A First Taste of Quantum Gravity Effects: Deforming Phase Spaces with the Heisenberg Double

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We present a well-defined framework to deform the phase spaces of classical particles. These new phase spaces, called Heisenberg doubles, provide a laboratory to probe the effects of quantum gravity. In particular, they allow us to equip momentum space with a non-abelian group structure by the introduction of a single deformation parameter. In order to connect Heisenberg doubles with classical phase spaces we begin with a review of Hamiltonian systems, symmetries and conservation laws in the classical framework. Next, we provide a comprehensive review of the theory behind Poisson-Lie groups, including Lie bialgebras and the construction of the Drinfeld double. Lastly, we build the Heisenberg double from Poisson-Lie group components. We then identify the Heisenberg double as a deformation of the cotangent bundle of Lie groups and extend many of the notions of classical Hamiltonian systems to this new picture with Poisson-Lie symmetries. As an example, we look at a new presentation of the deformed rotator.

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Nomenclature

$\circlearrowleft_{i,j,k}$ summation over the cyclic permutations of elements or indices i, j, k

δ a cocommutator

$\mathcal{F}(M)$ the algebra of functions on M

κ^L, κ^R the left and right Maurer-Cartan form, respectively

$\mathfrak{g}$ the Lie algebra of group G

$\mathfrak{g}^*$ the dual of $\mathfrak{g}$ as a vector space

$\langle \cdot, \cdot \rangle$ the canonical inner product between a vector space and its dual

$\mathbf{x}$ a column vector

$\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ a double Lie bialgebra

ω a symplectic form (closed non-degenerate 2-form)

$\Omega^n(M)$ the space of n-form fields on M

Φ a group action

Π a Poisson bivector

Π_+ The Heisenberg double Poisson structure

Π_- the Drinfeld double Poisson structure

π_Q the projection map onto Q

$\mathrm{pr}_{\mathfrak{g}}$ the projection map onto the Lie algebra $\mathfrak{g}$

θ the Liouville form

$\mathfrak{X}(M)$ the algebra of vector fields on M

$\{\, ,\, \}$ a Poisson bracket

$Ad_g = L_g \circ R_{g^{-1}}$ the conjugate action of a group by an element g

$D = G \bowtie G^*$ the double Lie group

D the intrinsic derivative operator

F^*, F_* the pullback and push forward of a function F between manifolds, respectively, i.e. if $F : M \to N$ then $F^* : T^*N \to T^*M$ and $F_* : T_*M \to T_*N$, it can be extended to multivector or n-form fields

G a Lie group

G^{op} G with opposite product, i.e. $a *^{op} b = ba$

H a Hamiltonian function

J a momentum map

L_g, R_g the left and right action of a group by element g, respectively

M a manifold

m^L, m^R the left and right 'inverse' Maurer-Cartan forms, respectively

p_R, p_L, p_R^*, p_L^* factorization maps of the double group D

$r_{\mathfrak{d}}$ the canonical r-matrix of the Drinfeld double

T^*M the cotangent bundle of M

TM the tangent bundle of M

X^L, X^R the left and right invariant vector fields on G associated to $X \in \mathfrak{g}$

X^l, X^r the left and right invariant 1-forms on G^* associated to $X \in \mathfrak{g}$

X_f the Hamiltonian vector field of a function f

Chapter 1

Introduction

'As quantum mechanics is to classical mechanics, quantum groups are to Lie groups' [1]

In classical mechanics phase space (most of the time) is the cotangent bundle of configuration space and symmetries are captured by Lie groups. In the following we present a well-defined method to deform this picture based on the theory of Poisson-Lie groups. Most notably this new phase space, called the Heisenberg double, allows us to introduce curvature in momentum space controlled by a single deformation parameter. Usual momentum spaces are flat and consist of the collection of cotangent planes of a configuration space. This curvature is relevant in the context of constructing effective models for quantum gravity. In particular, this new type of phase space allows us to introduce a scale by hand in the momentum space to generate first order quantum gravity effects while maintaining some symmetries. Heisenberg doubles have already been used to investigate models of 3 dimensional loop gravity [14]. Some other approaches where this development is relevant include relative locality [7, 24, 13] and κ-Minkowski space [8, 19]. A particular case of interest is the dynamics of deformed particles in (2+1) dimensional gravity with Lorentzian and Euclidean metric [12, 9, 35] and in (3+1) dimensional gravity [26]. The inclusion of spin presents a challenge to be tackled in future work.

The author aims to provide a consistent and comprehensive introduction to the theory of Poisson-Lie groups for physicists unfamiliar with the subject, since much of the literature on Poisson-Lie groups is directed towards the study of integrable systems and intended for mathematicians. This material will also provide the background necessary to understand

[1] This statement is paraphrased from the introductory sentence of Chapter 1 of [16].

the construction of the Heisenberg double. Further, the Heisenberg double will be introduced and presented as a well-defined framework for deforming the phase spaces of classical particles. In particular, it has been known that Heisenberg doubles are a generalization of cotangent bundles of Lie groups (the phase spaces of classical particles) [36, 3] and it will be shown that many familiar notions, like momentum maps and conservation laws, carry over to this new framework. New contributions in this thesis include the extension of body and space coordinates to Heisenberg doubles and the momentum maps of the left and right action of the Drinfeld double (the symmetry structure of the Heisenberg double) and it subgroups on the Heisenberg double. Several new examples are presented along the way, including a new representation of the deformed spinning top, whose phase space corresponds with a (2+1) gravity model with Euclidean signature and negative cosmological constant.

Poisson-Lie groups are of interest, not only to those studying integrable systems, but to the greater physics community because they are the classical limit of quantum groups [2]. Quantum groups are seen as a generalization of the symmetries given by Lie groups and are utilized in order to encompass quantum gravitational effects. Hence they provide new types of symmetries relevant in the study of the quantum gravity regime. Since Poisson-Lie group symmetries are the classical analog of these symmetries it is hoped that the study of classical systems with Poisson-Lie group symmetries will provide insight into the corresponding quantized system where quantum gravitational effects will be apparent [32]. This thesis provides a stepping stone between classical mechanics and quantum groups via classical systems with Poisson-Lie group symmetries.

The approach presented brings together a number of important features. Firstly, Heisenberg doubles are mathematically well-defined objects and so provide a concrete foundation to develop and study physically interesting models. Secondly, they provide a bridge between quantum groups and classical mechanics by new types of symmetries. Poisson-Lie group symmetries are not canonical symmetries in that they don't need to preserve the symplectic structure of the phase space but are defined so that the Poisson brackets are invariant once the symmetry group is equipped with a non-trivial Poisson bracket itself. Further, with Heisenberg doubles we are able to introduce curvature into momentum space by introducing a deformation parameter to ideally encode some quantum gravitational effects. In the limit of this parameter we recover the classical 'cotangent bundle picture' with flat momentum space. This curvature corresponds to momentum space

[2]Quantum groups are an active area of research in the physics community in the context of non-commutative spacetimes. They can be understood as deformations of the algebra of functions on a Lie group, or equivalently, as a deformation of the universal enveloping algebra of the associated Lie algebra [16, 27, 36].

being equipped with a non-abelian group structure and implies that the corresponding quantized theory would be inherently non-local, a feature desired by some approaches to quantum gravity. This introduces new and interesting problems, such as, how does one add momenta? This thesis doesn't aim to answer these questions but aims to provide a comprehensive foundation to build and investigate models with these novel features that are anchored to a classical system, particularly in the context of quantum gravity theories.

This thesis begins with a review of classical mechanics in the modern Hamiltonian formalism. We describe the important roles that the phase space and Hamiltonian function play in determining the equations of motion. We review symmetries and the characterization of classical particles, group actions and their corresponding momentum maps. Then we present a version of Noether's theorem that holds in this picture. Next we take a special look at the cotangent bundles of Lie groups. We cover in detail their canonical symplectic form and Poisson structure as well as their global trivializations. Lastly, we conclude by presenting Euler's conservation law. All of these pieces are connected to familiar physical examples throughout.

Next we introduce Poisson-Lie groups and outline the theory behind them beginning with Poisson manifolds. We define Poisson-Lie groups by equipping a Lie group with a Poisson structure compatible with its product. At the infinitesimal level we find Poisson-Lie groups have a corresponding Lie bialgebra structure. These are Lie algebras together with a special linear map called a cocommutator. The relation between Poisson-Lie groups and Lie bialgebras is captured by a theorem of Drinfeld, which effectively states there is a one to one relation between Lie bialgebras and connected, simply connected Poisson-Lie groups. We will see that the cocommutator defines a dual Lie bialgebra structure so that Lie bialgebras and, in turn, Poisson-Lie groups always come in dual pairs. Next we look at a special class of Lie bialgebras, called coboundary Lie bialgebras. They have cocommutators of a special form defined by r-matrices. Using this theory we show how to construct a double Lie bialgebra from a dual pair of Lie bialgebras and a canonical r-matrix. Then, for the first time, it is shown that the group corresponding to the double Lie bialgebra admits factorizations that correspond exactly to the trivializations of cotangent bundles of Lie groups seen in the first section. Equipping the double group with the Poisson structure associated to the r-matrix we construct the Drinfeld double which is a Poisson-Lie group built out of a pair of dual Poisson-Lie groups. Poisson-Lie groups are best studied in terms of the Drinfeld double since it captures all the 'symmetries' present in the construction of Poisson-Lie groups. Since we are concerned in this presentation only with physics at the classical level we don't cover the quantization of Poisson-Lie groups and its relation to quantum groups.

In the last section we finally reveal the Heisenberg double. It is simply the double group with an affine Poisson structure. Heisenberg doubles are not Poisson-Lie groups themselves but nonetheless are built out of Poisson-Lie group components. We show how a Heisenberg double is related to cotangent bundles of Lie groups. We will see that a Heisenberg double constructed from a Lie group with trivial Poisson structure shares the same manifold and Poisson structure as a cotangent bundle on that group with its canonical Poisson structure. Hence introducing a scale parameter into the Poisson-structure of a Poisson-Lie group allows us to go from a Heisenberg double to its associated cotangent bundle in the limit. Further, we see that equivalent notions of momentum maps and Noether's theorem hold on the Heisenberg double. Also, for the first time, Euler's conservation law is presented in this setting, as well as the momentum maps of the full symmetry structure (the Drinfeld double) acting on the Heisenberg double. We conclude by investigating a new representation of the deformed rotator.

In the following we assume all objects are real, smooth and of finite dimension, unless stated otherwise. All groups considered are Lie groups. We have also used Einstein summation convention throughout the thesis.

Chapter 2

Hamiltonian Systems, Particles and Symmetry

The following is a review of the contemporary formulation of Hamiltonian mechanics that will be relevant in the upcoming discussion. We do not go into full details here. The interested reader should consult [1].

We also give a definition of a classical particle based on space-time symmetry groups. Using this outlook we examine the consequent phase spaces with examples.

2.1 Hamiltonian Mechanics

Hamiltonian mechanics is built from the notions of a phase space and energy function. Phase space is captured by a symplectic manifold (M, ω) (the symplectic form ω is a closed non-degenerate 2-form). The energy function $H \in \mathcal{F}(M)$ is the Hamiltonian of the system. Together these form a **Hamiltonian system** (M, ω, H) from which the equations of motion can be found. Hamilton's equations, which can be presented succinctly in geometric terms as

$$\omega(X_H, \cdot) = dH \tag{2.1}$$

define the **Hamiltonian vector field** $X_H \in \mathfrak{X}(M)$ where $\mathfrak{X}(M)$ is the space of vector fields on the manifold M. The trajectories of the system are the flow lines of the Hamiltonian vector field. The flow parameter is identified as time and the Hamiltonian vector field is said to generate time translation.

Example 2.1. Consider a free particle of mass m travelling in one dimension with position and momentum given by $(q, p) \in \mathbb{R}^2$. Then $M = \mathbb{R}^2$ is our phase space with canonical symplectic form $\omega = dq \wedge dp$. Letting $H = \frac{p^2}{2m}$, the associated Hamiltonian vector field is $X_H = \frac{p}{m}\frac{\partial}{\partial q}$. Since the Hamiltonian vector field generates time translation, i.e. $X_H = \frac{d}{dt}$, we find $\frac{dp}{dt} = 0$ and $\frac{dq}{dt} = \frac{p}{m}$. This tells us that the particle has constant momentum $p = p_0$ and motion given by $q(t) = \frac{p}{m}t + q_0$ where $p_0, q_0 \in \mathbb{R}$ are constants dependent on initial conditions.

Since the phase space (M, ω) is symplectic there always exist, by Darboux's theorem [1], local coordinates (q^i, p_i) called **canonical coordinates** where the symplectic form can be written $\omega = dq^i \wedge dp_i$. In these coordinates

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i}$$

and Hamilton's equations (2.1) take on a familiar form

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}.$$

Both the Hamiltonian H and symplectic form ω determine the systems behaviour. Different choices of Hamiltonian clearly lead to different behaviours of the system, but ω is free to be chosen as well as long as it remains a symplectic form. For example, the introduction of a magnetic field changes the kinematics of a charged particle which corresponds to a change in ω [38]. The freedom in our choice of symplectic form will be exploited and expanded upon later in this paper. The symplectic form is in fact where our deformation parameter will lie.

The classical algebra of observables of a Hamiltonian system is the algebra of functions $\mathcal{F}(M)$ on the phase space. This algebra is associative and commutative with pointwise product. Using the symplectic form we can equip this algebra with an extra structure, a Poisson bracket. The Poisson bracket formalism provides one of the clearest connections between classical and quantum physics. This formalism will be relied upon extensively in the following and so will be reviewed first in the context of our classical Hamiltonian formalism.

The **Poisson bracket** of two functions $f, h \in \mathcal{F}(M)$ is the function

$$\{f, h\} = \omega(X_f, X_h). \tag{2.2}$$

Note that we can also write $\{f, h\} = df(X_h)$ since $df = \omega(X_f, \cdot)$. We may immediately note the that the bracket is skew-symmetric and satisfies the Jacobi identity since the symplectic form is skew-symmetric and closed (i.e. $0 = d\omega(X_f, X_g, X_h) = \circlearrowleft_{f,g,h} \{f, \{g, h\}\}$ for all $f, g, h \in \mathcal{F}(M)$). A more general way to get a Poisson bracket is to equip the manifold with a Poisson bivector $\Pi \in \wedge^2 TM$ such that the bracket

$$\{f, h\} = \Pi(df, dh)$$

is a Lie bracket on $\mathcal{F}(M)$. The details of this construction and the properties of Poisson structures will be explained in detail in section 3.1.

In canonical coordinates the bracket takes the familiar form

$$\{f, h\} = \frac{\partial f}{\partial p_i}\frac{\partial h}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial h}{\partial p_i}.$$

From this expression, it is easy to see that $X_H = \{H, \cdot\}$ so that the Hamiltonian vector field is easily defined using the Poisson structure. The evolution of any observable $f \in \mathcal{F}(M)$ may be written

$$\frac{d}{dt}(f \circ F_t) = \{H, f \circ F_t\}$$

where F_t is the solution of Hamilton's equations (i.e. F_t is the flow of the Hamiltonian vector field). In particular, if f is a constant of motion then $\{H, f\} = 0$.

It is important to note that the symplectic form defines a Poisson bracket and so these two elements are intimately related. An important fact that will be relevant later is that symplectic maps preserve the Poisson brackets, i.e. for $f, h \in \mathcal{F}(N)$, $\{f, h\}_N \circ F = \{f \circ F, h \circ F\}_M$ where a **symplectic map** is a diffeomorphism between symplectic manifolds $F : (M, \omega) \to (N, \rho)$ such that $F^*\rho = \omega$ and F^* is the pullback of F.

2.2 Symmetries and Momentum

Symmetries provide deep insights into the workings of nature. They provide unity and clarity to muddied paths once uncovered. A symmetry of a system is a transformation that preserves the systems action and so preserves the Hamiltonian and 'respects' the phase space structure. Symmetries naturally form a group. Noether's theorem tells us that to each symmetry of a system corresponds a conserved quantity, i.e. an element in the algebra of functions on the phase space whose value is constant over time. This well-known result applies to the Lagrangian formulation of mechanics and the calculus of variations. In the modern geometric formulation of Hamiltonian mechanics, the relation between symmetries and conserved quantities is expressed in terms of group actions and momentum maps.

A **symmetry** of a Hamiltonian system (M, ω, H) is a function $S : M \to M$ that preserves both ω and H, i.e. $S^*\omega = \omega$ and $H \circ S = H$ [38]. These symmetries are captured by group actions. An **action of the group** G on M is a smooth function $\Phi : G \times M \to M$ such that

- $\Phi(e, p) = p$, for all $p \in M$ where e is the identity element of G
- $\Phi(g, \Phi(h, p)) = \Phi(gh, p)$, for all $p \in M$ and $g, h \in G$.

Often we write $\Phi(g, p) = \Phi_g(p)$. So the above two conditions may be written as $\Phi_e = id$ and $\Phi_g \circ \Phi_h = \Phi_{gh}$. A group action is a **symplectic action** if $\Phi_g : M \to M$ is a symplectic map for all $g \in G$.

Given a group G acting on M we can define a vector field corresponding to $X \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the group G, by

$$\Phi_X(p) = \left.\frac{d}{dt}\right|_{t=0} \Phi(\exp tX, p).$$

Example 2.2. The infinitesimal generator of the left (right) action of G on itself, $L_g(h) = gh$ ($R_g(h) = hg$) for $g, h \in G$ is given by the right (left) invariant vector field on G. The invariant vector fields of a group are those related to the elements of the Lie algebra by the pushforward of the group action. The left (right) invariant vector fields can be written as $X^L(g) = (L_g)_*X$ ($X^R(g) = (R_g)_*X$) for $X \in \mathfrak{g}$. More explicitly, for $X \in \mathfrak{g}$, the corresponding generator of the left action is

$$L_X(h) = \left.\frac{d}{dt}\right|_{t=0} L_{\exp(tX)}h = (R_h)_*X = X^R(h).$$

The same is the case for the right action with R and L switched.

