]

[10] Since the *Mathematische Annalen* was, at the time, the pre-eminent mathematical journal, this conflict ended up involving many of the world's major mathematicians, most of whom did not return to the editorial board after the journal's reforming. This included the physicist Albert Einstein who, in private letters, wrote that the whole affair had the air of: "one of the most funny and successful farces performed by people who take themselves deadly seriously" (Einstein in letter of November 27, 1928 to Max Born, quoted in van Dalen, 1990, p.26). Einstein also referred to the affair as a "War of Mice and Frogs" (*Froschmäusekrieg*) after the pseudo-Homeric *Batrachomyomachia* a comic epic poem parodying the *Iliad*.

3.4 Intuitionistic Logic

It has been argued that intuitionisitic logic is one of the two logical systems that "stand out as having a solid philosophical-mathematical justification" *i.e.* for which, one can provide a good argument that they are "the" correct logic to describe the foundations of mathematics—for classical logic because it provides an "ontological basis" and for intuitionistic logic due to its "epistemic motivation" (van Dalen, 2002, p.1). Intuitionistic logic is technically a subsystem of classical logic, that is, all intuitionistic principles are classically correct, but it lacks certain principles such as the excluded middle (or *tertium non datur*) ($A \vee \neg A$). However, as van Dalen points out, while intuitionisitic logic is subsystem of classical logic:

> the matter changes, however, in higher-order logic and in mathematical theories. In those cases specific intuitionistic principles come into play, e.g. in the theory of choice sequences the meaning of the prefix $\forall\xi\exists x$ derives from the nature of the mathematical objects concerned (van Dalen, 2002, p.1).

So while intuitionistic logic can be seen as a subsystem of classical logic, the same principles lead, in number theory to intuitionistic theories that seem to contradict classical theories. In Troelstra's words:

> choice sequences, when taken seriously as mathematical objects, enforced the use of intuitionistic logic, since some of the principles valid for choice sequences contradicted classical logic (Troelsta, 1998, p.199).

A choice sequence is constructed in one of two manners, either by the application of a rule, a lawlike sequence, or in some random manner, a lawless sequence, *e.g.* the roll of a die.

> A lawless sequence α is thought of as a process of choosing values $\alpha(0), \alpha(1), \alpha(2), \alpha(3), ...$ such that:
>
> (a) at any moment only an initial segment is known, and no restrictions are imposed on future choices,
>
> (b) initially one may specify that α starts with an initial segment $(\alpha(0), \alpha(1), ..., \alpha(n))$. (Troelstra, 1983, p. 208)

Since the choice is free, such constructions describe dynamic rather than static states of information. Hence we should not be surprised that such cases require intuitionistic logic to describe them correctly. That is, the law of excluded middle fails for lawless sequences because there is no fact of the matter as to what the next segment of a lawless sequence is until it is chosen.

So while it is true in one sense that intuitionistic logic is a subset of classical logic, that is, that intuitionistic logic uses a subset of the axioms that classical logic uses, because it has fewer restrictions (*i.e.* axioms) intuitionistic logic has more models than classical logic. And because intuitionistic logic is valid in both intuitionist and classical models, it is the classical models which are a subset of intuitionistic models. So which is a subset of the other is not as clear cut as those who make this rhetorical point may wish it to be.

3.4.1 Formalizations of intuitionistic logic

As is well known Arend Heyting, Brouwer's student, axiomatized and formalized intuitionistic logic in (Heyting 1930a,b,c and Heyting 1931) and then more completely in (Heyting, 1934). Much of Heyting's work, however, had been anticipated by Andrei Kolmogorov (Kolomogorov, 1925), who explain intuitionistic logic in terms of "problems" and "solutions" rather than proofs.

As Per Martin Löf points out, Kolomogorov felt that he had provided "an objective verdict in this controversy" between Brouwer and Hilbert (Martin-Löf, 2008, p.248). By ignoring Brouwer's dislike of language and logic, Kolomogorov simply removed all the axioms of classical logic, the formal system he labels $\mathfrak{H}$ for Hilbert, which seemed non-constructive to him and retained the ones he felt were justified for an intutionistic logic, labelled $\mathfrak{B}$ for Brouwer (Kolomogorov, 1925, p.422).[11]

As noted above, aside from terminology the main difference between Heyting's and Kolomogorov's interpretations of intuitionistic logic was the acceptance of explosion. The similarity of the two versions indicates why this understanding of proof as primary is often refereed to as the Brouwer-Heyting-Kolmogorov (BHK) interpretation.[12] Troelstra explains:

> Where classical semantics describes how the truth-value of a logically compound statement is determined by the truth-values of its components, the BHK-interpretation describes what it means to prove a logically compound statement in terms of what it means to prove the components. In this explanation, "constructive proof," and "constructive method" appear as primitive

[11]Kolomogorov's logic is often referred to as minimal logic because in addition to removing the law of excluded middle $\vdash \varphi \wedge \neg\varphi$ he also removed the principle of explosion: $\varphi \wedge \neg\varphi \vdash \psi$, traditionally known as *ex falso quodlibet* or *ex contradictione sequitur quodlibet*

[12]The BHK-interpretation can also be formalized in Kleene's realizability theory or Per Martin Löf's typed lambda calculus, also known as the Curry-Howard isomorphism.

> notions. Kolmogorov interpreted statements as problems, and the solving of the problem associated with a logically compound statement is explained in terms of what it means to solve the problems represented by the components. The connection with Heyting's formulation becomes clear if we think of "solving the problem associated with statement A" as "proving A" (Troelstra and van Ulsen, 1999, p.2)

The following table compares several interpretations of the propositional meaning for intuitionistic systems. Each of these "interpretations" describes the semantic value of a proposition in a different manner, either in terms of the proof of that proposition or in terms of the construction of that proposition.

Interpretation	$a \in A$ **means**:	**A true**
Gentzen	a is a proof of proposition A	A is true
Heyting	a fulfils the expectation A	A is fulfilled
Kolmogorov	a is a solution to problem A	A has a solution
Martin-Löf	a is an element of A	A has an element

(table from, Ranta 1994, p. 40)

This list is by no means exhaustive. For instance we could add Kleene's interpretation of intuitionistic logic in terms of realizability, which is understood by a mapping of the natural numbers onto an Heyting arithmetic (HA), such that if a sentence is true in HA there is some natural number which is mapped onto that sentence that realises it.

Note that the term "interpretation" above does not denote what is normally meant in logic as the assignment of meaning to the symbols of a formal language by the definition of a domain and the mappings of functions and predicate symbols onto that domain, but a more informal understanding of the meaning of the symbols and their semantic interpretations.

3.4.2 Axiom Systems for Intuitionistic Logic

The full axiomatization of what we now know as intuitionistic propositional and predicate logic were thus first presented in Heyting (1930a), and then in Gentzen (1935).[13] The following is taken from standard presentation of Hilbert style axioms for intuitionistic first-order predicate logic, or intuitionistic predicate calculus (IPC) presented by Kleene (1952, pp.81-82, see also p.101, and pp.441ff):

[13] In addition Kolomogorov (1925) and Gödel (1933) proved the equiconsistency of intuitionistic and classical theories, meaning that all classical theorems expressed in the classical predicate calculus are not false in intuitionistic logic.

Rules and Axiom Schemata for Propositional Calculus

(1a) $\varphi \to (\psi \to \varphi)$
(1b) $(\varphi \to \psi) \to ((\varphi \to (\psi \to \chi)) \to (\varphi \to \chi))$
(2) $\dfrac{\varphi \to \psi \quad \varphi}{\psi}$
(3) $\varphi \to (\psi \to \varphi \wedge \psi)$
(4a) $\varphi \wedge \psi \to \varphi$
(4b) $\varphi \wedge \psi \to \psi$
(5a) $\varphi \to \varphi \vee \psi$
(5b) $\psi \to \varphi \vee \psi$
(6) $(\varphi \to \chi) \to ((\psi \to \chi) \to (\varphi \vee \psi \to \chi))$
(7) $(\varphi \to \psi) \to ((\varphi \to \neg\psi) \to \neg\varphi)$
(8) $\neg\varphi \to (\varphi \to \psi)$

Rules and Axiom Schemata for Predicate Calculus

(9) $\dfrac{(\psi \to \varphi(x))}{(\psi \to \forall x\varphi(x))}$
(10) $\forall x\varphi(x) \to \varphi(t)$
(11) $\varphi(t) \to \exists x\varphi(x)$
(12) $\dfrac{(\varphi(x) \to \psi)}{(\exists x\varphi(x) \to \psi)}$

3.4.3 Introduction and Elimination Rules

Gerhard Gentzen introduced natural deduction (Gentzen, 1934, 1935) along with the sequent calculus, though Jaśkowski had already made the first attempt at providing a structural rules based form of natural deduction in 1926 (see Jaśkowski, 1934). Dag Prawitz (Prawitz, 1965) provided a full presentation of natural deduction and translated the proof of the *Hauptsatz*, or the normal form theorem, which Prawitz notes was first "established by Gentzen for the calculi of sequents" but for which he feels his proof using "natural deduction is in many ways simpler and more illuminating" (Prawitz, 1965, p.10).

Gentzen style natural deduction emphasises how such constructions take place by relying on inference rules rather than axioms. Such a system reflects the nature of constructive reasoning, instead of focusing on known truths, and applying one rule (*modus ponens*) there are rules for the introduction and elimination of each of the logical operators (see van Dalen, 2002, pp.10-11). Dummett's arguments about the nature of intuitionisitic logic, which we will discuss below in sections 3.5 and 3.6, turn on the nature of the harmony between introduction and elimination rules.

$\wedge$–Introduction and Elimination Rules

$$\frac{A \qquad B}{A \wedge B} \qquad\qquad \frac{A \wedge B}{A} \qquad\qquad \frac{A \wedge B}{B}$$

$\vee$–Introduction and Elimination Rules

$$\frac{B}{A \vee B} \qquad \frac{A}{A \vee B} \qquad \frac{A \vee B \quad \begin{matrix}[A] \\ \vdots \\ C\end{matrix} \quad \begin{matrix}[B] \\ \vdots \\ C\end{matrix}}{C}$$

$\rightarrow$–Introduction and Elimination Rules

$$\frac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \rightarrow B} \qquad \frac{A \rightarrow B \quad A}{B}$$

$\forall$–Introduction and Elimination Rules

$$\frac{\begin{matrix}[x \in A] \\ \vdots \\ B(x)\end{matrix}}{\forall x_{\in A} B(x)} \qquad \frac{(\forall x_{\in A}) B(x) \quad t \in A}{B(t)}$$

$\exists$–Introduction and Elimination Rules

$$\frac{t \in A \quad B(t)}{(\exists x_{\in A}) B(x)} \qquad \frac{(\exists x_{\in A}) B(x) \quad \begin{matrix}[x \in A] \quad [B(x)] \\ \vdots \\ C\end{matrix}}{C}$$

$\perp$–Elimination Rule

$$\frac{\perp}{\mathrm{A}}$$

Classical logic can be presented in a natural deduction system with the addition of a classical $\perp$ rule:

$$\frac{\begin{matrix}[\neg A] \\ \vdots \\ \bot\end{matrix}}{A}$$

Since $\neg A$ is an abbreviation for $A \rightarrow \bot$ the above gives us double negation (see PRAWITZ, 1965, pp.20-21).

Note that the universal and the existential quantifiers, as shown above, explicitly quantify over-defined domains [e.g. $(\exists \mathrm{x} \in \mathrm{A})\mathrm{B}(\mathrm{x})$], not all interpretations formalize the domains of the quantifiers and instead often employ some form of arbitrary object; we shall discuss debates over arbitrary objects in natural deduction in more detail in Chapter 6.

3.4.4 Intermediate Logics

Extensions to intuitionisitic logics are termed *superintuitionistic* logics, those between intuitionistic logic and classical logic are termed *intermediate* logics.[14]

Dan van Dalen in his introduction to intuitionistic logic notes that:

> Among these logics that deal with the familiar connectives and quantifiers two stand out as having a solid philosophical—mathematical justification. On the one hand there is a classical logic with its ontological basis and on the other hand intuitionistic logic with its epistemic motivation. The case for other logics is considerably weaker; although one may consider intermediate logics with more or less plausible principles from certain viewpoints none of them is accompanied by a comparably compelling philosophy (VAN DALEN, 2002, p.1).

While van Dalen is certainly correct that the vast majority of the infinite number of intermediate logics are philosophically uninteresting, there are several that do stand out as more interesting than others for various reasons. For our purposes, let us consider three: De Morgan logic as an example of a logic that includes all four of De Morgan's laws, and in which there is a more complete duality between conjunction and disjunction; Dummett-Gödel logic where the linearity axiom holds; and thirdly all of the logics between Dummett-Gödel logic and classical logic, not individually but taken as a whole.[15]

[14] We will use the terms interchangeably as we are not considering any inconsistent logics: "An intermediate propositional logic is the same as a consistent superintuitionistic logic" (GABBAY and MAKSIMOVA, 2011, p.103).

[15] These logics can be understood as:

What we will find in the next part of the thesis is that the addition of existence principles, specifically the ε-operator to intuitionistic logic enables us to prove DeMorgan's intuitionistically invalid law and Dummett's scheme — depending on the addition of certain decidability criteria. While this is not as obviously philosophically interesting as motivations for classical or intuitionistic logic, it does show that there is a path between the two involving the addition of ontological principles not as strong as the excluded middle. This is important to what follows, because the normal method of increasing the strength of logics is to add logical axioms. For example the DeMorgan's law that is not valid in intutionistic logic ($\neg(B \wedge C) \vdash \neg B \vee \neg C$) when added to intutionistic logic gives us a DeMorgan logic, (*aka* KC or Jankov's logic). De Morgan logics are weaker than Gödel–Dummett logic, (*aka* LC, G) for which Dummett's scheme, also known as the linearity axiom ($(B \to C) \vee (C \to B)$), holds. And there are in fact a countably infinite number of intermediate logics between Gödel–Dummett logic and classical logic.

Intermediate logics are more often studied from the point of view of modal logics. That is intermediate logics are often compared to their cognate modal logics. The McKinsey–Tarski translation theorem (McKinsey and Tarski, 1948) shows how S4 and S5 are equivalent to intuitionistic and classical logic respectively. Dummett and Lemmon further proved that DeMorgan's and Gödel–Dummett logics are respectively translatable to their modal versions S4.2 and S4.3.[16]

Gödel proves two points about intuitionistic logic in his short paper "On the intuitionistic propositional calculus" (Gödel, 1932): first that intuitionistic propositional logic is not equivalent to any many-valued logic; and second that:

> Infinitely many systems lie between H [of intuitionisitic predicate calculus] and the system A [of classical logic] of the ordinary propositional calculus, that is, there is a monotonically decreasing sequence of systems all of which include H as a subset and are included in A as subsets (Gödel, 1932, p.223).

Dummett extended this work describing the system LC, which is defined by adding the following axiom to intuitionistic logic:

$$\vdash (\varphi \to \psi) \vee (\psi \to \varphi)$$

the class of rooted linearly ordered Kripke models, or alternatively, as many-valued logics whose connectives are interpreted as functions over subsets of the real interval [0,1] Tiu (2011).

[16]S4.3 is formed by adding $\Box(\Box p \to \Box q) \vee \Box(\Box q \to \Box p)$ to S4. Under Dummett's translation T of the modal into the propositional language that is the translation of and is translatable into $(p \to q) \vee (q \to p)$. More generally, Dummett proves that $\vdash_{IL} \alpha$ iff $\vdash_{S4.3} T(\alpha)$, while S4.2 is formed by adding $\Box(\Diamond\Box p \to \Box \Diamond \Box p)$ or $\Box(\Diamond\Box \Diamond p \to \Box \Diamond p)$ to S4. The first of which is translatable into $\neg p \vee \neg\neg p$ and so $\vdash_{KC} \alpha$ iff $\vdash_{S4.2} T(\alpha)$ (Dummett and Lemmon, 1959)

Toshio UMEZAWA (1955, 1959a,b) coined the term "intermediate logics" to describe the class of logics between intuitionsitic and classical logic. He showed that the logics between classical logic and Dummett-Gödel logic can be enumerated in several ways. By extending the law of the excluded middle:

$$R_n = \alpha_1 \vee (\alpha_1 \to \alpha_2) \vee (\alpha_2 \to \alpha_3) \vee ... \vee (\alpha_{n-1} \to \alpha_n) \vee \neg\alpha_n \ (n \geq 2)$$
$$\text{such that } R_n \vdash R_{n+1} \text{ but } R_{n+1} \nvdash R_n$$

and providing a proof that $R_\omega \equiv (\varphi \to \psi) \vee (\psi \to \varphi)$ (UMEZAWA, 1955, pp.188-189); or by extending other versions of the excluded middle. Tsutomo Hosoi (HOSOI, 1966a,b, 1967a,b) surveys much of this material and proves a similar result using Peirce's law:

$$(P_1) = ((\alpha_1 \to \alpha_0) \to \alpha_1) \to \alpha_1 \qquad \text{(Peirce's Law)}$$
$$(P_{i+1}) = ((\alpha_{i+1} \to P_i) \to \alpha_{i+1}) \to \alpha_{i+1} \quad (i \leq 1)$$

in this case $P_\omega \equiv (\varphi \to \psi) \vee (\psi \to \varphi)$ (HOSOI, 1967a, pp.1-2).

These examples of intermediate logics, including Hosoi's modifications of Pierce's law, will feature in our discussion of algebraic semantics in later chapters. In particular to the question of "how many truth values are there?" Each of the P_i corresponds to a Heyting algebra in the sense that P_i is valid if the truth values are a linear ordering with $i + 1$ elements in it, but invalid if there are $i + 2$ linearly ordered truth values. That is, P_1, Pierce's law, is valid if the truth values form the traditional two valued Boolean algebra of classical logic, but it can fail to be true if there are three truth values. P_2 is valid if there are 3 truth values, but can fail if there are four. And so on.

This means that in a language where Dummett's scheme is true every pair of elements is ordered by implication. Recall one standard definition of implication on a Heyting algebra:

$$(y \to z) \equiv \bigvee \{x | x \wedge y \leq z\}$$

Hence "ordered by implication" does not mean that every truth value object is greater or equal than or less than or equal to every other element, simply that for any two elements ordered by implication the join ($\vee$) of the meets($\wedge$) of every antecedent is less than or equal to every consequence. This is why Dummett's scheme is valid, for example, not only in linear partial orderings but also in all Boolean algebras.

3.5 Dummett and Intuitionistic Logic

While Dummett is well known for extolling the virtues of constructivist approaches in mathematics and logic, and while he certainly owes important debts to Brouwer, it would be a mistake to think of him as an intuitionist of Brouwer's sort. Dummett writes that he is not interested in preforming an "exegesis of the writings of Brouwer or of Heyting" or studying how they argued for intuitionism. Instead for Dummett "the question is what forms of justification of intuitionistic mathematics will stand up," and more specifically this question can be narrowed down to what he referred to as, "the most fundamental feature of intuitionistic mathematics, its underlying logic" (Dummett, 1975, p.215).

Dummett asserts that the only good justification for the choice of intuitionistic logic over classical is one based on a question of meaning and the demand that the meaning of a statement be defined by the use of that statement and not our intuitions about what entities are real or not (Dummett 1975, p.5). In fact, in the 'The Philosophical Basis of Intuitionistic Logic' Dummett argues that only an approach "turning on the answers given to general questions in the theory of meaning" can justify an anti-realist, specifically an intuitionistic, interpretation of mathematical statements. He thus writes that, "the route to a defence of an intuitionistic interpretation of mathematical statements which begins from the ontological status of mathematical objects is closed" unless one is willing to accept "the most resolute scepticism concerning subjunctive conditionals." Specifically, to follow such a line to anti-realism, one must be willing to "deny that there exists any proposition which is now true about what the result of a computation which has not yet been performed would be if it were to be performed" (Dummett, 1975, p.247). Thus Dummett would view traditional philosophical intuitionism as näive, or in his own words "hard headed".

How might general considerations about meaning lead to anti-realism? Dummett's view is that to know the meaning of a statement is to know its role in language. Indeed it follows that we understand a mathematical calculation or construction when we can calculate it or follow the proof of it. Dummett describes this process as "learning the use" and thus as the "learning of the meaning of a mathematical language":

> When we learn a mathematical notation, or mathematical expressions, or, more generally, the language of a mathematical theory, what we learn to do is to make use of the statements of that language: we learn when they may be established by computation, and how to carry out the relevant computations, we learn from what they may be inferred and what may be inferred from them, that is what role they play in mathematical proofs... (Dummett 1975, p.7).

Worth mentioning here that Dummett himself identifies the first aspect with a version of verificationism, and the second with a kind of pragmatism.

Dummett writes further that use has two aspects. The first aspect is the conditions for the use of a statement in a language, which he identifies with a version of verificationism, and the second aspect is the consequences of the use of a statement which he identifies with a kind of pragmatism (DUMMETT 1975, p.11). Of course, the plausibility of such a view is called into question if what is required to know the meaning of a term is every possible inferential relationship it might have to any other bit of the language. Dummett therefore defends a "molecularist" view under which the grasp of a bit of language involves mastery of a small, "canonical" subset of its uses (DUMMETT, 1975, p.12). His view therefore contrasts with earlier non-realist views like idealism which were holist about language (such as F.H. Bradley).[17] This conception of language does put restrictions on the two aspects of use:

> If a linguistic system as a whole is to be coherent there must be harmony between these two aspects: [for instance] it must not be possible to deduce observation statements from which the perceptual stimuli require dissent (DUMMETT 1975, pp.10-11).

Dummett goes on to argue that these two key aspects of use of logical words are reflected in the introduction- and elimination-rules of intuitionistic logic. These rules preserve the harmony that is required in the two aspects of language use, and thus, according to Dummett, intuitionistic logic correctly models use as a primary to meaning. Classical logic, on the other hand, is consistent but not harmonious, because in classical logic one can infer a disjunctive statement by means of double negation elimination without ever calling upon the disjunctive introduction rule (DUMMETT 1975, pp.12-13). While in intuitionistic logic the introduction-rules describe the manner in which logical constants are introduced in a proof, and the elimination-rules the removal of these constants, in fact,

[17] Bradley asserted that facts and relations must be taken as a whole not as discrete parts arguing:

> If relations are facts that exist between facts, then what comes between the relations and the other facts? The real truth is that the units on one side, and on the other side the relation existing between them, are nothing actual (BRADLEY, 1883, Ch. 2 §65).

In the second edition of 1922 he adds the following note, to contrast his views with the logical atomism of RUSSELL (1918), stating that:

> This is the doctrine for which I have now for so many years contended... Relations exist only in and through a whole which can not in the end be resolved into relations and terms. "And," "together" and "between," are all in the end senseless apart from such a whole. The opposite view is maintained (as I understand) by Mr. Russell, and was perhaps at last tacitly adopted by Prof. Royce. But, for myself, I am unable to find that Mr. Russell has ever really faced this question (BRADLEY, 1922, Ch. 2 §65 en. 50).

Russell to his credit did face this question in his 1913 manuscript *Theory of Knowledge* (RUSSELL, 1984) especially in the Chapter IX entitled "Logical Data". Though Bradley could not have known this as, though much of this work was published in a series of papers in *The Monist* between 1914 and 1915 (see RUSSELL, 1914b,c,d,a, 1915b,a), the parts that addressed the question of the relation between facts and relations remained unpublished until 1984.

the elimination rules never let you extract, from a logical sentence with a particular logical term as its main operator, any content that was not required to be established before that instance of the operator could be introduced into the proof. But the elimination rules do allow you to extract *all* the content. One sees, then, the outline of Dummett's grounds for the view that intuitionistic logic is logic properly so-called, and is metaphysically neutral. Use of a harmonious set of inference rules yields nothing that is not contained in the premises one brings in from elsewhere, but one never loses any of that content either. For a non-harmonious system we have no such guarantee.

What the introduction rules describe is how constituent propositions of a proof, each of which have appropriate proofs, are placed in relation to each other to form a logical construction. The relations required between these constituents to construct a proposition with the operator in question as the main logical operator are described by the rule. This act of constructive proof is dependent on each step following from the previous ones and there being a harmony in the rules. It is not merely that the correct application of the introduction and elimination rules does not produce a contradiction, but rather, that we are guaranteed that nothing not contained in the premises is contained in the conclusion and that all content used therefore constructing the conclusion can be extracted from it. Lacking harmony means that classical logic can provide no such guarantee.

With this in view, we can see how Dummett came to make his influential proposal to reconsider disputes about realism and anti-realism—not only in mathematics, but more generally—as disputes about meaning.

> In trying to describe the general form of disputes of the kind in which I was interested, I needed some generic means of referring to the particular subject-matter of any one such dispute. Very often, realism of a particular variety is referred to as realism about some particular class of putative entities—mental events, for example, or mathematical objects. I chose to speak instead of the "disputed class of statements" rather than of the "disputed class of objects" (Dummett, 1993c, p.465).

The suggestion is that instead of beginning our investigations of the metaphysical nature of the objects of a domain of discourse with a metaphysical argument about the nature of those objects; based on, perhaps, our intuitions about the objects, we should look at what principles of reasoning are correct for the domain. Intuitionistic logic, because of the harmony between the introduction and elimination rules, is "harmless" in the sense that we can be sure it will introduce no new content into our information about the domain, and will not lose any of it either. But if we accept non-harmonious rules, and in particular if we accept the reasoning principles of classical logic, we must be doing so for some non-logical reason. In particular, the grounds for accepting classical logic are, if they are ever legitimate, to be found in grounds for accepting realism for the discourse in question.

However things are not as clear as Dummett's method of beginning with logical principles and deciding what metaphysics fits these principles. This line of reasoning is certainly suggestive. In the coming chapters, I hope to usefully elaborate on it. There are other grounds, not obviously the same as lack of harmony, for declaring a principle non-logical. One is explicit ontological content. A standard objection to Russell and Whitehead's suggestion that their view in *Principia Mathematica* should count as logicism is their use of the axiom of infinity—a principle most would not count as logical because it postulates the existence of any objects at all.[18] As David DeVidi puts it "The prevailing view among philosophers, I think, is that if a principle of reasoning depends on existence assumptions for its correctness, then it's not, properly speaking, a logical principle" (DeVidi, 2011, p.1). The results below show that the relationship between logic and metaphysics is not merely an all or nothing equation of intuitionistic logic with anti-realism and full classical logic with realism. As we will see, there are principles that we have good reason to regard as non-logical principles yet that imply principles that classical logic declares 'logical', without implying all of classical logic.

3.6 Dummett and Logical Laws

In the *Logical Basis of Metaphysics* Dummett argues that we can justify logical laws on the basis of semantic theory but that a semantic theory needs to be justified in terms of a meaning theory. He gives the following argument for why debates about metaphysics reduce to debates about meaning theory:

> The realist argues that an independently existing material universe is the only hypothesis that explains the regularities in our experience. The idealist retorts by asking, with Berkeley, what content the belief in an autonomous realm of matter can have. It is, however, useless to carry on a debate in favour of one or other of these competing pictures as if they were rival hypotheses to be supported by evidence. What we need to do is to formulate theses which are no longer in pictorial language but which embody the intended applications of these pictures. If we do that, those theses will be found to be theses belonging to the theory of meaning, theses about the correct meaning-theory for statements of one or another kind. When we have resolved the issue about the correct meaning-theory, then we shall surely find that one or another of the rival pictures will force itself on us, unless it proves that we want to reject all the competing pictures (Dummett, 1991, p. 339).

[18]The problem with this axiom isn't that it postulates infinitely many, but that it postulates any. The axiom of empty set in standard set theory is non-logical for the same reason.

However this does not mean that the reasoning that a semantic theory uses to justify logical laws is without basis. Rather Dummett asserts that some forms of deduction are self-justified. He accepts that this may be "pragmatically circular" but there is not a problem. He states:

> Since a justification of a logical law will take the form of a deductive argument, there can be no justification that appeals to no other laws whatever; but that does not matter, since there is no sceptic who denies the validity of all principles of deductive reasoning, and, if there were, there would obviously be no reasoning with him (Dummett, 1991, pp203-204).

For Dummett, there are two ways of justifying a logical law: through self justification, and *via* a semantic theory. Those that Dummett refers to as self-justifying he evaluates using methods he refers to as proof theoretic justification of the second (Dummett, 1991, pp.252 ff.) and third grade (Dummett, 1991, pp. 259 ff.). We will look at the methods with which Dummett evaluates these logical laws and consider his criteria when evaluating Hilbert's choice operators.

3.6.1 Harmony and Introduction Rules

It is clear that there are things that are terribly wrong with Arthur Prior's 'tonk' rule—which uses the introduction rules of 'or' and the elimination rules of 'and' (Prior, 1960). The most obvious problem is inconsistency. But the rules governing the uses of a bit of language ought to satisfy conditions stricter than mere consistency. As we've seen each bit of language will have two sorts of principles and these principles must be harmonious.

Natural language is not immune to contradiction and paradox. Such things arise due to the "multiplicity of principles governing out linguistic practices" (Dummett, 1991, p.210). However Dummett asserts that we can "distinguish two general categories of such principles," the first set deal with the "circumstances that warrant an assertion" what Dummett refers to generally as "principles of verification," and the second the "principles determining the consequences of possible utterences" (Dummett, 1991, pp.210-212). What Dummett calls the "twin notions of verification and of consequences" (Dummett, 1991, p.214) must be in harmony with each other, though he admits that "there is no automatic mechanism to ensure that they will be" (Dummett, 1991, p.215). That is harmony implies consistency but consistency does not necessarily imply harmony.

The introduction and elimination rules for the logical constants are paradignmatic of the verificationist and pragmantist meaning theories and hence of Dummett's two aspects of language use. Thus Dummett writes that the "canonical verification of a statement ⌜A and

$B^{\urcorner}$ will proceed by verifying both A and B, and then applying the standard introduction rule for 'and'." Likewise he continues the "canonical means for arriving at the consequences of a conjunctive statement $^{\ulcorner}A$ and $B^{\urcorner}$ will consist in applying either or both of the standard elimination rules for 'and', and then drawing consequences from A or from B or both" (Dummett, 1991, p.216).

Hence Dummett asserts that :

> we have for the logical constants a hope that a verificationist account of their meanings can be given in terms of a familiar type of logical law, allowing us, in their case, a gratifyingly sharp notion of what those meanings consist in. Just the same holds good for pragmatist meaning-theories (Dummett, 1991, p.216).

The introduction of new rules of inference, Dummett writes, is modulated by a "fear of disharmony" hence:

> A weakening of the introduction rules, while leaving the elimination rules unchanged, or a strengthening of the elimination rules, while leaving the introduction rules unchanged, must upset a harmony that prevailed previously: we can now draw conclusions not warranted by our methods of arriving at the premisses (Dummett, 1991, p.217).

Thus for Dummett, reference to both pragmatist or verificationist theories of meaning is important for understanding the appropriate use, and so the meanings of the logical constants. He notes that someone, giving as an example Wittgenstein, who, without reference to both theories of meaning, wishes to present a theory of meaning focused only on "mastery of use" of sentences containing the various logical constants "is likely to invoke the introduction rule for the existential quantifier and the elimination rule for the universal one;" but Dummett cautions this is not enough. One cannot be said to completely understand the use of a quantifier or connective without knowing both rules (Dummett, 1991, p.217).

Dummett introduces the notion of a conservative extension to a formal language as a "more precise characterisation of the notion of harmony." A conservative extension involves adding "new primitive predicates, terms, or functors, and introducing new axioms or rules of inference to govern expressions formed by means of the new vocabulary" in such a manner that in the new theory one can prove no statement, that does not contain the new vocabulary, which they could not prove in the original (Dummett, 1991, pp.217–218). But a conservative extension is restricted to adding elements and rules *etc.* to those that are already accepted.

Harmony, Dummett has argued, "*within a language as a whole*" is a requirement for a "compositional meaning-theory" (*cf.* DUMMETT, 1991, p.215-220). But as applied to an element of a language, *e.g.* a logical constant, harmony means that for a connective $*$:

> the canonical ways of establishing a statement $\ulcorner A * B \urcorner$ as true should match, and be matched by, the consequences which accepting that statement as true is canonically treated as having

This means that the introduction and elimination rules for the connective are in harmony. Dummett suggests that non-conservativeness follows when "within the restricted domain of logic" and "for an arbitrary logical constant c" if we can apply the introduction rules for the constant c and then immediately draw a consequence "that we could not otherwise have drawn" by then applying elimination rule to the conclusion of the introduction rule, using the conclusion as a major premise (DUMMETT, 1991, p.247).

For an example of a connective that fails this requirement we need go no further than the above mentioned *tonk*:

$$\dfrac{\dfrac{A}{A\text{-}tonk\text{-}B}}{B}$$

Dummett gives as an example of a harmonious pair of rules: the introduction rule and elimination rules for '$\wedge$". Here we apply the introduction and then the elimination rule:

$$\dfrac{\dfrac{A \qquad B}{A \wedge B}}{A}$$

which shows that, in this instance the "detour through $\ulcorner A \wedge B \urcorner$ was superfluous", unlike the above with Prior's *tonk* where we start simply with A and end up with B, the conclusion in this case gives use nothing we did not already have. Dummett then considers the $\vee$-introduction and elimination rules. The standard left and right introduction rules for the connective $\vee$ are:

$$\frac{A}{A \vee B} \qquad \frac{B}{A \vee B}$$

The elimination rule for $\vee$ is:

$$\dfrac{\begin{array}{c} \\ \vdots \\ A \vee B \end{array} \qquad \begin{array}{c} [A] \\ \vdots \\ C \end{array} \qquad \begin{array}{c} [B] \\ \vdots \\ C \end{array}}{C}$$

The square brackets around the hypotheses A and B, on which the two intermediate premises C depend, are discharged by the application of the rule. That is, given that one can derive C from either the hypotheses A and B, one can then eliminate the $\vee$-statement $A \vee B$ to conclude C. Note though if the elimination rule is immediately preceded by the introduction rule we have the following:

$$\dfrac{\dfrac{\begin{array}{c}\ddots\ (\alpha)\ \cdot^{\cdot^{\cdot}} \\ A\end{array}}{A \vee B} \qquad \begin{array}{c}[A] \\ (\beta) \\ \ddots\quad \cdot^{\cdot^{\cdot}} \\ C\end{array} \qquad \begin{array}{c}[B] \\ (\gamma) \\ \ddots\quad \cdot^{\cdot^{\cdot}} \\ C\end{array}}{C}$$

where (α), (β), and (γ) are the labels for the sub-arguments that lead to A and the two Cs respectively. But in such a case the use of the introduction and elimination rules are superfluous, as we can move directly *via* (α) to A and thence through (β) to C skipping the introduction of the disjunct sentence $\ulcorner A \vee B \urcorner$ and its elimination (*cf.* Dummett, 1991, p.249).

Dummett argues that harmony may be "provisionally" identified with this procedure, which he refers to as "levelling local peaks." It is a "fundamental type of reduction step used in the process of normalizing natural deduction proofs" introduced by Dag Prawitz in line with Gentzen's removal of the cut rule in the sequent calculus. If a proof can be normalized with respect to a logical constant c it means that the sentence using that constant has been introduced and eliminated in the course of that proof. This Dummett notes implies a "relative consistency: if an arbitrary atomic sentence can be proved using the rules governing c, it could have been proved without using those rules" (Dummett, 1991, p.250).

Dummett then makes the distinction between "intrinsic harmony" and "total harmony" (*i.e.* "harmony in context"). By "total harmony" he means that the addition of a logical constant produces a conservative extension to the logical theory to which it has been added, whereas "intrinsic harmony" is to be understood as involving simply the "eliminability of local peaks" (Dummett, 1991, pp.250–251).

Rules of inference can be termed self-justifying if no proof is needed that they are in order. According to Dummett therefore it is "essential to develop a characterisation that will allow it to recognize a set of logical laws as self-justifying by their very form" (Dummett, 1991, p.251). Dummett refers to Gerhard Gentzen's dissertation on logical deduction where he argues that such rules can be understood as in some sense self justifying:

> The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'. An example may clarify what is meant: We were able to introduce the formula $\mathfrak{A} \supset \mathfrak{B}$ when there existed a derivation of $\mathfrak{B}$ from the assumption formula $\mathfrak{A}$. If we then wished to use that formula by eliminating the $\supset$-symbol (we could, of course, also use it to form longer formulae, *e.g.*, $(\mathfrak{A} \supset \mathfrak{B} \vee \mathfrak{C}, \vee\text{–}I)$, we could do this precisely by inferring $\mathfrak{B}$ directly, once $\mathfrak{A}$ has been proved, for what $\mathfrak{A} \supset \mathfrak{B}$ attests is just the existence of a derivation from $\mathfrak{B}$ from $\mathfrak{A}$ (Gentzen, 1934, pp.80–81).

Dummett prefers to say that the elimination rules are "justified" rather than consequences of the introduction rules as there is no aspect of logical consequence in Gentzen's explanation (Dummett, 1991, p.252).

3.6.2 Proof Theoretic Justification

Let us continue our investigation of Dummett's argument of how intuitionistic logic can be understood as metaphysically neutral. We will now look at Dummett's analysis of what he calls proof theoretic justification, specifically with regard to the standard logical operators, and how this relates to what he calls canonical arguments.

Dummett separates proof theoretic justification into several degrees; the first grade he describes as follows:

> A proof-theoretic justification of the first grade assumes, of the logical laws it takes as its base, only that they are valid (Dummett, 1991, p.252).

Prawitz (Prawitz, 1974, 1973) provides a method which Dummett refers to as "proof theoretic justification of the second grade". This, Dummett asserts, is good enough for simple introduction rules (*e.g.* $\vee$ and $\wedge$) because "for the standard introduction rules governing these constants involve neither free variables nor the discharge of hypotheses" (Dummett, 1991, p.259). In second grade proof theoretic justification:

> the assumption it makes concerning the introduction rules it takes as a base is correspondingly stronger, namely, that they are collectively in a certain sense complete (Dummett, 1991, p.252).

However Dummett's version differs from Prawitz's in that Prawitz considers only "the standard introduction rules for the standard logical constants" whereas Dummett wishes to "achieve a quite general formulation for all conceivable logical constants, provided that they are govemed by introduction rules of a restricted type" (Dummett, 1991, p.252).

What then is this sort of justification? Dummett gives the following answer:

> The strategy of proof-theoretic justifications of the second grade is that of all proof-theoretic justifications, namely, to show that we can dispense with the rule up for justification: if we have a valid argument for the premisses of a proposed application of it, we already have a valid argument, not appealing to that rule, for the conclusion (Dummett, 1991, p.254).

