Partition Algebras and Kronecker Coefficients

by

Cameron Marcott

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Abstract

Classical Schur-Weyl duality relates the representation theory of the general linear group to the representation theory of the symmetric group via their commuting actions on tensor space. With the goal of studying Kronecker products of symmetric group representations, the partition algebra is introduced as the commutator algebra of the diagonal action of the symmetric group on tensor space. An analysis of the representation theory of the partition offers results relating reduced Kronecker coefficients to Kronecker coefficients.

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Chapter 1

Introduction

This thesis uses Schur-Weyl duality between the symmetric group algebra and the partition algebra to study Kronecker products of symmetric group representations. The problem of finding a combinatorial rule for the decomposition of the Kronecker product of symmetric group representations into irreducible representations is a long standing problem in combinatorial representation theory. The structure constants appearing in this decomposition are known as Kronecker coefficients. Kronecker coefficients have been studied by, for example, Garcia and Remmel in [13], James and Kerber in [19], and Littlewood in [23]. They have applications to the quantum marginal problem [22], Potts models in statistical mechanics [20] and [24], and geometric complexity theory [3].

Classically, Schur-Weyl duality uses commuting actions of the general linear group and the symmetric group on tensor space to relate tensor products of general linear group representations to a different product of symmetric group representations. Our goal is to mimic this construction using a different action of the symmetric group algebra on tensor space. The commutator algebra of this action is an algebra known as the partition algebra, which was introduced independently by Martin in [24] and Jones in [20]. Schur-Weyl duality then relates Kronecker products of symmetric group algebra representations to products of representations of the partition algebra.

The bulk of this thesis will be devoted to study the representation theory of partition algebra. The partition algebra depends on two parameters and for certain choices of these parameters the partition algebra fails to be semisimple. The notions of recollements of derived categories and of quasi-heredity algebras are introduced to study the partition algebra at these parameters. This thesis provides an example driven introduction to these concepts, which is more intuitive than the standard introductions appearing in the literature.

Our analysis of the non-semisimple representation theory of the partition algebra allows us to prove a stability result about Kronecker coefficients originally due to Murnaghan in [27]. These stable Kronecker coefficients are known as reduced Kronecker coefficients. Following [4], we are able to interpret reduced Kronecker coefficients as the dimensions of standard modules of the partition algebra, and then use a homological resolution of simple modules of the partition algebra in terms of standard modules to express Kronecker coefficients as alternating sums of reduced Kronecker coefficients. These stability results about Kronecker coefficients relate to a more generalized notion of stability of Kronecker coefficients due to Stembridge in [35], which has been of recent interest.

Throughout this thesis, all algebras are associative, unital, finite dimensional, complex, and Noetherian. Algebras are denoted by calligraphic letters, such as $\mathcal{A}, \mathcal{B}, \mathcal{C}$. The identity element of the algebra $\mathcal{A}$ is denoted $Id_{\mathcal{A}}$ or sometimes just Id if the algebra is clear from context.

Chapter 2

Representation Theory Background

This section provides basic background from representation theory. Our primary reference for the material appearing in this section is [8]

2.1 Representations of Algebras

Representation theory is born out of the philosophical point that the best way to study an object is to study what it does. Specifically, the goal is to study an algebra by asking how it can act on a vector space.

Definition 2.1.1. Let $\mathcal{A}$ be an algebra. A *representation* of $\mathcal{A}$ is an $\mathcal{A}$-module. That is, if V is a complex vector space, then a representation is a map

$$\begin{array}{rccc} \rho: & \mathcal{A} \times V & \to & V \\ & a \times \mathbf{v} & \mapsto & a\mathbf{v}, \end{array}$$

such that for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $a_1, a_2 \in \mathcal{A}$

- $Id\mathbf{v}_1 = \mathbf{v}_1$,
- $a_1(\mathbf{v}_1 + \mathbf{v}_2) = a_1\mathbf{v}_1 + a_1\mathbf{v}_2$,
- $(a_1 + a_2)\mathbf{v}_1 = a_1\mathbf{v}_1 + a_2\mathbf{v}_1$, and
- $a_1(a_2\mathbf{v}_1) = (a_1a_2)\mathbf{v}_1$.

The words "representation" and "module" will be used interchangeably. Morphisms of representations are simply morphisms of $\mathcal{A}$-modules. Subrepresentations are submodules.

Example 2.1.2. For any algebra $\mathcal{A}$, the *left regular representation* of $\mathcal{A}$ is the action of $\mathcal{A}$ (considered as an algebra) on $\mathcal{A}$ (considered as a vector space) given by multiplication on the left,

$$\begin{array}{rrcl} \ell: & \mathcal{A}\times\mathcal{A} & \to & \mathcal{A} \\ & a\times\mathbf{b} & \mapsto & a\mathbf{b}. \end{array}$$

The left regular representation is denoted $\ell(\mathcal{A})$.

Example 2.1.3. For any algebra $\mathcal{A}$, the *right regular representation* of $\mathcal{A}$ is the action of $\mathcal{A}$ (considered as an algebra) on $\mathcal{A}$ (considered as a vector space) given by multiplication on the right,

$$\begin{array}{rrcl} r: & \mathcal{A}\times\mathcal{A} & \to & \mathcal{A} \\ & a\times\mathbf{b} & \mapsto & \mathbf{b}a. \end{array}$$

The right regular representation is denoted $r(\mathcal{A})$. The right regular representation is not a representation of $\mathcal{A}$, but rather a representation of the opposite algebra $\mathcal{A}^{\mathrm{op}}$. Often, there is a simple isomorphism between $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$ and thus $r(\mathcal{A})$ may be viewed as a representation of $\mathcal{A}$. For instance, if $\mathcal{A} = \mathrm{End}\,(\mathbb{C}^n)$, transposition provides an isomorphism between $\mathcal{A}$ and $\mathcal{A}^{\mathrm{op}}$. Then, the map

$$\begin{array}{rrcl} r^t: & \mathcal{A}\times\mathcal{A} & \to & \mathcal{A} \\ & a\times\mathbf{b} & \mapsto & \mathbf{b}a^t \end{array}$$

is a representation of $\mathcal{A}$.

Though these two examples are trivial, they will play an important role later in the discussion of Jones' basic construction.

The perhaps more familiar notion of a group representation can be recovered by taking representations of the group's group algebra.

Definition 2.1.4. Let G be a finite group. The *group algebra* $\mathbb{C}[G]$ is the complex vector space with a basis indexed by the elements of G whose multiplication is the bilinear extension of the multiplication on G.

Example 2.1.5. The group algebra $\mathbb{C}[\mathfrak{S}_3]$ is a six dimensional complex vector space spanned by $Id, (12), (13), (23), (123), (132)$. Multiplication of basis elements is given by composition of permutations, so $(12)(132) = (13)$ and

$$((23) + (321))\,((12) - (13)) = ((132) + (23) - (123) - (12))\,.$$

Definition 2.1.6. A *representation of a group* G is defined to be a representation of the group algebra $\mathbb{C}[G]$.

One of the main problems in representation theory is to find a complete collection of irreducible representations of a given algebra, then show how to break larger representations into these irreducible components.

Definition 2.1.7. A representation $\rho : \mathcal{A} \times V \to V$ is said to be *irreducible* if the only subspaces $W \subseteq V$ such that $a\mathbf{w} \in W$ for all $a \in \mathcal{A}$ and all $\mathbf{w} \in W$ are V and $\{\emptyset\}$. Equivalently, a representation is irreducible if it is a simple $\mathcal{A}$-module. The terms "irreducible representation" and "simple module" are used interchangeably.

Any algebra $\mathcal{A}$ has only finitely many irreducible representations. If $\mathcal{A}$ is semisimple (that is $J(\mathcal{A})$, the Jacobson radical of $\mathcal{A}$, is trivial), $\rho : \mathcal{A} \times V \to V$ is a representation, and $W \subset V$ is such that $\rho|_W$ is irreducible, then there is some representation ρ' such that $\rho = \rho|_W \oplus \rho'$. If $\mathcal{A}$ is a group algebra $W = \mathbb{C}^n$ equipped with the usual inner product, we will have $\rho' = \rho|_{W^\perp}$. Thus, any representation of a semisimple algebra may be written as a direct sum of irreducible representations. Details about these results can be found in any introductory text on representations of algebras, such as [8]. The following example shows that if $\mathcal{A}$ is not a semisimple algebra, then orthogonal complements (under the usual inner product) of irreducible representations are not necessarily representations.

Example 2.1.8. Let $\mathcal{A}$ be the algebra of matrices of the form $\begin{pmatrix} x & y \\ 0 & x \end{pmatrix}$ and consider the representation $\rho : \mathcal{A} \times \mathbb{C}^2 \to \mathbb{C}^2$ where $\mathcal{A}$ acts by left multiplication. The subspace W of $\mathbb{C}^2$ consisting of vectors of the form $\begin{pmatrix} * \\ 0 \end{pmatrix}$ is invariant under the action of $\mathcal{A}$. Since W is one dimensional, $\rho|_W$ is trivially irreducible. The orthogonal compliment of W consists of vectors of the form $\begin{pmatrix} 0 \\ * \end{pmatrix}$. However, the restriction of ρ to this subspace is not a representation because

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin W^\perp.$$

If $\mathcal{A}$ is not semisimple, then an element is in $J(\mathcal{A})$ if and only if it acts as zero in every irreducible representation of $\mathcal{A}$. Hence, the irreducible representations of $\mathcal{A}$ are exactly the irreducible representations of $\mathcal{A}/J(\mathcal{A})$, which can be shown to be semisimple [8]. Unfortunately, the tasks of analyzing $J(\mathcal{A})$ and $\mathcal{A}/J(\mathcal{A})$ are not always straightforward. Much of this thesis will be focused on developing technology for describing the irreducible representations of certain non-semisimple algebras.

2.2 Isotypic Decompositions

Isotypic decompositions provide a basis free way of breaking a representation into irreducible components. They come from a series of simple observations involving Schur's Lemma.

Lemma 2.2.1 (Schur's Lemma). *Let A^λ and A^μ be non-zero simple $\mathcal{A}$-modules and let $\phi \in \mathrm{Hom}_{\mathcal{A}}(A^\lambda, A^\mu)$. Then,*

$$\phi = \begin{cases} \alpha \cdot Id_{A^\lambda} & \text{for some } \alpha \in \mathbb{C} \text{ if } A^\lambda \cong A^\mu, \\ 0 & \text{if } A^\lambda \ncong A^\mu. \end{cases}$$

Proof. Let $\phi \in \mathrm{Hom}_{\mathcal{A}}(A^\lambda, A^\mu)$. Since $\ker(\phi)$ is a submodule of A^λ and A^λ is simple, $\ker(\phi) = A^\lambda$ or 0. Since $\mathrm{Im}(\phi)$ is a submodule of A^μ and A^μ is simple, $\mathrm{Im}(\phi) = A^\mu$ or 0. If $\ker(\phi) = 0$, then $\mathrm{Im}(\phi) \cong A^\lambda$ and so ϕ is non-zero only when it is an isomorphism.

Since $\mathbb{C}$ is algebraically closed, ϕ has an eigenvalue α. Then, $\phi - \alpha \cdot Id_{A^\lambda} \in \mathrm{Hom}_{\mathcal{A}}(A^\lambda, A^\mu)$ and by the above argument $\phi - \alpha \cdot Id_{A^\lambda}$ is either 0 or an isomorphism. However, $\phi - \alpha \cdot Id_{A^\lambda}$ has 0 as an eigenvalue and thus cannot be an isomorphism. So, $\phi - \alpha \cdot Id_{A^\lambda} = 0$ and thus $\phi = \alpha \cdot Id_{A^\lambda}$. □

Schur's Lemma gives a tool for keeping track of the multiplicity of an irreducible representation occurring in an irreducible representation. If A^λ and A^μ are irreducible representations of $\mathcal{A}$, then

$$\mathrm{Hom}_{\mathcal{A}}\left(A^\lambda, (A^\mu)^{\oplus k}\right) \cong \begin{cases} \mathbb{C}^k & \text{if } A^\lambda \cong A^\mu, \\ 0 & \text{otherwise.} \end{cases}$$

So, if a module M is written as a direct sum of simple modules where A^λ occurs with multiplicity k, then $\mathrm{Hom}_{\mathcal{A}}\left(A^\lambda, M\right) = \mathbb{C}^k$. The invariant subspace of M associated to A^λ is isomorphic to

$$\mathrm{Hom}_A\left(A^\lambda, M\right) \otimes A^\lambda.$$

These observations yield the following definition.

Definition 2.2.2. Let M be an $\mathcal{A}$-module and A^λ be a simple $\mathcal{A}$-module. The *isotypic component associated to* A^λ is the $\mathcal{A}$-module $\mathrm{Hom}_A\left(A^\lambda, M\right) \otimes A^\lambda$. A factorization of M into submodules of this form is called the *isotypic decomposition of M.*

An isotypic decomposition of a representation can profitably be thought of as a basis free way of writing a representation as a direct sum of irreducible representations; the following trivial example illustrates this.

Example 2.2.3. Consider the representation of the algebra $\mathbb{C}$ on the vector space $\mathbb{C}^2$ given by $x \times (y, z) \mapsto (xy, xz)$. Picking any basis, there are continuumly many ways to write $\mathbb{C}^2$ as the direct sum of two $\mathbb{C}$ invariant subspaces. Simply calling all of $\mathbb{C}^2$ the isotypic subspace associated to one dimensional irreducible representation of $\mathbb{C}$ does away with this nonsense.

2.3 Induction and Restriction

Restriction gives a way of relating a representation of an algebra to a representation of a subalgebra.

Definition 2.3.1. Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}$ and let M be a representation of $\mathcal{B}$. Then, M is also a representation of $\mathcal{A}$ letting $\mathcal{A}$ act via its inclusion in $\mathcal{B}$. We call M considered as a representation of $\mathcal{A}$ *the restriction of* M. The restriction of M from a $\mathcal{B}$-module to an $\mathcal{A}$-module is denoted by $\mathrm{Res}^{\mathcal{B}}_{\mathcal{A}}(M)$.

A natural question to ask is whether this operation has an invserve. Given a representation of $\mathcal{A}$, is it possible to build a representation of $\mathcal{B}$.

Definition 2.3.2. Let $\mathcal{A}$ be a subalgebra of $\mathcal{B}$ and let M be a representation of $\mathcal{A}$. Then, $\mathcal{B} \otimes_{\mathcal{A}} M$ is a representation of $\mathcal{B}$ and is called the *induced representation of* M. The induction of M from an $\mathcal{A}$-module to a $\mathcal{B}$-module is denoted by $\mathrm{Ind}^{\mathcal{B}}_{\mathcal{A}}(M)$.

These two operations are adjoint.

Theorem 2.3.3 (Frobenius Reciprocity)**.** *Let* $\mathcal{A}$ *be a subalgebra of* $\mathcal{B}$*, let* M *be a representation of* $\mathcal{A}$*, and let* N *be a representation of* $\mathcal{B}$*. Then,*

$$\mathrm{Hom}_{\mathcal{A}}\left(M, \mathrm{Res}^{\mathcal{B}}_{\mathcal{A}}(N)\right) \cong \mathrm{Hom}_{\mathcal{B}}\left(\mathrm{Ind}^{\mathcal{B}}_{\mathcal{A}}(M), N\right).$$

Proof. Define a map

$$\begin{array}{rrcl} f: & \mathrm{Hom}_{\mathcal{A}}\left(M, \mathrm{Res}^{\mathcal{B}}_{\mathcal{A}}(N)\right) & \to & \mathrm{Hom}_{\mathcal{B}}\left(\mathrm{Ind}^{\mathcal{B}}_{\mathcal{A}}(M), N\right) \\ & \phi & \mapsto & f(\phi), \end{array}$$

where $f(\phi) \in \mathrm{Hom}_{\mathcal{B}}\left(\mathrm{Ind}_{\mathcal{A}}^{\mathcal{B}}(M), N\right)$ acts on the element $b \otimes_{\mathcal{A}} m \in \mathcal{B} \otimes_{\mathcal{A}} M$ by

$$f(\phi)(b \otimes_{\mathcal{A}} m) = b\phi(m).$$

Now, define a map

$$\begin{array}{rrcl} g: & \mathrm{Hom}_{\mathcal{B}}\left(\mathrm{Ind}_{\mathcal{A}}^{\mathcal{B}}(M), N\right) & \rightarrow & \mathrm{Hom}_{\mathcal{A}}\left(M, \mathrm{Res}_{\mathcal{A}}^{\mathcal{B}}(N)\right) \\ & \psi & \mapsto & g(\psi), \end{array}$$

where $g(\psi)$ acts on $m \in M$ by

$$g(\psi)(m) = \psi(1 \otimes_{\mathcal{A}} m).$$

It is clear that f and g are inverses. So, $\mathrm{Hom}_{\mathcal{B}}\left(\mathrm{Ind}_{\mathcal{A}}^{\mathcal{B}}(M), N\right) \cong \mathrm{Hom}_{\mathcal{A}}\left(M, \mathrm{Res}_{\mathcal{A}}^{\mathcal{B}}(N)\right)$, as desired. □

2.4 Bratteli Diagrams

Many interesting families of algebras naturally form towers by inclusion. For instance, the group algebra $\mathbb{C}[\mathfrak{S}_3]$ may be thought of as the subalgebra of $\mathbb{C}[\mathfrak{S}_4]$ generated by permutations of $\{1, 2, 3, 4\}$ which fix 4. Likewise $\mathbb{C}[\mathfrak{S}_4]$ may be thought of as a subalgebra of $\mathbb{C}[\mathfrak{S}_5]$, and $\mathbb{C}[\mathfrak{S}_5]$ as a subalgebra of $\mathbb{C}[\mathfrak{S}_6]$, and so on. Bratteli diagrams provide a combinatorial picture to help analyze the representation theory of such a tower of algebras.

Definition 2.4.1. Let $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$ be a tower of semisimple algebras, let $\Lambda_{\mathcal{A}_i}$ be an index set for the irreducible representations of $\mathcal{A}_i$, and for $\lambda \in \Lambda_{\mathcal{A}_i}$ let A_i^λ be the irreducible representation of $\mathcal{A}_i$ indexed by λ. For $\lambda \in \Lambda_{\mathcal{A}_{i+1}}$ and $\mu \in \Lambda_{\mathcal{A}_i}$, define integers $m_{\mu,\lambda}$ by

$$m_{\mu,\lambda} = \dim\left(\mathrm{Hom}_{\mathcal{A}_i}\left(A_i^\mu, \mathrm{Res}_{\mathcal{A}_i}^{\mathcal{A}_{i+1}}\left(A_{i+1}^\lambda\right)\right)\right).$$

The *Bratteli diagram* of the tower is the directed graph graph which has vertices labeled by $\Lambda_{\mathcal{A}_i}$ on its i-th level. For a vertex μ on the ith level of the Bratteli diagram and λ on the $(i+1)$st level, $m_{\mu,\lambda}$ directed edges are drawn between μ and λ. These are the only edges in the diagram.

An example of a Bratteli diagram is given in Section 2.5, where we analyze the tower $\mathbb{C}[\mathfrak{S}_0] \subseteq \mathbb{C}[\mathfrak{S}_1] \subseteq \mathbb{C}[\mathfrak{S}_2] \subseteq \cdots$. The next theorem shows how Bratteli diagrams may be used to compute the dimensions of irreducible representations of the algebras in the tower.

Theorem 2.4.2. *Using the notation of Definition 2.4.1, for $\lambda \in \Lambda_{\mathcal{A}_i}$ and $\mu \in \Lambda_{\mathcal{A}_j}$ let $p_{\mu,\lambda}$ denote the number of directed paths from μ to λ in the Bratteli diagram. Then, for any $i > j$,*

$$\dim\left(A_i^{\lambda}\right) = \sum_{\mu \in \Lambda_{\mathcal{A}_j}} p_{\mu,\lambda} \dim\left(A_j^{\mu}\right).$$

Proof. Note that $\dim\left(A_i^{\lambda}\right) = \dim\left(\mathrm{Res}_{\mathcal{A}_j}^{\mathcal{A}_i}\left(A_i^{\lambda}\right)\right)$. We prove the theorem by induction on $i-j$. Suppose that $i - j = 1$. Taking the isotypic decomposition of $\mathrm{Res}_{\mathcal{A}_j}^{\mathcal{A}_i}\left(A_i^{\lambda}\right)$,

$$\mathrm{Res}_{\mathcal{A}_j}^{\mathcal{A}_i}\left(A_i^{\lambda}\right) \cong \bigoplus_{\mu \in \Lambda_{\mathcal{A}_j}} \mathrm{Hom}_{\mathcal{A}_j}\left(A^{\mu_j}, \mathrm{Res}_{\mathcal{A}_j}^{\mathcal{A}_i}\left(A_i^{\lambda}\right)\right) \otimes A^{\mu_j}.$$

Computing the dimensions of each side,

$$\dim\left(A_i^{\lambda}\right) = \sum_{\mu \in \Lambda_{\mathcal{A}_j}} m_{\mu,\lambda} \dim\left(A^{\mu_j}\right).$$

Since $m_{\mu,\lambda}$ is the number of directed paths from μ to λ in the Bratteli diagram, the fact holds.

Suppose for some $k \in \mathbb{N}$ that the theorem holds for $i - j = k$. For each $\lambda \in \mathcal{A}_{i+1}$, it follows from the case when $i - j = 1$ that

$$\dim\left(A_{i+1}^{\lambda}\right) = \sum_{\nu \in \Lambda_{\mathcal{A}_i}} m_{\nu,\lambda} \dim\left(A_i^{\nu}\right).$$

Then,

$$\dim\left(A_{i+1}^{\lambda}\right) = \sum_{\nu \in \Lambda_{\mathcal{A}_i}} m_{\nu,\lambda} \dim\left(A_i^{\nu}\right) \sum_{\mu \in \Lambda_{\mathcal{A}_j}} p_{\mu,\nu}.$$

Rearranging this double sum,

$$\dim\left(A_i^{\lambda}\right) = \sum_{\nu \in \Lambda_{\mathcal{A}_i}} \sum_{\mu \in \Lambda_{\mathcal{A}_j}} p_{\mu,\nu} m_{\nu,\lambda} \dim\left(A_j^{\mu}\right).$$

Since any path from μ to λ in the Bratteli diagram can be broken up into a path from μ

to ν followed by a path from ν to λ for some $\mu \in \Lambda_{\mathcal{A}_j}$,

$$p_{\mu,\lambda} = \sum_{\nu \in \Lambda_{\mathcal{A}_i}} p_{\mu,\nu} m_{\nu,\lambda}.$$

Thus,

$$\dim\left(A_i^{\lambda}\right) = \sum_{\mu \in \Lambda_{\mathcal{A}_j}} p_{\mu,\lambda} \dim\left(A^{\mu_j}\right),$$

and the theorem holds. $\square$

An example of the use of this theorem is given in the next section where we analyze the tower $\mathbb{C}[\mathfrak{S}_0] \subseteq \mathbb{C}[\mathfrak{S}_1] \subseteq \mathbb{C}[\mathfrak{S}_2] \subseteq \cdots$.

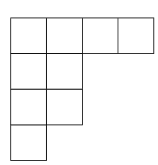
2.5 Representation Theory of $\mathbb{C}[\mathfrak{S}_n]$

This section describes the irreducible representations of the group algebra of the symmetric group. These representations are described combinatorially, using partitions and tableaux.

Definition 2.5.1. A *partition* is a finite string of positive integers, $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. If $n = \lambda_1 + \lambda_2 + \cdots + \lambda_m$, we say that λ is a partition of n and write $\lambda \vdash n$.

Definition 2.5.2. The *Ferrers diagram* or *Ferrers shape* of a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$ is a left justified array of boxes with λ_1 boxes in the top row, λ_2 boxes in the 2nd row,..., λ_m boxes in the m-th row. Often, λ is used to refer to both a partition and its Ferrers shape.

Example 2.5.3. $\lambda = (4, 2, 2, 1)$ is a partition of 9 with the following Ferrers shape.



Definition 2.5.4. Given a partition $\lambda \vdash n$, a *standard Young tableau* is a filling of the n boxes of λ's Ferrers diagram with the integers $1, \ldots, n$ such that rows are increasing when read from left to right, columns are increasing when read from top to bottom, and each

integer appears exactly once. The set of standard Young tableau of shape λ is denoted by $\mathrm{SYT}(\lambda)$.

Example 2.5.5. The figure on the left is a standard Young tableau; the other two figures are not.

$$
\begin{array}{|c|c|c|c|}\hline 1&2&4&8\\\hline 3&6\\\cline{1-2} 5&9\\\cline{1-2} 7\\\cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline 1&2&4&8\\\hline 2&6\\\cline{1-2} 5&9\\\cline{1-2} 7\\\cline{1-1}\end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline 1&4&2&8\\\hline 3&6\\\cline{1-2} 5&9\\\cline{1-2} 7\\\cline{1-1}\end{array}
$$

The following lemma gives an efficient means of computing the number of standard Young tableaux of a given shape. A probabilistic proof of this lemma may be found in [16].

Lemma 2.5.6. *For each box $\square$ in a Ferrers shape λ, let*

$$h_\lambda(\square) = \begin{pmatrix} \#\ \textit{of boxes} \\ \textit{right of}\ \square \end{pmatrix} + \begin{pmatrix} \#\ \textit{of boxes} \\ \textit{below of}\ \square \end{pmatrix} + 1$$

be the hook number of the box. Then,

$$|\mathrm{SYT}(\lambda)| = \frac{n!}{\prod_{\square\in\lambda} h_\lambda(\square)}.$$

This formula is known as the hook formula.

Given partitions λ and μ, define the partial order relation $\lambda \leq \mu$ if λ's Ferrers diagram is a sub diagram of μ's Ferrers diagram. The Hasse diagram of partitions under this partial ordering is known as Young's lattice. Adhering with the conventions in the partition algebra literature, we draw Young's lattice as well as any Bratteli diagram upside down from the standard convention. So, the smallest element appears at the top of the diagram, and elements get bigger as one goes down. The first several levels of Young's lattice are drawn in Figure 2.5. Consider a path from $\emptyset$ to λ in Young's lattice. At each step in the path, a new box is added. Filling the box added at step i in a path with the integer i gives a bijection between standard Young tableau of shape λ and paths from $\emptyset$ to λ in Young's lattice.

The symmetric group $\mathfrak{S}_n$ acts on a tableau with n boxes by permuting the entries (the result is not necessarily a standard Young tableau). Extending this action linearly, $\mathbb{C}[\mathfrak{S}_n]$ acts on the $n!$ dimensional vector space spanned by all fillings of a Ferrers shape $\lambda \vdash n$ with the integers $1, \cdots, n$.

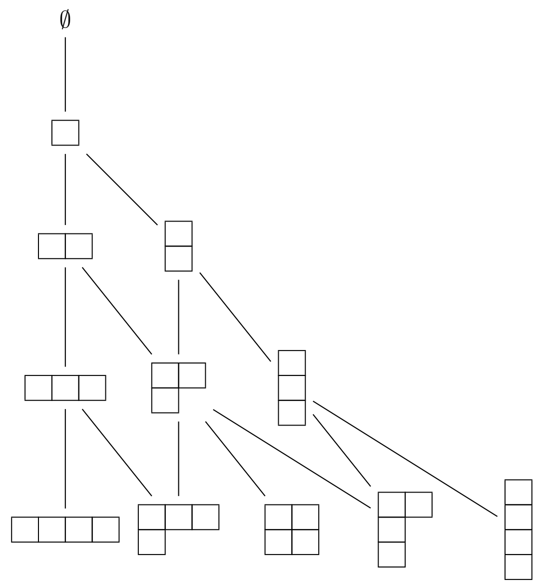


Figure 2.1: Young's lattice.

Given a standard Young tableau T of shape $\lambda \vdash n$, we define $R(T)$ and $C(T)$ to be subgroups of $\mathfrak{S}_n$ consisting of permutations whose action preserves all of the rows and columns of T respectively. The only permutation which fixes both every row and every column is the identity, so $R(T) \cap C(T) = Id$.

Example 2.5.7. Let

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}.$$

Then, $R(T)$ is the subgroup of $\mathfrak{S}_5$ generated by the transpositions $(1, 2), (2, 5)$, and $(3, 4)$. So, $R(T)$ is isomorphic to $\mathfrak{S}_3 \times \mathfrak{S}_2$. The subgroup $C(T)$ is generated by the transpositions $(1, 3)$ and $(2, 4)$. So, $C(T)$ is isomorphic to $\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_1$.

Definition 2.5.8. Let T be a standard Young tableaux of shape $\lambda \vdash n$. The element

$$v_T = \sum_{\sigma \in C(T)} \sum_{\tau \in R(T)} \operatorname{sgn}(\sigma)\sigma\tau.$$

in $\mathbb{C}[\mathfrak{S}_n]$ is called the *Young symmetrizer associated to T*.