Example 2.3. The coadjoint action of G on the dual of the Lie algebra $\mathfrak{g}^*$ is a group action defined by

$$Ad^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$$

$$(g, \alpha) \mapsto (Ad_{g^{-1}})^*\alpha$$

where $(Ad_{g^{-1}})^* : \mathfrak{g}^* \to \mathfrak{g}^*$ satisfies $\langle (Ad_{g^{-1}})^* \alpha, X\rangle = \langle \alpha, (Ad_{g^{-1}})_* X\rangle$ where $(Ad_g)_* X = (L_g \circ R_{g^{-1}})_* X$ for $X \in \mathfrak{g}, \alpha \in \mathfrak{g}^*$ and $\langle \cdot, \cdot \rangle$ is the canonical scalar product between elements of $\mathfrak{g}$ and $\mathfrak{g}^*$. The infinitesimal generator of the coadjoint action is

$$Ad^*_X = -ad^*_X \tag{2.3}$$

so $Ad^*_X(Y)(\alpha) = -\langle ad^*_X \alpha, Y\rangle = -\langle \alpha, ad_X Y\rangle = -\langle \alpha, [X, Y]\rangle$ for $X, Y \in \mathfrak{g}, \alpha \in \mathfrak{g}^*$. Note that the subscript X in (2.3) plays different roles on the left and right side of the equation.

Given a symplectic action Φ of G on (M, ω) then $J : M \to \mathfrak{g}^*$ is a **momentum map** for the action Φ given that

$$d\hat{J}(X) = \omega(\Phi_X, \cdot) \tag{2.4}$$

for all $X \in \mathfrak{g}$ where $\hat{J}(X) : M \to \mathbb{R}$ is given by $\hat{J}(X)(p) = \langle J(p), X\rangle$. The above condition is equivalent to

$$\Phi_X = \{\hat{J}(X), \cdot\}. \tag{2.5}$$

Momentum maps are the modern geometric generalization of conserved quantities. The map $\hat{J}(X)$ is a real valued function on the phase space and can be seen as a Hamiltonian associated to the momentum map J through the similarity between (2.1) and (2.4). Note by (2.5) for each $X \in \mathfrak{g}$ the associated momentum map $\hat{J}(X)$ is conserved under the group action in the direction X, i.e. $\Phi_X(\hat{J}(X)) = 0$. Thus we see that the Hamiltonian vector field of the momentum map must equal the generator of the symplectic group action. We shall also see that for each $X \in \mathfrak{g}$, $\hat{J}(X)$ will be a conserved quantity if the Hamiltonian H is invariant under the group action Φ.

It is important to note that symplectic group actions are not guaranteed to have corresponding momentum maps. Group actions with momentum maps are called **Hamiltonian actions**. Some group actions fail to have momentum maps because not all locally Hamiltonian vector fields are globally Hamiltonian [1]. For example, the following action of $\mathbb{R}$ on $S^1 \times S^1$, $\Phi_t(\theta, \phi) = (\theta, \phi + t)$ has no associated momentum map, where θ, ϕ are the angle parameterization of $S^1 \times S^1$ [38]. However, for some special cases there are guaranteed momentum maps. For example, the case where the symplectic manifold is a cotangent bundle $M = T^*Q$ with canonical symplectic form and the group action of G on M is the lift Φ^{T^*} of the group action Φ on the base manifold Q. This will be explained in the next section. Below we present the two most common examples of momentum maps, the linear and angular momentum associated with translational and rotational invariance.

Example 2.4 (Linear momentum)**.** Consider the translational action of $\mathbb{R}^n$ on $M = \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. We may view M as the phase space of two particles in $\mathbb{R}^n$. We already expect the momentum map to be the total linear momentum of the two particles.

First we choose coordinates $(\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \mathbf{p}_{(1)}, \mathbf{p}_{(2)})$ where $(\mathbf{q}^{(i)}, \mathbf{p}_{(i)}) = (q^{i1}, \dots, q^{in}, p_{i1}, \dots, p_{in})$. The canonical symplectic form can be written $\omega = d\mathbf{q}^{(1)} \wedge d\mathbf{p}_{(1)} + d\mathbf{q}^{(2)} \wedge d\mathbf{p}_{(2)}$ where $d\mathbf{q}^{(i)} \wedge d\mathbf{p}_{(i)} = \sum_j dq^{ij} \wedge dp_{ij}$ (no summation over i!). The symplectic action of $\mathbb{R}^n$ is given by

$$\begin{aligned} \Phi : \mathbb{R}^n \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} &\to \mathbb{R}^{2n} \times \mathbb{R}^{2n} \\ \mathbf{s} \times (\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \mathbf{p}_{(1)}, \mathbf{p}_{(2)}) &\mapsto (\mathbf{q}^{(1)} + \mathbf{s}, \mathbf{q}^{(2)} + \mathbf{s}, \mathbf{p}_{(1)}, \mathbf{p}_{(2)}) \end{aligned}$$

Now for $\mathbf{x} = (x_1, \dots, x_n) \in \mathrm{Lie}(\mathbb{R}^n) \simeq \mathbb{R}^n$, the associated infinitesimal generator is

$$\Phi_{\mathbf{x}}(\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \mathbf{p}_{(1)}, \mathbf{p}_{(2)}) = \mathbf{x}\partial_{\mathbf{q}^{(1)}} + \mathbf{x}\partial_{\mathbf{q}^{(2)}}$$

where $\mathbf{x}\partial_{\mathbf{q}^{(i)}} = \sum_j x_j \frac{\partial}{\partial q^{ij}}$. By equation 2.4, we require the momentum map's associated Hamiltonian $\hat{J}(\mathbf{x}) : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ to satisfy

$$d\hat{J}(\mathbf{x}) = \mathbf{x}d\mathbf{p}_{(1)} + \mathbf{x}d\mathbf{p}_{(2)}$$

where $\mathbf{x}d\mathbf{p}_{(i)} = \sum_j x_j dp_{ij}$. Thus $\hat{J}(\mathbf{x}) = \mathbf{x}\mathbf{p}_{(1)} + \mathbf{x}\mathbf{p}_{(2)}$. And hence,

$$\begin{aligned} J : \mathbb{R}^{2n} \times \mathbb{R}^{2n} &\to \mathrm{Lie}(\mathbb{R}^n)^* \simeq \mathbb{R}^n \\ (\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \mathbf{p}_{(1)}, \mathbf{p}_{(2)}) &\mapsto \mathbf{p}_{(1)} + \mathbf{p}_{(2)} \end{aligned}$$

is the associated momentum map, as expected [38]. In this example, we can see that

$$\Phi_{\mathbf{x}}(\hat{J}(\mathbf{y})) = 0, \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Thus the total linear momentum of the system (i.e. of the two particles) is conserved under all translations.

Example 2.5 (Angular Momentum)**.** Consider the rotational action of $SO(3)$ on $M = \mathbb{R}^3 \times \mathbb{R}^3$ with canonical symplectic form. This action is given by

$$\begin{aligned} \Phi : SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 &\to \mathbb{R}^3 \times \mathbb{R}^3 \\ R \times (\mathbf{q}, \mathbf{p}) &\mapsto (R\mathbf{q}, \mathbf{p}R^T) \end{aligned}$$

and is symplectic. We can identify elements of $\mathfrak{so}(3)$ with $\mathbb{R}^3$ via the map

$$X_{\mathbf{v}} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \in \mathfrak{so}(3) \mapsto \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3.$$

Using this identification we can write the product $X_{\mathbf{v}}\mathbf{z} = \mathbf{v} \times \mathbf{z} \in \mathbb{R}^3$, where $\times$ is the cross product and $\mathbf{z} \in \mathbb{R}^3$. Further we may identify the dual $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ using $\langle X_{\mathbf{v}}, X_{\mathbf{u}} \rangle = -\frac{1}{2}\mathrm{Tr}(X_{\mathbf{v}}X_{\mathbf{u}}) = \mathbf{u}^T\mathbf{v}$ as the natural inner product. Thus, we can identify $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$ and our momentum map can be written as a map $J : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

The infinitesimal generator of the rotational action is given by

$$\Phi_{X_{\mathbf{v}}}(\mathbf{q}, \mathbf{p}) = (X_{\mathbf{v}}\mathbf{q}, -\mathbf{p}X_{\mathbf{v}}) = (\mathbf{v} \times \mathbf{q})\partial_{\mathbf{q}} + (\mathbf{v} \times \mathbf{p}^T)\partial_{\mathbf{p}}$$

for $X_{\mathbf{v}} \in \mathfrak{so}(3)$, since $X_{\mathbf{v}}^T = -X_{\mathbf{v}}$ where $\mathbf{z}\partial_{\mathbf{y}} = \sum_i z_i \frac{\partial}{\partial y^i}$. Then by (2.4),

$$\begin{aligned} d\hat{J}(\mathbf{v})(\mathbf{q}, \mathbf{p}) &= (\mathbf{v} \times \mathbf{q})d\mathbf{p} - (\mathbf{v} \times \mathbf{p}^T)d\mathbf{q} \\ &= d((\mathbf{q} \times \mathbf{p}^T) \cdot \mathbf{v}) \end{aligned}$$

where we have used the scalar triple product identity. Thus we see that $J(\mathbf{q}, \mathbf{p}) = \mathbf{q} \times \mathbf{p}^T$, the angular momentum, is the momentum map of the rotational action [38, 1]. Note that the momentum map $\hat{J}(X_{\mathbf{w}})$ is not conserved under rotations in all directions since

$$\begin{aligned} \Phi_{X_{\mathbf{v}}}(\hat{J}(X_{\mathbf{w}}))(\mathbf{q}, \mathbf{p}) &= [(\mathbf{v} \times \mathbf{q})\partial_{\mathbf{q}} + (\mathbf{v} \times \mathbf{p}^T)\partial_{\mathbf{p}}]((\mathbf{q} \times \mathbf{p}^T) \cdot \mathbf{w}) \\ &= (\mathbf{v} \cdot \mathbf{p}^T)(\mathbf{w} \cdot \mathbf{q}) - (\mathbf{w} \cdot \mathbf{p}^T)(\mathbf{v} \cdot \mathbf{q}) \end{aligned}$$

is zero only when $\mathbf{v} = a\mathbf{w}$ for some $a \in \mathbb{R}$ assuming $\mathbf{v}, \mathbf{w}$ are non-zero. The quantity $\hat{J}(X_{\mathbf{w}}) = (\mathbf{q} \times \mathbf{p}^T) \cdot \mathbf{w}$ is conserved only under rotations generated by $X_{\mathbf{w}}$.

Using the terminology developed thus far we can state the following fundamental conservation law[1] that captures the significance of momentum maps:

Theorem 1. *If $H \in \mathcal{F}(M)$ is invariant under the symplectic action Φ of G, i.e. $H(p) = H(\Phi_g(p))$ for all $p \in M, g \in G$ and supposing this action has a momentum map J, then J is an integral for X_H, i.e. if F_t is the flow of X_H then $J(F_t(p)) = J(p)$ for all $p \in M$ and $t \in \mathbb{R}$ where F_t is defined.*

[1]This theorem is sometimes referred to as the Noether's theorem for Poisson manifolds [15].

Since H is invariant we know $H(\Phi_{\exp tX}(p)) = H(p)$ for all $X \in \mathfrak{g}$. Differentiating at $t = 0$,

$$0 = dH(\Phi_X) = -\omega(\Phi_X, X_H) = -d\hat{J}(X)(X_H) = -\{\hat{J}(X), H\}.$$

This proves the theorem above. Thus momentum maps are conserved quantities of Hamiltonian systems with symmetries. In this formulation we see that symmetries are paramount. Given a symmetry captured by a group action Φ we can (sometimes) find conserved quantities even before defining our choice of Hamiltonian H as long as H co-operates with the symmetries, i.e. is invariant under the group action Φ.

2.3 Classical Particles and the Cotangent Bundle Formulation

The phase space M of many mechanical systems can be identified with a cotangent bundle T^*Q where Q is the configuration space and the collection of cotangent planes makes up the momentum space. A cotangent bundle is naturally equipped with a symplectic form $\omega = -d\theta \in \Omega^2(T^*Q)$ where $\theta \in \Omega^1(T^*Q)$ is the Liouville form. The **Liouville form** is related to the projection map $\pi_Q : T^*Q \to Q$ and is defined by

$$\theta(X)(\alpha_q) = \langle \alpha_q, (\pi_Q)^*(X) \rangle$$

where $\alpha_q \in T_q^*Q$, $X \in T_{\alpha_q}(T^*Q)$, $q \in Q$ and $\langle \cdot, \cdot \rangle$ is the canonical pairing of 1-forms and tangent vectors on Q. In canonical coordinates $\theta = p_i dq^i$ so $\omega = dq^i \wedge dp_i$ as expected.

Inspired by the definition in [34] we consider a **classical particle** to be a Hamiltonian system whose configuration space Q is a homogeneous space of the group G of space-time transformations that relate inertial reference frames according to some relativity principle. The group G acts transitively on the configuration space meaning any two points q, q' in the configuration space are related by the action of some group element, i.e. there exists a $g \in G$ such that $\Phi(g, q) = q'$. This means all points in Q share the same local properties and that Q is in fact the orbit of any point in Q under the action of G. Choosing such a point is equivalent to choosing an origin of the configuration space. Homogeneous spaces of the group G can always be constructed as a quotient of G by some continuous subgroup. Thus the largest and most versatile homogeneous space of G is G itself. Using this we can make the space-time symmetries manifest by identifying the configuration space with the space-time symmetry group itself. For example, a particle with spin may have its configuration space identified with the Poincaré group $ISO(3,1)$, the symmetry group of special relativity [25, 28] or an isotropic body may have its configuration space identified with the rotation group $SO(3)$ [1]. Hence once we choose a space-time symmetry group G to work with the particles in this space-time will be Hamiltonian systems on T^*G.

In order to better grasp the canonical symplectic structure and corresponding Poisson bracket we introduce the following maps and notations on a Lie group G with Lie algebra $\mathfrak{g}$. The details of the following exposition can be found in [5]. First we denote left and right translation by $L_g, R_g : G \to G$, respectively, so that $gh = L_g(h) = R_h(g)$. Further, the left and right Maurer-Cartan forms $\kappa^L, \kappa^R : \mathfrak{X}(G) \to \mathfrak{g}$, respectively, are given by

$$\kappa^L_g, \kappa^R_g : T_gG \to \mathfrak{g}$$

$$\kappa^L_g(X_g) = (L_{g^{-1}})_*(X_g)$$
$$\kappa^R_g(X_g) = (R_{g^{-1}})_*(X_g)$$

for $X_g \in T_gG$. Notice that the left and right invariant vector fields are given by $X^L_g = (L_g)_*(X)$ and $X^R_g = (R_g)_*(X)$, respectively for $X \in \mathfrak{g}$ so that $\kappa^L(X^L) = X$ and $\kappa^R(X^R) = X$. The left and right 'inverse' Maurer-Cartan forms $m^L, m^R : \Omega^1(M) \to \mathfrak{g}^*$, respectively, are given by

$$m^L_g, m^R_g : T^*_gG \to \mathfrak{g}^*$$
$$m^L_g(\alpha_g) = ((\kappa^R_g)^{-1})^*(\alpha_g) = (R_g)^*(\alpha_g)$$
$$m^R_g(\alpha_g) = ((\kappa^L_g)^{-1})^*(\alpha_g) = (L_g)^*(\alpha_g)$$

for $\alpha_g \in T^*_gG$. Using this notation the Liouville form of the cotangent bundle of a Lie group can be written

$$\theta = \frac{1}{2}\left(\langle m^L \circ \pi_{T^*G}, (\pi_G)^*\kappa^R\rangle + \langle m^R \circ \pi_{T^*G}, (\pi_G)^*\kappa^L\rangle\right) \in \Omega(T^*G)$$

where $\pi_M : TM, T^*M \to M$ is a projection map onto the base manifold. Notice, $m^A \circ \pi_{T^*G} : TT^*G \to \mathfrak{g}^*$ and $(\pi_G)^*\kappa^A : TT^*G \to \mathfrak{g}$ for $A = R, L$. The canonical symplectic form can then be written

$$\omega = -d\theta = -\frac{1}{2}\left(\langle dm^L \overset{\wedge}{,} (\pi_G)^*\kappa^R\rangle + \langle dm^R \overset{\wedge}{,} (\pi_G)^*\kappa^L\rangle\right) \in \Omega^2(T^*G)$$

where we have used the short form $\langle A \overset{\wedge}{,} B\rangle(X, Y) = \langle A(X), B(Y)\rangle - \langle A(Y), B(X)\rangle$.

Letting X_i be a basis of $\mathfrak{g}$ and ξ^i the dual basis of $\mathfrak{g}^*$ we can write for $A = R, L$,

$$m^A(\cdot) = \sum_i m^A_i(\cdot)\xi^i, \quad m^A_i = \langle m^A(\cdot), X_i\rangle \in \mathcal{F}(T^*G).$$

So $dm^A = \sum_i dm_i^A(\cdot)\xi^i$ with $dm_i^A \in \Omega^1(T^*G)$. Similarly,

$$\kappa^A(\cdot) = \sum_i \kappa_i^A(\cdot)X_i, \quad \kappa_i^A = \langle \xi^i, \kappa^A(\cdot)\rangle \in \Omega^1(G).$$

Then in these coordinates we uncover,

$$\omega = -\frac{1}{2}\sum_i (dm_i^L \wedge (\pi_G)^*\kappa_i^R + dm_i^R \wedge (\pi_G)^*\kappa_i^L).$$

This expression can be inverted to find the corresponding Poisson structure. Let X_i^L, X_i^R be the left and right invariant vector fields, respectively, on G corresponding to the basis element $X_i \in \mathfrak{g}$. They generate right and left translation on G and their corresponding flows are given by $F_t^L(g) = g\exp(tX_i)$ and $F_t^R(g) = \exp(tX_i)g$. These flows have lifts $(F_t^L)^*$ and $(F_t^R)^*$ on T^*G with generators given by

$$X_i^{L*}(\alpha_g) = \frac{d}{dt}\bigg|_{t=0}(F_t^L)^*(\alpha_g) \in \mathfrak{X}(T^*G), \quad X_i^{R*}(\alpha_g) = \frac{d}{dt}\bigg|_{t=0}(F_t^R)^*(\alpha_g) \in \mathfrak{X}(T^*G).$$

These satisfy

$$(\pi_G)^*\kappa_i^A(X_j^{A*}) = \kappa_i^A((\pi_G)_*X_j^{A*}) = \kappa_i^A(X_j^A) = \delta_{ij}$$

for $A = R, L$. Since it can be verified that these flows preserve the Liouville form, $\omega(\cdot, X_j^{R*}) = -dm_i^L$ and $\omega(\cdot, X_j^{L*}) = -dm_i^R$. Lastly, we define the vector fields $Z_i^L, Z_i^R \in \mathfrak{X}(T^*G)$ satisfying

$$\omega(\cdot, Z_i^L) = -(\pi_G)^*\kappa_i^L, \quad \omega(\cdot, Z_i^R) = -(\pi_G)^*\kappa_i^R$$

The Poisson structure can now be written as,

$$\Pi = -\omega^{-1} = \frac{1}{2}\sum_i X_i^{R*} \wedge Z_i^R + X_i^{L*} \wedge Z_i^L.$$

This expression is rather opaque and unenlightening but can be simplified greatly by trivializing the cotangent bundle [5, 3].