Such a justification is certainly dependent on what Dummett refers to as the "fundamental assumption." The fundamental assumption is the view Dummett adapts from Belnap that whenever we have a sentence with a logical constant c as the principal operator, we could have in principle derived that sentence using the introduction rule for c (Dummett, 1991, p.251). As Dummett puts it:

> The fundamental assumption is that, whenever we are entitled to assert a complex statement, we could have arrived at it by means of an argument terminating with at least one of the introduction rules governing its principal operator (Dummett, 1991, p.257).

Dummett uses as an example the distributive law to demonstrate this:

$$\frac{A \wedge (B \vee C)}{(A \wedge B) \vee (A \wedge C)}$$

The premises which could get us the antecedent $A \wedge (B \vee C)$ by means of the introduction rule for the primary operator $\wedge$, are A and $B \vee C$. The introduction rule for the primary operator $\vee$ of $B \vee C$ gives us two options either B or C. Hence we will have as premises A and B or A and C which is enough to justify the conclusion $(A \wedge B) \vee (A \wedge C)$ (Dummett, 1991, p.253).

Types of Introduction Rules and the Complexity Condition

In considering the formal properties that an introduction rule needs to have to be considered self-justifying, Dummett considers first what is the general definition of a rule of inference, and secondly introduces a vocabulary to describe the types of possible introduction rules. In general he notes that:

> A rule of inference may be called an introduction rule for a logical constant **c** if its conclusion is required to have **c** as principal operator; it may be called an elimination rule for **c** if one of its premisses is required to have **c** as principal operator, relative to which that will be the 'major premiss' (Dummett, 1991, p.256).

Rules of inference can be divided into several types:

- "single-ended" rules are those not simultaneously both introduction and elimination rules. We may call double ended those rules that do not meet this condition.
- "pure" are those in which only one logical constant appears. We will refer to rules that are not pure as 'impure'.
- "simple" rules are those where the logical operator that appears in the rule is the primary operator in the sentence. We will refer to rules that are not simple as 'complex'.
- "sheer" rules are those where for any introduction rule for a logical constant, that constant does not appear in any hypothesis, and conversely in the elimination rules for the constant, the logical constant does not appear in the conclusion. We will refer to those that are not sheer as 'restricted'.
- "oblique" or indirect rules are those where a logical constant appears in a hypothesis discharged by that rule. "Direct" rules are those which are not "oblique".

Double ended rules, those which can "simultaneously be an introduction rule for one logical constant and an elimination rule for another," Dummett does not consider self justifying. As examples Dummett gives the "distributive law", "the law of transitivity for '$\rightarrow$'" which is an introduction rule and elimination rule, and "*modus tollendo ponens*" which is an elimination rule for two different constants.

Examples of rules typed by Dummett's system:

- The classical double negation rule $\neg\neg\varphi \vdash \varphi$ is pure, single-ended and complex.

- *modus tollendo ponens* $\varphi \to \psi, \neg\psi \vdash \neg\varphi$ is simple, single-ended and impure
- the transitivity law $\varphi \to \psi, \psi \to \chi \vdash \varphi \to \chi$ is pure, simple and double-ended.

Recall the "fundamental assumption" means that " whenever we are entitled to assert a complex statement, we could have arrived at it by means of an argument terminating with at least one of the introduction rules governing its principal operator." So the meaning of a logical constant is all the introduction rules "governing it as a whole".

Thus according to this line of argument Dummett states that:

> To determine the meaning of a logical constant in a compositional meaning-theory it is necessary and sufficient to determine that of a sentence of which that constant is principal operator, relative to the meanings of the subsentences. Hence, what the introduction rules for a constant **c** are required collectively to do is to display all the canonical ways in which a sentence with principal operator **c** can be inferred (Dummett, 1991, p.257).

This he asserts might lead one to think "we should follow the example of Gentzen by restricting our rules, at least for constants other than negation, to those that are pure, simple, and single-ended," but Dummett finds such a condition to be too onerous.

Impure rules should not be objected to, according to Dummett, so long as they are not cyclical. Cyclicality would not occur if the constants were ordered in such a manner that their ordering depended on those preceding it.[19] But even such an ordering, Dummett notes is not necessary as "the principle of compositionality in no way demands this [ordering]; all that is essentially presupposed for the understanding of a complex sentence is the understanding of the subsentences." That is:

> Hence the minimal demand we should make on an introduction rule intended to be self-justifying is that its form be such as to guarantee that, in any application of it, the conclusion will be of higher logical complexity than any of the premisses and than any discharged hypothesis. We may call this the 'complexity condition'. In practice, it is evident that there will be no loss of generality if we require the rule to be single-ended, since, for a premiss with the same principal operator as the conclusion, we may substitute the hypotheses from which that premiss could be derived by the relevant introduction rule. We may accordingly recognise as an introduction rule a single-ended rule satisfying the complexity condition (Dummett, 1991, p. 258).

[19]Such an ordering would also mean complex rules too should be permitted, if the main operator in such complex rules was already justified by a constant preceding it in the ordering.

It is in satisfying this "complexity condition" that the simple rules satisfy the conditions for what Dummett calls proof-theoretic justifications of the second grade. This second grade we noted above was that they show that: " if we have a valid argument for the premisses of a proposed application of it, we already have a valid argument, not appealing to that rule, for the conclusion" (Dummett, 1991, p.254). Proof theoretic justification of the third grade is what Dummett asserts we need for rules that employ either "free variables" or "the discharge of hypotheses" (Dummett, 1991, p.259). To prove proof-theoretic justifications of the third grade we will have to have a broader "notion of an argument" that admits examples of open sentences to which the fundamental assumption does not apply (Dummett, 1991, p.259). Let us look first at the differences between the simple and complex constants, and then see how Dummett rewrites the definition of a canonical argument.

The simple constants: $\vee$ and $\wedge$ and $\exists$

Dummett defined "valid arguments" as those which can be transformed into canonical arguments. And for statements with only simple connectives such as $\vee$ and $\wedge$ and the existential quantifier $\exists$, the "antecedent of the conclusion" is simply the "union of the antecedents of the premises," and the definition of a canonical argument is simple.

A canonical argument, Dummett defines for these simple cases as:

> one in which no initial premiss is a complex sentence (no complex sentence stands at a topmost node) and in which all the transitions are in accordance either with one of the boundary rules or with one of the given set of introduction rules (Dummett, 1991, pp.254-255).

By a boundary rule, Dummett means, those "certain rules of inference, which we recognise as valid, for deriving atomic sentences from one or more other atomic sentences" (Dummett, 1991, p.254).

For an arbitrary argument then, we can term that argument valid if we have a method that can transform "any supplementation of it into a canonical argument with the same final conclusion and no new initial premisses" (Dummett, 1991, p.255).

> If we say that we are canonically entitled to assert a sentence when we have a canonical argument for it, then our definition deems an argument valid when we can effectively show ourselves canonically entitled to assert the conclusion whenever we are canonically entitled to assert the premisses (Dummett, 1991, p.255)

Taking as an example the distributive law discussed above, the reduction of the argument into simple atomic sentences is direct as it is clear that the "antecedent of the conclusion" is simply the "union of the antecedents of the premises" (Dummett, 1991, p.259).

Dummett notes that there is a problem with introduction rules which involve a hypothesis that is discharged, such as the introduction rule for $\rightarrow$ or $\forall$.

$\forall$ and $\rightarrow$

The introduction rules for the logical constants $\forall$ and $\rightarrow$ are not as simple as $\vee$, $\wedge$, and $\exists$, as they employ free variables and the discharge of hypotheses respectively.

The standard introduction rule for $\forall$ is:

$$\frac{A(a)}{\forall x A(x)}$$

with the proviso that the free variable a does not occur in any hypothesis on which the open sentence $A(a)$ depends, or in the conclusion (*e.g.* where A is a complex sentence containing the free variable a for example $A \equiv P(a) \vee Q(x)$).

This means a definition of a canonical argument for the universal quantifier needs to handle open sentences. Dummett notes that we have to impose several constraints on introduction rules for constants that employ open sentences. First we must "assume that the language contains a constant term for each element of the domain" such that any free variable in an argument can be replaced by any of the constant terms. Secondly:

> for it to be self-justifying... if one or more of the premisses of an application of the rule contains a free variable, but the conclusion does not, that free variable should not occur in any of the hypotheses on which the conclusion depends. This condition, which is satisfied by the universal quantifier-introduction rule, is needed in order to guarantee that, if we are entitled to assert a closed sentence, it should be possible to derive it by means of the given introduction rules from closed initial premisses (Dummett, 1991, pp.259-260).

The use of discharged hypotheses also create problems in the introduction rule for $\rightarrow$. Between the premise and the conclusion, we cannot necessarily avoid the use of elimination rules. Dummett gives as an example:

$$\dfrac{A \qquad \dfrac{B \wedge C}{B}}{\dfrac{A \wedge B}{(B \wedge C) \rightarrow (A \wedge B)}}$$

The proof unavoidably uses the elimination rule for $\wedge$. Therefore because of the use of free variables or discharged hypotheses a canonical argument cannot be "automatically valid," because firstly "we can place no general restriction upon the derivations of open sentences" and because it is"impossible to demand that a subordinate deduction be capable of being framed so as to appeal only to introduction rules" where a subordinate deduction is any step between the hypothetical premise and the conclusion (Dummett, 1991, p.260).

Canonical Argument

Given the above considerations about the introduction rules that employ free variables and the discharge of hypotheses, we can now refine the definition of a canonical argument to be such only if the following four conditions hold:

> (a) its final conclusion is a closed sentence;
>
> (b) all its initial premisses are closed atomic sentences;
>
> (c) every atomic sentence in the main stem is either an initial premiss or is derived by a boundary rule;
>
> (d) every closed complex sentence in the main stem is derived by means of one of the given set of introduction rules (Dummett, 1991, p.260).

A 'valid' argument remains one that "we can effectively transform any supplementation of an instance of it into a valid canonical argument" but since not all canonical arguments can be guaranteed to be *prima facie* 'valid'[20] we define those that are as those in which "every critical sub-argument it contains is valid" (Dummett, 1991, p.261).[21]

Dummett now defines validity for both arbitrary and canonical arguments. Arbitrary arguments "will be said to be 'valid' if we can effectively transform any supplementation of an instance of it into a valid canonical argument with the same final conclusion and initial premisses". A canonical argument, on the other hand, will be deemed 'valid,' "just in case every critical subargument it contains is valid" (Dummett, 1991, p.260).

[20] *E.g.* those that depend on the derivations of open sentences.

[21] Dummett defines a critical subargument of an argument (α) as one where

> ...if its conclusion stands, in (α), immediately above a closed sentence in the main stem of (α), but is itself either an open sentence or a closed sentence not in the main stem.

He defines a supplementation of an argument as what "results from replacing every initial premiss by a valid canonical argument with that premiss as final conclusion" (Dummett, 1991, p.261).

3.6.3 Canonical Proofs and Demonstrations

Dummett had written in his paper "The Philosophical Basis of Intuitionistic Logic" that, "it is in the terms of a canonical proof that the meanings of the logical constants are given." The definitions of the introductions of the logical constants can then be each seen as the most basic canonical proofs (DUMMETT, 1975, p.122).

According to Dummett, in intuitionistic logic a true statement is a statement of which one has a proof. This conception of truth means that the proof of a statement is tied to the meaning of a statement, which in turn is tied to the network of conditions and consequences for its correct use. A statement is correctly or truthfully asserted when there is a proof of the statement. While any statement that has a proof can be asserted Dummett draws a distinction between different types of proof, that is between a proof proper and a cogent argument. A proof proper is a 'proof', "in the sense of 'proof' used in the explanation of the logical constants" (DUMMETT, 1975, pp.120–122).

Here he is drawing a distinction between what he calls a canonical proof and what he calls a demonstration. A demonstration provides a method for obtaining a canonical proof. Prawitz agrees with Dummett on the difference between what he calls direct or canonical proofs and indirect proofs, or what Dummett refers to as demonstrations. Direct proofs provide a proof of a mathematical statement and indirect ones provide a method for finding the direct proof (PRAWITZ, 1977, pp.21–22). Dummett describes the difference between the two in the following manner:

> A demonstration is just as cogent a ground for the assertion of its conclusion as is a canonical proof, and is related to it in this way: that a demonstration of a proposition provides an effective means for finding a canonical proof. But it is in the terms of a canonical proof that the meanings of the logical constants are given (DUMMETT, 1975, p.122)

However at this point Dummett writes that:

> The notion of canonical proof thus lies in some obscurity; and this state of affairs is not indefinitely tolerable, because, unless it is possible to find a coherent and relatively sharp explanation of the notion, the viability of the intuitionist explanations of the logical constants must remain in doubt. But, for present purposes, it does not matter just how the notion of canonical proof is to be explained; all that matters is that we require some distinction between canonical proofs and demonstrations (DUMMETT, 1975, p.124).

In DUMMETT (1991) he attempts to make the distinction between valid canonical arguments and valid arbitrary arguments, as discussed above. Others have argued that the distinction between canonical and non–canonical proofs can be made clear by making a distinction between proof processes and proofs as objects. Arne Ranta writes:

> DUMMETT (1975) and PRAWITZ (1977) made a distinction between canonical and non-canonical proofs and suggested that propositions can be explained by telling what their canonical proofs are. But they did not distinguish between proof objects and proof processes (RANTA, 1994, p.41)

Ranta goes on to claim Per Martin-Löf united the informal intuitionistic explanation of propositions and the formal interpretation of formulae as type in an identification of propositions and types (RANTA 1994, p.41, see also MARTIN-LÖF 1975, pp.73-118). Generally this view can be understood as a tension that needs to be resolved between the formal representation of proofs, which conceives of them as objects, and that of their use which understands them as act. Proofs are both. When we say that the act finds the object we define two levels of meaning, the philosophical and the formal. The proof object is not the meaning of the proof in the philosophical sense, rather the use is. Nicolaas Govert de Bruijn's discusses the difference between meaning as is philosophically understood and meaning as a machine, which operates on formal language:

> In our relation with the machine there is language and nothing but language. There is no concern for meaning in the usual philosophical sense, relating words to things in the real world. For our machines the word "meaning" cannot refer to anything else other than to a mapping from one language system to another (DE BRUIJN, 1998, p.42).

That is we can see here there are two types of meaning. One the philosophical that always inhabits our metalanguage and another, a formal definition of meaning that is a significantly rich language system, like constructive type theory, that does not always, but at times is metalinguistic. Further discussion of how constructive type theory applies to the definition of proof is beyond the scope of this chapter.

Though one cannot assert that infinitely lengthy proofs can be surveyed or understood as canonically expressed by presenting all their steps as introduction rules, they are potentially describable. Furthermore, if a method can be described that, though infinite in length, describes how one would reach a proof if it were possible, one ends up in a situation where one is not appealing to some third realm, or unknowable truth. Such a demonstration is still grounded in use. It is this conception of use in principle which extends the criterion of meaningfulness, that might otherwise place a quite radical verificationist limit on the assertible statements of mathematics, and hence in the meaning of the content of the proof. As Dummett puts it:

> That is why such a proof may be an infinite structure: a proof of a universally quantified statement will be an operation which, applied to each natural number, will yield a proof of the corresponding instance; and, if this operation is carried out for each natural number, we shall have proofs of denumerably many statements. The conception of the mental construction which is the fully analysed proof as being an infinite structure must, of course, be interpreted in the light of the intuitionist view that all infinity is potential infinity: the mental construction consists of a grasp of general principles according to which any finite segment of the proof could be explicitly constructed... Indeed, it might reasonably be said that the standard intuitionistic meanings of the universal and conditional quantifiers involve that a proof is such a potentially infinite structure (Dummett, 1975, p.242).

Having surveyed Dummett's arguments about the nature of logcial laws, and his evaluation of the introduction and elimination rules, it seems appropriate to consider what the introduction and elimination rules would be for the epsilon calculus.

3.7 The Introduction and Elimination Rules for Intuitionistic Logic $+\varepsilon$

There have been various suggestions for the introduction and elimination rules for the epsilon operator first presented by Mints (1977) and Dragalin (1974):

$$\frac{A(a)}{(\forall x)A(x)} \qquad \frac{(\forall x)A(x)}{A(t)}$$

$$\frac{A(b)}{\exists x Ax} \qquad \frac{\exists x A(x)}{A\varepsilon x A(x)}$$

$$\frac{((\forall x)B(x) \wedge (\exists x)A(x))}{B\varepsilon x Ax} \qquad \frac{((\exists x)Ax \wedge B\varepsilon x Ax)}{(\exists x)B(x)}$$

where a is again a free variable, a does not occur in any hypothesis on which the open sentence $A(a)$ depends, and b is a term other than an epsilon term (Mints, 1977). Smirnov (1971) uses a similar presentation except his epsilon elimination rule is simply:

$$\frac{B\varepsilon x A(x)}{\exists x A(x)}$$

In addition there have been various attempts to provide conservative epsilon extensions to intuitionistic logic (*cf.* Smirnov, 1979; Leivant, 1973; Dragalin, 1974; Mints, 1974, 1977, 1991; Mints *et al.*, 1996; Mints and Tupailo, 1999; Mints and Sarenac, 2003; Mints, 2012; Meyer-Viol, 1995a,b). Slater (2009, p.398) notes that Mints' system requires that all sequents be what he refers to as formally "meaningful." That is "an epsilon term, say, $\varepsilon x Axyz$', appearing in an occurrence of some sequent is only 'meaningful', according to Mints (Mints, 1977, 318), if the formula $(\forall z)(\forall y)(\exists x)Axyz$' is either in the antecedent of the sequent, or some sequent below that occurrence". Likewise Mints himself notes that Leivant's system (Leivant, 1973) does not allow what Mints terms "quasiterms". Quasiterms and quasiformulae are:

> expressions obtained, respectively, from terms and formulas by replacing some free variables by bound ones. For example, if a is a free variable and y is a bound one, then $\varepsilon x Pxx$ and $\varepsilon x Pxa$ are terms and $\varepsilon x Pxy$ is only a quasiterm (Mints, 1974, pp.317-318).

Mints (2012) elaborated on the intuitionistic ε-calculus first presented in his previous work (Mints, 1974, 1991). In this paper he presented an intuitionistically epsilon calculus in a Gentzen style natural deductive, and sequent calculus without identity that is conservative over intuitionistic predicate logic. By placing restrictions on the introduction and elimination rules for epsilon terms, he restricted his IPC+ε from being able to prove anything that you can't prove already in IPC by itself (except for statements that include epsilon terms). For instance, ignoring the fact that Mints used a version without identity, for the sake of the example, if you had identity you could prove $\exists x(x = \varepsilon x.A(x))$ for any epsilon term $\varepsilon x.A(x)$, which obviously can't be proven if you don't have epsilon terms.

Mints resorted to weakening the rules for quantifiers, specifically the introduction rule for $\exists$ and the elimination rule for $\forall$ requiring that the terms be defined. That means that epsilon terms cannot be used in either rule, as they remain partially defined in the language, that is if you can prove $\exists x.A(x)$, then you can invoke the epsilon rule to get $A(\epsilon x.A(x))$, but if you can't prove the existential, there is no epsilon term for A— the epsilon operator is only "partially defined", in much the same manner a function can be partially defined *i.e.*, it's not defined for the whole language, like a operation that does not operate on say zero, like division.

We present in the next chapter several proofs that $IPC + \varepsilon$ with and without identity show how adding the ε axiom along with various decidability criteria is non-conservative over intuitionistic logic.

Part II

Formal Results

Chapter 4

Deriving Logical Axioms from Choice Principles

4.1 Introduction: The Intuitionistic ε-calculus

In 1993 John Bell published two papers where he showed how the epsilon operator is non-conservative when added to intuitionistic logic. In the first (Bell, 1993a) he showed how Markov's principle and DeMorgan's law can be derived in the intuitionistic epsilon calculus plus some "modest" decidability conditions, and then that the law of excluded middle can be proved by a adding an extensional epsilon axiom. He also presented a simple, sound formal semantics which allowed him to demonstrate that these assumptions were essential to each of these proofs. In the second paper, Bell showed that linearity (aka Dummett's Scheme) can be derived in intuitionistic type theory without extensionality (Bell, 1993b). These results are suggestive, since choice is obviously an "ontological" principle, but it yields, in interesting and distinct ways, logical principles [1].

The intuitionistic ε-calculus is defined as a first-order intuitionistic language $\mathscr{L}$. We start with the standard axioms and rules of inference for our first order intuitionistic logic to which we add a choice operator by introducing the ε-axiom schema:

$$(\varepsilon)\ \ \exists x\varphi\ \rightarrow\ \varphi(x/\varepsilon x\varphi)$$

We call the resulting language $\mathscr{L}_\varepsilon$.

In "Intuitionistic ε- and τ-calculi" David DeVidi presented the first semantics for intuitionistic logic with the epsilon operator that is not just sound, but also complete (DeVidi

[1] David DeVidi describes this as "buying logical principles with ontological coin" (DeVidi, 2011).

1995 *cf.* DeVidi 1994). In addition he provides independence results showing that the following sentences, true in classical first order logic, are not provable in the intuitionistic ϵ-calculus:

$$\begin{array}{ll} \vdash & \neg\forall x\varphi \to \exists x\neg\varphi \\ \vdash & \exists x(\varphi \to \forall x\varphi) \\ \vdash & (\forall x\varphi \to \psi) \to \exists x(\varphi \to \psi) \\ \vdash & \varphi \vee \neg\varphi \end{array}$$

This means that the intuitionistic ϵ-calculus is a super-intuitionistic logic, and when one adds decidability conditions, one can prove several standard intermediate sentential logics. [2]

Bell's proofs of DeMorgan's intuitionistically invalid law, and of Dummett's scheme involve two "decidability conditions." First, there is a decidable constant a *i.e.* every other object is provably either equal to a or distinct from a; and secondly, there is an object b that is distinct from a.

In addition to reviewing these results we present some new proofs which are a modest improvement on these proofs. In the new proofs that we present below, we use decision conditions that relate all elements of a domain to a predicate and constant using implication but not a full identity relation, *e.g.* in the DeMorgan's proof the decision condition asserts that there is a predicate P and a constant a such that: $\forall x((P(x) \to P(a))) \vee \neg(P(x) \to P(a)))$. This condition is substantially weaker than the one using identity, but it is still a decidability condition—all elements can be judged in relation to the constant a and the single placed predicate P.

[2]Note that there are many quantified sentences true in the classical predicate calculus that are not true in intuitionistic predicate calculus. The following propositions, true in classical predicate logic, are not intuitionistically valid but are true in the ε-calculus:

$$\vdash (\varphi \to \exists x.\psi) \to \exists x.(\varphi \to \psi) \quad \text{and} \quad \vdash \exists x.(\exists x.\psi \to \psi)$$

The following propositions, also true in classical predicate logic, are not intuitionisitically valid, but are true in the intuitionistic τ-calculus:

$$\begin{array}{ll} \vdash \neg\forall x.\psi \to \exists x.\neg\psi & \vdash (\forall x.\psi \to \varphi) \to \exists x.(\psi \to \varphi) \\ \vdash \exists x.(\psi \to \forall x.\psi) & \vdash \forall x.(\varphi \vee \psi) \to (\varphi \vee \forall x.\psi) \end{array}$$

(see DeVidi, 1995, pp.538-539, for details and proofs.)

4.2 Syntax

The syntax we present will work with either ε and τ axioms. Instead of separate ε and τ-calculus languages. We follow DeVidi in defining a σ-calculus to indicate how we will treat any term-forming operators (DeVidi, 1995, pp.524-527).

We will sketch the language noting the standard features and making careful note where the language deviates from standard first order logic.

The language we will be using needs all the standard notational conventions of normal first order logic.

However, we also need some extra tools. Syntactically speaking, term forming operators are, from the point of view of standard logic, an unusual hybrid. They take, as input, a variable and a formula and return a *term*; in standard logic the quantifiers take the same input and return a *formula*, while function symbols take terms as input and return terms as output. This means that the syntax of languages with term forming operators take a bit of extra attention. We will describe the syntax of a language without identity in some detail; the additions necessary to make a language with identity are straightforward, so we include fewer details.

We start with a first order language $\mathcal{L}_\sigma$ (referred to as $\mathcal{L}_\varepsilon$ if is has the ε axiom or $\mathcal{L}_\tau$ if it has the τ axiom) with the connectives, standard punctuation, and the general term-forming operator σ which will be written ε or τ depending on which axiom scheme is used.

The variables form a countably infinite sequence $\text{Var} = \{v_1, v_2, v_3, ...\}$ and the set of *constants*: $\text{Con} = \{c_1, c_2, c_3, ...\}$ which may be empty, finite or countably infinite. Likewise the predicates form a set $\text{Pred} = \bigcup_{1 \leq n \leq \omega} \text{Pred}_n$ where for each n Pred_n is a possibly empty set of n-ary predicate symbols, but which is not empty for at least one n.

It may also include any number of function symbols of each finite "arity," and may include the special two place predicate $=$.

4.2.1 Notational Conventions

As usual, we will describe certain notational conventions for our logic, the following will be used as arbitrary variables (x, y, z, u, v, w) and arbitrary constants (**a, b, c**) possibly with subscripts. In addition we will describe conventions for arbitrary terms (s, t, r) and arbitrary formulas (φ, ψ, α, β, γ), and arbitrary expressions (M and N). Though we shall also present predicates ($A(x), B(x), C(x), ..., A(x,y), B(x,y), C(x,y), ...$) indicating that x (or x and y) are among the free variables in the predicate.

Normally we can define the terms in two separate recursive definitions, but given the syntactically hybrid nature of term forming operators we cannot do the same thing here. Instead, we must use a single recursive definition including the clauses for the normal definitions of formulas and of terms, and the additional clause that:

$$x \in Var, \varphi \in \text{ Wff } \Longleftarrow \sigma x.\varphi \in Term$$

We will assume normal definitions of *atomic formula, identity free formula* and note that the usual notion of scope applies both to quantifiers ($\forall, \exists$) and term forming operators (σ *i.e.* ε, and τ). In addition, will assume that c and d are replacing the same variable in $\varphi(c)$ and $\varphi(d)$, *i.e.* $\varphi(c/x)$ $\varphi(d/x)$, unless we explicitly note otherwise.

Free and Bound Variables in a Term Forming Calculus

The notion of free and bound variables in a logic with a term forming operator must include definitions for terms as well as formulas. We follow DeVidi (1995) in the following definitions:

A given occurrence of a variable x is free in the following cases:

in any term that is (a) a variable or (b) a function term $ft_1...t_n$ and $x \equiv t_i$ or x occurs in t_i and is free there

$\varphi \equiv Pt_1...t_n$ and $x \equiv t_i$ or that occurrence of x is free in t_i for some $1 \leq i \leq n$

$\varphi \equiv \neg\psi$ and the occurrence of x is free in ψ

$\varphi \equiv \alpha \vee \beta$, $\varphi \wedge \alpha \vee \beta$, or $\varphi \rightarrow \alpha \vee \beta$ and the occurrence of x is free in α or β

$\varphi \equiv \forall y\psi$ or $\varphi \equiv \exists y\psi$ and $x \not\equiv y$ and the occurrence of x is free in ψ

$t \equiv \sigma y\psi$, and $y \not\equiv x$ and the occurrence of x is free in ψ then that occurrence is free in t

A given occurrence of a variable x is bound in the following cases:

$\varphi \equiv \forall x\psi$ or $\varphi \equiv \exists x\psi$ then every occurrence of x is bound in φ

$t \equiv \sigma x\psi$ then every occurrence of x is bound in t

Moreover, each free occurrence of x in ψ is said to be bound by the occurrence of $\forall x$, $\exists x$, or σx, respectively (for details see DeVidi, 1995, pp.525-526).

Free Variable in an expression

The set of variables with at least one free occurrence in an expression M, which we denote by $FV(M)$. More generally, a free occurrence of a term t in M, is such, that there is no variable $x \in FV(t)$ nor $x \notin FV(N)$ for any subterm N of M.

The set of free terms of M, which are terms that have a free occurrence in M, and are not proper subterms of terms that have a free occurrence in M, is denoted by $FT(M)$. DeVidi notes that:

> this means that it will not in general be the case that $FV(M) \subseteq FT(M)$. This will give us some needed flexibility later when discussing various interpretations of the σ-terms. (DeVidi, 1995, p.526)

Definition of substitution

Now we need to define substitution of free variables in well formed expressions *i.e* in formulas and terms. To the usual definition (see Kleene 1952, p.78) we add the following clause to deal with our "hybrid" terms:

> if $s \equiv \sigma y\beta$ and either (i) $x \notin FV(s)$ or (ii) $x \in FV(s)$ (and so $x \not\equiv y$) and y does not occur in t and t is free for x in β, then t *is free for* x *in* s. (DeVidi, 1995, p.526)

Since Var is infinite, for any finite set $\{y_1, ..., y_n\} \subseteq Var$ in any formula φ or term t, we can always find n "fresh" variables with which to uniformly replace them, thus generating a variant φ' or t' for which any particular term t is free for x; so, as usual, we shall often assume that such substitutions have been made as necessary without further comment except when such details are salient to the discussion in question.

4.3 DeMorgan's Law and the Intuitionistic ε-calculus

Bell's proof shows that we can prove DeMorgan's intuitionistically invalid law: $\neg(B \wedge C) \vdash \neg B \vee \neg C$, in intuitionistic ε-calculus with an a and b such that $\forall x(x = a \vee x \neq a)$ and $(a \neq b)$. While Bell's is obviously an interesting result, and he was breaking new ground in his results, this formulation of the decidability condition is potentially misleading if we are interested in the philosophical implications of this result. The use of the identity predicate is mathematically natural, but identity is in itself a metaphysically freighted notion. After reviewing Bell's results we will therefore include in the thesis a new version of this proof, which replaces Bell's decision condition with one that uses a version of apartness defined by the following decision condition: $\forall x((Px \rightarrow Pa) \vee \neg(Px \rightarrow Pa))$ and two constants such that $\neg(Pb \rightarrow Pa)$.

4.3.1 Bell's Second Proof of DeMorgan's Laws from Epsilon

Bell gives two distinct proofs of DeMorgan's law from the ε-axiom. In the first he shows that ε implies what he calls "Markov's principle:"

$$\forall x[\neg\neg A(x) \rightarrow A(x)] \rightarrow [\neg\forall x A(x) \rightarrow \exists x \neg A(x)] \text{ for a decidable predicate } A(x).$$

The principle can be characterized as showing that the "infinitary" version of the De Morgan law holds for "decidable" predicates $A(x)$. Using the aforementioned modest decidability conditions, Bell is able to derive De Morgan's law in its propositional form. We leave aside the details of this proof to look instead at the second proof where Bell moves more directly from the epsilon principle to De Morgan's law.[3]

Recall that DeMorgan's intuitionistically invalid law is: $\neg(B \wedge C) \vdash \neg B \vee \neg C$. To prove DeMorgan's intuitionistically invalid law Bell assumes what he refers to as a "modest 'decidibility' condition" **D**: $\vdash_I \forall x(x = \mathbf{a} \vee x \neq \mathbf{a})$, and a constant **b** such that $\vdash_I \mathbf{a} \neq b$.

Theorem 4.3.1 (Bell's Second Proof of DeMorgan's Laws)**.** *Given the following decidibility condition:*

$$(\boldsymbol{D}) \vdash_I \forall x(x = a \vee x \neq a)$$

*and a constant **b** such that*

$$\vdash_I a \neq b$$

then

$$\neg(B \wedge C) \vdash \neg B \vee \neg C$$

[3]Beeson descibes Markov's constructivism, giving a slightly different formalization of his eponymous superintuitionistic principle as follows:

> The reasoning by the Markov school is formalized in the intuitionistic predicate calculus, with one additional principle. The additional principle has been given Markov's name. *Markov's principle* is
>
> $$\forall x \in \mathbb{R}(\neg x \leq 0 \rightarrow x > 0)$$
>
> (Beeson, 1980, p.47)

Proof. We define a formula $A(x)$ in term of B and C:

$$(*) \quad A(x) \leftrightarrow [(x = a \wedge B) \vee (x \neq a \wedge C)].$$

Given the decidability conditions we have:

$$\vdash_I A(a) \leftrightarrow B$$

and

$$\vdash_I A(b) \leftrightarrow C,$$

and hence:

$$(**) \quad \vdash_I \neg A(x) \leftrightarrow [(x = a \to \neg B) \wedge (x \neq a \to \neg C)]$$

Recall the epsilon axiom:

$$\varphi(x) \to \varphi(\varepsilon_\varphi)$$

Which is equivalent to:

$$\vdash \exists x \varphi(x) \leftrightarrow \varphi(\varepsilon_\varphi)$$

Appyling this to the $A(a)$ and $A(b)$,

$$\neg A(a) \vdash_I \neg A(\varepsilon_{\neg A})$$

and

$$\neg A(b) \vdash_I \neg A(\varepsilon_{\neg A})$$

Since $\neg A(a) \leftrightarrow B$, we have, $\neg B \vdash \neg A(a)$, and so,

$$\neg B \vdash_I \neg A(\varepsilon_{\neg A})$$

and similarly:

$$\neg C \vdash_I \neg A(\varepsilon_{\neg A})$$

And so we can say:

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_I \neg\neg B$$

and

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_I \neg\neg C$$

Combining these two definitions we can state:

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_I \neg\neg B \wedge \neg\neg C$$

Since it holds intuitionistically that:

$$\neg\neg B \wedge \neg\neg C \vdash_I \neg\neg(B \wedge C)$$

We can say that:

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_I \neg\neg(B \wedge C)$$

Hence:

$$\neg\neg\neg(B \wedge C) \vdash_I \neg\neg\neg A(\varepsilon_{\neg A})$$

But since it is true in intuitionistic logic that for any φ:

$$\vdash_I \neg\neg\neg\varphi \leftrightarrow \neg\varphi$$

We have:

$$\neg(B \wedge C) \vdash_I \neg A(\varepsilon_{\neg A})$$

Recall above that we were able to show:

$$(**) \quad \vdash_I \neg A(x) \leftrightarrow [(x = a \rightarrow \neg B) \wedge (x \neq a \rightarrow \neg C)]$$

Now by substituting "$\varepsilon_{\neg A}$" for "x" in (**) we obtain:

$$(***) \quad \neg(B \wedge C) \vdash_I [(\varepsilon_{\neg A} = a \rightarrow \neg B) \wedge (\varepsilon_{\neg A} \neq a \rightarrow \neg C)]$$

Recall the decision condition for objects **D**:

$$(\mathbf{D}) \quad \vdash_I \forall x(x = a \vee x \neq a)$$

Now we apply the decision condition:

$$\vdash_I \varepsilon_{\neg A} = a \vee \varepsilon_{\neg A} \neq a$$

This with (***) gives us

$$\neg(B \wedge C) \vdash_I \neg B \vee \neg C$$

□

4.4 Proof of DeMorgan's Laws in Intuitionisitic Logic + ε without Identity

While the decidability condition in theorem 4.3.1 seems a reasonably weak principle, the proof can in fact be done using a weaker principle. We do not need a decidability condition that includes equality of objects. We can prove it using apartness $\forall x(\neg(x \neq a) \vee x \neq a)$ and $(a \neq b)$, or with the following decidability condition: $\forall x((Px \to Pa) \vee \neg(Px \to Pa))$ and two constants such that $\neg(Pb \to Pa)$.

Theorem 4.4.1. *In intuitionistic predicate calculus with the ε axiom:*

$$(\varepsilon) \quad \vdash \exists x\varphi(x) \leftrightarrow \varphi(\varepsilon_\varphi)$$

and a decidability condition:

$$(\mathbf{D}^*) \quad \forall x((Px \to Pa) \vee \neg(Px \to Pa))$$

and two constants a and b such that:

$$\neg(Pb \to Pa)$$

we can derive De Morgan's intuitionistically invalid law

$$\neg(B \wedge C) \vdash_I \neg B \vee \neg C$$

Proof. We begin with our decision principle:

$$(\mathbf{D}^*) \quad \vdash_I \forall x((Px \to Pa) \vee \neg(Px \to Pa))$$

Then we define the predicate A in the following manner assuming that x is not free in either B or C:

$$(*') \quad A(x) \equiv ((Px \to Pa) \wedge B) \vee (\neg(Px \to Pa) \wedge C))$$

It follows that:

$$\vdash_I A(a) \leftrightarrow B$$

and

$$\vdash_I A(b) \leftrightarrow C$$

Hence:

$$\vdash_I \neg A(x) \leftrightarrow \neg[((Px \to Pa) \wedge B) \vee (\neg(Px \to Pa) \wedge C))]$$

and by De Morgan's intuitionisitically valid law:

$$\vdash_I \neg A(x) \leftrightarrow [\neg((Px \to Pa) \wedge B) \wedge \neg(\neg(Px \to Pa) \wedge C)]$$

which by the intuitionistic valid axiom: $(\varphi \to \neg\psi) \leftrightarrow \neg(\varphi \wedge \psi)$ gives us:

$$(**') \vdash_I \neg A(x) \leftrightarrow [(Px \to Pa) \to \neg B) \wedge (\neg(Px \to Pa) \to \neg C)]$$

By the epsilon axion we can now say:

$$\neg A(a) \vdash_I \neg A(\varepsilon_{\neg A})$$

and

$$\neg A(b) \vdash_I \neg A(\varepsilon_{\neg A})$$

And since $\neg B \vdash_I \neg A(a)$, we have $\neg B \vdash_I \neg A(\varepsilon_{\neg A})$, and so:

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_I \neg\neg B.$$

Similarly,

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_I \neg\neg C$$

so we have:

$$\neg\neg A(\varepsilon_{\neg A}) \vdash_I \neg\neg B \wedge \neg\neg C$$

and:

$$\neg\neg\neg(B \wedge C) \vdash_I \neg\neg\neg A(\varepsilon_{\neg A})$$

Hence since intuitionistically $\neg\neg\neg\varphi \vdash \neg\varphi$ is true, we can say:

$$\neg(B \wedge C) \vdash_I \neg A(\varepsilon_{\neg A}).$$

By (**') we get:

$$\neg(B \wedge C) \vdash_I [(P\varepsilon_{\neg A} \to Pa) \to \neg B) \wedge (\neg(P\varepsilon_{\neg A} \to Pa) \to \neg C)]$$

and by $\mathbf{D}^*$ we get:

$$\neg(B \wedge C) \vdash_I \neg B \vee \neg C$$

$\square$

4.5 Linearity Axiom and Intuitionistic Predicate Calculus + ε

Bell also provides a proof of Dummett's scheme, also known as linearity, in intuitionistic type theory (Bell, 1993a). A similar proof can be made for first order intuitionistic logic $+\varepsilon$ and first order intuitionistic logic $+\varepsilon$ without identity.