Example 2.5.9. Let $T = \ytableaushort{12,3}$. Then,

$$\begin{aligned} v_T &= \sum_{\sigma \in C\left(\ytableaushort{12,3}\right)} \sum_{\tau \in R\left(\ytableaushort{12,3}\right)} \operatorname{sgn}(\sigma)\sigma\tau \\ &= \sum_{\sigma \in C\left(\ytableaushort{12,3}\right)} \operatorname{sgn}(\sigma)\sigma\left(Id + (1,2)\right) \\ &= Id + (1,2) - (1,3) - (1,2,3). \end{aligned}$$

The Young symmetrizers are eigenvectors for automorphism of $\mathbb{C}[\mathfrak{S}_n]$ given by left multiplication by $((1,k) + (2,k) + \cdots + (k-1,k))$, where $1 \le k \le n$. The associated eigenvalue can be read directly off a tableau. This fact was first written down by the Lithuanian physicist Jucys in [21] and first written down in the West by Murphy in [29]. Hence, the elements $((1,k) + (2,k) + \cdots + (k-1,k))$ are called *Jucys-Murphy elements*.

Remark 2.5.10. Though we will not need the fact, we mention that the importance of the Jucys-Murphy elements comes from the facts that symmetric polynomials in the Jucys-Murphy elements generate the center of $\mathbb{C}[\mathfrak{S}_n]$, and that the algebra generated by the Jucys-Murphy elements commutes with the natural inclusion of $\mathbb{C}[\mathfrak{S}_{n-1}]$ in $\mathbb{C}[\mathfrak{S}_n]$. Using these facts, one can recover all of the classical results about the representation theory of the symmetric group using the Jucys-Murphy elements. Details about this construction, as well as a proof of 2.5.13 may be found in [29].

Definition 2.5.11. The *content* of a box in a standard Young tableaux of shape is the number of boxes to its left in its row minus the number of boxes above it in its column. The content of the box containing k in the tableau T is denoted $c_T\left(\boxed{k}\right)$.

Example 2.5.12. Let T be the standard Young tableau from Example 2.5.7. Then, $c_T\left(\boxed{4}\right) = 0$, and $c_T\left(\boxed{5}\right) = 2$.

Lemma 2.5.13. *Let $\lambda \vdash n$, and let T be a standard Young tableau of shape λ. Then,*

$$((1,k) + (2,k) + \cdots + (k-1,k))v_T = c_T\left(\boxed{k}\right) v_T.$$

The Young symmetrizers may also be used to directly construct a family of idempotents which project $\mathbb{C}[\mathfrak{S}_n]$ onto its simple modules.

Definition 2.5.14. For, $\lambda \vdash n$, define the element $F_\lambda \in \mathbb{C}[\mathfrak{S}_n]$ by

$$F_\lambda = \frac{1}{\prod_{\square \in \lambda} h_\lambda(\square)} \sum_{T \in \mathrm{SYT}(\lambda)} v_T.$$

Example 2.5.15. Let $\lambda = (2,1)$. Then,

$$F_\lambda = \frac{1}{3}\left(\sum_{\sigma \in C\left(\begin{smallmatrix}1&2\\3\end{smallmatrix}\right)} \sum_{\tau \in R\left(\begin{smallmatrix}1&2\\3\end{smallmatrix}\right)} \mathrm{sgn}(\sigma)\sigma\tau \right) + \frac{1}{3}\left(\sum_{\sigma \in C\left(\begin{smallmatrix}1&3\\2\end{smallmatrix}\right)} \sum_{\tau \in R\left(\begin{smallmatrix}1&3\\2\end{smallmatrix}\right)} \mathrm{sgn}(\sigma)\sigma\tau \right)$$

$$= \frac{1}{3}\left(\sum_{\tau \in R\left(\begin{smallmatrix}1&2\\3\end{smallmatrix}\right)} Id\tau \right) - \frac{1}{3}\left(\sum_{\tau \in R\left(\begin{smallmatrix}1&2\\3\end{smallmatrix}\right)} (1,3)\tau \right) + \frac{1}{3}\left(\sum_{\tau \in R\left(\begin{smallmatrix}1&3\\2\end{smallmatrix}\right)} Id\tau \right) - \frac{1}{3}\left(\sum_{\tau \in R\left(\begin{smallmatrix}1&3\\2\end{smallmatrix}\right)} (1,2)\tau \right)$$

$$= \frac{1}{3}(Id + (1,2)) - \frac{1}{3}((1,3) + (1,2,3)) + \frac{1}{3}(Id + (1,3)) - \frac{1}{3}((1,2) + (1,3,2))$$

$$= \frac{2}{3}Id - \frac{1}{3}(1,2,3) - \frac{1}{3}(1,3,2).$$

The F_λ's are used to build the irreducible representations of $\mathbb{C}[\mathfrak{S}_n]$. To accomplish this task, one shows that the F_λ's are *minimal idempotents* (that is, idempotents which may not be written as a sum of nonzero idempotents). It follows that the modules $F_\lambda \mathbb{C}[\mathfrak{S}_n] F_\lambda$ obtained by the two sided projection of $\mathbb{C}[\mathfrak{S}_n]$ by the F_λ's are simple. Proofs of the following two facts may be found in [19].

Theorem 2.5.16. *For each $\lambda \vdash k$, F_λ is a minimal idempotent in $\mathbb{C}[\mathfrak{S}_n]$. If $\mu \vdash n$ and $\lambda \neq \mu$, $F_\lambda F_\mu = 0$.*

Corollary 2.5.17. *For any $\lambda \vdash n$, $F_\lambda \mathbb{C}[\mathfrak{S}_n] F_\lambda$ is an irreducible representation of $\mathfrak{S}_n$. Distinct partitions yield distinct representations, and all irreducible representations of $\mathfrak{S}_n$ are obtained in this way.*

Definition 2.5.18. The simple module $F_\lambda \mathbb{C}[\mathfrak{S}_n] F_\lambda$ is called a *Specht module*, and is denoted by S^λ.

We define the Littlewood-Richardson numbers as the structure constants for a certain product on representations of $\mathbb{C}[\mathfrak{S}_n]$. The Littlewood-Richardson numbers appear in many different contexts, including representation theory, geometry, and symmetric function theory. In Chapter 3, we will see that they are also the structure constants for the decomposition of the tensor product of certain irreducible representations of $\mathrm{End}\left(\mathbb{C}^n\right)$.

Definition 2.5.19. Let $\lambda \vdash n_1$, $\mu \vdash n_2$, and $\nu \vdash n_1 + n_2$. The *Littlewood-Richardson number* $c^{\nu}_{\lambda,\mu}$ is given by

$$c^{\nu}_{\lambda,\mu} = \dim\left(\mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_{n_1+n_2}]}\left(S^{\nu}, \mathrm{Ind}^{\mathbb{C}[\mathfrak{S}_{n_1+n_2}]}_{\mathbb{C}[\mathfrak{S}_{n_1}\times\mathfrak{S}_{n_2}]}\left(S^{\lambda}\otimes S^{\mu}\right)\right)\right).$$

Littlewood-Richardson numbers can be computed combinatorially in terms of the partitions λ, μ, and ν. A proof of this fact may be found in [12].

Theorem 2.5.20 (The Littlewood-Richardson Rule)**.** *Let $\lambda \vdash n_1$, $\mu \vdash n_2$, and $\nu \vdash n_1+n_2$. Suppose $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$. Then, $c^{\nu}_{\lambda,\mu}$ is the number of ways to fill the skew shape ν/λ with μ_1 1's, μ_2 2's, ..., μ_m m's such that*

- *each row is weakly increasing from right to left,*
- *each column is strictly increasing from top to bottom, and*
- *if as the entries of the boxes are read, from right to left from top to bottom, then the number i always occurs at least as many times than the number $i+1$.*

If λ is not a subdiagram of ν, then $c^{\nu}_{\lambda,\mu} = 0$.

From Definition 2.5.19 we see that $c^{\nu}_{\lambda,\mu} = c^{\nu}_{\mu,\lambda}$, though this fact is not at all obvious from Theorem 2.5.20.

Example 2.5.21. The tableau on the left satisfies the conditions of the theorem. The tableau in the middle does not satisfy the conditions of the theorem because the third column is not strictly increasing when read from top to bottom. The tableau on the right satisfy the conditions of the theorem because the result of reading the entries from right to left from top to bottom is 1112233431, and at the second to last digit three 3's have been read while only two 2's have been read.

$$\begin{array}{cccccc} & & & 1 & 1 & 1 \\ & & 2 & 2 & & \\ & & 3 & 3 & & \\ 1 & 2 & 4 & & & \end{array} \qquad \begin{array}{cccccc} & & & 1 & 1 & 1 \\ & & 2 & 2 & & \\ & & 2 & 3 & & \\ 1 & 2 & 4 & & & \end{array} \qquad \begin{array}{cccccc} & & & 1 & 1 & 1 \\ & & 2 & 2 & & \\ & & 3 & 3 & & \\ 1 & 3 & 4 & & & \end{array}$$

Example 2.5.22. Let $\lambda = (2, 1, 1)$, $\mu = (3, 2, 1)$, and $\nu = (4, 3, 2, 2)$. Then, $c^{\nu}_{\lambda,\mu}$ is the number of ways to fill the shape

$$\begin{array}{cccc} & & \square & \square \\ & \square & \square & \\ & \square & & \\ \square & & & \end{array}$$

with three 1's, two 2's, and one 3 subject to the conditions of Theorem 2.5.20. The possible ways to do this are

$$\begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \\ & \boxed{2} & & \\ \boxed{3} & & & \end{array} \qquad \begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \\ & \boxed{3} & & \\ \boxed{2} & & & \end{array}$$

and so $c^{\nu}_{\lambda,\mu} = 2$.

Example 2.5.23. Let $\lambda \vdash n_1$, $\mu = (n_2)$, and $\nu \vdash n_1 + n_2$. Then, $c^{\nu}_{\lambda,\mu} = 1$ if ν/λ has no two boxes in the same column and $c^{\nu}_{\lambda,\mu} = 0$ otherwise.

As a special case, the Littlewood-Richardson rule allows us to compute restrictions of irreducible representations of $\mathbb{C}[\mathfrak{S}_n]$ to $\mathbb{C}[\mathfrak{S}_{n-1}]$. Note that while this fact follows directly from the Littlewood-Richardson rule, it can be proved independently and is in fact much easier to prove than the Littlewood-Richardson rule.

Special Case 2.5.24. *Let $\nu \vdash n$ and $\lambda \vdash n - 1$. Then,*

$$\dim\left(\mathrm{Hom}\left(\mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}(S^{\nu}), S^{\lambda}\right)\right) = \begin{cases} 1 & \textit{if } \lambda \textit{ and } \nu \textit{ differ by a box,} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Let $\mu = (1)$. By Frobenius reciprocity,

$$c^{\nu}_{\lambda,\mu} = \dim\left(\mathrm{Hom}\left(\mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}\times\mathfrak{S}_1]}(S^{\nu}), S^{\lambda} \otimes S^{\mu}\right)\right).$$

Note that $\mathfrak{S}_{n-1} \times \mathfrak{S}_1 \cong \mathfrak{S}_{n-1}$ and thus. Since $S^{\mu} = \mathbb{C}$,

$$S^{\lambda} \otimes S^{\mu} \cong S^{\lambda}.$$

Thus,

$$\dim\left(\mathrm{Hom}\left(\mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}(S^{\nu}), S^{\lambda}\right)\right) = c^{\nu}_{\lambda,\mu}.$$

Then, the corollary follows immediately from Theorem 2.5.20. $\square$

Example 2.5.25. Consider the tower of algebras $0 \subseteq \mathbb{C}[\mathfrak{S}_1] \subseteq \mathbb{C}[\mathfrak{S}_2] \subseteq \cdots$. Since the irreducible representations of $\mathbb{C}[\mathfrak{S}_n]$ are indexed by partitions of n, it follows from Special Case 2.5.24 that the Bratteli diagram for the tower is exactly Young's lattice.

Knowing the Bratteli diagram for the tower $0 \subseteq \mathbb{C}[\mathfrak{S}_1] \subseteq \mathbb{C}[\mathfrak{S}_2] \subseteq \cdots$ allows us to easily compute the dimension of any Specht module.

Corollary 2.5.26. *For $\lambda \vdash n$, the dimension of the Specht module S^λ is the number of standard Young tableaux with Ferrers shape λ. By Lemma 2.5.6, this number may be computed using the hook formula.*

2.6 Kronecker Products of $\mathbb{C}[\mathfrak{S}_n]$ Representations

A major problem in the representation theory of the symmetric group is to describe the decomposition of the tensor product of two Specht modules into irreducible components.

Definition 2.6.1. Let λ, μ and ν be partitions of n. Consider the decomposition

$$S^\lambda \otimes S^\mu \cong \bigoplus_{\nu \vdash n} g^\nu_{\lambda,\mu} S^\nu.$$

The integers $g^\nu_{\lambda,\mu}$ appearing in the decomposition above are called *Kronecker coefficients.*

Ideally, one would like something akin to the Littlewood-Richardson Rule (Theorem 2.5.20) to describe the Kronecker coefficients. Unfortunately, such a combinatorial interpretation remains elusive. In [27] and [28], Murnaghan provides a stability result for the Kronecker coefficients. Computing these reduced Kronecker coefficients is believed to be a more tractable problem.

Definition 2.6.2. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{m_\lambda})$ is a partition, define $\lambda_{>1}$ to be the partition $(\lambda_2, \ldots, \lambda_{m_\lambda})$. That is, $\lambda_{>1}$ is the Ferrers shape obtained by removing the top row of boxes from λ.

Theorem 2.6.3 (Stability of Kronecker Coefficients)**.** *Fix partitions $\lambda_{>1}, \mu_{>1}$, and ν_1 with $|\nu_{>1}| \leq |\lambda_{>1}| + |\mu_{>1}|$. Denote by $\lambda_{[N]}$ the unique partition of N such that $(\lambda_{[N]})_{>1} = \lambda_{>1}$. There exists some N such that for all $n \geq N$,*

$$g^{\nu_{[n]}}_{\lambda_{[n]},\mu_{[n]}} = g^{\nu_{[N]}}_{\lambda_{[N]},\mu_{[N]}}.$$

Definition 2.6.4. The integer $g^{\nu_{[N]}}_{\lambda_{[N]},\mu_{[N]}}$ from the previous theorem is called a *reduced Kronecker coefficient* and is denoted by $\bar{g}^{\nu}_{\lambda,\mu}$.

We will study the reduced Kronecker coefficients and their relationship to the Kronecker coefficients. Chapter 9.1 provides a proof of Theorem 2.6.3 using the partition algebra. The reduced Kronecker coefficients are interpreted as the dimensions of certain modules in the partition algebra. We show that any Kronecker coefficient can be written as an alternating sum of reduced Kronecker coefficients. Using these techniques, we will also reprove another classical result of Murnaghan in [28] and Littlewood in [23] relating reduced Kronecker coefficients to Littlewood-Richardson numbers.

Theorem 2.6.5. *Let $\lambda, \mu, \nu \vdash n$ and suppose that $|\nu_{>1}| = |\lambda_{>1}| + |\mu_{>1}|$. Then, $\bar{g}^{\nu}_{\lambda,\nu} = c^{\nu_{>1}}_{\lambda_{>1},\mu_{>1}}$.*

Proofs of these theorems using the partition algebra are already known, [4]. A generalized version of these stability results about Kronecker coefficients has been the subject of recent attention, [35]. A possible avenue of further research would be to see if these generalized stability results admit an interpretation in terms of the partition algebra, or some other related algebra.

Chapter 3

Centralizer Algebras and Schur-Weyl Duality

This chapter introduces centralizer algebras, which are used to prove Schur-Weyl duality in its general form. Schur-Weyl duality and the closely related seesaw reciprocity theorem will be among the primary tools used throughout this thesis. The presentation of much of the material in this chapter follows [14].

3.1 Centralizer Algebras

Thinking of a representation as a homomorphism from an algebra into a general linear group, decomposing a representation into irreducible components corresponds to writing the matrices in the algebra's image in block form. Given an algebra of matrices in block form, the algebra of matrices that commute with them will have the same block form. So, if we want to extract combinatorial data about a representation's decomposition into irreducible components, we can get the same combinatorial data by studying the algebra of commuting matrices. This motivation leads to the study of centralizer algebras.

Definition 3.1.1. Consider a representation $\rho : \mathcal{A} \times V \to V$. The algebra commuting with this action,

$$\mathrm{End}_{\mathcal{A}}(V) = \{b \in \mathrm{End}(V) : ba\mathbf{v} = ab\mathbf{v} : \text{for all } \mathbf{v} \in V\},$$

is called the *centralizer algebra of $\mathcal{A}$'s action on V*.

It is clear that everything in $\mathcal{A}$ will commute with the action of $\mathrm{End}_{\mathcal{A}}(V)$ on V. The next theorem tells us that $\mathcal{A}$ is in fact the entire centralizer algebra of $\mathrm{End}_{\mathcal{A}}(V)$.

Theorem 3.1.2 (The Double Commutant Theorem)**.** *Let* $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(V)$. *If* $\mathcal{A}$ *is semisimple, then* $\mathcal{A} = End_{\mathcal{B}}(V)$. *In the case,* $\mathcal{A}$ *and* $\mathcal{B}$ *are called mutual centralizers.*

Proof. This theorem will follow from Theorem 3.2.1, appearing in the next section. □

We conclude this section with some examples of centralizer algebras.

Example 3.1.3. Let $\mathrm{End}\left(\mathbb{C}^n\right)$ act on $\left(\mathbb{C}^n\right)^{\otimes k}$ by

$$g\left(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_k\right) = g\mathbf{v}_1 \otimes g\mathbf{v}_2 \otimes \cdots \otimes g\mathbf{v}_k.$$

This action is centralized by $\mathbb{C}[\mathfrak{S}_k]$ acting via

$$\sigma\left(\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \cdots \otimes \mathbf{v}_k\right) = \mathbf{v}_{\sigma(1)} \otimes \mathbf{v}_{\sigma(2)} \otimes \cdots \otimes \mathbf{v}_{\sigma(k)}.$$

Classically, the term "Schur-Weyl duality" refers to this specific example.

The following lemma provides another example of centralizer algebras.

Lemma 3.1.4. *Let* $\mathcal{A} \subseteq \mathcal{B}$ *and let* $\mathcal{A}$ *act on the vector space* $\mathcal{B}$ *via its inclusion in the left regular representation of* $\mathcal{B}$ *and let* $\mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{A})$ *denote the algebra of* $\mathcal{A}$*-linear maps from* $\mathcal{B}$ *to* $\mathcal{A}$. *Then,* $\mathrm{End}_{\mathcal{A}}(\mathcal{B}) \cong \mathcal{B} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{A}) \cong \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$. *The algebra* $\mathcal{B} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{A})$ *acts on the vector space* $\mathcal{B}$ *by*

$$(b_1 \otimes \phi)(b_2) = b_1\phi(b_2),$$

where $b_1, b_2 \in \mathcal{B}$, *and* $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{A})$.

Proof. Let $a \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$, and $\phi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{A})$. Then,

$$(b_1 \otimes \phi)(ab_2) = b_1\phi(ab_2)$$

$$= b_1 a\phi(b_2) = (b_1 \otimes a\phi)b_2$$

$$= a(b_1 \otimes \phi)b_2$$

So, $\mathcal{B} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{A}) \subseteq \mathrm{End}_{\mathcal{A}}(\mathcal{B})$. Note that $\mathrm{End}_{\mathcal{A}}(\mathcal{B}) = \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{B})$, which has dimension $(\dim(\mathcal{B}) - \dim(\mathcal{A}))^2$ as an $\mathcal{A}$-module. Since $\mathcal{B} \otimes_{\mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(\mathcal{B}, \mathcal{A})$ also has dimension $(\dim(\mathcal{B}) - \dim(\mathcal{A}))^2$ as an $\mathcal{A}$-module, it must be the whole space. □

Setting $\mathcal{A} = \mathbb{C}$ in the above lemma, one recovers the familiar fact that $\mathrm{End}(\mathcal{B}) \cong \mathcal{B} \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\mathcal{B}, \mathbb{C})$. The next example of centralizer algebras will be useful in our discussion of Jones' basic construction in Chapter 6.

Lemma 3.1.5. *Suppose $\mathcal{A}$ is semisimple. Let $l(\mathcal{A})$ denote the left regular representation and let $r^t(\mathcal{A})$ denote the transpose of the right regular representation of $\mathcal{A}$, as defined in Examples 2.1.2 and 2.1.3. Then, $\mathrm{End}_{l(\mathcal{A})}(\mathcal{A}) = r^t(\mathcal{A})$.*

Proof. Let $a_1, a_2, \mathbf{v} \in \mathcal{A}$. Then,

$$l(a_1)r^t(a_2)\mathbf{v} = a_1\mathbf{v}(a_2)^t = r^t(a_2)l(a_1)\mathbf{v}.$$

So, $r^t(\mathcal{A}) \subseteq \mathrm{End}_{l(\mathcal{A})}(\mathcal{A})$. Now, let $\phi \in \mathrm{End}_{l(\mathcal{A})}(\mathcal{A})$. Then, ϕ can be represented by right multiplication by $(\phi(Id))^t$. Then, $\phi = r^t(\phi(Id))$, and so $\mathrm{End}_{l(\mathcal{A})}(\mathcal{A}) \subseteq r^t(\mathcal{A})$. Thus, $\mathrm{End}_{l(\mathcal{A})}(\mathcal{A}) = r^t(\mathcal{A})$. □

3.2 Schur-Weyl Duality and Seesaw Reciprocity

Throughout this section, $\mathcal{A}$ will be assumed to be semisimple and $\mathcal{B}$ will be the centralizer algebra of the representation $\rho : \mathcal{A} \times V \to V$. The previous section introduced centralizer algebras with the motivation of studying combinatorial data about the decomposition of a representation into irreducible representations by analyzing the algebra of commuting matrices. Schur-Weyl duality and seesaw reciprocity tell us how to transfer data about irreducible representations between $\mathcal{A}$ and $\mathcal{B}$.

Theorem 3.2.1 (Schur-Weyl Duality)**.** *There is a bijection between the simple $\mathcal{A}$-modules and the simple $\mathcal{B}$ appearing in V. If Λ is a set indexing the simple $\mathcal{A}$-modules appearing in V, then*

$$V \cong \bigoplus_{\lambda \in \Lambda} B^\lambda \otimes A^\lambda,$$

as a $(\mathcal{B} \times \mathcal{A})$-module.

It will follow from the proof of Theorem 3.2.1 that the dimension of B^λ is the multiplicity of A^λ in the $\mathcal{A}$-module V and that the dimension of A^λ is the multiplicity of B^λ in the $\mathcal{B}$-module V. To prove Theorem 3.2.1, we employ the following lemma.

Lemma 3.2.2. *Let $X \leq V$. If $\rho|_X$, the restriction of ρ to the subspace X, is a subrepresentation of ρ, then $\mathcal{B}|_X = Hom_{\mathcal{A}}(X, V)$.*

Proof of Lemma 3.2.2. Let $\phi \in \mathcal{B}$. Then, for any $a \in \mathcal{A}$ and $\mathbf{x} \in X$,

$$\phi(a\mathbf{x}) = a\phi(\mathbf{x}).$$

So, $\phi|_X \in \mathrm{Hom}_{\mathcal{A}}(X, V)$ and hence $\mathcal{B}|_X \subseteq \mathrm{Hom}_{\mathcal{A}}(X, V)$.

Now, let $\phi \in \mathrm{Hom}_{\mathcal{A}}(X, V)$. Pick a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of X, and extend it to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_k, \ldots, \mathbf{v}_n$ of V. Define the map $\phi' : V \to V$ to be the linear extension of the map given by

$$\phi'(\mathbf{v}_i) = \begin{cases} \phi(\mathbf{v}_i) & \text{if } i \leq k, \\ \mathbf{v}_i & \text{otherwise.} \end{cases}$$

It is clear that ϕ' commutes with the action of $\mathcal{A}$ and that $\phi'|_X = \phi$. So, $\mathrm{Hom}_{\mathcal{A}}(X, V) \subseteq \mathcal{B}|_X$. Thus, $\mathcal{B}|_X = \mathrm{Hom}_{\mathcal{A}}(X, V)$. □

Proof of Theorem 3.2.1. Consider the isotypic decomposition

$$V = \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_{\mathcal{A}}(A^\lambda, V) \otimes A^\lambda.$$

We show that $\mathrm{Hom}_{\mathcal{A}}(A^\lambda, V)$ is an irreducible $\mathcal{B}$-module for each λ, and that $\mathrm{Hom}_{\mathcal{A}}(A^\lambda, V) \ncong \mathrm{Hom}_{\mathcal{A}}(A^\mu, V)$ for $\lambda \neq \mu$. Note that $\mathrm{Hom}_{\mathcal{A}}(A^\lambda, V)$ is indeed a $\mathcal{B}$-module because

$$b\phi(a\mathbf{v}) = ba\phi(\mathbf{v}) = ab\phi(\mathbf{v}),$$

for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, $\mathbf{v} \in V$, and $\phi \in \mathrm{Hom}_{\mathcal{A}}(A^\lambda, V)$.

By Lemma 3.2.2, $\mathrm{Hom}_{\mathcal{A}}(A^\lambda, V) \cong \mathcal{B}|_{A^\lambda}$. By Schur's Lemma, $\mathrm{Hom}_{\mathcal{A}}(A^\lambda, V) \cong \mathbb{C}^k$ for some k. Since $\mathrm{Hom}_{\mathcal{A}}(A^\lambda, V)$ is simple, $\mathcal{B}|_{A^\lambda}$ is a simple $\mathcal{B}$-module.

To show that $\mathrm{Hom}(A^\lambda, V) \ncong \mathrm{Hom}(A^\mu, V)$ for $\lambda \neq \mu$, consider some $\mathcal{B}$-module homomorphism

$$\phi : \mathrm{Hom}(A^\lambda, V) \to \mathrm{Hom}(A^\mu, V).$$

Let $f \in \mathrm{Hom}(A^\lambda, V)$. Since isotypic subspaces are disjoint, $f\left(A^\lambda\right)$ is disjoint from $\phi(f)\left(A^\mu\right)$. Let p be the projection from $f\left(A^\lambda\right) \oplus \phi(f)\left(A^\mu\right)$ to $\phi(f)\left(A^\mu\right)$. By Lemma 3.2.2, there is some element $b \in \mathcal{B}$ which restricts to this projection. Then,

$$0 = \phi\left(bf\left(A^\lambda\right)\right) = b\phi(f)\left(A^\mu\right) = \phi(f)\left(A^\mu\right).$$

So, ϕ must be the zero morphism and thus $\mathrm{Hom}(A^\lambda, V) \ncong \mathrm{Hom}(A^\mu, V)$. □

Example 3.2.3. Continuing Example 3.1.3, this theorem tells us that $\mathrm{Hom}\left(S^\lambda, (\mathbb{C}^n)^{\otimes k}\right)$ is a distinct irreducible representation of $\mathrm{End}\,(\mathbb{C}^n)$ for each $\lambda \vdash k$.

In addition to giving a bijection between simple $\mathcal{A}$-modules and simple $\mathcal{B}$-modules, Schur-Weyl duality gives a way of relating tensor products of $\mathcal{A}$-modules to tensor products of $\mathcal{B}$-modules.

Definition 3.2.4. A *seesaw pair* is a pair

$$\begin{array}{ccc} \mathcal{A} & \times & \mathcal{B}, \\ \mathcal{A}' & \times & \mathcal{B}', \end{array}$$

where

- $\mathcal{A}, \mathcal{A}', \mathcal{B}$, and $\mathcal{B}'$ are all algebras acting on a vector space V;
- $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{B}' \subseteq \mathcal{B}$;
- $\mathcal{A}$ and $\mathcal{B}$ are mutual centralizers; and
- $\mathcal{A}'$ and $\mathcal{B}'$ are mutual centralizers.

In this definition, the cartesian products $\mathcal{A} \times \mathcal{B}$ and $\mathcal{A}' \times \mathcal{B}'$ are used to highlight the $(\mathcal{A} \times \mathcal{B})$-module and $(\mathcal{A}' \times \mathcal{B}')$-module structures on V guaranteed by Theorem 3.2.1 respectively. The following theorem is the primary workhorse of this thesis.