Cotangent bundles of Lie groups are trivial bundles and admit two global trivializations. By abuse of terminology, these trivializations are commonly referred to as choices of coordinates and are called

- body coordinates:

$$\begin{aligned} B = (\pi_G, m^R) : T^*G &\to G \times \mathfrak{g}^* \\ \alpha &\mapsto (g, (L_g)^*\alpha) \end{aligned}$$

- space coordinates:

$$
\begin{aligned}
S = (m^L, \pi_G) : T^*G &\to \mathfrak{g}^* \times G \\
\alpha &\mapsto ((R_h)^*\alpha, h)
\end{aligned}
$$

We will see that these coordinate choices are analogous to the Iwasawa decomposition of double Lie groups in section 3.7. Note that in the above expressions $h = g$. The name of these trivializations originates from the study of the rigid body. If we think of the left action L_g of a group on itself as the 'natural' action then once we lift it to $(L_{g^{-1}})^*$, an action on T^*G, this action is realized as

$$
\begin{aligned}
L^B_{g'}(g,\mu) &= B \circ (L_{g'^{-1}})^* \circ B^{-1}(g,\mu) = (g'g,\mu) \\
L^S_{g'}(\nu,h) &= S \circ (L_{g'^{-1}})^* \circ S^{-1}(\nu,h) = ((Ad_{g'^{-1}})^*\nu, g'h).
\end{aligned}
$$

where (g,μ) are body and (ν,h) space coordinates. So if we think of μ as the angular momentum vector attached to the body it would not vary relative to an observer fixed to that body under the action but would vary to an observer in space since the left action is the group action relating inertial observers. As we see above μ in body coordinates is unchanged but ν in space coordinates does vary, hence their names.

Expanding on the bases introduced earlier, we let the structure constants of $\mathfrak{g}$ be given by $[X_i, X_j] = c^k_{ij}X_k$ and choose coordinates x^i on the dual space $\mathfrak{g}^*$ so that any $\xi \in \mathfrak{g}^*$ can be written as $\xi = \sum_i x^i\xi^i$. Then in body and space coordinates

$$
\begin{aligned}
\omega^B = (B^{-1})^*\omega = -\sum_i dx^i \wedge \kappa^L_i + \frac{1}{2}\sum_{ijk} x^i c^i_{jk}\kappa^L_j \wedge \kappa^L_k \\
\omega^S = (S^{-1})^*\omega = -\sum_i dx^i \wedge \kappa^R_i - \frac{1}{2}\sum_{ijk} x^i c^i_{jk}\kappa^R_j \wedge \kappa^R_k
\end{aligned}
$$

and

$$
\Pi^B = (B^{-1})_*\Pi = \sum_i \frac{\partial}{\partial x^i} \wedge X^L_i + \frac{1}{2}\sum_{ijk} x^i c^i_{jk}\frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} \tag{2.6}
$$

$$
\Pi^S = (S^{-1})_*\Pi = \sum_i \frac{\partial}{\partial x^i} \wedge X^R_i - \frac{1}{2}\sum_{ijk} x^i c^i_{jk}\frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k}. \tag{2.7}
$$

From these expressions we note that the Poisson bracket of two functions on the underlying group G will always be zero and thus the Poisson structure is trivial on the configuration space. Equipping the configuration space G with a non-trivial Poisson structure will be the first step when constructing our deformed phase space. The above expressions are a special case of the Heisenberg double Poisson structure. This will be detailed in section 4.2.

Example 2.6 (The rigid body)**.** An isotropic rigid body is symmetric under rotations. Thus we can identify the phase space of this system with $T^*SO(3)$. Working with the following matrix representation, the generators of $SO(3)$ are given by

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.8}$$

which satisfy $[E_i, E_j] = \epsilon_{ij}^k E_k$. In body coordinates $SO(3) \times \mathbb{R}^3 = \{(R, \mathbf{u})\}$ the bracket structure follows from (2.6)

$$\{u_i, u_j\} = \epsilon_{ij}^k u_k, \quad \{u_i, R\} = RE_i, \quad \{R, R\} = 0.$$

In space coordinates $\mathbb{R}^3 \times SO(3) = \{(\mathbf{v}, R)\}$, the bracket structure follows from (2.7)

$$\{v_i, v_j\} = -\epsilon_{ij}^k v_k, \quad \{v_i, R\} = E_i R, \quad \{R, R\} = 0.$$

Note the above expressions are in terms of matrix elements, so $E_i^L R_{jk} = (RE_i)_{jk}$ and $E_i^R R_{jk} = (E_i R)_{jk}$.

Example 2.7 (The particle with spin)**.** The Poincaré group $PG = ISO(3,1) = R^{3,1} \rtimes SO(3,1)$ is the symmetry group of a classical particle with spin [25, 28]. The associated Lie algebra $\mathfrak{pg}$ has 4 generators P_μ of translation, 3 generators J_i of rotations and 3 generators K_i of boosts. These satisfy the relations

$$\begin{gathered}[J_i, P_0] = 0, \quad [J_i, P_j] = \epsilon_{ij}^k P_k, \quad [K_i, P_0] = P_i, \quad [K_i, P_i] = P_0, \\ [J_i, J_j] = \epsilon_{ij}^k J_k, \quad [J_i, K_j] = \epsilon_{ij}^k K_k, \quad [K_i, K_j] = -\epsilon_{ij}^k J_k.\end{gathered}$$

If we choose coordinates on $\mathfrak{pg}^*$ so an element ξ can be written $\xi = p_\mu P^{*\mu} + j_i J^{*i} + k_i K^{*i}$ where $A^* \in \mathfrak{pg}^*$ is dual to $A \in \mathfrak{pg}$, then in space coordinates $\mathbb{R}^{10} \times ISO(3,1) \ni (\xi, h)$ the Poisson bivector can be read off as

$$\begin{aligned}\Pi^B(\xi, h) = {} & \frac{\partial}{\partial p_\mu} \wedge P_\mu^R + \frac{\partial}{\partial j_i} \wedge J_i^R + \frac{\partial}{\partial k_i} \wedge K_i^R \\ & - \epsilon_{ij}^k p_k \frac{\partial}{\partial j_i} \otimes \frac{\partial}{\partial p_j} - p_i \frac{\partial}{\partial k_i} \otimes \frac{\partial}{\partial p_0} - p_0 \frac{\partial}{\partial k_i} \otimes \frac{\partial}{\partial p_i} \\ & - \epsilon_{ij}^k j_k \frac{\partial}{\partial j_i} \otimes \frac{\partial}{\partial j_j} - \epsilon_{ij}^k k_k \frac{\partial}{\partial j_i} \otimes \frac{\partial}{\partial k_j} + \epsilon_{ij}^k j_k \frac{\partial}{\partial k_i} \otimes \frac{\partial}{\partial k_j}.\end{aligned} \tag{2.9}$$

Using more familiar terms we write $h = (x^\mu, \Lambda^{\mu\nu}) \in ISO(3,1)$ where $x^\mu \in \mathbb{R}^4, \Lambda^{\mu\nu} \in SO(3,1)$ the Poisson brackets then read

$$\begin{gathered}
\{x^\mu, x^\nu\} = 0, \quad \{x^\rho, \Lambda^{\mu\nu}\} = 0, \quad \{\Lambda^{\mu\nu}, \Lambda^{\rho\sigma}\} = 0, \\
\{p_\mu, x^\nu\} = \delta^\nu_\mu, \quad \{p_\mu, \Lambda^{\rho\sigma}\} = 0, \\
\{j_i, x^\nu\} = 0, \quad \{j_i, \Lambda^{\mu\nu}\} = J_i \Lambda^{\mu\nu}, \\
\{k_i, x^\nu\} = 0, \quad \{k_i, \Lambda^{\mu\nu}\} = K_i \Lambda^{\mu\nu}, \\
\{j_i, p_0\} = 0, \quad \{j_i, p_j\} = -\epsilon^k_{ij} p_k, \quad \{k_i, p_0\} = -p_i, \quad \{k_i, p_i\} = -p_0, \\
\{j_i, j_j\} = -\epsilon^k_{ij} j_k, \quad \{j_i, k_j\} = -\epsilon^k_{ij} k_k, \quad \{k_i, k_j\} = \epsilon^k_{ij} j_k.
\end{gathered}$$

As stated previously certain symplectic actions on cotangent bundles have guaranteed momentum maps. Given a symplectic action Φ on the configuration space Q, $\Phi : G \times Q \to Q$ we can define its **lift** onto the full cotangent bundle T^*Q by

$$\Phi^{T*} : G \times T^*Q \to T^*Q$$

$$(g, \alpha) \mapsto (\Phi_{g^{-1}})^* \alpha$$

This lifted action is also symplectic and has momentum map $J : T^*Q \to \mathfrak{g}^*$ given by

$$\langle J(\alpha_q), X \rangle := \theta(\Phi_X)(q) = \langle \alpha_q, \Phi_X(q) \rangle$$

for $X \in \mathfrak{g}$ and where Φ_X is the infinitesimal generator of the non-lifted action Φ on Q.

The momentum map of the lifted left action $(L_{g^{-1}})^*$ is a map $J : T^*G \to \mathfrak{g}^*$ given by

$$J(\alpha_g) = (R_g)^* \alpha_g \tag{2.10}$$

for $X \in \mathfrak{g}$. If the Hamiltonian $H : T^*G \to \mathbb{R}$ is left invariant (i.e. $H \circ (L_g)^* = H$ for all $g \in G$) then $\hat{J}(X)$ is constant on the orbits of X_H (i.e. $\{\hat{J}(X), H\} = 0$) for all $X \in \mathfrak{g}$ and X_H is left invariant. This is called the Euler conservation law [1].

We now restate the Euler conservation law in terms of our coordinate systems. In body coordinates the momentum map associated to the left action (i.e. (2.10) in body coordinates) is

$$\begin{aligned}
J^B : G \times \mathfrak{g}^* &\to \mathfrak{g}^* \\
(g, \mu) &\mapsto (Ad_{g^{-1}})^* \mu
\end{aligned}$$

So if $H : G \times \mathfrak{g}^* \to \mathbb{R}$ is left invariant (i.e. $H(g'g, \mu) = H(g, \mu)$ for all $g', g \in G, \mu \in \mathfrak{g}^*$) then $\hat{J}^B(X)$ is constant on the orbits of the left invariant vector field X_H for all $X \in \mathfrak{g}$. And in space coordinates (2.10) is given by

$$\begin{aligned} J^S : \mathfrak{g}^* \times G &\to \mathfrak{g}^* \\ (\nu, h) &\mapsto \nu \end{aligned}$$

So if $H : \mathfrak{g}^* \times G \to \mathbb{R}$ is left invariant (i.e. $H((Ad_{g'^{-1}})^*\nu, g'h) = H(\nu, h)$ for all $g', h \in G, \nu \in \mathfrak{g}^*$) then $\hat{J}^S(X)$ is constant on the orbits of the left invariant vector field X_H for all $X \in \mathfrak{g}$.

Example 2.8 (The rigid body cont'd)**.** We continue with example 2.6 using space coordinates. We let our Hamiltonian be $H(\mathbf{v}, R) = \frac{1}{2}\sum_i v_i^2 = \frac{1}{2}\mathbf{v}^T\mathbf{v}$, the total angular momentum. The coadjoint action of $SO(3)$ on $\mathfrak{so}(3)^*$ is simply the left action of $SO(3)$ on $\mathbb{R}^3$, i.e. $(Ad_S)^*\xi_{\mathbf{v}} = S\mathbf{v}$ where we have identified $\xi_{\mathbf{v}} \in \mathfrak{so}(3)^*$ with $\mathbf{v} \in \mathbb{R}^3$, as in the example on page 10. Hence, it is clear to see that this Hamiltonian is left invariant,

$$H(S^{-1}\mathbf{v}, SR) = \frac{1}{2}(S^{-1}\mathbf{v})^T S^{-1}\mathbf{v} = \frac{1}{2}\mathbf{v}^T\mathbf{v} = H(\mathbf{v}, R)$$

as $S \in SO(3)$ so $S^{-1} = S^T$. The momentum map is then given by $J^S(\mathbf{v}, R) = \mathbf{v}$. So we see that the angular momentum is conserved under time evolution, as expected. Recall that the angular momentum was conserved under rotational symmetry and now that the Hamiltonian is invariant under the same rotational action the angular momentum is also constant in time. Using the Poisson brackets found in example 2.6 this is expected since Hamilton's equations tell us

$$\begin{aligned} &\frac{d}{dt}v_i = \{H, v_i\} = 0, \\ &\frac{d}{dt}R = \{H, R\} = \sum_i v_i E_i R = \xi_{\mathbf{v}} R. \end{aligned}$$

From the first equation it is clear to see that $\mathbf{v}$ is conserved on the orbits of H.

Chapter 3

Poisson-Lie Groups and Lie Bialgebras

In the following we introduce the theory of Poisson-Lie groups. We begin by generalising symplectic manifolds to Poisson manifolds which increases the class of possible Poisson algebras of observables under consideration. It also allows us to expand the class of symmetries under consideration to those beyond the canonical symplectic structure preserving symmetries. We saw in section 2.1 that a symplectic structure naturally defines a Poisson bracket, however the converse is not always true. A Poisson bracket structure on a manifold is equivalent to equipping the manifold with a Poisson bivector. In the symplectic case the bivector can be seen as the negative of the 'inverse' of the symplectic form.

Next, we equip a Lie group with a compatible Poisson structure and so define a Poisson-Lie group. This is the first step to defining our new deformed phase spaces. We now have a way to attach a non-trivial Poisson structure to our particles configuration space. This will change the symplectic structure of the resulting phase space and introduce some quantum gravitational effects. The compatibility is with respect to the groups product and in turn defines strict relations at the infinitesimal level. The infinitesimal counterpart to a Poisson-Lie group is called a Lie bialgebra. These two components, Poisson-Lie groups and Lie bialgebras, determine each other as given by a theorem of Drinfeld. A special class of Lie bialgebras have their structure determined by an r-matrix that satisfies a special relation called the generalized Yang-Baxter equation which has a quantum counterpart the quantum Yang-Baxter equation.

Further, the study of Lie bialgebras naturally leads to the realization that their duals define compatible Lie bialgebras. A Lie bialgebra and its dual can be glued together by choice of canonical r-matrix to form a double Lie bialgebra. This double in turn has a corresponding unique (connected, simply-connected) Poisson-Lie group associated to it called the Drinfeld double. The underlying double group of this Poisson-Lie group is the manifold of our new deformed phase space. It consists of a compatible pair of Poisson-Lie groups of equal dimension that can be identified with configuration and momentum space. We shall see that both spaces can be non-abelian groups and that the cotangent bundle is a limiting case where the configuration space is equipped with a trivial Poisson structure. At this stage we have not yet equipped the double group with the proper Poisson structure to construct our deformed phase spaces. That will be done in the following section where we define Heisenberg doubles. Much of this material can be found in [16, 27].

3.1 Poisson Manifolds

A Poisson structure on a manifold M is a bilinear map

$$\{\ ,\ \} : \mathcal{F}(M) \times \mathcal{F}(M) \to \mathcal{F}(M)$$

on the algebra of functions $\mathcal{F}(M)$ on M called the **Poisson bracket** satisfying,

- skew-symmetry: $\{f, g\} = -\{g, f\}$
- Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
- Leibniz identity: $\{fg, h\} = f\{g, h\} + \{f, h\}g$

for all $f, g, h \in \mathcal{F}(M)$. Note that $\{\ ,\ \}$ is a Lie bracket on $\mathcal{F}(M)$ by the first two properties above. This bracket is equivalent to having a skew-symmetric contravariant 2-tensor $\Pi \in \bigwedge^2 TM$ called the **Poisson bivector** such that

$$\{f, g\} = \Pi(df, dg)$$

where $df, dg \in T^*M$. In coordinates we can write,

$$\Pi(p) = \frac{1}{2}\Pi^{ij}(p)\frac{\partial}{\partial x^i}\bigg|_p \otimes \frac{\partial}{\partial x^j}\bigg|_p = \sum_{i,j}\{x^i, x^j\}(p)\frac{\partial}{\partial x^i}\bigg|_p \otimes \frac{\partial}{\partial x^j}\bigg|_p$$

for $p \in M$ so that Π is a Poisson bivector if and only if,

$$\Pi^{il}\frac{\partial \Pi^{jk}}{\partial x^l} + \Pi^{jl}\frac{\partial \Pi^{ki}}{\partial x^l} + \Pi^{kl}\frac{\partial \Pi^{ij}}{\partial x^l} = 0 \tag{3.1}$$

for all cyclic permutations of i, j, k. We can see that this condition is equivalent to the bracket $\{\ ,\ \}$ satisfying the Jacobi identity. A **Poisson manifold** denoted $(M, \{\ ,\ \})$ or (M, Π) is a manifold M with a Poisson structure given by a Poisson bracket $\{\ ,\ \}$ or equivalently, a Poisson bivector Π (sometimes the Poisson structure will be denoted $\{\ ,\ \}_M$ or Π_M if confusion is possible). Equation 3.1 is an instance of a special type of product on multivector fields called the **Schouten bracket** $[|\ ,\ |]$ (see Appendix B). Equation 3.1 is equivalent to

$$[|\Pi, \Pi|] = 0,$$

the vanishing of the Schouten bracket of Π with itself.

In the previous section we saw that the Poisson bracket defines the Hamiltonian vector field X_f of a function $f \in \mathcal{F}(M)$ via the equation $X_f = \{f, \cdot\}$. This can now be equivalently written as $X_f = \Pi(df, \cdot)$. Thus we see that the Poisson bivector Π provides a map

$$\begin{aligned} \Pi : T^*M &\to TM \\ df &\mapsto X_f \end{aligned}$$

The mapping of a function to its Hamiltonian vector field $f \mapsto X_f$ is a homomorphism of the Lie algebra $\mathcal{F}(M)$ with Poisson bracket and the Lie algebra of vector fields [23]. This gives us an alternative necessary and sufficient condition for Π to be a Poisson bivector which is that the following holds

$$X_{\{f,g\}} = [X_f, X_g]$$

for all $f, g \in \mathcal{F}(M)$. This equation hints at the relations that will be found at the infinitesimal level of a Poisson-Lie group.