4.5.1 Dummett's Scheme

Bell provides a proof of Dummett's scheme in a intuitionistic type theory (Bell, 1993a, p 334) which he remarks can be modified to be a proof for an intuitionistic ε–calculus. Here we provide a proof in intuitionistic ε–calculus with identity.

Theorem 4.5.1 (Dummett's Scheme from Intuitionistic ε–Calculus with Identity)**.** *Assuming a intuitionistic–ε calculus with constants* 0 *and* 1 *such that* $\forall x(x = 0 \vee x \neq 0)$ *and* $(0 \neq 1)$ *then for any sentence* B *and* C*:*

$$\vdash (B \to C) \vee (C \to B)$$

Proof. Define:

$$A(x) \equiv (x = 0 \wedge B) \vee (x = 1 \wedge C)$$

Then $\vdash A(0) \leftrightarrow B$ and $\vdash A(1) \leftrightarrow C$, so:

$$\vdash \exists x\, A(x) \leftrightarrow B \vee C$$

Also since $(x \neq 0 \wedge A(x)) \vdash C$, we have:

$$x \neq 0 \vdash A(x) \to C$$

Recall the epsilon axiom yeilds:

$$\vdash \exists x A(x) \leftrightarrow A(\varepsilon_A)$$

so we can state:

$$\vdash A(\varepsilon_A) \leftrightarrow B \vee C$$

and

$$\vdash A\varepsilon_A \leftrightarrow (\varepsilon_A = 0 \wedge B) \vee (\varepsilon_A \neq 0 \wedge C)$$

hence:

$$\begin{aligned}
&\vdash B \vee C \to A(\varepsilon_A) \\
&\vdash ((B \vee C) \to A(\varepsilon_A)) \wedge (\varepsilon_A = 0 \vee \varepsilon_A \neq 0) \\
&\vdash (((B \vee C) \to A(\varepsilon_A)) \wedge \varepsilon_A = 0) \vee (((B \vee C) \to A(\varepsilon_A)) \wedge \varepsilon_A \neq 0) \\
&\vdash ((B \vee C) \to A(0)) \vee (((B \vee C) \to A(\varepsilon_A)) \wedge A(\varepsilon_A) \to C) \\
&\vdash ((B \vee C) \to B) \vee ((B \vee C) \to C) \\
&\vdash (C \to B) \vee (B \to C)
\end{aligned}$$

as required. □

Theorem 4.5.2 (Dummett's Scheme from from Intuitionistic ε–Calculus without Identity)**.** *Assuming a intuitionistic–ε calculus with a predicate P, and constants a and b such that $\forall x\ ((Px \to Pa) \vee (Px \to Pb))$ and $\neg(Pb \wedge Pa)$*[4] *then for any sentences B and C:*

$$\vdash (B \to C) \vee (C \to B)$$

Proof. Define:

$$A(x) \equiv ((Px \to Pa) \wedge B) \vee ((Px \to Pb) \wedge C)$$

Then by the definition of A and the decidability principle we have $\vdash A(a) \leftrightarrow B$ and $\vdash A(b) \leftrightarrow C$, so:

$$\vdash (B \vee C) \to \exists x\, A(x)$$

the rest of the proof follow *ceteris paribus* from the ε-axiom.

Recall the epsilon axiom:

[4] One could also say $\forall x\ ((Px \to Pa) \vee (Px \to \neg Pa))$ and $\neg Pa \to Pb$.

$$\vdash \exists x A(x) \leftrightarrow A(\varepsilon_A)$$

so we can state:

$$\vdash (B \vee C) \rightarrow A(\varepsilon_A)$$

hence:

$$\begin{aligned}
&\vdash (B \vee C) \rightarrow A(\varepsilon_A)\\
&\vdash (B \vee C) \rightarrow A(\varepsilon_A) \wedge ((A\varepsilon_A \rightarrow Pa) \vee (A\varepsilon_A \rightarrow Pb))\\
&\vdash (((B \vee C) \rightarrow A(\varepsilon_A)) \wedge ((A\varepsilon_A \rightarrow Pa)) \vee (((B \vee C) \rightarrow A(\varepsilon_A)) \wedge (A\varepsilon_A \rightarrow Pb))\\
&\vdash ((B \vee C) \rightarrow Pa) \vee ((B \vee C) \rightarrow Pb)\\
&\vdash ((B \vee C) \rightarrow B) \vee ((B \vee C) \rightarrow C)\\
&\vdash (C \rightarrow B) \vee (B \rightarrow C)
\end{aligned}$$

as required.

□

4.6 Extensional Epsilon and the Law of Excluded Middle

Bell provides a proof that the addition of an axiom of "epsilon extensionality" (*i.e.*, that the identity of epsilon terms is determined by the extension of the predicates out of which the epsilon term is formed) to the intuitionistic epsilon calculus makes the law of excluded middle (and so all of classical logic) provable (Bell, 1993a). We will first present a version of Bell's proof, then show once again that a weaker principle not involving identity is sufficient to get the result.

4.6.1 The Proof of the Law of Excluded Middle from Intuitionisitic Logic+ε and the Axiom of Epsilon Extensionality

The axiom of epsilon extensionality further extends the "decision" condition on objects.

$$\forall x[A(x) \leftrightarrow B(x)] \rightarrow \varepsilon_A = \varepsilon_B$$

(Bell, 1993a, p6)

In his proof Bell proves a Lemma:

Lemma 4.6.1. *Let T be a theory in a first order language $\mathcal{L}$ with constants **0** and **1** such that $\vdash_T 0 \neq 1$; and for any predicate $A(x)$ define:*

$$B(x, y) \equiv y = 0 \vee A(x)$$
$$C(x, y) \equiv y = 1 \vee A(x)$$

Suppose that in $\mathcal{L}$ there are terms s and t such that:

$$\textit{(1)} \quad \vdash_T B(x, s) \wedge C(x, t)$$

and

$$\textit{(2)} \quad A(x) \vdash_T s = t$$

then

$$\vdash_T A(x) \vee \neg A(x)$$

Proof. From (1) we know that $\vdash_T [s = 0 \vee A(x)] \wedge [t = 1 \vee A(x)]$

by distributivity we get:

$$\vdash_T [s = 0 \wedge t = 1] \vee A(x)$$

so we can say:

$$(3) \qquad \vdash_T s \neq t \vee A(x)$$

however we know:

$$(2) \quad A(x) \vdash_T s = t$$

so:

$$s \neq t \vdash_T \neg A(x)$$

so by (3) we get:

$$\vdash_T A(x) \vee \neg A(x)$$

□

Theorem 4.6.2. *Suppose that for each formula B(x,y) of $\mathcal{L}$ such that* $\vdash \exists y B(x,y)$

(1') $B(x, \varepsilon_{B(x)})$
(2') $\forall y[B(x,y) \leftrightarrow C(x,y)] \rightarrow \varepsilon_{B(x)} = \varepsilon_{C(x)}$

Then for any formula $A(x)$*,*

$\vdash A(x) \vee \neg A(x)$

Proof. Consider $B(x,y)$ and $C(x,y)$ as defined in the lemma 0 and 1 respectively ensure that:

$\vdash \exists y B(x,y)$ and $\vdash \exists y C(x,y)$. Taking s and t as $\varepsilon_{B(x)}$ and $\varepsilon_{C(x)}$ (1') yields (1) and (2') yields (2). The conclusion follows from Lemma 4.6.1. □

4.6.2 Proof of LEM from Intutionistic Logic+ε and a Weakened Extensional Axiom

This proof, like Bell's of DeMorgan's Law, can be done without identity. We can replace the ε-extensionality axiom using identity with the following principle:

For all predicates $A(x)$, $B(x)$ and $P(x)$:

$$\forall x(A(x) \rightarrow B(x)) \rightarrow (P\varepsilon_A \rightarrow P\varepsilon_B).$$

Now we can revise lemma 4.6.1 as follows:

Lemma 4.6.3. *Given a formula $A(x)$, define $B(x,y)$ and $C(x,y)$:*

$$B(x,y) \equiv Py \vee A(x) \text{ and } C(x,y) \equiv \neg Py \vee A(x)$$

Suppose we have s and t such that $\vdash B(x,s) \wedge C(x,t)$, and $A(x) \vdash Ps \to Pt$. Then

$$\vdash A(x) \vee \neg A(x).$$

Proof. By distributivity on $\vdash B(x,s) \wedge C(x,t)$ and the definitions of $B(x,y)$ and $C(x,y)$ we have:

$$\vdash (Ps \wedge \neg Pt) \vee A(x)$$

From $A(x) \vdash Ps \to Pt$ we have:

$$\neg(Ps \to Pt) \vdash \neg A(x)$$

Since

$$(Ps \wedge \neg Pt) \vdash \neg(Ps \to Pt)$$

we have:

$$\vdash \neg A(x) \vee A(x)$$

as required. □

The theorem and proof follow *ceteris parabis*:

Theorem 4.6.4. *Suppose that for each formula B(x,y) of $\mathcal{L}$ such that $\vdash \exists y B(x,y)$ we have:*

(1") $B(x, \varepsilon_{B(x)})$
(2") $\forall y[B(x,y) \to C(x,y)] \to (P\varepsilon_{B(x)} \to P\varepsilon_{C(x)})$

Then for any formula $A(x)$

$$\vdash A(x) \vee \neg A(x)$$

Proof. $B(x,y)$ and $C(x,y)$ are defined by the lemma 4.6.3, so from $\vdash \exists y B(x,y)$ and $\vdash \exists y C(x,y)$. Taking s and t as $\varepsilon_{B(x)}$ and $\varepsilon_{C(x)}$ $(1'')$ yields $(B(x)\varepsilon_{B(x)} \wedge C(x)\varepsilon_{C(x)})$ and $(2'')$ yields $A(x) \vdash P\varepsilon_{B(x)} \to P\varepsilon_{C(x)}$. The conclusion follows from Lemma 4.6.3. □

Chapter 5

Semantics for Intuitionistic Logic $+\,\varepsilon$

In this chapter we will discuss two semantics for intuitionistic ε-calculus. The first is developed by John Bell in his 1993 paper "Hilbert's ϵ-operator and classical logic"(Bell, 1993a) and is sound but not complete. The second sound and complete semantics is presented by David DeVidi in his 1994 thesis (DeVidi, 1994) and summarized in his 1995 paper "Intuitionistic ε- and τ-calculi" (DeVidi, 1995). We will first present Bell's semantics as it is a development of the standard semantics presented for classical ε-calculus by Asser (Asser, 1957; Leisenring, 1969). We will then sketch DeVidi's semantics, specifically focusing only on differences between Bell's and DeVidi's semantics. These differences will present some insight for why the addition of ε to intuitionistic logic makes a logic super-intuitionistic in an interesting manner.

5.1 Semantic Considerations

There are three issues that must be dealt with in creating a semantics for an intuitionisitic ε-calculus. The first is dealing with the problem of multiple truth values, the second is dealing with non-linear truth sets and the third is the problem of trying to avoid extensionality.

The first problem is that Heyting algebras, which make up sets of truth values of intuitionisitic logics, typically have more than two truth values. Hence while there may be nothing that, when substituted for x in $\varphi(x)$, makes $\varphi(x)$ *true*, it doesn't follow that all the things we might substitute for x make it *false*. That is, there will be some elements that make $\varphi(x)$ truer than do others. This is a problem with the Asser-Leisenring semantics: call the "truth set" for $\varphi(x)$ the set of things that, when substituted for x, make $phi(x)$ true. The semantics uses a choice function in the metalanguage to select a member of the truth set for $\varphi(x)$, and allow it to be the referent of $\varepsilon x\varphi(x)$. But if nothing makes that formula completely true, then the truth set for φ is the empty set. The semantics assigns an arbitrary member of the domain to the empty set (since there is nothing that can be selected from within it). One thing that makes this problematic in the presence of more than two truth values is that the epsilon formula is not made valid, since $\exists x.\varphi(x)$ can have a non-true, non-false value, but there is no guarantee that $\varphi(x)$ will have the same value when the arbitrarily chosen element is substituted in for x. Bell's answer to this is to put a restriction on the kinds of Hetying algebras that can serve as lattices of truth values: they must be inversely well-ordered sets. This guarantees that while the truth set must be open, there is always a "greatest truth value" that is attained $\varphi(x)$ when the various elements of the domain are substituted for x. He then has the choice function select from the set of elements that "achieve" that greatest value, thus making the epsilon principle valid.

Bell's solution however does not work for all Heyting algebras, only the subset of them which are inversely well ordered sets. This is the key reason that his semantics is sound but not complete. To derive a semantics which is valid for all Heyting algebras DeVidi semantics puts a restriction on what sorts of interpretations are permitted, rather than what sort of algebras are permitted.

One of the main problems with creating a choice function is avoiding making the epsilon extensionality principle (Ack) true.[1] This is a problem that DeVidi (DeVidi, 1995, 1994) and Bell (Bell, 1993b) approach in different ways. As noted above, in a classical ε-calculus the choice function picks out an element of the truth set for φ. The solution works for non-empty sets but with empty sets the method simply picks out an arbitrary element.

This means that for say three propositions φ, ψ, and χ all of which have empty truth sets, that $[\![\varepsilon x\varphi]\!] = [\![\varepsilon x\psi]\!] = [\![\varepsilon x\chi]\!]$. But what if all three are not intensionally the same? For example let φ be "the smallest prime that is composite", ψ be the "dog that is also a cat" and χ be "a square circle". The extension of all three of these generally is taken to be the empty set. However they are not empty for the same reasons. Certainly if we want our ε-operator to pick out the "most likely element to have a property," even if the property is impossible it is reasonable that they should pick out different elements, even if there are no elements that can possibly fit the criteria. Even though in the case of χ no squares are round, and hence the set of round squares is empty, an argument can be made that the most likely element to be a round square should either be round or be square, or, if not that, at least it should be a two dimensional shape and not say a shade of the colour blue or the state bird of Western Australia[2].

Likewise in the case of φ, the smallest prime that is composite, since primes are by definition not composite, it is just as likely (or unlikely) for any particular prime to be

[1]Known as (Ack) after Wilhelm Ackerman in whose 1924 PhD thesis ε made one of its first appearances (Ackermann, 1924, p.8).

[2]*i.e.*the Black Swan

composite, but again the candidates are all natural numbers, not geometric shapes or mammals. Perhaps the case of "dogs that are cats" is the easiest to solve, while it may be taxomologically impossible one could argue that it is not logically impossible, a common ancestor, some sort of hybrid, or perhaps just a rating of behaviour characteristics might be how one would assert that a certain type or breed of dog is most cat like.

While the examples given above are all varieties of "impossible predicates," under any particular interpretation the epsilon term for *contingently empty* predicates will also receive the same referent—the "likeliest square circle" and the "likeliest female Canadian Prime Minister before 1990" are the same thing.

This is really an instance of a more general problem. By selecting an element from the truth sets for formulas, one is likely to make valid the extensionality principle appealed to in the proof of the law of excluded middle in Chapter 4, which states that if $\forall x(\varphi(x) \leftrightarrow \psi(x))$, then φ and ψ have the same truth set. The proof in Chapter 4, however, shows that this extensionality principle is not something we might want to assume to be generally correct, since it seems to pack a proof-theoretic wallop. To avoid this, DeVidi's semantics makes the reference of epsilon terms depend not only on the semantic values a formula gets for each possible substitution, but also on the syntactic structure of the formula. Predictably, the results are somewhat messy. They do, however, yield a sound and complete semantics for the intuitionistic epsilon calculus.

5.2 Classical Semantics for ε -Calculus

Considering why the standard semantics for the classical ε calculus can be fairly simple will be instructive for what follows. First, it is useful to have in hand the notion of a truth set for a formula under an interpretation (and, for simplicity, assume that the formula has exactly one free variable). This is, as the name suggests, precisely the subset of the domain of interpretation that includes all and only those elements that make the formula true when the free variable refers to that element—it is the set of things of which the formula is true. If φ is such a formula, then if the epsilon principle is to be satisfied, $\varepsilon\varphi$ must be something that makes φ true if anything does. If nothing makes φ true, then it really doesn't matter to what $\varepsilon\varphi$ refers. In short, what is needed to turn an ordinary interpretation of a predicate language into an interpretation that satisfies the ε calculus is simply that we be able to select an element of the truth set for each formula φ, and assign that element as the interpretation of $\varepsilon\varphi$. This is precisely what Asser and Leisenring do—a classical ε interpretation is simply a classical interpretation to which a choice function on the domain has been added (Asser, 1957; Leisenring, 1969).[3] The definition of

[3]The function f is defined as a map from the powerset of the domain D of the structure $\mathcal{L}$ to that domain, *i.e.* $f : \mathcal{P}(D) \longrightarrow D$: specifically f is a function, such that, for $\emptyset \neq X \subseteq D$, $f(X) \in X$,

interpretation is then amended so that $\varepsilon\varphi$ is interpreted by the element selected from the truth set of φ if it is non-empty, and to the element of the domain the function assigns to the empty set otherwise.

It is important to note a few features of classical, two-valued logic that this simple semantics depends on. First, it is clear that this semantics will satisfy Ackermann's extensionality principle. This is harmless in classical logic, where adding the epsilon operator is conservative over the classical predicate calculus, whether the epsilon terms are determined extensionally or not. However the extensionality assumption is not harmless in the intuitionistic case. The second thing to note is that this simple semantics depends on two-valuedness. But suppose there are intermediate truth values between F and T [4]. In such a semantics, there can be interpretations under which the truth set of φ is *empty*, and yet not every element of the domain, when assigned to the free variable in φ, gives it the truth value F. In this case, the epsilon principle requires that the truth value of $\phi(\varepsilon\varphi)$ be at least as great as the truth value of $\exists x.\varphi$.

DeVidi considers classical semantics with more than two truth values (*i.e.* Boolean valued models larger than the linear algebra **2**) for epsilon (see DeVidi, 1994, pp.201*ff.*). Of course, Boolean valued models can be reduced to two valued models—the addition of extra models doesn't really generate any new counter-examples, so the Boolean-valued and two valued epsilon semantics have the same classes of validities so that aspect of DeVidi's work is of less interest for present purposes. However, no similar reduction is possible in the intuitionistic case, so the techniques developed in the Boolean valued case are essential for developing a (Heyting valued) semantics for the intuitionistic epsilon calculus.

5.3 Existing Semantics for Inutitionistic ε-Calculus

A complete semantics for first order intuitionistic logic with identity and the ε-axiom is provided by (DeVidi, 1994, 1995). However a simpler model theory is provided by Bell (1993a), one which is sound but not complete. This soundness proof for first order intuitionistic logic with identity and the ε-axiom (Bell, 1993a, pp.13-17) can be easily modified for the intuitionisitic ε-calculus with or without identity.

---

and $f(\emptyset) = d$ for an arbitrary fixed element $d \in D$. An evaluation function ϱ that maps variables onto the members of the the domain D defined as: $\varrho : Var \longrightarrow D$ and $\varrho(x/d) : Var \longrightarrow D$ be such that $\varrho(z/d)(z) = d$ and $\varrho(z/d)(y) = \varrho(y)$ for $y \neq x$. Epsilon terms such as $\varepsilon x.\varphi$ are then interpreted in the following manner: take the subset $X \subseteq D$ that makes φ true on the the valuation $\varrho(x/d)$, the interpretation of $\varepsilon x.\varphi$ under ϱ is $f(X)$. The truth-set for φ and x under ϱ is X, and $\varepsilon x.\varphi$ is understood to be a member of that truth-set, except when it is empty, in which case $\varepsilon x.\varphi$ refers to $f(\emptyset) = d$ the arbitrary element of the domain. Defining the semantics for a classical epsilon calculus in the above manner means that ε-terms for formulas that have co-extensive truth-sets receive the same interpretation.

[4] Or whatever your two truth values may be, *e.g.* $\perp$ and $\top$, *true* and *false*, or 1 and 0

The difference between these two show interestingly how the structure of intuitionistic ε-calculus is different from intuitionistic logic. Some of the limitations of the model, *i.e.* that certain subsets of the domain have to be non-empty, indicate how the models are limited and hence might highlight how choice principles strengthen logical principles by limiting the acceptable structures their domains may take.

5.4 Bell's Semantics for Intuitionistic First Order Logic + ε

It is useful to begin with a definition of a standard Heyting algebra semantics for intuitionistic logic. Let $\mathscr{L}$ be a standard first order language. The semantics for intutionistic logic are similar to those for classical logic. However for non-classical logics instead of assuming that the only two truth values are $\top$ and $\bot$ (or T and F, or 1 and 0), we specify an "algebra of truth values" and make this an explicit part of the interpretation. For intuitionistic logic we require that the algebra be a Heyting algebra.

A Heyting Algebra (HA) is a lattice with a bottom element and relative pseudo-complementation[5]. We will define a lattice, pseudo-complementation, and finally a Heyting algebra in the following manner:

Definition 5.4.1: We can define a *lattice* in the following way. Let $S =< S, \leq>$ be an ordered set, *i.e.* $\leq$ is a reflexive, transitive, and anti-symmetric relation on S. For $T \subseteq S$, we now define the upper and lower bounds in the following manner: if $\forall t \in T[t \leq s]$ then s is an upper bound for T, $\bigvee T$, if it exists, denotes the least upper bound of T, also called the join of T. Dually, if it exists, $\bigwedge T$ denotes the greatest lower bound, or the meet of T. We write $x \vee y$ for $\bigvee\{x, y\}$, and $x \wedge y$ for $\bigwedge\{x, y\}$.

Definition 5.4.2: *Pseudo-complementation* can be defined in the following manner: for a lattice L and $a, b \in L$; a is said to have a pseudo-complement relative to b if the set $\{c \in L \mid c \wedge a \leq b\}$ has a maximal element.

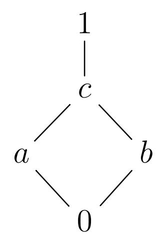
Definition 5.4.3: A *Heyting Algebra* is an ordered quadruple $\langle H, \leq, \Rightarrow, 0\rangle$ such that $\langle H, \leq\rangle$ is a lattice, where 0 is the minimal element, and $\Rightarrow$ is a binary operator on H such that for all $x, y, z \in H$, $x \leq y \Rightarrow z \Leftrightarrow x \wedge y \leq z$. In other words the $\Rightarrow$ operator picks out a join, *i.e.* $y \Rightarrow z = \bigvee \{x \mid x \wedge y \leq z\}$.

[5] A Heyting algebra is sometimes defined as a Brouwerian Lattice with a bottom element. A Brouwerian Lattice, or implicative lattice, is a lattice with relative pseudo-complementation. However the terminology is not uniform, in some of the literature a Brouwerian Algebra is taken to mean the same thing as a Co-Heyting Algebra, the dual of Heyting Algebra. Heyting Algebras, Co-Heyting Algebras, and Brouwerian Algebras are all also referred to collectively as Pseudo-Boolean Algebras.

I will not present a comprehensive explication of Heyting algebras here. For more detail see Rasiowa and Sikorski (1963) pp.58*ff.* or Davey and Priestly (2002) pp.33 *ff.* and pp.128 *ff.*

Note that all Boolean algebras are Heyting algebras, and the set $F < T$ is a Boolean algebra, often referred to as **2**. It is worth noting a few different Heyting algebras that are important in the discussion below: all linear orders, in particular **2**, the only Boolean algebra that is linear:

Also the three element linear ordering (**3**), the natural numbers, the negative integers, *etc...* are Heyting algebras. However not all Heyting algebras are linear. For instance, all finite distributive lattices are Heyting algebras; as are the lolipop, and the inverted lolipop *i.e.*:



Other examples of Heyting algebras will be explained as need arises.

An intuitionistic model $\mathfrak{M}$ is an ordered triple $< D, H, I >$, where D is a non-empty set, and I assigns each n-ary basic predicate of $\mathscr{L}$ to a function $D^n \to H$, each constant c of $\mathscr{L}$ to an element $c^{\mathfrak{M}}$ of D, and each function symbol to a function $D^n \to D$. One should note that this describes the standard classical definition of a model, if we require that $\mathscr{L}$ be **2**. We now let α be an assignment of values to all the variables of $\mathscr{L}$: α: $Var \to D$.

Recall the language $\mathscr{L}_\varepsilon$ from the previous chapter. Bell presents a version of Liesenring/Asser semantics for $\mathscr{L}_\varepsilon$. That is he appends a choice function to an intuitionistic model so that it can be used to interpret the ε terms.

However, as noted above, there is the problem that "truth sets" do not work so simply in the presence of intermediate truth values. That is to put it more precisely, the problem is that the intepretation of $\exists x.\varphi$, *i.e.* $\bigvee_{x \in D}[\varphi]$ will not necessarily be equal to $[\varphi]^{\alpha(x/d)}$ for any particular d, so there may not be any element of D that could render the epsilon principle satisfiable.

Bell's intuitionistic epsilon model includes a choice function on the powerset of the domain with a particular sort of complete Heyting algebra as its truth set—specifically an inversely well-ordered set. Thus we follow Bell and define a $\mathscr{L}_\varepsilon - structure$ as a system of the form:

$$\mathfrak{M} = < D, L, e, eq_{\mathfrak{M}}, P_{\mathfrak{M}} >$$

where:

D is a non-empty set

L is an inversely well-ordered set with least element, $L =< L, \leq >$, such sets are a form of Heyting algebra, but also have the useful property that every subset has a maximal element and that maximal element is a member of the set. This neatly solves the problem of "unattained joins".

e is a choice function for $\mathscr{P}D$, *i.e.*, e is a map $\mathscr{P}D - \{\emptyset\} \to D$ such that $e(X) \in X$ for all $X \neq \emptyset$ in $\mathscr{P}D$

$\mathcal{P}_{\mathfrak{M}}$ is a map from each n-ary basic predicate in $\mathscr{L}_\varepsilon$ to a function from $D^n \to L$

$eq_{\mathfrak{M}}$ $D \times D \to L$ satisfies the standard equity axioms of reflexivity, transitivity, symmetry, the substitution for functions and the substitution for formulas

The choice function e defines a map from the non-empty subsets of D onto D (*i.e.* $\mathscr{P}D - \{\emptyset\} \to D$). From it we can define $\bar{e} : L^D \to D$, from the set of all functions from D to the truth set L (*i.e.* $\{f : D \to L\}$ which we write as L^D), in the following manner: for each $f \in L^D$, that is for each function $D \to L$ the set $f[D] = \{f(d) : d \in D\}$ has the maximal element $l = \bigvee f[D]$. Because L is inversely well ordered, meaning that each subset has a top element, the choice function picks the top element of each subset of L for each predictate in $\mathscr{L}_\varepsilon$.

Now we see how $\bar{e}$ is defined in terms of e in the following manner:

$$\bar{e}(f) = e(f^{-1}(\{l\})).$$

That is the choice function $\bar{e}$, is applied to $f : (D \to L)$, which picks out a member of D.

Hence we see that:

$$(*) \quad f(\bar{e}(f)) = l = \vee f[D]$$

And so we have defined a choice function on L that will give us a sound semantics if interpret each function φ and term t of our language $\mathscr{L}_\varepsilon$ in the following manner:

> given an $\mathscr{L}_\varepsilon$-structure and a map α from the set Var of variables of $\mathscr{L}$ to D (*i.e.* a *valuation* in D) we define, in the standard manner, for each formula φ and each term t of $\mathscr{L}_\varepsilon$, the *value*

$$[\![\varphi]\!]^\alpha_{\mathfrak{M}} \in L \quad [t]^\alpha_{\mathfrak{M}} \in M$$

of φ and t under α in $\mathfrak{M}$ recursively as follows, (noting that $\wedge$, $\vee$, $\rightarrow$, and $*$ on the right hand side indicate the, meet, join, relative pseudo-complement, and pseudo-complement operations in the Heyting algebra of truth values):

$$[\![t_1 = t_2]\!]^\alpha_{\mathfrak{M}} = eq_{\mathfrak{M}}([t_1]^\alpha_{\mathfrak{M}}, [t_2]^\alpha_{\mathfrak{M}})$$

$$[x]^\alpha_{\mathfrak{M}} = \alpha(x) \text{ for } x \in \mathrm{Var}$$

$$[\![Pt]\!]^\alpha_{\mathfrak{M}} = P_{\mathfrak{M}}([t]^\alpha_{\mathfrak{M}})$$

$$[\![\varphi \wedge \psi]\!]^\alpha_{\mathfrak{M}} = [\![\varphi]\!]^\alpha_{\mathfrak{M}} \wedge [\![\psi]\!]^\alpha_{\mathfrak{M}}$$

$$[\![\varphi \vee \psi]\!]^\alpha_{\mathfrak{M}} = [\![\varphi]\!]^\alpha_{\mathfrak{M}} \vee [\![\psi]\!]^\alpha_{\mathfrak{M}}$$

$$[\![\varphi \rightarrow \psi]\!]^\alpha_{\mathfrak{M}} = [\![\varphi]\!]^\alpha_{\mathfrak{M}} \Rightarrow [\![\psi]\!]^\alpha_{\mathfrak{M}}$$

$$[\![\varphi \leftrightarrow \psi]\!]^\alpha_{\mathfrak{M}} = [\![\varphi]\!]^\alpha_{\mathfrak{M}} \Leftrightarrow [\![\psi]\!]^\alpha_{\mathfrak{M}}$$

$$[\![\neg\varphi]\!]^\alpha_{\mathfrak{M}} = ([\![\varphi]\!]^\alpha_{\mathfrak{M}})^*$$

$$[\![\exists x\varphi]\!]^\alpha_{\mathfrak{M}} = \bigvee_{d \in D} [\![\varphi]\!]^{\alpha(x/d)}_{\mathfrak{M}}$$

$$[\![\forall x\varphi]\!]^\alpha_{\mathfrak{M}} = \bigwedge_{d \in D} [\![\varphi]\!]^{\alpha(x/d)}_{\mathfrak{M}}$$

where $\alpha(x/d)$ is the map which coincides with α except possibly at x where it assigns the value d.

Finally,

$$[\varepsilon x\varphi]^\alpha_{\mathfrak{M}} = \bar{e}(h_\varphi),$$

where $h_\varphi : D \rightarrow L$ is defined by:

$$h_\varphi(d) = [\![\varphi]\!]^{\alpha(x/m)}_{\mathfrak{M}}$$

> A formula φ is said to be $\mathfrak{M}$-*valid* if $[\![\varphi]\!]^{\alpha}_{\mathfrak{M}} = 1$ for every valuation φ in M, and ε-*valid*, written $\vDash_{\varepsilon} \varphi$, if φ, is $\mathfrak{M}$-*valid* for all $\mathfrak{M}$.
>
> (Bell, 1993a, p.14)

With our modified account of Bell's description above of $\mathscr{L}_{\varepsilon}$-structure, we can prove the soundness for ε-calculus.

Theorem 5.4.1 (Bell's ε-Soundness Theorem)**.**

$$\vdash_{\varepsilon} \varphi \Rightarrow \vDash_{\varepsilon} \varphi$$

for any formula φ

Proof. Since the axioms and rules of inference for intuitionistic logic are valid in any complete Heyting algebra valued structure (*cf.* Kleene 1952, pp.412ff, or Rasiowa and Sikorski 1963, pp.383–385), to prove soundness we only need to show that ε-axiom is valid in any $\mathscr{L}_{\varepsilon}$-structure (*cf.* Bell, 1993a, pp.14-15).

Observe that:

$$\bigvee h_{\varphi}[D] = \bigvee_{d \in D} [\![\varphi]\!]^{\alpha(a/d)}_{\mathfrak{M}} = [\![\exists x \varphi]\!]^{\alpha}_{\mathfrak{M}}$$

and

$$\begin{aligned} \bigvee h_{\varphi}[D] \ &= h_{\varphi}(\bar{e}(h_{\varphi})) = h_{\varphi}([\varepsilon x \varphi]^{\alpha}_{\mathfrak{M}} \\ &= [\![\varphi]\!]^{\alpha(x/[\varepsilon x \varphi]^{\alpha}_{\mathfrak{M}})}_{\mathfrak{M}} \\ &= [\![\varphi(x/\varepsilon x \varphi)]\!]^{\alpha}_{\mathfrak{M}} \end{aligned}$$

Therefore,

$$\begin{aligned} [\![\exists x \varphi \leftrightarrow \varphi(x/\varepsilon \varphi)]\!]^{\alpha}_{\mathfrak{M}} &= [\![\exists x \varphi]\!]^{\alpha}_{\mathfrak{M}} \Leftrightarrow [\![\varphi(x/\varepsilon \varphi)]\!]^{\alpha}_{\mathfrak{M}} \\ &= 1, \end{aligned}$$

as required.

□

5.5 Bell's Model and Completeness

As we have seen, Bell's semantics for the intuitionistic epsilon calculus is sound. He is therefore able to use it to establish a number of *independence results*, *i.e.* to show that certain things—in particular, the law of excluded middle—are *not* provable in the intuitionistic epsilon calculus. However, he does not present a completeness proof for the very good reason that the semantics he offers is very far from complete. That is, there are principles valid in this semantics that are not provable in the intuitionistic epsilon calculus. One glaring example is the principle of ε-extensionality, *i.e.*, Ackermann extensionality. As we have seen, if this principle *were* provable in the intuitionistic epsilon calculus, then the law of excluded middle would also be provable in the intuitionistic epsilon calculus. Yet this is precisely what Bell's independence result shows us cannot be the case.

It was considerations of this sort that led DeVidi to pursue the question of what a sound and complete semantics for various term forming operators, including epsilon, would look like. The problem with Bell's semantics shows why much of his attention was devoted to ways to give a semantics that gradually reduces the degree of extensionality validated by the semantics.

As we have noted above Bell's model theory for ε-calculus, while it is sound it is not complete for all Heyting algebras.

First note that from the (ε)-axiom ($\exists x\varphi \to \varphi(x/\varepsilon x\varphi)$) and $\exists$-introduction ($\varphi(t) \to \exists x\varphi(x)$) we can easily prove that $\exists x\varphi \leftrightarrow \varphi(x/\varepsilon x\varphi)$ when we add the ε-scheme to our language $\mathscr{L}$. And since we have more than two truth values, *i.e.* 0 or 1, we open up the possibility that though we may have $[\![\exists x\varphi]\!]^{\alpha}_{\mathfrak{M}} \neq 0$ we may also have:

$$[\![\exists\varphi]\!]^{\alpha}_{\mathfrak{M}} \neq 1$$

and hence

$$\{d \in D : [\![\varphi]\!]^{\alpha(x/d)}_{\mathfrak{M}} = 1\} = \emptyset$$

but the definition of $[\varepsilon x\varphi]^{\alpha}_{\mathfrak{M}}$ is $\bar{e}(h_{\varphi})$ an arbitrary fixed element of M. So we can have in this case:

$$[\![\exists x\varphi]\!]^{\alpha}_{\mathfrak{M}} \neq [\![\varphi^{[x/\varepsilon x\varphi]}]\!]^{\alpha}_{\mathfrak{M}}$$

if, for example,

$$[\varepsilon x\varphi]^{\alpha}_{\mathfrak{M}} \in \{m \in \mathfrak{M} : [\![\varphi]\!]^{\alpha(x/m)}_{\mathfrak{M}} = 0\}.$$

And so in a non-classical case, Asser-Leisenring semantics clearly will not do. For according to this semantics, when the "truth set" for φ is empty, $\epsilon x\varphi$ is an arbitrary element of M, so we clearly can have:

$$[\![\exists x\varphi]\!]^{\alpha}_{\mathfrak{M}} \Leftrightarrow [\![\varphi[x/\varepsilon x\varphi]]\!]^{\alpha}_{\mathfrak{M}} \neq 1$$

5.6 DeVidi Semantics for Intuitionistic ε-Calculus

While Bell (1993a) sketches a sound semantics for the intuitionistic epsilon calculus very much like those constructed for classical epsilon semantics (*cf.* Asser, 1957; Leisenring, 1969), as we have seen it is not complete. Moreover, unlike in the classical case, it is not appropriate to simply swallow this defect in the semantics by adding an extensionality axiom to the epsilon calculus, *e.g.* Ack, that will *make* it complete because in the epsilon case this addition is not harmless and conservative as it is in the classical case. DeVidi (1994, 1995) sets out to fix this problem by providing a general purpose semantics for operators such as epsilon and tau.[6] He approaches the task by investigating various methods for gradually scaling back the degree of extensionality validated by the semantics. In the next few sections we will highlight certain features of DeVidi's semantics without giving details—which are provided in abundance in DeVidi (1994).

The first distinguishing feature of DeVidi's semantics compared to Bell's is how he sets out to accomplish what Bell does by restricting the algebras of truth values to inversely well-ordered sets. As noted above, the key virtue of this for Bell is that it assures us that all "joins are attained". That is we must ensure that under any interpretation, for every $\varphi(x)$, the join of the set of truth values that φ takes for each value of x is attained by one of the values that φ takes. That means that if Ω is the four element Boolean algebra, it can't happen that $\varphi(x)$ sometimes takes a, sometimes takes $a*$ (its compliment), but that it never takes 1, since in that case the join would not be attained.

Making sure that "joins are attained" is automatic, of course, in an inversely well ordered set of truth values. The cost, though, is that Bell has restricted the class of Heyting algebras available in ways that risk giving rise to validities in the semantics that are not provable in the calculus. In particular, inversely well ordered sets are linearly ordered, and most Heyting algebras are not. DeVidi therefore adopts a different approach to guaranteeing that joins are attained by building it into the definition of an interpretation. The algebra of truth values can be any complete Heyting algebra, but an interpretation

[6] These include what he refers to as very extensional (which includes the ε-extensionality axiom), quasi-extensional (which includes a slightly weaker quantified version of the ε extensionality axiom), Hilbertian (with the substitution of identicals but lacking Leibniz's law) and intuitionistic ε-calculi (that is intuisionistic logic + ε without an extensionality axiom).

is only ε-acceptable if every join corresponding to a predicate in the language is attained, i.e., if and only if the truth value of $\exists x.P(x)$ is equal to the truth value of $P(x)$ when some individual in the domain is substituted for x.

What this approach does is require that for each interpretation there will be a non-empty set of elements of the domain that make each formula 'as true as possible.' Unfortunately, this creates an additional problem: two predicates might have the same "as true as possible" set, but the truth value in question might be different for both those predicates.