Theorem 3.2.5 (Seesaw Reciprocity)**.** *If $\mathcal{A} \times \mathcal{B}$ and $\mathcal{A}' \times \mathcal{B}'$ is a seesaw pair, then*

$$\dim\left(\mathrm{Hom}_{\mathcal{A}}(A^\lambda, \mathrm{Res}^{\mathcal{A}'}_{\mathcal{A}}(A^\mu))\right) = \dim\left(\mathrm{Hom}_{\mathcal{B}'}(B^\mu, \mathrm{Res}^{\mathcal{B}}_{\mathcal{B}'}(B^\lambda))\right).$$

If A^μ is a representation that does not appear in V, then $\dim\left(\mathrm{Hom}_{\mathcal{A}}(A^\lambda, \mathrm{Res}^{\mathcal{A}'}_{\mathcal{A}}(A^\mu))\right) = 0$. *A similar fact holds for $\mathcal{B}$.*

Proof. Let A^λ be a simple $\mathcal{A}'$-module. Then,

$$A^\lambda \cong \bigoplus_{\mu \in \Lambda_{\mathcal{A}'}} \mathrm{Hom}_{\mathcal{A}}\left(A^\mu, \mathrm{Res}^{\mathcal{A}'}_{\mathcal{A}}(A^\lambda) \otimes A^\mu\right).$$

Taking the isotypic decomposition of V as a $\mathcal{B}$-module and applying Schur-Weyl duality,

$$V \cong \bigoplus_{\lambda \in \Lambda_{\mathcal{B}}} \mathrm{Hom}_{\mathcal{B}'}(B^\lambda, V) \otimes B^\lambda \cong \bigoplus_{\lambda \in \Lambda_{\mathcal{B}}} A^\lambda \otimes B^\lambda.$$

Then, by the above observation,

$$\bigoplus_{\lambda \in \Lambda_{\mathcal{B}}} A^\lambda \otimes B^\lambda \cong \bigoplus_{\mu \in \Lambda_{\mathcal{A}'}} \bigoplus_{\lambda \in \Lambda_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(A^\mu, \mathrm{Res}^{\mathcal{A}'}_{\mathcal{A}}(A^\lambda)) \otimes A^\mu \otimes B^\lambda.$$

Performing the same computation beginning with the isotypic decomposition of V as an $\mathcal{A}$-module,

$$\begin{aligned} V &\cong \bigoplus_{\mu \in \Lambda_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(A^\mu, V) \otimes A^\mu \\ &\cong \bigoplus_{\mu \in \Lambda_{\mathcal{A}}} B^\mu \otimes A^\mu \cong \bigoplus_{\lambda \in \Lambda_{\mathcal{B}'}} \bigoplus_{\mu \in \Lambda_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{B}'}(B^\lambda, \mathrm{Res}^{\mathcal{B}}_{\mathcal{B}'}(B^\mu)) \otimes B^\lambda \otimes A^\mu. \end{aligned}$$

So, we see that

$$\mathrm{Hom}_{\mathcal{A}}(A^\mu, \mathrm{Res}^{\mathcal{A}'}_{\mathcal{A}}(A^\lambda)) \cong \mathrm{Hom}_{\mathcal{B}'}(B^\lambda, \mathrm{Res}^{\mathcal{B}}_{\mathcal{B}'}(B^\mu)).$$ □

Example 3.2.6. We continue Examples 3.1.3 and 3.2.3. Let $\lambda \vdash l$, $\mu \vdash k-l$, and $\nu \vdash k$. Let V^λ, V^μ, and V^ν be the irreducible representations of $\mathrm{End}\,(\mathbb{C}^n)$ centralizing the actions of the Specht modules S^λ on $(\mathbb{C}^n)^{\otimes l}$, S^μ on $(\mathbb{C}^n)^{\otimes k-l}$, and S^ν on $(\mathbb{C}^n)^{\otimes k}$ respectively. Let $\mathbb{C}[\mathfrak{S}_l] \times \mathbb{C}[\mathfrak{S}_{k-l}]$ act on $(\mathbb{C}^n)^{\otimes k}$ where $\mathbb{C}[\mathfrak{S}_l]$ acts by permuting the first l factors and $\mathbb{C}[\mathfrak{S}_{k-l}]$ acts by permuting the final $k-l$ factors. This action is centralized by $\mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n)$, where the first copy of $\mathrm{End}(\mathbb{C}^n)$ acts diagonally on the first l factors of $(\mathbb{C}^n)^{\otimes k}$ and the second copy of $\mathrm{End}(\mathbb{C}^n)$ acts diagonally on the final $k-l$ factors of $(\mathbb{C}^n)^{\otimes k}$. Then, $\mathrm{End}(\mathbb{C}^n) \times \mathbb{C}[\mathfrak{S}_k]$ and $(\mathrm{End}(\mathbb{C}^n) \times \mathrm{End}(\mathbb{C}^n)) \times (\mathbb{C}[\mathfrak{S}_l] \times \mathbb{C}[\mathfrak{S}_{k-l}])$ is a seesaw pair. So, Theorem 3.2.5 tells us

$$\begin{aligned} &\dim\left(\mathrm{Hom}_{\mathrm{End}(\mathbb{C}^n)}\left(V^\nu, \mathrm{Res}^{\mathrm{End}(\mathbb{C}^n)\times\mathrm{End}(\mathbb{C}^n)}_{\mathrm{End}(\mathbb{C}^n)}\left(V^\lambda \otimes V^\mu\right)\right)\right) \\ &= \dim\left(\mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_l]\times\mathbb{C}[\mathfrak{S}_{k-l}]}\left(S^\lambda \otimes S^\mu, \mathrm{Res}^{\mathbb{C}[\mathfrak{S}_k]}_{\mathbb{C}[\mathfrak{S}_l]\times\mathbb{C}[\mathfrak{S}_{k-l}]}\left(S^\nu\right)\right)\right) = c^\nu_{\lambda,\mu}. \end{aligned}$$

So, we see that the structure constants for the tensor products of certain $\mathrm{End}\,(\mathbb{C}^n)$-modules are given by Littlewood-Richardson numbers.

3.3 A Centralizer for the Symmetric Group Algebra

Let $\mathfrak{S}_n$ act on $\mathbb{C}^n$ by permuting the standard basis vectors and extend this action to an action of $\mathbb{C}[\mathfrak{S}_n]$. We call this representation the *permutation representation* of $\mathbb{C}[\mathfrak{S}_n]$. Then, $\mathbb{C}[\mathfrak{S}_n]$ acts on $(\mathbb{C}^n)^{\otimes k}$ diagonally. This section gives an inductive description of $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$.

Let 1_n denote the trivial representation of $\mathbb{C}[\mathfrak{S}_n]$. That is, the representation obtained by a linear extension of the group action

$$\begin{array}{rcl} \mathfrak{S}_n \times \mathbb{C} & \to & \mathbb{C} \\ \sigma \times z & \mapsto & z. \end{array}$$

Observe that $\mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}(1_n) = 1_{n-1}$.

Lemma 3.3.1. *The representation* $\mathrm{Ind}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}\mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}(1_n)$ *is isomorphic to the permutation representation of* $\mathbb{C}[\mathfrak{S}_n]$.

Proof. It is clear that

$$\mathrm{Ind}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}\mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}(1_n) = \mathbb{C}[\mathfrak{S}_n] \otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]} 1_{n-1}.$$

For any $\sigma \in \mathfrak{S}_n$, there is some $\sigma' \in \mathfrak{S}_{n-1}$ and some transposition (i, n) such that

$$\sigma = \sigma'(i, n).$$

Pulling σ' through the tensor product, we see that $\mathbb{C}[\mathfrak{S}_n] \otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]} 1_{n-1}$ has a basis given by transpositions (i, n) where $1 \leq i \leq n$. Then, for any $\tau \in \mathfrak{S}_n$, τ acts on this basis by

$$\tau(i, n) \mapsto (\tau(i), n).$$

Extending this action to $\mathbb{C}[\mathfrak{S}_n]$, we see that $\mathbb{C}[\mathfrak{S}_n] \otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]} 1_{n-1}$ is isomorphic as a $\mathbb{C}[\mathfrak{S}_n]$-module to the permutation representation of $\mathbb{C}[\mathfrak{S}_n]$. $\square$

The following theorem shows how the induction and restriction functors may be used to construct the diagonal representation of $\mathbb{C}[\mathfrak{S}_n]$ on $(\mathbb{C}^n)^{\otimes k}$.

Theorem 3.3.2. *The representation* $\left(\mathrm{Ind}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}\mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}\right)^k (1_n)$ *is given by the diagonal action of* $\mathbb{C}[\mathfrak{S}_n]$ *on* $(\mathbb{C}^n)^{\otimes k}$.

Proof. The result is proved inductively. The base case is given by by Lemma 3.3.1. Suppose the result is true for a fixed k. Then,

$$\left(\mathrm{Ind}_{\mathbb{C}[\mathfrak{S}_{n-1}]}^{\mathbb{C}[\mathfrak{S}_n]}\mathrm{Res}_{\mathbb{C}[\mathfrak{S}_{n-1}]}^{\mathbb{C}[\mathfrak{S}_n]}\right)^{k+1}(1_n) = \mathrm{Ind}_{\mathbb{C}[\mathfrak{S}_{n-1}]}^{\mathbb{C}[\mathfrak{S}_n]}\mathrm{Res}_{\mathbb{C}[\mathfrak{S}_{n-1}]}^{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$$

$$= \mathrm{Ind}_{\mathbb{C}[\mathfrak{S}_{n-1}]}^{\mathbb{C}[\mathfrak{S}_n]}\left(\mathrm{Res}_{\mathbb{C}[\mathfrak{S}_{n-1}]}^{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)\otimes_{\mathbb{C}} 1_{n-1}\right).$$

We claim that this module is isomorphic to

$$(\mathbb{C}^n)^{\otimes k}\otimes_{\mathbb{C}}\mathrm{Ind}_{\mathbb{C}[\mathfrak{S}_{n-1}]}^{\mathbb{C}[\mathfrak{S}_n]}(1_{n-1})$$

as a $\mathbb{C}[\mathfrak{S}_n]$-module. Writing out the induction and restriction functors explicitly, this claim becomes

$$\left(\mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_n]}\left(\mathbb{C}[\mathfrak{S}_n],(\mathbb{C}^n)^{\otimes k}\right)\otimes_{\mathbb{C}} 1_{n-1}\right)\otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]}\mathbb{C}[\mathfrak{S}_n]\cong(\mathbb{C}^n)^{\otimes k}\otimes_{\mathbb{C}}\left(1_{n-1}\otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]}\mathbb{C}[\mathfrak{S}_n]\right).$$

Since $\mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_n]}\left(\mathbb{C}[\mathfrak{S}_n],(\mathbb{C}^n)^{\otimes k}\right)\cong(\mathbb{C}^n)^{\otimes k}$, the claim is true. From Lemma 3.3.1,

$$\left(1_{n-1}\otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]}\mathbb{C}[\mathfrak{S}_n]\right)\cong\mathbb{C}^n,$$

as an $\mathbb{C}[\mathfrak{S}_n]$-module. So,

$$(\mathbb{C}^n)^{\otimes k}\otimes_{\mathbb{C}}\left(1_{n-1}\otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]}\mathbb{C}[\mathfrak{S}_n]\right)\cong(\mathbb{C}^n)^{\otimes k+1},$$

which is exactly what we wanted to show. □

Theorem 3.3.2 will allow us to construct the Bratteli diagram for the tower of $\mathbb{C}[\mathfrak{S}_n]$-modules

$$0\subseteq\mathbb{C}^n\subseteq(\mathbb{C}^n)^{\otimes 2}\subseteq(\mathbb{C}^n)^{\otimes 3}\subseteq(\mathbb{C}^n)^{\otimes 4}\subseteq\cdots,$$

and thus allow us to inductively compute the multiplicity of the Specht module S^λ occurring in the diagonal representation of $\mathbb{C}[\mathfrak{S}_n]$ on $(\mathbb{C}^n)^{\otimes k}$ for any k. Then, the note following Theorem 3.2.1 tells us that the multiplicity of S^λ in the diagonal representation of $\mathbb{C}[\mathfrak{S}_n]$ on $(\mathbb{C}^n)^{\otimes k}$ is exactly the dimension of the irreducible $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$-module which is paired with S^λ via Schur-Weyl duality. We cannot expect computing the dimensions of irreducible $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ in this fashion to give us any special insight into the Kronecker problem. This computation only involves analysis of $\mathbb{C}[\mathfrak{S}_n]$, and thinking of the Kronecker problem purely in terms of $\mathbb{C}[\mathfrak{S}_n]$ is historically a dead end street. In order to make any real progress, a deeper analysis of $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ is needed. In the coming chapters, we will provide a description of an algebra $P_k(n)$. This algebra contains $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$, and in the case where $P_k(n)$ is semisimple equals $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$. In order to determine which simple $P_k(n)$-modules are paired with which Specht modules via Schur-Weyl duality, all that will be required is a simple comparison of dimensions. For instance, if we know that exactly one simple $P_k(n)$-module centralizes the action of S^λ on $(\mathbb{C}^n)^{\otimes k}$, we know the dimension of the module which centralizes the action of S^λ, and we know that $P_k(n)$ has only one simple module of the prescribed dimension, then we know exactly which $P_k(n)$-module is paired with S^λ without needing to do any work to verify that they actually are centralizers.

Theorem 3.3.3. *The irreducible representations of $\mathbb{C}[\mathfrak{S}_n]$ which appear in the diagonal representation of $\mathbb{C}[\mathfrak{S}_n]$ on $(\mathbb{C}^n)^{\otimes k}$ are indexed by partitions $\lambda \vdash n$ with $|\lambda_{>1}| \leq k$. Letting m_k^λ denote the multiplicity of S^λ in the diagonal representation of $\mathbb{C}[\mathfrak{S}_n]$ on $(\mathbb{C}^n)^{\otimes k}$,*

$$m_{k+1}^\lambda = \sum_{\mu \vdash n} p_{\lambda,\mu} m_k^\mu,$$

where $p_{\lambda,\mu}$ is 1 if λ and μ differ by two boxes, the number of corners in λ (that is, the number of boxes in λ such that the shape obtained by removing the box from λ is a Ferrers shape) if $\lambda = \mu$, and 0 otherwise.

Proof. The result will follow directly from Theorem 2.4.2 and Special Case 2.5.24.

Let $\nu \vdash n-1$ and let $m_{k+\frac{1}{2}}^\nu$ denote the multiplicity of S^ν in $\mathrm{Res}_{\mathbb{C}[\mathfrak{S}_{k-1}]}^{\mathbb{C}[\mathfrak{S}_k]}\left((\mathbb{C}^n)^{\otimes k}\right)$. Then, Special Case 2.5.24 tells us that

$$m_{k+1}^\lambda = \sum_{\substack{\nu \vdash n-1 \\ \lambda/\nu = \square}} m_{k+\frac{1}{2}}^\nu,$$

where the sum is over all possible ways to delete a box from λ. Special Case 2.5.24 also tells us that

$$m_{k+\frac{1}{2}}^\nu = \sum_{\substack{\lambda \vdash n \\ \lambda/\nu = \square}} m_k^\lambda,$$

where the sum is over all possible ways to add a box to ν. Thus,

$$m_{k+1}^\lambda = \sum_{\mu \vdash n} p_{\lambda,\mu} m_{k-1}^\mu,$$

where $p_{\lambda,\mu}$ is the number of ways to delete a box from the Ferrers shape λ, then add a box back on to arrive at the Ferrers shape μ. As in the statement of the theorem, $p_{\lambda,\mu}$ is 1 if λ and μ differ by a box, the number of corners in λ if $\lambda = \mu$, and 0 otherwise. $\square$

Example 3.3.4. The preceding theorem allows us to draw the Bratteli diagram for the tower

$$0 \subseteq \mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}(\mathbb{C}^n) \subseteq \mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}\left(\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}(\mathbb{C}^n)\right) \subseteq \mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes 2}\right) \subseteq \cdots$$

The first several levels of the Bratteli diagram in the case where $n = 6$ are given in Figure 3.3.4.

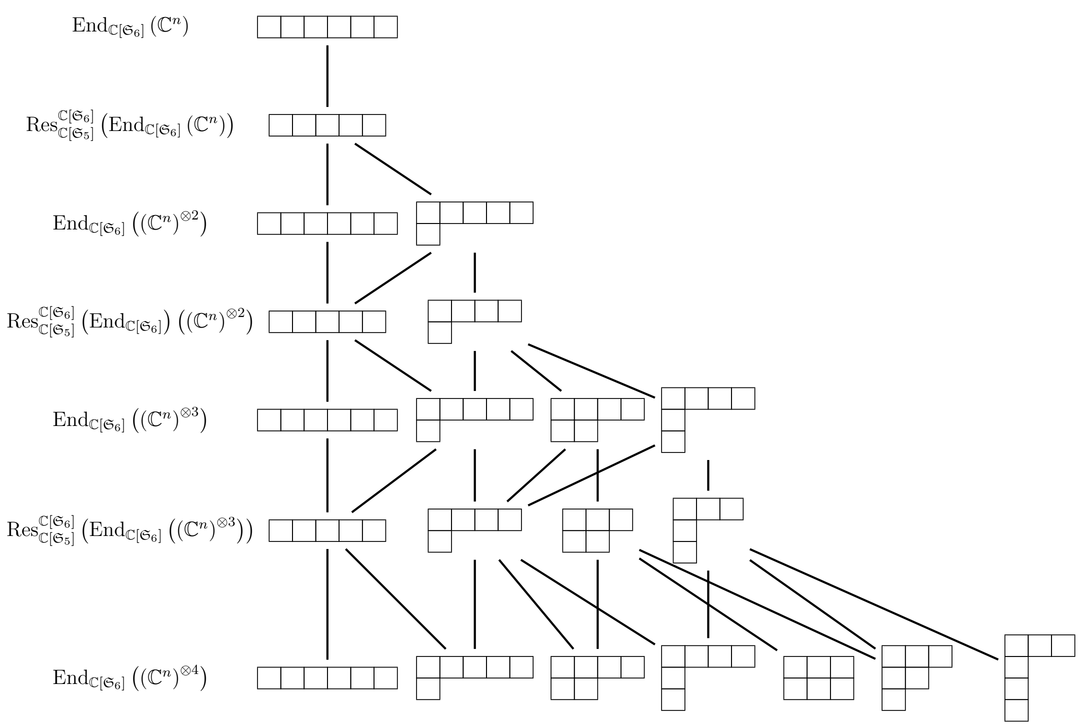


Figure 3.1: Brattelli Diagram for Centralizers of the Symmetric Group.

Chapter 4

The Partition Algebra

We have witnessed Schur-Weyl duality's power to relate Kronecker products of $\mathrm{End}\left(\mathbb{C}^n\right)$ representations to certain products of $\mathbb{C}[\mathfrak{S}_k]$ representations. The goal of this chapter is to give an explicit construction of $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left(\left(\mathbb{C}^n\right)^{\otimes k}\right)$, which is an algebra known as the partition algebra. Section 4.1 constructs the partition algebra. Section 4.2 offers a diagrammatic presentation of this algebra. Section 4.3 describes a collection of distinguished elements, ideals, and quotient algebras of the partition algebra, which we shall find useful later. Section 4.4 equips the partition algebra with a trace.

4.1 Building the Partition Algebra

Let $\mathbb{C}[\mathfrak{S}_n]$ act on $\mathbb{C}^n$ by permuting the standard basis vectors. Specifically, if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is the standard basis of $\mathbb{C}^n$, and $\sigma \in \mathfrak{S}_n$, then $\sigma \cdot \mathbf{v}_i = \mathbf{v}_{\sigma(i)}$. The symmetric group acts diagonally on $\left(\mathbb{C}^n\right)^{\otimes k}$ by the same action. If $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$, and $\sigma \in \mathfrak{S}_n$, then

$$\sigma \cdot \left(\mathbf{v}_{i_1} \otimes \mathbf{v}_{i_2} \otimes \cdots \otimes \mathbf{v}_{i_k}\right) = \left(\mathbf{v}_{\sigma(i_1)} \otimes \mathbf{v}_{\sigma(i_2)} \otimes \cdots \otimes \mathbf{v}_{\sigma(i_k)}\right).$$

This action extends to an action of $\mathbb{C}[\mathfrak{S}_n]$. Our goal is to find the centralizer algebra this action. Let $\psi \in \mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left(\left(\mathbb{C}^n\right)^{\otimes k}\right)$. Then,

$$\psi\left(\mathbf{v}_{i_1} \otimes \mathbf{v}_{i_2} \otimes \cdots \otimes \mathbf{v}_{i_k}\right) = \sum_{i_1', i_2', \cdots, i_k' \in \{1, \ldots, n\}} \psi_{i_1', \ldots, i_k'}^{i_1, \ldots, i_k}\left(\mathbf{v}_{i_1'} \otimes \mathbf{v}_{i_2'} \otimes \cdots \otimes \mathbf{v}_{i_k'}\right). \tag{4.1}$$

for some values $\psi^{i_1,\ldots,i_k}_{i'_1,\ldots,i'_k} \in \mathbb{C}$. On the level of basis elements, the requirement that $\sigma\psi = \psi\sigma$ for all $\sigma \in \mathfrak{S}_n$ manifests itself as

$$\psi^{\sigma(i_1),\ldots,\sigma(i_k)}_{i'_1,\ldots,i'_k}\left(\mathbf{v}_{i'_1} \otimes \cdots \otimes \mathbf{v}_{i'_k}\right) = \psi^{i_1,\ldots,i_k}_{\sigma^{-1}(i'_1),\ldots,\sigma^{-1}(i'_k)}\left(\mathbf{v}_{\sigma^{-1}(i'_1)} \otimes \cdots \otimes \mathbf{v}_{\sigma^{-1}(i'_k)}\right).$$

Multiplying on the left by σ^{-1},

$$\psi^{\sigma(i_1),\ldots,\sigma(i_k)}_{\sigma(i'_1),\ldots,\sigma(i'_k)}\left(\mathbf{v}_{i_1} \otimes \cdots \otimes \mathbf{v}_{i_k}\right) = \psi^{i_1,\ldots,i_k}_{i'_1,\ldots,i'_k}\left(\mathbf{v}_{i'_1} \otimes \cdots \otimes \mathbf{v}_{i'_k}\right).$$

So, $\psi \in \mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ if and only if

$$\psi^{i_1,\ldots,i_k}_{i'_1,\ldots,i'_k} = \psi^{\sigma(i_1),\ldots,\sigma(i_k)}_{\sigma(i'_1),\ldots,\sigma(i'_k)}. \tag{4.2}$$

for all $\sigma \in \mathfrak{S}_n$. Specializing $i_1,\ldots,i_k,i'_1,\ldots i'_k$ to integers from 1 to n gives a partition of $2k$ distinct elements into n distinct (possibly empty) parts. The identity in Equation 4.2 suppresses the distinctness of the parts. So, the algebra $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ has a basis indexed partitions of the set of symbols $\{i_1,\ldots,i_k,i'_1,\ldots,i'_k\}$ into $\leq n$ non-distinct parts. Equivalently, we could think of the basis as being indexed by equivalence relations on the set of symbols $\{i_1,\ldots,i_k,i'_1,\ldots,i'_k\}$ which consist of $\leq n$ equivalence classes. For now, we stick with the original inspiration and write partitions as maps from $\{i_1,\ldots,i_k,i'_1,\ldots,i'_k\}$ to $\{1,\ldots,n\}$, where it is understood that the labeling of the parts is superfluous data.

We describe multiplication in this algebra. Let

$$\begin{array}{rcl} \phi: \; \{i_1,\ldots,i_k,i'_1,\ldots,i'_k\} & \to & \{1,\ldots,n\}, \\ \psi: \; \{i'_1,\ldots,i'_k,i''_1,\ldots,i'_k\} & \to & \{1,\ldots,n\} \end{array}$$

be two partitions of $2k$ element sets into $\leq n$ parts. We also use ϕ and ψ to denote the basis element of $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ corresponding to these partitions. That is,

$$\phi^{i_1,\ldots,i_k}_{i'_1,\ldots,i'_k} = \begin{cases} 1 & \text{if } i_a = i_b \text{ whenever } \phi(a) = \phi(b), \\ & i'_a = i_b \text{ whenever } \phi(a') = \phi(b), \\ & i'_a = i'_b \text{ whenever } \phi(a') = \phi(b'), \text{and} \\ 0 & \text{otherwise.} \end{cases}$$

The basis element of $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ corresponding to ψ is defined similarly. Composing

ψ and ϕ,

$$(\psi \circ \phi)^{i_1,\ldots,i_k}_{i''_1,\ldots,i''_k} = \sum_{(i'_1,\ldots,i'_k)\in\{1,\ldots,n\}^k} \psi^{i'_1,\ldots,i'_k}_{i''_1,\ldots,i''_k}\phi^{i_1,\ldots,i_k}_{i'_1,\ldots,i'_k}$$

$$= \begin{cases} 1 & \text{if } i_a = i_b \text{ whenever } \phi(a) = \phi(b), \\ & i'_a = i_b \text{ whenever } \phi(a') = \phi(b), \\ & i'_a = i'_b \text{ whenever } \phi(a') = \phi(b'), \\ & i'_a = i'_b \text{ whenever } \psi(a') = \psi(b'), \\ & i''_a = i'_b \text{ whenever } \psi(a'') = \psi(b''), \\ & i''_a = i''_b \text{ whenever } \psi(a'') = \psi(b''), \text{and} \\ & \\ 0 & \text{otherwise.} \end{cases}$$

While somewhat intuitive, this formula looks messy. It will be simplified a little bit, and in the next section will receive a pictorial description. Define a partition

$$(\psi \circ \phi): \quad \{i_1, \ldots, i_k, i''_1, \ldots, i''_k\} \quad \to \quad \{1, \ldots, n\}$$

where

$$\begin{array}{rll} (\psi \circ \phi)(i_a) = (\psi \circ \phi)(i_b) & \text{if and only if} & \phi(i_a) = \phi(i_b), \\ (\psi \circ \phi)(i''_a) = (\psi \circ \phi)(i''_b) & \text{if and only if} & \psi(i''_a) = \psi(i''_b), \text{and} \\ (\psi \circ \phi)(i_a) = (\psi \circ \phi)(i''_b) & \text{if and only if} & \text{there exists some } c \text{ such} \\ & & \text{that } \phi(i_a) = \phi(i'_c) \text{ and} \\ & & \psi(i'_c) = \psi(i''_b). \end{array}$$

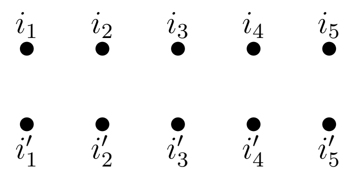
Define the statistic $\#(\psi \circ \phi)$ as follows. Consider the set $S \subset \{i'_1, \ldots, i'_k\}$ consisting of the i'_a for which $\phi(i'_a) \neq \phi(i_b)$ for any $b \in \{i_1, \ldots, i_k\}$ and $\psi(i'_a) \neq \psi(i''_c)$ for any $c \in \{i''_1, \ldots i''_k\}$. Define an equivalence relation on S which is the transitive closure of the union of the equivalence relations $\phi|_S$ and $\psi|_S$. This relation gives a partition of S, and $\#(\psi \circ \phi)$ is the number of parts in this partition. With this definition, the formula for $(\psi \circ \phi)^{i_1,\ldots,i_k}_{i''_1,\ldots,i''_k}$ is

$$(\psi \circ \phi)^{i_1,\ldots,i_k}_{i''_1,\ldots,i''_k} = \begin{cases} n^{\#(\psi\circ\phi)} & \text{if } i_a = i_b \text{ whenever } (\psi \circ \phi)(i_a) = (\psi \circ \phi)(i_b), \\ & i_a = i''_b \text{ whenever } (\psi \circ \phi)(i_a) = (\psi \circ \phi)(i''_b), \\ & i''_a = i''_b \text{ whenever } (\psi \circ \phi)(i''_a) = (\psi \circ \phi)(i''_b), \text{and} \\ 0 & \text{otherwise.} \end{cases}$$

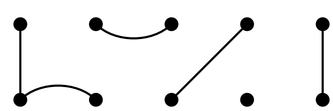
4.2 Pretty Pictures

This section gives this algebra $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ described in the previous section a slick pictorial description. Commonly, this algebra is first described in terms of these diagrams and it is later shown that it is isomorphic to $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$. However, we wanted emphasize that this diagrammatic presentation is a simply mnemonic for something we already care about rather than giving a set of seemingly arbitrary rules and later explaining why one might care.

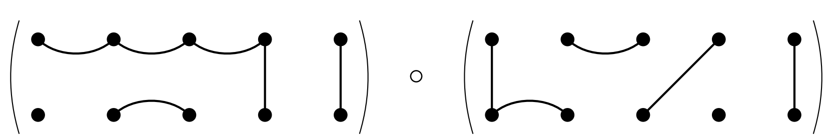
Begin by representing the elements of $\{i_1, \ldots, i_k, i'_1, \ldots, i'_k\}$ as little dots arranged in two rows like so.