Symplectic manifolds are always Poisson manifolds, however the converse is not in general true. A Poisson manifold is symplectic if the Poisson bivector is everywhere non-degenerate, i.e. $\Pi^{ij}(p)$ is invertible for all $p \in M$. In this case we can define the symplectic form by $\omega = -\Pi^{-1}$ (more specifically, $\omega_{ik}\Pi^{kj} = -\delta_i^j$). We will see that the Poisson structure of a Poisson-Lie group is never symplectic since Π must vanish at the identity.

Example 3.1. If $M = \mathbb{R}^n \times \mathbb{R}^n$ with global coordinates (q^i, p_i) the canonical Poisson bivector is given by $\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}$ with corresponding brackets given by $\{q_i, q_j\} = \{p_i, p_j\} = 0, \{p_j, q^i\} = \delta_j^i$. This Poisson bivector can be represented by the constantmatrix

$$\Pi^{ij} = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ -\mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \end{pmatrix}.$$

Example 3.2 (The linear Poisson structure)**.** Noting that a Poisson bracket is a Lie bracket on the algebra of functions we can see that the dual $\mathfrak{g}^*$ of a finite dimensional Lie algebra $\mathfrak{g}$ is naturally a Poisson manifold. Let $\zeta \in \mathfrak{g}^*$ and $Z \in \mathfrak{g}$. As vector spaces $\mathfrak{g}$ is isomorphic to $T^*_\zeta \mathfrak{g}^*$ so we can consider $Z \in T^*_\zeta \mathfrak{g}^* \subset \mathcal{F}(\mathfrak{g}^*)$. Then

$$\Pi(Z, \cdot)(\zeta) = -ad^*_Z \zeta \in T_\zeta \mathfrak{g}^* \simeq \mathfrak{g}^*$$

where ad^* satisfies $\langle \zeta, ad_Z Y \rangle = -\langle ad^*_Z \zeta, Y \rangle$. This is called the **linear Poisson structure** or sometimes the **Berezin-Kirillov-Kostant-Souriau Poisson structure** (or some subset of these names). In particular, if $X_1, \ldots, X_n$ is a basis of $\mathfrak{g}$ then

$$\Pi^{ij}(\xi) = \langle \xi, [X_i, X_j] \rangle$$

As we shall see below this is part of a bigger picture involving Poisson-Lie groups and their duals.

Example 3.3. Consider the Lie algebra $\mathfrak{su}(2)$ with basis σ_i satisfying the relations $[\sigma_i, \sigma_j] = \epsilon^k_{ij} \sigma_k$. Let ξ^i be the basis of $\mathfrak{su}(2)^*$ with coordinate functions x^i so an element ξ in the dual can be written $\xi = \sum_i x^i \xi^i$. The linear Poisson structure on the dual then reads,

$$\Pi(\xi) = \sum_{i,j,k} \epsilon^k_{ij} x^k \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}.$$

In order to compare Poisson manifolds and to build new ones we define the following notions. The direct product of two Poisson manifolds $M \times N$ has a **product Poisson structure** given by

$$\{f, h\}_{M \times N}(x, y) = \{f(\cdot, y), h(\cdot, y)\}_M(x) + \{f(x, \cdot), h(x, \cdot)\}_N(y), \tag{3.2}$$

where $x \in M, y \in N$ and $f(\cdot, y)$ denotes f with y fixed. This product Poisson structure has an associated bivector field $\Pi_{M \times N}$ on $M \times N$ which under the projection of $M \times N$ onto M and N gives Π_M and Π_N, respectively. Identifying $\mathcal{F}(M \times N)$ with an appropriate tensor product $\mathcal{F}(M) \otimes \mathcal{F}(N)$ we may write the product Poisson bracket as

$$\{f_1 \otimes f_2, h_1 \otimes h_2\}_{M \times N} = \{f_1, h_1\}_M \otimes f_2 h_2 + f_1 h_1 \otimes \{f_2, h_2\}_N.$$

A map $\phi : M \to N$ between Poisson manifolds M, N is a **Poisson map** if

$$\{f \circ \phi, h \circ \phi\}_M = \{f, h\}_N \circ \phi \tag{3.3}$$

or equivalently,

$$\phi_* \Pi_M(x) = \Pi_N(\phi(x)).$$

That is, the map ϕ is consistent with the Poisson structure in the sense that the induced map $\phi_* : \mathcal{F}(M) \to \mathcal{F}(N)$ is a homomorphism of Poisson brackets [23]. Poisson maps are a generalisation of symplectic maps. The product Poisson structure is the unique bracket on $M \times N$ such that the natural projections $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ are Poisson maps.

Before we define Poisson-Lie groups let us consider the following scenario. A group G acts on a Poisson manifold M

$$\begin{aligned} \Phi : G \times M &\to M \\ (g, p) &\mapsto \Phi(g, p). \end{aligned}$$

Just as the group actions on symplectic manifolds we considered were symplectic in order to respect the phase space structure we can demand the group action Φ be a Poisson map so it is consistent with the Poisson bracket. Physically, this is equivalent to demanding that the Poisson brackets be invariant under the group action. For example, we may require this if we wish the Poisson brackets be consistent under rotations or Poincaré transformations. In order to maintain consistency of the brackets we have the freedom to equip the group G with a non-trivial Poisson structure. This is an important first step towards constructing our deformed phase spaces. By (3.2) and (3.3) we require

$$\begin{aligned} \{f, h\}_M \circ \Phi(g, p) &= \{f \circ \Phi, h \circ \Phi\}_{G \times M}(g, p) \\ &= \{f \circ \Phi(\cdot, p), h \circ \Phi(\cdot, p)\}_G(g) + \{f \circ \Phi(g, \cdot), h \circ \Phi(g, \cdot)\}_M(p) \end{aligned} \tag{3.4}$$

for $f, h \in \mathcal{F}(M)$.

Example 3.4. (A motivating example) Let $G = SU(2)$ and $M = \mathbb{C}^2$ with global coordinates (z_1, z_2). The canonical Poisson structure is

$$\{z_i, \bar{z}_j\}_M = -i\delta_{ij} \tag{3.5}$$

with all other brackets zero. We may write an element of $SU(2)$ as $g = \begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$ where $\alpha, \gamma \in \mathbb{C}$ and $\alpha\bar{\alpha} + \gamma\bar{\gamma} = 1$. Under the group action the coordinates of $\mathbb{C}^2$ transform as

$$(g, z_i) \mapsto \sum_j g_{ij} z_j = z_i', \qquad (g, \bar{z}_i) \mapsto \sum_j \bar{g}_{ij} \bar{z}_j = \bar{z}_i'.$$

Now by demanding consistency of the group action with the bracket, from the left hand side of (3.4) we desire

$$\{z'_i, \bar{z}'_j\}_M = \sum_{l,k} \{g_{il} z_l, \bar{g}_{jk} \bar{z}_k\}_M = \sum_{l,k} g_{il} \bar{g}_{jk} \{z_l, \bar{z}_k\}_M = -i \sum_{l,k} g_{il} \bar{g}_{jk} \delta_{lk} = -i\delta_{ij}$$

and from the right hand side of (3.4) we also have that

$$\{z'_i, \bar{z}'_j\}_M = \sum_{l,k} \{g_{il}, \bar{g}_{jk}\}_G z_l \bar{z}_k - i\delta_{ij}.$$

Thus it follows that $\{g_{il}, \bar{g}_{jk}\}_G = 0$ maintains the Poisson brackets under the group action. That is, a trivial Poisson structure on $SU(2)$ is required to maintain a consistent Poisson structure with the canonical Poisson structure (3.5). Now, if we equip $\mathbb{C}^2$ with a non-standard Poisson structure let us see what happens to the compatible Poisson structure on $SU(2)$.

Consider $\mathbb{C}^2$ with the following non-standard Poisson bracket, as given in [37],

$$\{z_1, z_2\}_M = \frac{i}{\beta} z_1 z_2, \tag{3.6}$$

$$\{z_1, \bar{z}_2\}_M = \frac{i}{\beta} z_1 \bar{z}_2, \tag{3.7}$$

$$\{z_1, \bar{z}_1\}_M = -i \left(1 - \frac{2}{\beta}(z_1 \bar{z}_1)\right), \tag{3.8}$$

$$\{z_2, \bar{z}_2\}_M = -i \left(1 - \frac{2}{\beta}(z_1 \bar{z}_1 + z_2 \bar{z}_2)\right) \tag{3.9}$$

where β is a deformation parameter and all other brackets are zero. Demanding consistency we require these equations to hold with all variables primed. Similar to our previous example the first bracket relation (3.6) gives us the following two equations,

$$\{z'_1, z'_2\}_M = \frac{i}{\beta} z'_1 z'_2 = \sum_{l,k} \frac{i}{\beta} g_{1l} g_{2k} z_l z_k$$

$$\{z'_1, z'_2\}_M = \sum_{l,k} \{g_{1l}, g_{2k}\}_G z_l z_k + \frac{i}{\beta} z_1 z_2$$

using the fact $g_{11} g_{22} - g_{12} g_{21} = \alpha\bar{\alpha} + \gamma\bar{\gamma} = 1$. Putting these two together

$$\sum_{l,k} \{g_{1l}, g_{2k}\}_G z_l z_k = \frac{i}{\beta} \left(\sum_{l,k} g_{1l} g_{2k} z_l z_k - z_1 z_2 \right).$$

Since this holds for all z_l, z_k we can read off the following three equations;

$$\{g_{12}, g_{22}\}_G = \frac{i}{\beta} g_{12} g_{22}, \tag{3.10}$$

$$\{g_{11}, g_{21}\}_G = \frac{i}{\beta} g_{11} g_{21}, \tag{3.11}$$

$$\{g_{11}, g_{22}\}_G + \{g_{12}, g_{21}\}_G = \frac{i}{\beta}(g_{11} g_{22} + g_{12} g_{21} - 1) \tag{3.12}$$

Notice (3.11) is just the conjugate of (3.10), thus it contains no new information since $\overline{\{a, b\}} = \{\bar{a}, \bar{b}\}$.

In a similar manner the second and third bracket relations, (3.7) and (3.8), tell us that

$$\{g_{11}, \bar{g}_{21}\}_G = \frac{i}{\beta} g_{11} \bar{g}_{21}, \tag{3.13}$$

$$\{g_{11}, \bar{g}_{11}\}_G = -\frac{2i}{\beta} g_{12} \bar{g}_{12}. \tag{3.14}$$

Combining (3.14) with (3.12) we get,

$$\{g_{12}, g_{21}\}_G = \frac{i}{\beta}(g_{11} g_{22} + g_{12} g_{21} - 1) + \frac{2i}{\beta} g_{12} \bar{g}_{12} \tag{3.15}$$

Writing (3.10), (3.13), (3.14), (3.15) in terms of the matrix entries and simplifying, the Poisson brackets on $SU(2)$ consistent with the Poisson structure given by (3.6)-(3.9) on $\mathbb{C}^2$ are the following:

$$\{\alpha, \bar{\alpha}\} = -\frac{2i}{\beta}\gamma\bar{\gamma}, \quad \{\alpha, \gamma\} = \frac{i}{\beta}\alpha\gamma$$
$$\{\alpha, \bar{\gamma}\} = \frac{i}{\beta}\alpha\bar{\gamma}, \quad \{\gamma, \bar{\gamma}\} = 0.$$

This is in fact (up to isomorphism) the only non-trivial Poisson structure on $SU(2)$ that is compatible with its group product. In other words this is the only non-trivial Poisson-Lie structure on $SU(2)$ [29]. Also note the limit $\beta \to \infty$ recovers the canonical (trivial) Poisson bracket.

3.2 Poisson-Lie Groups

A **Poisson-Lie group** is a Lie group G with a compatible Poisson structure $\{\ ,\ \}$ so that group multiplication $\mu : G \times G \to G$ is a Poisson map,

$$\{f, h\}_G \circ \mu = \{f \circ \mu, h \circ \mu\}_{G \times G}$$

for all $f, h \in \mathcal{F}(G)$ where $G \times G$ is endowed with the product Poisson structure. In more explicit terms this condition reads, using (3.2) and (3.3),

$$\{f, h\}(gg') = \{f \circ L_g, h \circ L_g\}(g') + \{f \circ R_{g'}, h \circ R_{g'}\}(g) \tag{3.16}$$

for all $g, g' \in G$. This compatibility relation is not strong enough to single out a unique compatible Poisson structure for a given Lie group since, for example, the trivial Poisson structure is always an option. Classifying all such compatible Poisson structures for Lie groups is an open area of research.

The above compatibility relation is equivalent to the Poisson bivector Π on G being **multiplicative**. That is, Π satisfies

$$\Pi(gg') = (L_g)_*\Pi(g') + (R_{g'})_*\Pi(g) \tag{3.17}$$

where $(L_g)_*$ and $(R_{g'})_*$ denote the differential maps of left and right translations by g and g' extended to multivector fields [1]. We may note immediately that $\Pi(e) = 0$ by setting g, g' to e in (3.17). Thus Poisson-Lie groups cannot be symplectic since the Poisson structure is degenerate at the identity.

Example 3.5 (Poisson-Lie structures on $\mathbb{R}^n$)**.** We can already find all the compatible Poisson structures of the abelian group $\mathbb{R}^n$. We begin by considering the possible compatible Poisson bivectors. Using global coordinates we let $\mathbf{x} = (x^i), \mathbf{y} = (y^i) \in \mathbb{R}^n$. For a bivector Π to be multiplicative

$$\Pi(\mathbf{x} + \mathbf{y}) = (L_{\mathbf{x}})_*\Pi(\mathbf{y}) + (R_{\mathbf{y}})_*\Pi(\mathbf{x}) = \Pi(\mathbf{y}) + \Pi(\mathbf{x})$$

since the group product is simple addition. This tells us that the component functions Π^{ij} are linear in the coordinates so $\Pi^{ij}(\mathbf{x}) = \sum_l \Pi^{ij}_l x^l$. Using this fact and equation 3.1 Π is a Poisson bivector if and only if

$$\Pi^{il}_m \Pi^{jk}_l + \Pi^{jl}_m \Pi^{ki}_l + \Pi^{kl}_m \Pi^{ij}_l = 0.$$

This is exactly the relation the structure constants of an n dimensional Lie algebra. Thus the compatible Poisson bivectors on $\mathbb{R}^n$ are given by

$$\Pi(\mathbf{x}) = \Pi^{ij}_l x^l \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

[1] Explictly, $(L_g)_*\Pi(df, dh)(g') = \Pi(d(f \circ L_g), d(h \circ L_g))(g') = \{f \circ L_g, h \circ L_g\}(g')$. Thus it is clear to see the connection between (3.16) and (3.17).

where Π_l^{ij} are the structure constants of any n dimensional Lie algebra. The corresponding Poisson bracket is given by

$$\{f, g\}(\mathbf{x}) = \frac{1}{2}\Pi_l^{ij} x^l \left(\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j} - \frac{\partial f}{\partial x^j}\frac{\partial g}{\partial x^i} \right)(\mathbf{x}).$$

From this we can see that a flat Poisson-Lie group (i.e. an abelian Poisson-Lie group) is intimately connected with another Lie group through its Lie algebra structure which comes from the compatibility condition. We will see that this holds for general Poisson-Lie groups when we study Poisson-Lie groups at the infinitesimal level in the next section.

Example 3.6 ($\mathfrak{g}^*$ as a Poisson-Lie group)**.** The dual $\mathfrak{g}^*$ of Lie algebra $\mathfrak{g}$ regarded as an abelian Lie group is a Poisson-Lie group when equipped with the linear Poisson structure. See example 3.2.

We note the above two examples are in fact the same, just approached from different perspectives. In the first, we begin with $\mathbb{R}^n$ and find, through the compatibility relations, a related Lie group G. In the second, we begin with Lie group G and its Lie algebra $\mathfrak{g}$. Then on its dual $\mathfrak{g}^*$ we define a Poisson structure using the structure constants of $\mathfrak{g}$. These are equivalent since as we shall see there exists a symmetry between a Poisson-Lie group and its dual. In particular, it is important to note that any Lie group G, which we may identify as a configuration space, can be paired with a flat group of equal dimension. This, we will see, is exactly a copy of the groups cotangent bundle T^*G.

Let (G, Π_G) be a Poisson-Lie group and (M, Π_M) a Poisson manifold with $\Phi : G \times M \to M$ an action of G on M the this action is a **Poisson action** if Φ is a Poisson map, i.e. it satisfies

$$\{f, h\}_M \circ \Phi(g, x) = \{f \circ \Phi(\cdot, x), h \circ \Phi(\cdot, x)\}_G(g) + \{f \circ \Phi(g, \cdot), h \circ \Phi(g, \cdot)\}_M(x). \quad (3.18)$$

Note if the Poisson structure on G is trivial (i.e. $\Pi_G = 0$) then the action simply preserves the bracket on M much like symplectic actions considered in chapter [**?**]. Using a general Poisson-Lie group we can introduce an action that does not necessarily preserve the Poisson structure on M. We saw this in example 3.4 which motivated our consideration of Poisson-Lie groups.

Example 3.7. For any Lie group G with Lie algebra $\mathfrak{g}$, the action of $\mathfrak{g}^*$ on T^*G defined by

$$\begin{aligned} \Phi : \mathfrak{g}^* \times T^*G &\to T^*G \\ (\xi, \alpha_g) &\mapsto \alpha_g + (L_{g^{-1}})^*\xi \end{aligned}$$

is a Poisson action.

An important class of Poisson-Lie structures is given by bivectors of the form

$$\Pi(g) = (L_g)_*\Lambda - (R_g)_*\Lambda$$

where $\Lambda \in \mathfrak{g} \otimes \mathfrak{g}$. It is simple to check this bivector is multiplicative

$$\begin{aligned}
\Pi(gg') &= (L_{gg'})_*\Lambda - (R_{gg'})_*\Lambda \\
&= (L_g)_*(L_{g'})_*\Lambda - (R_{g'})_*(R_g)_*\Lambda \\
&= (L_g)_*[(L_{g'})_*\Lambda - (R_{g'})_*\Lambda] - (R_{g'})_*[(R_g)_*\Lambda - (L_g)_*\Lambda] \\
&= (L_g)_*\Pi(g') + (R_{g'})_*\Pi(g).
\end{aligned}$$

Poisson-Lie structures of this type are in fact quite natural and intimately related to Lie theory cohomology. They arise from the r-matrix formalism which will be presented in section 3.4. The following is a corollary of Whitehead's Lemma (see Appendix A) given by Drinfeld

Theorem 2. *Every multiplicative Poisson structure on a connected semi-simple or compact Lie group G is of the form:*

$$\Pi(g) = (L_g)_*\Lambda - (R_g)_*\Lambda$$

where $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ and $[|\Lambda, \Lambda|] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is invariant under the adjoint action of G.