For example in the most extreme case, all values of x might make $\varphi(x)$ completely true and all of them might make $\psi(x)$ completely false (e.g., $x = x$ and $not(x = x)$). The simplest "as true as possible" semantics will assign the same reference to the epsilon terms for both of those predicates. The result is that this semantics validates something even stronger than Ack. DeVidi introduces the principle HAck to get completeness in an intuitionistic system. We can compare the two axioms as follows:

(Ack) $\quad \forall x(\varphi \leftrightarrow \psi) \to (\varepsilon x.\varphi = \varepsilon x.\psi)$
(HAck) $\quad \forall x[(\exists y.\varphi \to \varphi[y/x]) \leftrightarrow (\exists y.\psi \to \psi[y/x])] \vdash \varepsilon y.\varphi = \varepsilon y.\psi$

Note that in HAck there is an extra condition, that for all x the existential quantified statement must imply the free version of the statement, making sure that the the epsilon term will get the same referent (DeVidi, 1994, pp.201 *ff.*). A logic with HAck DeVidi terms as quasi-extensional and shows that is it sound and complete (see DeVidi, 1995, p.530).

The bigger task, though, is to eliminate the need for extensionality assumptions not merely diagnose which ones are made valid by the existing semantics. This obviously requires making changes to the semantics. We turn to that question now.

5.7 Skeletons and Ground Terms

In his seminal paper, Günther Asser offers three different sorts of semantics for classical epsilon calculus (for detail see Asser, 1957, pp.53 *ff.*). The first is the simple semantics much discussed above, and given its fullest treatment by Leisenring. The second is such that $\varepsilon x\varphi$ only gets an interpretation when $\exists x\varphi$ is established, and hence can be seen more as a semantics for Hilbert's η-operator.[7] The third semantics, though, is a first step towards giving a semantics for epsilon that does not make Ack valid. He does so by making the referent of $\varepsilon x\varphi$ depend not only on the truth values φ takes, but on its syntax. The semantics DeVidi offers is explicitly a modification of Asser's approach.[8]

[7]Hilbert's η-operator is briefly discusses above in subsection 2.8.2.

[8]Both Asser and DeVidi present versions of the semantics which are suitable for interpreting terms with free variables, which requires making the interpretations depend also on finite sequences of elements

Asser introduced the idea of "ground terms" as his method for allowing the referents of epsilon terms to be distinct even if the truth sets of the predicates are identical. The idea is that the choice function will be a two-place function depending not only on the truth set, but also on the ground term underlying the epsilon term.[9] Since, for instance, the ground term of "ideal featherless biped" will be different from that for "ideal rational animal," the epsilon terms can refer to different objects.

Asser's ground term makes $\varepsilon x.\varphi(x, \alpha)$ where α is either a free term, a free variable or constant: $\varepsilon v_1.\varphi v_1 v_2$ indicating that the syntactical form of the epsilon term for φ is a bound variable and an unbound element of some sort or another. [10]

The result of all of this in the classical setting is a semantics for what Asser calls the "Hilbertian calculus." It is actually a creature of Asser's paper, not of Hilbert. In it, the substitutivity of identicals holds (*i.e.* that Frege's "the morning star" and "the evening star" may be substituted for one another in sentences), but Leibniz's law does not (*i.e.* that we can establish then identity of two elements by the fact that they are truthmakers for the all the same predicates.)

DeVidi makes two modifications to Asser's semantics (beyond placing them in an intuitionistic rather than a classical setting). The first is to eliminate the validity of the substitutivity of identicals, which is unmotivated in the case of epsilon terms. The second, on the other hand, is designed to prevent the intensionality from going too far. As once we allow that the reference of an epsilon term $\varepsilon\varphi$ will depend not only on the truth value of φ with x interpreted in various ways, but also on the syntax of φ, we are faced with some choices. It would be possible to allow, for instance, $\varepsilon x P(x)$ to differ from $\varepsilon y P(y)$, but this DeVidi judges a step too far. For the first, he offers *skeleton terms* as a modification of Asser's ground terms (see DeVidi, 1994, pp. 172-190). For the latter he introduces what he calls the "alpha axiom."

5.7.1 Skeleton Terms

Asser's third semantics depicts the syntactic structure of a term by replacing variables and free terms, *i.e.* terms t whose variables are not bound in any formula or term for which t is a subterm, with numbered variables. For example:

$$\varepsilon x.P(x)(\varepsilon y.Q(y)) \text{ has as its ground term } \varepsilon v_1.P(v_1)(v_2)$$

of the domain (to serve as referents of those variables). We set those details aside here as not essential for the present discussion.

[9] Again, leaving aside details about interpretation of any free variables in the terms.

[10] Corcoran *et al.* (1972, p.179) refers to ground terms as "canonical variable binding term operators (*cvbt*)".

DeVidi's skeleton terms preserve more information by not translating free terms into variables but keeping more of the structure of the original epsilon term, for example:

$$\varepsilon x.P(x)(\varepsilon y.Q(y)) \text{ has as its skeleton term } \varepsilon v_1.P(v_1)(\varepsilon v_2.Qv_2)$$

Unlike ground terms, skeleton terms preserve in the syntactic structure all non-bound terms and constants, so that one can distinguish as part of an element, not only its referent, but its name. Hence the following examples:

Original term	**Ground term**	**Skeleton term**
$\varepsilon x.\varphi(x, y)$ where y is a free variable.	$\varepsilon v_1.\varphi(v_1, v_2)$	$\varepsilon v_1.\varphi(v_1, v_2)$
$\varepsilon x.\varphi(x, \varepsilon y.\psi y)$ where $\varepsilon y.\psi y$ is a free term.	$\varepsilon v_1.(\varphi v_1, v_2)$	$\varepsilon v_1.\varphi(v_1, \varepsilon v_2.\psi v_2)$
$\varepsilon x.\varphi(x, c)$ where c is a constant.	$\varepsilon v_1.\varphi(v_1, v_2)$	$\varepsilon v_1.\varphi(v_1, c)$

As can be seen from the above examples skeleton terms preserve more type information about the structure of epsilon terms.

5.7.2 The Alpha Axiom

It is important to note that simply by always being false, or false under the same conditions, does not mean that propositions should always have the same epsilon term. Hence the most likely entity to be a talking dog should not be identical to the most likely flying pig, even though neither talking dogs nor flying pigs exist.

Adding an alpha axiom allows us identify the likeliest x to be a pig and to fly with the likeliest y to be a pig and to fly. Thus we need a way to distinguish between equivalence and syntactical equivalence with respect to propositions. Of course, while some syntactic differences in predicates should result in the epsilon terms differing, not all syntactic differences should.

The α-axiom defines α-convertibility in lambda calculus. It is written:

$$\lambda x.M = \lambda y.M[x/y] \text{ where } y \text{ does not appear free in } M$$

Having such an axiom means that if two propositions M and N are alpha convertible *i.e.* $M \equiv_\alpha N$ they are syntactically identical. Hence the equivalent for term forming operators is given by DeVidi as:

$$\sigma x.\varphi = \sigma y.\varphi[x/y]$$

where σ is any term forming operator.

With these definitions, DeVidi is able to provide a variety of completeness proofs for the various intuitionistic epsilon calculi. In particular, For the "quasi-extensional calculus", one adds "HAck" as an axiom, and uses a simple choice function on the "as true as possible" sets for the formula from which the epsilon term was formed. But if one makes the choice function depend not only on the truth set but also on the skeleton term, one gets a complete semantics for the epsilon calculus to which the only addition is the alpha axiom.

5.8 Identity

In my discussion, I have frequently referred to the epsilon calculus without identity. DeVidi's machinery also functions well for this calculus, with a few simple modifications. Without identity we can write the rule in the following manner, for every formula χ in the language $\mathscr{L}_\varepsilon$:

$$(**) \quad \frac{\forall x[\exists y.\varphi \Rightarrow \varphi[y/x] \Leftrightarrow (\exists y.\psi \Rightarrow \psi[y/x])]}{\chi[x/\varepsilon y.\varphi] \Leftrightarrow \chi[x/\varepsilon y.\psi]}$$

In lemma 4.6.3 and theorem 4.6.4 we use the following even weaker formulation of axiom (**) where for the proposition P:

$$\forall x(Ax \rightarrow Bx) \rightarrow (P\varepsilon_A \rightarrow P\varepsilon_B)$$

which replaces Bell's axiom of ε extensionality which we describe in lemma 4.6.1 and theorem 4.6.2:

$$\forall x[A(x) \leftrightarrow B(x)] \rightarrow \varepsilon_A = \varepsilon_B$$

Our version is an obviously valid rule, as it replaces the two place identity relation, with a weaker one place predicate ordered by implication.

Part III

Philosophical Implications

Chapter 6

Metaphysical Implications of Choice

6.1 Introduction

In the third part of this thesis we will draw lessons from both the technical results presented in Part Two and the philosophical debates presented in Part One. In Part One we discussed the development of Hilbert's programme, specifically with regard to his choice operators; Brouwer's intuitionistism and the origin and development of intuitionistic logic; and Michael Dummett's arguments about the connection between metaphysical positions such as realism and logical strength. In Part Two of the thesis we looked at the formal results derived by combining intuitionisitic logic and choice operators. After introducing Bell's results we presented several new results, about how choice operators strengthen intuitionistic logic, and then presented the semantic systems of Bell and DeVidi for intuitionistic epsilon calculus.

In this chapter we will try then to reconcile the results and semantics presented in Part two, with the insights about the connections between ontology and logical strength we can draw from our discussion of Dummett's arguments presented at the end of part one. To this end we will consider the sources of the ontological commitments surrounding the adoption of choice axioms and various ancillary principles about the nature of terms, and what those ontological commitments are. Hence we will draw some conclusions about the metaphysical implications of the formal results about intuitionisitic ε-calculus, discussed and presented in Chapter 4. One manner in which the ontological commitments can in part be determined is by looking at the nature of the intermediate logics which choice operators allow us to prove, *e.g.* to what degree such logics are the correct logics for different domains of different epistemic or ontological warrant, *i.e.* whether we are realist about such a domain or not and to what degree we are. To assess what those commitments are we will look at the proofs, for the addition of ε (or τ) does not strengthen the logic alone but requires the addition of decision principles and epsilon extensionality (Ack) to get the results shown in Chapter 4. We will discuss how choice operators combine with decidability principles to strengthen logics and will investigate whether the ideal objects picked out by epsilon terms are best understood to be abstract objects and if so what sort of objects. The proofs presented in part two using intuitionist logic plus the ε-operator and of intuitionist logic plus extensional ε-operator can help us break down further the metaphysical implications of the strengthening of intuitionistic logic. Thus first we will discuss how choice works in proofs of superintuitionistic logics. Secondly we will look at the role of extensionality in the proofs of the law of excluded middle, and the role of what Bell calls "modest decidability" conditions. Finally we will discuss the debate over arbitrary objects referring to Kit Fine's defence, for many of the same criticisms can be made of choice objects as are made of arbitrary objects.

6.2 From Choice to Excluded Middle

Let us first consider effects of choice on intuitionistic logic, then discuss the proofs of superintuitionistic logics presented in Chapter 4 and compare the non-constructive nature of choice with the criticisms of quantifier rules. After this rather abstract discussion it would perhaps be best to introduce and discuss some examples of various properties that will help draw into focus the issues at hand.

Choice principles not only strengthen intuitionistic predicate logic by enabling the proof of certain quantified statements[1] not provable in intuitionistic logic (see Rasiowa and Sikorski, 1963, p. 427), but as Bell notes (Bell, 1993a) they also enable us to prove well known intermediate sentential axioms: *i.e.* linearity and weakened excluded middle (which is equivalent to De Morgan's intuitionistically invalid law).[2] However choice axioms

[1]The following predicate axioms do not need decision principles to prove and can be proven directly from intuitionisitic logic $+\varepsilon$ without appeal to any extra axioms, but cannot be proven in normal intuitionisitic predicate logic (see DeVidi 1994 pp.256-259 for details):

$$\begin{array}{llll} \vdash & \neg\forall x\varphi \to \exists x\neg\varphi & \vdash & \exists x(\varphi \to \forall x\varphi) \\ \vdash & (\forall x\varphi \to \psi) \to \exists x(\varphi \to \psi) & \vdash & \forall x(\psi \vee \varphi) \to (\psi \vee \forall x\varphi) \end{array}$$

[2]The derivation of DeMorgan's to WLEM is obvious from the inspection of the following: $\neg(\varphi \wedge \neg\varphi) \vdash \neg\varphi \vee \neg\neg\varphi$. The other direction is a bit more complicated. We note first that identity gives us: $\neg(\varphi \wedge \psi) \vdash \neg(\varphi \wedge \psi)$ and we have WLEM: $\vdash \neg\varphi \vee \neg\neg\varphi$. Now we consider the two cases:

$$\neg(\varphi \wedge \psi) \vdash \neg(\varphi \wedge \psi) \wedge \neg\varphi \text{ or } \neg(\varphi \wedge \psi) \vdash \neg(\varphi \wedge \psi) \wedge \neg\neg\varphi.$$

Taking the first case, by DeMorgan's intuitionistically valid law we get: $\neg(\varphi \wedge \psi) \vdash \neg((\varphi \wedge \psi) \vee \varphi)$, and by absorption we get:$\neg(\varphi \wedge \psi) \vdash \neg\varphi$. The second case is more complicated, we have: $\neg(\varphi \wedge \psi) \vdash \neg\neg\varphi \wedge \neg(\varphi \wedge \psi)$ and since $\varphi \vdash \neg\neg\varphi$ we can write: $\neg(\varphi \wedge \psi) \vdash \neg\neg\varphi \wedge \neg(\neg\neg\varphi \wedge \psi)$ and so we have: $\neg(\varphi \wedge \psi) \vdash \neg(\neg\varphi \vee (\neg\neg\varphi \wedge \psi))$ by De Morgan's intuitionistically valid law. From here by distribution in some intuitionistic theories like Martin-Löf type theory[3] and Bell's weak set theory[4] do not imply excluded middle, which shows that there is more than one ingredient in proofs in intuitionistic logic of classical or superintuitionisitic principles than choice principles. Such proofs require the principles themselves and either a decidability condition on terms or an extensionality axiom for choice terms.

Recall that, in the classical case, due to Herbrand's theorem, which shows that epsilon is conservative (see Herbrand, 1930), one may argue that term forming operators are either simply place holders, or constructed names, since adding them to the language does not enable any new formulas to be proved, and so requires no new truthmakers.[5] That is, when treated as mere placeholders, we regard the 'namelikeness' of the epsilon terms as illusory, and the seeming ontological commitment as something to be explained away, so we are committed to the existence of nothing beyond what we were already committed to before epsilon was introduced. Those who wish to argue for a classical epsilon calculus must address this criticism directly, perhaps arguing that since Hebrand's theorem applies only proofs that start without epsilon terms and end without such terms and hence that the claim of conservation is limited. However since we are considering the addition of epsilon to intuitionistic logic where it is not conservative we need not answer such criticisms directly.

In a context where we are not willing to assert the law of excluded middle, the use of a device like choice operators is obviously more philosophically interesting. Such epsilon terms are ontologically potent in two ways: First, the *prima facie* ontological commitment due to their name-likeness has to be taken more seriously, because their inclusion in the language makes a difference. It is one thing to explain away something as a mere mode of speaking if speaking that way makes a difference. If speaking that way changes one's logic, presumably it's more than a mere mode. But the strengthening of the logic also increases our ontological commitments. That is, given the multiplicity of truth values in our semantics, an increase in logical strength will raise the truth value of some sentences, because there will be fewer possible truth values in a stronger logic. Changing logics in such a manner as to increase the truth value of, for instance, an existential claim, or an atomic sentence with a name in it, indicates that we are committed to a position that asserts that

---

we have:$\neg(\varphi \wedge \psi) \vdash \neg((\neg\varphi \vee \neg\neg\varphi) \wedge (\neg\varphi \vee \psi))$ since we know: $(\neg\varphi \vee \neg\neg\varphi) = 1$ we can simplify to: $\neg(\varphi \wedge \psi) \vdash \neg(\neg\varphi \vee \psi)$ and again by De Morgan's intuitionistically valid law we get:$\neg(\varphi \wedge \psi) \vdash \neg\neg\varphi \wedge \neg\psi$ which we can simplify to: $\neg(\varphi \wedge \psi) \vdash \neg\psi$ and combining with the first case and WLEM, we have derived DeMorgan's intuitionistically invalid law: $\neg(\varphi \wedge \psi) \vdash \neg\psi \vee \neg\varphi$.

[3]Martin-Löf type theory is also referred to as the Propositions-as-Types Theory or the Curry-Howard theory. The defining characteristic of such constructive type theories is the conception of propositions as types, which is also known as the Curry-Howard Isomorphism (DeVidi, 2004, p.223 n.1)

[4]Bell's weak set theory can be understood as a fragment of intuitionistic Δ_0 Zermelo set theory and Aczel's constructive set theory lacking an extensionality axiom (Bell, 2009, pp.122 n.2)

[5]ε-terms of course do allow the creation of new names distinct from the names in the language, hence sentences with epsilon terms that cannot be reduced to non-epsilon sentences could appear as premises in proofs.

such objects (or types of objects) have a 'greater degree' or 'more definite existence.'

In the proofs in Chapter 4, we saw how choice operators strengthen intuitionistic logic. Though it has been argued by some that the axiom of choice is constructive (*e.g.* Bishop, 1967, p.9),[6] acceptance of the ε-principle implies that for every property P, if we acquire conclusive evidence that some object has the property P, we thereby acquire conclusive evidence that an object εP exists, and it is the most likely object to have that property. Stated in this manner there is an obvious way that the ε-principle can be construed as non-constructive. There is no requirement for one to examine all the objects in the domain to know whether any of them have a property P. Rather, because of the ordering implicit in the epsilon axiom, if I show that ε_P does not have the property P nothing has that property: *i.e.* $\neg P(\varepsilon_P) \vdash \forall x \neg P(x)$.

Similarly, what the τ principle tells us is this: when we have evidence that the least likely object to have property P has that property, *i.e.* $(\varphi \tau_P)$, we can infer that all objects have property P, *i.e.* $(\forall x P x)$. This too is non-constructive, in perhaps a much stronger manner, for it tells us that all objects have a particular property, again by only examining one. This is stronger because of the distinction in the intuitionistic case between a property and a negative property, *i.e.* not every property is equivalent to a negative property in the intuitionistic case.[7] In the classical case P is equivalent to $\neg\neg P$, so the ε case yields the τ case, but that a result holds for $\neg\neg P$ does not tell us that it holds for P in the intuitionistic case. That is the most likely not P is the same as the least likely P in the classical case, and τ and ε are inter-definable in that case. In the intuitionistic case $\phi(\tau_{\phi(x)}) \vdash \forall x.\phi(x)$ and since $\forall x.\phi(x) \vdash \exists x.\phi(x)$ we have $\phi(\tau_{\phi(x)}) \vdash \phi(\varepsilon_{\phi(x)})$ but not the inverse.

What the strengthening effects of the decidability principles show us is that one must give up certain assumptions about objects if we wish to keep choice and not strengthen our logic.[8] In many domains, even if one were not inclined to accept the law of excluded middle, one might want to accept the ε principle. Perhaps, for example, because we want to accept the principle that we can talk in general about certain types of objects. But there is a sense in which one moves away from the epistemic norms of intuitionistic logic when one accepts the abstraction involved in the move from speaking in general about the fact that some object has a property to speaking specifically about some ideal object. Evidence

[6] Arguments about the status of the axiom of choice have often come down to one's intuitions about it. Eric Schechter quotes Jerry Bona's quip, that: "the axiom of choice is obviously true; the well ordering principle is obviously false; and who can tell about Zorn's lemma?" which draws attention to the fact that different formulations of this axiom can seem intuitive or counter-intuitive (personal communication quoted in Schechter, 1997, p.145).

[7] Were this the case then we would have $\vdash_{\text{INT}} \varphi \leftrightarrow \neg\psi$ and since $\vdash_{\text{INT}} \neg\neg\neg\psi \leftrightarrow \neg\psi$ we would then be able to show that $\vdash_{\text{INT}} \neg\neg\varphi \leftrightarrow \varphi$ which is principle of double negation and equivalent to the law of excluded middle.

[8] As in the above mentioned cases of Per Martin Löf's typed lambda calculus and John Bell's weak set theory.

that there is some x or other that has a property A does not necessarily give one evidence that a specific x, has A, nor even that there is a most likely x with that property A.

The argument for the non-constructive nature of the ε axiom is that once one begins to employ choice objects one is moving another step away epistemically. The quantifier introduction rules for $\exists$ and the ε-axiom enables one to move from statements asserting that a particular object has a particular property $A(a)$, to a quantified statement that some object has a particular property $\exists x A(x)$, to a statement that a ideal object, named ε_A, has the property A; and is in fact a member of the equivalence set of objects that are most likely to be A-objects. This allows one to make general statements about objects of a type without talking about a specific object. That is, the epsilon axiom lets one name an arbitrary or ideal object. One moves from making assertions about an object using a unique name, represented by a constant, to assertions about all objects of a class, named by variables, to an ideal representative of that class named by the ε-term.

DeVidi compares the non-constructive nature of choice operators to constructive criticisms of certain uses of the universal quantifier introduction: if for an arbitrary c we have $A(c/y) \vdash \perp$, then we can conclude $\neg\exists x A(x/y)$, *i.e.* $\forall x \neg A(x/y)$, but a constructive proof of such a formula would require method for proving $\neg A$ for any t we are presented with, and this need not be supplied by a demonstration that $\neg A(c/x)$ (DeVidi, 2004, pp.226).[9]

Likewise the ε-principle lets us translate the notion that for any property A there is a most likely A-object (ε_A), *i.e.* a thing that is most likely to have the property A. Some, *e.g.*, Tait (1994), have argued that the constructive nature of quantifiers in intuitionistic logic, means that we can accept the axiom of choice as constructive as well due to the relationship between choice and quantifier introduction (see Tait, 1994, pp.59-60). DeVidi argues that this is putting the cart before the horse:

> Employing a temporal metaphor, we might say that the reasoning which suggested that a constructivist ought to accept (ε) went wrong in supposing that we could, so to speak, wait around until *after* discovering a proof of $\exists x$, then use εx as a label for the t such that (t) is employed in the proof. But (ε) claims that we can find εx before we know whether $\exists x$ is provable, and this is what makes it non-constructive (DeVidi, 2004, pp.226-7).

Taking the above into consideration, let us now ask what an introduction rule for epsilon would look like. The preconditions for the introduction of epsilon are that there is a property A and that some object must be most likely to have that property (either the

[9]DeVidi and Korté (2014) present one solution, instead of including rules for $\exists$E and $\forall$I they provide a proof theoretic method, employing a Fitch bar for reasoning within the scope of various quantifiers, which does away with the "cumbersome restrictions placed on these inference rules" (For an example see footnote 35 below.)

most likely or a most likely). However the axiom does not provide a method of defining that element. In this sense it commits one to accepting a fact to which one does not have immediate epistemic access. In domains where one might reject realism, epsilon too may be rejected for similar reasons. And lack of epistemic warrant is one of the key reasons many have given for rejecting realism.

One way to see the link between realism and the ordering which choice imposes is to think in terms of properties. An epsilon term for A selects an object from the equivalence class of the most likely objects to have A even if there no object x such that $[\![A(x)]\!] = 1$. In terms of our algebraic semantics this is because all the joins corresponding to values of predicates must be attained when we have epsilon in our language since:

$$[\![\exists x.A]\!] = [\![A(\varepsilon_A)]\!] = \bigvee_{x \in D} [\![A(x)]\!];$$

that is for any such subset, there is at least a maximal element. This, however, may not be appropriate for certain properties.

Now let us introduce some examples that may make the above points more perspicuous. Consider the example of 'overall intelligence' as a property for which epsilon may not apply: there are many types of intelligence, and while we may say when it comes to the spatial intelligence of a Wayne Gretzky, the creative intelligence of a Picasso, or the mathematical reasoning ability of a Fields medallist there may not be a way to rank such intelligences in some obviously linear manner. The sentence "x is intelligent" may give rise to an ordering, but one with many maximal elements, and no maximum.

On the other hand, if there is some reason for us to accept the idea that there is an ordering in our domain, whether or not we can know what exactly that ordering is, then we may want to accept the ε operator. Say perhaps we restrict ourselves to only one component of intelligence, *e.g.* memory or even more specifically long term memory, we may see this as a property more suitable to being understood as ordered or orderable. Likewise in mathematical domains, certain entities or claims about entities may be orderable in some manner, even when we cannot directly construct them, while others, especially in transfinite cases, may not be.

Like Hilbert's example of the just man and the τ-axiom,[10] we can imagine a similar example for the ε-axiom. Consider the example of a predicate $H(x)$ indicating that a joke is funny. Some jokes are funny to some people, but can ε be said to pick out a joke most likely to be funny? There are several options: there could be several funniest jokes, a set of funniest jokes, or jokes may not be orderable; there may be no ideal joke. If there was a set of funniest jokes ε would then pick one from an equivalence class at the top of the ordering of funniest jokes, but jokes would still have to be ordered. This ordering feature

[10]see Chapter 1

seems to apply to some predicates better than to others and "funny" is likely a problematic predicate for such an ordering. So ε only seems to be applicable to certain domains where the predicates attach to certain sorts of properties.

It is no accident that the obvious examples of domains where the predicates don't have the sort of ordering that determines a class of "most likely" objects for each predicate are the ones where, intuitively, anti-realism is appealing. The most obvious domain would be the one that Dummett has discussed in his arguments against a Platonistic interpretation of mathematical objects (*cf.* DUMMETT 1975 DUMMETT 1977). It is suggestive that the two examples given above, the first, humour, given famously by Crispin Wright (WRIGHT, 1992), and the second, intelligence, of where epsilon is not plausible, are both examples of different sorts of cases where anti-realism has proved a tempting option. Humour is a property that we do not doubt certain things have, but which, at least in some manner, lacks full objectivity. Wright makes a distinction between "something which we conceive as apt to map a feature of the world which is independent of human minds—of their existence, cognitive standards, powers and reactive propensities," and those things, or properties, that do not seem to be independent of human minds (WRIGHT, 1992, pp.7-8).[11] Likewise "overall intelligence" is also a notion regarded with significant scepticism by some, and for reasons that parallel those given for why epsilon might not apply to it—the notion is an attempt to amalgamate a variety of different subsidiary notions, and there is arguably no non-arbitrary way to combine the subsidiary notions into a "combined score". Intelligence is not the only notion like this. At least sometimes, similar considerations lead people to deny that "beauty" is something real, for instance.

6.3 Logical Strength and Metaphysical Commitments

Now let us turn our attention to the connection between logical strength and metaphysical commitments. We will first review the general argument Dummett gives that connects the two and then discuss how from a model theoretic approach, strengthening a logic means that it is true for fewer models. Then I will discuss how this applies to the logics that make De Morgan's laws and Dummett's scheme true, and briefly consider how their models differ from those of intuitionistic and classical logic.

Broadly stated, Dummett argued that acceptance of mind independent reality in a domain implies the law of excluded middle. This is of course because mind independent reality means the existence of real stuff—*e.g.* objects, concepts, ideas *etc...*—that can fix the truth values of our claims irrespective of our ability to verify those truth values; so realism implies bivalence, since reality fixes truth or falsehood to well formed claims.

[11]Wright's pluralism about realism and objectivity is complex and we need not go into it in detail for our purposes. For more details see WRIGHT (2013).

Bivalence likewise implies the excluded middle [12]. So realism implies excluded middle. Contrapositively, rejection of excluded middle implies anti-realism. On the other hand if there is no mind-independent reality to fix the truth values of sentences, *i.e.* if anti-realism is true, truth can only mean verifiability. And if truth is only verifiability we are left with no reason to believe in bivalence, and hence no grounds for asserting the law of excluded middle in general.

Hence in the realist case, that is, in a domain in which we accept mind independence, all objects about which one can make an existence claim, whether simple or complex, physical or theoretical, either exist or do not. Classical logic cannot model cases where existence is more subtle or problematic.[13] This is a problem for modelling more epistemically dependent domains, where we have two competing yet contradictory theories.

The ε-operator lets one construct objects (or rather terms which refer to objects) and one might think that the simplest method of investigating the ontological effects of adding them to our logic would be to inquire about the type of objects these are. However before we do that we should consider that our proofs show that the addition of such objects also strengthen the logic and strengthen not only the quantified logic, but also strengthen the sentential logic. Recall that Dummett's argued that the most productive method for investigating "realism about some particular class of putative entities" is not to investigate "disputed class of objects" but to look at the "disputed class of statements" (Dummett, 1993c, p.465). This strategy replaces discussion of the nature of particular objects or entities with a discussion of the meaning of sentences. It is reasonable to begin our investigation by looking at the effects that adding such operators have simply by virtue of strengthening the underlying logic to which they are added. For the addition of these operators will change what the "disputed class of statements" will be in as much as they add to the "disputed class of objects" in the language.

The effects of being able to prove De Morgan's laws and linearity give us a manner to analyse disputed classes of sentences with reference to these axioms. We can use Dummett's examples of the metaphysical assumptions of the law of excluded middle, and hence of classical logic, to elucidate the metaphysical assumptions of De Morgan's laws and linearity, hence of De Morgan and Gödel-Dummett logics, and how these relate to realism about specific entities, or aspects of certain entities.

[12] The story is not as simple as this of course, LEM and bivalence are not the same. Bivalence is semantic or metalinguistic and LEM is syntactic or in the object language. We need at least a commitment to a truth theory + LEM to entail bivalence. There are counterexamples: Quantum logic allows LEM without bivalence and John Bell's "weakly classical" logic allows bivalence but makes $\vdash \omega \vee \neg\omega$ unprovable from within the language.

[13] This slightly overstates the case, as Boolean-valued or supervaluation semantics for classical logic make the matter somewhat more complex than this statement suggests. But we will set aside such complexities for the present.

6.3.1 Models and Strength

The natural way to look at the question of how metaphysical changes are entailed by weakening (or strengthening) a logic is to consider the problem model theoretically. The weaker a logic, and the fewer the validities, the more models of which it is true. If possible models are considered to map the possible states of affairs, then weaker logics allow for more possibilities. Conversely, strengthening a logic reduces the number of models and hence the possible states of affairs that might obtain.

Intuitionistic logic allows many, in fact an infinite number of, models which are not valid in classical logic. This is true even in the propositional case, even though in propositional logic if φ is classically valid then $\neg\neg\varphi$ is intuitionistically valid. That is, there cannot be intuitionistic models in which $\neg\varphi$ is true, when we have a classically valid φ because in that case $\neg\neg\varphi$ is intuitionistically valid and $\neg\neg\varphi$ and $\neg\varphi$ cannot both be true. Nevertheless for many classically valid φ, there will be non-classical models in which $\neg\neg\varphi$ is true but φ is not. Classical validities are more complicated in first order logic, *e.g.* $\neg\neg\forall x(\varphi x \vee \neg\varphi x)$ is classically but not intuitionistically valid, and there are models of intuitionistic logic in which $\neg\forall x(\varphi x \vee \neg\varphi x)$ is true (see RASIOWA and SIKORSKI, 1963, pp.427-429).

In general, as we move up the hierarchy of logics from intuitionistic logic to intuitionistic logic + De Morgan's laws (or its equivalent intuitionistic logic + weakened excluded middle) to Gödel-Dummett logic (i.e. intuitionistic logic + linearity) to classical logic, we see a reduction of possibilities, because we reduce the supply of models.[14]

Intuitionistic logic is generally taken to be best understood as an epistemic logic. A fairly natural semantics that captures this epistemic character is sometimes called a state of information semantics, with the idea that for φ to be true at a particular information state it must be *definitively established* by the information available at that state. Thus it is monotonic and adding information will never move the truth value of φ from true to not true. It is useful to consider a few examples of what models must look like for the key principles of some intermediate logics to fail. First, for the LEM to fail, we must allow for the possibility that there is a statement which is indeterminate at one state of information, but where the addition of further information definitely establishes it. For in this semantics, $\neg A$ is definitely established only if the available information rules out ever establishing A.

This is because in an epistemic logic when a proposition φ is true, there is evidence or a construction that shows that φ is known to be true, while when there is not such evidence

[14]One of the standard translations (Gödel–Gentzen translation) between intuitionisitic and classical logics is simply to write $\neg\neg\varphi$ for all φ, since $\neg\neg\neg\varphi \vdash_{int} \neg\varphi$ we obtain an intuitionistic embedding of classical proofs (see GÖDEL, 1933; GENTZEN, 1936). Likewise one could obtain a similar embedding of classical logic in Gödel-Dummett logic by changing all references to the law of excluded middle to Peirce's law and replacing that with the weaker Dummett's scheme.

it does not necessarily mean that $\neg\varphi$ is established. Whereas in classical logic the law of excluded middle implies that there are no such middle grounds.[15] The information state semantics yields counterexamples at each level of the hierarchy of intermediate logics and classical logic. As noted to make the law of excluded middle fail you need a proposition A that is indeterminate and a later state where it is true, because in classical logics: $[\![A]\!] \neq 1 \rightarrow [\![\neg A]\!] = 1$. But in information semantics this isn't so, and so $[\![A \vee \neg A]\!]$ at any such state. On the other hand, to get linearity to fail you need two different extensions from your beginning state; one with A true and B indeterminate, while in the other B is true and A indeterminate. Finally to get weakened excluded middle to fail what you need is that there be two possible extensions of our knowledge with A true in one and false in the other. So if weakened excluded middle fails the other two fail, and if linearity fails so too does the law of excluded middle.

As noted above the law of excluded middle fails when our logic is in some manner dependent on knowledge, as knowledge increases, so does the number of things that can be proved. So at some state t_n perhaps we cannot say $A \vee \neg A$, but if at some state t_m we have a proof of either of these disjunctions we can make the statement.

Consider again the example of humour. Some might say that a joke is either funny or it is not, but one could also make the argument that there is a middle ground, and that a joke is only funny relative to an audience, and until an audience reacts to it, there is no matter of fact about it. To keep with the example of humour, perhaps a joke, presented in a stand-up comedy routine is funny only in a context, and that such contexts consist of all the other jokes said, and the audience, and other factors. Could we not then get a context where linearity fails, the telling of joke A or joke B in some context, say with a particular type of audience, makes the other joke B or A possibly less funny. To extend this further, to get WLEM to fail we have two jokes that are both 'killer' openers, but both would set the routine off in a particular direction, changing the context and making the other joke unfunny.

More generally, what the semantics suggests is that for the law of excluded middle to fail what is required is that there be states of information that are insufficient to definitively establish A, while likewise being insufficient to rule it out. Humor is one example, but arguably there are many others; Goldbach's conjecture certainly seems to be a mathematical example where our current state of information is of this sort. Other cases will require more detailed philosophical scrutiny before we can say whether this applies or not—it seems that linguistic decisions to make our linguistic usage more precise (and so determine a definite truth value where one previously did not apply) are not extensions of information in the same sense as would be a new discovery. (In fact, the other semantics, in terms of algebras of truth values that are not Boolean seems more illuminating of failures of excluded middle of this sort.) What about empirical claims that have not yet been investigated? Much will

[15] I continue to elide the difference between bi-valance and excluded middle to simply exposition.

turn on what counts as "available information"; presumably we want to allow that there is a fact about whether the cat is sleeping in the next room right now, even if nobody has checked, so we are going to have to include such facts as available information despite our not having looked; and so someone who hopes to be a realist about cats and their locations but not about mathematical truths like Goldbach's conjecture must make a distinction between "not having looked" in the next room and "not having looked" in the areas of mathematics where the proof of the conjecture actually resides.

Similarly, in the case of linearity, the semantic requirement is quite plausibly satisfied in the case of many pairs of unsolved mathematical conjectures. Given what we know now, it seems that a proof of Goldbach's conjecture need not yield a proof of the Riemann Hypothesis, not vice versa, so there seem to be extensions of our current state of mathematical knowledge that establish each without the other.

It will be useful in what follows to be able to move fairly freely back and forth between talk of information states and algebraic semantics, these conditions translate fairly naturally into the lattices of truth values we introduced in Chapter 5. For example the law of excluded middle rules out **3** as a lattice of truth values. Linearity is valid in **3**, but invalid in the inverted lollipop (see below). For linearity to fail, it suffices to have a pair of propositions whose conjunction has a truth value strictly lower than either but whose disjunction is strictly greater without being equal to 1. That is, there is some important degree of incompatibility between the propositions (so the conjunction reduces the truth value of both). A useful way to grasp this idea, though one to be treated with caution, would be to think in terms of likelihoods for being true: "This shirt is red (all over)" and "This shirt is blue (all over)" rule each other out, so their conjunction is zero; their disjunction is more likely than either claim alone, but it is not equal to 1.

Likewise weakened excluded middle fails if we have two propositions that are non-comparable (*i.e.*, neither entails the other), incompatible, and yet which fail to have a true disjunction. Now let us look more closely at what De Morgan's (*i.e.* weakened excluded middle) and linearity entail.

6.3.2 DeMorgan's Laws

De Morgan logic,[16] as noted above, admits all of De Morgan's laws. Consider an example of a Heyting algebra where De Morgan's intuitionistically invalid law fails, *e.g.* the inverted lolipop, which we introduced in Chapter 5:

[16]also known as, KC, Jankov's logic, or simply IPC + $\neg\neg p \vee \neg p$

$$\begin{tikzcd}[column sep=small]
& 1 \arrow[d, dash] & \\
& c \arrow[dl, dash] \arrow[dr, dash] & \\
a \arrow[dr, dash] & & b \arrow[dl, dash] \\
& 0 &
\end{tikzcd}$$

We can let a and b be the truth values of incompatible sentences as $\neg a = b$ and $\neg b = a$, and so $\neg\neg a = a$ where c is the value of another sentence which though not true *i.e.* $c < 1$, has a higher truth value than a or b, and is in this example the value of the disjunction of $\neg a \vee \neg\neg a$, and thus we a have a counter-example to WLEM.

In the above example, if a sentence A has the value c, then $\neg\neg A$ in this case has the value of $[\![\neg\neg A]\!] = \neg\neg c = 1$.[17] Likewise a sentence with a value of c is implied by the double negation of sentence of value either a or b. While De Morgan's intuitionistically invalid law is not true for the above lattice, it does hold for the lolipop:

$$\begin{tikzcd}[column sep=small]
& 1 \arrow[dl, dash] \arrow[dr, dash] & \\
a \arrow[dr, dash] & & b \arrow[dl, dash] \\
& c \arrow[d, dash] & \\
& 0 &
\end{tikzcd}$$

Indeed it holds in any or in any linearly ordered lattice where any non-zero element is replaced by any of the the boolean algebras (*e.g.* **2**, **4**, or **8**, *etc...*) as **4** is below in the two stick lollipop though linearity still fails:

$$\begin{tikzcd}[column sep=small]
& 1 \arrow[d, dash] & \\
& a \arrow[dl, dash] \arrow[dr, dash] & \\
b \arrow[dr, dash] & & c \arrow[dl, dash] \\
& d \arrow[d, dash] & \\
& 0 &
\end{tikzcd}$$

This means that there can be sentences that are not directly comparable, *i.e.* there are

[17] It should be noted that one of the the standard translations of S4 to intuitionistic logic is to define the modal operator $\diamond$ as double negation.