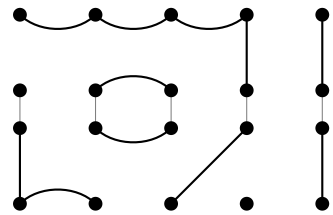
To make the pictures easier to look at, the labels are omitted and the indices are supposed to be in increasing order from left to right with non-primed entries on the top and primed entries on the bottom. A partition is drawn by connecting elements which are in the same part (there might be more than one way to draw this, but they are all equivalent). For example, the partition $\{i_1, i'_1, i'_2\}, \{i_2, i_3\}, \{i_4, i'_3\}, \{i_5, i'_5\}, \{i'_4\}$ is drawn below.



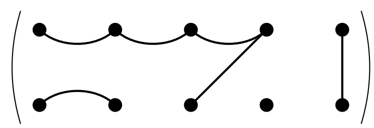
To multiply two diagrams, line them up one above the other, identify the middle nodes, trace out the partition between the top and the bottom rows, and multiply by a factor of n for each connected component within the middle nodes which was eliminated. The reader is encouraged to check that this is indeed the same multiplication as was described in the previous subsection. An example of multiplication is given below.



$=$



$= n$



Working diagrammatically, the value n is no longer necessarily an integer tied to $\mathbb{C}[\mathfrak{S}_n]$. It could take on any complex value or could even be considered to be a formal variable. To avoid degeneracy, we simply suppose that n is non-zero. Once n is allowed to take on any value, the restriction that a partition consist of only n parts becomes meaningless. So, we allow partitions with any number of parts. With these observations in mind, we are finally ready to define the partition algebra.

Definition 4.2.1. For $k \in \mathbb{Z}_{>0}$ and $n \in \mathbb{C} \setminus \{0\}$, the *partition algebra* is the algebra over $\mathbb{C}$ with basis given by partitions of $[2k]$ and multiplication as described above. The partition algebra is denoted by $P_k(n)$.

Note that in the case where $n \geq 2k$ is an integer, $P_k(n)$ is exactly $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$. If $n < 2k$, then $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ consists only of partitions with $\leq n$ parts while $P_k(n)$ consists of any partition. Hence, $P_k(n)$ strictly contains $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$, but is a distinct algebra. In later chapters, we will see that $P_k(n)$ exhibits singular behavior at these parameters, which we will be able to exploit to study the Kronecker problem.

Remark 4.2.2. The partition algebra $P_k(0)$ is a well defined object. However, it exhibits its own brand of singular behavior which requires a slightly different statement of every single theorem. For the sake of simplicity, we thus assume that $n \neq 0$ throughout this thesis. Results about the partition algebra $P_k(0)$ may be found in [11].

The following will be a useful statistic for partitions in $P_k(n)$.

Definition 4.2.3. Let d be a partition diagram. A block of the partition is called a *propagating block* if it contains vertex both the top and bottom rows of the diagram. The *propagating number* of a diagram is the number of propagating blocks in the diagram.

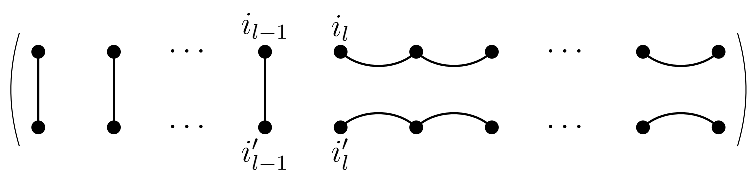
Note that the multiplication of two diagrams cannot increase the propagating number.

4.3 Special Elements, Ideals, and Quotients

This subsection provides a collection of elements, ideals, and quotients of the partition algebra which we will find useful later on.

- For any $0 \leq l \leq k+1$, the following element is an idempotent in $P_k(n)$.

$$E_l = \frac{1}{n}$$



  By convention, set $E_{k+1} = Id_{P_k(n)}$ and $E_0 = 0$.

- The algebra $E_{l+1}P_k(n)E_{l+1}$ consists of partitions with propagating number $\leq l$ where only the leftmost l dots on the top or bottom row may propagate. Ignoring the rightmost $k-l$ dots provides an isomorphism $E_lP_k(n)E_l \cong P_l(n)$.

- For each E_l, $P_k(n)E_lP_k(n)$ is the two sided ideal generated by E_l. Since multiplication in the partition algebra cannot increase the propagating number and E_l has propagating number $l-1$, $P_k(n)E_lP_k(n)$ consists of all partitions with propagating number $< l$.

- The quotient $P_k(n)/(P_k(n)E_lP_k(n))$ may be treated as either a left or a right quotient, since $P_k(n)E_lP_k(n)$ is a two sided ideal. In the quotient $P_k(n)/(P_k(n)E_lP_k(n))$, all partitions with propagating number $< l$ are identified with 0. Since multiplication in the partition algebra cannot increase the propagating number, no two elements with propagating number $\geq l$ are related by multiplication by an element in $P_k(n)E_lP_k(n)$. Thus, the quotient $P_k(n)/(P_k(n)E_lP_k(n))$ has a basis indexed by partitions with propagating number $\geq l$. Let $x, y \in P_k(n)$ be basis partitions with propagating number $\geq l$. If the product xy has propagating number $\geq l$, the product of x and y in $P_k(n)/(P_k(n)E_lP_k(n))$ agrees with the product of x and y in $P_k(n)$. If xy has propagating number $< l$, the product of x and y in $P_k(n)/(P_k(n)E_lP_k(n))$ is 0.

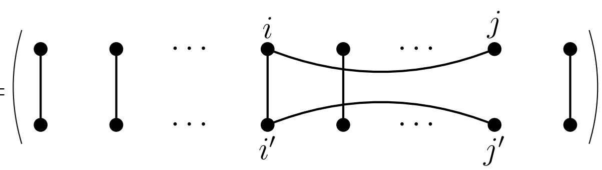
- The subquotient $E_{l+1}\frac{P_k(n)}{P_k(n)E_lP_k(n)}E_{l+1}$ has a basis consisting of partitions of propagating number exactly l where exactly the leftmost l dots on the top and bottom row propagate. Such partitions may naturally be viewed as permutations of the leftmost l dots and their multiplication is identical to multiplication in $\mathbb{C}[\mathfrak{S}_l]$. Thus, $E_l\frac{P_k(n)}{P_k(n)E_{l-1}P_k(n)}E_l \cong \mathbb{C}[\mathfrak{S}_l]$.

- The left ideal $P_k(n)E_l$ has a basis consisting of partitions with propagating number $\leq l$ where only the leftmost l dots on the bottom row may propagate.

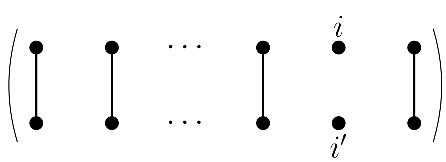
- The subquotient $\frac{P_k(n)}{P_k(n)E_lP_k(n)}E_{l+1}$ has a basis consisting of partitions of propagating number exactly l where exactly the leftmost l dots on the bottom row propagate. Such partitions may be viewed as set partitions of the k dots on the top row into at least l parts where the first l parts are ordered and any other parts are unordered. As in $\frac{P_k(n)}{P_k(n)E_{l-1}P_k(n)}$, the multiplication of two basis elements is 0 if the result of the multiplication causes the propagating number to drop below l. The symmetric group $\mathfrak{S}_l$ acts on a basis element of $\frac{P_k(n)}{P_k(n)E_lP_k(n)}E_{l+1}$ by permuting the leftmost l dots on the bottom row. This action gives $\frac{P_k(n)}{P_k(n)E_lP_k(n)}E_{l+1}$ the structure of a right $\mathbb{C}[\mathfrak{S}_l]$-module.

- The subquotient $E_{l+1}\frac{P_k(n)}{P_k(n)E_lP_k(n)}$ is identical to the subquotent described in the previous bullet point where the words "top" and "bottom" are interchanged. Then, $E_{l+1}\frac{P_k(n)}{P_k(n)E_lP_k(n)}$ the structure of a left $\mathbb{C}[\mathfrak{S}_l]$-module.

- The following element may be used to fuse two parts in a partition into a single part.

$A^{i,j} =$ 

- The following element may be used to disconnect a dot on either the top or bottom row.

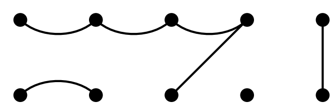
$A^{i} =$ 

4.4 A Trace for the Partition Algebra

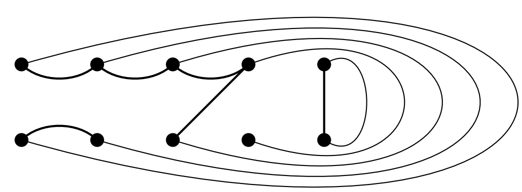
This section defines a trace on the partition algebra $P_k(n)$, which will be necessary when analyzing the representation theory of $P_k(n)$ via Jones' basic construction in Chapter 6. The trace is best viewed pictorially, as in Example 4.4.1. To compute the trace of a partition diagram, begin by joining the dot i to the dot i' in a diagram for each i. Let

c be the number of connected components in this diagram. The trace of a basis element in $P_k(n)$ is defined to be n^c. The linear extension of this map gives a trace on $P_k(n)$. It is clear from construction that this map cyclically invariant and thus may legitimately be called a trace.

Example 4.4.1. To compute the trace of the basis element



in $P_5(n)$, begin by connecting each dot i in the top row to i' in the bottom row.



Since there are two connected components in this image, the trace is n^2.

It is immediate from its definition that this map is cyclically invariant and thus may be called a trace. Having a definition on the basis elements of $P_k(n)$, we may extend it linearly to obtain a trace on the entire partition algebra.

Chapter 5

Representation Theory of the Partition Algebra

Across the next several chapters, we will analyze the irreducible representations of the partition algebra $P_k(n)$. The idea is to use the idempotents E_i from Section 4.3 to obtain a stratification of $P_k(n)$ by symmetric groups. Section 5.1 shows that any irreducible representation of $P_k(n)$ is either an irreducible representation of $E_kP_k(n)E_k$ or an irreducible representation of $P_k(n)/(P_k(n)E_kP_k(n))$. So, we may analyze the restriction to the algebra $E_kP_k(n)E_k$ and the quotient $P_k(n)/(P_k(n)E_kP_k(n))$ separately; no extra work will be required to figure out how to mix the data coming from the restriction and the quotient. Since any representation of a quotient of $P_k(n)$ is already a representation of $P_k(n)$, and $P_k(n)/(P_k(n)E_kP_k(n)) \cong \mathbb{C}[\mathfrak{S}_k]$, the information coming from the quotient algebra is well understood.

While any irreducible representation of $E_kP_k(n)E_k$ is the restriction of an irreducible representation of $P_k(n)$, the task of recovering an irreducible representation of $P_k(n)$ given an irreducible representation of $E_kP_k(n)E_k$ is nontrivial. The problem stems from the fact that while induction and restriction are in some sense inverse operations, and while restriction maps simple modules to simple modules, it is not true that induction always maps simple modules to simple modules. If $P_k(n)$ is not semisimple, then it is possible to take a simple $P_k(n)$-module, restrict it to a simple $E_kP_k(n)E_k$-module, then induct this module back up to $P_k(n)$ and obtain a non-simple module.

Chapter 6 introduces an inductive tool, known as Jones' basic construction, to analyze $P_k(n)$ in the semisimple case. Though we will eventually need to analyze the non-semisimple case, this tool is useful for several reasons: it is a direct application of the

Seesaw Reciprocity Theorem (Theorem 3.2.5); it shows how a tool originally developed to handle questions about functional analysis and statistical mechanics can be applied to representation theory; and it provides some intuition for the more obtuse inductive tool we will employ in the non-semisimple case. Chapter 7 introduces a categorical tool with a similar flavor similar to Jones' basic construction for analyzing the irreducible representations of $P_k(n)$ in the non-semisimple case. Though this tool also does not produce an explicit description of the irreducible representations of $P_k(n)$, it does pick up a homological fingerprint of the irreducible representations. Chapter 8 examines this homological fingerprint and finally yields a procedure for computing the dimension of the irreducible representations of $P_k(n)$ in both the semisimple and non-semisimple cases.

The main results about the representation theory of the partition algebra are given below. All of these results are originally due to Martin in **??**.

Theorem 5.0.2. *The irreducible representations of $P_k(n)$ are indexed by (integer) partitions of size $\leq k$.*

We prove this result at the end of Section 5.1. The simple $P_k(n)$-module labeled by a partition λ will be denoted by $L_k(\lambda)$. The pairing between $\mathbb{C}[\mathfrak{S}_n]$-modules and $P_k(n)$-modules guaranteed by Schur-Weyl duality has a particularly nice description in terms of Ferrers shapes. Recall that for a Ferrers shape λ, $\lambda_{>1}$ is the shape obtained by removing the top row of λ.

Theorem 5.0.3. *Let $\lambda \vdash n$ such that $|\lambda_{>1}| \leq k$. Under Schur-Weyl duality, the Specht module S^λ is paired with the simple $P_k(n)$-module $L_k(\lambda_{>1})$. That is,* $\mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_n]}\left(S^\lambda, (\mathbb{C}^n)^{\otimes k}\right) \cong L_k(\lambda_{>1})$.

This result will be proved at the end of Section 6.2 in the semisimple case by computing the dimensions of the simple $P_k(n)$-modules and comparing them to the dimensions of the simple $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$-modules which we computed in Theorem 3.3.3. Though we will not give a proof of this result in the non-semisimple case, we mention that the Schur-Weyl duality pairing works in exactly the same fashion in the non-semisimple case and may also be proved by directly comparing the dimensions of the simple $P_k(n)$-modules with the dimensions of the simple $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$-modules. Our next result shows when $P_k(n)$ is semisimple.

Theorem 5.0.4. *$P_k(n)$ is semisimple if and only if $n \notin \{1, 2, \ldots, 2k-2\}$.*

For integer valued $n \geq 2k$, it is obvious that $P_k(n)$ is semisimple because $P_k(n)$ is exactly $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ and centralizers of group algebras are semisimple. If n is an integer and $n < 2k$, $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ consists of partition diagrams which have $\leq n$ parts and thus $P_k(n)$ contains, but is not equal to $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$. So, we should expect singular behavior at these parameters, and indeed that is the case for $n < 2k-1$. The fact that $P_k(n)$ is semisimple for all other values of n comes as a bonus. The somewhat surprising fact that $P_k(n)$ is semisimple when $n = 2k-1$ can be seen by noting that the exact sequence in Theorem 5.0.7 is trivial when $n = 2k-1$.

We will delay proving Theorem 5.0.4 until Section 7.3, where we will demonstrate a class of $P_k(n)$-modules $\Delta_k(\lambda)$ such that each $\Delta_k(\lambda)$ contains a unique maximal submodule $U_k(\lambda)$ and such that $\Delta_k(\lambda)/U_k(\lambda)$ is a simple $P_k(n)$-module. In Chapter 8, we will demonstrate that $U_k(\lambda) \neq 0$ if and only if $n \notin \{1, 2, \ldots, 2k-2\}$, which is equivalent to Theorem 5.0.4.

The singular behavior of $P_k(n)$ at its non-semisimple parameters should be viewed a blessing. The problem of analyzing Kronecker coefficients using only semisimple partition algebras is a priori exactly as difficult as the problem of analyzing Kronecker coefficients by looking only at the symmetric group algebra. The additional complexity of $P_k(n)$ at its non-semisimple parameters gives us a structure to exploit which was not present in the symmetric group algebra to begin with. This added structure has an elegant combinatorial description.

Definition 5.0.5. Given two Ferrers shapes $\mu \subset \lambda$, we call (μ, λ) an *n-pair* if λ/μ is a horizontal row whose rightmost box has content $n - |\mu|$. A chain of n-pairs $\lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subseteq \lambda^{(t)}$ is maximal if it cannot be extended on the right.

Note that for any Ferrers shape λ, that there exists at most one Ferrers shape μ such that (μ, λ) is an n-pair, and there exists at most one Ferrers shape μ' such that (λ, μ') is an n-pair.

Example 5.0.6. The following is a chain of 8-pairs. The chain is not maximal since it may be extended on the right. The content of each box in each diagram is written in the box.

$$\begin{array}{|c|c|} \hline 0 & 1 \\ \hline -1 & 0 \\ \hline -2 \\ \cline{1-1} \end{array} \subset \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 \\ \cline{1-2} -2 \\ \cline{1-1} \end{array} \subset \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 \\ \cline{1-3} -2 \\ \cline{1-1} \end{array}$$

Theorem 5.0.7. *If $\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(t)}$ is a maximal chain of n-pairs, then there is an exact sequence of $P_k(n)$-modules*

$$0 \to \Delta_k\left(\lambda^{(t)}\right) \to \cdots \to \Delta_k\left(\lambda^{(i+1)}\right) \to \Delta_k\left(\lambda^{(i)}\right) \to L_k\left(\lambda^{(i)}\right) \to 0$$

The image of each morphism is a distinct simple $P_k(n)$-module. Every simple $P_k(n)$-module can be obtained in this fashion.

We will prove Theorem 5.0.7 in the abstract in Section 7.2, and sketch Doran and Wales' proof of it in the case of the partition algebra in Chapter 8. We will see in Section 9.1 that the Kronecker coefficients may be obtained from the dimensions of simple $P_k(n)$-modules while reduced Kronecker coefficients may be obtained from the dimensions of the $\Delta_k(\lambda)$'s. Thus, the exact sequence in 5.0.7 will allow us to express Kronecker coefficients as an alternating sum of reduced Kronecker coefficients.

5.1 Green's Trick

Let $\mathcal{A}$ be an algebra, and $e \in \mathcal{A}$ be an idempotent. Let $\Lambda_{\mathcal{A}}$ be a set indexing the simple $\mathcal{A}$-modules, and for each $\lambda \in \Lambda_{\mathcal{A}}$ let A^λ be the simple $\mathcal{A}$-module with index λ. This section uses e to reduce the study of the simple $\mathcal{A}$-modules to the study of the simple modules of the quotient $\mathcal{A}/(\mathcal{A}e\mathcal{A})$ and the restriction $e\mathcal{A}e$. Simple modules of the quotient $\mathcal{A}/(\mathcal{A}e\mathcal{A})$ are naturally simple modules of $\mathcal{A}$. Simple modules of $e\mathcal{A}e$ may be induced to $\mathcal{A}$-modules which contain unique simple $\mathcal{A}$-modules. All simple $\mathcal{A}$-modules can be obtained by exactly one of these two procedures; no work is required to mix the simple $\mathcal{A}$-modules obtained from $\mathcal{A}/(\mathcal{A}e\mathcal{A})$ and those obtained from $e\mathcal{A}e$.

In the case of the partition algebra, $P_k(n)/(P_k(n)E_kP_k(n)) \cong \mathbb{C}[\mathfrak{S}_k]$, which is well understood. Further, $E_kP_k(n)E_k \cong P_{k-1}(n)$, so once the irreducible representations of one partition algebra are known, that work can be recycled to study every partition algebra. The main theorem of this section comes from Green in [15]; our proof follows Green's very closely.

Theorem 5.1.1. *Let $\Lambda^e_{\mathcal{A}} = \{\lambda \in \Lambda_{\mathcal{A}} : eA^\lambda \neq 0\}$. Then, $\{eV^\lambda : \lambda \in \Lambda^e_{\mathcal{A}}\}$ is the complete set of simple $e\mathcal{A}e$-modules, and $\{A^\lambda : \lambda \in \Lambda_{\mathcal{A}} \setminus \Lambda^e_{\mathcal{A}}\}$ is the complete set of simple $\mathcal{A}/\mathcal{A}e\mathcal{A}$-modules.*

A couple of lemmas are required to prove this theorem. The following lemmas establish slightly more than is needed to prove Theorem 5.1.1; these extra results will be used later. The first set of lemmas establish a bijection between $\Lambda^e_{\mathcal{A}}$ and the set of simple $e\mathcal{A}e$-modules.

For an $\mathcal{A}$-module M, define $M_{(e)}$ to be the largest submodule of M contained in $(Id - e)M$. Equivalently, $M_{(e)}$ is the sum of all submodules of M which are annihilated by e on the left. Define functors

$$\begin{array}{rrcl} F: & \mathcal{A}-\text{mod} & \to & e\mathcal{A}e-\text{mod} \\ & M & \mapsto & eM \end{array}$$

$$\begin{array}{rrcl} G: & e\mathcal{A}e-\text{mod} & \to & \mathcal{A}-\text{mod} \\ & N & \mapsto & \mathcal{A}e \otimes_{e\mathcal{A}e} N/(\mathcal{A}e \otimes_{e\mathcal{A}e} N)_{(e)} \end{array}$$

Throughout the rest of this thesis, F and G will denote these functors. Technically, different symbols should be used for the F and G functors associated to different algebras and idempotents. This detail will be left to context.

Lemma 5.1.2. *F is an exact functor. For all $\lambda \in \Lambda_{\mathcal{A}}^{e}$, $F\left(A^{\lambda}\right)$ is simple. For all $\mu \in \Lambda_{\mathcal{A}} \setminus \Lambda_{\mathcal{A}}^{e}$, $F\left(A^{\mu}\right) = 0$.*

Proof. Suppose the following sequence of $\mathcal{A}$-modules is exact.

$$0 \longrightarrow M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3 \longrightarrow 0$$

Consider the sequence

$$0 \longrightarrow eM_1 \xrightarrow{e\phi} eM_2 \xrightarrow{e\psi} eM_3 \longrightarrow 0.$$

Suppose for some $m, n \in M_1$ that $e\phi(em) = e\phi(en)$. Then,

$$\phi(em) = e\phi(em) = e\phi(en) = \phi(en).$$

Because ϕ is injective, $em = en$. Hence, $e\phi$ is injective. For any $m \in M_1$,

$$e\psi(e\phi(em)) = e\psi(\phi(m)) = 0.$$

Finally, let $x \in M_3$ and $y \in M_2$ be such that $\psi(y) = x$. Then,

$$e\psi(y) = e\psi(ey) = ex,$$

and $e\psi$ is surjective. Thus, F is an exact functor.

Consider some simple $\mathcal{A}$-module A^{λ}. Since $eA^{\lambda} \subseteq A^{\lambda}$, we have $F(A^{\lambda}) = A^{\lambda}$ or 0. If $F(A^{\lambda}) \neq 0$, it remains to be shown that A^{λ} is simple as an $e\mathcal{A}e$-module. Let V be a nonzero $e\mathcal{A}e$-submodule of A^{λ}. Since $\mathcal{A}V$ is a nonzero submodule of A^{λ}, $\mathcal{A}V = A^{\lambda}$. Then, since $V = eV$,

$$eA^{\lambda} = e\mathcal{A}V = e\mathcal{A}(eV) = (e\mathcal{A}e)V = V.$$

So, the only nonzero $e\mathcal{A}e$-submodule of eA^λ is eA^λ, and hence eA^λ is simple. Thus, $F(A^\lambda)$ is either zero or a simple $e\mathcal{A}e$-module. $\square$

Lemma 5.1.2 shows that F induces a map from $\Lambda_{\mathcal{A}}^e$ to the set simple $e\mathcal{A}e$-modules. The following two lemmas establish that this map is a bijection.

Lemma 5.1.3. *$F \circ G$ is the identity.*

Proof. Let M be an $e\mathcal{A}e$-module. Then,

$$(F \circ G)(M) = F\left((\mathcal{A}e \otimes_{e\mathcal{A}e} M) \,/\, (\mathcal{A}e \otimes_{e\mathcal{A}e} M)_{(e)}\right)$$

Since $(\mathcal{A}e \otimes_{e\mathcal{A}e} M)_{(e)}$ is the sum of all of the submodules of $(\mathcal{A}e \otimes_{e\mathcal{A}e} M)$ which are annihilated by e on the left, it is exactly the kernel of F. So,

$$F\left((\mathcal{A}e \otimes_{e\mathcal{A}e} M) \,/\, (\mathcal{A}e \otimes_{e\mathcal{A}e} M)_{(e)}\right) = e\mathcal{A}e \otimes_{e\mathcal{A}e} M = M. \qquad \square$$

Lemma 5.1.4. *If M is a simple $e\mathcal{A}e$-module, then $G(M)$ is a simple $\mathcal{A}$-module.*

Proof. Recall that a module's quotient by a submodule is simple if and only if the submodule is maximal. So, it suffices to show that $(\mathcal{A}e \otimes_{e\mathcal{A}e} M)_{(e)}$ is a maximal submodule of $(\mathcal{A}e \otimes_{e\mathcal{A}e} M)$. In fact, we show that it is the unique maximal submodule of $(\mathcal{A}e \otimes_{e\mathcal{A}e} M)$. Toward this end, let M' be a proper submodule of $(\mathcal{A}e \otimes_{e\mathcal{A}e} M)$. We show that $M' \subseteq (\mathcal{A}e \otimes_{e\mathcal{A}e} M)_{(e)}$ by showing that M' is annihilated on the left by e. Note that $F(M') \subset F(M)$ is a submodule. By the previous lemma, $F(M) = M$. Since M is simple, $F(M') = 0$. Since M' is annihilated by e, it is contained in $(\mathcal{A}e \otimes_{e\mathcal{A}e} M)_{(e)}$. Thus, $(\mathcal{A}e \otimes_{e\mathcal{A}e} M)_{(e)}$ is the unique maximal submodule of $(\mathcal{A}e \otimes_{e\mathcal{A}e} M)$ and the functor G takes simple modules to simple modules. $\square$

Together with the previous two lemmas, this lemma establishes a bijection between $\Lambda_{\mathcal{A}}^e$ and the set of simple $e\mathcal{A}e$-modules. The next set of lemmas establish a bijection between $\Lambda_{\mathcal{A}} \setminus \Lambda_{\mathcal{A}}^e$ and the set of simple $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-modules. Let

$$q : \mathcal{A} \to \mathcal{A}/(\mathcal{A}e\mathcal{A})$$

be the quotient map.

Lemma 5.1.5. *The set $\Lambda_{\mathcal{A}}^e$ is exactly the set of $\lambda \in \Lambda$ such that A^λ is nonzero and simple considered as an $\mathcal{A}e\mathcal{A}$-module. Equivalently, $q\left(A^\lambda\right) = 0$. So, $eA^\lambda = A^\lambda$, and hence $q(A^\lambda) = 0$.*

Proof. Suppose that A^λ is non-trivial when considered as an $\mathcal{A}e\mathcal{A}$-module. Then there is some $a_1, a_2 \in \mathcal{A}$ such that $a_2ea_1A^\lambda \neq 0$. Then, $ea_1A^\lambda \neq 0$ and thus $eA^\lambda \neq 0$.

Suppose now that $eA^\lambda \neq 0$. Since $e \in \mathcal{A}$, $eeeA^\lambda = eA^\lambda$ is contained in the restriction of A^λ to an $\mathcal{A}e\mathcal{A}$-module. Thus, A^λ is a nonzero $\mathcal{A}e\mathcal{A}$-module. Let M be an $\mathcal{A}e\mathcal{A}$-submodule of A^λ. Then, letting $a \in \mathcal{A}$ act as ae, we may consider M as an $\mathcal{A}$-submodule of A^λ. Since A^λ is simple, $M = 0$ or A^λ. So, the restriction of A^λ to an $\mathcal{A}e\mathcal{A}$-module is simple. Thus, $eA^\lambda \neq 0$ if and only if A^λ is simple and nonzero as an $\mathcal{A}e\mathcal{A}$-module. □

Lemma 5.1.6. *If $\lambda, \mu \in \Lambda_\mathcal{A} \setminus \Lambda^e_\mathcal{A}$ and $\lambda \neq \mu$, then $q(A^\lambda)$ and $q(A^\mu)$ are inequivalent simple $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-modules.*

Proof. Since $q(A^\lambda)$ is a nonzero submodule of A^λ, it is isomorphic to A^λ. Likewise, $q(A^\mu) \cong A^\mu$ and so A^λ and A^μ are inequivalent. Any $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-submodule of A^λ is also an $\mathcal{A}$-submodule of A^λ and is thus isomorphic to zero or A^λ. □

Lemma 5.1.7. *Every simple $\mathcal{A}/(\mathcal{A}e\mathcal{A})$ is the quotient of a simple $\mathcal{A}$-module.*

Proof. Let M be a $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-module. Then, M is also an $\mathcal{A}$-module. Suppose that N is a $\mathcal{A}$-submodule of M. Since M is annihilated by e, N is annihilated by e as well and hence N is also an $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-submodule of M. Hence M is simple as an $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-module if and only if M is simple as an $\mathcal{A}$-module. □

Proof of Theorem 5.1.1. Lemmas 5.1.2, 5.1.3, and 5.1.4 establish a bijection between $\Lambda^e_\mathcal{A}$ and the set of simple $e\mathcal{A}e$-modules. Given a simple $e\mathcal{A}e$-module, we are able to compute a simple $\mathcal{A}$-module by applying the G functor. Lemmas 5.1.5, 5.1.6, and 5.1.7 establish a bijection between $\Lambda_\mathcal{A} \setminus \Lambda^e_\mathcal{A}$ and the set of simple $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-modules. Any simple $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-module is also a simple $\mathcal{A}$-module. □

Example 5.1.8. Consider the partition algebra $P_k(n)$ and the idempotent E_k. Recall that

$$P_k(n)/(P_k(n)E_kP_k(n)) \cong \mathbb{C}[\mathfrak{S}_k]$$

Then, all of the Specht modules indexed by partitions of size k are irreducible $P_k(n)$-modules. This constitutes the entire set of $P_k(n)$-modules which are annihilated by E_k.