Example 3.8 (SU(2) as a Poisson-Lie group)**.** Let us consider the group $G = SU(2)$. Since $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is one dimensional and $(Ad_g)_*$ invariant for all $g \in G$, any $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$, $[|\Lambda, \Lambda|]$ is $(Ad_g)_*$ invariant for all $g \in G$ [29]. Let $\sigma_1, \sigma_2, \sigma_3$ be a basis of $\mathfrak{g} = \mathfrak{su}(2)$ where

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

and $[\sigma_i, \sigma_j] = \epsilon_{ij}^k \sigma_k$. Now letting $\Lambda = -\frac{2}{\beta}(\sigma_2 \wedge \sigma_3)$ the Poisson structure on $SU(2)$ is given by

$$\Pi(g) = -\frac{2}{\beta}((L_g)_*(\sigma_2 \wedge \sigma_3) - (R_g)_*(\sigma_2 \wedge \sigma_3)).$$

This yields the same Poisson bracket as given in example 3.4.

3.3 Lie Bialgebras

A Lie bialgebra is the linearization of a Poisson-Lie structure at the identity of a group. Just as Lie algebras enlighten us about groups, Lie bialgebras play a fundamental role in understanding Poisson-Lie groups.

Recall $\{\,,\,\}$ is a Lie bracket on the algebra $\mathcal{F}(G)$ and that in order for Π to be a Poisson bivector the following must hold

$$X_{\{f,h\}} = [X_f, X_h] \tag{3.19}$$

for all $f, h \in \mathcal{F}(G)$. Considering this statement about the group identity, we expect a relation between the group's Lie algebra and Poisson structure to hold. Let us look at what happens infinitesimally about the identity 1 of a simple Poisson-Lie group (G, Π). We let $X \in \mathfrak{g} = T_1 G$ and consider the element $g = 1 + X \in G$ close to the identity. Then to first order

$$\begin{aligned}\Pi(1+X) &= (1+X)\otimes(1+X)\Lambda - \Lambda(1+X)\otimes(1+X)\\ &= [X\otimes 1 + 1\otimes X, \Lambda] + \dots\end{aligned}$$

The map $\delta(\Lambda) : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g} : X \mapsto [X\otimes 1 + 1\otimes X, \Lambda]$ is an example of 1-cocycle and is in fact the extra structure we are looking for.

There is more to say about equation 3.19. It also tells us we can expect there to be a relation between the Poisson structure and the dual of the group's Lie algebra. Since $X_f = \Pi(df, \cdot)$ for $f \in \mathcal{F}(G)$ and $df \in T^*G$ we can, again by considering (3.19) about the identity, define a bracket on $T_e^*G \simeq \mathfrak{g}^*$ by the following,

$$d\{f,h\}(e) = [df_e, dh_e]^*.$$

From this we see that $[\,,\,]^*$ is skew symmetric and satisfies the Jacobi identity. So provided $[\,,\,]^*$ is well-defined this defines a canonical Lie-bracket on $\mathfrak{g}^*$.

As we shall later see these two insights are intimately related and reveal an underlying symmetry of Lie bialgebras. This symmetry will be later used in the study of Poisson-Lie groups and the notion of the double. At this point, we have enough motivation to define Lie bialgebras.

A **Lie bialgebra** $(\mathfrak{g}, \delta)$ is a Lie algebra $\mathfrak{g}$ with a linear map $\delta : \mathfrak{g} \to \mathfrak{g}\otimes\mathfrak{g}$ called a **cocommutator** such that

1. the transpose of δ defines a Lie bracket $[\,,\,]^*$ on $\mathfrak{g}^*$ via the canonical pairing $\langle\cdot,\cdot\rangle$ of $\mathfrak{g}$ and its dual $\mathfrak{g}^*$, i.e.

$$\langle \delta(X), \xi\otimes\zeta\rangle = \langle X, [\xi,\zeta]^*\rangle \tag{3.20}$$

for $X \in \mathfrak{g}$ and $\xi, \zeta \in \mathfrak{g}^*$.

2. δ is a 1-cocycle of $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$, i.e. $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is linear and

$$\delta([X, Y]) = [X \otimes 1 + 1 \otimes X, \delta(Y)] + [\delta(X), Y \otimes 1 + 1 \otimes Y] \tag{3.21}$$

where it is understood that $[X \otimes 1 + 1 \otimes X, Y \otimes Z] = [X, Y] \otimes Z + Y \otimes [X, Z]$ for all $X, Y, Z \in \mathfrak{g}$.

The preceeding discussion and expression 3.20 hints at the symmetry between the structures on $\mathfrak{g}$ and $\mathfrak{g}^*$. The **dual** of a Lie bialgebra $(\mathfrak{g}, \delta)$ is a Lie bialgebra denoted $(\mathfrak{g}^*, \delta^*)$ where δ^* is the transpose of the Lie bracket of $\mathfrak{g}$, i.e. $\langle X \otimes Y, \delta^*(\xi) \rangle = \langle [X, Y], \xi \rangle$. The symmetry between a Lie bialgebra and its dual is now readily apparent. The bracket of $\mathfrak{g}$ defines a cocommutator on $\mathfrak{g}^*$ and the cocommutator of $\mathfrak{g}$ defines a Lie bracket on $\mathfrak{g}^*$, and vice-versa. This is why bialgebras are often denoted $(\mathfrak{g}, \mathfrak{g}^*)$ since knowledge of the dual and its bracket is equivalent to knowing the cocommutator δ on $\mathfrak{g}$. The relations between the Lie brackets and cocommutators are captured by the arrows in Figure 3.1. Further we note that the dual of the dual of a Lie bialgebra is the original Lie bialgebra. Also if $(\mathfrak{g}, \delta)$ is a Lie bialgebra then $(\mathfrak{g}, k\delta)$ is a Lie bialgebra for any scalar k. This will be important since we can introduce our deformation parameter here.

$$\begin{tikzcd} \big((\mathfrak{g}, [\ ,\ ]), \arrow[dr, leftrightarrow] & \delta\big) \arrow[dl, leftrightarrow] \\ \big((\mathfrak{g}^*, [\ ,\ ]^*), & \delta^*\big) \end{tikzcd}$$

Figure 3.1: The relations of a Lie bialgebra $(\mathfrak{g}, \delta)$ and its dual $(\mathfrak{g}^*, \delta^*)$

If we choose a particular basis of $(\mathfrak{g}, \delta)$ that satisfies

$$[X_i, X_j] = c_{ij}^k X_k, \quad \delta(X_l) = f_l^{nm} X_n \otimes X_m$$

then the dual basis of $(\mathfrak{g}^*, \delta^*)$ satisfies [11],

$$[\xi^i, \xi^j] = f_k^{ij} \xi^k, \quad \delta^*(\xi^l) = c_{nm}^l \xi^n \otimes \xi^m$$

and the 1-cocycle condition (3.21) reads as a compatibility condition between the structure constants

$$c_{ij}^k f_k^{mn} = c_{ik}^m f_j^{kn} + c_{ik}^n f_j^{mk} + c_{kj}^m f_i^{kn} + c_{kj}^n f_i^{mk}.$$

The infinitesimal structure of Poisson-Lie groups are Lie bialgebras. The Poisson structure Π and the cocommutator δ are connected via the intrinsic derivative. The **intrinsic derivative** at the identity of a contravariant k-tensor P on G which is trivial at the identity is a linear map

$$\begin{aligned} DP : \mathfrak{g} &\to \bigotimes^{k} \mathfrak{g} \\ X &\mapsto [\mathcal{L}_{\hat{X}} P](e) \end{aligned}$$

for all $X \in \mathfrak{g}$, where $\hat{X}$ is an arbitrary vector field on G satisfying $\hat{X}(e) = X$ and $\mathcal{L}$ is the Lie derivative. The Poisson bivector of a Poisson-Lie group (G, Π) defines the cocommutator of its associated Lie bialgebra $(\mathfrak{g}, \delta)$ via the formula

$$\delta(X) := D\Pi(X) = \mathcal{L}_{X^L}\Pi$$

for all $X \in \mathfrak{g}$. In a basis, this expression reads

$$\begin{aligned} \delta(X_k) &= (\mathcal{L}_{X_k^L}\Pi)(e) \\ &= [\mathcal{L}_{X_k^L}(\Pi^{ij} X_i^L \otimes X_j^L)](e) \\ &= (X_k^L \Pi^{ij})(e) X_i \otimes X_j + \Pi^{ij}(e)[X_k, X_i] \otimes X_j + \Pi^{ij}(e) X_i \otimes [X_k, X_j] \\ &= (X_k^L \Pi^{ij})(e) X_i \otimes X_j \\ &= \frac{d}{dt}\bigg|_{t=0} \Pi^{ij}(\exp t X_k) X_i \otimes X_j \\ &= f_k^{ij} X_i \otimes X_j \end{aligned}$$

where we take advantage of the fact $\Pi(e) = 0$. The skew symmetry of the structure constants f_k^{ij} of $\mathfrak{g}^*$ clearly comes from the skew-symmetry of Π^{ij}. Further, the fact that constants f_k^{ij} satisfy the Jacobi identity comes from the condition $[|\Pi, \Pi|] = 0$, as mentioned earlier. The following is a theorem of Drinfeld's [31] which makes the correspondence between Lie bialgebras and Poisson-Lie groups clear. This relation is captured in Figure 3.2.

Theorem 3. *Let (G, Π) be a Poisson-Lie group with Lie$(G) = \mathfrak{g}$ and $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}, X \mapsto D\Pi(X)$. Then $(\mathfrak{g}, \delta)$ is a Lie bialgebra which is said to be associated to the Poisson-Lie group (G, Π).*

Conversely, let $(\mathfrak{g}, \delta)$ be a Lie bialgebra and let G be the connected, simply connected Lie group with Lie$(G) = \mathfrak{g}$. Then there exists a unique Poisson structure Π on G such that (G, Π) is a Poisson-Lie group whose associated Lie bialgebra is isomorphic to $(\mathfrak{g}, \delta)$.

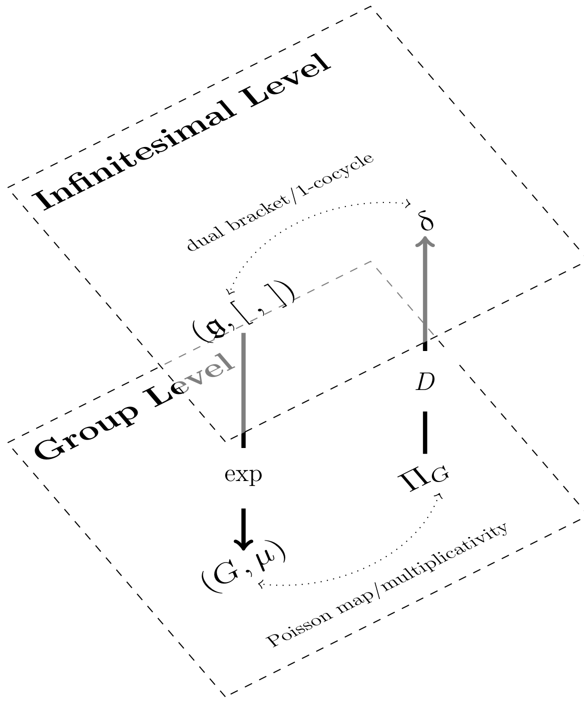

Figure 3.2: The relation of a Poisson-Lie group (G, Π_G) to its Lie bialgebra $(\mathfrak{g}, \delta)$

Thus to a Poisson-Lie group (G, Π) there is an associated Lie bialgebra $(\mathfrak{g}, \delta)$ which has a natural dual $(\mathfrak{g}^*, \delta^*)$ that in turn is related to a unique connected, simply connected Poisson-Lie group (G^*, Π^*) called the **dual Poisson Lie group**. These relations are captured by the arrows in Figure 3.3. Note that in general the dual of (G^*, Π^*) may not be (G, Π) because G need not be connected and simply connected to begin with.

Example 3.9. There is one non-abelian 2 dimensional Lie algebra up to isomorphism which we call $\mathfrak{e}^2$. Denoting the generators by X, Y it can identified by the relations

$$[X, X] = [Y, Y] = 0, \quad [X, Y] = Y.$$

There are only two possible commutators (up to isomorphism and scalar multiples) on $\mathfrak{e}^2$ given by $\delta^\alpha(X) = 0, \delta^\alpha(Y) = \alpha X \wedge Y$ and $\delta^\beta(X) = \beta X \wedge Y, \delta^\beta(Y) = 0$ [16]. If $\alpha = 0$ or $\beta = 0$ the dual Lie bialgebra is the 2-dimensional abelian Lie algebra, for $\alpha \neq 0$ the dual is a scaled version of $\mathfrak{e}^2$ with $[\tilde{X}, \tilde{Y}]^* = \alpha \tilde{Y}$ and for $\beta \neq 0$ the dual is a scaled isomorphic copy of $\mathfrak{e}^2$ with $[\tilde{X}, \tilde{Y}]^* = \beta \tilde{X}$ where $\tilde{X}, \tilde{Y}$ are the generators of the dual

Example 3.10 (Poisson-Lie structures on SU(2))**.** Consider the group $G = SU(2)$. The Lie algebra $\mathfrak{su}(2)$ has basis σ_i satisfying $[\sigma_i, \sigma_j] = \epsilon_{ij}^k \sigma_k$ (see example 3.8). From cohomological

$$\begin{tikzcd}
(G, \arrow[rr, leftrightarrow, bend left] & \Pi_G) \arrow[r, twoheadleftrightarrow] \arrow[rr, leftrightarrow, bend left] & ((\mathfrak{g},[\ ,\ ]), \arrow[dr, leftrightarrow] & \delta) \arrow[dl, leftrightarrow] \\
(G^*, \arrow[rr, leftrightarrow, bend right] & \Pi_{G^*}) \arrow[r, twoheadleftrightarrow] \arrow[rr, leftrightarrow, bend right] & ((\mathfrak{g}^*,[\ ,\ ]^*), & \delta^*)
\end{tikzcd}$$

Figure 3.3: The relations of a Poisson-Lie group (G, Π_G) and its dual (G^*, Π_{G^*})

arguments it is known this Lie algebra can be equipped with two (non-isomorphic) bialgebra structures.

The simplest is the trivial bialgebra structure $(\mathfrak{su}(2), 0)$ where $\delta(\sigma_i) = 0$ for all $i = 1, 2, 3$. The dual Lie bialgebra then has a corresponding trivial bracket and can be identified with $\mathbb{R}^3$. The Poisson-Lie dual of $(SU(2), 0)$ is then $\mathbb{R}^3$ with the linear Poisson structure inherited from $\mathfrak{su}(2)$.

The second bialgebra structure (up to isomorphism and scaling) corresponds to the element $\Lambda = -\frac{2}{\beta}(\sigma_2 \wedge \sigma_3)$ where we can interpret β as a scale relevant to quantum gravity. As we shall see in section **??** Λ is an example of an r-matrix. The commutator defined by Λ on the basis elements is

$$\delta(\sigma_1) = 0, \quad \delta(\sigma_2) = \frac{2}{\beta}\sigma_1 \wedge \sigma_2, \quad \delta(\sigma_3) = \frac{2}{\beta}\sigma_1 \wedge \sigma_3.$$

If we denote the dual basis by ρ^i then the bracket on the dual is given by

$$[\rho^1, \rho^2]^* = \frac{2}{\beta}\rho^2, \quad [\rho^1, \rho^3]^* = \frac{2}{\beta}\rho^3, \quad [\rho^2, \rho^3]^* = 0.$$

The dual $\mathfrak{su}(2)^*$ can be identified with the Lie algebra of the book group $SB(2, \mathbb{C})$. The basis of $\mathfrak{sb}(2, \mathbb{C})$ can be represented by

$$\rho_1 = \frac{1}{\beta}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho_2 = \frac{1}{\beta}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho_3 = \frac{1}{\beta}\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}.$$

Thus the dual of $SU(2)$ with non-trivial Poisson structure is $SB(2, \mathbb{C})$. Note that in the limit $\beta \to \infty$ we recover the previous trivial case. Thus we can see that the introduction of a scale β relevant to the quantum gravity regime deforms the dual structures.

3.4 Coboundary Lie Bialgebras and r-matrices

We now look at an important class of Lie bialgebras. These bialgebras have cocommutators that are given by an r-matrix. The study of r-matrices originated from the study of inverse scattering methods in the theory of integrable systems. The reader is encouraged to read Appendix A to familiarize themselves with the basics of Lie theory cohomology. From Lie algebra cohomology we know $\{\text{coboundaries}\} \subset \{\text{cocycles}\}$. Elements $r \in \mathfrak{g} \otimes \mathfrak{g}$ whose coboundary define a cocommutator $\delta(r)$ are called **r-matrices**. Many cocommutators of Lie bialgebras are of the coboundary type and have corresponding r-matrices. In the following we will see how we can transform the conditions on Lie bialgebra cocommutators to conditions on r itself. This will lead to the Yang-Baxter equation, its variants and the corresponding classes of r-matrices.

A Lie bialgebra $(\mathfrak{g}, \delta)$ is called a **coboundary Lie bialgebra** if there exists an r-matrix $r \in \mathfrak{g} \otimes \mathfrak{g}$ such that $\delta = \delta(r)$. Letting $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$ and $\xi^1, \ldots, \xi^n$ be the dual basis of $\mathfrak{g}^*$ with relations $[X_i, X_j] = c_{ij}^k X_k$ and $\langle X_i, \xi^j \rangle = \delta_i^j$ we can then write $r = r^{ij} X_i \otimes X_j$ and the action of $\delta(r)$ on the basis of $\mathfrak{g}$

$$\begin{aligned}
\delta(r)(X_l) &= [X_l \otimes 1 + 1 \otimes X_l, r] \\
&= r^{ij}([X_l, X_i] \otimes X_j + X_i \otimes [X_l, X_j]) \\
&= r^{ij}(c_{li}^k X_k \otimes X_j + X_i \otimes c_{lj}^k X_k) \\
&= (r^{kj} c_{lk}^i + r^{ik} c_{lk}^j) X_i \otimes X_j
\end{aligned}$$

The Lie bracket on $\mathfrak{g}^*$ defined by the r-matrix is called the **Sklyanin bracket** and can be found using the symmetry relations between a Lie bialgebra and its dual. It is given by

$$[\xi^i, \xi^j]^* = (r^{kj} c_{lk}^i + r^{ik} c_{lk}^j) \xi^l.$$

The Sklyanin bracket is often written $[\,,\,]^r$ instead of $[\,,\,]^*$ to make the r-matrix dependence clear.