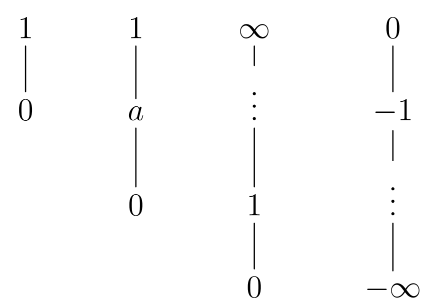
two sentences with values b and c such that neither $c \leq b$ nor $b \leq c$, but whose meet is not equal to 0. In the diagram above, consider an interpretation of $\neg(B \wedge C) \to (\neg B \vee \neg C)$ setting B and C to the values of b and c. Under such an interpretation we see that:

$$((b \wedge c) \to 0) \to ((b \to 0) \vee (c \to 0)) = (d \to 0) \to (0 \vee 0) = 0 \to 0 = 1$$

That is, to get De Morgan's intuitionistically invalid law to fail one needs an incomparable but incompatible A and B whose disjunction is not 1. We can compare this to the law of excluded middle which for failure requires a single A and it's negation whose disjunction is not 1. Looking at the example of different intelligences, take the claims that 'John is the smartest in the class' and 'Jane is the smartest in the class'. The conjunction is absurd, by definition of the superlative. But the disjunction, while more likely to be true than either disjunct, is not entirely true because it's still possible that nobody is smartest. To see this, assume for the purposes of this argument that grades do correlate with intelligence, and they have the same final grade but achieved in two very different manners, for example, assume two very different assignments testing very different aspects of intelligence, say writing a poem and a literary critique of a poem. In this example John got an A+ on a poem and a B on their essay, and Jane got an A+ on their essay, and a B on their poem, while Cletus got solid B+s on both assignments, all received the same final grade, but no one can reasonably be said to be in general smarter than the others.

6.3.3 Linearity

Linearity, or Dummett's scheme, $(a \to b) \vee (b \to a)$ holds in Boolean algebras and linearly ordered Heyting algebras, including, but not limited to obviously linear algebras such as: **2**, the only linearly ordered Boolean algebra, the three element linear ordering, the natural numbers (with a top), and the negative integers (with a bottom) *e.g.:*



These examples, though, suggest that the lattice must be a linear ordering, *i.e.* that for any y and z in the lattice, either $x \leq y$ or $y \leq x$. The requirement, algebraically speaking, is slightly weaker. The property required to validate Dummett's scheme, from a algebraic point of view, is that for any two element y and z of an algebra we have the following:

$$\bigvee\{x|x \wedge y \leq z\} \vee \bigvee\{x|x \wedge z \leq y\} = 1$$

This is a complicated looking condition: the join of all the elements which meet y below z, joined with the join of all the elements that meet z below y is 1. Let us pause to consider the condition a bit more closely. First, it's clearly going to be satisfied in a linearly ordered algebra, since if $z \leq y$, then $1 \wedge z \leq y$, so one of the two large joins is itself 1. In a Boolean algebra it will also be satisfied because $\neg y \wedge y \leq x$ and $y \wedge x \leq y$, so the join on the left is at least $\neg x$ and that on the right at least x. But the condition doesn't reduce to either of these.

Horn gives the following example that is neither Boolean nor a simple linear ordering:

> The algebra A is described as follows. Let A be the set of all functions on ω to $\{0, \frac{1}{2}, 1\}$ such that $f(0) \neq 0$ and $f(x) = 1$ for all sufficiently large x, or $f(x) = 0$ for all $x \in \omega$. Then A is a sublattice of the set B of all functions on ω to $\{0, \frac{1}{2}, 1\}$. For $f, g \in A$ and both not identically 0, $f \rightarrow g$ exists in A and has the same meaning as in B. Also for $f \in A$, $f \rightarrow 0 = 0$ and $0 \rightarrow f = 1$. Therefore A is an L-algebra (Horn, 1969, p.404).

The elements of this lattice are (eventually constantly 1) infinite sequences of 0s, $\frac{1}{2}$s and 1s which, other than the constant 0 sequence (which is 0 in the lattice), start with either $\frac{1}{2}$ or 1.[18] The ordering here is the "component wise" one, where $f \leq g$ if and only if at each place n in the sequences it is the case that $f(n) \leq g(n)$.[19] Clearly, there are non-comparable sequences, *e.g.*, $1, 0, 1, 1, 1, 1, \ldots$ and $1, 1, 0, 1, 1, 1, 1, \ldots$, so this is not a linear ordering. Nor is it Boolean, obviously, since $\neg f = 0$ for all non-zero f, while $f \vee \neg f = f$ (since joins and meets are also determined component wise). Why does it satisfy the linearity condition? Either $f(n) \leq g(n)$ or vice versa. If the former, then the join of all the $h \in B$ such that $f \wedge h \leq g$ will take value 1 at spot n, and so since joins are calculated point-wise the join will be 1 at every spot.

Horn's example is interesting because of the structure of the lattice. Of course because it satisfies the condition of linearity, it means that every sentence is implied by every other. But there are interesting examples that might correspond to a lattice supporting

[18] There are no reasons for not starting with 0, except ones related to simplifying other proofs in Horn (1969).

[19] Component wise ordering works in following manner: take two sequences $f =< 1, 0, 0, 0, \frac{1}{2}, 1, 1, 1, \ldots >$ and $g =< 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 1, 1, 1, \ldots >$ their component wise join is $f \vee g =< 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1, 1, 1, \ldots >$, and their meet is $f \wedge g =< 1, 0, 0, 0, 0, 0, 1, 1, 1, \ldots >$.

truth values for claims in a domain with potentially infinitely many dimensions on which to compare something—that is, allowing for infinitely many "yes", "maybe", "no" scores, and of course since the values that we map ω onto can be a series of 0s, $\frac{1}{2}$s, and 1s , or it could be a series of members of any size set. Consider again the example of intelligence. It's not true that "a person is either smart or they're not", because "Beth is smart" has some non-zero, non-1 truth value f, and $f \vee \neg f = f$. But "if Beth is smart, then Cletus is smart or if Cletus is smart, then Beth is smart" gets truth value 1. Neither of these people is uniformly smarter than the other. But for every dimension on which Beth is less smart than Cletus we get the truth value 1 in "if B, then C", and for every one where Cletus is less smart than Beth, we get 1 at that dimension, so when we disjoin we get 1 at every slot. If one claims that Gödel-Dummett logic is the correct one in a domain, say humour, a proposition asserting, for example, that a certain joke is funny, say $H(a)$, implies, or is implied by, all assertions that any other joke, say $H(b)$, is funny, that is:

$$\vdash (H(a) \rightarrow H(b)) \vee (H(b) \rightarrow H(a)).$$

6.4 Choice and Decidability Conditions

The addition of a choice operator like epsilon is obviously metaphysically loaded, simply because it adds singular terms to the language, hence it's adding things that purport to name objects. At least it does so, assuming that we treat the new terms as we do the other terms in the language.[20] But choice alone does not imply excluded middle, for example in systems like Martin Löf's typed calculus or John Bell's weak set theory because of the lack of an extensionality axiom. This means that if a language permits certain abstractions, *e.g.* ideal terms, one can to some degree balance these abstractions by restricting terms in some other manner. However one can still prove new quantificational laws, if not new propositional ones.

Adding the epsilon-axiom is of course trivially non-conservative in that it add terms, and it allows us to prove certain quantifier laws that can't be proved constructively, *e.g.*, $\exists x(\exists y Ay \rightarrow Ax)$, (see DeVidi 1994, DeVidi 1995, and Slater 2009, p.398). Adding it, plus decidability principles make certain important axioms true as seen in the proofs in part two of this thesis.

In Bell's proof of DeMorgan's intuitionistically invalid law what he refers to as a "modest 'decidability' condition" simply asserts that there is some a among all objects, that is well defined (Bell, 1993a, p.4). That is, all other objects have a relation to the object a,

[20] Some people have considered various restrictions on choice terms as a method of reducing logical strengthening for example see Mints (1991) p.2.

they are either equal to it or not (or in the new proof, presented in this thesis, indistinguishable from it or not). This may seem very modest in many contexts, for example in the natural numbers every number is either zero or it isn't. [21] What Bell shows is that the decidability conditions used to prove DeMorgan's intuitionisitically invalid law (equivalent to the WLEM) and Dummett's scheme (linearity), turn out to be not so modest after all, because in the presence of choice axioms these 'facts' about terms, enable you to prove intuitionistically invalid axioms.

An inquiry into how these proofs work must look not only at the results and the functioning of the operator rules but also at the other assumptions. As we noted above the proof of DeMorgan's and linearity in intuitionistic predicate calculus+ε requires not only ε but also a decidability condition, and the proof of the law of excluded middle requires ε-extensionality (Ack) or at least the weakened form presented in Chapter 4. These decidability conditions are needed in combination with the ε-axiom to increase the strength of the logic. The ε-axiom then enables us to take these facts about terms and enables us to prove facts about sentences.

For example to prove the law of excluded middle we employ epsilon and the extensionality, or in theorem 4.6.4, what we call the weakened extensionality axiom. These work in much the same manner as the decidability conditions connecting assertions about objects with assertions about propositions, such as the axioms we are trying to prove. When we add extensionality for ε:

$$\forall x[A(x) \leftrightarrow B(x)] \rightarrow \varepsilon_A = \varepsilon_B$$

we add a further ontological claim that these expressions do not have some sort of other intensional meaning, and are only accidentally coextensive, because they have the same most likely element. This axiom then is a claim both about statements and about objects. It is one thing to say that all featherless bipeds are rational beings and all rational beings are featherless bipeds, but quite another to say that the most likely members of these two sets are the same members. This perhaps becomes even clearer when we consider empty predicates: should the most likely square circle also be the most likely golden mountain?

6.4.1 Decidability Conditions for DeMorgan's Laws

Let us first quickly recap the proofs of DeMorgan intuitionistically invalid law (DEM) presented in theorems 4.3.1 and 4.4.1.

[21] At least this is the case in the standard classical understanding of the continuum. However this is not true in in all contexts, for example in a smooth continuum which admits elements such as 'zero-square' infinitesimals (see Bell, 1998, pp.5-6).

The decidability condition for proving DeMorgan's intuitionistically invalid law is:[22]

$$\vdash_I \forall x(x = a \vee x \neq a) \text{ and constants } a \text{ and } b \text{ such that: } \vdash_I a \neq b$$

The proof of theorem 4.3.1 proceeds by defining the sentence B and C in terms of objects, and it is these objects in particular that are constrained by the conditions : $\forall x(x = \mathbf{a} \vee x \neq \mathbf{a})$, and there is a $\mathbf{b}$ such that $\vdash_I \mathbf{a} \neq \mathbf{b}$.

When DeMorgan's laws are true, we can define a sentence to compare to all other sentences in reference to decidable object and at least one other object that is not equal to the former. The trick to Bell's proof is that for any pair of sentences B and C, we can define a predicate $A(x)$ such that $B \leftrightarrow A(a)$ and $C \leftrightarrow A(b)$, namely $A(x) =_{df} (x = a \wedge B) \vee x \neq a \vee B)$.

So, of course, $\neg B \vdash \neg A(a)$, and so $\neg B \vdash \neg A(\varepsilon_{\neg A})$. Contraposing, then running the same reasoning for C, we have that $\neg\neg A(\varepsilon_{\neg A}) \vdash \neg\neg B \wedge \neg\neg C$. Since, intuitionistically, a conjunctions of double negations is equivalent to the double negation of the conjunction, using contraposition and the fact that triple and single negations are equivalent we have $\neg(B \wedge C) \vdash \neg A(\epsilon_{\neg A})$

Now recall how $A(x)$ was defined. For instance, in the original form $\neg A(\epsilon_{\neg A})$ is $\neg[(\epsilon_{\neg A} = a \wedge A) \vee (\epsilon_{\neg A} \neq a \wedge B).]$ Since the decidability of a means that the identity claim in one of these disjuncts is true, we have that the second conjunct of one of the disjuncts is false, *i.e.*, $\neg B \vee \neg C$.

Clearly both the epsilon principle and the decidability conditions are essentially involved in this proof. But the proof is complicated enough that it might have the flavour of someone having pulled a rabbit out of a hat, rather than really illuminating the metaphysical situation. To address this, consider the proof again, but this time getting the weakened law of excluded middle (WLEM) from it by considering the case where C is $\neg B$.

In that case, $A(x)$ is defined in the following manner:

$$A(x) =_{df} (x = a \wedge B) \vee (x \neq a \wedge \neg B)$$

.

And we have seen how to get to:

$$\neg(B \wedge \neg B) \vdash \neg A(\epsilon_{\neg} A).$$

Since the formula on the left is simply an instance of non-contradiction, we can focus on the formula to the right of the turnstile.

[22] In Chapter 4 we get these results with a weaker condition than Bell's.

The formula that really does the work in the proof, in the sense that we apply the epsilon principle to it, is $\neg A$, which is the denial of a disjunction. The epsilon principle tells us that there is a likeliest object to make this negated disjunction true, which is to say there is an object most likely to falsify both disjuncts. The decidability of a tells us that this likeliest object is either a or it is not a, and so that the likeliest object can only falsify the disjunction if at least one of B and $\neg B$ is false.

So implication is not the key; rather it is the ability to create a disjunction between two propositions which imply the same ε-term but which are incompatible when filled with certain defined terms. Hence in theorem 4.4.1 the predicative decidability principle shows that we can derive DeMorgan's intuitionistically invalid law from intuitionistic logic plus the epsilon axiom without appealing to identity.

Epsilon lets us assert that if some sentence is true of some object then we may assume that there is an 'ideal' object of this type. Hence whatever we assert to be true of all objects is true of our ideal object as well. This fact enables us to translate 'decidability conditions' on terms to conditions on sentences.

6.4.2 Decidability Conditions for Dummett's Scheme

The proofs of Dummett's scheme, 4.5.1 and 4.5.2 from Chapter 4, use essentially the same decidability conditions, and defines $A(x)$ in the same way as it is defined in the proof of DeMorgan's law. As in the De Morgan case, we assume there are at least two provably distinct objects, *i.e.*, $\vdash 0 \neq 1$; and a single object 0, that is decidable, *i.e.*, $\forall x(x = 0 \vee x \neq 0) \wedge (1 \neq 0)$.

In this proof, instead of applying epsilon to $\neg A$, we apply it directly to the disjunction A. For from it we have that $A(\varepsilon_A) \leftrightarrow (\varepsilon_A = 0 \wedge B) \vee (\varepsilon_A \neq 0 \wedge C) \leftrightarrow B \vee C$.

The formal version of the proof involves application of the distributive law and some symbolic manipulation. Once again, it may be helpful to unpack the language a bit. $B \vee C$ is equivalent to $A(\varepsilon A)$, so the likeliest A is the thing likeliest to make $B \vee C$ true.

Dummett's scheme is not true in intuitionistic logic because, in intuitionistic logic, two sentences A and B can be non-comparable in such a manner as to make the scheme not true,[23] for example in the lollipop and inverted lollipop algebras presented above in section 6.3.2. In Dummett's scheme all sentences need to be comparable, *modulo* their joins, though not necessarily linear, as we noted above in our discussion of Horn algebras.

[23]That is such that the join of the join of the set of their respective meets $\neq 1$.

6.4.3 Extensional Epsilon

We present both Bell's proof of law of excluded middle from intuitionisitic logic+ε and the ε-extensionality axiom:

$$\forall x[A(x) \leftrightarrow B(x)] \rightarrow \varepsilon_A = \varepsilon_B$$

and a proof from what we refer to as the weakened or implicative extensionality axiom scheme:

$$\forall x(Ax \rightarrow Bx) \rightarrow (P\varepsilon_A \rightarrow P\varepsilon_B)$$

Rather than requiring the epsilon terms to equivalent formulae to be identical, this schema requires that if $A(x)$ implies $B(x)$, the ε_B has any property "nameable," in the languages that ε_A has.

The implicative extensionality axiom is perhaps weaker than one would expect a simple logical translation of the identity statement in the original axiom, something perhaps like:

$$\forall x[A(x) \leftrightarrow B(x)] \rightarrow (P\varepsilon_A \leftrightarrow P\varepsilon_B)$$

This schema is still weaker than Ack because the identity of indiscernibles is a second order principle.

Both the ε and τ principles under the above conditions license the law of excluded middle. If excluded middle is indeed related, as Dummett argues, to "mind independence" then it is not entirely surprising to see "extensionality" playing a key role in establishing the law of excluded middle. That is, one part of the "mind independence" is that we can speak in general about the objects of propositions and furthermore that an object or term either has or does not have a property. Thus any talk that implies indeterminacy with regard to truth or an object only partially having a property is either not well formed or false. Thus if two properties are true of all the same objects then they are the same property, this is what extensionality states.

Extensionality is generally understood to correspond with our intuitions about what is "real" and what isn't. There is a sense in which extensionality and objectivity are commonly understood to be linked. That is, the distinction between intensional and extensional, it is sometimes suggested, is actually the distinction between something "subjective", or at least merely linguistic, and the actual facts of the matter. As we noted in Chapter 5 in mathematics when we deal with intensional content we work on the linguistic object, either the length or structure of the actual formulae. And so treating $2 + 2$ and $1+1+1+1$ intensionally allows us to treat them as having in some sense different meanings, but despite being different expressions, extensionally have the same value.

Consider as an example something like preferences. Is there a fact of the matter about an individual's preferences? Is there a fact of the matter as to whether, say, someone prefers one food over another? That most people have preferences for food they like over food they dislike would make it seem that there is evidence that this is the case. So if you asked someone if they preferred “steak to broccoli for dinner” you would not be surprised to hear them (perhaps a majority of them) say “steak,” but if you asked the question differently, “do you prefer something that will give you a heart attack to broccoli for dinner”, you would not be surprised to hear their answer change. Likewise if one were to ask a ex-smoker who still craves cigarettes, if they would like one, you would not be surprised to hear them say “yes, but don’t give me one”. It is common in discussion of preferences to take this as evidence that preferences are not objective in so far as they depend on the way the situation is described.

So, how does it relate to epsilon? Well, if we consider what sorts of cases are left open by the L-algebra condition, it included things like “intelligence”, which has many dimensions that might be objective but there is no objective way of combining them. One thing that you might try is to explain why extensionality for epsilon terms rules out that sort of thing. Epsilon by itself (*modulo* those modest decidability conditions) makes the realm at least as “real” as intelligence is, but now we also require that the likeliest person to prove Goldbach’s conjecture is also the likeliest to be the next Bob Dylan. In the case of intelligence, maybe epsilon picks out different “most likelys” depending on whether you give a description that focuses on mathematical or linguistic brilliance—a top member along different dimensions, perhaps. But in the case of something that is more naturally regarded as objective we can rule out such phenomena—the likeliest dog to bite my leg is also the likeliest domestic canine to bite my leg.

Taking some arbitrary sentence A the classical assertion $\vdash \neg A \vee A$ can be taken to mean more than that A is true or false, but propositions are about objects, and if a proposition asserts that an object has a property, specifically a property that implies existence, the law of excluded middle (LEM) implies then that all objects that can be described by a proposition either exist or do not exist.

This is not how the LEM is usually expressed and does not at first glance appear to be the most problematic aspect of law of excluded middle. But consider some of the more usual critiques of law of excluded middle — vague descriptions and other sentences that do not seem to have clear, or worse to have paradoxical truth values. Choice operators approach such sentences from the point of view of truth-makers for a sentence that has a vague or paradoxical truth value — a choice operator states that we can always pick out the most or least likely truth-maker.

Choice operators are thus related to logical principles through this mechanism of reification—since you can assert the existence of the ideal type, the object more (or least) likely to have the property and choose the object in the domain with said properties it is clear that the ontological force of the existence principle is related to that of logical principles like law of excluded middle.

6.5 Choice Objects

Up to this point we have focused on the ontological interpretation and the effects of various contested principles which when added to intuitionistic logic strengthen it and form various intermediate logics. What we have learnt is that these various superintuitionistic logics do have ontological implications that are not clear without looking carefully at them and their models. But if this is the case, and these logics are formed by the addition of choice terms, and decidability principles on terms, we need now to look carefully at the nature of these terms. We will now switch focus to the topic of abstraction in general and ideal objects in particular and consider what others have said on the topic, and consider how what we have said above applies to these subjects.

In a formal language constants represent proper names and hence pick out objects. Variables, on the other hand range across sets. The ε operator functions as a sort of hybrid. If, for simplicity, we consider predicates with one free variable, the epsilon operator binds that variable and yields a term, i.e., it names a definite object. But which object it is depends on the predicate P(x)—if anything has the property P, then ε_P names one of the things with P. That is, epsilon terms are generally said to pick out one of the set of "the most likely object to have P".[24]

Choice terms (*e.g.* $\varepsilon x \varphi$) are thus said to pick out "ideal elements" (HILBERT, 1926, p.383), but 'ideal' can mean several things. Semantically they pick out the most likely object to possess a property. This seems quite natural for some choices of P: when we say "if anyone wins the race it will be Anne", or "if anyone passes the exam it will be Bill", it seems easy to see a ground for the choice, namely that Anne is the fastest runner and Bill the most diligent studier. This suggests that there is some abstraction about the entities that will show up in the extension of P.

But this fact seems also to imply there is an abstraction of the entities in the set of things with the property in question — that is the 'ideal' A somehow exemplifies the properties of an object with the property A. Finding the ideal seems at first to be different in kind when we talk of finite concrete objects, because it seems that what is needed in this case is a good definition and method of comparison to derive the set of objects most likely to have the property A. But even with easily denumerable concrete objects the problem really becomes one of whether there is a natural way of ordering complex properties that make it obvious which member is 'likeliest'. Hence it gets more complicated when we speak of abstract objects, numbers, geometric shapes, impossible objects *etc...* and perhaps worse still in descriptions of ideals that are disjunctive, and exclusive.

There are various ways in which it is puzzling to interpret exactly what epsilon terms

[24] τ-terms likewise are defined as the objects which are defined by universally quantified statements—recall Hilbert's definition from the introduction.

refer to. The notion of "most likely thing to have P" has often been used as the interpretation of "ε_P". But we may ask: what sort of thing is that? For predicates that pick out fairly concrete things, it might seem fairly clear what is going on: they pick out the "ideal" P, in the sense of naming the object which has the key characteristics of being a P to the greatest extent, but with finite and concrete objects the actual item(s) picked out by epsilon should still have many other properties. Picking out an ideal, of course, is trickier for abstract objects like triangles or square circles. There is a further complication: do epsilon terms really pick out objects at all? For, arguably, the referent of an epsilon term should be less determinate since predicated epsilon terms are logically equivalent with existentially quantified formulae:

$$\vdash \exists x\varphi \leftrightarrow P\varepsilon_\varphi$$

While our formal semantics has so far involved simply picking a suitable element of the domain of discourse to interpret each epsilon term, perhaps this is philosophically misleading—these considerations that suggests epsilon terms must pick out ideal or indeterminate objects suggest that epsilon terms "really" pick out objects that are more abstract.

We will approach these questions indirectly, first by clarifying an important point about the context of the debate, *i.e.* that it is a post-Fregean discussion, then looking at Lewis's taxonomy of ways of distinguishing abstract objects from concrete ones.

As we have noted above, what "ideals" are will be different for a set defined by concrete and abstract properties. When considering ideals for abstract properties that may share the character of Finean arbitrary objects, ranging over sets, than some other form of abstract object. The difference between an "ideal" member of a set and an "arbitrary" member seems intuitively that the "ideal" members are members with all and only the characteristics that best exemplify the property, whereas "arbitrary" seems to imply a randomly chosen element. However this characterization doesn't really match Fine's account of an arbitrary object, which acts as a sort of place holder. It is an abstract object that has all the properties of members of the set, that make the members of the set members of the set, and none of their accidental properties.

There is a sense in which both quantification and choice are in some sense processes of abstraction. The normal introduction rule for an existential statement, for example, starts with a named term, while the quantified statement is not specific as to the name of the object(s) having the property. If we consider a normal term denoting an object that has a property, say Pa *i.e.* the object a has the property P, we can assert that there is at least one object x that has a property P, $\exists xPx$, in that it loses specificity, but we cannot say that the opposite, that is we cannot specify the name of the object or object(s) with the property P given the assertion $\exists xPx$ is true. The movement from specific to general is arguably a type of abstraction, that is the loss of specificity is one way of understanding from concrete to abstract (see LEWIS, 1986, pp.84-86). The statement $P\varepsilon xPx$ also lacks this specificity, since the ε-axiom tells us that the sentence $P\varepsilon xPx$ is logically equivalent to $\exists xPx$.

If we want to claim "ideal objects" are abstract to be informative we should first decide what we mean by "abstract" and consider Lewis's taxonomy of definitions of abstract objects.[25] However, we will see that the abstract objects that choice objects resemble most are those often termed arbitrary or random objects. Even there however the semantics that have been suggested for arbitrary objects and those for choice do not fully correspond. Yet our discussion of arbitrary objects will not be in vain as it is obvious that they serve cognate purposes and have similar applications. For example, Fine offers as a virtue of his account of arbitrary objects that they illuminate aphoristic reference in what are commonly called "donkey sentences" (FINE, 1983, pp.75-77), an application that has also been suggested for choice operators. Thus the various criticisms and defences of arbitrary objects must be considered, and an understanding of their ontological implications should go a long way towards understanding the implications of choice objects. We shall discuss such examples in more detail in the next chapter.

6.5.1 Defining Abstract objects

The most obvious interpretation of the ideal version of some thing is that it is an abstraction of that thing. That is, an ideal object lacks some quality (perhaps only that of specificity) that an actual object has. When we consider an ideal, though, it stands in for a class of objects, so instead of a particular triangle, which is either right or not, we consider the class of all triangles, and our ideal triangle should have all the properties that triangles have but none, or rather must have none, that only some triangles have. The nature of triangles are such that it is possible that we see all triangles as being members of the set of most like triangles.

Goodman and Quine describe what they call a "nominalist position." Their view is that nothing that is not explained in terms of concrete entities is valid, hence they reject

[25] The question for concrete objects seems in some ways to reveal a somewhat easier possible solution at first. For example, there are many items that one can use as a table, but there are some objects that are definitely made to be tables as their primary functions and others that are only used as tables but it is obviously a secondary manner. Take for example entering a room with a normal kitchen table (say the ubiquitous Ikea Ingo table found in student apartments) and a house door propped up on plastic milk cartons (a sight also very common in student apartments). If one were asked to "leave the keys on the table" would there be many people who would have trouble realising on which object to leave the keys? Obviously there are many objects that one can argue are the archetypical example of some type, so while borderline cases are likely not the object referred in the use of ε-terms, choosing one specific object is still problematic.

the existence of abstract entities (*cf.* Goodman and Quine, 1947).[26] In their words:

> By renouncing abstract entities, we of course exclude all predicates which are not predicates of concrete individuals or explained in terms of predicates of concrete individuals. Moreover, we reject any statement or definition—even one that explains some predicates of concrete individuals in terms of others—if it commits us to abstract entities (Goodman and Quine, 1947, p.106).

Under such an interpretation of nominalism, platonism or realism about abstract objects would consist in holding that there is at least one abstract object that is not explained in terms of concrete entities. Obviously for Quine and Goodman, a "nominalist position" does not stop one from talking about mathematics; rather, they assert the formulas of mathematics are "like the beads of an abacus, convenient computational aids which need involve no question of truth" (Goodman and Quine, 1947, p.122). Similarly Quine deems the use of arbitrary terms in his presentation of natural deduction necessary, as we shall see later in our discussion of arbitrary terms.

Dummett discusses the development and the nature of abstract objects in Chapter fourteen of *Frege: Philosophy of Language*, where he notes that the use of the terms 'abstract object' and 'concrete object' are "modern" and began with Frege, and that the traditional distinction was between universal and particular. The distinction is different because under the traditional distinction "the difference lies precisely in the fact that we can predicate universals of other things" but we cannot predicate individuals of other things (Dummett, 1973, p.471). "A universal" can traditionally, Dummett asserts, "be alluded to ... in two different ways: both by a predicative expression, by means of which we predicate the universal of something else; and by a term, by means of which we refer

[26] Rosen notes that Goodman and Quine's use of "nominalism" has "little to do with" the traditional use of the term. Rosen goes on to note that Platonism with regard to abstract objects too needs to be understood as a modern usage:

> In this connection, it is essential to bear in mind that modern platonists (with a small p') need not accept any of the distinctive metaphysical and epistemological doctrines of Plato, just as modern nominalists need not accept the distinctive doctrines of the medieval Nominalists. Insofar as these terms are useful in a contemporary setting, they stand for thin doctrines: platonism is the thesis that there is at least one abstract object; nominalism is the thesis that the number of abstract objects is exactly zero (Rosen, 2012).

Rosen cites Field's definition of nominalism:

> Nominalism is the doctrine that there are no abstract entities. The term "abstract entity" may not be entirely clear, but one thing that does seem clear is that such alleged entities as numbers, functions, and sets are abstract—that is, they would be abstract if they existed. In defending nominalism therefore I am denying that numbers, functions, sets,or similar entities exist (Field, 1980).

to the universal in the course of predicating something of it;" an example of the former is the sentence "x is wise" and the latter "x has wisdom" (DUMMETT, 1973, p.471).

For Frege though: "Terms (proper names) and predicates are expressions of such radically different kinds... that it is senseless to suppose that the same thing could be alluded to both by some predicate and by some term" (DUMMETT, 1973, p.472). Frege's criticism then was simply that a workable semantics is not possible on the traditional distinction. Dummett gives the example of "wisdom" and the predicate "ξ is wise", to say that one is just a reconstrual of the other, *e.g.* "Wisdom is not confined to the old" is simply another version of "Not only the old are wise" (DUMMETT, 1973, p.472). The use of reconstrual simply denies the "status of genuine term or proper name" to abstract terms. Frege, because he wished to define numbers, wanted abstract entities to have the status of object and could not accept that numbers were simply the reconstrual of predicates

Dummett sums up, noting that the most basic ontological question is "what is there?" On the traditional conception this question is broken into two questions "What particulars are there" and "Are there universals, and, if so, what universals are there?" Traditional nominalism answers the first part of the second question with a negative response. According to Dummett, the modern counterparts of this view appear "only against the background of a Fregean ontological perspective." The first question is thus "What objects are there" and "'Are there concepts', 'Are there relations', 'Are there functions' and 'Are there truth-values?"' are its "companion" questions (DUMMETT, 1973, p.473).

While Quine holds the thesis that, for any segment of language which commits us to the existence of particular objects we must "enquire how to analyse that language in terms of predicate logic" only then will we understand to which objects we are committed (DUMMETT, 1973, p.476). This thesis is according to Dummett predicated on a Fregean understanding of language and the presumption that "the notion of 'object' which we are using has been given in the first place." (DUMMETT, 1973, pp.477-478). What Dummett means is that Fregean semantics objects play a dual role, they are "the referents of proper names" and "are what predicates are true or false of" (DUMMETT, 1973, p.474).

The Fregean view that Dummett identifies in Quine is that "the ontological commitment embodied in a language depends upon its quantificational structure, as revealed by logical analysis" (DUMMETT, 1973, p.479). Unlike Frege though, for Quine the question: "'What objects are there?', exhausts the content of the general ontological query, 'What is there?'" while for Frege concepts are required as well, because classes are objects for Frege, and to explain what a class is one needs to be able to quantify over concepts (DUMMETT, 1973, p.479).

As well as rejecting concepts, Goodman and Quine's modern nominalism rejects other abstract objects. While Frege's admits abstract objects into his domain and adjudges them harmless,[27] to evaluate Frege's acceptance of such entities Dummett considers how they are distinguished. Rejecting first the criterion of accessibility or rather lack of accessibility to human senses as such a distinction makes light-waves concrete but radio waves abstract (DUMMETT, 1973, p.480).

The wide variety of abstract objects (*e.g.* centres of mass, the capital of ξ, conventions in bridge, and chess openings, games themselves, colour-words used as nouns *etc...*) "naturally reinforces Frege's contention that the distinction between concrete and abstract objects is not of fundamental logical significance" according to Dummett (DUMMETT, 1973, pp.486–487).

In fact Dummett asserts that the "exact line of demarcation" between concrete and abstract objects is not "clearly marked," but he believes a distinction can be made by analysis of the use of language. Though the distinction is muddled because "pure abstract objects" are "reflections of certain linguistic expressions" these sorts of expressions, terms, we treat formally in the same manner as we treat proper names of concrete objects. The distinction to Dummett is that the sense of these expressions "cannot be represented as consisting in our capacity to identify objects as their bearers" (DUMMETT, 1973, pp.493–494). This is because, according to Dummett:

> Frege is denying that it is possible, on the traditional basis, to construct a workable semantics for a language: we can do nothing with the suggestion that a certain term —say, "wisdom" — should be regarded as standing for the very same thing as that which a certain predicate — in this case, 'ξ is wise' —stands for.

Frege's view that objects are the referent's of names could be accepted and the existence of abstract object rejected. In fact what Dummett refers to as "nominalism in the latter-day sense" by which he means the nominalism of Goodman and Quine, is exactly this, not the denial of universals, but Fregean abstract objects. Such a modern understanding of what is meant by 'objects' even to reject them, Dummett notes, requires this Fregean distinction between object and concept (DUMMETT, 1973, p.473). It is this strict demarcation between concept and object that in certain cases requires a large amount of rewording of what seems to be perfectly clear language, and which at times seems involve a loss of meaning, that attracts some to the use of term forming operators as appropriate formalizations.

Where then do choice objects sit under such a picture? It is clear that they are in some sense abstract, but not merely because they are traditional universals. Epsilon will

[27]Dummett notes that while "Frege recognizes the possibility of drawing a distinction between concrete and abstract objects: that is to say, he employs, in *Grundlagen*, the notion of 'concrete' (*wirklich*, literally 'actual') objects, though only in the course of arguing that not every object is concrete"(DUMMETT, 1973, p.480).

not give us “wisdom.” Rather it refers to a most likely “wise thing”. Choice terms are an attempt to develop something that enables one to have both something less general than “universal term”, but that serves much the same purpose.

6.5.2 Lewis on Abstract Objects

David Lewis too discusses the various ways we might distinguish abstract objects, describing four possible methods for defining abstract from concrete objects, with the goal of presenting worlds as concrete objects. He describes these methods as the “way of example,” the “way of abstraction,” the “way of conflation,” and the “way of negation.” He also notes Dummett’s method based on use but does not consider it in detail (Lewis, 1986, p.82).

Lewis’s “way of example” is simply the listing of entities that provide examples of abstract objects and those that are concrete objects. For example, his notes that: “concrete entities are things like donkeys and puddles and protons and stars, whereas abstract entities are things like numbers” (Lewis, 1986, p.82). He rejects the “way of example” as provides no method to judge any borderline cases.

The “way of conflation” is according to Lewis simply the assertion that the distinction between concrete and abstract entities is simply the same as some other distinction already made, *e.g.* the distinction between set and individual, or between universals and particulars. The reasons for regarding this way as deficient have been covered above.

The method of distinguishing abstract from concrete entities by negation is simply by noting properties that abstract entities lack a “spatiotemporal location; they do not enter into causal interaction; they are never indiscernible one from another” (Lewis, 1986, p.83). The method Lewis settles on is actually a more sophisticated version of the method of negation, which Lewis terms the “way of abstraction”. To Lewis this ‘way’ of describing abstract objects as the “historically and etymologically correct thing to mean if we talk of ‘abstract entities,’ ” but not perhaps the “dominant meaning in contemporary philosophy” (Lewis, 1986, p.85).

According to Lewis abstract entities are “abstractions from concrete entities” which means that abstract entities are derived by “somehow subtracting specificity, so that an incomplete description of the original concrete entity would be a complete description of the abstraction” (Lewis, 1986, pp.84-85). This version has the advantage of being epistemologically grounded in some sort of experience. Lewis distinguishes this from the “negative way”:

> ...if we can abstract the spatio–temporal location of something, that abstraction will not be unlocated; rather, there will be nothing to it except location.

> Likewise if we can abstract the causal role of something, then the one thing the abstraction will do is enter into causal interactions.

Lewis's "way of abstraction" is about negating, or subtracting, all but one quality.[28] Choice operators work on a predicate, presenting a ideal term that stands in place of a the predicate, so in one sense they subtract specificity, but work not on individual objects, but on the set that the predicate defines.

What are often referred to as arbitrary objects work in a similar manner. In the most sophisticated understanding of such objects they are some sort of abstractions that represent a set, while not strictly being members of the set. The orthodox view, following Frege, rejects their existence, and asserts that natural language usage that seems to refer to such objects must be reinterpreted properly into quantified sentences. However there are some proponents of arbitrary objects, whose arguments in their favour we should consider, since certain of the criticisms of arbitrary objects may also be made of choice objects. An abstract object, defined by the way of abstraction, does not quite have what we need for it to perform the role of an epsilon object. To see why this is the case consider the standard rejections of "arbitrary objects".

One of the most famous examples of the criticism of arbitrary objects is Berkeley's criticism of Locke's "abstract triangles" that are "neither oblique nor rectangle, neither equilateral, isosceles, nor scalene, but all and none of these at once" (cf. Berkeley, 1843, p.79 §13).[29] Of course a triangle must have one of these properties, but Locke asserts that the idea of a general triangle is "all and none of these at once" which seems to mean that such an object has a contradictory nature. These are perhaps what we might refer to as disjunctive objects, or more accurately as exclusive disjunctive objects, *i.e.* the choice object of a disjunctive existential proposition such as $\exists x Px$ where $P \equiv Ax \vee Bx \vee Cx$ and where $\forall x_{x \in D}(Ax \vee Bx \vee Cx) \wedge (\neg(Ax \wedge Bx) \wedge \neg(Bx \wedge Cx) \wedge \neg(Cx \wedge Ax))$. The example of different types of triangles are but just one example of such objects. Natural numbers can too be understood in this manner; they are prime or composite, even or odd, divisible by a certain prime or not. When we wish to talk about natural numbers using these properties we often treat them as arbitrary objects, *e.g.* "some number ξ is prime or composite." This seems more natural in ordinary language than using an extensionally quantified statement.