Since $E_kP_k(n)E_k \cong P_{k-1}(n)$, to obtain a description of the irreducible representations of $P_k(n)$, we need to understand how to induct a $P_{k-1}(n)$ representation up to a $P_k(n)$ representation. That is, given some simple $P_{k-1}(n)$-module $L_{k-1}(\lambda)$, we need to describe the simple $P_k(n)$-module

$$G(L_{k-1}(\lambda)) = \left(P_k(n)E_k \otimes_{P_{k-1}(n)} L_{k-1}(\lambda)\right) / \left(P_k(n)E_k \otimes_{E_kP_k(n)E_k} L_{k-1}(\lambda)\right)_{(E_k)}.$$

Taking the philosophy that it is easier to do things one step at a time, we will first concern ourselves with describing the modules of the form $P_k(n)E_k \otimes_{E_kP_k(n)E_k} L_{k-1}(\lambda)$, and worry about analyzing the quotient later. Once some algebraic formalism has been put in place in Chapter 7, modules of the form $P_k(n)E_k \otimes_{E_kP_k(n)E_k} L_{k-1}(\lambda)$ will be known as standard modules. Since every simple $P_{k-1}(n)$-module can be obtained by applying the F functor to some simple $P_k(n)$-module, all the standard modules of $P_k(n)$ will be contained in the $P_k(n)$-module

$$P_k(n)E_k \otimes_{E_kP_k(n)E_k} E_kP_k(n).$$

Hence, $P_k(n)E_k \otimes_{E_kP_k(n)E_k} E_kP_k(n)$ will play a major role in our analysis of the set of $P_k(n)$-modules which are not annihilated by E_k.

Even without giving an explicit description of all of the irreducible representations of $P_k(n)$, Theorem 5.1.1 and the commentary above yield an import result.

Theorem 5.0.2. *The irreducible representations of $P_k(n)$ are indexed by (integer) partitions of size $\leq k$.*

Proof. The theorem is proved inductively. If $k = 0$, then $P_k(n)$ is trivial. This trivial module is indexed by the empty set, which may also be viewed as the unique partition of 0.

Suppose then that $k > 0$ and that the irreducible representations of $P_{k-1}(n)$ are indexed by partitions of size $\leq k - 1$. Applying Theorem 5.1.1 to $P_k(n)$ using the idempotent E_k, the set of irreducible representations of $P_k(n)$ is indexed by union the index sets of the irreducible representations of $\mathbb{C}[\mathfrak{S}_k]$ and the irreducible representations of $P_{k-1}(n)$. So, there is exactly one irreducible representation of $P_k(n)$ for each $\lambda \vdash k$ as well as one irreducible representation of $P_{k-1}(n)$ for each irreducible representation of $P_{k-1}(n)$. All of these representations are distinct. Thus, there is a distinct irreducible representation of $P_k(n)$ for each partition of size $< k$.. $\square$

Chapter 6

Jones' Basic Construction

In the previous chapter, the task of describing the irreducible representations of $\mathcal{A}$ was reduced to describing the irreducible representations of $e\mathcal{A}e$ and $\mathcal{A}/(\mathcal{A}e\mathcal{A})$. The goal of both this and the next chapter is to give a description of the irreducible representations of $\mathcal{A}$ which are obtained by applying the G functor from the previous section to an irreducible representation of $e\mathcal{A}e$.

This chapter introduces a technique known as Jones' basic construction, which, given an inclusion of algebras $\mathcal{A} \subseteq \mathcal{B}$, describes the algebra $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$. To apply this tool to study the representation theory of the partition algebra, we introduce an algebra $P_{k-\frac{1}{2}}(n)$ which lies between $P_k(n)$ and $P_{k-1}(n)$. That is, $P_{k-1}(n) \subseteq P_{k-\frac{1}{2}}(n) \subseteq P_k(n)$. We show that

$$P_k(n)E_k \otimes_{E_k P_k(n) E_k} E_k P_k(n) \cong P_{k-\frac{1}{2}}(n) \otimes_{P_{k-1}(n)} P_{k-\frac{1}{2}}(n).$$

There exists an idempotent $E_{k-\frac{1}{2}} \in P_{k-\frac{1}{2}}(n)$ such that

$$P_{k-\frac{1}{2}}(n)E_{k-\frac{1}{2}} \otimes_{E_{k-\frac{1}{2}} P_{k-\frac{1}{2}}(n) E_{k-\frac{1}{2}}} E_{k-\frac{1}{2}} P_{k-\frac{1}{2}}(n) \cong P_{k-1}(n) \otimes_{P_{k-\frac{3}{2}}(n)} P_{k-1}(n).$$

Applying Jones' basic construction and Green's theorems inductively allow us to construct the Bratteli diagram for the tower

$$P_0(n) \subset P_{\frac{1}{2}}(n) \subset P_1(n) \subset P_{1+\frac{1}{2}}(n) \subset P_2(n) \subset \cdots$$

in the semisimple case. From here, Theorem 2.4.2 combined with our knowledge of the representation theory of $\mathfrak{S}_k$ allows us to compute the dimensions of the irreducible representations of $P_k(n)$. Comparing this Bratteli diagram with the Bratteli diagram for the tower

$$0 \subseteq \mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left(\mathbb{C}^n\right) \subseteq \mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_{n-1}]}\left(\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left(\mathbb{C}^n\right)\right) \subseteq \mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left(\left(\mathbb{C}^n\right)^{\otimes 2}\right) \subseteq \cdots$$

from Section 3.3, we are able to show which simple $P_k(n)$-modules are paired with which Specht modules via Schur-Weyl duality.

6.1 General Theory

Throughout this section, all algebras are assumed to be semisimple. Our presentation of the following material closely resembles Hans Wenzl's in [38].

Let $\mathcal{A} \subseteq \mathcal{B}$ such that and suppose that $\mathcal{B}$ is equipped with a trace function

$$\mathrm{tr} : \mathcal{B} \otimes \mathcal{B}^* \to \mathbb{C}.$$

Identifying $\mathcal{B}$ with its left regular representation in $\mathcal{B} \otimes \mathcal{B}^*$, the trace may be thought of as a map from $\mathcal{B}$ to $\mathbb{C}$. Taking this perspective, the trace gives an isomorphism between $\mathcal{B}$ and $\mathcal{B}^*$ via

$$b \mapsto \mathrm{tr}\left(b-\right).$$

Restricting the map $\mathrm{tr}\left(b-\right) : \mathcal{B} \to \mathbb{C}$ to $\mathcal{A}$, $\mathrm{tr}\left(b-\right)$ may be viewed as an element of $\mathcal{A}^*$. Then, using the isomorphism between $\mathcal{A}$ and $\mathcal{A}^*$ induced by the trace, $\mathrm{tr}\left(b-\right)|_{\mathcal{A}}$ maps to an unique element $\epsilon_{\mathcal{A}}(b) \in \mathcal{A}$. Combining these maps gives a map

$$\begin{array}{rccc} \epsilon_{\mathcal{A}} : & \mathcal{B} & \to & \mathcal{A} \\ & b & \mapsto & \epsilon_{\mathcal{A}}(b). \end{array}$$

The map $\epsilon_{\mathcal{A}}$ is known as the *trace preserving conditional expectation* from $\mathcal{B}$ onto $\mathcal{A}$. This name comes from the fact that $\mathrm{tr}\left(\epsilon_{\mathcal{A}}(b)a\right) = \mathrm{tr}\left(ba\right)$ for all $b \in \mathcal{B}$ and all $a \in \mathcal{A}$. Note that $\epsilon_{\mathcal{A}}$ acts as the identity on $\mathcal{A}$.

Remark 6.1.1. Though it will not be important for our discussion of the partition algebra, it is interesting to note that this construction and vocabulary have their nascence in statistical mechanics. A common problem in statistical mechanics is to compute the expected energy of a system of particles arranged in a lattice. One method of computing this expected energy is to introduce a matrix algebra whose generators correspond to either adding a new particle to the lattice or introducing a bond between two adjacent particles. In this framework, one computes the expected energy of a lattice by multiplying the appropriate generating matrices to build the lattice, then taking the trace to measure the expected energy. Given a lattice that is 6 particles wide and infinitely long it is natural to ask what one could learn by probing only a 5-particle-wide sub-lattice. The conditional expectation map can be thought of as mapping this 6-particle-wide system to a 5-particle-wide system whose expected energy is identical to the larger system it was originally living inside. Taking this perspective, Jones' basic construction will give a method of adding a row to a lattice. Given a matrix algebra describing a 6-particle-wide lattice and knowledge about how inclusion of lattices should behave (via the conditional expectation map), it allows us to build a matrix algebra corresponding to a 7-particle-wide lattice. Details about this statistical mechanical perspective may be found in [24].

Suppose that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ and that there exists some idempotent $e \in \mathcal{C}$ such that

- $ebe = e\epsilon_{\mathcal{A}}(b) = \epsilon_{\mathcal{A}}(b)$ for all $b \in \mathcal{B}$, and
- the map
$$\begin{array}{ccc} \mathcal{A} & \to & \mathcal{C} \\ a & \mapsto & ae \end{array}$$
is an injective homomorphism from $\mathcal{A}$ into $\mathcal{C}$.

Definition 6.1.2. Given algebras $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{C}$ and an idempotent $e \in \mathcal{C}$ as above, *Jones' basic construction* (occasionally just the *basic construction*) is the algebra $\langle \mathcal{B}, e\rangle$, the subalgebra of $\mathcal{C}$ generated by $\mathcal{B}$ and e.

Given an inclusion of algebras $\mathcal{A} \subseteq \mathcal{B}$ and a conditional expectation map, the following example shows that there always exists an algebra $\mathcal{C}$ satisfying the requirements of Definition 6.1.2.

Example 6.1.3. Let $\mathcal{C} = \operatorname{End}(\mathcal{B})$ and let $e = \epsilon_{\mathcal{A}}$. Then, e trivially satisfies all of the properties above. This example will be fundamental in all of the proofs that follow. However, in the wild, we will more often encounter the case where $\mathcal{C}$ is some general algebra, not necessarily $\operatorname{End}(\mathcal{B})$.

Recall that $\operatorname{End}_{\mathcal{A}}(\mathcal{B})$ is the algebra of endomorphisms of $\mathcal{B}$ commuting with the action of $\mathcal{A}$ on $\mathcal{B}$. Often (in [17] for example), Jones' basic construction is defined to be the algebra $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$. Using the trace on $\mathcal{B}$ to establish an isomorphism $\mathcal{B} \cong \mathcal{B}^*$, the following theorem together with Lemma 3.1.4, which says that $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B} \cong \operatorname{End}_{\mathcal{A}}(\mathcal{B})$, shows the equivalence of these two definitions.

Theorem 6.1.4. *Let $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathrm{End}(\mathcal{B})$ and let $\langle \mathcal{B}, e\rangle$ be a basic construction. View $\mathcal{B}$ as an $\mathcal{A}$-module by letting $\mathcal{A}$ act via its inclusion in the left regular representation of $\mathcal{B}$. View $\mathcal{B}$ as a $\langle \mathcal{B}, e\rangle$-module by letting $\langle \mathcal{B}, e\rangle$ act on the right. Then, $\mathcal{A}$ and $\langle \mathcal{B}, e\rangle$ are full mutual centralizers. That is, $\langle \mathcal{B}, e\rangle \cong \mathrm{End}_{\mathcal{A}}(\mathcal{B})$.*

To prove this theorem, we need the following lemma describing elements of the basic construction.

Lemma 6.1.5. *Any element in $\langle \mathcal{B}, e\rangle$ can be written as a linear combination of elements in $\mathcal{B}$ and elements of the form $b_1 e b_2$ where $b_1, b_2 \in \mathcal{B}$.*

Proof of Lemma 6.1.5. Applying the identity $ebe = \epsilon_A(b)$ for all $b \in \mathcal{B}$, any element in $\langle \mathcal{B}, e\rangle$ can be written as a linear combination of elements of the form $b_1 e^k b_2$ for $k \in \mathbb{Z}_{\geq 0}$ and $b_1, b_2 \in \mathcal{B}$. Since e is an idempotent, the lemma holds. $\square$

Proof of Theorem 6.1.4. Let $\ell : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the left regular representation of $\mathcal{B}$ defined in Example 2.1.3. This representation induces a map from $\mathcal{B}$ to $\mathrm{End}(\mathcal{B})$ given by

$$b \mapsto \ell(b, -).$$

Abusing notation, this map will also be called ℓ. The inclusion $\mathcal{A} \subseteq \mathcal{B}$ gives an inclusion $\ell(\mathcal{A}) \subseteq \ell(\mathcal{B})$.

Let $r : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ be the right regular representation defined in Example 2.1.3. Since $\mathcal{B}$ is assumed to be semisimple, $\mathcal{B}$ is isomorphic to a direct sum of matrix algebras and the transposition map provides an isomorphism between $\mathcal{B}$ and $\mathcal{B}^{\mathrm{op}}$. Then, $r^t : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$, as defined in Example 2.1.3, is a representation of $\mathcal{B}$. As with the left regular representation, r^t induces a map from $\mathcal{B}$ to $\mathrm{End}(\mathcal{B})$ given by

$$b \mapsto r^t(b, -)$$

Again abusing notation, this map will be denoted by r^t. The inclusion $\mathcal{A} \subseteq \mathcal{B}$ gives an inclusion $r^t(\mathcal{A}) \subseteq r^t(\mathcal{B})$.

We begin by showing that $\langle \mathcal{B}, e\rangle \subseteq \mathrm{End}_{\mathcal{A}}(\mathcal{B})$. By Lemma 3.1.5, the left and right regular representations of $\mathcal{B}$ are full mutual centralizers. Since $r^t(\mathcal{B})$ commutes with $\ell(\mathcal{B})$ and $\ell(\mathcal{A}) \subseteq \ell(\mathcal{B})$, $r^t(\mathcal{A})$ certainly commutes with $\ell(\mathcal{A})$. So, by Lemma 6.1.5, it remains to show that

$$\ell(b_1 e b_2) r^t(a) = r^t(a) \ell(b_1 e b_2)$$

for all $a \in \mathcal{A}$ and all $b_1, b_2 \in \mathcal{B}$. By the definition of the idempotent e,

$$b_1 e b_2 = b_1 \epsilon_{\mathcal{A}}(b_2).$$

Since ℓ is a homomorphism and $\ell(b_2)$ commutes with $r^t(a)$,

$$\ell(b_1 e b_2) r^t(a) = \ell(b_1 e) r^t(a) \ell(b_2).$$

Because $\ell(b_1 \epsilon_{\mathcal{A}}(b_2)) \in \ell(\mathcal{B})$ and $r^t(a) \in r^t(\mathcal{B})$,

$$\ell(b_1 \epsilon_{\mathcal{A}}(b_2)) r^t(a) = r^t(a) \ell(b_1 \epsilon_{\mathcal{A}}(b_2)).$$

Then,

$$r^t(a) \ell(b_1 \epsilon_{\mathcal{A}}(b_2)) = r^t(a) \ell(b_1 e b_2).$$

In total,

$$\ell(b_1 e b_2) r^t(a) = r^t(a) \ell(b_1 e b_2).$$

Thus, $\langle \mathcal{B}, e \rangle \subseteq \mathrm{End}_{\mathcal{A}}(\mathcal{B})$.

It remains to show that $\mathrm{End}_{\mathcal{A}}(\mathcal{B}) \subseteq \langle \mathcal{B}, e \rangle$, where $\langle \mathcal{B}, e \rangle$ acts on the right. Since taking centralizers is an involution which reverses inclusions, it is equivalent to show that

$$\mathrm{End}_{\langle \mathcal{B}, e \rangle}(\mathcal{B}) \subseteq \mathcal{A}.$$

Since $r^t(\mathcal{B}) \subseteq \langle \mathcal{B}, e \rangle$ and taking centralizers reverses inclusions, $\mathrm{End}_{\langle \mathcal{B}, e \rangle}(\mathcal{B}) \subseteq \mathrm{End}_{r^t(\mathcal{B})}(\mathcal{B})$. By the now well worn fact that the centralizer of the right regular representation of $\mathcal{B}$ is the left regular representation of $\mathcal{B}$,

$$\mathrm{End}_{\langle \mathcal{B}, e \rangle}(\mathcal{B}) \subseteq \ell(\mathcal{B}).$$

Let $b \in \mathcal{B}$ be such that $l(b)$ commutes with the action of $\langle \mathcal{B}, e \rangle$ on $\mathcal{B}$. In particular, $l(b)$ must commute with the action of e. So,

$$\ell(b) e = e \ell(b).$$

Since the identity is in $\mathcal{A}$ and e acts trivially on $\mathcal{A}$, $eId = Id$. So,

$$\ell(b) e(Id) = \ell(b)(Id) = b.$$

On the other hand,

$$e \ell(b)(Id) = eb = \epsilon_{\mathcal{A}}(b).$$

Thus, $\ell(b) \in \mathrm{End}_{\langle \mathcal{B},e\rangle}(\mathcal{B})$ only if $b = \epsilon_{\mathcal{A}}(b)$. But, the only b fixed by $\epsilon_{\mathcal{A}}$ are those which were living in $\mathcal{A}$ to begin with. So, $\mathrm{End}_{\langle \mathcal{B},e\rangle}(\mathcal{B}) \subseteq \mathcal{A}$. Taking centralizers again,

$$\mathrm{End}_{\mathcal{A}}(\mathcal{B}) \subseteq \langle \mathcal{B}, e\rangle.$$

Finally, $\langle \mathcal{B}, e\rangle = \mathrm{End}_{\mathcal{A}}(\mathcal{B})$. □

Corollary 6.1.6. *The pair of inclusions $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \langle \mathcal{B}, e\rangle$ is a seesaw pair acting on the vector space $\mathcal{B}$. Here, the first pair of algebras act on the left and the second pair of algebras act on the right.*

Proof. By the previous theorem, $\mathcal{A}$ and $\langle \mathcal{B}, e\rangle$ are mutual centralizers. $l(\mathcal{B})$ and $r^t(\mathcal{B})$ are mutual centralizers by Lemma 3.1.5. □

Together with Theorems 3.2.1 and 3.2.5, this corollary gives us a description of the irreducible representations of the basic construction. The irreducible representations of $\langle \mathcal{B}, e\rangle$ are indexed by the same set as the irreducible representations of $\mathcal{A}$. Restriction from the $\langle \mathcal{B}, e\rangle$-module labeled by λ to the $\mathcal{B}$-module labeled by μ will be identical to induction from the $\mathcal{A}$-module labeled by λ to the $\mathcal{B}$-module labeled by μ. Pictorially, the Brattelli diagram for the inclusion of algebras $\mathcal{A} \subseteq \mathcal{B} \subseteq \langle \mathcal{B}, e\rangle$ will have reflection symmetry about the level associated to $\mathcal{B}$.

6.2 Jones' Construction and the Partition Algebra

This section provides a description of the Bratteli diagram of the tower

$$0 \subseteq P_1(n) \subseteq P_2(n) \subseteq \cdots$$

in the case where each $P_k(n)$ is semisimple. Recall from Theorem 5.0.2 that the irreducible representations of $P_k(n)$ are indexed by the set of integer partitions of size $\leq k$. The main result of this section is the following theorem.

Theorem 6.2.1. *Let λ and μ be partitions with $|\lambda| \leq k$ and $|\mu| \leq k-1$. Let $L_k(\lambda)$ denote the the simple $P_k(n)$-module indexed by λ and $L_{k-1}(\mu)$ denote the simple $P_{k-1}(n)$-module indexed by μ. Let $c(\lambda)$ denote the number of corners of the Ferrers shape λ. Then,*

$$\dim\left(\mathrm{Hom}_{P_{k-1}(n)}\left(\mathrm{Res}^{P_k(n)}_{P_{k-1}(n)}\left(L_k(\lambda)\right), L_{k-1}(\mu)\right)\right) = \begin{cases} 1 & \text{if } \lambda \text{ and } \mu \text{ differ by a box,} \\ c(\lambda)+1 & \text{if } \lambda = \mu, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

This theorem allows us to recursively compute the dimension of each irreducible representation of $P_k(n)$ in the cases where $P_k(n)$ is semisimple. The resemblance of this theorem to Theorem 3.3.3 will allow us to explicitly describe the pairing between simple $P_k(n)$-modules and simple $\mathbb{C}[\mathfrak{S}_n]$-modules guaranteed by Schur-Weyl duality.

Recall that Example 5.1.8 used Green's theorems to reduce the study of the representation theory of the partition algebra to the study of the representation theory of $\mathbb{C}[\mathfrak{S}_k]$ and of $P_k(n)E_k \otimes_{E_kP_k(n)E_k} E_kP_k(n)$. Unfortunately, studying $P_k(n)E_k \otimes_{E_kP_k(n)E_k} E_kP_k(n)$ directly is tricky using the technology we have developed so far because $E_kP_k(n)$ is not an algebra (among other problems, its multiplication lacks a right identity). The next two lemmas describe an algebra $P_{k-\frac{1}{2}}(n)$ such that

$$P_k(n)E_k \otimes_{E_kP_k(n)E_k} E_kP_k(n) \cong P_{k-\frac{1}{2}}(n) \otimes_{P_{k-1}(n)} P_{k-\frac{1}{2}}(n).$$

The algebra on the right can be described by applying Jones' basic construction to the inclusion of algebras $P_{k-1}(n) \subseteq P_{k-\frac{1}{2}}(n)$.

Lemma 6.2.2. *The map*

$$\begin{array}{rcl} P_k(n)E_k \otimes_{E_kP_k(n)E_k} E_kP_k(n) & \to & P_k(n)E_kP_k(n) \\ xE_k \otimes E_ky & \mapsto & xE_ky \end{array}$$

is an isomorphism of algebras.

Proof. A basis for $P_k(n)E_k$ consists of partitions where no two dots from the bottom row are in the same part and only the leftmost $k-1$ dots on the bottom row may propagate. Similarly, a basis for $E_kP_k(n)$ consists of partitions where no two dots from the top row are in the same part and only the leftmost $k-1$ dots on the top row may propagate. Taking the tensor product over $E_kP_k(n)E_k$ corresponds to gluing the bottom dots of a partition in $P_k(n)E_k$ to the top row dots of a partition in $E_kP_k(n)$. It is clear that the result can be any partition with propagatating number $< k$. $\square$

Definition 6.2.3. The algebra $P_{k-\frac{1}{2}}(n)$ is the subalgebra of $P_k(n)$ where the dots k and k' are in the same block.

Lemma 6.2.4. *The map*

$$\begin{array}{rcl} P_{k-\frac{1}{2}}(n) \otimes_{P_{k-1}(n)} P_{k-\frac{1}{2}}(n) & \to & P_k(n)E_kP_k(n) \\ x \otimes y & \mapsto & xE_ky \end{array}$$

is an isomorphism of algebras.

Proof. The proof of this fact is similar to the proof of Lemma 6.2.2 $\square$

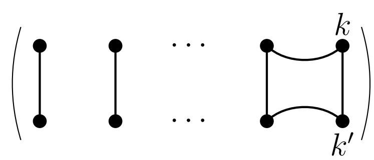
From Lemmas 6.2.2 and 6.2.4, it follows that

$$P_k(n)E_k \otimes_{E_k P_k(n) E_k} E_k P_k(n) \cong P_{k-\frac{1}{2}}(n) \otimes_{P_{k-1}(n)} P_{k-\frac{1}{2}}(n).$$

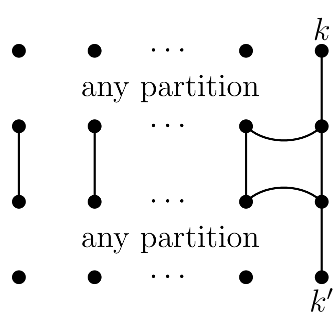
Then, Jones' basic construction says that

- the irreducible representations of $P_k(n)E_k \otimes_{E_k P_k(n) E_k} E_k P_k(n)$ are indexed by the same set as the irreducible representations of $P_{k-1}(n)$ (that is, they are indexed by partitions of size $\leq k-1$), and
- the restrictions from irreducible representations of $P_k(n)E_k \otimes_{E_k P_k(n) E_k} E_k P_k(n)$ to irreducible representations of $P_{k-\frac{1}{2}}(n)$ are identical to the restrictions from irreducible representations of $P_{k-\frac{1}{2}}(n)$ to irreducible representations of $P_{k-1}(n)$.

The task at hand is now to analyze the irreducible representations of $P_{k-\frac{1}{2}}(n)$ and to describe their restrictions to $P_{k-1}(n)$. Fortunately, no new techniques are needed for this analysis; the same techniques already used to study $P_k(n)$ may be employed. Begin by using Green's trick again, this time with the idempotent

$E_{k-\frac{1}{2}} =$ 

in $P_{k-\frac{1}{2}}(n)$. From the following picture, it is clear that $P_{k-\frac{1}{2}}(n)E_{k-\frac{1}{2}}P_{k-\frac{1}{2}}(n)$ consists of all partitions of $P_{k-\frac{1}{2}}(n)$ which have propagating number $< k$.

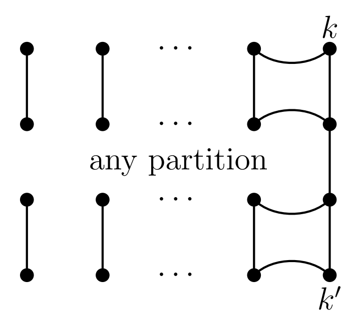


Then, $P_{k-\frac{1}{2}}(n)/\left(P_{k-\frac{1}{2}}(n)E_{k-\frac{1}{2}}P_{k-\frac{1}{2}}(n)\right)$ consists of all partitions with propagating number exactly k where the dot labeled k is connected to the dot labeled k'. This algebra is isomorphic to $\mathbb{C}[\mathfrak{S}_{k-1}]$. Thus, the Specht modules labeled by partitions of $k-1$ form a subset of the irreducible representations of $P_{k-\frac{1}{2}}(n)$. Since these same Specht modules are also irreducible representations of $P_{k-1}(n)$, the restriction from $P_{k-\frac{1}{2}}(n)$ to $P_{k-1}(n)$ of one of these Specht module is the exact same Specht module.

To describe the irreducible representations of $P_{k-\frac{1}{2}}(n)$, it then remains to describe the algebra

$$(G \circ F)\left(P_{k-\frac{1}{2}}(n)\right) = P_{k-\frac{1}{2}}(n)E_{k-\frac{1}{2}} \otimes_{E_{k-\frac{1}{2}}P_{k-\frac{1}{2}}(n)E_{k-\frac{1}{2}}} E_{k-\frac{1}{2}}P_{k-\frac{1}{2}}(n).$$

Any basis element of $E_{k-\frac{1}{2}}P_{k-\frac{1}{2}}(n)E_{k-\frac{1}{2}}$ is of the following form.



In every such partition, the dots labeled $k-1, k, (k-1)'$, and k' are always in the same block. Any such partition with this property is admissible. So, the map from $P_{k-\frac{1}{2}}(n)$ to $P_{k-\frac{3}{2}}(n)$ given by gluing the dots $k-1$ and k together and gluing the dots $(k-1)'$ and k' together is an isomorphism. Then, employing lemmas similar to Lemmas 6.2.2 and 6.2.4,

$$P_{k-\frac{1}{2}}(n)E_{k-\frac{1}{2}} \otimes_{E_{k-\frac{1}{2}}P_{k-\frac{1}{2}}(n)E_{k-\frac{1}{2}}} E_{k-\frac{1}{2}}P_{k-\frac{1}{2}}(n) \cong P_{k-1}(n) \otimes_{P_{k-\frac{3}{2}}(n)} P_{k-1}(n).$$

Applying Jones' basic construction to the inclusion $P_{k-\frac{3}{2}}(n) \subseteq P_{k-1}(n)$ shows that

- the irreducible representations of $P_{k-1}(n) \otimes_{P_{k-\frac{3}{2}}(n)} P_{k-1}(n)$ are indexed by the same set as the irreducible representations of $P_{k-\frac{3}{2}}(n)$, and
- the restrictions from irreducible representations of $P_{k-1}(n) \otimes_{P_{k-\frac{3}{2}}(n)} P_{k-1}(n)$ to irreducible representations of $P_{k-1}(n)$ are mirror images of the restrictions from irre-

ducible representations $P_{k-1}(n)$ to irreducible representations of $P_{k-\frac{3}{2}}(n)$.