The above paragraph gives a brief synopsis of the structures defined by an r-matrix, but when do we know an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is an r-matrix? An element $r \in \mathfrak{g} \otimes \mathfrak{g}$ is a 0-cochain on $\mathfrak{g}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$. Thus the coboundary of r is a 1-cochain that is necessarily a 1-cocycle. The only necessary and sufficient condition remaining is that $\delta(r)$ defines a Lie bracket on $\mathfrak{g}^*$. Thus $\delta(r)^*$ must be skew-symmetric and satisfy the Jacobi identity, where $\delta(r)^*(\xi \otimes \eta) = [\xi, \eta]^*$.

Before we examine these conditions we introduce some new notation. Let 1 denote the identity map from $\mathfrak{g}$ to itself. We define the following elements in the tensor algebra of $\mathfrak{g}$

$$\begin{aligned}r_{12} &= r^{ij} X_i \otimes X_j \otimes 1\\ r_{23} &= r^{ij} 1 \otimes X_i \otimes X_j\\ r_{13} &= r^{ij} X_i \otimes 1 \otimes X_j.\end{aligned}$$

Then in $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ we have

$$[r_{12}, r_{13}] = r^{ij} r^{kl} [X_i, X_k] \otimes X_j \otimes X_l = r^{jk} r^{lm} c^i_{jl} X_i \otimes X_k \otimes X_m,$$

and similarly,

$$\begin{aligned}[r_{12}, r_{23}] &= r^{ij} r^{lm} c^k_{jl} X_i \otimes X_k \otimes X_m,\\ [r_{13}, r_{23}] &= r^{ij} r^{kl} c^m_{jl} X_i \otimes X_k \otimes X_m.\end{aligned}$$

Further we define the algebraic Schouten bracket of r with itself (see appendix B),

$$\begin{aligned}[|r, r|] &:= [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]\\ &= (r^{jk} r^{lm} c^i_{jl} + r^{ij} r^{lm} c^k_{jl} + r^{ij} r^{kl} c^m_{jl}) X_i \otimes X_k \otimes X_m\end{aligned} \tag{3.22}$$

and the symmetric element

$$r_{12} + r_{21} = (r^{ij} + r^{ji}) X_i \otimes X_j.$$

The bracket $[\, ,\,]^*$ is skew symmetric if and only if $\delta(r)$ takes values in $\wedge^2 \mathfrak{g}$. This occurs precisely when $r_{12} + r_{21}$ is invariant under the adjoint action of $\mathfrak{g}$ (i.e. $ad^{(2)}_X [r_{12} + r_{21}] = 0$) . Writing $r = a + s$ where a, s are the skew-symmetric and symmetric parts of r, respectively, the adjoint action of an element $X_k \in \mathfrak{g}$ on $r_{12} + r_{21}$ looks as follows

$$\begin{aligned}ad^{(2)}_{X_k}[r_{12} + r_{21}] &= [X_k \otimes 1 + 1 \otimes X_k, r_{12} + r_{21}]\\ &= (r^{ij} + r^{ji})([X_k, X_i] \otimes X_j + X_i \otimes [X_k, X_j])\\ &= (r^{ij} + r^{ji})(c^l_{ki} X_l \otimes X_j + c^l_{kj} X_i \otimes X_l)\\ &= ((r^{lj} + r^{jl}) c^i_{kl} + (r^{il} + r^{li}) c^j_{kl}) X_i \otimes X_j\\ &= ((a^{lj} + a^{jl}) c^i_{kl} + (a^{il} + a^{li}) c^j_{kl} + (s^{lj} + s^{jl}) c^i_{kl} + (s^{il} + s^{li}) c^j_{kl}) X_i \otimes X_j\\ &= 2(s^{lj} c^i_{kl} + s^{li} c^j_{kl}) X_i \otimes X_j.\end{aligned}$$

This condition is clearly equivalent to the symmetric component s being ad invariant. Thus the skew symmetry of $\delta(r)^*$ is equivalent to the condition

$$s^{lj} c^i_{kl} + s^{li} c^j_{kl} = 0$$

for all i, j, k. We note when this is satisfied $\delta(s)(X) = 0$ for all $X \in \mathfrak{g}$. Thus if r defines a Lie bialgebra structure and $r' = r + s'$ where s' is symmetric (equivalently, ad-invariant) then r' defines the *same* Lie bialgebra structure as r. That is, only the skew-symmetric component of an r-matrix contributes to the corresponding cocommutator. All coboundary Lie bialgebras can be obtained from a purely skew-symmetric r-matrix. The symmetric component does however play a role in the classification of r-matrices.

Given that this first condition has been satisfied we only need focus on the anti-symmetric component of r since by the above comments $\delta(r) = \delta(a)$. Now $\delta(a)$, $a \in \wedge^2\mathfrak{g}$ defines a Lie bracket on $\mathfrak{g}^*$ if and only if $[\ ,\ ]^a$ satisfies the Jacobi identity. This is true if and only if the algebraic Schouten bracket of a with itself $[|a, a|] \in \wedge^3\mathfrak{g}$ is ad-invariant (Note: $[|a, a|]$ is ad-invariant iff $[|r, r|]$ is ad-invariant, provided the first condition is satisfied [2]). The condition $[|a, a|]$ be ad-invariant (or equivalently, $[|r, r|]$ ad-invariant) is called the **generalized Yang-Baxter equation**. In conclusion, r is an r-matrix if and only if s and $[|a, a|]$ are ad-invariant (or equivalently, $r_{12} + r_{21} \in \otimes^2\mathfrak{g}$ and $[|r, r|] \in \otimes^3\mathfrak{g}$ are ad-invariant).

The simplest way to satisfy the second condition is to assume $[|r, r|] = 0$. This is called the **Classical Yang-Baxter Equation** (CYBE). If r satisfies the CYBE and s is ad-invariant then it follows that $[|a, a|]$ is ad-invariant and so r is an r-matrix. A solution of the CYBE is called a **classical r-matrix**. A Lie bialgebra arising from a solution of the CYBE is called **quasitriangular** (also r-matrices satisfying the CYBE may be called quasitriangular). Further, if r satisfies the CYBE and is skew-symmetric the associated Lie bialgebra is called **triangular** (similarly, skew-symmetric r-matrices satisfying the CYBE may be called triangular). Lastly, if r is quasitriangular and s is invertible so that it defines a non-degenerate symmetric bilinear form on $\mathfrak{g}^*$, then r is called **factorizable**.

Example 3.11. The only coboundary bialgebra structure on $\mathbb{R}^n$ is the trivial bialgebra corresponding to the trivial r-matrix.

Example 3.12 (The Weyl-Heisenberg algebra and triangular r-matrices)**.** Consider the Weyl-Heisenberg algebra $\mathfrak{wh}$ generated by Q, P, Z with relations

$$[Q, P] = i\hbar Z, \quad [Q, Z] = [P, Z] = 0.$$

Now let's try and find a triangular r-matrix to turn $\mathfrak{wh}$ into a Lie bialgebra. We require

$$0 = [|r, r|]^{kmi} = (r^{jm}r^{li}c^k_{jl} + r^{kj}r^{li}c^m_{jl} + r^{kj}r^{ml}c^i_{jl})$$

for all indices $k, m, i = q, p, z$. Since the only non-zero structure constants of $\mathfrak{wh}$ are $c^z_{qp} = i\hbar = -c^z_{pq}$ we only need to consider the cases where $jl = pq, qp$. So

$$0 = [|r, r|]^{kmi} = (r^{qk}r^{pm} - r^{pk}r^{qm})c^i_{qp} + (r^{iq}r^{pm} - r^{ip}r^{qm})c^k_{qp} + (r^{iq}r^{kp} - r^{ip}r^{kq})c^m_{qp}.$$

This yields 12 independent equations corresponding to the choices of index kmi. In particular the choices $zzz, zzp, zzq, zpz, zqz, pzz, qzz, zqq, zpp, zqp, zpq,$ and qzp lead to the independent set of equations. Solving these we find

$$r^{qq} = r^{pp} = r^{zz} = r^{pq} = r^{qp} = 0$$

and

$$r^{qz} = -r^{zq} = A, \quad r^{pz} = -r^{zp} = B$$

for any $A, B \in \mathbb{R}$. Thus the possible triangular r-matrices that turn $\mathfrak{wh}$ into a Lie bialgebra have two free parameters. If we consider the dual bialgebra we shall see these parameters do not determine any structure at the bialgebra level. We denote the dual by $\mathfrak{wh}^*$ with dual basis $\tilde{Q}, \tilde{P}, \tilde{Z}$. The brackets on $\mathfrak{wh}^*$ are given by the Sklyanin bracket. Since the only nonzero terms are $c^z_{qp} = i\hbar = -c^z_{pq}$ and $r^{qz} = -r^{zq} = A, r^{pz} = -r^{zp} = B$. The brackets on $\mathfrak{wh}^*$ are in fact all zero. For example

$$\begin{aligned}
[\tilde{Q}, \tilde{Z}]^r &= (r^{kz}c^q_{qk} + r^{qk}c^z_{qk})\tilde{Q} + (r^{kz}c^q_{pk} + r^{qk}c^z_{pk})\tilde{P} + (r^{kz}c^q_{zk} + r^{qk}c^z_{zk})\tilde{Z} \\
&= r^{qk}c^z_{pk}\tilde{P} + r^{qk}c^z_{qk}\tilde{Q} \\
&= r^{qq}c^z_{pq}\tilde{P} + r^{qp}c^z_{qp}\tilde{Q} \\
&= 0.
\end{aligned}$$

Thus $\mathfrak{wh}^* = \mathbb{R}^3$ and we see the parameters A, B don't determine any particular structure at the infinitesimal level.

3.5 Coboundary Poisson-Lie Groups

Coboundary Poisson-Lie groups, as their name suggests, are Poisson-Lie groups whose Poisson structure are associated to an r-matrix. In other words, they are Poisson-Lie groups whose tangent Lie bialgebras are coboundary. We may assume without loss of generality that the r-matrices in the following discussion are all skew-symmetric, as the skew component is what determines the cocommutator and the subsequent Poisson structure uniquely.

Suppose $(\mathfrak{g}, \delta(r))$ is a coboundary Lie bialgebra with corresponding Poisson-Lie group (G, Π). The Poisson structure Π is a 1-cocycle of G with values in $\mathfrak{g} \otimes \mathfrak{g}$ obtained by 'integrating out' $\delta(r)$. It is easy to see that

$$\Pi^R(g) = (Ad_g \otimes Ad_g)_*(r) - r$$

has the correct derivative δ at the identity and satisfies the cocycle property

$$\begin{aligned}\Pi^R(gg') &= (Ad_{gg'} \otimes Ad_{gg'})_*(r) - r \\ &= (Ad_g \otimes Ad_g)_*[(Ad_{g'} \otimes Ad_{g'})_*(r) - r + r] - r \\ &= (Ad_g \otimes Ad_g)_*(\Pi^R(g')) + \Pi^R(g).\end{aligned}$$

The Poisson bivector Π at g is given by the right translate of $\Pi^R(g)$ to g.

If we write the r-matrix as $r = r^{ij} X_i \otimes X_j$ then the corresponding bracket is given by

$$\begin{aligned}\{f, h\}(g) &= r^{ij}((X_i^L f)(X_j^L h) - (X_i^R f)(X_j^R h)) && (3.23)\\ &= r^{ij}(((Ad_g)_* X_i^R f)((Ad_g)_* X_j^R h) - (X_i^R f)(X_j^R h)) && (3.24)\\ &= r^{ij}((X_i^L f)(X_j^L h) - ((Ad_{g^{-1}})_* X_i^L f)((Ad_{g^{-1}})_* X_j^L h)). && (3.25)\end{aligned}$$

We can simplify this if G is a matrix group of $n \times n$ matrices. Then r is a $n^2 \times n^2$ matrix and there is a straightforward formula to calculate the Poisson brackets of the matrix elements, which is enough to determine the Poisson bracket on the group completely. If $M \in G$ then

$$X^L M_{ij} = (MX)_{ij}, \text{ and } X^R M_{ij} = (XM)_{ij}$$

where M_{ij} is the ij^{th} entry of the matrix M. So,

$$\begin{aligned}\{M_{ij}, M_{kl}\} &= r^{ab}((X_a^L M_{ij})(X_b^L M_{kl}) - (X_a^R M_{ij})(X_b^R M_{kl})) \\ &= r^{ab}((MX_a)_{ij}(MX_b)_{kl} - (X_a M)_{ij}(X_b M)_{kl}) \\ &= ((M \otimes M)(r^{ab} X_a \otimes X_b) - (r^{ab} X_a \otimes X_b)(M \otimes M))_{ijkl} \\ &= [M \otimes M, r]_{ijkl}\end{aligned}$$

This is captured succintly by the **second Russian formula**

$$\{M \overset{\otimes}{,} M\} = [M \otimes M, r] \qquad (3.26)$$

where $\{M \overset{\otimes}{,} M\}_{ijkl} = \{M_{ij}, M_{kl}\}$. This formula makes it clear that the Poisson brackets of any two entries of M is a quadratic function of the entries of M, justifying the name quadratic bracket which is commonly seen in the literature. This equation is the basic equation of classical inverse scattering theory. Sometimes $\{M \overset{\otimes}{,} M\}$ is also written as $\{M_1, M_2\}$ where $M_1 = M \otimes 1, M_2 = 1 \otimes M$.

Example 3.13 (SU(2) cont'd)**.** Let $G = SU(2)$ and recall the r-matrix $\Lambda = -\frac{2}{\beta}(\sigma_2 \wedge \sigma_3)$ given example 3.8. Writing an element of $SU(2)$ as $g = \begin{pmatrix} \alpha & -\bar{\gamma} \\ \gamma & \bar{\alpha} \end{pmatrix}$ with $\alpha, \gamma \in \mathbb{C}$ and

$\alpha\bar{\alpha} + \gamma\bar{\gamma} = 1$ the second Russian formula allows us to compute the Poisson brackets using only matrix multiplication. We obtain

$$\{g \overset{\otimes}{,} g\} = \begin{pmatrix} \{\alpha, \alpha\} & \{\alpha, -\bar{\gamma}\} & \{-\bar{\gamma}, \alpha\} & \{-\bar{\gamma}, -\bar{\gamma}\} \\ \{\alpha, \gamma\} & \{\alpha, \bar{\alpha}\} & \{-\bar{\gamma}, \gamma\} & \{-\bar{\gamma}, \bar{\alpha}\} \\ \{\gamma, \alpha\} & \{\gamma, -\bar{\gamma}\} & \{\bar{\alpha}, \alpha\} & \{\bar{\alpha}, -\bar{\gamma}\} \\ \{\gamma, \gamma\} & \{\gamma, \bar{\alpha}\} & \{\bar{\alpha}, \gamma\} & \{\bar{\alpha}, \bar{\alpha}\} \end{pmatrix} \text{ and } \Lambda = \frac{i}{\beta} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From which we use the second Russian formula to find,

$$\{\alpha, \bar{\alpha}\} = -\frac{i}{\beta}\gamma\bar{\gamma}, \quad \{\alpha, \gamma\} = \frac{i}{\beta}\alpha\gamma$$

$$\{\alpha, \bar{\gamma}\} = \frac{i}{\beta}\alpha\bar{\gamma}, \quad \{\gamma, \bar{\gamma}\} = 0.$$

Example 3.14 (The Weyl-Heisenberg group and triangular r-matrices)**.** The Weyl-Heisenberg group WH is a 3-dimensional manifold with elements given by $g = (q, p, z)$ and product $gg' = (q, p, z) \cdot (q', p', z') = (q + q', p + p', zz' \exp \frac{i}{2\hbar}(qp' - pq'))$. The right and left invariant vector fields on WH are

$$X^R_{(q)} = \frac{\partial}{\partial q} + \frac{i}{2\hbar} pz \frac{\partial}{\partial z}, \quad X^R_{(p)} = \frac{\partial}{\partial p} - \frac{i}{2\hbar} qz \frac{\partial}{\partial z}, \quad X^R_{(z)} = z\frac{\partial}{\partial z},$$
$$X^L_{(q)} = \frac{\partial}{\partial q} - \frac{i}{2\hbar} pz \frac{\partial}{\partial z}, \quad X^L_{(p)} = \frac{\partial}{\partial p} + \frac{i}{2\hbar} qz \frac{\partial}{\partial z}, \quad X^L_{(z)} = z\frac{\partial}{\partial z}.$$

Using the triangular r-matrix computed in example 3.12 we can use (3.23) to compute the corresponding coboundary Poisson structure on WH. The only nonzero elements of the r-matrix are $r^{qz} = -r^{zq} = A, r^{pz} = -r^{zp} = B$ for $A, B \in \mathbb{R}$. Since

$$X^L_{(q)} f X^L_{(z)} h - X^R_{(q)} f X^R_{(z)} h = \left(-\frac{i}{\hbar} pz^2 \frac{\partial f}{\partial z}\frac{\partial h}{\partial z}\right) = X^L_{(z)} f X^L_{(q)} h - X^R_{(z)} f X^R_{(q)} h$$
$$X^L_{(p)} f X^L_{(z)} h - X^R_{(p)} f X^R_{(z)} h = \left(\frac{i}{\hbar} qz^2 \frac{\partial f}{\partial z}\frac{\partial h}{\partial z}\right) = X^L_{(z)} f X^L_{(p)} h - X^R_{(z)} f X^R_{(p)} h$$

for $f, h \in \mathcal{F}(WH)$ the Poisson structure on WH is trivial, by the skew-symmetry of the r-matrix. Hence, the free parameters A, B are really meaningless since they determine no structure at the group level or infinitesimal level on WH.

Thus the only Poisson-Lie structure on WH arising from a solution of the CYBE is the trivial Poisson structure. Considering the similarity between WH and $\mathbb{R}^3$ we observe a trend that the more ‘abelian-ess’ a group exhibits the less likely it will have a non-trivial coboundary Poisson-Lie structure.

3.6 The Classic Double Lie Bialgebra

Lie bialgebras always come in pairs due to the symmmetry in their duality relations. We can construct a factorizable Lie bialgebra out of any such pair of Lie bialgebras with a canonical choice of r-matrix.

Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra and $(\mathfrak{g}^*, \delta^*)$ its dual. The **double Lie algebra** denoted $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ is a Lie algebra on $\mathfrak{g} \oplus \mathfrak{g}^*$ with brackets arising from $\mathfrak{g}, \mathfrak{g}^*$ and their adjoint actions on each other. It is the unique Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras and that the natural scalar product on $\mathfrak{g} \oplus \mathfrak{g}^*$ is invariant.