[28]Lewis argues that "the way of abstraction" enables one to assert that possible worlds are concrete because:

> They lack no specificity, and there is nothing for them to be abstractions from. As for the parts of worlds, certainly some of them are concrete, such as the other-worldly donkeys and protons and puddles and stars. But if universals or tropes are non-spatio-temporal parts of ordinary particulars that in turn are parts of worlds, then here we have abstractions that are parts of worlds(Lewis, 1986, p.86).

[29]Berkeley is quoting Locke (1700) IV Chapter 7 section 9.

Should we consider such objects, arbitrary objects, as reasonable furniture of our universe? Kit Fine puts the standard argument against arbitrary numbers as follows:

> A typical version goes as follows. Take an arbitrary number. Then it is odd or even, since each individual number is odd or even. But it is not odd, since some individual number is not odd; and it is not even, since some individual number is not even. Therefore it is odd or even, yet not odd and not even. A contradiction (Fine, 1983, p.59).

Obviously no concrete object holds contradictory properties. However if one is choosing the most likely object to hold a disjunctive property one might choose a concrete object that was most likely to hold the opposite property than the one it does. Take for example a legislative body with members of only two parties where, according to the rules of the body, no member can be a member of more than one party, but on occasion members defect. At any time no member could be in both, but at any time there are definitely some members who may change allegiance, and there are members who at any time are more likely to defect and most likely to be members of the empty set of those who are members of both parties.[30]

Are choice objects arbitrary objects? In the case of a proposition which the choice operation is being applied to a contradictory conjunction, or an exclusive disjunction, it certainly seems to run into many of the same problems as arbitrary objects. Consider the classical semantics for epsilon; if we pick an element from the truth set, of some sentence say $A \vee B$ for an epsilon term $\varepsilon_{(A \vee B)}$, we may pick an element that makes one disjunct true and the other false, but makes the disjunction true. But this does not seem arbitrary in the manner we want. Fine's arbitrary objects are said to represent a set, but in such a manner in which one cannot refer to specific properties (see Fine, 1983, 1985). For example with an arbitrary natural number one cannot assert that it is even or odd, when one uses it in reasoning. Likewise choice objects are not resolved. That is, while some object $\varepsilon_{(A \wedge \neg B) \vee (B \wedge \neg A)}$ represents an object that could have both conjuncts it is treated much as an arbitrary object is, as not specifically having a set property or A or B-ness.

Thought they differ from epsilon-objects we should still consider arbitrary objects, because in some they seem required to make sense of our inferential practices. Could epsilon terms represent such object, in the intuitionistic case, if not in the classical case? Recall that a proposition $P\varepsilon_P$ is logically equivalent to $\exists x Px$ and hence should in some

[30]Likewise with triangles, obviously a triangle cannot by definition be both obtuse and right, but is not a triangle that has an angle of 89.9° more likely to be a right triangle than one with three angles of 60° (under some definition of "likely"), if one had to rank triangles that were not right but were most likely to be right triangles one would likely rank a triangle with a greatest angle of 70° higher than one whose greatest angle was 65°?

manner retain the nature of that proposition with regard to x being unspecific. Let us first consider what people have meant by arbitrary objects and why they have rejected them, secondly why Fine tries to rehabilitate them, and finally how they seem required to make sense of our inferential practices with specific focus on natural deduction.

6.5.3 The Rejection of Arbitrary Objects

Kit Fine in his "Defence of Arbitrary Objects" notes that Frege rejected "indeterminate numbers" (*i.e.* arbitrary numbers) as "unnecessary" and "unwise" in light of his theory of quantification and what he termed the "absurdities in the notion of a variable number". Fine continues that this view has become standard "amongst subsequent philosophers"; as he puts it: "If more philosophers of the present day have not added their voices to the protest, it is probably because they have not thought it worth the bother" (Fine, 1983, p.55).[31]

While the former view is the contemporary orthodoxy, the view that the referents of variables should be treated as arbitrary individuals did of course have its adherents. The most commonly cited example of Frege rejection of such numbers is his reply to the mathematician Czuber, who wrote in his textbook on analysis published in 1898, "By a real variable, we understand a number that is indeterminate at the outset, and which, depending on the problem in which it occurs, can assume indefinitely many real values" (Czuber 1898, quoted in Frege 1979 p.160). In response to this Frege writes that:

> The author [Czuber] obviously distinguishes two classes of numbers: the determinate and the indeterminate. We may then ask, say, to which of these classes

[31]Quine notes that Frege's rejection of the "variable number" was not completely novel at the time writing that, "the pronominal character of the variable was clear to Peano" (Quine, 1951, p. 71). To be clear Peano actually suggests that two different interpretations of variable can be understood. In the introduction to volume three of his *Formulaire de Mathématiques* Peano writes that:

> In natural language the words 'this, that, the same, first, second, ... ' play the role of variables letters. These could be replaced by the numbers 1, 2, ... by making the appropriate conventions so as not to produce ambiguities in Arithmetic'" (Peano 1901, p.2)

My translation. The original is as follows:

> Dans le langage commun les mots ≪ceci, cela, le même, premier, deuxième,...≫jouent le rôle des lettres variables. On pourrait les remplacer par les nombres 1, 2,... en faisant des conventions opportunes pour ne pas produire des ambiguïtés dans l'Arithmétique

Peano means that instead of attaching variables letters, we could introduce arbitrary terms indexed to the natural numbers. Peano then cites an earlier volume where he discusses two possible systems of variables and his preference for the pronominal version (*e.g.* $a, b \in N \subset a + b = b + a$) over one that instantiates a new indexed variable for each member of a class (*e.g.* $N1 + N2 = N2 + N1$ where an instance of the class is indicated by adding an index) (Peano 1897 pp.26-27).

> the primes belong, or whether maybe some primes are determinate numbers and others indeterminate. We may ask further whether in the case of indeterminate numbers we must distinguish between the rational and the irrational, or whether this distinction can only be applied to determinate numbers. How many indeterminate numbers are there? How are they distinguished from one another? Can you add two indeterminate numbers, and if so, how? How do you find the number that is to be regarded as their sum? The same questions arise for adding a determinate number to an indeterminate one. To which class does the sum belong? Or maybe it belongs to a third? (Frege, 1979, p.160)

Thus rejection of arbitrary or indeterminate objects has often been put into terms of the nature of the variable in mathematics. Russell notes in his *Principles of Mathematics* that "The variable is, from the formal standpoint, the characteristic notion of Mathematics" (Russell, 1903, p.90). Russell accepted that phrases like "any number is a number" do pose, at the very least, a need for interpretation, and that in such cases treating "any number" as a definite term raises the issue, that it is not a number like any particular number. However, he prefers to deal with it by rewording such expressions foreshadowing his method for dealing with non-existent kings of France.[32]

Tarski too does not admit arbitrary terms into his system, making the distinction between constants and variables in the following manner in his *Introduction to Logic*:

> As opposed to the constants, the variables do not possess any meaning by themselves. Thus, the question:
>
> *does zero have such and such a property ?*

[32] Russell thus rejects arbitrary numbers with an argument very similar to Frege's in the *Principles of Mathematics* writing that:

> The terms included in the object denoted by the defining concept of a variable are called the values of the variable: thus every value of a variable is a constant. There is a certain difficulty about such propositions as "any number is a number." Interpreted by formal implication, they offer no difficulty, for they assert merely that the propositional function "x is a number implies x is a number" holds for all values of x. But if "any number" be taken to be a definite object, it is plain that it is not identical with 1 or 2 or 3 or any number that may be mentioned. Yet these are all the numbers there are, so that "number" cannot be a number at all. The fact is that the any concept "any number" does denote one number, but not a particular one. This is just the distinctive point about any, that it denotes a term of a class, but in an impartial distributive manner, with no preference for one term over another. Thus although x is a number, and no one number is x, yet there is here no contradiction, so soon as it is recognized that x is not one definite term." (Russell, 1903, p.91)

e.g.

is zero an integer?

can be answered in the affirmative or in the negative; the answer may be true or false, but at any rate it is meaningful. A question about x, on the other hand, for example the question:

is x an integer?

cannot be answered meaningfully. (Tarski, 1946, pp.4-5).[33]

Indeed, Alonzo Church specifically cites Frege's objection to the division of the reals into classes of "constant real numbers" and "variable real numbers". Though Church notes that some mathematical writers speak of "variable real numbers" or "oftener 'variable quantities' " he does not believe that these should be taken "literally". Church asserts that "a satisfactory theory has never been developed on this basis and it is not easy to see how it might be done."

In a footnote he mentions an example which he describes as "parallel to one of Frege's examples" that he sees as an argument against arbitrary objects:

> Shall we say that the usual list of seventeen names is a complete list of the Saxon kings of England, or only that it is a complete list of the constant Saxon kings of England, and that account must be taken in addition of an indefinite number of variable Saxon kings? One of these variable Saxon kings would appear to be a human being of a very striking sort, having been say, a grown man named

[33] Tarski also felt the need to warn students of earlier views of variables:

> In some textbooks of elementary mathematics, particularly in the less recent ones, one does occasionally come across formulations which convey the impression that it is possible to attribute an independent meaning to variables One might find an explanation that the symbols "x", "y", ... also denote certain numbers or quantities, not "constant numbers" however (which are denoted by constants such as "0", "1", ...), but so-called "variable numbers" or "variable quantities". Statements of this kind stem out of a gross misunderstanding. The "variable number" x which one tries to envisage could not possibly have any specified property, for instance, it could be neither positive nor negative nor equal to zero; or rather, the properties of such a number would have to change from case to case, that is to say, the number would sometimes be positive, sometimes negative, and sometimes equal to zero. But entities of such a kind are not to be found in our world at all; their existence would contradict the fundamental laws of thought (Tarski, 1946, p.5).

> Alfred in A.P. 876, and a boy named Edward in A.D. 976. According to the doctrine we would advocate (following Frege), there are just seventeen Saxon kings of England, from Egbert to Harold, and neither a variable Saxon king nor an indeterminate Saxon king is to be admitted to swell the number. And the like holds for the positive integers, for the real numbers. and for all other domains abstract and concrete. Variability or indeterminacy, where such exists, is a matter of language and attaches to symbols or expressions (Church, 1956, p.13 fn.33).

Nicholas Rescher sums up the argument against arbitrary objects or as he refers to them as "random individuals" in the following passage:

> To regard a "random element" as an element or a "random individual" as an individual is to commit what Whitehead terms the "fallacy of misplaced concreteness" and involves what philosophers have come to call a category mistake. A statement like "φy" does not say something about a peculiar "random individual" y: it says that the property φ characterizes every particular element of our universe of discourse. There are no "random" or "arbitrarily selected" individuals, just individuals. The "arbitrariness" or"randomness" resides not in individuals, but in the deliberate ambiguity of the notation by which reference to them is made. To talk of "random" or "arbitrarily selected individuals" is to reify a notational device. And this, in the present instance, is not merely unwarranted, it is demonstrably absurd. (Rescher, 1958, p.117)

Thus the orthodox view of arbitrary objects and specifically numbers has, since Frege, been that they are "unwise", "unnecessary" and "weird". There is no more compelling argument than that made by Frege which is tied up entirely with his views on concepts and objects. In fact Peano's view seems not to be as Quine insisted of one of being committed to the "pronominal nature" of the variable but more that either approach is simple a convention, one as good as the other.

6.5.4 Natural Deduction and the Rehabilitation of Arbitrary Objects

Though, as we have seen, many have rejected the use of arbitrary objects, or random individuals, there are several cases where their use seems to be most natural. For example arbitrary terms have often been employed to provide Gentzen style versions of the introduction and elimination rules for the universal and existential quantifiers, respectfully, in first order predicate logic. In this case arbitrary terms seem both natural and useful to produce rules that mirror those of the other logical connectives.

The universal elimination (∀E) and existential introduction (∃I) rules are obviously correct. However to provide their opposites one must introduce "restrictions and conditions" that seem both "unnatural and complicated" (DeVidi and Korté, 2014, p.4).

For example, Quine despite his rejection of abstract objects, presents (Quine, 1959, pp.159-166) his rules for ∃ elimination and ∀ introduction (which he refers to as Existential Instantiation and Universal Generalization) with his method of flagging variables. He begins by noting that, though they are not strictly correct they "are not wholly alien to unformalized thinking" and hence "can, under certain restrictions, be used as steps in trustworthy deductions". Quine thus notes that:

> Intervening use of UG or El does not impair the implication of conclusion by premiss, as long as no flagged variable retains free occurrences in premiss or conclusion....
>
> The restriction sought may accordingly be formulated, for deductions generally, as follows:
>
> (A) If there are $n(> 1)$ flagged steps in a deduction, then the flagged variable of some step must be free in the step-conditionals of none of the remaining $n-1$; the flagged variable of another step must be free in the step-conditionals of none of the remaining $n-2$; and so on... (Quine, 1959, p.163).

Which he rephrases as the following two restrictions:

> (B) No variable may be flagged twice in a deduction.
>
> (C) It must be possible to list the flagged variables of a deduction in some order $V_1, ..., V_n$ such that, for each number i from 1 to $n-1$, V_i is free in no line in which $V_{i+1}, ..., V_n$ is flagged (Quine, 1959, p.164).

Thus he presents his rules as follows:

> *Rule of universal generalization (UG)*: We may subjoin a universal quantification to a line which is a conservative instance of it.
>
> *Rule of existential instantiation (EI)*: To a line which is an existential quantification we may subjoin a conservative instance of it.
>
> *Flagging*: Off to the right of each line subjoined by UG or El, we must flag the instantial variable by writing it in the margin.
>
> *Restrictions*: (B) and (C) above. (Quine, 1959, p.165)

Francis Jeffry Pelletier notes that Fitch style systems, which follow more closely the original formulation provided by Gentzen and insist that all applications of these rules happen in sub-proofs which are marked graphically, are more easily restricted than Quine style systems of natural deduction (Pelletier, 1999, p.13). Quine style systems, because they do not cordon off the use of such rules into sub-proofs, require complex restrictions and flagging. This Pelletier points out led to several logic texts of the 1950s presenting incorrect versions of both EI and UG. This is a problem that Quine ran into, as he acknowledges in the forward to the revised edition (Quine, 1959, p.vi):

> In §28 there are two convenient deductive rules that cannot be directly justified, for the good reason that they serve to deduce conclusions from premisses insufficient to imply them. In past printings of §28 these rules have been indirectly justified by proving that deductions in which they are used will still turn out all right in the end, as long as certain arbitrary-looking restrictions are respected. In this new edition, §28 is rewritten. The rules and restrictions are now explained and justified in a way that dispels the old air of artificiality.

In §28 Quine notes that while "the basis for UI [universal instantiation] and EG [existential generalization] was that '$(x)Fx$' implies 'Fy' and 'Fy' implies ($\exists x$)Fx. For UG and EI we cannot plead that 'Fy' implies '$(x)Fx$' nor that '$(\exists x)Fx$' implies 'Fy'." (Quine, 1959, p.160). Quine of course then maintains that "under certain restrictions' these rules can be used. These restrictions were changed between the 1950 version of the book and the 1959 as noted above.

Irving Anellis discusses what Pelletier calls the "mysterious stew of ideas" that surrounded natural deduction systems with incorrect restrictions presented in Quine 1950, and Copi 1954. These could be contrasted with the correct but "radically different" Fitch-Gentzen sub-proof methods that were also current in the mid-century, but known for the most part only to specialists (Pelletier, 1999, p.13 fn.13). Anellis (1991) notes that the system in Copi (1954) came under attack by Hugues Leblanc who provided an obviously incorrect derivation well within Copi's rules (Leblanc, 1965, p.210). Leblanc provided a revised set of restrictions for $\exists$ elimination and $\forall$ introduction (see Leblanc, 1965) the former of which was adapted by Copi in his 1967 revision of his text (Copi, 1967, p.iv).[34]

The version of these natural deduction rules presented by Suppes is the closest to the use of a choice operator. Instead of flags or graphical represented sub-proofs, he presents a "method of handling existential quantifiers by the use of *ambiguous names*" and notes that "the central idea of this approach is related to Hilbert's ε symbol" (Suppes, 1957,

[34]Copi's thanks to LeBlanc led to speculation that Copi had paid for the rights to include LeBlanc's formulation of the rule. The going rate for an elimination rule in 1967 was $300, according to rumours cited by Anellis (Anellis, 1991, p.142 fn.4)

p.vi). In his system, universal generalization ($\forall$ introducation), from free variables which are not flagged as premises, is allowed, and existential specification ($\exists$ elimination) is done by means of special constants which he symbolizes using Greek letters.

Anellis notes that Suppes' method received a second criticism, that LeBlanc had already levelled at Copi's version of EI, due to its use of *ambiguous names*. Gupta argues, like LeBlanc, that Suppes version of EI means that it cannot be considered an inference rule but rather should be considered a rule that introduces "auxiliary premises for the purpose of proof strategy".

Free variables are treated in two ways in the various formulations of natural deduction systems in textbooks: "about half treat them [free variables] semantically as existentially quantified and the other half as universally quantified" (PELLETIER, 1999, p.24). Suppes' versions of EI and UG are also interesting because he uses two different methods, both his ambiguous names for EI and flagging of free variables of UG. Suppes method of flagging free variables in the assumptions and his introduction of ambiguous names in EI means that he has the best of both worlds, free variables treated as universally quantified and ambiguous names which work more simply than another set of free variables that are treated as existentially quantified (see SUPPES, 1957).

Recent developments in natural deduction have shown that there is no need for arbitrary objects in natural deduction proofs (see DEVIDI and KORTÉ, 2014). The apparent need for them led to philosophers like Quine to both reject their existence, and use arbitrary terms in his logic textbooks.[35] Quine notes that variables should not be considered in anyway as names:

[35] DeVidi and Korté's method uses sub-proofs, but is a refinement of the Gentzen-Fitch proof method in which the flagged free variables or ambiguous names in propositions created by $\exists$-elimination and used in $\forall$-introduction are replaced with variables bound by what they refer to as "commonizing quantifiers", which range over sub-proofs (see DEVIDI and KORTÉ, 2014). For example, the following two proofs, one in the standard Fitch style and the other in their method, show how the commonizing quantifier replaces the arbitrary object that is flagged in the Fitch proof:

Fitch style Example:

1	$\forall y \neg Py$	
2	$\quad \exists x Px$	
3	$\quad\quad {}^{e}\ Pe$	
4	$\quad\quad\quad \forall y \neg Py$	R, 1
5	$\quad\quad\quad \neg Pe$	$\forall$E, 4
6	$\quad\quad\quad \bot$	$\neg$E, 3,5
7	$\quad\quad \bot$	$\exists$E, 2, 3–6
8	$\neg \exists x Px$	$\neg$I, 2–7

DeVidi-Korté Example:

1	$\forall y \neg Py$	A
2	$\quad \exists x Px$	A
3	$\quad\quad \exists x$	UC
4	$\quad\quad\quad Px$	A
5	$\quad\quad\quad \neg Px$	$\forall$E, 1
6	$\quad\quad\quad \bot$	$\neg$E, 3,5
7	$\quad\quad \bot$	$\exists$E, 2, 3-6
8	$\neg \exists x Px$	$\neg$I, 2-7

> Variables have no meaning beyond the pronominal sort of meaning which is reflected in translations such as ("Whatever you may select, it = it."); they serve merely to indicate cross-references to various positions of quantification (Quine, 1951, pp. 68-70).

and insists on the Fregean orthodoxy about the nature of variables:

> Care must be taken, however, to divorce this traditional word of mathematics from its archaic connotations. The variable is not best thought of as somehow varying through time, and causing the sentence in which it occurs to vary with it. Neither is it to be thought of as an unknown quantity, discoverable by solving equations. The variables remain mere pronouns for cross-reference to quantifiers (Quine, 1959, pp.127-128).

However, as we noted above, in his natural deduction system, flagged variables enable us in his words to "subjoin a conservative instance" of existential statements. Here the variables no longer simply provide a "cross-reference to quantifiers" but represent arbitrary instances of the propositions previously quantified.

6.5.5 Fine's Defence of Arbitrary Objects

Kit Fine attempts to address both Frege's and other's criticisms of arbitrary objects in Fine (1983). Fine thinks that arbitrary objects are "extremely valuable" and attempts to work out a non-naïve theory of them which he sees as answering the obvious criticisms (Fine, 1983, p.56).

The four main criticisms of the existence of arbitrary objects that he sees as needing answering to develop a more sophisticated theory are in Fine's words as follows:

1. There are no arbitrary objects;
2. the principles governing them are incoherent;
3. the theory leads to questions with no answers;
4. and it is, in any case, of no use. (Fine, 1983, p.56)

Fine deals quickly with the first point, which he breaks up into the sub-questions which we can call (1a) "Are there actually any such objects?" and assuming that the answer to the former is yes (1b) "What are they like?" He continues that the answer to (1a) all depends on what we mean by "there are."[36] In any "ontologically significant sense" Fine says that arbitrary objects are not real:

> I have a sufficiently robust sense of reality not to want to people my world with arbitrary numbers or arbitrary men. Indeed, I may be sufficiently robust not even to want individual numbers or individual men in my world. But if the intended sense is ontologically neutral, then my answer is a decided 'yes'. (Fine, 1983, p.57)

Fine asserts that he is "happy to go along with the most ardent reductionist and have him reduce the whole theory of arbitrary objects to one that trades in more respectable entities." He insists that this is not odd, a nominalist who rejects the reality of numbers, does not refuse to "speak with the mathematician or common man and say that there is a solution to the equation '$x + 5 = 12$' or that there are prime numbers greater than 12" (Fine, 1983, pp. 56-57). Rather he wishes to assert only that they 'exist' at what he calls an "intermediate level of theorizing" (Fine, 1983, p.57).

That done, Fine turns to the question (1b) "What are they like?" and to this he first gives the general answer that they "belong, like sets or propositions, to the category of abstractions" (Fine, 1983, p.58). The mistake that critics of arbitrary objects make, he asserts, is to assume they are on ontological parity with individuals. This is a problem he feels is rooted in a, "certain metaphysical or psychological picture that may have been suggested by the more zealous advocates of arbitrary objects," specifically the view that abstract entities are simply normal objects "shorn or particular features" as in Lewis's "Way of Negation" (see Lewis, 1986, p.83). The other sort of reason that Fine wishes to reject is that it seems normal to say that an arbitrary member of a set should have all the properties common to members of that set so that say an arbitrary number should be either odd or even (Fine, 1983, p.58). Instead he holds that objects are more akin to functions than individuals. Hence much like choice functions they range over sets, and seem to pick out elements, but are treated as if they lack specificity.

The incoherence of the principles governing arbitrary objects is the second objection which Fine addresses. Again he breaks this up into two sub-questions: the problem of complex properties and the problem of special properties (Fine, 1983, p.59).

The problem of complex properties arises from the idea that members of a set may be divided by an exclusive disjunction, for example the natural numbers are broken into odds and evens, and every member of that set is one or the other. What then of arbitrary

[36]Or in the words of US president Bill Clinton "It depends upon what the meaning of the word 'is' is." Though of course Clinton's distinction was about the reality of past and future events and Fine's point is slightly more prosaic.

numbers? Do they fall to the criticism of Locke's triangles? Fine asserts that no, arbitrary objects fall under what he calls "the principle of generic attribution" a principle which he gives several more and more refined definitions G1 through G4 (Fine, 1983, pp.59ff).

Fine's first formulation, (G1) $\varphi(a) \equiv \forall i \varphi i$ (a φ's iff every individual φ's), of his generic applicability principle fails because it does not apply to the whole context in which the arbitrary object applies (Fine, 1983, p.60). This gives us a problem because if our predicate is $E(x)$ ('x is even') and a is an arbitrary number then the claim $\neg E(a)$ follows from the fact that $\neg\forall x E(x)$, but we can also get the opposite result—since we clearly can get $\neg O(a)$, and $\forall(\neg Ox \rightarrow Ex)$.

Fine's G2 and G3 are similarly stages which solve a problem with the previous version, but fail, as G1 does, to get everything right. Arbitrary objects do not have the properties that all typical individuals do (*e.g.* "all individuals are individuals"), so they should not satisfy certain classical properties, but only 'generic' ones. And there may be interdependence between arbitrary objects when more than one of them is in question in a particular formula. Eventually, Fine arrives at a final generic applicability principle:

> (G4) If $\varphi(x_1, x_2, ..., x_n)$ is a generic condition containing no names for arbitrary objects, then $\varphi(a_1, a_2, ...a_n)$ is true iff $\varphi(a_1, a_2, ...a_n)$ is true for all admissible assignments of individuals $i_1, i_2, ...i_n$ to the objects $a_1, a_2, ...a_n$

which also takes into account the fact that there may be interdependence among arbitrary objects and though a single arbitrary object represents a class of individuals a number of them may represent a relation between individuals (Fine, 1983, pp.67-68).

Fine uses this principle to avoid the problem of complex properties, *e.g.* $\psi(x) \vee \chi(x)$. Rather than having a different rule for disjunction of arbitrary objects, Fine suggests that to evaluate a disjunction applied to some arbitrary object a:

> $\psi(a) \vee \chi(a)$, we first apply the rule of generic attribution. This tells us that $\psi(i) \vee \chi(i)$ is true iff each of the statements $\psi(i) \vee \chi(i)$ is true for an individual in the range of a. We then apply the standard rule of disjunction to each of the statements $\psi(i) \vee \chi(i)$ (Fine, 1983, p.62).

More interesting for our purposes and something that shows a clear relation between Fine's arbitrary objects and choice is his alternate formulation of his principle of generic attribution. Instead of applying the principle of generic attribution and then the rule for the operator, he suggests that instead one should understand the statement $\psi(a) \vee \chi(a)$ to be "syntatically ambigious", that is it may be understood to be formed in two different ways. It may either be taken to be simply the disjunction of $\psi(a)$ and $\chi(a)$ or more interestingly it can be understood to be that application of $\lambda x(\psi(x) \vee \chi(x))$ to a. The latter is not a simple disjunction and "the extensions of the property abstracts $\lambda x\varphi(x)$ may be evaluated in the usual way for individuals in a way analogous to [the generic attribution principle] for arbitrary objects" (Fine, 1983, p.62). Of course to make the application of the abstract property sync with his final version (G4) of his principle of generic attribution it should be written more properly as:

$$[\lambda(x_1, x_2, ..., x_n)\varphi(x_1, x_2, ..., x_n)](a_1, a_2, ..., a_n)$$

The third criticism that Fine addresses is actually several concerns brought up by Frege. The main two are how one establishes whether or not arbitrary numbers have certain properties, and what the identity criterion for arbitrary numbers is. To Frege's question:

> 'to which of these classes [of the determinate and indeterminate numbers] do the primes belong?' Some of the determinate (individual) numbers are prime. But what of the indeterminate (arbitrary) numbers?

Fine answers that arbitrary numbers can have a restricted range and so could be either restricted to the primes or have a range of all numbers (Fine, 1983, p.66). Fine's point is that this is not the sort of question that is appropriate to ask of arbitrary numbers.

To the question of whether an identity criterion for arbitrary objects can be provided, Fine introduces the idea of dependence among arbitrary objects (Fine, 1983, p.68). An object is to be considered "independent if it depends upon no other objects and that otherwise it is dependent." Identity is then determined in one of two ways:

> Suppose first that a and b are independent objects. Then we say that $a = b$ iff their ranges are the same.

In the case that they are dependent objects, however:

> Then we shall say that $a = b$ iff two conditions are satisfied. The first is that they should depend upon the same arbitrary objects; their 'dependency range' should be the same. The second is that they should depend upon these objects in the same way.

The final criticism that Fine addresses is that there is no need for such entities. Fine suggests four applications: "(1) the logic of generality; (2) mathematical logic; (3) language; and (4) the history of ideas" (Fine, 1983, p.73). Of course by the logic of generality he means exactly the semantics of natural deduction and the rules of universal generalization (UG) and existential instantiation (EI) which we have covered above. With regard to mathematical logic, Fine notes that besides the "modest formal development" of the general theory presented in his book, the theory may also shed light, on more traditional puzzles in mathematical logic. He gives as examples nul-potent infinitesimals which might be treated as arbitrary objects "whose values tend closer and closer to zero" This would be an improvement on Robinson's non-standard analysis approach to infinitesimals which can make room for nilpotent infinitesimals. He also suggests that arbitrary objects might illuminate Skolem functions, where arbitrary objects "correspond loosely to multi-valued Skolem functions" (Fine, 1983, p.74).

With regard to language Fine suggests that arbitrary objects have use in understanding mathematical language, which though it purports to treat variables as 'signs of generality' informally holds to a usage that is much closer to that of traditional understanding of variables (Fine, 1983, p.75). Another interesting application to language is that of understanding pronouns.

> Every farmer owns a donkey. He beats it. He feeds it rarely ...

Fine notes that this is difficult to parse into quantified statements, and is at the least ambiguous.

> How are we to interpret the pronouns 'he' and 'it'? The simple and natural view of the pronouns is that they are used to refer, in a given discourse, to objects that in some sense are 'in play'. But if we stick to an ontology of individuals, this view can hardly be maintained; for there is no individual farmer or individual donkey to which the pronouns can sensibly be taken to refer.

Some would say that the example Fine gives translates as: $\forall x \exists y (Fx \rightarrow (Oxy \wedge Dy \wedge Bxy \wedge Rxy...))$ but this does not reflect the natural language grammars, nor does it let us get from this sentence and the sentence "Rex is the donkey owned by Bob the farmer" (say $Fb \wedge Orb \wedge Dr$) to the conclusion that therefore "Bob beats Rex" (Bbr).

While some like Fine see entities that would represent pronouns as useful, and a semantics using some sort of operator worth proposing, Peter Geach has criticised investigating the use of pronouns in this manner, and the desire by some for a formalism more complicated than that of quantified propositional logic. Geach referred to such an effort as exploring "the labyrinth of idiom" and believed that a proper theory of reference would focus on only what he termed the "logically important features" of language (Geach, 1962, p.132).

The example about the inability to infer "Bob beats Rex" suggests that Geach's charge is not on target. The inference is clearly a correct one, so it seems that the representation of pronouns is "logically important."

Fine's suggestion is that his theory of arbitrary objects is useful for dealing with pronouns because of its ability to handle the dependence and independence of objects:

> 'He' refers to the arbitrary farmer, 'it' to the arbitrary donkey that he owns. Note that this arbitrary donkey is a dependent object and that for a given farmer as value for the arbitrary farmer, the arbitrary donkey can only take as a value a donkey that the farmer owns. Thus the statement 'He beats it' will be true, just as it should be, iff for all values i and j simultaneously assumed by the arbitrary farmer and donkey, it is true that i beats j (Fine, 1983, p.76).

Fine is not alone in is view that arbitrary or ideal objects can help deal with inter- and intra-sentential reference. Gareth Evans (Evans, 1977, p.470) in his response to Geach (1962, 1975) deals with the so-called donkey pronouns (*e.g.*,the 'it' in the sentence "Every farmer who owns a donkey beats it.") which he refers to as e-type pronouns, with the use of term forming operator λ, in much the same manner Fine does. However there is a whole literature of applying the ε-operator to not only e-type pronouns, but also to a variety of linguistic phenomena. Further discussion of this and other applications of choice is unfortunately beyond the scope of this thesis.

6.6 The Nature of Epsilon Terms

Let us now return to epsilon objects, and in particular to the question of what they must be like given their role in "strengthening" the logic when added to intuitionistic logic. The strengthening of the logic in part is due to the properties of the sentences that epsilon abstracts in conjunctions with the facts about terms that are encoded in the decidability principles. Arbitrary objects like Locke's triangles are one extreme, defining objects that entail exclusive disjunctive properties, like being acute, or obtuse; or right, or oblique.

How does this pan out in the proofs presented in Chapter 4? Recall that Bell's decidability condition was: $\forall x(x = a \vee x \neq a) \wedge (b \neq a)$. In this condition the term a is well defined in reference to all terms, that is all objects x either are equal a or not. Furthermore the second part indicates that at least one term b is definitely not equal to a as well. Then we define a proposition $A(x) \equiv [(x = a \wedge B) \vee (x \neq a \wedge C)]$ and propositions $B \leftrightarrow A(a)$ and$C \leftrightarrow A(b)$ follow, given decidability.

Thus in the proofs of linearity and DeMorgan's intuitionistically invalid law, we rely on the fact that, given the existence of epsilon terms and the existence of one decidable object, for any pair of sentences B and C we can define a predicate $A(x)$ so that the following two conditions hold:

$$A(\varepsilon_A) \text{ iff } B \vee C$$

$$\neg A(\varepsilon_{\neg A}) \text{ iff } \neg B \vee \neg C.$$

Roughly, given the decidability conditions and the epsilon principle, we have an "ideal" element, a likeliest element to make any disjunctive property hold, and an ideal that rules both disjuncts out. The nature of such objects is obviously connected to issues around the nature of disjunctive properties.

In a language with identity, we can define the properties $A(x)$ and $\neg A(x)$. That is, any first order with identity language will allow us to describe the "property" in the following manner:

> $A(x)$: The property of being a if B is true or of being something else if C is true.

Given the decidability of the identity of a (meaning that we can move from "not failing to be a" to "being a"), we have:

> $\neg A(x)$: The property of not being a if B is true and of being a if C is true.

What does the addition of epsilon add to the mix? It gives us two objects: one that has the property $A(x)$ to precisely the extent that $B \vee C$ is true, and another that has the property $\neg A(x)$ precisely to the extent that $\neg B \vee \neg C$ is true. Here it is tempting to let "classical", in particular bivalent, thinking infect our description: since "$A(x)$" has a "disjunctive" property, if B is true it will be a, if C is true it will be something else such as b, if both B and C are true, well it can be either, and if neither is true x can be anything at all.

But here, recall we don't have bivalence. The truth value of $B \vee C$ is the join of the truth value of B and the truth value of C, and so in general is greater than the truth value of either disjunct. What epsilon is adding is that there is an object that has the property A to precisely the extent that $B \vee C$ is true.

Consider a simple toy example, using numbers we should not take too seriously because we're not treating them as meets and joins: suppose one third of all animals are each of dogs, pigs and humans. For any particular animal there is a one third chance that it's a dog, a third it is a pig, and a one third chance that it's a human. So, Bd is "Derek is a pig", Cd is that "Derek is a dog", and we write C to abbreviate Cd, B to abbreviate Bd. Then ε_A is some element of the domain that has the property A to exactly the extent that $B \vee C$ is true, *i.e.*,$\frac{2}{3}$. It's not Derek, because the truth value of B is $\frac{1}{3}$ and the truth value of C is $\frac{1}{3}$. So what is ε_A? It seems like it's going to have to be part dog and part pig, doesn't it. And what, indeed, will $\varepsilon_{\neg A}$ be?

It seems, then, that in a non-classical context, the epsilon operator is compelling us to take some of the talk that occurs in discussions of arbitrary objects somewhat more seriously.

In a classical, but not bivalent setting, we might consider the natural numbers example: $Ox \vee Ex$ has the truth value 1, though each of Ox and Ex truth values n and $\neg n$ in the middle of the Boolean Algebra 4. What epsilon requires is that, for these sentences Ox and Ex, ε_A has the property "odd or even" to the extent 1, though it doesn't have either disjunct completely.

Of course, in the classical case, there is a temptation to explain this away: the number is one or the other, so you really only need to be sure that you've picked from the winning side one of the disjuncts must really have the truth value 1.

The lesson of the proofs, though, is that while we can explain these things away in the classical setting, it makes a difference in non-classical settings. Let us consider an ordinary language case where classical logic is at the least doubtful. If we choose a class of properties that don't have sharp boundaries, such as colour properties, things look different. So if we use green or not green, for instance, and assume that there are objects which fail to be green while also failing to be not green, so that $Gx \vee \neg Gx$ can have a truth value less than 1, though greater than either Gx or $\neg Gx$, then having the property of being "green or not green" is something besides "picking from the winning side".

We can take from this a metaphysical lesson. In non-classical settings, accepting these "ideal objects" has consequences. We can characterize different metaphysical categories based on how harmless the abitrary objects for disjunctive properties are. In objective domains, where bivalence reigns, the "ideality" can be regarded as merely a matter of us not knowing which category of a disjunction epsilon comes out of. This is because in such objective domains all truth values are either 0 or 1, therefore we can simply assume that it comes from one of the pots that has value 1 if any do.

For domains where, perhaps, De Morgan is correct but bivalence is not correct—*e.g.* the example of intelligence given above—we should get something different. If Bob is good at math, but can't cook a meal without burning his house down, we can't say that Bob is smart, nor can we say that he's not. But the extent to which Bob is smart or not smart is greater than the extent to which he is either. So there is something that has the property "Bob is smart or Bob is not smart" to this greater extent—which is not valued 1. However, as we've seen, the willingness to accept these ideal objects, and correspondingly to take these odd properties seriously, also commits us to accepting that the property "Bob is not smart or Bob is not not smart" is a different property again, and something has that property completely. That is, some explaining needs to be done that accounts for the taxonomy of properties imposed by these formal results.

Finally, for truth apt domains where objectivity and realism seem unlikely, we should find the idea of such properties and ideal objects to be a non-starter. At a very abstract level of description, this seems plausible enough. If, for instance, beauty really is in the eye of the beholder, the notion that there is an ideal object for the predicate "x is pretty" seems unlikely—we are confronted not with issues like the disjunctive properties of being even or odd for natural numbers, but with something much worse, namely the absence of typical characteristic of the non-objective property in question. Lack of objectivity seems likely to rob us of grounds for judging likeliness. What if we translate this down to the level of the plausibility of the property $A(x)$ corresponding to "Bob is pretty or Bob is not pretty"? I suggest that what makes it more reasonable to suggest that there could be such an ideal object in the case of a property like intelligence, if intelligence does indeed have the multi-factorial structure we've supposed as we've used it as an example, is that we are presuming the compatibility of the various factors that make someone intelligent, and the possibility of their varying independently. To adapt the example slightly, if Yotam Ottolenghi has more cooking smarts than Bob but less mathematical intelligence, then what epsilon is asking us to buy an $A(x)$ that has Ottolenghi's degree of kitchen smarts and Bob's level of math smarts. If prettiness really is un-objective, there are no similar factors on which to pin the judgements, or to arrange ideals, so there is nothing be said in favour of there being an ideal object that has the $A(x)$ property that captures "Picasso pretty or sunset pretty".