At this point, the irreducible representations of $P_k(n)$ and $P_{k-\frac{1}{2}}(n)$ have descriptions in terms of the irreducible representations of partition algebras on strictly fewer dots, and of symmetric group representations. A simple induction then yields the following theorem.

Theorem 6.2.5. *The irreducible representations of $P_k(n)$ are indexed by partitions of size $\leq k$, and the irreducible representations of $P_{k-\frac{1}{2}}(n)$ are indexed by partitions of size $\leq k-1$. If λ is a partition of size $\leq k$, μ and ν are partitions of size $\leq k-1$, and $L_k(\lambda), L_{k-\frac{1}{2}}(\mu)$, and $L_{k-1}(\nu)$ are the irreducible representations of $P_k(n), P_{k-\frac{1}{2}}(n)$, and $P_{k-1}(n)$ indexed by λ, μ, and ν, then*

$$\dim\left(\mathrm{Hom}\left(\mathrm{Res}^{P_k(n)}_{P_{k-\frac{1}{2}}(n)}\left(L_k(\lambda)\right), L_{k-\frac{1}{2}}(\mu)\right)\right) = 1$$

if $\lambda = \mu$ or if λ can be obtained by adding a corner to μ. Otherwise $L_{k-\frac{1}{2}}(\mu)$ is not a constituent of the restriction of $L_k(\lambda)$. Similarly,

$$\dim\left(\mathrm{Hom}\left(\mathrm{Res}^{P_{k-\frac{1}{2}}(n)}_{P_{k-1}(n)}\left(L_{k-\frac{1}{2}}(\mu)\right), L_{k-1}(\nu)\right)\right) = 1$$

if $\mu = \nu$ or if μ can be obtained by removing a corner from ν. Otherwise $L_{k-1}(\nu)$ is not a constituent of the restriction of $L_{k-\frac{1}{2}}(\mu)$.

Proof. The irreducible representations of $P_k(n)$ are indexed by the union of the indexing set of the irreducible representations of $P_{k-1}(n)$ and the indexing set for the irreducible representations of $\mathbb{C}[\mathfrak{S}_k]$. It follows inductively that the irreducible representations of $P_k(n)$ are indexed by partitions of size $\leq k$.

Likewise, the irreducible representations of $P_{k-\frac{1}{2}}(n)$ are indexed by the union of the indexing set of the irreducible representations of $P_{k-\frac{3}{2}}(n)$ and the indexing set for the irreducible representations of $\mathbb{C}[\mathfrak{S}_{k-1}]$. It follows inductively that the irreducible representations of $P_{k-\frac{1}{2}}(n)$ are indexed by partitions of size $\leq k-1$.

If $|\lambda| = k$, then Special Case 2.5.24 tells us that

$$\dim\left(\mathrm{Hom}\left(\mathrm{Res}^{P_k(n)}_{P_{k-\frac{1}{2}}(n)}\left(L_k(\lambda)\right), L_{k-\frac{1}{2}}\right)\right) = 1$$

if λ and μ differ by a box, and is 0 otherwise. If $|\lambda| < k$, then

$$\dim\left(\mathrm{Hom}\left(\mathrm{Res}^{P_k(n)}_{P_{k-\frac{1}{2}}(n)}\left(L_k(\lambda)\right), L_{k-\frac{1}{2}}(\mu)\right)\right)$$

$$= \dim\left(\mathrm{Hom}\left(\mathrm{Res}^{P_{k-\frac{1}{2}}(n)}_{P_{k-1}(n)}\left(L_{k-\frac{1}{2}}(\mu)\right), L_{k-1}(\lambda)\right)\right),$$

and the result follows inductively.

If $|\mu| = k-1$ and $\mu = \nu$, then $L_{k-\frac{1}{2}}(\mu) \cong L_{k-1}(\nu) \cong S^\mu$ and hence

$$\dim\left(\mathrm{Hom}\left(\mathrm{Res}^{P_{k-\frac{1}{2}}(n)}_{P_{k-1}(n)}\left(L_{k-\frac{1}{2}}(\mu)\right), L_{k-1}(\nu)\right)\right) = 1.$$

Since $L_{k-\frac{1}{2}}(\mu) \cong L_{k-1}(\nu)$, no other simple $P_{k-1}(n)$-module appears in the decomposition of $\mathrm{Res}^{P_{k-\frac{1}{2}}(n)}_{P_{k-1}(n)}\left(L_{k-\frac{1}{2}}(\mu)\right)$. If $|\mu| < k-1$, then

$$\dim\left(\mathrm{Hom}\left(\mathrm{Res}^{P_{k-\frac{1}{2}}(n)}_{P_{k-1}(n)}\left(L_{k-\frac{1}{2}}(\mu)\right), L_{k-1}(\nu)\right)\right)$$

$$= \dim\left(\mathrm{Hom}\left(\mathrm{Res}^{P_{k-1}(n)}_{P_{k-\frac{3}{2}}(n)}\left(L_{k-1}(\nu)\right), L_{k-\frac{3}{2}}(\mu)\right)\right),$$

and the result follows inductively. □

Proof of Theorem 6.2.1. Theorem 6.2.1 follows immediately from Theorem 6.2.5 and Theorem 2.4.2. The exact details of this proof are identical to the details of Theorem 3.3.3, where we gave a method for recursively computing the multiplicity of a given Specht module in the diagonal representation of $\mathbb{C}[\mathfrak{S}_n]$ on $(\mathbb{C}^n)^{\otimes k}$. □

Example 6.2.6. If $P_k(n)$ is semisimple for each k, the first several levels of the Bratteli diagram for the tower

$$0 \subseteq P_{\frac{1}{2}}(n) \subseteq P_1(n) \subseteq P_{1\frac{1}{2}}(n) \subseteq \cdots$$

is drawn in Figure 6.2.6. In this figure, the edges obtained by using Jones' basic construction are drawn in blue while the edges coming from the quotients $P_k(n)/\left(P_k(n)E_kP_k(n)\right)$ are drawn in red.

This Bratteli diagram allows us to compute the dimension of each irreducible $P_k(n)$-module in the semisimple case. Comparing these dimensions with the dimensions obtained from the Bratteli diagram for $\mathrm{End}_{\mathbb{C}[\mathfrak{S}_n]}\left((\mathbb{C}^n)^{\otimes k}\right)$ in Section 3.3, we see that Schur-Weyl must pair up the Specht module S^λ with the simple $P_k(n)$-module $L_k(\lambda_{>1})$. This fact remains true in the non-semisimple case and is still proved by a direct comparison of the dimensions of the simple $P_k(n)$-modules to the multiplicity of S^λ in the diagonal representation of $\mathbb{C}[\mathfrak{S}_n]$. Such a direct comparison of dimensions yields a proof of Theorem 5.0.3.

Theorem 5.0.3. *Consider the diagonal representation of $\mathbb{C}[\mathfrak{S}_n]$ on $(\mathbb{C}^n)^{\otimes k}$. Under Schur-Weyl duality, the Specht module S^λ is paired with the $P_k(n)$-module $L_k(\lambda_{>1})$.*

Remark 6.2.7. As is suggested by comparing Figure 6.2.6 to Figure 3.3.4, the algebra $P_{k-\frac{1}{2}}(n)$ is the centralizer algebra for the diagonal representation $\mathbb{C}[\mathfrak{S}_{n-1}]$ on $(\mathbb{C}^n)^{\otimes k}$, where a permutation in $\mathfrak{S}_{n-1}$ acts on $\mathbb{C}^n$ by permuting the first $n-1$ standard basic vectors. Details about this fact may be found in [17].

Remark 6.2.8. It can be shown that the number of pairs of paths from $\emptyset$ to a fixed partition on the $P_k(n)$ level of the Bratteli diagram in Figure 6.2.6 is a Sterling number of the second kind (that is the number of partitions of $2k$ distinct elements into a fixed number nonempty parts). Since $P_k(n)$ has a basis indexed by partitions, its dimension is the Bell number B_{2k}. Since the dimension of an algebra is the sum of the squares of the dimensions of its irreducible representations, one obtains an extremely obtuse proof of the trivial fact that a Bell number is the sum of Sterling numbers. Details about this fact, as well as a version of the Robinson-Schensted procedure for the partition algebra, may be found in [25]

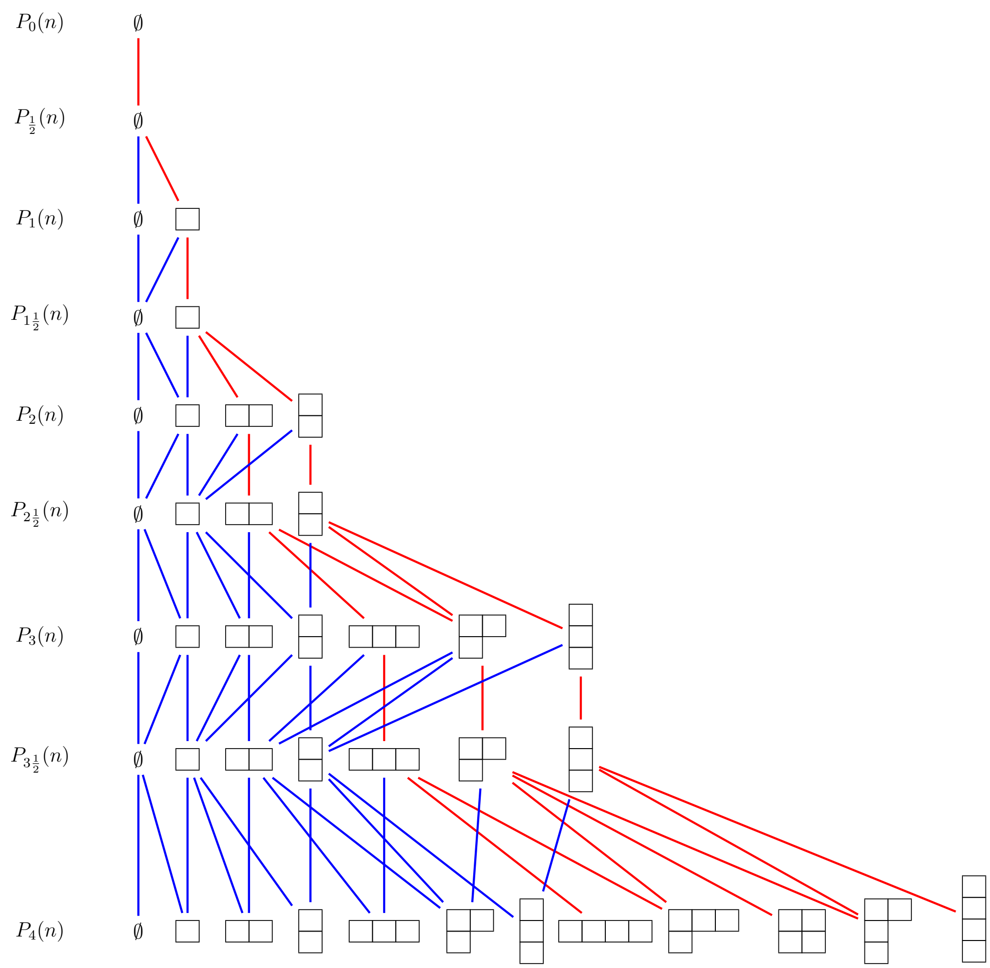


Figure 6.1: Bratteli Diagram for a Tower of Partition Algebras.

Chapter 7

Quasi-hereditary Algebras

This chapter gives a brief introduction to quasi-hereditary algebras and shows that $P_k(n)$ is quasi-hereditary. These results give a homological description of the simple $P_k(n)$-modules, even in the case that $P_k(n)$ is not semisimple. Section 7.1 provides an intuitive, but complete, introduction to the concepts of derived categories and recollement, which are necessary for motivating and analyzing abstract quasi-hereditary algebras. Section 7.2 provides an introduction to general quasi-heredity algebras. Section 7.3 proves that the partition algebra is quasi-hereditary, providing a concrete example of the concepts from Section 7.2.

7.1 Hand waving some hard math

The concepts of derived categories and recollement appearing in this section are fairly dense. Making matters more inaccessible, this material has a tendency to be presented in full generality, with little motivation, and in French. To avoid diverging too far from our discussion of the representation theory of the partition algebra and its relation to the Kronecker problem, we will not present proofs or even definitions in their full form in this section. The interested reader can read [36] for a somewhat intuitive presentation of derived categories similar to this one, [37] for a detailed presentation of derived categories, [2] for further discussion of recollement, and [31] for an explanation of how these concepts relate to quasi-hereditary algebras.

Recapitulating our analysis to this point: recall that $e \in \mathcal{A}$ is an idempotent, and that we began with the observation that every simple module of $\mathcal{A}$ is either a simple module of

$\mathcal{A}/(\mathcal{A}e\mathcal{A})$ or $e\mathcal{A}e$. If $\mathcal{A}$ is semisimple, then $\mathcal{A}$ is a direct sum of its simple modules, and hence $\mathcal{A}$ fits in the middle of a split exact sequence

$$0 \longrightarrow \mathcal{A}/(\mathcal{A}e\mathcal{A}) \longrightarrow \mathcal{A} \longrightarrow e\mathcal{A}e \longrightarrow 0.$$

Here $\mathcal{A}/(\mathcal{A}e\mathcal{A})$ is isomorphic to the direct sum of the simple modules of $\mathcal{A}$ which are annihilated by the idempotent e and $e\mathcal{A}e$ is isomorphic to the direct sum of the simple modules of $\mathcal{A}$ which are not annihilated by e. Since $\mathcal{A}$ is isomorphic to the direct sum of these two submodules, the above split exact sequence must exist. This split exact sequence is written backwards from the usual fashion. We choose to write it in this way because it more closely reflects the categorification of this sequence which is to come. Similar to the sequence above, for any $\mathcal{A}$-module M, the exact sequence

$$0 \longrightarrow \mathrm{Res}^{\mathcal{A}}_{\mathcal{A}/(\mathcal{A}e\mathcal{A})}(M) \longrightarrow M \longrightarrow eM \longrightarrow 0$$

is split. Here, $\mathrm{Res}^{\mathcal{A}}_{\mathcal{A}/(\mathcal{A}e\mathcal{A})}(M)$ is the portion of M which is annihilated by e and eM is the portion of M which is not. We can functorialize this observation about split sequences of modules to the following sequence of categories,

$$\begin{tikzcd} \mathcal{A}/(\mathcal{A}e\mathcal{A})\text{-}\mathtt{mod} \arrow[r, bend left, "i"] & \mathcal{A}\text{-}\mathtt{mod} \arrow[l, bend left, "\mathrm{Res}"] \arrow[r, bend right, "F"'] & e\mathcal{A}e\text{-}\mathtt{mod}. \arrow[l, bend right, "G"'] \end{tikzcd}$$

Here, i is the inclusion functor (since any $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-module is also an $\mathcal{A}$-module), and F and G are Green's functors from Section 5.1. The correct functorial analog for a split exact sequence has the properties:

- all the functors are exact,
- F and G are adjoint and $(F \circ G) = Id_{e\mathcal{A}e-\mathtt{mod}}$,
- Res and i are adjoint and $(\mathrm{Res} \circ i) = Id_{\mathcal{A}/(\mathcal{A}e\mathcal{A})-\mathtt{mod}}$,
- $(F \circ i) = 0$ and $(\mathrm{Res} \circ G) = 0$, and
- any object M of $\mathcal{A}$-`mod` fits into a split exact sequence $0 \to \mathrm{Res}(M) \to M \to F(M) \to 0$.

When these conditions are met, the categories $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-`mod` and $e\mathcal{A}e$-`mod` are both embedded in the category $\mathcal{A}$-`mod`. The third bullet point tells us that the embedding of $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-`mod` and the embedding of $e\mathcal{A}e$-`mod` are disjoint. The final bullet point says that the structure of $\mathcal{A}$-`mod` is completely determined by the structure of $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-`mod` and the structure of $e\mathcal{A}e$-`mod`. This situation occurs when $\mathcal{A}$ is semisimple; our discussion up to this point has focused on analyzing this case.

Unfortunately, if $\mathcal{A}$ is not semisimple, the above sequence of categories does not split. However, we will be able to replace $\mathcal{A}$-`mod` with *the derived category* $D^-(\mathcal{A})$ and show that there is a sequence

$$\begin{tikzcd}
D^-(\mathcal{A}/(\mathcal{A}e\mathcal{A})) \arrow[r, bend left, "i"] & D^-(\mathcal{A}) \arrow[l, bend left, "\mathrm{Hom}(\mathcal{A}/(\mathcal{A}e\mathcal{A}),-)"] \arrow[r, bend right, "F"'] & D^-(e\mathcal{A}e) \arrow[l, bend right, "G"']
\end{tikzcd}$$

called a *recollement of* $D^-(\mathcal{A})$ *relative to* $D^-(\mathcal{A}/(\mathcal{A}e\mathcal{A}))$ *and* $D^-(e\mathcal{A}e)$, which is close enough to being a split sequence to tell us about the structure of any $\mathcal{A}$-module in terms of $\mathcal{A}/(\mathcal{A}e\mathcal{A})$-modules and $e\mathcal{A}e$-modules.

The construction of the category $D^-(\mathcal{A})$ is most sensibly thought of in terms of simplicial cohomology. Given a triangulizable topological space X, we can compute the simplicial cohomology groups of X by triangulating X as some simplicial complex S, associating to S an algebraic complex $S^\bullet$, then computing the cohomology groups of $S^\bullet$. Here, morphisms between algebraic complexes are only considered up to homotopy. The idea behind the derived category is to work directly with these complexes for as long as possible before computing the cohomology groups.

If T is a finer triangulation of X, there is a natural map $T^\bullet \to S^\bullet$ which induces an isomorphism on cohomology (maps of complexes which induce isomorphisms on cohomology will be referred to as *quasi-isomorphisms*). Since the goal is to compute invariants of X and not of triangulations, quasi-isomorphisms are declared to be an isomorphisms and thus $T^\bullet$ and $S^\bullet$ are thus thought of as indistinguishable objects. The objects of the derived category will be algebraic complexes considered up to quasi-isomorphism.

It remains to describe morphisms in the derived category. Keeping in mind the goal of computing invariants of topological spaces, morphisms in the derived category should come from continuous maps $\sigma : X \to Y$ of triangulated topological spaces. Suppose S is a simplicial complex triangulating X and R is a simplicial complex triangulating Y. Then, the map σ may be realized on these complexes by picking a fine enough subtriaglation T

of S and mapping T into R. Expressing this intuition in terms of algebraic complexes, morphisms from $S^\bullet$ to $R^\bullet$ in the derived category will come from diagrams in the category of complexes of the form

$$\begin{tikzcd} & T^\bullet \arrow[dl, "\simeq"'] \arrow[dr] & \\ S^\bullet & & R^\bullet \end{tikzcd}$$

where the map from $T^\bullet$ to $S^\bullet$ is a quasi-isomorphism (and thus $S^\bullet$ and $T^\bullet$ are considered to be the same object after passing to the derived category).

Sadly, inverting all of the quasi-isomorphisms in a category of complexes destroys the abelian nature of the category of complexes. That is, kernels, cokernels, and short exact sequences will no longer exist in general. At this point, our discussion should feel like DIY repair project gone horribly awry. We noticed that short exact sequences weren't splitting, and promised we could fix the problem by taking everything apart by some triangulation. By taking everything apart, we managed to mess things up even worse than they were when we started: not only do short exact sequences fail to split, but they don't even exist in general. Continuing in this DIY project vein, we will proceed to duct tape the short exact sequences together as best we can, then argue that the end product works well enough. In the derived category, the notion of short exact sequences is replaced with distinguished triangles, the definition of which requires the mapping cone construction from topology.

Let $[i]$ be the shift operator which acts on the complex $S^\bullet$ by

$$\begin{aligned} S^j[i] &= S^{j+i} \\ d^j_{S^\bullet}[i] &= (-1)^i d^{j+i}_{S^\bullet} \end{aligned}$$

where $d^j_{S^\bullet}$ is the differential map from S^j to S^{j+1}.

Definition 7.1.1. Let $f : S^\bullet \to T^\bullet$ be a morphism of complexes. Then, *the mapping cone of f* is the complex $\kappa(f)^\bullet = S^\bullet[1] \oplus T^\bullet$ with differential map

$$d_{\kappa(f)^\bullet} = \begin{pmatrix} d_{S^\bullet[1]} & 0 \\ f & d_{T^\bullet} \end{pmatrix}$$

Definition 7.1.2. Given a morphism of complexes $f : S^\bullet \to T^\bullet$, a *distinguished triangle* is defined to be the collection of complexes and morphisms given in the following diagram.

$$
\begin{tikzcd}
& \kappa(f) \arrow[dl, "{[1]\circ w}"'] & \\
S^\bullet \arrow[rr, "f"'] & & T^\bullet \arrow[ul, "v"']
\end{tikzcd}
$$

Here, $\kappa(f)$ is the mapping cone of f, $v : T^\bullet \to \kappa(f)$ is the inclusion map, and $w : \kappa(f) \to S^\bullet[1]$ is the projection map.

To see why distinguished triangles are a decent substitute for exact sequences, recall that short exact sequences of complexes should give rise to long exact sequences in cohomology. Passing to the derived category, we destroyed the short exact sequences of complexes. Distinguished triangles carry the data of the long exact sequences in cohomology without presupposing the existence of a short exact sequence of complexes. Since the goal of the derived category is to think of cohomology groups purely in terms of complexes, the data of long exact sequences in cohomology is really all we need.

It is not a hard exercise to verify that we can rotate distinguished triangles. That is, given a distinguished triangle as in Definition 7.1.2,

$$
\begin{tikzcd}
& T^\bullet \arrow[dl, "v", "{[1]}"'] & \\
\kappa(f) \arrow[rr, "w"'] & & S^\bullet[1] \arrow[ul, "f"']
\end{tikzcd}
$$

is a distinguished triangle as well. This observation gives an interesting philosophical interpretation of the information carried by distinguished triangles (see [33] for further discussion of these ideas). A short exact sequence $0 \to A \to B \to C \to 0$ expresses the idea that the B contains copies of A and C and is in some sense built out of these pieces. Distinguished triangles replace the atomist building block philosophy with a philosophy of interconnectedness. If A, B, and C are in a distinguished triangle, the structure of B can be expressed in terms of its relationship to the structures of A and C. However, rotating the triangle gives an expression of the structure of A in terms of B and C. So, we are no longer describing modules in terms of how they are built out of smaller blocks, but instead in terms of their relationships to each other.

With distinguished triangles replacing exact sequences, we are ready to introduce the derived category analog of a split exact sequence.

Definition 7.1.3. Suppose that $D^-(\mathcal{A}), D^-(\mathcal{B})$, and $D^-(\mathcal{C})$ are derived categories as described above. Then $D^-(\mathcal{B})$ *admits a recollement relative to* $D^-(\mathcal{A})$ *and* $D^-(\mathcal{C})$ if there is a diagram

$$\begin{tikzcd} D^-(\mathcal{A}) \arrow[r, "i_*"] & D^-(\mathcal{B}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & D^-(\mathcal{C}) \arrow[l, bend right, "j_*"'] \arrow[l, bend left, "j_!"] \end{tikzcd}$$

such that:

(i) $i_*, j^*, i^*, i^!, j_*$, and $j_!$ are all exact functors;

(ii) i^* is left adjoint to i_*, $i^!$ is right adjoint to i_*, $j_!$ is left adjoint to j^*, and j_* is right adjoint to j^*;

(iii) letting $\cong$ denote natural equivalence of functors $i^*i_* \cong i^!i_* \cong \mathrm{Id}_{D^-(\mathcal{A})}$, and $j^*j_* \cong j^*j_! \cong \mathrm{Id}_{D^-(\mathcal{C})}$ (and thus, i_*, j_*, and $j_!$ are all full embeddings);

(iv) $j^*i_* = i^!j_* = i^*j_! = 0$ (by adjointness, if any one of these three compositions of functors is zero, the other two must be as well);

(v) and for any object X in $D^-(\mathcal{B})$, there are distinguished triangles

$$\begin{tikzcd} & j_*j^*X \arrow[dl, "{[1]}"'] & \\ i_*i^!X \arrow[rr] & & X \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & i_*i^*X \arrow[dl, "{[1]}"'] & \\ j_!j^*X \arrow[rr] & & X \arrow[ul] \end{tikzcd}$$

in $D^-(\mathcal{B})$.

Observe that a recollement of derived categories is almost identical to our notion of categorical split exact sequences, only that exact sequences have been replaced by distinguished triangles. Points (ii) and (iii) say that $D^-(\mathcal{B})$ contains embedded copies of $D^-(\mathcal{A})$ and $D^-(\mathcal{C})$; (iv) says that these two copies are disjoint. The only difference is (v) where instead of having a spilt exact sequence saying every object in $D^-(\mathcal{B})$ is built out of an object in $D^-(\mathcal{A})$ and an object in $D^-(\mathcal{C})$, there are distinguished triangles saying every object in $D^-(\mathcal{B})$ is determined by its relationship with objects $D^-(\mathcal{A})$ and objects in $D^-(\mathcal{C})$.

7.2 Quasi-hereditary Algebras

While the constructions in the previous section are topological in intuition, they are purely algebraic in formulation. Thus, it makes sense to define the derived category for complexes of $\mathcal{A}$-modules for an any algebra $\mathcal{A}$. In [7], Cline, Parshall, and Scott give conditions for when the derived category $D^-(\mathcal{A})$ admits a recollement.

Definition 7.2.1. An idempotent $e \in \mathcal{A}$ is a *heredity idempotent* if $\mathcal{A}e\mathcal{A}$ is projective as a right $\mathcal{A}$-module, and $(\mathcal{A}e\mathcal{A})J(\mathcal{A})(\mathcal{A}e\mathcal{A}) = 0$, where $J(\mathcal{A})$ is the Jacobson radical of $\mathcal{A}$. If e is a heredity idempotent, $\mathcal{A}e\mathcal{A}$ is called a *heredity ideal*.

Without proof, Cline, Parshall, and Scott's theorem is the following.

Theorem 7.2.2. *Let $e \in \mathcal{A}$ be an idempotent. Then, $D^-(\mathcal{A})$ admits a recollement of the form*

$$\begin{tikzcd} D^-(\mathcal{A}/(\mathcal{A}e\mathcal{A})) \arrow[r] & D^-(\mathcal{A}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D^-(e\mathcal{A}e) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

if and only if e is a heredity idempotent.

In this theorem, the maps most relevant to us will come from the inclusion functor from $\mathcal{A}/(\mathcal{A}e\mathcal{A}) - \texttt{mod}$ to $\mathcal{A} - \texttt{mod}$, the quotient functor from $\mathcal{A} - \texttt{mod}$ to $\mathcal{A}/(\mathcal{A}e\mathcal{A}) - \texttt{mod}$ to $\mathcal{A} - \texttt{mod}$, the F functor from $\mathcal{A} - \texttt{mod}$ to $e\mathcal{A}e - \texttt{mod}$, and the G functor from $e\mathcal{A}e - \texttt{mod}$ to $\mathcal{A} - \texttt{mod}$.

While the stated definition of heredity idempotents is more useful for proving structural facts about heredity ideals, it is often easier to verify an idempotent is heredity via the following lemma. A proof of this lemma, as well as other structural facts about heredity ideals are collected in [10].

Lemma 7.2.3. *If $e \in \mathcal{A}$ is an idempotent such that $e\mathcal{A}e$ is semisimple and the multiplication map*

$$\begin{array}{rcl} \mathcal{A}e \otimes_{e\mathcal{A}e} e\mathcal{A} & \to & \mathcal{A}e\mathcal{A} \\ a_1 e \otimes e a_2 & \mapsto & a_1 e a_2 \end{array}$$

is bijective, then e is a heredity idempotent.

Heredity idempotents make analysis of $D^-(\mathcal{A})$ in terms of $D^-(\mathcal{A}/(\mathcal{A}e\mathcal{A}))$ and $D^-(e\mathcal{A}e)$ behave nicely. After expressing $D^-(\mathcal{A})$ in terms of $D^-(\mathcal{A}/(\mathcal{A}e\mathcal{A}))$ and $D^-(e\mathcal{A}e)$, the hope then is to find a similar idempotent in $e\mathcal{A}e$ and apply this analytical tool inductively, just as we did when using Jones' basic construction to analyze the partition algebra in Section 6.2. This desire motivates the following two definitions.