The natural scalar product on $\mathfrak{g} \oplus \mathfrak{g}^*$ is defined by

$$\langle (X, \xi), (Y, \eta) \rangle_{\mathfrak{d}} = \langle \xi, Y \rangle + \langle \eta, X \rangle$$

for $X, Y \in \mathfrak{g}$ and $\xi, \eta \in \mathfrak{g}^*$. The invariance condition tells us the bracket structure on the double $\mathfrak{d}$ is given by

$$[(X, \xi), (Y, \eta)] = ([X, Y] - ad^*_\eta X + ad^*_\xi Y, [\xi, \eta]^* + ad^*_X \eta - ad^*_Y \xi).$$

In particular, if the bases of $\mathfrak{g}$ and $\mathfrak{g}^*$ satisfy

$$[X_i, X_j] = c^k_{ij} X_k, \quad [\xi^i, \xi^j] = f^{ij}_k \xi^k,$$

then the bracket on $\mathfrak{d}$ is given by [16, 33],

$$\begin{aligned} [X_i, X_j] &= c^k_{ij} X_k \\ [X_i, \xi^j] &= -c^j_{ik} \xi^k + f^{jk}_i X_k \\ [\xi^i, \xi^j] &= f^{ij}_k \xi^k. \end{aligned}$$

The double $\mathfrak{d}$ can be equipped with a factorizable bialgebra structure by a canonical r-matrix. We call this bialgebra $(\mathfrak{d}, \delta_{\mathfrak{d}})$ the **Drinfeld** or **classic double Lie bialgebra**. The canonical choice of r-matrix is [33]

$$r_{\mathfrak{d}} = \xi^i \otimes X_i \in \mathfrak{g}^* \otimes \mathfrak{g} \subset \mathfrak{d} \otimes \mathfrak{d}$$

so that the commutator of the double bialgebra is

$$\delta(r_{\mathfrak{d}})(Y) = [Y \otimes 1 + 1 \otimes Y, r_{\mathfrak{d}}], \quad \text{for all } Y \in \mathfrak{d}.$$

In terms of the basis

$$\begin{aligned}\delta(r_{\mathfrak{d}})(X_k+\xi^l) &= [X_k+\xi^l\otimes 1+1\otimes X_k+\xi^l, \xi^i\otimes X_i] \\ &= -f_k^{ni}X_n\otimes X_i + c_{in}^l \xi^i\otimes\xi^n\end{aligned}$$

and in particular

$$\delta(r_{\mathfrak{d}})(X_k) = -\delta(X_k) = -f_k^{ij}X_i\otimes X_j, \quad \delta(r_{\mathfrak{d}})(\xi^k) = \delta^*(\xi^k) = c_{ij}^k\xi^i\otimes\xi^j.$$

Thus we see that $(\mathfrak{g},-\delta),(\mathfrak{g}^*,\delta^*)$ are sub-Lie bialgebras of $(\mathfrak{d},\delta(r_{\mathfrak{d}}))$. We may note

$$\hat{r}_{\mathfrak{d}} = \frac{1}{2}(\xi^i\wedge X_i) = \frac{1}{2}(\mathrm{pr}_{\mathfrak{g}} - \mathrm{pr}_{\mathfrak{g}^*})$$

where $\mathrm{pr}_{\mathfrak{g}}:\mathfrak{g}\oplus\mathfrak{g}^*\to\mathfrak{g}$, $\mathrm{pr}_{\mathfrak{g}^*}:\mathfrak{g}\oplus\mathfrak{g}^*\to\mathfrak{g}^*$ are the projection operators and we consider $\hat{r}_{\mathfrak{d}}\in\mathfrak{d}\otimes\mathfrak{d}$ as a linear map from $\mathfrak{d}$ to $\mathfrak{d}$ by contracting against the first element.

The dual $\mathfrak{d}^*\simeq\mathfrak{g}^*\oplus\mathfrak{g}$ of the Drinfeld double can be read off using our duality relations. We denote the basis of $\mathfrak{d}^*$ by Y^i,γ_i such that the following duality relations hold

$$\langle X_i, Y^j\rangle = \langle\xi^j,\gamma_i\rangle = \delta_i^j, \quad \langle X_i,\gamma_j\rangle = \langle\xi^i, Y^j\rangle = 0.$$

The brackets are then given by

$$\begin{aligned}[][Y^i,Y^j] &= -f_l^{ij}Y^l \\ [Y^i,\gamma_j] &= 0 \\ [\gamma_i,\gamma_j] &= c_{ij}^k\gamma_k.\end{aligned}$$

Thus we see that the dual of $\mathfrak{d}$ is $\mathfrak{d}^* = \mathfrak{g}^{*op}\times\mathfrak{g}$ where $\mathfrak{g}^{op}$ is the Lie algebra associated to G^{op} where G^{op} is G with opposite product, i.e. for $a,b\in G$, $ab = b *^{op} a$.

Example 3.15. Any Lie algebra $\mathfrak{g}$ is a Lie bialgebra when equipped with the trivial commutator. The dual of $(\mathfrak{g},\delta=0)$ is then abelian with commutator coming from the Lie bracket on $\mathfrak{g}$. The corresponding double of this pair is the semidirect product $\mathfrak{g}\ltimes\mathfrak{g}^*$.

Example 3.16 (The double of $SU(2)$)**.** Continuing example 3.10. The double corresponding to the pair $\mathfrak{su}(2)$ and $\mathfrak{sb}(2,\mathbb{C})$ is precisely $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2)\bowtie\mathfrak{sb}(2,\mathbb{C})$.

Example 3.17 (The Lorentz algebra as a double Lie algebra)**.** The Lorentz algebra $\mathfrak{so}(3,1)$ has a number of decompositions into double Lie algebras which can be found in [11]. One of them is the decomposition $\mathfrak{so}(3,1) = \mathfrak{so}(3)\bowtie\mathfrak{an}(2)$ where $\mathfrak{an}(2)$ is the 3 dimensional hyperbolic Lie algebra. This decomposition is isomorphic to the previous example.

We will now realize the double $\mathfrak{so}(3,1) = \mathfrak{so}(3) \bowtie \mathfrak{an}(2)$ using a new 4×4 representation. $\mathfrak{so}(3,1)$ is generated by boosts P_1, P_2, P_3 and rotations J_1, J_2, J_3 with the following relations

$$[J_i, J_j] = \epsilon_{ij}^k J_k, \quad [P_i, P_j] = -\epsilon_{ij}^k J_k, \quad [P_i, J_j] = \epsilon_{ij}^k P_k.$$

They have the following (non-standard) 4×4 matrix representation,

$$P_i = \begin{bmatrix} 0 & -\mathbf{e}_i^T \\ -\mathbf{e}_i & \mathbf{0}_{3\times3} \end{bmatrix}, \quad J_i = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & E_i \end{bmatrix}$$

where E_i are as defined in example 2.6 and $\mathbf{e}_i \in \mathbb{R}^3$ are 0 column vectors with a 1 in the i^{th} entry. We now introduce a deformation parameter β into the boosts.

$$P_i \to P_i^\beta = \begin{bmatrix} 0 & -\mathbf{e}_i^T \\ -\frac{\mathbf{e}_i}{\beta} & \mathbf{0}_{3\times3} \end{bmatrix}$$

These satisfy,

$$[J_i, J_j] = \epsilon_{ij}^k J_k, \quad [P_i^\beta, P_j^\beta] = -\epsilon_{ij}^k \frac{J_k}{\beta^2}, \quad [P_i^\beta, J_j] = \epsilon_{ij}^k P_k^\beta.$$

Notice if $\beta = 1$ then we recover the original relations, as expected. Further, if $\beta \to \infty$ then the boosts P_i^β commute and the Lie algebra is that of $SO(3) \ltimes \mathbb{R}^3$.

We now define a new basis

$$X_1 = \frac{1}{\sqrt{2}}\left(J_1 + J_2\right), \qquad X_2 = \frac{1}{\sqrt{2}}\left(-J_1 + J_2\right), \qquad X_3 = J_3$$
$$x_1 = \frac{1}{\sqrt{2}}(P_1^\beta + P_2^\beta + \frac{1}{\beta}(-J_1 + J_2)), \quad x_2 = \frac{1}{\sqrt{2}}(-P_1^\beta + P_2^\beta - \frac{1}{\beta}(J_1 + J_2)), \quad x_3 = P_3^\beta.$$

These satisfy the relations

$$[X_i, X_j] = \epsilon_{ij}^k X_k,$$

and

$$[x_1, x_2] = 0, \quad [x_1, x_3] = \frac{1}{\beta} x_1, \quad [x_2, x_3] = \frac{1}{\beta} x_2,$$

with mixed brackets given by

$$[x_1, X_1] = -\frac{1}{\beta}X_3, \qquad [x_1, X_2] = x_3, \qquad [x_1, X_3] = -x_2$$
$$[x_2, X_1] = -x_3, \qquad [x_2, X_2] = -\frac{1}{\beta}X_3, \qquad [x_2, X_3] = x_1$$
$$[x_3, X_1] = x_2 + \frac{1}{\beta}X_1, \qquad [x_3, X_2] = -x_1 + \frac{1}{\beta}X_2, \qquad [x_3, X_3] = 0$$

We see that the X_i satisfy the relations of $\mathfrak{so}(3)$ and the x_i satisfy the relations of $\mathfrak{an}(2)$. Further the mixed brackets satisfy the relations of a double Lie algebra. Hence we have realized $\mathfrak{so}(3,1) = \mathfrak{so}(3) \bowtie \mathfrak{an}(2)$.

We introduced the deformation parameter β at the level of the generators of $\mathfrak{an}(2)$. Due to the relations on Lie bialgebras summarized in Figure 3.1 this is equivalent to introducing β in the commutator of the Lie bialgebra $(\mathfrak{so}(3), \delta)$. In particular, the above is equivalent to defining the commutator on $\mathfrak{so}(3)$ by the r-matrix $\frac{1}{\beta}X_1 \wedge X_2$. In the limit $\beta \to \infty$ we can see that the double corresponds to $\mathfrak{iso}(3) = \mathfrak{so}(3) \ltimes \mathbb{R}^3$.

3.7 Double Poisson-Lie Groups

Corresponding to the double of a Lie bialgebra, there is a double of a Poisson-Lie group. A Poisson-Lie group (G, Π) has a tangent Lie bialgebra $(\mathfrak{g}, \delta)$. This Lie bialgebra has a double Lie bialgebra $(\mathfrak{d}, \delta(r_{\mathfrak{d}}))$ which is factorizable. The connected and simply-connected Poisson-Lie group (D, Π_-) with Lie bialgebra $(\mathfrak{d}, \delta(r_{\mathfrak{d}}))$ is called the **Drinfeld double** of (G, Π) where $D = G \bowtie G^*$ is the **double** Lie group associated to the Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ (the reason for the $-$ on the Poisson structure will become more apparent when we consider affine Poisson structures in section 4.1). The relations of a Drinfeld double to its building blocks is given in Figure 3.4.

The Poisson structure of a Drinfeld double is given conveniently by (3.23)

$$\Pi_-(a) = \xi^{iL} \otimes X_i^L - \xi^{iR} \otimes X_i^R \tag{3.27}$$

where $a \in D$. Since the Poisson structure is determined uniquely by the skew-symmetric component of the r-matrix the above is equivalent to either [6]

$$\Pi_-(a) = X_i^R \otimes \xi^{iR} - X_i^L \otimes \xi^{iL}$$

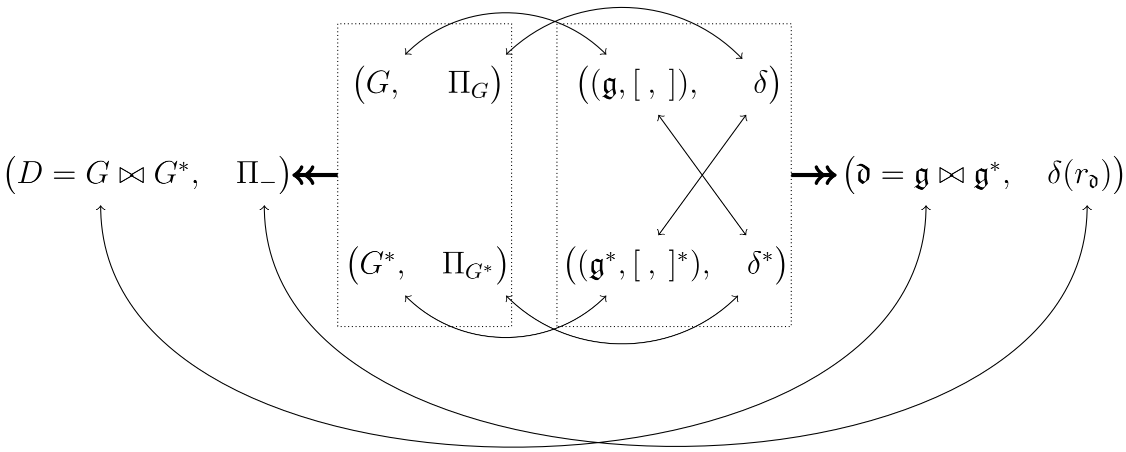

Figure 3.4: The relation of a Drinfeld double (D, Π_-) to its subgroups.

or,

$$\Pi_-(a) = \frac{1}{2}\left((\xi^{iL} \otimes X_i^L - X_i^L \otimes \xi^{iL}) - (\xi^{iR} \otimes X_i^R - X_i^R \otimes \xi^{iR})\right).$$

Since $(\mathfrak{g}, -\delta), (\mathfrak{g}^*, \delta^*)$ are sub-Lie bialgebras of $(\mathfrak{d}, \delta^D)$ $(G, -\Pi_G)$ and (G^*, Π_{G^*}) are Poisson-Lie subgroups of (D, Π_-). The following product maps are diffeomorphisms about the identity of the double group D

$$\varphi : G \times G^* \to D, \quad (g, u) \mapsto \varphi(g, u) = gu \in D,$$

$$\phi : G^* \times G \to D, \quad (v, h) \mapsto \phi(v, h) = vh \in D.$$

This means that in an open neighbourhood of the identity we can factor the elements into products on GG^* or G^*G by an Iwasawa-type decomposition [16, 6]. The following are projections that map onto the different factors about the identity,

$$\begin{aligned}
p_R &: D \sim G^*G \to G, vh \mapsto h \in G,\\
p_L &: D \sim GG^* \to G, gu \mapsto g \in G,\\
p_L^* &: D \sim G^*G \to G^*, vh \mapsto v \in G^*,\\
p_R^* &: D \sim GG^* \to G^*, gu \mapsto u \in G^*.
\end{aligned}$$

We often call, with some abuse of terminology, g, u and v, h different coordinates choices. If the maps φ, ϕ are globally defined diffeomorphisms then D is a **complete double group** and the projections are also globally defined. If either G or G^* is compact then the double group D is complete [29].

We can now see how a cotangent bundle is a special case of a double group. Given a group G we can equip it with a trivial Poisson structure to construct the Poisson-Lie group $(G, \Pi = 0)$. This has an associated Lie bialgebra $(\mathfrak{g}, \delta = 0)$ which has a dual Lie bialgebra $(\mathfrak{g}^*, \delta^*)$. The associated double Lie algebra is $\mathfrak{d} = \mathfrak{g} \ltimes \mathfrak{g}^*$ with bracket given by

$$[(X, \xi), (Y, \eta)] = ([X, Y], ad_X^* \eta - ad_Y^* \xi).$$

The double group is then $D = G \ltimes \mathfrak{g}^*$ with multiplication

$$(g, \mu) \cdot (h, \nu) = (gh, (Ad_{h^{-1}})^* \mu + \nu), \quad (g, \mu)^{-1} = (g^{-1}, -(Ad_{g^{-1}})^* \mu).$$

The elements of D can be factorized as

$$(g, \mu) = (g, 0)(e, \mu) = (e, (Ad_{g^{-1}})^* \mu)(g, 0)$$

where we have identified the subgroups G, $\mathfrak{g}^*$ with elements of the form $(g, 0)$ and (e, μ), respectively. This pair of factorizations corresponds with the pair of body and space coordinates of the cotangent bundle T^*G. In particular, if an element α of T^*G is (g, μ) in body coordinates then in space coordinates α is written $((Ad_{g^{-1}})^* \mu, g)$ since the coadjoint map relates body and space coordinates. We can thus straightforwardly extend the definition of body and space coordinate to general double groups,

- body coordinates:

$$\begin{aligned} B = \varphi^{-1} = (p_L, p_R^*) : D &\to G \times G^* \\ gu &\mapsto (g, u) \end{aligned}$$

- space coordinates:

$$\begin{aligned} S = \phi^{-1} = (p_L^*, p_R) : D &\to G^* \times G \\ vh &\mapsto (v, h) \end{aligned}$$

It is important to note that these projection maps satisfy some simplifying relations. For example, if $a \in D$ and $g \in G$ then $p_R^*(ga) = p_R^*(a)$ since we are projecting onto G^* on the right. Thus we have for example $p_R^*(vh) = p_R^*(Ad_{h^{-1}} v)$ where define the Ad map to be the conjugate map, e.g. $Ad_{h^{-1}} v = h^{-1} vh$. This will make it easier to relate the coordinates on the double $D = G \bowtie G^*$ and cotangent bundle T^*G. These relations can be found in Table 3.1.