6.7 Conclusion

To summarize the lessons of this discussion, it is useful to recall what we might regard as a fairly standard view of intuitionistic logic, which is that it is somehow a "epistemic" logic in a way that classical logic is not. We have seen this idea expressed in slightly different ways by different authors above: it is sometimes claimed that intuitionistic logic is said to have an "epistemic motivation" and classical logic is said to have an "ontological basis" (van Dalen, 2002, p.1); and, as discussed at length in Chapter 3, Michael Dummett contends that intuitionistic logic is *metaphysically neutral*, largely because it is compatible with a close link between truth conditions and knowability, while the correctness of classical logic presupposes a commitment to realism. It is therefore not a new view in philosophy that as we commit ourselves to a stronger logic by moving from intuitionistic to classical logic, if we do so legitimately, it is because our ontological commitments have increased.

What our investigations have shown us is that if we take this picture seriously as a starting point there is much more philosophical insight to be gained by the investigation of the relationship between increased ontological commitment and principles of logic. We have seen that there are degrees of strengthening—that it is not merely a matter of "neutral" intuitionistic logic and "realist" classical logic, but that there are interesting logical levels in between.

We have seen that what degree of strengthening beyond intuitionistic logic is achieved can be connected to assumptions we make about the existence of objects of particular sorts, as represented by our accepting into our languages terms of various sorts.

The addition of abstract or ideal objects to intuitionistic logic through the means of choice operators, which allow us to use names of individuals to talk about types of objects, not specific objects, involves the acceptance of generic objects, i.e. for each property that there is a most likely object to have that property. This assumption alone already moves us away from the epistemic or constructive justification of intuitionistic logic—for instance, we certainly have no recipe for constructing the likeliest ϕ for every ϕ. As we have seen, making this move already changes the logic. By itself, though, it comes nowhere near taking us all the way to classical logic.

Let us review this in a bit more detail. In this chapter we have discussed how epsilon terms are ontologically potent, i.e., accepting them involves an ontological commitment. They act like names and hence represent some sort of objects, and because their inclusion in an intuitionistic logical language makes a difference to the strength of the language we cannot dismiss them as mere place-holders or "manners of speaking" that can be explained away via contextual definition in the manner Russell and others sometimes suggested for definite descriptions.

However, by themselves the epsilon terms only change the logic in the sense of making valid certain quantificational laws that are not valid in intuitionistic logic. I have suggested in this chapter that more metaphysically interest lies when certain propositional "laws" that are not valid in intuitionistic logic are made valid. Choice operators do not do this alone. We saw that accepting epsilon together with some modest-seeming additional assumptions—that there are two objects a and b that we can prove to be distinct, and that for one of them a there is always a fact of the matter for any object whether it is a or it isn't—allows us to arrive at a way station between classical and intuitionistic logic. We were also able to isolate some other, perhaps less modest, assumptions that are enough to yield classical logic—for instance, that coextensive predicates have identical epsilon terms. These results suggest this philosophical picture: The sense in which each of these assumptions involves acceptance of an existence claim or an assumption about the nature of the objects referred to by the epsilon terms is clear, so these results in a minimal sense make quite clear that ontological assumptions can "buy logical principles". Where the assumptions are warranted, we are paying cash rather than credit. Where we accept logical laws such as excluded middle, or the fourth De Morgan's law, without inquiring into what grounds our acceptance of the existence of epsilon terms and the assumptions leaves us with an outstanding philosophical debt.

I suggest that much of the discussion in this chapter usefully suggests that this philosophical picture links up with what is already often thought and said about questions of realism and anti-realism, objectivity and subjectivity, and so on, but that the way it links can teach us something. I think that there is much more to be learned by further investigation and discussion of the philosophical lessons of these results and related ones yet to be discovered, and that this discussion is only a start. But it is an interesting start. Some of the points made above are perhaps, in retrospect, unsurprising, but they had not yet been remarked. In a domain for which epsilon terms are legitimate, if we make an extensionality assumption we get all of classical logic. So, given the present framework, those assumptions are tantamount to the assumption of realism. But, as we noted, the link between extensionality and realism is not news in philosophy. Arguably, what the present discussion does is lay bare the contribution it makes to the acceptance of "full-blooded realism."

More interesting, perhaps, are the issues about existential commitments being sufficient to justify logics between classical and intuitionistic logic. What then are the sort of domains that the addition of choice seems right, but not extensionality? Given the "modest" decidability conditions, this makes "Dummett's scheme" valid, but not excluded middle, which from the semantic point of view means that the "truth values" in the domain must be linearly ordered (or rather, have the property of linearity which we have noted is not exactly the same thing).

What I have suggested above is that such domains are, metaphysically speaking just as the logic suggests they ought to be, perched somewhere between full-blooded realism and those for which anti-realism seems most appropriate. Considering what a domain must be like for linearity to hold for it and yet for it not be classical, we considered cases where there are composed of many properties that are themselves composed of other properties that themselves are "objective" (and so for which the is always a matter of fact about which of two objects has each subsidiary property to a greater extent, for instance) but where there is no non-arbitrary way to compile them into a single overall ranking. We used intelligence as an intuitive example of a property that is, arguably, of this sort, but there are plenty of others. What I also tried to suggest was that those who contend that intelligence is not a real thing have a point, but so do those who insist that it's not (as humor seems to be) unreal either. This account nicely explains why there are sometimes facts of the matter—e.g. Betty is more intelligent than Bob when she exceeds him on every dimension—but not always. Intelligence has a kind of objectivity that humor seems not to have.

The second major line of discussion in this chapter takes the discussion of the metaphysical lessons of these results about term forming operators in a new direction. Analytical philosophers, especially those with an interest in logic, have of course spent considerable time investigating metaphysical matters of many sorts. All the work done so far in this thesis put us in a position to make a useful contribution to some longstanding discussions in this literature, namely those involving the inter-related issues of abstraction and ideal (or generic or arbitrary) objects.

The lesson is easiest to state briefly with reference to Kit Fine's influential discussion of arbitrary objects. As I tried to show, the challenges Fine has to answer in defending arbitrary objects are similar to if not the same as those confronting someone wishing to advocate for epsilon. It is thus unsurprising to find a term forming operators (λ) playing an important role in Fine's account. One indirect lesson of the discussion of arbitrary objects for this project, then, is that it helps clarify the nature of the ontological commitments one seems to need to defend be taking aboard when one defends accepting something like epsilon. The central lesson, though, is that Fine's discussion fails to account for the important role of the logic presupposed to be running in the background of his discussion. As we have seen (and repeatedly noted), both epsilon and tau are conservative over classical logic, so while there may be some practical advantages (for instance in terms of computational or representational efficiency) to including them in our languages, including them need not commit one to the existence of additional entities.

In intuitionistic logic where these operators are not conservative, the terms these terms purport to refer to are no longer subject to being "explained away." Thus taking seriously the role of intuitionistic logic in metaphysics adds a layer of complexity to these discussions. For one thing, It makes it likely that the question "are there arbitrary objects" is one that is not going to have a uniform answer, if answering "yes" involves at the same time committing oneself to a stronger logic than one otherwise would accept. Alternatively, one might try to make clear why a commitment to arbitrary objects is not the same thing as a commitment to epsilon objects (or tau objects), in spite of their seeming likeness, and so the kinds of reasoning that leads from epsilon to strengthened logic. Either way, a defender of arbitrary objects has more work to do than had previously been thought.

Conclusion

Chapter 7

Conclusion

7.1 Hilbert and Brouwer

We began this thesis by discussing Hilbert's famous reaction to early constructive views, and his reaction to the *Ignorabimusstreit*, the debate over whether there were problems that, in principle, humans would never be able to solve. Hilbert's optimism in asserting that "there was no *ignorabimus* in mathematics" is not important simply as a reaction to a the general *fin-de-siècle zeitgeist*, but more specifically it represented his rejection of a certain type of conservative proto-constructivism prevalent among mathematicians like Kronecker, who represented an important faction in the German mathematical establishment at the time.

Hilbert's programme was an attempt to develop a finitist foundation for mathematics, that is, one that would be acceptable in the terms of those who opposed his existence proofs. Hilbert saw this as part of his position of defending the consistency, autonomy and completeness of mathematics. The development of Hibert's foundational work, from his early attempts at axiomatising arithmetic, to his response to Russell and Frege's logicism, informed the logical programme that led Hilbert, Ackermann, and Bernays to introduce first the τ and then the ε axioms in the early 1920s.

While Brouwer and Hilbert agreed on the autonomy of mathematics, they disagreed on how to provide foundations for mathematics. Hilbert wanting to build foundations that would support the parts of mathematics dependant on principles, like excluded middle, that in his mind were essential to mathematics; while constructivists like Brouwer were willing to pare the tree of mathematics down to those parts that could be well supported by the logical principles he saw as acceptable.

The τ and the ε axioms were introduced into Hilbert's programme as being obvious formal cognates to informal mathematical practices where mathematicians often speak of an arbitrary object of some type or another. Hilbert's use of these operators was not focused on the use of ideal elements themselves, but rather the derivation of finitist consistency proofs. Hilbert always meant for the ideal elements to be removed in the final results. Hence Albert Leisenring notes that Hilbert and Bernays use the ε-operator only in what he calls a "subsidiary role," that is, they use it to prove certain theorems which could be proved using the predicate calculus, but which are more easily proved using the ε-calculus (Leisenring, 1969, p.5) .

It is perhaps ironic that τ and ε-calculi were introduced by Hilbert, Ackermann and Bernays to help to find a finitist, and hence supposedly constructive, consistency and completeness proof for arithmetic. In fact the τ and ε-operators are not constructive, and of course cannot help prove completeness for something that has been shown, by Gödel's famous proofs, to be incomplete. Part of the problem is simply until Heyting formalised intuitionistic logic there was no good account of what constructive or finitist foundations should look like.

7.2 The Development of Intuitionism

As we saw in Chapter 2, intuitionism can be can be broken up into several related movements. We discussed philosophical intuitionism as a precursor to L.E.J. Brouwer's intuitionistic mathematics. The principles of Brouwer's mathematics were formalised into intuitionistic logic by Kolomgorov and Heyting. Then in the second half of the 20th century intuitionistic logic was put on better philosophical foundations by Michael Dummett and other philosophically minded logicians, or logically minded philosophers.

For the purposes of the present investigation, Dummett's foundation for intuitionistic logic is of central importance. Intuitionistic logic is metaphysically neutral, he argues, because the relevant introduction and elimination rules are "harmonious", while stronger systems are not.

The law of exclude middle is not logical, but rather an ontological principle according to Dummett's argument that intuitionistic logic is metaphysically neutral and hence 'logic proper' and that the acceptance of classical logic is legitimate only in case one accepts realism. This is interesting when we take into account that we can prove the law of excluded middle by adding the ε-operator and extensionality for epsilon to intuitionistic logic. Extensional ideal elements are enough to get you to a realist position.

There is, of course, another approach one can take in the face of the irrefutable evidence the non-conservativeness of Hilbert's operators and so their unsuitability to his original philosophical intentions. There have been some attempts to provide "harmonious" versions of the epsilon principle which can be found in the literature. Such approaches always require limitations on the epsilon principle that seem *ad hoc*, and so merely render the principle unsuitable to Hilbert's original intentions in another way—recall, the principle is supposed to be a formal version of a very natural and common form of reasoning in mathematics. In any case, we set that approach aside for the present investigation as there is ample reason for philosophers to be interested in what happens when the most natural versions of the epsilon principle are added to intuitionistic logic.

In fact the addition of the ε-operator to intuitionistic predicate logic lets us prove several intermediate principles in predicate logic. But what is even more interesting is that the addition of ε-operator and weaker decidability principles for terms to intuitionistic logic result in sentential intermediate logics.

7.3 Formal Matters

In Chapter 4 reviewed the proofs given by John Lane Bell in Bell (1993a,b) and provided several new proofs of both intermediate logics and the law of excluded middle with decidability conditions that lack identity.

In Chapter 5, we reviewed several different systems of formal semantics for epsilon, largely because doing so is an efficient way to lay bare some of the complications of giving an adequate semantics in intuitionistic logic and so to make clear where its addition is making a difference to the logic. We began with the standard semantics for classical epsilon, Asser-Leisenring semantics which call the "truth set" for $\varphi(x)$ the set of things that, when substituted for x, make $\varphi(x)$ true. These semantics use a choice function in the metalanguage to select a member of the truth set for $\varphi(x)$, and allow it to be the referent of $\varepsilon x \varphi(x)$.

Günther Asser also developed more complicated semantic machinery to handle classical epsilon operators when we do not make the simplifying assumption of extensionality. Asser in fact offers three different sorts of semantics for classical epsilon calculus (for detail see Asser, 1957, pp.53 *ff.*). The first is the Asser-Leisenring semantics given its fullest treatment by Leisenring. But it is the third semantics, that is a first step towards giving a semantics for epsilon that does not make epsilon extensionity valid. He does so by making the referent of $\varepsilon\varphi$ depend not only on the truth values φ takes, but on its syntax.

There are three issues that must be dealt with in creating a semantics for an intuitionisitic ε-calculus. The first is dealing with the problem of multiple truth values, the second is dealing with non-linear truth sets and the third is the problem of trying to avoid extensionality.

DeVidi's semantics differs from Asser's semantics in two ways. DeVidi eliminates the validity of the substitutivity of identicals and introduces what he calls the "alpha axiom" to rename bound variables. The elimination of substitutivity of identicals he accomplishes by introducing what he calls skeleton terms, a modification of what Asser referred to as ground terms.

Asser's ground term for the expression $\varepsilon x.\varphi(x, \alpha)$ where α is either a free term, such as $\varepsilon y.\psi y$; a free variable, say y; or constant, for example c, is $\varepsilon v_1.\varphi v_1 v_2$, indicating that the syntactical form of the epsilon term. The skeleton term shows the syntactic form in more detail, the expression $\varepsilon x.\varphi(x, \varepsilon y.\psi y)$ where $\varepsilon y.\psi y$ is a free term, has the ground term $\varepsilon v_1.(\varphi v_1, v_2)$ but its skeleton term is $\varepsilon v_1.\varphi(v_1, \varepsilon v_2.\psi v_2)$.

The alpha axiom simply allows for the renaming of bound variables so that $\sigma x.\varphi = \sigma y.\varphi[x/y]$ where σ is a choice operator, otherwise every variable would demand a different skeleton term.

In both Asser's type three and DeVidi semantics the choice function is a two-place function depending not only on the truth set, but also on the ground or skeleton term underlying the epsilon term.

7.4 Philosophical Conclusions

These technical results offer the prospect of philosophical, and in particular metaphysical, illumination if we consider them while taking seriously the view arrived at in the first three chapters of the thesis, that due to its verificationist nature intuitionistic logic is metaphysically neutral while the stronger classical logic is not. We saw in Chapter 6 that by increasing our commitment to the existence of entities we increase the strength of our logic. That is we can see that by first adding simply the epsilon axiom, and then stronger and stronger principles about terms, first the decidability conditions then the extensionality, we get stronger and stronger logics. We get degrees of strengthening, not merely intuitionistic logic and then classical logic, but logical levels in between.

The addition of choice operators involves a certain acceptance of generic objects. Instead of speaking circuitously that there is some x that has a property φ, the epsilon operator lets us name it: ε_φ. While this is in some manner non-constructive, it involves an ontological commitment, it does not alone demand a full blown realist interpretation. Without further conditions on terms, epsilon terms only make valid certain quantificational laws that are not valid in intuitionistic logic. But with relatively minor conditions on terms one can prove intermediate principles. These conditions are not obviously ontologically burdensome, they simply say that there are distinct objects, at least two, and of one of them that it can be distinguished from any other. To put it another way, all objects that exist either are distinguishable from a canonical term or not, if they are not, then all other terms are distinguishable from them or not *via* what we know about the canonical term. This is what make objects in some limited sense discrete, they either are or are not equivalent to the canonical term.

Other, perhaps less modest assumptions, strengthen intuitionistic+ε logic even more, for example, the Ack principle, that coextensive predicates have identical epsilon terms, or the following principle:

$$\forall x(\varphi \to \psi) \to (\varrho(\varepsilon_\varphi/x) \to \varrho(\varepsilon_\psi/x))$$

are enough to yield classical logic.

As we have noted the fact that extensionality and realism are connected is not news, but what is interesting here is that we have a gradation of existential commitments that justify logics between classical and intuitionistic logic. This is interesting from two directions: The first is that intermediate logics, which hitherto have not been thought particularly interesting philosophically, can be connected with particular ontological commitments. And secondly in that it may give us a way of looking at a gradation of metaphysical commitments between anti-realism and full blown realism.

This enables us to make something of the discussion of generic, or abstract, objects and the nature of abstraction itself. We discussed Kit Fine's defence of arbitrary objects which are similar to, if not the same as, the sort of ideal objects defined by epsilon, in fact Fine discussed the term-forming operator lambda in his account. We noted that what Fine misses out on is the role of the underlying logic. There must be some logical system running in the background, and it has certain ontological commitments attached to it whether we want it to or not. If the logic is classical since epsilon and tau are to a large extent conservative over classical logic including them your logic need not commit you to the existence of additional entities. But if your logic is constructive, the addition of them puts you in a very different place.

7.5 Further Directions

There are many untrodden paths and potentially philosophically interesting aspects of choice operators, that could not be covered here for reasons of space and time.

7.5.1 Modal Objects

Since the choice operators ε and τ define terms that reference most or least likely objects to have a property, the connection between this and modality is quite obvious.[1] It seems

[1] In fact in intuitionistic modal logic there may be an even tighter connection, as in intuitionistic modal logic, box is better understood as a preciseness operator, rather than a necessity operator (see DeVidi

intuitive that the most likely object to have a property say ε_φ would also be the object that would most likely make the sentence $\exists x \Box \varphi(x)$ true. Likewise it seems likely that τ_φ would also be the object that would most likely make the sentence $\exists x \diamond \varphi(x)$ false.

Furthermore there is the question of possible objects and their ontological status. Dummett argues, using the discussion of Quine and Kripke on the question of possible unicorns, that one must use a modal logic weaker than S5 to describe the logic of possible fictional objects like unicorns, because they can be possible in different ways (DUMMETT, 1993a, p. 333-346). This raises the question of the status of the most likely thing to have a possible property $\diamond\varphi\varepsilon_{\diamond\varphi}$ considering that the ε-operator strengthens the logic.

7.5.2 Abstract Objects and Identity in First Order Logic

Several of the proofs given in Chapter 4 are given in a language which lacks identity. Such formulations are often seen as being more logical, as the inclusion of identity in one's language seems to be making an ontological claim.

The technical results of Jakko Hintikka and Kai Wehmeier (HINTIKKA, 1956; WEHMEIER, 2008, 2004, 2012) show that first order logic without identity is equal in expressiveness to first order logic with identity and provide a transformation algorithm between the two. This has not yet been given for intuitionistic logic and it is not immediately clear if their proof would translate. In addition, as Wittgenstein foresaw, to maintain this logical strength an exclusive interpretation of variables is necessary. It is not immediately clear how such variables work with regard to term forming operations.

7.5.3 van Heijenoort on Sortal and Mass Terms

Quine (QUINE, 1960, pp.90-95) and Strawson (STRAWSON, 1959, pp.202-209) both assert that mass terms, in comparison with the use of sortal terms "represent a primitive, archaic survival of a preparticular level of thought" (VAN HEIJENOORT, 1973, p.32). On the other hand van Heijenoort points out that because the use of mass terms, *i.e.* stuff talk, leads to "magnitude talk" and that "magnitude talk" has supplanted "sortal talk" in scientific discourse, for example, the sortal terms "light and heavy" have been replaced with the notion of weight in scientific discourse. Hence he asserts that one could easily take "stuff talk" to represent "a higher level of thought"(VAN HEIJENOORT, 1973, p.32). Van Heijenoort further notes that there is a systematization of "stuff ontology" in modern physics. "Energy" or "Matter" is spoken of rather than objects.

---

and SOLOMON, 1997).

> ...modern logic is thoroughly based on the subject-predicate analysis, both by the form of its prime sentences and the ontology of its semantic. But there, are, in Western-European languages, words, like mass terms, that suggest another ontology (or is it naïve physics?)(van Heijenoort, 1973, p.33)..

Quine is chided by van Heijenoort for, when considering other methods of talking than of objects, he discusses only obtuse examples of "temporal segments of rabbits" or "local manifestations of rabbithood", while not noticing that both natural languages and some "systematic" languages, such as the Aristotelian syllogistic and theoretical physics, present an view that "escapes an ontology of individuals" (van Heijenoort, 1973, p.33).

Epslion terms seem to force an "object" rather than a "stuff" approach. Even magnitude predicates are treated in a sortal manner if Mx is "x is 5 kg of gold" the ε_M is the name of the object that is best decribed by M *i.e.* the object most likely to be "5 kg of gold". Hence if we add an epsilon axiom to our logic we are supposing a sortal ontology rather than a stuff ontology.

7.5.4 Natural Language Applications

In their 2000 book *Reference and Anaphoric Relations* Klaus von Heusinger and Urs Egli survey the application of choice functions in formal semantics including: using choice functions to represent *in situ* wh-expressions (Engdahl, 1986; Reinhart, 1992)[2] representing specific indefinites (cf. Reinhart, 1992, 1997; Kratzer, 1998; Winter, 1997), formalizing E-type pronouns (Ballmer, 1978; Hintikka and Kulas, 1985; Slater, 1986; Chierchia, 1992; van der Does, 1993; Egli and von Heusinger, 1995) and definite NPs (von Heusinger, 1997) (von Heusinger and Egli, 2000). Of course since von Heusinger's and Egli's survey there has been even more recent work by linguists who have discussed using epsilon to reference nouns in article-less languages such as Japanese (Nishiguchi, 2015), and computer scientists studying the application of epsilon calculi in automated proof solvers (Wirth, 2015; Baaz and Weller, 2015).

The most famous of all these examples are the donkey sentences, examples of sentential anaphora so named for the example Peter Geach adopts from a set of mediaeval examples of problematic sentence schemata (Geach, 1962).[3] Donkey pronouns (*e.g.*,the 'it' in the

[2]Wh-expressions are clauses or sentences using interrogative words; the five Ws in English *who, what, where, why, how*, or the rhetorical interrogatives *quis, quid, quando, ubi, cur* and *quem ad modum, quibus adminiculis* or *quomodo, quibus auxiliis* (Robertson, 1946, p.7).

[3]Geach states the example was supplied to him by William Kneale. Later in *The Development of Logic* William and Martha Kneale discuss the example which comes from an appendix of rules at the end of the *Modernorum Summulae Logicales.* The appendix contained "a list of miscellaneous rules *ad discerendos syllogismos a fallaciarum paralogismis*" (Kneale and Kneale, 1971, p.273) (on how to discern syllogisms

sentence "Every farmer who owns a donkey beats it.") are also referred to as either d-type or e-type pronouns.[4]

If certain aspects of language need to be modelled by a term forming operator, and then since we know that epsilon strengthens constructive logics then you need take care of the parts of language modelled by epsilon in a domains where there are good arguments for taking a non-realist approach: *e.g.* theoretical objects, possible objects, fictional objects, mathematical objects.

7.6 Final Thoughts

This thesis should be regarded as programmatic—as is perhaps appropriate for dissertations which are both a culmination and a beginning of a next stage. I have tried to demonstrate a few related things: first I have tried to make the case that there is the potential for rich philosophical rewards from further investigation of the effects of adding term forming operators to intuitionistic logic, a process that required a good deal of historical and technical stage setting. Secondly, I have tried to draw from those lessons. Finally, I briefly tried to show ways in which this investigation is a long way from finished. I intend to push these investigations forward in the future, and hope other philosophers will join me.

---

from fallacious arguments) the first of which includes the example of when terms are distributed in one part of a syllogism and later undistributed. Geach gives the example in the following manner:

> Every donkey that belongs to a villager is running in the race;
> Brownie is not running in the race;
> Ergo, Brownie is not a donkey that belongs to a villager

The original reads:

Cuiuslibet huis asinus currit.	Any asses that run are theirs.
Brunellus est huis asinus.	Burnel is their ass.
Ergo Brunellus currit	Therefore Burnel runs. (Diel 1489. My Translation.)

[4] First named such by Gareth Evans (Evans, 1977, p.470) in his response to Geach (1962, 1975).

References

References are given by name of author and date of first publication. Translations are given separate entries unless part of collections. If there is more than one work for a given author in a given year then works will be denoted by the year and a lower case alphabetic character, *e.g.* Bloggins (1900a, 1900b, 1900c,*etc*...). This applies for all works, except those by L. E. J. Brouwer which are numbered according to the scheme presented in van Dalen (2008b), *e.g.*, Brouwer (1911E).

Abrusci, V. M. (1989). "David Hilbert's 'vorlesungen' Logic and Foundations of Mathematics." In G. Corsi, C. Mangione, and M. Mugnai, eds., *Atti Del Convegno Internazionale Di Storia Della Logica, San Gimignano, 1987*. Editrice Cooperativa Libraria Universitaria Editrice, 1989, pp. 333–338.

Ackermann, W. (1924). "Begründung des "tertium non datur" mittels der Hilbertschen Theorie der Widerspruchsfreiheit." *Mathematische Annalen*, vol. 93:pp. 1–36

Ajdukiewicz, K. (1928). *Glówne zasady metodologji nauk i logiki formalnej (Main principles of the methodology of the sciences and of fomal logic).* Warszawa

Akama, S. (1997). *Logic, Language and Computation.* Springer

Anellis, I. (1991). "Forty Years of "Unnatural" Natural Deduction and Quantification: A History of First-Order Systems of Natural Deduction from Gentzen, to Copi." *Modern Logic*, vol. 2:p. 113

Aristotle (1894). *Ethica Nicomachea.* Oxford, Clarendon Press

Aristotle (1985). *Nicomachean Ethics.* Hackett Publishing Company. Translated by T. Irwin

Asser, G. (1957). "Theorie der Logischen Auswahlfunktionen." *Mathematical Logic Quarterly*, vol. 3(1-5):pp. 30–68

Avigad, J. and Zach, R. (2013). "The Epsilon Calculus." In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy.* The Metaphysics Research Lab, Center for the Study of Language and Information, Stanford University, spring 2013 edn.

BAAZ, M. and WELLER, D. (2015). "Cut-free epsilon-calculus allows a non-elementary speed-up." In *Hilbert's Epsilon and Tau in Logic, Informatics and Linguistics*

BALLMER, T. (1978). *Logical Grammar*. North Holland

BARTOCCI, C., BETTI, R., GUERRAGGIO, A., and LUCCHETTI, R. (2011). *Mathematical Lives: Protagonists of the Twentieth Century from Hilbert to Wiles*. Springer. Translated by Kim WILLIAMS

BEESON, M. J. (1980). *Foundations of Constructive Mathematics*. Metamathematical studies. Springer-Verlag

BELL, J. L. (1993a). "Hilbert's ε-operator and classical logic." *Journal of Philosophical Logic*, vol. 22(1):pp. 1–18

BELL, J. L. (1993b). "Hilbert's ε-operator in intuitionistic type theories." *Mathematical Logic Quarterly*, vol. 39(1):pp. 323–337

BELL, J. L. (1998). *A Primer of Infinitesimal Analysis*. Cambridge University Press

BELL, J. L. (2009). *The Axiom of Choice*, vol. 22 of *Studies in Logic: Mathematical Logic and Foundations*. College Publications

BENACERRAF, P. and PUTNAM, H. (1983). *Philosophy of Mathematics: Selected Readings*. Cambridge University Press

BERKELEY, G. (1843). *The works of George Berkeley, Bishop of Cloyne*. Past masters. Printed for Thomas Tegg, London

BERNAYS, P. (1922b). "Die Bedeutung Hilberts für die Philosophie der Mathematische." *Die Naturwissenschaften*, vol. 10:pp. 93–99. Excerpted and translated in HALLETT (1996).

BERNAYS, P. (1923). "Erwiderung auf die Note von Herrn Aloys Müller: Uber Zahlen als Zeichen." *Mathematische Annalen*, vol. 90:pp. 159–163. English translation in MANCOSU 1998a pp.223–226.

BERNAYS, P. (1928). "Zusatz zu Hilberts Vortrag über "Die Grundlagen der Mathematik"." *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 6:pp. 88–92. English translation in: VAN HEIJENOORT (1967a) pp.485-489

BISHOP, E. (1967). *Foundations of Constructive Analysis*. McGraw-Hill

BOTTAZZINI, U. (2011). "Hilbert's Problems: A Research Program for "Future Generations"." In BARTOCCI *et al.* (2011). Springer, pp. 1–10

BRADLEY, F. H. (1883). *The Principles of Logic*. GE Stechert

BRADLEY, F. H. (1922). *The Principles of Logic*. Oxford University Press, second edition edn.

BROUWER, L. E. J. (1905A). *Leven, Kunst en Mystiek*. Waltman Delft. Translated as BROUWER (1996)

BROUWER, L. E. J. (1907). *Over de Grondslagen der Wiskunde*. Ph.D. thesis, Universiteit van Amsterdam. Advisor: D.J. KORTEWEG. Translated as "On the Foundations of Mathematics" in BROUWER, L.E.J., 1975, *Collected Works 1. Philosophy and Foundations of Mathematics*, A. HEYTING (ed.), Amsterdam: North-Holland

BROUWER, L. E. J. (1908C). "De onbetrouwbaarheid der logische principes." *Tijdschrift voor Wijsbegeerte*, vol. 2:pp. 152–158. Translated as "The unreliability of the logical principles" in (BROUWER, 1975, pp.107-111)

BROUWER, L. E. J. (1909A). "The Nature of Geometry." In BROUWER (1975). North-Holland Elseiver, pp. 112–120

BROUWER, L. E. J. (1911A). "Mannoury, Methodologisches und Philosophisches zur Elementar-Mathematik (Haarlem 1909)." *Nieuw Archief voor Wiskunde*, vol. 2(9):pp. 199–201. Excerpted and translated in BROUWER (1975) p.121–122

BROUWER, L. E. J. (1911E). "Beweis der Invarianz des n-dimensionalen Gebiets." *Mathematische Annalen*, vol. 71:pp. 305–313

BROUWER, L. E. J. (1912A). *Intuïtionisme en Formalisme*. Clausen, Amsterdam. Translated into English as (BROUWER, 1913C). Polish translation in (MURAWSKI, 1986). French translation in (LARGEAULT, 1992)

BROUWER, L. E. J. (1913C). "Intuitionism and Formalism." *Bulletin of the American Mathematical Society*, vol. 20:pp. 81–96. Also in BENACERRAF and PUTNAM (1983), pp. 77-89. Original BROUWER (1912A).

BROUWER, L. E. J. (1949C). "Consciousness, Philosophy and Mathematics." In E. Beth, H. Pos, and H. Hollak, eds., *Library of the Tenth International Congress in Philosophy, August 1948*, vol. 1. North-Holland, pp. 1235–1249. Also appears in BROUWER (1975), pp. 480–494.

BROUWER, L. E. J. (1975). *Collected Works 1. Philosophy and Foundations of Mathematics*. v. 1. North Holland Elseiver, Amsterdam

Brouwer, L. E. J. (1996). "Life, Art, and Mysticism." *Notre Dame Journal of Formal Logic*, vol. 37(3):pp. 389–429. Translation of Brouwer (1905A).

Cantini, A., Casari, E., and Minari, P., eds. (1999). *Selected Contributed Papers of the Tenth International Congress of Logic, Methodology and Philosophy of Science, Florence, August 1995*. Springer-Science+Business Media

Carnap, R. (1942). *Introduction to semantics*. Harvard University Press

Chierchia, G. (1992). "Anaphora and Dynamic Logic." *Linguistics and Philosophy*, vol. 15:pp. 111–183

Chimakonam, J. (2012). *Proof in Alonzo Church's and Alan Turing's Mathematical Logic: Undecidability of First Order Logic*. Universal-Publishers

Church, A. (1956). *Introduction to Mathematical Logic*. Princeton University Press

Copi, I. (1954). *Symbolic Logic*. Macmillan

Copi, I. (1967). *Symbolic Logic (Second Edition)*. Macmillan

Corcoran, J., Hatcher, W., and Herring, J. (1972). "Variable Binding Term Operators." *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 18:pp. 177–182

Corry, L. (2004). *David Hilbert and the Axiomatization of Physics (1898-1918): From Grundlagen Der Geometrie to Grundlagen Der Physik*. Archimedes (Dordrecht). Springer

Curry, H. B. (1963). *Foundations of Mathematical Logic*. Mcgraw Hill

Curry, H. B. (1966). *A Theory of Formal Deducibility*, vol. 6 of *Notre Dame Mathematical Lectures*. University of Notre Dame, Notre Dame, Indiana, 2nd edn.

Czuber, E. (1898). *Vorlesungen über Differential und Integralrechnung,*, vol. 1. Teubner, Leipzig

Dauben, J. W. (1979). *Georg Cantor: his mathematics and philosophy of the infinite*. Harvard University Press, Boston

Davey, B. and Priestly, H. (2002). *Introduction to Lattices and Order*. Cambridge University Press, second edn.

de Bruijn, N. G. (1998). "Type-theoretical checking and the philosophy of mathematics." In Sambin and Smith (1998). Clarendon Press, Oxford

Dekker, P., Stokhof, M., Institute for Logic, L., and Computation (1992). *Proceedings of the Eighth Amsterdam Colloquium: December 17-20, 1991*. Proceedings of the Eight Amsterdam Colloquium. ILLC, University of Amsterdam

Descartes, R. (1954). *Descartes Philosophical Writings*. Nelson. Translated byAnscombe, G.E.M., Geach, Peter T. and Koyré, Alexandre

DeVidi, D. (1994). *Term Forming Operators in First Order Logic*. Ph.D. thesis, University of Western Ontario

DeVidi, D. (1995). "Intuitionistic ε- and τ-calculi." *Mathematical Logic Quarterly*, vol. 41(4):pp. 523–546

DeVidi, D. (2004). "Choice Principles and Constructive Logics." *Philosophia Mathematica*, vol. 12:pp. 222–243

DeVidi, D. (2006). "Assertion Proof and the Axiom of Choice." In DeVidi and Kenyon (2006). Springer-Verlag

DeVidi, D. (2011). "Buying Logical Principles with Ontological Coin." In *Canadian Philosophical Association Conference*

DeVidi, D. and Kenyon, T. (2006). *A Logical Approach to PhilosophyEssays in Honour of Graham Solomon*. Springer-Verlag

DeVidi, D. and Korté, H. (2014). "Naturalizing Natural Deduction." De Gruyter

DeVidi, D. and Solomon, G. (1997). "$\Box$ In Intuitionistic Logic." *Australasian Journal of Philosophy*, vol. 75(2):pp. 201–213

Diaconescu, R. (1975). "Axiom of Choice and Complementation." *Proceedings of the American Mathematical Society*, vol. 51(1):pp. 176–178

Diel, F. (1489). *Modernorum summulae logicales cum notabilibus topicorum ac disputatis elenchorum librorum ex aristotele / boetio / beato augustino / marsilio / et ab aliis / subtilioribus sententiis / viris doctissimis fideliter enucleatae / ac a magistris collegii moguntini regentibus de modernorum doctrina sunt studiosissime innouatae*. Peter Drach

Dragalin, A. G. (1974). "Интуиционисцкая Логика и ε-Символ Гилберта." *История и Методология Естественных Наук : Математика, Механика*, vol. XVI:pp. 78–84

du Bois-Reymond, E. (1872). "Über die Grenzen des Naturerkennens." In *Ein Vortrag in der zweiten öffentlichen Sitzung der 45. Versammlung deutscher Naturforscher und Ärzte zu Leipzig am 14. August 1872*. Translated by J. Fitzgerald as du Bois-Reymond (1874) du Bois-Reymond, E. (1874). "The Limits of our Knowledge of Nature." *The Popular Science Monthly*, vol. 5:pp. 17–32. Translated by J. Fitzgerald.

du Bois-Reymond, E. (1880). "Die sieben Welträtsel." In *der Leibniz-Stiftung der Akademie der Wissenschaften am 8. Juli 1880*, vol. 20. pp. 433–447. Translated by J. Fitzgerald as du Bois-Reymond (1882a).

du Bois-Reymond, E. (1882a). "The Seven World-Problems." *The Popular Science Monthly*, vol. 20:pp. 433–447. Translated by J. Fitzgerald.

du Bois-Reymond, P. (1882b). *Die allgemeine Functionentheorie*. Tübingen. Translated as du Bois-Reymond (1887)

du Bois-Reymond, P. (1887). *Théorie générale des fonctions*. Imprimiere Ni coise, Nice. Translated by G Milhaud and A Girot

Dugac, P. (1973). "Elements d'analyse de Karl Weierstrass." *Archive for History of Exact Sciences*, vol. 10:pp. pp. 41–176.

Dummett, M. (1973). *Frege: Philosophy of Language*. Duckworth

Dummett, M. (1975). "The Philosophical Basis of Intuitionistic Logic." In H. Rose and J. Shepherdson, eds., Rose *et al.* (1975). Elsevier, pp. 5–40. Reprinted in Benacerraf and Putnam (1983) pp. 97–129.

Dummett, M. (1977). *Elements of Intuitionism*. Oxford University Press, Oxford

Dummett, M. (1978a). "A Defence of McTaggart's Proof of the Unreality of Time." In *Truth and Other Enigmas*. Duckworth, pp. 351–357

Dummett, M. (1978b). "The Reality of the Past." In *Truth and Other Enigmas*. Duckworth, pp. 358–374

Dummett, M. (1991). *The Logical Basis of Metaphysics*. Harvard University Press

Dummett, M. (1993a). "Could There Be Unicorns?" In *The Seas of Language*. Oxford: Clarendon Press, pp. 328–348

Dummett, M. (1993b). *Origins of Analytical Philosophy*. Duckworth

Dummett, M. (1993c). *The Seas of Language*. Clarendon Press, Oxford

Dummett, M. (2008). "Concluding remarks at the Cerisy conference." In M. v. Atten, P. Boldini, M. Bourdeau, and G. Heinzmann, eds., van Atten *et al.* (2008). Birkhäuser Basel, pp. 341–342

Dummett, M. A. E. and Lemmon, E. J. (1959). "Modal Logics Between S 4 and S 5." *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, vol. 5:p. 250

Egli, U., Pause, P. E., Schwarze, C., von Stechow, A., and Wienold, G. (1995). *Lexical Knowledge in the Organization of Language*. Benjamins

Egli, U. and von Heusinger, K. (1995). "The Epsilon Operator and E-Type Pronouns." In Egli *et al.* (1995), no. 114 in Current Issues in Linguistic Theory. Benjamins

Engdahl, E. (1986). *Constituent Questions: the Syntax and Semantics of Questions with Special Reference to Swedish*. Studies in linguistics and philosophy. D. Reidel Pub. Co.