Definition 7.2.4. A list $(e_1, \cdots, e_l)$ is a heredity chain if

(i) Each e_i is an idempotent in $\mathcal{A}$,

(ii) $\mathcal{A}e_1\mathcal{A} = \mathcal{A}$,

(iii) $\mathcal{A}e_i\mathcal{A} \subseteq \mathcal{A}e_{i-1}\mathcal{A}$ for each i, and

(iv) e_i is a heredity idempotent in $\mathcal{A}/(\mathcal{A}e_{i+1}\mathcal{A})$ (by convention, $e_{l+1} = 0$).

Definition 7.2.5. An algebra $\mathcal{A}$ is *quasi-hereditary* if it has a heredity chain.

A diagram helps elucidate what exactly this definition is saying. Suppose $\mathcal{A}$ is a quasi-hereditary algebra with heredity chain $(e_1, \ldots, e_l)$. Setting $\mathcal{A}_i = \mathcal{A}/(\mathcal{A}e_{i+1}\mathcal{A})$. The third isomorphism theorem implies

$$\frac{\mathcal{A}_i}{\mathcal{A}_i e_i \mathcal{A}_i} \cong \mathcal{A}_{i-1}.$$

Then, repeated application of Theorem 7.2.2 yields the diagram in Figure 7.1.

At each level, the map from $D^-(\mathcal{A}_i/(\mathcal{A}_i e_i \mathcal{A}_i))$ to $D^-(\mathcal{A}_i)$ is simply the inclusion functor. So, to understand $D^-(\mathcal{A})$, all that is necessary is to understand $D^-(e_i\mathcal{A}_i e_i)$ and to understand how to apply Green's G functor from $D^-(e_i\mathcal{A}_i e_i)$ to $D^-(\mathcal{A}_i)$. Since $e_i\mathcal{A}_i e_i$ is semisimple, the analysis of the relevant portion of $D^-(e_i\mathcal{A}_i e_i)$ is relatively straight forward (the only objects that will matter are complexes that consist of a single non-zero term, which is a simple $e_i\mathcal{A}_i e_i$-module).

At a glance, this diagram looks backwards from our inductive application Jones' basic construction in Section 6.2. In that application, the algebra obtained by taking a quotient was simple to analyze, and an inductive procedure was employed to analyze the algebra obtained by projecting by an idempotent. Bearing in mind that a quotient is built into the notation $\mathcal{A}_i$, we see that the inductive step is in fact applied to the algebra obtained by projecting by an idempotent. The next lemma makes this observation explicit; a proof can be found in [10].

Lemma 7.2.6. *Let $\mathcal{A}$ be a quasi-hereditary algebra with heredity chain $(e_1, \ldots, e_l)$. Then, $(e_m, \ldots, e_l)$ is a heredity chain for the algebra $e_m\mathcal{A}e_m$ for each $m \leq l$.*

Remark 7.2.7. A heredity chain should be thought of as putting a last man standing type ordering on the set of $\mathcal{A}$-modules. For any i, $\mathcal{A}_i$ is the subalgebra of $\mathcal{A}$ consisting of elements which are annihilated by e_{i+1}. Since $\mathcal{A}e_i\mathcal{A} \subseteq \mathcal{A}e_{i-1}\mathcal{A}$, anything that is annihilated

$$
\begin{tikzcd}
D^-(\mathcal{A}/(\mathcal{A}e_l\mathcal{A})) \arrow[r] \arrow[dr, <->, "\cong"] & D^-(\mathcal{A}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D^-(e_l\mathcal{A}e_l) \arrow[l, bend right] \arrow[l, bend left] \\
D^-(\mathcal{A}_{l-1}/(\mathcal{A}_l e_{l-1}\mathcal{A}_{l-1})) \arrow[r] \arrow[dr, <->, "\cong"] & D^-(\mathcal{A}_{l-1}) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D^-(e_{l-1}\mathcal{A}_{l-1}e_{l-1}) \arrow[l, bend right] \arrow[l, bend left] \\
{} \arrow[dr, <->, "\cong"] & \vdots & \\
D^-(\mathcal{A}_1/(\mathcal{A}_1 e_1\mathcal{A}_1)) \arrow[r] \arrow[dr, <->, "\cong"] & D^-(\mathcal{A}_1) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & D^-(e_1\mathcal{A}_1e_1) \arrow[l, bend right] \arrow[l, bend left] \\
& D^-(0) &
\end{tikzcd}
$$

Figure 7.1: Recollements Along a Heredity Chain.

by e_i is also annihilated by e_{i+1}. Then, $e_i\mathcal{A}_ie_i$ is the algebra of elements which are not annihilated by e_i, but are annihilated by e_{i+1}. Taking this perspective, the chain puts an ordering on the elements of $\mathcal{A}$ as well as on the set of $\mathcal{A}$-modules. In fact, [7] proves that the existence of a heredity chain is sufficient for an algebra to admit a Kazhdan-Lusztig type basis and representation theory.

We mentioned that data about $\mathcal{A}$-modules could be obtained by applying Green's G functor to simple $e_i\mathcal{A}_ie_i$-modules. The remainder of this subsection is devoted to describing that process.

Definition 7.2.8. Given quasi-heredity algebra $\mathcal{A}$ with heredity chain $(e_1, \ldots, e_l)$ and a simple module $A_i^\lambda \subseteq e_i\mathcal{A}_ie_i$, the *standard module* $\Delta(\lambda)$ is defined to be the $\mathcal{A}$-module obtained by applying Green's G functor to A_i^λ. That is,

$$\Delta(\lambda) = \mathcal{A}_i e_i \otimes_{e_i \mathcal{A}_i e_i} A_i^\lambda$$

The algebra $\mathcal{A}$ acts on $\Delta(\lambda)$ by $a(x \otimes y) = (ax \otimes y)$.

Remark 7.2.9. This definition of standard modules is slightly different than the one usually found in the literature. Typically, one shows that a heredity chain can be refined to a heredity chain where each $e_i \mathcal{A}_i e_i$ is simple (not merely semisimple). Given such a heredity chain, the standard modules are of the form

$$G(e_i \mathcal{A}_i e_i) = \mathcal{A}_i e_i \otimes_{e_i \mathcal{A}_i e_i} e_i \mathcal{A}_i e_i \cong \mathcal{A}_i e_i.$$

To achieve the refinement of the heredity chain, one notes that if $e_i \mathcal{A}_i e_i$ is semisimple there are idempotents $f_{i,j}$ which are minimal, orthogonal, and nonzero, such that

$$e_i = \sum_j f_{i,j}.$$

Then, it is shown that e_i in chain $(e_1, \ldots, e_l)$ can be replaced with the $f_{i,j}$'s in any order. While this perspective gives the standard modules a more elegant description, it does not necessarily make life easier. To refine a chain $(e_1, \ldots, e_l)$, it is effectively necessary to find all of the simple modules of $e_i \mathcal{A}_i e_i$. So, no work is saved at that step. Then, after the chain is refined, it becomes necessary to compute more quotient algebras than using Definition 7.2.8.

Lemma 7.2.6 showed that the quasi-hereditary structure of $\mathcal{A}$ is inherited by the algebra $e_m \mathcal{A} e_m$. The following lemma shows that using Green's F and G functors to pass between these two algebras respects the set of standard modules. This fact shows an advantage to working with standard modules over working with simple modules; the result of applying the G functor to a simple $e_m \mathcal{A} e_m$-module is not necessarily a simple $\mathcal{A}$-module. In the context of $P_k(n)$, the F and G functors cause the k parameter to vary while keeping n fixed. Thus, the dimension of a $P_k(n)$ standard module will depend only on k and not on n.

Lemma 7.2.10. *Let $\mathcal{A}$ be a quasi-hereditary algebra with heredity chain $(e_1, \ldots, e_l)$ and let $\Delta(\lambda) = \mathcal{A}_i e_i \otimes_{e_i \mathcal{A}_i e_i} A_i^\lambda$ be a standard module of $\mathcal{A}$. If $m \leq i$, then $e_m \Delta(\lambda)$ is a standard module of $e_m \mathcal{A} e_m$ and*

$$\mathcal{A} e_m \otimes_{e_m \mathcal{A} e_m} e_m \Delta(\lambda) = \Delta(\lambda).$$

If $m > i$, then $e_m \Delta(\lambda) = 0$.

The following set of theorems describe some useful properties of standard modules. They are proved by Cline, Parshall, and Scott in [6] and [7] by relating quasi-hereditary algebras to highest weight categories. The proofs of these facts rely on the recollement of derived categories induced by heredity idempotents.

Theorem 7.2.11. *Let $\mathcal{A}$ be a quasi-hereditary algebra. The standard module $\Delta(\lambda)$ contains a unique maximal submodule $U(\lambda)$. If $\Delta(\lambda)/U(\lambda)$ is nonzero, it is a simple $\mathcal{A}$-module. Distinct standard modules give rise to distinct simple $\mathcal{A}$-modules.*

Because $F(\Delta(\lambda)) = e_i\Delta(\lambda)$ is simple, the ideal $U(\lambda)$ must be annihilated by e_i. Then, $U(\lambda)$ must also be annihilated by the ideal of $\mathcal{A}$ generated by e_i, which is $\mathcal{A}e_i\mathcal{A}$. Since the subset of $\Delta(\lambda)$ which is annihilated by $\mathcal{A}e_i\mathcal{A}$ forms an ideal and Theorem 7.2.11 asserts that $\Delta(\lambda)$ contains a unique maximal submodule, $U(\lambda)$ is exactly the ideal of $\Delta(\lambda)$ which is annihilated by $\mathcal{A}e_i\mathcal{A}$.

The next theorem constrains the morphisms of distinct standard modules. It should be thought of as a fancy version of Schur's Lemma (Lemma 2.2.1). In the case where $U(\mu) = 0$, one recovers an extremely overpowered statement and proof of the portion Schur's Lemma which says that there are no nonzero morphisms between distinct simple modules.

Theorem 7.2.12. *Suppose $\lambda \neq \mu$. If there exists a nonzero homomorphism of standard modules $\Delta(\lambda) \to \Delta(\mu)$, the image of $\Delta(\lambda)$ in $\Delta(\mu)$ is contained in the ideal $U(\mu)$.*

Finally, we are able to construct homological resolutions of any simple $\mathcal{A}$-module using only standard modules. These resolutions respect the order the heredity chain placed on the set of standard modules.

Theorem 7.2.13. *Let $L(\lambda_1)$ be the unique simple $\mathcal{A}$-module contained in $\Delta(\lambda_1)$. Then, $L(\lambda_1)$ has a resolution of the form*

$$0 \longrightarrow \Delta(\lambda_m) \longrightarrow \cdots \longrightarrow \Delta(\lambda_2) \longrightarrow \Delta(\lambda_1) \longrightarrow L(\lambda_1) \longrightarrow 0.$$

If $\Delta(\lambda_i) = G\left(A_j^{\lambda_i}\right)$ and $\Delta(\lambda_{i+1}) = G\left(A_k^{\lambda_{i+1}}\right)$, then $k < j$.

7.3 $P_k(n)$ as a Quasi-hereditary Algebra

This section proves that the partition algebra $P_k(n)$ is quasi-hereditary and describes how the results from the previous subsection manifest themselves in the context of $P_k(n)$. This section employs many of the elements, ideals, and quotients described in Section 4.3.

Theorem 7.3.1. *The partition algebra $P_k(n)$ is quasi-hereditary with hereditary chain $(E_k, E_{k-1}, \ldots E_1)$.*

Proof. Points (i) and (ii) in Definition 7.2.4 are clear. Recall from Section 4.3 that $P_k(n)E_iP_k(n)$ is the subalgebra of $P_k(n)$ consisting of partitions with propagating number $\leq i$. Point (iii) follows immediately from this this observation. It remains to show point (iv), that E_i is a heredity idempotent in $P_k(n)/\left(P_k(n)E_{i-1}P_k(n)\right)$.

Recall that $P_k(n)/\left(P_k(n)E_{i-1}P_k(n)\right)$ is the algebra consisting of all partitions with propagating number $\geq i$, and that $E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i$ is the algebra consisting of partitions which have propagating number exactly i, where only the leftmost i dots on both the top and bottom row propagate. Hence,

$$E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i \cong \mathbb{C}\left[\mathfrak{S}_i\right],$$

which is semisimple.

It remains to show that the multiplication map provides a bijection between

$$\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i \otimes_{\mathbb{C}[\mathfrak{S}_i]} E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)$$

and

$$\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right).$$

Note that $\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)$ has a basis consisting of partitions with propagating number exactly i where it does not matter which dots propagate.

The quotient $\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i$ has a basis consisting of partitions with propagating number exactly i, where the only dots on the bottom row that propagate are the leftmost i dots and where the rightmost $k-i$ dots on the bottom row are in a single part. These partitions may be pictured as a distribution of some subset of the k top row dots into boxes labeled $1, 2, \ldots, i$ such that each box is nonempty, plus a distribution of the leftover top row dots into any number of additional unlabeled boxes. Recall from Section 4.3 that $\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i$ has the structure of a $\mathbb{C}[\mathfrak{S}_n]$-module. This action is the linear extension of the action obtained by letting permutation $\sigma \in \mathfrak{S}_i$ acts on a basis element of $\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i$ by applying the permutation σ to the labels of the labeled boxes.

Similarly, $E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)$ has a basis consisting of partitions with propagating number exactly i where the only dots on the top row that propagate are the leftmost i dots and where the rightmost $k-i$ dots on the top row form a single part. These partitions may be pictured as a distribution of some subset of the k dots on the bottom row into boxes labeled $1', 2', \ldots, i'$ such that each box is nonempty, plus a distribution of the leftover bottom row dots into any number of additional unlabeled boxes. Similarly, $E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)$ has the structure of a $\mathbb{C}[\mathfrak{S}_n]$-module given by letting a permutation $\sigma \in \mathfrak{S}_i$ acts on a basis element by applying the permutation σ^{-1} to the labels of the labeled boxes. This action extends linearly to an action of $\mathbb{C}[\mathfrak{S}_i]$ on $E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)$.

Let $x, y \in P_k(n)/\left(P_k(n)E_{i-1}P_k(n)\right)$ be basis elements such that $xE_i, E_iy \neq 0$. For all $\sigma \in \mathfrak{S}_i$,

$$xE_i \otimes_{\mathbb{C}[\mathfrak{S}_i]} E_iy = \sigma(xE_i) \otimes_{\mathbb{C}[\mathfrak{S}_i]} \sigma^{-1}(E_iy).$$

If a top row dot was in box j in xE_i, it will be in box $\sigma(j)$ in $\sigma(xE_i)$. Likewise, if a bottom row dot in box j' in E_iy, it will be in box $\sigma(j)'$ in $\sigma^{-1}(E_iy)$. So, taking the tensor product over $\mathbb{C}\left[\mathfrak{S}_i\right]$ may then be viewed combining the contents of box 1 and 1', of box 2 and 2', and so on and then forgetting about the labeling. The data from the non-propagating dots on the top row of xE_i and the bottom row of E_iy is retained. For basis elements $z, w \in P_k(n)/\left(P_k(n)E_{i-1}P_k(n)\right)$,

$$xE_i \otimes_{\mathbb{C}[\mathfrak{S}_i]} E_iy = zE_i \otimes_{\mathbb{C}[\mathfrak{S}_i]} E_iw$$

if and only if each side has the same image under the box combining operation.

The result the box combining operation is a distribution of top and bottom row dots into i unlabeled parts such that each part contains at least one top row dot and at least one bottom row dot, plus some non-propagating blocks. This is exactly a partition in $P_k(n)$ with propagating number i. In fact, it is exactly the same partition with propagating number i obtained by the multiplication xE_iy. Since distinct basis elements of $\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i \otimes_{\mathbb{C}[\mathfrak{S}_i]} E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)$ have distinct images under the box combining operation, the multiplication map is injective. Since for any basis element $aE_ib \in \left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)$, the element $aE_i \otimes_{\mathbb{C}[\mathfrak{S}_i]} E_ib$ is a basis element of $\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)E_i \otimes_{\mathbb{C}[\mathfrak{S}_i]} E_i\left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right)$, the multiplication is also sujective. Thus the multiplication map is a bijection and E_i is a heredity idempotent in $P_k(n)/(P_k(n)E_{i-1}P_k(n))$. This completes the proof that the partition algebra is quasi-hereditary. $\square$

Repeated application of Theorem 7.2.2 from the previous section yields the following stratification of $D^-\left(P_k(n)\right)$.

$$\begin{tikzcd}
D^-(0) \arrow[r, shift left=2] & D^-\left(\frac{P_k(n)}{P_k(n)E_{k-1}P_k(n)}\right) \arrow[l, shift left] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[d] & \cdots \arrow[l, shift left] \arrow[l, shift right=2] \arrow[r, shift left=2] & D^-\left(\frac{P_k(n)}{P_k(n)E_1P_k(n)}\right) \arrow[l, shift left] \arrow[l, shift right=2] \arrow[r, shift left=2] \arrow[d] & D^-\left(P_k(n)\right) \arrow[l, shift left] \arrow[l, shift right=2] \arrow[d] \\
 & D^-\left(\mathbb{C}\left[\mathfrak{S}_k\right]\right) \arrow[u, bend left] \arrow[u, bend right] & & D^-\left(\mathbb{C}\left[\mathfrak{S}_2\right]\right) \arrow[u, bend left] \arrow[u, bend right] & D^-\left(\mathbb{C}\left[\mathfrak{S}_1\right]\right) \arrow[u, bend left] \arrow[u, bend right]
\end{tikzcd}$$

In the top horizontal row, all of the arrow pointing to the right are inclusion functors and the arrows pointing to the right are the left and right adjoints to inclusion (one of these notably is restriction). The arrows pointing downward are all Green's F functor for some idempotent and the arrows pointing upward are F's left and right adjoints (one of these notably is G).

For any $\lambda \vdash i$ with $i \leq k$, the module

$$\Delta_k(\lambda) = \left(\frac{P_k(n)}{P_k(n)E_{i-1}P_k(n)}\right) E_i \otimes_{\mathbb{C}[\mathfrak{S}_i]} S^\lambda$$

is a standard module for $P_k(n)$, where S^λ is the Specht module indexed by λ. By convention, $E_0 = 0$. The results from the previous section say that each of standard module contains a unique maximal submodule $U_k(\lambda)$ and that $\Delta_k(\lambda)/U_k(\lambda)$ is a simple $P_k(n)$-module for each λ. This simple module will be denoted by $L_k(\lambda)$. The next chapter will be devoted to analyzing the homological resolution of every simple $P_k(n)$-module guaranteed by Theorem 7.2.13.

Chapter 8

The n-pair Condition

This chapter analyzes the homological resolution of a simple $P_k(n)$-module by standard modules guaranteed by Theorem 7.2.13. The result is a combinatorial description of these resolutions in terms of n-pairs. Recall from Definition 5.0.5 that a pair of Ferrers shapes (μ, λ) is an n-pair if λ/μ is a horizontal row whose rightmost box has content $n - |\mu|$. A chain of n-pairs $\lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots \subset \lambda^{(t)}$ is maximal if it cannot be extended on the right. The resolution of simple $P_k(n)$-modules is given by the following previously stated theorem.

Theorem 5.0.7. *If $\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(t)}$ is a maximal chain of n-pairs, then there is an exact sequence of $P_k(n)$ modules*

$$0 \to \Delta_k\left(\lambda^{(t)}\right) \to \cdots \to \Delta_k\left(\lambda^{(i+1)}\right) \to \Delta_k\left(\lambda^{(i)}\right) \to L_k\left(\lambda^{(i)}\right) \to 0$$

Where $L_k\left(\lambda^{(i)}\right)$ is the unique simple $P_k(n)$-module contained in the standard module $\Delta_k\left(\lambda^{(i)}\right)$.

This theorem is equivalent to the following theorem, which this chapter is devoted to sketching a proof of.

Theorem 8.0.2 (The n-pair Condition)**.** *Let λ and μ be partitions of size $\leq k$, and $\lambda \neq \mu$. There is a nonzero mapping of $P_k(n)$ standard modules $\Delta_k(\lambda) \to \Delta_k(\mu)$ if and only if (μ, λ) is an n-pair. If such a mapping exists, it is injective, and it is unique.*

Unfortunately, the proofs of the n-pair condition appearing in the literature involve an unavoidable quantity of technical manipulation. The original proof, due to Martin appears in [24], occupies a substantial percentage of a 40-page article and contains some errors.

A simplified proof due to Doran and Wales appears in [11], but their simplification of explanation comes at the cost of making the proof even longer than Martin's. Rather than diverge into a protracted technical manipulation, this section gives a high-level summary of Doran and Wales's proof of the n-pair condition. The reader interested in minutia is referred to [11].

Our goal is to explain when there is an embedding of the $P_k(n)$ standard module $\Delta_k(\lambda)$ into the standard module $\Delta_k(\mu)$. By Theorem 7.2.12, such an embedding can only exist if $|\lambda| > |\mu|$. By Lemma 7.2.10, we only need to consider the case when $\lambda \vdash k$.

Let $\lambda \vdash k$ and $\mu \vdash m$ for some $m \leq k$. Consider the standard module $\Delta_k(\mu)$ of $P_k(n)$. By Theorem 7.2.12, if there exists a nonzero homomorphism from $\Delta_k(\lambda)$ to $\Delta_k(\mu)$ the image of $\Delta_k(\lambda)$ will be contained in the ideal $U_k(\mu) \subseteq \Delta_k(\mu)$. We begin by exhibiting a series of necessary conditions, the aggregate of which is the n-pair condition, for such an embedding to exist.

Proposition 8.0.3. *If there is an embedding $\Delta_k(\lambda) \to \Delta_k(\mu)$, then $\mu \subset \lambda$ and all the boxes in the skew shape λ/μ lie in different columns. Further, the embedding is unique.*

By the comment following Theorem 7.2.11, the ideal $U_k(\mu) \subseteq \Delta_k(\mu)$ is exactly the subset of $\Delta_k(\mu)$ which is annihilated by $P_k(n)E_mP_k(n)$. Recall that $x \in P_k(n)$ acts on $\Delta_k(\mu)$ by

$$x\left(\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m \otimes_{\mathbb{C}[\mathfrak{S}_m]} S^\mu\right) = \left(x\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m\right) \otimes_{\mathbb{C}[\mathfrak{S}_m]} S^\mu.$$

Recall from Section 4.3 that multiplication by the element $A^{i,j} \in P_k(n)$ fuses two parts of whatever partition it is multiplied with.

Lemma 8.0.4. *If an element of $\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m$ is in $U_k(\mu)$, then it is annihilated by $A^{i,j}$ for every $1 \leq i < j \leq k$.*

Proof. Picking appropriate permutations $\sigma_1, \sigma_2, \ldots, \sigma_{k-m-1}$ of k elements, which may naturally be viewed as elements of $P_k(n)$,

$$A^{i,j}\sigma_1 A^{i,j}\sigma_2 \cdots A^{i,j}\sigma_{k-m-1}A^{i,j}$$

will have propagating number m. Thus, $A^{i,j}\sigma_1 A^{i,j}\sigma_2 \cdots A^{i,j}\sigma_{k-m-1}A^{i,j}$ annihilates every element of Ψ_k^m. Since permutations do not annihilate any element, $A^{i,j}$ must annihilate every element of Ψ_k^m. $\square$

Let Ψ_k^m be the subset of $\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m$ which is annihilated by each $A^{i,j}$. Then,

$$U_k(\mu) \subseteq \Psi_k^m \otimes_{\mathbb{C}[\mathfrak{S}_m]} S^\mu.$$

Recall that $\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m$ has a basis indexed by partitions of propagating number exactly m where exactly the leftmost m dots on the bottom row propagate. Let x be a basis element of $\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m$. Then, $A^{i,j}x$ is either 0 or another basis element. So, the $A^{i,j}$'s put a partial order on the set of basis elements of $\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m$ where $x \lessdot y$ if and only if $y = A^{i,j}x$ for some i and j. The minimal elements of this poset are partitions where each of the m leftmost bottom row dots are connected to exactly one distinct top row dot. This set of minimal elements is denoted by M. Define for each map $h : M \to \mathbb{C}$ the element in $\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m$

$$\chi(h) = \sum_x \sum_{y \in M} h(y)\mu(y,x)x,$$

where the first sum is across basis elements of $\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m$ and μ denotes the Möbius function on the poset defined by the $A^{i,j}$'s. An application of Weisner's Theorem about posets (see [34]) shows that Ψ_k^m is exactly the set of elements of the form $\chi(h)$ for some $h : M \to \mathbb{C}$. So, Ψ_k^m has a basis given by functions $h_y : M \to \mathbb{C}$ for each $y \in M$ where

$$h_y(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

Note that $\mathfrak{S}_k \times \mathfrak{S}_m$ acts on M by letting $\mathfrak{S}_k$ permute the k dots on the top row of a partition and letting $\mathfrak{S}_m$ acts by permuting the m dots on the bottom row. This action is transitive, and gives a $\mathfrak{S}_k \times \mathfrak{S}_m$ action on the set of h_y's. We extend this action to a transitive $\mathbb{C}[\mathfrak{S}_k \times \mathfrak{S}_m]$ action on Ψ_k^m.

The stabilizer of a given $y \in M$ of the $\mathfrak{S}_k \times \mathfrak{S}_m$ action is isomorphic to $\mathfrak{S}_m \times \mathfrak{S}_{k-m}$. Here, $\mathfrak{S}_m$ acts by permuting the leftmost m dots on the bottom row and applying the inverse permutation to the m dots on the top row, and $\mathfrak{S}_{k-m}$ acts by permuting the $k-m$ non-propagating dots on the the row. It follows that the stabilizer of the $\mathbb{C}[\mathfrak{S}_k \times \mathfrak{S}_m]$ on Ψ_k^m is isomorphic to $\mathbb{C}[\mathfrak{S}_m \times \mathfrak{S}_{k-m}]$. Thus,

$$\Psi_k^m \cong \operatorname{Ind}_{\mathbb{C}[\mathfrak{S}_m \times \mathfrak{S}_{k-m}]}^{\mathbb{C}[\mathfrak{S}_k \times \mathfrak{S}_m]} \left(1_{\mathbb{C}[\mathfrak{S}_m \times \mathfrak{S}_{k-m}]}\right),$$

where $1_{\mathbb{C}[\mathfrak{S}_m \times \mathfrak{S}_{k-m}]}$ denotes the trivial representation of $\mathbb{C}\left[\mathfrak{S}_m \times \mathfrak{S}_{k-m}\right]$. A series of computations shows that, as a $\mathfrak{S}_k \times \mathfrak{S}_m$ module,

$$\Psi_k^m \cong \bigoplus_{\xi \vdash m} \mathrm{Ind}_{\mathfrak{S}_m \times \mathfrak{S}_{k-m}}^{\mathfrak{S}_k} \left(S^\xi \otimes S^{(k-m)}\right) \times S^\xi$$

$$\cong \sum_{(\nu,\xi)} S^\nu \times S^\xi,$$

where the sum in the second line is across pairs of partitions (ν, ξ) such that $c^\nu_{\xi,(k-m)} = 1$. So,

$$\Psi_k^m \otimes S^\mu = \left(\sum_{(\nu,\xi)} S^\nu \times S^\xi\right) \otimes_{\mathbb{C}[\mathfrak{S}_m]} S^\mu.$$

Here, $\sum_{(\nu,\xi)} S^\nu \times S^\xi$ is considered as a left $\mathbb{C}[\mathfrak{S}_k]$-module and right $\mathbb{C}[\mathfrak{S}_m]$-module. The tensor product is over the $\mathbb{C}[\mathfrak{S}_m]$-module structure. For each ξ, $S^\xi = F_\xi \mathbb{C}[\mathfrak{S}_m] F_\xi$. We may pull the F_ξ on the right through the tensor product. Since $S^\mu = F_\mu \mathbb{C}[\mathfrak{S}_m] F_\mu$ and, by Theorem 2.5.16, the F_τ's are a family of orthogonal idempotents, the only nonzero term in the occurs when $\xi = \mu$. Hence,

$$\Psi_k^m \otimes S^\mu = \sum_\nu (S^\nu \times S^\mu) \otimes_{\mathbb{C}[\mathfrak{S}_m]} S^\mu,$$

where the sum is across $\nu \vdash k$ such that $c^\nu_{\mu,(k-m)} = 1$. If $\Delta_k(\lambda)$ embeds into $\Psi_k^m \otimes S^\mu$, its image must be isomorphic to S^λ as a left module. Hence, $\Delta_k(\lambda)$ can only be contained in the term of the above sum where $\nu = \lambda$. This term occurs only if $c^\lambda_{\mu,(k-m)} = 1$. By Example 2.5.23, $c^\lambda_{\mu,(k-m)} = 1$ if and only if λ/μ has no two boxes in the same column. Furthermore, it can be shown that the Littlewood-Richardson number $c^\lambda_{\mu,(k-m)}$ bounds the multiplicity of the embedding. So, if such an embedding exists, it is unique. This completes our sketch of Doran and Wales's proof of Proposition 8.0.3.

Note that Proposition 8.0.3 only tells us that the boxes of λ/μ lie in different columns, while the n-condition further requires that the boxes of λ/μ all lie in the same row. The fact that λ/μ is indeed a row of boxes is established inductively using tools developed in the proof of the next proposition, which shows in the case that $|\lambda/\mu| = 1$ that if there is an embedding of $\Delta_k(\lambda)$ into $\Delta_k(\mu)$, then λ and μ must be an n-pair.

Proposition 8.0.5. *If there is an embedding of $\Delta_k(\lambda)$ into $\Delta_k(\mu)$ where $\lambda \vdash k$ and $\mu \vdash k-1$, then λ/μ consists of a single box and $n - |\mu| - c(\lambda/\mu) = 0$. Here, $c(\lambda/\mu)$ denotes* *the content of the box λ/μ.*

The fact that λ/μ consists of a single box follows from Proposition 8.0.3. Recall from Section 4.3 that multiplication by the element $A^i \in P_k(n)$ disconnects a dot from whatever partition it is multiplied. Again, these elements annihilate $U_k(\lambda)$.

Lemma 8.0.6. *If an element of $\frac{P_k(n)}{P_k(n)E_{m-1}P_k(n)}E_m$ is annihilated by $P_k(n)E_mP_k(n)$, then it is annihilated by A^i for every $1 \leq i \leq k$.*

Proof. The proof of this lemma is identical to the proof of Lemma 8.0.4. □

Proposition 8.0.5 is proved using this lemma. From Proposition 8.0.3, we know that if there is an embedding of $\Delta_k(\lambda)$ in $\Delta_k(\mu)$, its image is in the space $\Psi_k^{k-1} \otimes S^\mu$. Consider the partial ordering on the basis of $\frac{P_k(n)}{P_k(n)E_{k-2}P_k(n)}E_{k-1}$ we used in our sketch of the proof of Proposition 8.0.3. The Hasse diagram for this poset is only two levels high. The minimal elements consist of a pairing of the leftmost $k-1$ dots on the bottom row of a partition with some $k-1$ dots on the top row. These minimal elements may be notated by a permutation and a specification of which element in the top row is isolated. Denote such a minimal element by $W_{\sigma,i}$, where i specifies which dot on the top row is isolated, and σ is a permutation matching the remaining $k-1$ dots on the top row to the leftmost $k-1$ dots on the bottom row. Additionally, let $W^j_{\sigma,i} = A^{i,j}W_{\sigma,i}$ in the cases where $A^{i,j}W_{\sigma,i}$ is nonzero. Applying the same Möbius inversion trick from before, the elements

$$X_{\sigma,i} = W_{\sigma,i} - \sum_{j \neq i} W^j_{\sigma,i}$$

form a basis for Ψ_k^{k-1}.

We examine the action of A^j on this basis. Unless $j = i$, the element $A^jW_{\sigma,i}$ will have propagating number $k-2$ and thus will be zero in $\frac{P_k(n)}{P_k(n)E_{k-2}P_k(n)}E_{k-1}$. If $j = i$, then $A^jW_{\sigma,i} = nW_{\sigma,i}$. If $j \neq i$ or l, then $A^jW^l_{\sigma,i}$ has propagating number $k-2$ and thus is zero in $\frac{P_k(n)}{P_k(n)E_{k-2}P_k(n)}E_{k-1}$. If $j = i$, then $A^jW^l_{\sigma,i} = W_{\sigma,l}$. Likewise, if $j = l$, then $A^jW^l_{\sigma,i} = W_{\sigma,i}$. Putting these facts together A^j acts on the Ψ_k^{k-1} basis element by

$$A^jX_{\sigma,i} = \begin{cases} -W_{\sigma,j} & \text{if } j \neq i, \\ (n-(k-1))W_{\sigma,i} & \text{if } j = i. \end{cases}$$

Since the image of $\Delta_k(\lambda)$ in $\Psi_k^{k-1} \otimes S^\mu$ is annihilated by each A^i, this image will additionally be annihilated by $T = \sum_{j=1}^k A^j$. The element T acts on a Ψ_k^{k-1} basis element by

$$\begin{aligned} TX_{\sigma,i} &= \sum_{j=1}^{k} A^j X_{\sigma,i} \\ &= (n-(k-1))W_{\sigma,i} - \sum_{j\neq i} W_{\sigma,j}. \end{aligned}$$

Note that Ψ_k^{k-1} has the structure of a left $\mathbb{C}[\mathfrak{S}_k]$-module (where $\mathfrak{S}_k$ acts by permuting the k dots on the top row). This action gives

$$(i,j)W_{\sigma,i} = W_{\sigma,j}.$$

Hence,

$$\sum_{j\neq i} W_{\sigma,j} = \sum_{i\neq j}(i,j)W_{\sigma,i}.$$

Notice that Ψ_k^{k-1} also has the structure of a right $\mathbb{C}[\mathfrak{S}_{k-1}]$-module (where $\mathfrak{S}_{k-1}$ acts by permuting the leftmost $k-1$ dots on the bottom row). Notice that for any $1 \leq i < l \leq k$ with $i, l \neq j$ there exists a unique transposition $\tau \in \mathfrak{S}_{k-1}$ such that

$$(i,l)W_{\sigma,j} = W_{\sigma,j}\tau.$$

Different transpositions (i,l) require different transpositions τ for the above identity to hold. Thus,

$$\sum_{\substack{1\leq i<l\leq k \\ i,l\neq j}} (i,l)W_{\sigma,i} = \sum_{1\leq i<l\leq k-1} W_{\sigma,i}(i,l).$$

This identity allows Doran and Wales to introduce some clever redundancy into their expression for $TX_{\sigma,i}$. Since

$$\sum_{i\neq j}(i,j)W_{\sigma,i} = \sum_{i\neq j}(i,j)W_{\sigma,i} + \sum_{\substack{1\leq i<l\leq k \\ i,l\neq j}} (i,l)W_{\sigma,i} - \sum_{1\leq i<l\leq k-1} W_{\sigma,i}(i,l),$$

they may write

$$TX_{\sigma,i} = (n-(k-1))W_{\sigma,i} - \sum_{1\leq i<l\leq k} (i,l)W_{\sigma,i} + \sum_{1\leq i<l\leq k-1} W_{\sigma,i}(i,l).$$

Notice that $\sum_{1\le i<l\le k}(i,l)$ and $\sum_{1\le i<l\le k-1}(i,l)$ are the sums of all of the Jucys-Murphy elements in $\mathbb{C}[\mathfrak{S}_k]$ and $\mathbb{C}[\mathfrak{S}_{k-1}]$ respectively. Consider some element x contained in the image of $\Delta_k(\lambda)$ in $\Delta_k(\mu)$. Since $\Delta_k(\mu) = \Psi_n^{n-1} \otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]} S^\mu$, we may pull the action of $\sum_{1\le i<l\le k-1}(i,l)$ through the tensor product so that

$$T\left(X_{\sigma,i} \otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]} S^\mu\right)$$

$$= \left((n-(k-1))W_{\sigma,i} - \sum_{1\le i<l\le k}(i,l)W_{\sigma,i} + W_{\sigma,i}\right) \otimes_{\mathbb{C}[\mathfrak{S}_{n-1}]} \left(\sum_{1\le i<l\le k-1}(i,l)S^\mu\right).$$

Since $\sum_{1\le i<l\le k-1}(i,l)$ is a sum of Jucys-Murphy elements,

$$\sum_{1\le i<l\le k-1}(i,l)S^\mu = \sum_{1\le i\le k-1} c_\mu\left(\boxed{i}\right).$$

Similarly, $\sum_{1\le i<l\le k}(i,l)$ acts on the image of $\Delta_k(\lambda) = \left(\frac{P_k(n)}{P_k(n)E_{k-1}P_k(n)}\right) E_k \otimes_{\mathbb{C}[\mathfrak{S}_k]} S^\lambda$ in $\Delta_k(\mu)$ by

$$\left(\sum_{1\le j<l\le k}(j,l)\frac{P_k(n)}{P_k(n)E_{k-1}P_k(n)}E_k\right)\otimes_{\mathbb{C}[\mathfrak{S}_k]}S^\lambda = \left(\frac{P_k(n)}{P_k(n)E_{k-1}P_k(n)}E_k\right)\otimes_{\mathbb{C}[\mathfrak{S}_k]}\left(\sum_{1\le j<l\le k}(j,l)S^\lambda\right)$$

$$= \sum_{1\le j\le k} c_\lambda\left(\boxed{j}\right).$$

Thus, T acts the image of $\Delta_k(\lambda)$ in $\Delta_k(\mu)$ by

$$T\Delta_k(\lambda) = \left((n-(k-1)) + \sum_{1\le i\le k-1} c_\mu\left(\boxed{i}\right) - \sum_{1\le j\le k} c_\lambda\left(\boxed{j}\right)\right)\Delta_k(\lambda).$$

However, T must annihilate the image of $\Delta_k(\lambda)$ in $\Delta_k(\mu)$. Thus,

$$(n-(k-1)) + \sum_{1\le i\le k-1} c_\mu\left(\boxed{i}\right) - \sum_{1\le j\le k} c_\lambda\left(\boxed{j}\right) = 0.$$

Since $|\lambda| - |\mu| = 1$ and $\mu \subseteq \lambda$,

$$\sum_{1\le i\le k-1} c_\mu\left(\boxed{i}\right) - \sum_{1\le j\le k} c_\lambda\left(\boxed{j}\right)$$

is exactly the content of the unique box in λ/μ, and we have established Proposition 8.0.5.

The fact that the n-pair condition is a necessary for the existence of an embedding of $\Delta_k(\lambda)$ into $\Delta_k(\mu)$ is established in the case when $|\lambda/\mu| = 2$ in a similar fashion as the proof of Proposition 8.0.5. Once again, they note that the image of $\Delta_k(\lambda)$ in $\Delta_k(\mu)$ must be annihilated by T and explicitly compute the action of T on these spaces. This computation however is much grittier than the previous computation due to the extra complexity of the poset structure of Ψ_k^{k-2} as compared to Ψ_k^{k-1}.

In the case where $|\lambda/\mu| \geq 3$, they are able to get away with a simpler computation by bootstrapping their results from the cases where $|\lambda/\mu| = 1$ and $|\lambda/\mu| = 2$ up inductively. To accomplish this task, they examine the action of T on the bottom two levels of the poset Ψ_k^l and establish the following weaker identity, which also holds in the cases where $|\lambda/\mu| = 1$ and $|\lambda/\mu| = 2$.

Proposition 8.0.7. *If $\lambda \vdash k$, $\mu \vdash l$, and there is an embedding $\Delta_k(\lambda)$ into $\Delta_k(\mu)$, then $n = l + \frac{1}{k-l}\left(\sum_{\square\in\lambda/\mu} c_\lambda(\square)\right) + \frac{k-l-1}{2}$.*

Note that in the case where $|\lambda/\mu| = 1$ that this proposition is identical to Proposition 8.0.5. Further, if λ/μ is known to be a row of boxes, then, letting b be the rightmost box in λ/μ,

$$\sum_{\square\in\lambda/\mu} c_\lambda(\square) = (k-l)c_\lambda(b) - \binom{k-l}{2}.$$

So,

$$\begin{aligned} n &= l + \frac{1}{k-l}\left((k-1)c_\lambda(b) - \frac{(k-l)(k-l-1)}{2}\right) - \frac{k-l-1}{2} \\ &= l + c_\lambda(b), \end{aligned}$$

which is exactly the n-pair condition. With Proposition 8.0.7 in hand, it only remains to establish that λ/μ is a row of boxes. This is done by computing the restrictions $\mathrm{Res}^{P_k(n)}_{P_{k-1}(n)}(\Delta_k(\lambda))$ and $\mathrm{Res}^{P_k(n)}_{P_{k-1}(n)}(\Delta_k(\mu))$. In the case where $P_k(n)$ is semisimple, the restriction rule may be read off from the Bratteli diagram construction in Chapter 6. In fact, Martin mistakenly uses this restriction rule when performing a similar computation in the non-semisimple case in [26]. The correct restriction rule in the non-semisimple case may be found in Section 8 of [11]. In essence, their computation shows that, if λ' and μ' index standard constituents of $\mathrm{Res}^{P_k(n)}_{P_{k-1}(n)}(\Delta_k(\lambda))$ and $\mathrm{Res}^{P_k(n)}_{P_{k-1}(n)}(\Delta_k(\mu))$ respectively, that λ'/μ' is a subdiagram of λ/μ. In fact, it differs by a box. Inducting on $|\lambda/\mu|$, the diagram λ'/μ' must be a row of boxes. A computation shows that the only place a box can be added to

λ'/μ' to build λ/μ is at the right end of the strip. It follows that λ/μ is a row of boxes, and then Proposition 8.0.7 tells us λ and μ form a n-pair.

It remains to show that the n-pair condition is sufficient for an embedding of $\Delta_k(\lambda)$ into $\Delta_k(\mu)$ to exist. Again, this fact is proved by induction on $|\lambda/\mu|$. The following proposition serves as the base case for this induction.

Proposition 8.0.8. *If $|\lambda/\mu| = 1$ and that λ and μ form an n-pair, then there is an embedding of $\Delta_k(\lambda)$ into $\Delta_k(\mu)$.*

Note that if we find a space inside of $U_k(\mu)$ which is isomorphic to S^λ as a $\mathbb{C}[\mathfrak{S}_k]$-module, then there will be an embedding of $\Delta_k(\lambda)$ in $\Delta_k(\mu)$.

Lemma 8.0.9. *If an element of $\Delta_k(\mu)$ is annihilated by each $A^{i,j}$ and each A^i, then it is in $U_k(\mu)$.*

Proof. This lemma follows from the facts that $U_k(\mu)$ is the subset of $\Delta_k(\mu)$ which is annihilated by $P_k(n)E_mP_k(n)$ and that any partition with propagating number $\leq m$ can be written as a product of A^i's, $A^{i,j}$'s, and permutations. $\square$

So, our task is to find a submodule of $\Delta_k(\mu)$ which is isomorphic to S^λ as a $\mathbb{C}[\mathfrak{S}_k]$-module, and which is annihilated by each $A^{i,j}$ and A^i. Since the submodule we are looking for is annihilated by each $A^{i,j}$, it must live inside the space

$$\Psi_n^{n-1} \otimes S^\mu = \sum_\sigma (S^\sigma \times S^\mu) \otimes S^\mu,$$

where again the sum is over σ such that $c^\sigma_{\mu,(1)} = 1$. Since the module we are looking for is isomorphic to S^λ as a $\mathbb{C}[\mathfrak{S}_k]$-module, it must be

$$\left(S^\lambda \times S^\mu\right) \otimes S^\mu.$$

This submodule of $\Psi_n^{n-1} \otimes S^\mu$ is given explicitly by $(F_\lambda F_\mu(\Psi_n^{n-1})) \otimes S^\mu$. Here, since $F_\mu \in \mathbb{C}[\mathfrak{S}_{k-1}]$, it acts on Ψ_n^{n-1} on the right. Since $F_\lambda \in \mathbb{C}[\mathfrak{S}_k]$, it acts on Ψ_n^{n-1} on the left. Using the basis of $X_{\sigma,i}$'s for Ψ_n^{n-1} that we employed previously, we may then find a basis for $\left(S^\lambda \times S^\mu\right) \otimes S^\mu \subseteq \Psi_n^{n-1} \otimes S^\mu$.

Since $\left(S^\lambda \times S^\mu\right) \otimes S^\mu$ lives inside $\Psi_n^{n-1} \otimes S^\mu$, it is annihilated by the $A^{i,j}$'s by construction. Further, $\left(S^\lambda \times S^\mu\right) \otimes S^\mu$ is clearly isomorphic to S^λ as a left $\mathbb{C}[\mathfrak{S}_k]$-module. So, all that is left to verify is that each basis element of $\left(S^\lambda \times S^\mu\right) \otimes S^\mu$ is annihilated by each A^i. This task is accomplished by associating a certain filling of the Ferrers shape λ to each basis element, then determining the action of A^j explicitly by performing a long computation using tableaux combinatorics.

This completes our sketch of the proof of Proposition 8.0.8. This proposition serves as a base case for an induction on $|\lambda/\mu|$ which shows that the n-pair condition is sufficient for there to exist a nontrivial homomorphism from $\Delta_k(\lambda)$ into $\Delta_k(\mu)$. As in establishing the n-pair condition as necessary, the inductive step requires computation of $\mathrm{Res}^{P_k(n)}_{P_{k-1}(n)}(\Delta_k(\mu))$.

This completes our sketch of Doran and Wales's proof of the n-pair condition. For fuller details, refer to [11]. In addition to giving us a means of computing the dimensions of the simple modules of $P_k(n)$, the n-pair condition finally gives us a characterization of the values of n for which $P_k(n)$ is semisimple.

Theorem 5.0.4. *If $n > 2k - 2$, then the partition algebra $P_k(n)$ is semisimple.*

Proof. If (μ, λ) is an n-pair, then $|\mu| + |\lambda| > n$. Since, $|\mu| < |\lambda| < k$, we see that $n < 2k - 1$. □

Chapter 9

Results about Kronecker Coefficients and Concluding Remarks

9.1 Results about Kronecker Coefficients

We are finally ready to apply our knowledge about the partition algebra to the Kronecker problem. The main results of this section are that reduced Kronecker coefficients are well defined, that Kronecker coefficients may be recovered as a sum of reduced Kronecker coefficients, and results about when reduced Kronecker coefficients are equal to either Kronecker coefficients or Littlewood-Richardson numbers. All of the results appearing in this section are previously known and have been proved using the partition algebra in [4].

The algebra $\mathbb{C}[\mathfrak{S}_n]$ acts diagonally on $(\mathbb{C}^n)^{\otimes k}$ and the algebra $\mathbb{C}[\mathfrak{S}_n] \times \mathbb{C}[\mathfrak{S}_n]$ acts on $(\mathbb{C}^n)^{\otimes l} \otimes (\mathbb{C}^n)^{\otimes k-l}$ by letting the first copy of $\mathbb{C}[\mathfrak{S}_n]$ act diagonally on $(\mathbb{C}^n)^{\otimes l}$ and letting the second copy of $\mathbb{C}[\mathfrak{S}_n]$ act diagonally on $(\mathbb{C}^n)^{\otimes k-l}$. These actions are centralized by $P_k(n)$ and $P_l(n) \times P_{k-l}(n)$ respectively. Thus,

$$\begin{array}{rcl} \mathbb{C}[\mathfrak{S}_n] & \times & P_k(n), \\ (\mathbb{C}[\mathfrak{S}_n] \times \mathbb{C}[\mathfrak{S}_n]) & \times & (P_l(n) \times P_{k-l}(n)) \end{array}$$

is a seesaw pair acting on $(\mathbb{C}^n)^{\otimes k}$. So, the Seesaw Reciprocity Theorem (Theorem 3.2.5) tells us that for $\lambda, \mu, \nu \vdash n$ with $|\lambda_{>1}| \leq l$, $|\mu_{>1}| \leq k-l$, and $|\nu_{>1}| \leq k$,

$$g^{\nu}_{\lambda,\mu} = \dim\left(\mathrm{Hom}_{\mathbb{C}[\mathfrak{S}_n]}\left(S^{\nu}, \mathrm{Res}^{\mathbb{C}[\mathfrak{S}_n]\times\mathbb{C}[\mathfrak{S}_n]}_{\mathbb{C}[\mathfrak{S}_n]}\left(S^{\lambda} \otimes S^{\mu}\right)\right)\right)$$

$$= \dim\left(\mathrm{Hom}_{P_l(n)\times P_{k-l}(n)}\left(L_l(\lambda_{>1})\otimes L_{k-l}(\mu_{>1}), \mathrm{Res}^{P_k(n)}_{P_l(n)\times P_{k-l}(n)}(L_k(\nu_{>1}))\right)\right).$$

For $n > 2k-2$, the algebras $P_k(n), P_l(n)$, and $P_{k-l}(n)$ are all semisimple and thus

$$\dim\left(\mathrm{Hom}_{P_l(n)\times P_{k-l}(n)}\left(L_l(\lambda_{>1})\otimes L_{k-l}(\mu_{>1}), \mathrm{Res}^{P_k(n)}_{P_l(n)\times P_{k-l}(n)}(L_k(\nu_{>1}))\right)\right)$$

$$= \dim\left(\mathrm{Hom}_{P_l(n+1)\times P_{k-l}(n+1)}\left(L_l(\lambda_{>1})\otimes L_{k-l}(\mu_{>1}), \mathrm{Res}^{P_k(n+1)}_{P_l(n+1)\times P_{k-l}(n+1)}(L_k(\nu_{>1}))\right)\right),$$

which proves that reduced Kronecker coefficients are well defined.

Theorem 2.6.3. *Fix partitions $\lambda_{>1}, \mu_{>1}$, and ν_1 with $|\nu_{>1}| \leq |\lambda_{>1}| + |\mu_{>1}|$. Denote by $\lambda_{[N]}$ the unique partition of N such that $(\lambda_{[N]})_{>1} = \lambda_{>1}$. There exists some N such that for all $n \geq N$,*

$$g^{\nu_{[n]}}_{\lambda_{[n]},\mu_{[n]}} = g^{\nu_{[N]}}_{\lambda_{[N]},\mu_{[N]}}.$$

The next result relates reduced Kronecker coefficients $\bar{g}^{\nu}_{\lambda,\mu}$ to restrictions of standard modules of the partition algebra.

Corollary 9.1.1. *Let $\lambda, \mu, \nu \vdash n$ with $|\lambda_{>1}| = l, |\mu_{>1}| = k-l$, and $|\nu_{>1}| \leq k$. Then,*

$$\bar{g}^{\nu}_{\lambda,\mu} = \dim\left(\mathrm{Hom}_{P_l(n)\times P_{k-l}(n)}\left(\mathrm{Res}^{P_k(n)}_{P_l(n)\times P_{k-l}(n)}\left(\Delta_k\left(\nu_{>1}\right)\right), L_l(\lambda_{>1})\otimes L_{k-l}(\mu_{>1})\right)\right).$$

Proof. From the comment preceding Lemma 7.2.10, the dimension of $\Delta_k(\nu_{>1})$ has no dependency on n. For n sufficiently large, $\Delta_k(\nu_{>1}) = L_k(\nu_{>1})$. Hence,

$$\dim\left(\mathrm{Hom}_{P_l(n)\times P_{k-l}(n)}\left(\mathrm{Res}^{P_k(n)}_{P_l(n)\times P_{k-l}(n)}\left(\Delta_k\left(\nu_{>1}\right)\right), L_l(\lambda_{>1})\otimes L_{k-l}(\mu_{>1})\right)\right)$$

$$= \dim\left(\mathrm{Hom}_{P_l(n)\times P_{k-l}(n)}\left(\mathrm{Res}^{P_k(n)}_{P_l(n)\times P_{k-l}(n)}\left(L_k\left(\nu_{>1}\right)\right), L_l(\lambda_{>1})\otimes L_{k-l}(\mu_{>1})\right)\right)$$

$$= \bar{g}^{\nu}_{\lambda,\mu},$$

as desired. □

If $n \leq 2k-2$, we may replace $L_k(\lambda_{>1})$ the homological approximation we obtained in Theorem 5.0.7 to obtain $g^{\nu}_{\lambda,\mu}$ as an alternating sum of reduced Kronecker coefficients.

Theorem 9.1.2. *Suppose that $\lambda, \mu, \nu \vdash n$ with $|\nu_{>1}| \leq |\lambda_{>1}| + |\mu_{>1}|$. Suppose that*

$$\nu_{>1} = \nu^{(0)} \subseteq \nu^{(1)} \subseteq \cdots \subseteq \nu^{(t)}$$

is chain of n pairs which cannot be extended on the right. Denote by $\nu_{[n]}^{(i)}$ *the unique partition of* n *such that* $\left(\nu_{[n]}^{(i)}\right)_{>1} = \nu^{(i)}$*. Then,*

$$g_{\lambda,\mu}^{\nu} = \sum_{i=0}^{t} (-1)^i \bar{g}_{\lambda,\mu}^{\nu_{[n]}^{(i)}}.$$

Proof. Consider the expression

$$g_{\lambda,\mu}^{\nu} = \dim\left(\mathrm{Hom}_{P_l(n)\times P_{k-l}(n)}\left(L_l(\lambda_{>1})\otimes L_{k-l}(\mu_{>1}), \mathrm{Res}_{P_l(n)\times P_{k-l}(n)}^{P_k(n)}(L_k(\nu_{>1}))\right)\right)$$

obtained via seesaw reciprocity. Replacing $L_k(\lambda_{>1})$ with the homological approximation of $L_k(\lambda_{>1})$ obtained in Theorem 5.0.7,

$$g_{\lambda,\nu}^{\nu} = \sum_{i=0}^{t}(-1)^i \dim\left(\mathrm{Hom}_{P_l(n)\times P_{k-l}(n)}\left(\mathrm{Res}_{P_l(n)\times P_{k-l}(n)}^{P_k(n)}\left(\Delta_n\left(\nu^{(i)}\right)\right), L_l(\lambda)\otimes L_{k-l}(\mu)\right)\right).$$

Then, the result follows from Corollary 9.1.1. □

Supposing that $|\lambda_{>1}| = l$, $|\mu_{>1}| = k-l$, $|\nu_{>1}| = k$, and n is large enough so that $P_k(n)$ is semisimple,

$$\begin{array}{rcccl} L_k(\lambda_{>1}) & \cong & \Delta_k(\lambda) & \cong & S^{\lambda} \\ L_k(\mu_{>1}) & \cong & \Delta_k(\mu) & \cong & S^{\mu} \\ L_k(\nu_{>1}) & \cong & \Delta_k(\nu) & \cong & S^{\nu}. \end{array}$$

Then,

$$\dim\left(\mathrm{Hom}_{P_l(n+1)\times P_{k-l}(n+1)}\left(L_l(\lambda_{>1})\otimes L_{k-l}(\mu_{>1}), \mathrm{Res}_{P_l(n+1)\times P_{k-l}(n+1)}^{P_k(n+1)}(L_k(\nu_{>1}))\right)\right) = c_{\lambda,\mu}^{\nu}.$$

This proves the following corollary.

Corollary 9.1.3. *If* $|\lambda_{>1}| = l$*,* $|\mu_{>1}| = k-l$*, and* $|\nu_{>1}| = k$*, then* $\bar{g}_{\lambda,\mu}^{\nu} = c_{\lambda,\mu}^{\nu}$*.*

9.2 Concluding Remarks

Mimicking the classical construction relating tensor products of representations of $\mathrm{End}(\mathbb{C}^n)$ to certain products of representations of $\mathbb{C}[\mathfrak{S}_n]$, we studied the representation theory of the partition algebra with the hope of learning about Kronecker products of representations of $\mathbb{C}[\mathfrak{S}_n]$. We briefly recapitulate some of the highlights of our analysis and point out some further avenues for research one might pursue by pushing each technique further.

Our first tool for analyzing the the irreducible representations of the partition algebra was Jones' basic construction. This tool allowed us to construct a Bratteli diagram for a tower of partition algebras and made the pairing between the irreducible representations of $P_k(n)$ and the irreducible representations of $\mathbb{C}[\mathfrak{S}_n]$ explicit. The study of other towers of algebras coming for the basic construction has been an area of active recent research, [1]. Additionally, since the Jones' basic construction is a tool originally designed to handle more general von Neumann algebras (specifically, type II_1 factors), it could possibly be adopted to study the infinite dimensional symmetric group. A centralizer algebra for the infinite symmetric group is studied in [9], though they do not employ the basic construction in the study of this algebra.

Our next tool for analyzing the the irreducible representations of the partition algebra was the theory of quasi-hereditary algebras. The standard modules of $P_k(n)$ coming from this analysis gave us a concrete interoperation of the reduced Kronecker coefficients, and the homological approximation of a simple $P_k(n)$-module in terms of standard modules gave a relationship between the Kronecker coefficients and the reduced Kronecker coefficients. It would be interesting to see if the more general notions of Kronecker stability introduced in [35] have a similar interpretation in terms of the standard modules of some other algebra. The homological approximations of simple modules of such algebra could elucidate the relationship between the Kronecker coefficients and generalized reduced Kronecker coefficients.

In addition to the results from [4] we mentioned relating reduced Kronecker coefficients to the Kronecker coefficients, they are able express the Kronecker coefficient $g^{\nu}_{\lambda,\mu}$ as a positive sum of products of Littlewood-Richardson numbers in the case when one of the three partitions is a hook or a two part partition. This result involves an explicit analysis of $\mathrm{Res}^{P_k(n)}_{P_l(n)\times P_{k-l}(n)}(\Delta_k(\nu))$. Further analysis of restrictions of the partition algebra could possibly yield positive formulas for reduced Kronecker coefficients in other cases.

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