Evans, G. (1977). "Quantifiers and Relative Clauses I." *Canadian Journal of Philosophy*, vol. 7:pp. 467–536

Ewald, W. (1996). *From Kant to Hilbert: a source book in the foundations of mathematics*. Oxford science publications. Clarendon Press

Field, H. (1980). *Science Without Numbers: A Defence of Nominalism*. Oxford University Press

Fine, K. (1983). "A Defence of Arbitrary Objects." *Proceedings of the Aristotelian Society*, vol. 57:pp. 55–77

Fine, K. (1985). "Natural Deduction and Arbitrary Objects." *Journal of Philosophical Logic*, vol. 14(1):pp. 57–107

Frege, G. (1884). *Die Grundlagen der Arithmetik: eine logisch-mathematische Untersuchung Über den Begriff der Zahl*. Verlag von Wilhelm Koebner. Translated as Frege (1980)

Frege, G. (1893). *Grundgesetze der Arithmetik I*, vol. 1. Jena: Verlag Hermann Pohle. Partially translated in Frege (1965)

Frege, G. (1903). *Grundgesetze der Arithmetik II*, vol. 2. Jena: Verlag Hermann Pohle.

Frege, G. (1965). *The Basic Laws of Arithmetic*. University of California Press. Translated by Montgomery Furth

Frege, G. (1979). *Posthumous Writings*. Basil Blackwell, Oxford. Translated by Peter Long and Roger White with the assistance of Raymond Hargreaves

Frege, G. (1980). *The Foundations of Arithmetic: A Logico-Mathematical Enquiry Into the Concept of Number*. Northwestern University Press

GABBAY, D. and GUENTHNER, F., eds. (2002). *Handbook of Philosophical Logic*, vol. 5 of *Handbook of Philosophical Logic*. Kluwer Academic Publishers

GABBAY, D. and GUENTHNER, F., eds. (2011). *Handbook of Philosophical Logic*, vol. 15 of *Handbook of Philosophical Logic*. Kluwer Academic Publishers

GABBAY, D. and MAKSIMOVA, L. (2011). "Interpolation and Definability." In GABBAY and GUENTHNER (2011). Springer

GABBAY, D. and WOODS, J., eds. (2009). *Handbook of the history of logic: Logic from Russell to Church*. Handbook of the History of Logic. Elsevier

GEACH, P. (1975). "Back-Reference." In A. Kasher, ed., *Language in Focus: Foundations, Methods and Systems*, vol. 43 of *Boston Studies in the Philosophy of Science*. Springer Netherlands, pp. 25–39

GEACH, P. T. (1962). *Reference and Generality: an Examination of some Medieval and Modern Theories*. Cornell University Press

GENTZEN, G. (1934). "Untersuchungen über das logische Schließen. I." *Mathematische Zeitschrift*, vol. 39:pp. 176–21. Translated as "Investigations into Logical Deduction: §I–III" in GENTZEN (1969) pp.68-103

GENTZEN, G. (1935). "Untersuchungen über das logische Schließen. II." *Mathematische Zeitschrift*, vol. 39:pp. 405–431. Translated as "Investigations into Logical Deduction: §IV" in GENTZEN (1969) pp.103-131

GENTZEN, G. (1936). "Die Widerspruchfreiheit der reinen Zahlentheorie." *Mathematische Annalen*, vol. 112:pp. 493–565. Translated as "The consistency of arithmetic" in GENTZEN (1969)

GENTZEN, G. (1969). *The Collected Papers of Gerhard Gentzen*. North Holland

GERSON, L. P. (1994). *Plotinus*. Routledge

GÖDEL, K. (1930). "Die Vollständigkeit der Axiome des logischen Funktionenkalküls." *Monatshefte für Mathematik und Physik*, vol. 37:pp. pp.349–360. Translated as "The completeness of the axioms of the functional calculus of logic," in VAN HEIJENOORT (1967a) pp.582–591

GÖDEL, K. (1931). "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I." *Monatshefte für mathematik und physik*, vol. 38:pp. 173–198. Translated as "On Formally Undecidable Propositions in Principia Mathematica and Related Systems I" in GÖDEL (1986) pp. 144–195

Gödel, K. (1932). "Zum intuitionistischen aussagenkalkül." *Anzeiger der Akademie der Wissenschaften in Wien*, vol. 69:pp. 65–66. Translated as "On the intuitionistic prepositional calculus" in Gödel (1986) pp.223-225

Gödel, K. (1933). "Zur intuitionistischen Arithmetik und Zahlentheorie." *Ergebnisse eines mathematischen Kolloquiums*, vol. 4:pp. 34–38. Translated as "On intuitionistic arithmetic and number theory" in Gödel (1986).

Gödel, K. (1986). *Collected works*. Oxford University Press, New York

Goodman, N. and Quine, W. V. (1947). "Steps Toward a Constructive Nominalism." *The Journal of Symbolic Logic*, vol. 12:pp. 105–122.

Goodman, N. D. and Myhill, J. (1975). "Choice Implies Excluded Middle." *Zeitschrift fur Mathematische Logik und Grundlaaen der Mathematik*, vol. 24:p. 461

Grattan-Guinness, I. (2000). *Search For Mathematical Roots, 1870–1940: Logics, Set Theories and The Foundations of Mathematics From Cantor through Russell to Gödel*. Princeton University Press

Gray, J. (1997). "Algebraic Geometry Between Noether and Noether — A Forgotten Chapter in the History of Algebraic Geometry." *Revue d'histoire des math'ematiques,*, vol. 3:pp. 1–48

Gray, J. (1998). "Mathematicians as Philosophers of Mathematics: Part 1." *For the Learning of Mathematics*, vol. 18:pp. 20–24

Gray, J. (1999). "Mathematicians as Philosophers of Mathematics: Part 2." *For the Learning of Mathematics*, vol. 19:pp. 28–91

Gray, J. (2000). *The Hilbert Challenge*. Oxford University Press

Groenendijk, J. (1993). *Plurals and Anaphora*. Institute for Logic, Language and Computation (ILLC), University of Amsterdam

Hallett, M. (1996). "Hilbert and Logic." In M. Marion and R. Cohen, eds., Marion and Cohen (1996), vol. 177 of *Boston Studies in the Philosophy of Science*. Springer Netherlands, pp. 135–187

Hardy, D., Richman, F., and Walker, C. (2009). *Applied Algebra: Codes, Ciphers and Discrete Algorithms, Second Edition*. Discrete Mathematics and Its Applications. CRC Press

Herbrand, J. (1930). *Recherches sur la thèorie de la dèmonstration*. Ph.D. thesis, Université de Paris. Translated in Herbrand (1971).

Herbrand, J. (1971). *Collected Works*. Harvard University Press.

Heyting, A. (1930a). "Die formalen Regeln der intuitionistischen Logik I." In *Sitzungsberichte der Preussischen Akademie der Wissenschaften*. Translated in Mancosu (1998a), pp. 311-327

Heyting, A. (1930b). "Die formalen Regeln der intuitionistischen Logik II." In *Sitzungsberichte der Preussischen Akademie der Wissenschaften*

Heyting, A. (1930c). "Die formalen Regeln der intuitionistischen Logik III." In *Sitzungsberichte der Preussischen Akademie der Wissenschaften*

Heyting, A. (1931). "Die intuitionistische Grundlegung der Mathematik." *Erkenntnis*, vol. 2:pp. 106–115. Translated in Benacerraf and Putnam (1983), pp. 52-60

Heyting, A. (1934). *Mathematische Grundlagenforschung: Intuitionismus, Beweistheorie.* Springer. Expanded and translated into French as Heyting (1955)

Heyting, A. (1955). *Les foundements des mathématiques: Intuitionnisme, Théorie de la démonstration*. Gauthier-Villars and E. Nauwelaerts, Paris and Louvain

Hilbert, D. (1885). *Über die invarianten Eigenschaften spezieller binärer Formen, insbesondere der Kugelfunktionen*. Ph.D. thesis, Königsberg. Translated as "On The Invariant Properties of Special Binary Forms, Especially Spherical Functions" in (Hilbert *et al.*, 1978)

Hilbert, D. (1887). "Über einen allgemeinen Gesichtspunkt für invarianten-theoretische Untersuchungen im binären Formengebiete." *Mathematische Annalen*, vol. 28:pp. 381–446. Translated as "On a General Point of View for Invariant-Theoretic Investigation of Binary Forms" in (Hilbert *et al.*, 1978)

Hilbert, D. (1890). "Über die Theorie der algebraischen Formen." *Mathematische Annalen*, vol. 36:pp. 473–534. Translated as "On the Theory of Algebraic Forms" in (Hilbert *et al.*, 1978)

Hilbert, D. (1893). "Über die vollen Invariantensysteme." *Mathematische Annalen*, vol. 42:pp. 3l3–373. Translated as "On the Complete Systems of Invariants" in (Hilbert *et al.*, 1978)

Hilbert, D. (1894). "Über die gerade Linie al küzeste Verbindung zwischen zweier Punkte." *Mathematische Annalen*, vol. 46:pp. 91–96. Translated as "Appendix I. The Straight Line as the Shortest Distance Between Two Points" as Hilbert and Bernays 1999, pp.108-112), translation of (Hilbert 1930, pp.241–132)

HILBERT, D. (1899). *Grundlagen der Geometrie*. Teubner, Leipzig. Translated as (HILBERT, 1902).

HILBERT, D. (1900a). "Über den Zahlbegriff." *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 8:pp. 180–184. Translated as "On the Concept of Number" in EWALD (1996) pp. 1089-1096

HILBERT, D. (1900b). "Mathematische Probleme." *Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Klasse*, vol. 1:pp. 253–297. Translated by MARY WINSTON NEWSON as "Mathematical Problems: Lecture Delivered Before The International Congress Of Mathematicians At Paris In 1900." in *Bulletin of the American Mathematical Society* 1902, pp. 437-479

HILBERT, D. (1902). *The Foundations of Geometery*. Open Court. Translated by E. J. TOWNSEND

HILBERT, D. (1905). "Über die Grundlagen der Logik und der Arithmetik." In *Verhandlungen des dritten Internationalen Mathematiker-Kongresses in Heidelberg vom 8. bis 13. August 1904*. Teubner, Leipzig, pp. 174–85. Translated as "On the Foundation of Logic and Arithmetic" in (VAN HEIJENOORT, 1967a, pp.129-138)

HILBERT, D. (1917). "Axiomatisches Denken." *Actes de la Societe Helvetique des Sciences Naturelles*, vol. 99:pp. 139–150. Full paper published as HILBERT (1918)

HILBERT, D. (1918). "Axiomatisches Denken." *Mathematische Annalen*, vol. 78:pp. 405–415. Longer version of HILBERT (1917). Translated as "Axiomatic Thinking" in *Philosophia Mathematica*, Vol. VII, No. 1-2, pp.1–12, and in EWALD (1996) pp. 1107-1115

HILBERT, D. (1920). "Probleme der mathematischen Logik." In EWALD (1996). Clarendon Press, pp. 943–946. Excerpt from lectures held in Gottingen in the Summer Semester of 1920, transcribed by Moses SCHÖNFINKEL and Paul BERNAYS. Translated by William EWALD.

HILBERT, D. (1922). "Neubegründung der Mathematik: Erste Mitteilung." *Abhandlungen aus dem Seminar der Hamburgischen Universität*, vol. 1:pp. 157–77. Talks given at the University of Hamburg, July 25-27, 1921. Translated as "The New Grounding of Mathematics" in MANCOSU 1998a pp.198–214 and EWALD 1996 pp.1115–1134

HILBERT, D. (1923). "Die logischen Grundlagen der Mathematik." *Mathematische Annalen*, vol. 88:pp. 151–165

HILBERT, D. (1926). "Über das Unendliche." *Mathematische Annalen*, vol. 95(1):pp. 161–190. Translated as "On the infinite " in VAN HEIJENOORT *(1967a)* pp.368-392

HILBERT, D. (1927). "The Foundations of Mathematics." In VAN HEIJENOORT (1967a). Harvard University Press

HILBERT, D. (1930). *Grundlagen der Geometrie*. Teubner, Leipzig, 7 edn.

HILBERT, D. (1931). "Die Grundlegung der elementaren Zahlenlehre." *Mathematische Annalen*, vol. 104:pp. 485–494. Translated as "The Grounding of Elementary Number Theory" in EWALD (1996) vol. II. pp.1148–1157

HILBERT, D., ACKERMANN, M., and HERMANN, R. (1978). *Hilbert's Invariant Theory Papers*, vol. VIII of *Lie Groups: History, Frontiers and Applications*. Math Science Press, Brockline, Massachusetts. Translated by M. ACKERMANN

HILBERT, D. and ACKERMANN, W. (1928). *Grundzüge der theoretischen Logik*. Springer-Verlag

HILBERT, D. and BERNAYS, P. (1934). *Grundlagen der Arithmetik I*. Springer-Verlag

HILBERT, D. and BERNAYS, P. (1939). *Gundlagen der Arithmetik II*. Springer-Verlag

HILBERT, D. and BERNAYS, P. (1999). *Foundations of Geometery*. Open Court, second edn. Translation of HILBERT (1930) by LEO UNGER.

HILBERT, D. and BERNAYS, P. (2001). *Fondements des mathématiques*, vol. I & II. Editions L'Harmattan. Translated by François GAILLARD and Marcel GUILLAUME, of HILBERT and BERNAYS (1934) and HILBERT and BERNAYS (1939)

HILBERT, D. and BERNAYS, P. (2004). "The Foundations of Arithmetic I & II." *Unpublished Manuscript*. Translated by Corinna ELSENBROICH of HILBERT and BERNAYS (1934) and HILBERT and BERNAYS (1939)

HILBERT, D. and BERNAYS, P. (2013). *The Foundations of Arithmetic I*. Springer. *Preprint*. Translated by Claus-Peter WIRTH of HILBERT and BERNAYS (1934) §1-7

HINTIKKA, J. (1956). "Identity, Variables, and Impredicative Defintions." *The Journal of Symbolic Logic*, vol. 21:pp. 225–245

HINTIKKA, J. and KULAS, J. (1985). *Anaphora and Definite Descriptions: Two Applications of Game-Theoretical Semantics*. Reidel

HINTIKKA, J., NIINILUOTO, I., and SAARINEN, E. (1979). *Essays on Mathematical and Philosophical Logic*. Boston Studies in the Philosophy of Science. Springer

HORN, A. (1969). "Logic with Truth Values in a Linearly Ordered Heyting Algebra." *Journal of Symbolic Logic*, vol. 34:pp. pp. 395–408

Hosoi, T. (1966a). “The axiomatization of the intermediate propositional systems S2 of Godel.” *Journal of the Faculty of Sciences, University of Tokyo Section 1*, vol. 13:pp. 183–187

Hosoi, T. (1966b). “The separation theorem on the classical system.” *Journal of the Faculty of Sciences, University of Tokyo Section 1*, vol. 12:pp. 223–230

Hosoi, T. (1967a). “On intermediate Logics I.” *Journal of the Faculty of Sciences, University of Tokyo Section 1*, vol. 14:pp. 293–312

Hosoi, T. (1967b). “On the axiomatic method and the algebraic method for dealing with propositional logics.” *Journal of the Faculty of Sciences, University of Tokyo Section 1*, vol. 14:pp. 131–169

Irvine, A. D. (2014). “Bertrand Russell.” In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Center for the Study of Language and Information, Stanford University, fall 2014 edn.

Jaśkowski, S. (1934). “On the Rules of Suppositions in Formal Logic.” In (McCall, 1967). Oxford at the Clarendon Press

Jespersen, O. (1924). *The Philosophy of Grammar*. George Allen & Unwin Ltd

Joyce, D. E. (1997). “The Mathematical Problems of David Hilbert.” Website, url:⟨http://aleph0.clarku.edu/˜djoyce/hilbert/⟩

Kant, I. (1724-1804). *Critique of Pure Reason*. Macmillan Company. Translation byFriedrich Max Müller 1881

Kleene, S. C. (1952). *Introduction to Metamathematics*. Bibl. Matematica. North-Holland, Amsterdam

Kleene, S. C. (1967). *Mathematical Logic*. Dover Publications

Kneale, W. and Kneale, M. (1971). *The Development of Logic*. Clarendon Press

Kneebone, G. T. (1963). *Mathematical Logic and the Foundations of Mathematics*. D. Van Nostrand

Kolomogorov, A. (1925). “On the principle of the excluded middle.” In van Heijenoort (1967a). Harvard University Press, pp. 416–437

Kratzer, A. (1998). “Scope or Pseudo scope? Are there Wide-Scope Indefinites?” In Rothstein (2001). Kluwer, pp. 163–196

KRONECKER, L. (1887). "Uber den Zahlbegriff." *Journal für die reine und angewandte Mathematik*, vol. 101:pp. 337–355. An shorter version of this paper is translated in EWALD (1996) pp. 941-955, full translation by DEAN, E.T. 2010, ⟨*http://www.andrew.cmu.edu/user/edean/papers/kronecker_zahlbegriff.pdf*⟩

KRONECKER, L. (1901). *Vorlesungen über Zahlentheorie*. Teubner

LARGEAULT, J. (1992). *Intuitionism*. Presses Universitaires de France

LARGEAULT, J. (1993). *Intuition et intuitionisme*. Librairie Philosophique J. Vrin, Paris

LEBLANC, H. (1965). "Minding One's X's and Y's." *Logique et Analyse*, vol. 8(31):pp. 209–210

LEISENRING, A. (1969). *Mathematical logic and Hilbert's ε-symbol*. University mathematical series. Macdonald Technical & Scientific

LEIVANT, D. (1973). *Existential Instantiation in a System of Natural Deduction for Intuitionistic Arithmetics*. Stichting Mathematisch Centrum

LEWIS, D. K. (1986). *On the Plurality of Worlds*. Blackwell, Oxford

LINK, G. (2004). *One Hundred Years of Russell's Paradox*. De Gruyter Series in Logic and Its Applications Series. Walter De Gruyter Incorporated

LOCKE, J. (1700). *An Essay Concerning Human Understanding*. Eliz. Holt

MANCOSU, P. (1998a). *From Brouwer to Hilbert: the debate on the foundations of mathematics in the 1920s*. Oxford University Press

MANCOSU, P. (1998b). "Hilbert and Bernays on Metamathematics." In MANCOSU (1998a). Oxford University Press, pp. 149–188

MARION, M. and COHEN, R. (1996). *Québec Studies in the Philosophy of Science*, vol. 177 of *Boston Studies in the Philosophy of Science*. Springer Netherlands

MARTIN-LÖF, P. (1975). "An Intuitionistic Theory of Types: Predicative Part." In ROSE *et al.* (1975). North Holland

MARTIN-LÖF, P. (2008). "The Hilbert-Brouwer Controversy Resolved?" In VAN ATTEN *et al.* (2008). North Holland, pp. 243–256

MCCALL, S., ed. (1967). *Polish Logic 1920–1939*. Oxford at the Clarendon Press

MCCARTY, D. C. (2004). "Hilbert and Paul Du Bois Reymond." In LINK (2004). Walter De Gruyter Incorporated, pp. 517–532

McKinsey, J. C. C. and Tarski, A. (1948). "Some Theorems About the Sentential Calculi of Lewis and Heyting." *The Journal of Symbolic Logic*, vol. 13(1):pp. pp. 1–15

Meyer-Viol, W. (1995a). *Instantial Logic*. Universtiteit Utrecht

Meyer-Viol, W. (1995b). "A Proof-theoretic Treatment of Assignments." *Logic Journal of IGPL*, vol. 3(2-3):pp. 223–242

Mints, G. and Tupailo, S. (1999). "Epsilon-substitution method for the ramified language and Δ^1_1-comprehension rule." In Cantini *et al.* (1999). Springer

Mints, G., Tupailo, S., and Buchholz, W. (1996). "Epsilon substitution method for elementary analysis." *Archive for Mathematical Logic*, vol. 35:pp. 103–130

Mints, G. E. (1974). "Геытинговскоые Исчислениые Предикатов с Епсилон-Символом." *Записки Научных Семинаров ПОМИ*, vol. 40:pp. 101–109. Translated as Mints (1977)

Mints, G. E. (1977). "Heyting predicate calculus with epsilon symbol." *Journal of Soviet Mathematics*, vol. 8(3):pp. 317–323. Originally published as Mints (1974)

Mints, G. E. (1991). "Gentzen-type systems and Hilbert's epsilon substitution method. I." In B. S. Dag Prawitz and D. Westerståhl, eds., Prawitz *et al.* (1994), vol. 134 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, pp. 91–122

Mints, G. E. (2012). "Intuitionistic Existential Instantiation and Epsilon Symbol." `arXiv:1208.0861v1 [math.LO]`

Mints, G. E. and Sarenac, D. (2003). "Completeness of indexed ε-calculus." *Archive for Mathematical Logic*, vol. 42(7):pp. 617–625

Mittag-Leffler, G. (1900). "Une page de la vie de Weierstrass." *Compte Rendu du Deuxième Congrès International des Mathématiciens*, vol. 2:pp. 131–152. Ed. E. Duporeq Paris: Gauthier-Villars, 1902

Moore, G. (1997). "Hilbert and the Emergence of Modern Mathematical Logic." *Theoria*, vol. 12(1):pp. 65–90

Murawski, R. (1986). *Filozofia matematyki. Antologia tekstów klasycznych*. Wydawnictwo Naukowe UAM. Poznan.

Nishiguchi, S. (2015). "Noun Phrases in Japanese and Epsilon-Iota Calculi." In *Hilbert's Epsilon and Tau in Logic, Informatics and Linguistics*

Noether, M. (1914). "Paul Gordan." *Mathematische Annalen*, vol. 75:pp. 1–41

PEANO, G. (1897). *Formulaire de Mathématiques*, vol. II. Paris

PEANO, G. (1901). *Formulaire de Mathématiques*, vol. III. Paris

PECKHAUS, V. (1990). *Hilbertprogramm und Kritische Philosophie.* Vandenhoeck und Ruprecht, Göttingen

PECKHAUS, V. (1994). "Logic in transition: The logical calculi of Hilbert (1905) and Zermelo (1908)." In PRAWITZ *et al.* (1994). Kluwer, Dordrecht, pp. 311–323

PEDERSEN, N. J. L. L. and WRIGHT, C. D. (2013). *Truth and Pluralism: Current Debates.* Oxford University Press

PELLETIER, F. J. (1999). "A Brief History of Natural Deduction." *History and Philosophy of Logic*, vol. 20(1):pp. 1–31

POINCARÉ, H. (1906). "Les Mathématiques et la Logique." *Revue de Métaphysique et de Morale*, vol. 14(1):pp. pp. 17–34. Translated as "Mathematics and logic: II", pp.1038–1051 in EWALD (1996)

POSY, C. J. (1974). "Brouwer's constructivism." *Synthese*, vol. 27:pp. 125–159

PRAWITZ, D. (1965). *Natural deduction: A proof-theoretical study.* Acta Universitatis Stockholmiensis, Stockholm studies in philosophy. Almqvist & Wicksell. Republished by Dover Publications in 2006.

PRAWITZ, D. (1973). "Towards a Foundation of General Proof Theory." In SUPPES *et al.* (1973). North Holland, Amsterdam

PRAWITZ, D. (1974). "On the Idea of a General Proof Theory." *Synthese*, vol. 27:pp. 63–77

PRAWITZ, D. (1977). "Meaning and Proofs: On the Conflict between Classical and Intuitionistic Logic." *Theroia*, vol. 43:pp. pp. 1–40.

PRAWITZ, D., SKYRMS, B., and WESTERSTÅHL, D. (1994). *Logic, Methodology, and Philosophy of Science IX: Proceedings of the Ninth International Congress of Logic, Methodology, and Philosophy of Science, Uppsala, Sweden, August 7-14, 1991.* International Congress of Logic Methodology and Philosophy of Science Proceedings. Elsevier

PRIOR, A. N. (1960). "The Runabout Inference Ticket." *Analysis*, vol. 21(2):pp. 38–39

QUINE, O., W. V (1959). "Meaning and Translation." In R. Brower, ed., *On translation*, vol. 23. Harvard University Press., pp. 148–72 of 297

QUINE, W. V. O. (1950). *Methods of Logic*. Harvard University Press, revised edition edn.

QUINE, W. V. O. (1951). *Introduction to Mathematical Logic*. Harvard University Press, revise edition edn.

QUINE, W. V. O. (1960). *Word and Object*. The MIT Press, Cambridge, MA

RANTA, A. (1994). *Type-Theoretical Grammar*. Clarendon Press

RASIOWA, H. and SIKORSKI, R. (1963). *The Mathematics of Metamathematics*. Monografie Matematyczne. Państwowe Wydawn. Naukowe

REID, C. (1986). *Hilbert—Courant*. Springer-Verlag. Originally published as two books: *Hilbert* 1970 and *Courant* 1976

REINHART, T. (1992). "Wh-in-situ: An Apparent Paradox." In (DEKKER *et al.*, 1992). ILLC, University of Amsterdam, pp. 483–491

REINHART, T. (1997). "Quantifier Scope: How labor is Divided Between QR and Choice Functions." *Linguistics and Philosophy*, vol. 20(4):pp. 335–397

RESCHER, N. (1958). "Can There be Random Individuals?" *Analysis*, vol. 18(5):pp. 114–117

ROBERTSON, J., D. W. (1946). "A Note on the Classical Origin of "Circumstances" in the Medieval Confessional." *Studies in Philology*, vol. 43(1):pp. pp. 6–14

ROSE, H., SHEPHERDSON, J., and FOR SYMBOLIC LOGIC, A. (1975). *Logic colloquium '73: proceedings of the Logic Colloquium, Bristol, July 1973*. Studies in logic and the foundations of mathematics. North-Holland Pub. Co.

ROSEN, G. (2012). "Abstract Objects." In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy*. The Metaphysics Research Lab, Center for the Study of Language and Information, Stanford University, spring 2012 edn.

ROTHSTEIN, S. (2001). *Events and Grammar*. Studies in linguistics and philosophy. Springer

ROWE, D. E. (1989). "Klein, Hilbert, and the Gottingen Mathematical Tradition." *Osiris*, vol. 5:pp. pp. 186–213

RUSSELL, B. (1897). *An Essay in the Foundations of Geometry*. Cambridge University Press

RUSSELL, B. (1903). *The Principles of Mathematics*. George Allen & Unwin, second edn.

Russell, B. (1908). "Mathematical Logic as Based on the Theory of Types." *American Journal of Mathematics*, vol. 30(3):pp. 222–262

Russell, B. (1914a). "Definitions and Methodological Principles in Theory of Knowledge." *The Monist*, vol. 24(4):pp. 582–593

Russell, B. (1914b). "On the Nature of Acquaintance, Part I." *The Monist*, vol. 24(1):pp. 1–16

Russell, B. (1914c). "On the Nature of Acquaintance. Part II. Neutral Monism." *The Monist*, vol. 24(2):pp. 161–187

Russell, B. (1914d). "On the Nature of Acquaintance. Part III. Analysis of Experience." *The Monist*, vol. 24(3):pp. 435–453

Russell, B. (1915a). "On the Experience of Time." *The Monist*, vol. 25(2):pp. 212–233

Russell, B. (1915b). "Sensation and Imagination." *The Monist*, vol. 25(1):pp. 28–44

Russell, B. (1918). "The Philosophy of Logical Atomism." In Russell (1986). Open Court, pp. pp. 193–210.

Russell, B. (1959). *My Philosophical Development*. Routledge

Russell, B. (1984). *Theory of knowledge: The 1913 manuscript*, vol. 7. Psychology Press. The first several chapters were originally published as Russell (1914b,c,d,a, 1915b,a).

Russell, B. (1986). *Collected Papers of Bertrand Russell, vol. 8, The Philosophy of Logical Atomism and Other Essays: 1914–1919*. Allen & Unwin

Russell, B. A. W. (1957). "Mr Strawson on Referring." *Mind*, vol. 66(263):pp. 385–389

Sambin, G. and Smith, J., eds. (1998). *Twenty-Five Years of Constructive Type Theory*. Clarendon Press

Saxe, K. (2002). *Beginning Functional Analysis*. Springer Undergraduate Texts in Mathematics and Technology. Springer

Schechter, E. (1997). *Handbook of Analysis and its Foundations*. Academic Press

Sieg, W. (1988). "Hilbert's Program Sixty Years Later." *The Journal of Symbolic Logic*, vol. 53(2):pp. 338–348

Sieg, W. (1999). "Hilbert's Programs: 1917–1922." *The Bulletin of Symbolic Logic*, vol. 5(1):pp. 1–44

Slater, B. H. (1986). "E-Type Pronouns and ε-Terms." *Canadian Journal of Philosophy*, vol. 16:pp. pp. 27–38

Slater, B. H. (2009). "Hilbert's Epsilon Calculus and its Successors." In Gabbay and Woods (2009). Elsevier, pp. 385–448

Smirnov, V. A. (1971). "Élimination des termes ε dans la logique intuitioniste." *Revue Internationale de Philosophie*, vol. 98(4):pp. 512–519

Smirnov, V. A. (1979). "Theory of Quantification and ε Calculi." In Hintikka *et al.* (1979). D.

Stolz, O. (1882). "du Bois-Reymond, P. Die allgemeine Functionentheorie. I. Teil. Metaphysik und Theorie der mathematischen Grundbegriffe: Grösse, Grenze, Argument und Function." *Jahrbuch uber die Fortschritte der Mathematik*, vol. 14:pp. 3–9. Translated as "Review of du Bois-Reymond's 'Die allgemeine Functionentheorie"' in the *The MacTutor History of Mathematics archive*, url:⟨http://www-history.mcs.st-andrews.ac.uk⟩

Strawson, P. F. (1959). *Individuals: An Essay in Descriptive Metaphysics.* Routledge

Sturmfels, B. (1993). *Algorithms in Invariant Theory.* Texts and Monographs in Symbolic Computation. Springer

Suppes, P. C. (1957). *Introduction to Logic.* Van Norstrand Reinhold Company

Suppes, P. C., Henkin, L., Joja, A., and Moisil, G. (1973). *Logic, Methodology and Philosophy of Science IV.* North Holland, Amsterdam

Tait, W. (1994). "The law of excluded middle and the axiom of choice." In A. George, ed., *Mathematics and Mind.* Oxford University Press, Oxford, pp. 45–70

Tarski, A. (1935). "Der Wahrheitsbegriff in den formalisierten Sprachen." *Studiaphilosophica*, (1):pp. 261–405. Translated as "The Concept of Truth in Formalized Languages" in Tarski (1956), pp. 152–274

Tarski, A. (1946). *Introduction to Logic: and to the Methodology of Deductive Sciences.* Oxford University Press, 2nd revised edn.

Tarski, A. (1956). *Logic, Semantics, Metamathematics Papers from 1923 to 1938.* Oxford at the Clarendon Press. Translation by J. H. Woodger

Tennenbaum, J. (1973). *A Constructive Version of Hilbert's Basis Theorem.* Ph.D. thesis, The University of California, San Diego. Advisor: Errett Albert Bishop

Tiu, A. (2011). "A Hypersequent System for Gödel-Dummett Logic with Non-constant Domains." In K. Brünnler and G. Metcalfe, eds., *Automated Reasoning with Analytic Tableaux and Related Methods*, vol. 6793 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, pp. 248–262

Troelsta, A. (1998). "Concepts and Axioms." *Philosophia Mathematica*:pp. 195–208

Troelstra, A. (1983). "Analysing Choice Sequences." *Journal of Philosophical Logic*, vol. 12(2):pp. 197–260

Troelstra, A. (1991). *History of Constructivism in the 20th Century*, vol. ML-91-05 of *ITLI Prepublication Series*. University of Amsterdam, Amsterdam

Troelstra, A. and van Ulsen, P. (1999). "The Discovery of E.W. Beth's Semantics for Intuitionistic Logic." In J. Gerbrandy, M. Marx, M. de Rijke, and Y. Venema, eds., *JFAK. Essays Dedicated to Johan van Benthem on the Occasion of his 50th Birthday*. Vossiuspers, Amsterdam University Press

Troelstra, A. S. (2003). *Constructivism and Proof Theory*. ILLC, University of Amsterdam

Umezawa, T. (1955). "Über die Zwischensysteme der Aussagenlogik." *Nagoya Mathematics Journal*, vol. 9:pp. 181–189

Umezawa, T. (1959a). "On Intermediate Propositional Logics." *Journal of Symbolic Logic*, vol. 24:pp. 20–36

Umezawa, T. (1959b). "On logics intermediate between intuitionistic and classical predicate logic." *Journal of Symbolic Logic*, vol. 24:pp. 141–153

van Atten, M., Boldini, P., Bourdeau, M., and Heinzmann, G. (2008). *One Hundred Years of Intuitionism (1907-2007): The Cerisy Conference*. Publications des Archives Henri-Poincaré. Birkhäuser Basel

van Dalen, D. (1990). "The War of the frogs and the mice, or the crisis of the *Mathematische Annalen*." *The Mathematical Intelligencer*, vol. 12(4):pp. 17–31

van Dalen, D. (2002). "Intuitionistic Logic." In D. Gabbay and F. Guenthner, eds., Gabbay and Guenthner (2002). Kluwer Academic Publishers, pp. 1–115

van Dalen, D. (2008b). "A bibliography of L.E.J. Brouwer." In van Atten *et al.* (2008), chap. 22. Birkhäuser Basel, pp. 343–390 van der Does, J. (1993). "The dynamics of sophisticated laziness." In Groenendijk (1993). Institute for Logic, Language and Computation (ILLC), University of Amsterdam, pp. 1–52

van Heijenoort, J. (1967a). *From Frege to Gödel: a source book in mathematical logic, 1879-1931*. Source books in the history of the sciences. Harvard University Press

van Heijenoort, J. (1973). "Subject and Predicate in Western logic." In van Heijenoort (1985). Bib, pp. 17–34. Originally published in *Philosophy of East and West 24* (1974) pp.253-268

van Heijenoort, J. (1985). *Selected Essays*. History of Logic. Bibliopolis

van Heijenoort, J. (2012). "Historical Development of Modern Logic." *Logica Universalis*, vol. 6:pp. 327–337. Also published in *Modern Logic* vol:2, pp. 242–255

van Stigt, W. P. (1998). "Brouwer's Intuitionist Programme." In P. Mancosu, ed., Mancosu (1998a). Oxford University Press. In Mancosu (1998a)

von Heusinger, K. (1997). "Definite Descriptions and Choice Functions." In S. Akama, ed., Akama (1997), vol. 5 of *Applied Logic Series*. Springer Netherlands, pp. 61–91

von Heusinger, K. and Egli, U. (2000). *Reference and Anaphoric Relations*, vol. 72 of *Studies in Linguistics and Philosophy*. Springer

Von Weber, H. (1891). "Leopold Kronecker." *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 2:pp. 5–32

Wehmeier, K. F. (2004). "Wittgensteinian Predicate Logic." *Notre Dame Journal of Formal Logic*, vol. 45(1):pp. 1–11

Wehmeier, K. F. (2008). "Wittgensteinian Tableaux, Identity, and Co-Denotation." *Erkenntnis*, vol. 69:pp. 363–376

Wehmeier, K. F. (2012). "How to Live Without Identity — and Why." *Australasian Journal of Philosophy*, vol. 90(4):pp. 761–777

Weierstrass, K. (1874-1888). "Briefe an Hermann Amandus Schwarz (Abschrift von Ludwig Bieberbach)/Letters to Hermann Amandus Schwarz (copied by Ludwig Bieberbach)." In Dugac (1973), chap. (Extraits) Appendice VII. *Archive for History of Exact Sciences*, pp. 140–143

Weyl, H. (1921). "Über die neue Grundlagenkrise der Mathematik." *Mathematische Zeitschrift*, vol. 10:pp. 39–79. Translated as Weyl (1998) in Mancosu (1998a)

WEYL, H. (1944). "David Hubert and his Mathematical Work." *Bulletin of the American Mathematical Society*, vol. 50(9):pp. 612–654

WEYL, H. (1998). "On the New Foundational Crisis of Mathematics." In MANCOSU (1998a). Oxford University Press

WHITEHEAD, A. N. and RUSSELL, B. (1912). *Principia Mathematica*, vol. II. Cambridge University Press

WHITEHEAD, A. N. and RUSSELL, B. (1913). *Principia Mathematica*, vol. III. Cambridge University Press

WHITEHEAD, A. N. and RUSSELL, B. A. W. (1910). *Principia mathematica*, vol. I. Cambridge University Press, Cambridge

WILLIAMSON, T. (2013). *Modal Logic as Metaphysics*. Oxford University Press

WINTER, Y. (1997). "Choice Functions and the Scopal Semantics of Indefinites." *Linguistics and Philosophy*, vol. 20(4):pp. 399–467

WIRTH, C.-P. (2015). "The descriptive operators iota, tau and epsilon - on their origin, partial and complete specification, model-theoretic semantics, practical applicability." In *Hilbert's Epsilon and Tau in Logic, Informatics and Linguistics*

WRIGHT, C. (1992). *Truth and Objectivity*. Harvard University Press

WRIGHT, C. (2013). "A Plurality of Pluralisms." In (PEDERSEN and WRIGHT, 2013). Oxford University Press, p. 123

ZACH, R. (1999). "Completeness Before Post: Bernays, Hilbert, and the Development of Propositional Logic." *The Bulletin of Symbolic Logic*, vol. 5(3):pp. 331–366

ZACH, R. (2003). "The practice of finitism. Epsilon calculus and consistency proofs in Hilbert's Program." *Synthèse*, vol. 137:pp. 211–259

ZACH, R. (2006). "Hilbert's Program Then and Now." In *Handbook of the Philosophy of Science. Volume 5: Philosophy of Logic*. Elsevier. Volume editor: Dale JACQUETTE. Handbook editors: Dov M. GABBAY, Paul THAGARD and John WOODS.

ZACH, R. (2009). "Hilbert's Program." In E. N. Zalta, ed., *The Stanford Encyclopedia of Philosophy*. The Metaphysics Research Lab, Center for the Study of Language and Information, Stanford University, spring 2009 edn.

ZARISKI, O., COHEN, I., and SAMUEL, P. (1958). *Commutative Algebra*. University Series in Higher Mathematics. Princeton

Zermelo, E. (1904). “Beweis, daßjede Menge wohlgeordnet werden kann.” *Mathematische Annalen*, vol. 59(4):p. 514ff. Translated as “Proof that every set can be well-ordered” in van Heijenoort (1967a) p. 139-141.

Zermelo, E. (1908). “Untersuchungen über die Grundlagen der Mengenlehre. I.” *Mathematische Annalen*, vol. 65(2):pp. 261–281. Translation as “Investigations in the foundations of set theory” in van Heijenoort (1967a) pp. 199–215