Table 3.1: Comparison of Coordinate Systems

Phase Space	Body	Space	Body $\leftrightarrow$ Space
T^*G	$T^*G \to G \times \mathfrak{g}^*$ $\alpha \mapsto (g, \mu)$ $g = \pi_G(\alpha)$ $\mu = (L_g)^*\alpha$	$T^*G \to \mathfrak{g}^* \times G$ $\alpha \mapsto (\nu, h)$ $h = \pi_G(\alpha)$ $\nu = (R_h)^*\alpha$	$(g, \mu) \mapsto ((Ad_{g^{-1}})^*\mu, g)$ $(\nu, h) \mapsto (h, (Ad_h)^*\nu)$
$D = G \bowtie G^*$	$D \to G \times G^*$ $a \mapsto (g, u)$ $g = p_L(a)$ $u = p_R^*(a)$	$D \to G^* \times G$ $a \mapsto (v, h)$ $h = p_R(a)$ $v = p_L^*(a)$	$(g, u) \mapsto (p_L^*(Ad_g u), p_R(Ad_{u^{-1}} g))$ $(v, h) \mapsto (p_L(Ad_v h), p_R^*(Ad_{h^{-1}} v))$

Example 3.18 (Identifying $T^*SO(3)$ with a Heisenberg double)**.** Consider the Poisson-Lie group $(SO(3), \Pi = 0)$. The Poisson-Lie dual is then $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ with the linear Poisson bracket. We denote the generators of $\mathfrak{so}(3)$ by X_i with relations $[X_i, X_j] = \epsilon_{ij}^k X_k$ and the generators of the dual by ξ^i with relations $[\xi^i, \xi^j] = 0$. The generators of the double Lie bialgebra $\mathfrak{so}(3) \oplus \mathbb{R}^3$ then satisfy

$$[X_i, X_j] = \epsilon_{ij}^k X_k, \quad [X_i, \xi^j] = \epsilon_{ki}^j \xi^k \quad [\xi^i, \xi^j] = 0.$$

We can represent this Lie algebra in terms of 4×4 matrices by

$$X_i = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & E_i \end{bmatrix}, \quad \xi^i = \begin{bmatrix} 0 & \mathbf{e_i}^T \\ \mathbf{0} & \mathbf{0}_{3\times3} \end{bmatrix}$$

where $E_i, \mathbf{e_i}$ are as defined in example 3.17. Exponentiating these generators we can see the double group is $ISO(3) = SO(3) \ltimes \mathbb{R}^3$ with elements,

$$g = \begin{bmatrix} 1 & \mathbf{u}^T \\ \mathbf{0} & R \end{bmatrix} = (R, \mathbf{u}) \in ISO(3)$$

and product

$$(R, \mathbf{u}) \cdot (S, \mathbf{w}) = (RS, S^{-1}\mathbf{u} + \mathbf{w}), \quad (R, \mathbf{u})^{-1} = (R^{-1}, -R^{-1}\mathbf{u})$$

where $R, S \in SO(3)$, $\mathbf{u}, \mathbf{w} \in \mathbb{R}^3$.

An element of the group can be factorized in terms of $SO(3) \cdot \mathbb{R}^3$ or $\mathbb{R}^3 \cdot SO(3)$

$$(R, \mathbf{u}) = (R, 0) \cdot (I, \mathbf{u}) = (I, R^{-1}\mathbf{u}) \cdot (R, 0)$$

where we identify elements $(R, 0)$ with $SO(3)$ and $(I, \mathbf{u})$ with $\mathbb{R}^3$. Since the coadjoint action of $SO(3)$ on $\mathfrak{so}(3)^*$ is simply the left action of $SO(3)$ on $\mathbb{R}^3$ we can write $R^{-1}\mathbf{u} = (Ad_{R^{-1}})^*\xi_{\mathbf{u}}$ where $\xi_{\mathbf{u}} = \sum_i u_i X_i \in \mathfrak{so}(3)^*$.

Double Poisson-Lie groups provide the full landscape in which to study Poisson-Lie groups since we can take advantage of all the 'symmetries' present in their construction. Decomposing (3.27) the Poisson structures on the subgroups G, G^* of the Drinfeld double can be written in terms of the the generators $X_i \in \mathfrak{g}, \xi^i \in \mathfrak{g}^*$ of the double lie algebra $\mathfrak{d}$

$$-\Pi_G(g) = -\sum_i (L_g)_* X_i \otimes (R_g)_* \mathrm{pr}_{\mathfrak{g}}((Ad_g)_* \xi^i) = \sum_i (R_g)_* X_i \otimes (L_g)_* \mathrm{pr}_{\mathfrak{g}}((Ad_{g^{-1}})_* \xi^i) \tag{3.28}$$

$$\Pi_{G^*}(u) = \sum_i (L_u)_* \xi^i \otimes (R_u)_* \mathrm{pr}_{\mathfrak{g}^*}((Ad_u)_* X_i) = -\sum_i (R_u)_* \xi^i \otimes (L_u)_* \mathrm{pr}_{\mathfrak{g}^*}((Ad_{u^{-1}})_* X_i) \tag{3.29}$$

where $g \in G$ and $u \in G^*$. This decomposition takes some effort and the details can be found in [6]. Using this decomposition the Poisson-structure of the Drinfeld double can be conveniently written in body and space coordinates. Following the decomposition we have

$$\begin{aligned}\Pi_-(gu) &= \sum_i (L_{gu})_* \xi^i \otimes (L_{gu})_* X_i - (R_{gu})_* \xi^i \otimes (L_{gu})_* X_i \\ &= -(R_u)_* \Pi_G(g) + (L_g)_* \Pi_{G^*}(u)\end{aligned}$$

so in body coordinates the Poisson structure is given by

$$\Pi_-^B(g, u) = -\Pi_G(g) + \Pi_{G^*}(u) \in \wedge^2 T(G \times G^*).$$

Similarly, the Poisson structure in space coordinates is given by

$$\Pi_-^S(v, h) = -\Pi_G(h) + \Pi_{G^*}(v) \in \wedge^2 T(G^* \times G).$$

The Poisson-Lie group dual to the Drinfeld double $(D = G \bowtie G^*, \Pi_-)$ is $G^{*op} \times G$ with Poisson structure

$$\Pi_-^*(v, h) = \Pi_G(h) + \Pi_{G^*}(v) - \sum_i (R_v)_* Y^i \wedge (R_h)_* \gamma_i + \sum_i (L_v)_* Y^i \wedge (L_h)_* \gamma_i$$

where $(R_v)_* Y^i, (L_v)_* Y^i$ are right and left invariant vector fields with respect to the opposite group structure on G^*.

Example 3.19 (The Lorentz group as a double group)**.** As we saw in example 3.17 we have $\mathfrak{so}(3,1) = \mathfrak{so}(3) \bowtie \mathfrak{an}(2)$. An element of $AN(2)$ can be written as [20]

$$\begin{aligned} v &= \exp(t_1 x_1 + t_2 x_2)\exp(t_3 x_3) \\ &= \begin{bmatrix} \frac{w(v_1^2+v_2^2)}{2\beta^2} + \cosh(v_3/\beta) & v_1 & v_2 & -\frac{w(v_1^2+v_2^2)}{2\beta} + \beta\sinh(v_3/\beta) \\ \frac{wv_1}{\beta^2} & 1 & 0 & -\frac{wv_1}{\beta} \\ \frac{wv_2}{\beta^2} & 0 & 1 & -\frac{wv_2}{\beta} \\ \frac{w(v_1^2+v_2^2)}{2\beta^3} + \frac{\sinh(v_3/\beta)}{\beta} & \frac{v_1}{\beta} & \frac{v_2}{\beta} & -\frac{w(v_1^2+v_2^2)}{2\beta^2} + \cosh(v_3/\beta) \end{bmatrix} \end{aligned}$$

where $w = \exp\frac{t_3}{\beta}, v_1 = \frac{t_2-t_1}{\sqrt{2}}, v_2 = -\frac{t_2+t_1}{\sqrt{2}}, v_3 = -t_3$ and elements of $SO(3)$ can be written as

$$h = \exp(\theta_1 X_1)\exp(\theta_2 X_2)\exp(\theta_3 X_3) = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & R \end{bmatrix}.$$

Then,

$$\lim_{\beta\to\infty} v = \begin{bmatrix} 1 & \mathbf{v}^T \\ \mathbf{0} & \mathbf{1}_{3\times 3} \end{bmatrix} \text{ and } \lim_{\beta\to\infty} h = h.$$

Thus in the limit $\beta \to \infty$ we have $SO(3,1) = SO(3) \bowtie AN(2) \to SO(3) \ltimes \mathbb{R}^3$. The deformation parameter β was originally introduced in the cocommutator of $\mathfrak{so}(3)$ which lead to the deformation of the dual group. We will see that this can be interpreted as a deformation of the momentum space of the rigid rotator.

We can, for the sake of interest, introduce another deformation parameter in this example which we will see under some interpretion is relevant to quantum gravity. Again we introduce the scale α at the level of the Lie algebras. We only inject α into three of the generators

$$P_1^\beta \to P_1^{\beta+\alpha} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -\frac{1}{\alpha\beta^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, J_2 \to J_2^\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\alpha} \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, J_3 \to J_3^\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\alpha} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The algebra then satisfies

$$
\begin{array}{lll}
[J_1, J_2^\alpha] = J_3^\alpha, & [J_1, J_3^\alpha] = -J_2^\alpha, & [J_2^\alpha, J_3^\alpha] = \dfrac{1}{\alpha} J_1 \\
[P_1^{\beta+\alpha}, P_2^\beta] = -\dfrac{1}{\beta^2} J_3^\alpha, & [P_1^{\beta+\alpha}, P_3^\beta] = \dfrac{1}{\beta^2} J_2^\alpha, & [P_2^\beta, P_3^\beta] = -\dfrac{1}{\beta^2} J_1 \\
[P_1^{\beta+\alpha}, J_1] = 0, & [P_1^{\beta+\alpha}, J_2^\alpha] = \dfrac{1}{\alpha} P_3^\beta, & [P_1^{\beta+\alpha}, J_3^\alpha] = -\dfrac{1}{\alpha} P_2^\beta \\
[P_2^\beta, J_1] = -P_3^\beta, & [P_2^\beta, J_2^\alpha] = 0, & [P_2^\beta, J_3^\alpha] = P_1^{\beta+\alpha} \\
[P_3^\beta, J_1] = P_2^\beta, & [P_3^\beta, J_2^\alpha] = -P_1^{\beta+\alpha}, & [P_3^\beta, J_3^\alpha] = 0.
\end{array}
$$

Notice if $\beta = 1$ and $\alpha \to \infty$ then the rotations J_i contract to $\mathfrak{iso}(2)$ and the double Lie algebra is that of $ISO(2) \bowtie AN(2)$. Thus this scale deforms the top's configuration space. When both $\beta, \alpha \to \infty$ the double reduces to $ISO(2) \ltimes \mathbb{R}^3$.

We define the basis X_i, x_i the same as before but replacing the generators appropriately with their newly scaled counterparts. These then have the relations

$$
\begin{aligned}
[X_1, X_2] &= X_3, \\
[X_1, X_3] &= \frac{1}{2}\left(\left(\frac{1}{\alpha} - 1\right) X_1 - \left(\frac{1}{\alpha} + 1\right) X_2\right), \\
[X_2, X_3] &= \frac{1}{2}\left(\left(\frac{1}{\alpha} + 1\right) X_1 - \left(\frac{1}{\alpha} - 1\right) X_2\right),
\end{aligned}
$$

and

$$
[x_1, x_2] = 0, \quad [x_1, x_3] = \frac{1}{\beta} x_1, \quad [x_2, x_3] = \frac{1}{\beta} x_2,
$$

and the mixed brackets are

$$
\begin{array}{ll}
[x_1, X_1] = \dfrac{1}{2}\left(\dfrac{1}{\alpha} - 1\right) x_3 - \dfrac{1}{\beta} X_3, & [x_1, X_2] = \dfrac{1}{2}\left(\dfrac{1}{\alpha} + 1\right) x_3, \\
[x_1, X_3] = -\dfrac{1}{2}\left(\left(\dfrac{1}{\alpha} - 1\right) x_1 + \left(\dfrac{1}{\alpha} + 1\right) x_2\right), & [x_2, X_1] = -\dfrac{1}{2}\left(\dfrac{1}{\alpha} + 1\right) x_3, \\
[x_2, X_2] = -\dfrac{1}{2}\left(\dfrac{1}{\alpha} - 1\right) x_3 - \dfrac{1}{\beta} X_3, & [x_2, X_3] = \dfrac{1}{2}\left(\left(\dfrac{1}{\alpha} + 1\right) x_1 + \left(\dfrac{1}{\alpha} - 1\right) x_2\right), \\
[x_3, X_1] = x_2 + \dfrac{1}{\beta} X_1, & [x_3, X_2] = -x_1 + \dfrac{1}{\beta} X_2, \\
[x_3, X_3] = 0. &
\end{array}
$$

Group elements $v \in AN(2), h \in SO(3)$ now take on the form

$$v = \begin{bmatrix} \frac{wv_1^2}{2\beta^2\alpha} + \frac{wv_2^2}{2\beta^2} + \cosh(-v_3/\beta) & v_1 & v_2 & \frac{wv_1^2}{2\beta\alpha} - \frac{wv_2^2}{2\beta} - \beta\sinh(-v_3/\beta) \\ \frac{wv_1}{\beta\alpha} & 1 & 0 & -\frac{wv_1}{\beta\alpha} \\ \frac{wv_2}{\beta^2} & 0 & 1 & -\frac{wv_2}{\beta} \\ \frac{wv_1^2}{2\beta^3\alpha} + \frac{wv_2^2}{2\beta^3} - \frac{\sinh(-v_3/\beta)}{\beta} & \frac{v_1}{\beta} & \frac{v_2}{\beta} & -\frac{wv_1^2}{2\beta^2\alpha} - \frac{wv_2^2}{2\beta^2} + \cosh(-v_3/\beta) \end{bmatrix},$$

$$h = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & R(\alpha) \end{bmatrix}$$

where $R(\alpha)$ is a rotation matrix dependent on α. Then

$$\lim_{\alpha\to\infty} v = \begin{bmatrix} \frac{wv_2^2}{2\beta^2} + \cosh(-v_3/\beta) & v_1 & v_2 & -\frac{wv_2^2}{2\beta} - \beta\sinh(-v_3/\beta) \\ 0 & 1 & 0 & 0 \\ \frac{wv_2}{\beta^2} & 0 & 1 & -\frac{wv_2}{\beta} \\ \frac{wv_2^2}{2\beta^3} - \frac{\sinh(-v_3/\beta)}{\beta} & \frac{v_1}{\beta} & \frac{v_2}{\beta} & -\frac{wv_2^2}{2\beta^2} + \cosh(-v_3/\beta) \end{bmatrix},$$

$$\lim_{\alpha\to\infty} h = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (\theta_3+1)\cos\frac{\theta_1-\theta_2}{\sqrt{2}} - 2\cos\frac{\theta_1}{\sqrt{2}} + 1 & \cos\frac{\theta_1-\theta_2}{\sqrt{2}} & -\sin\frac{\theta_1-\theta_2}{\sqrt{2}} \\ 0 & (\theta_3+1)\sin\frac{\theta_1-\theta_2}{\sqrt{2}} - 2\sin\frac{\theta_1}{\sqrt{2}} & \sin\frac{\theta_1-\theta_2}{\sqrt{2}} & \cos\frac{\theta_1-\theta_2}{\sqrt{2}} \end{bmatrix}.$$

If we interpret $SO(3,1)$ as the phase space of a deformed top the introduced α parameter then flattens part of the configuration space (i.e. $SO(3) \to ISO(2)$ as $\alpha \to \infty$) and leaves the curved momentum space intact. This is possible since the algebras before and after contraction happen to be compatible. Really though this interpretation breaks down when we take the limit $\alpha \to \infty$ since a particle with phase space $ISO(2)$ is no longer the top. We can, however, interpret this double deformation as a deformation of the phase space of a 2 dimensional particle with spin (i.e. $T^*ISO(2)$) where we have added curvature to the configuration space with α representing the cosmological constant and added curvature to momentum space with β to acquire quantum gravitational effects.

Chapter 4

Heisenberg Doubles

We are now in a position to define a new type of deformed phase spaces, Heisenberg doubles. Heisenberg doubles are double Lie groups equipped with a Poisson structure closely related to the multiplicative structures of Poisson-Lie groups. This makes them closely related to the Drinfeld double, but unlike the Drinfeld doubles they are not Poisson-Lie groups. Their Poisson structure is non-degenerate in a neighbourhood about the identity and we can thus define a corresponding symplectic form in such a neighbourhood.

Cotangent bundles are a special type of Heisenberg double. In this section we generalize the concepts introduced on cotangent bundles, like body/space coordinates, momentum maps and Noether's theorem to Heisenberg doubles. Heisenberg doubles allow us to define, in a well-defined manner, phase spaces with a momentum space that has a non-abelian group structure. The non-abelian group structure is achieved by introducing a deformation parameter dependent on the quantum gravity scale. Hence they are a natural lab in which to study possible quantum gravitational effects.

4.1 The Heisenberg Double an Affine Poisson Structure

Recall that the Drinfeld double (D, Π_-) is a Poisson-Lie group built from a pair of dual Poisson-Lie groups $(G, \Pi_G), (G^*, \Pi_{G^*})$ and a canonical r-matrix $r_{\mathfrak{d}} = \xi^i \otimes X_i \in \mathfrak{d} \otimes \mathfrak{d}$. A Heisenberg double is built from the same pieces. A Heisenberg double has the same underlying group D but with Poisson structure Π_+ given by

$$\{a \overset{\otimes}{,} a\}_+ = (a \otimes a) r_{\mathfrak{d}} - r_{\mathfrak{d}}^* (a \otimes a) \tag{4.1}$$

where $r_{\mathfrak{d}}^* = X_i \otimes \xi^i$ and $a \in D$. We can decompose (4.1) using the coordinates $a = gu = vh \in D, g, h \in G, u, v \in G^*$ to find the brackets between the elements of the subgroups

$$\begin{array}{llll}
\{g_1, g_2\}_+ = -[g_1 g_2, r], & \{u_1, u_2\}_+ = -[u_1 u_2, r^*], & \{g_1, u_2\}_+ = g_1 r u_2, & \{u_1, g_2\}_+ = g_2 r^* u_1, \\
\{h_1, h_2\}_+ = -[h_1 h_2, r^*], & \{v_1, v_2\}_+ = -[v_1 v_2, r], & \{v_1, h_2\}_+ = v_1 r h_2, & \{h_1, v_2\}_+ = v_2 r^* h_1, \\
\{g_1, h_2\}_+ = 0, & \{g_1, v_2\}_+ = -r^* g_1 v_2, & \{u_1, v_2\}_+ = 0, & \{u_1, h_2\}_+ = u_1 h_2 r,
\end{array}$$

where we have used the tensor notation $g_1 = g \otimes 1, g_2 = 1 \otimes g$.

The relation between the Poisson structure of a Drinfeld double Π_- and Heisenberg double Π_+ can be easily deduced from the following formula

$$\Pi_\pm = \frac{1}{2} \left((\xi^{iL} \otimes X_i^L - X_i^L \otimes \xi^{iL}) \pm (\xi^{iR} \otimes X_i^R - X_i^R \otimes \xi^{iR}) \right). \tag{4.2}$$

In terms of the second Russian formula this is the same as

$$\{a \overset{\otimes}{,} a\}_\pm = [a \otimes a, \hat{r}_{\mathfrak{d}}]_\pm \tag{4.3}$$

where $\hat{r}_{\mathfrak{d}} = \frac{1}{2} \xi^i \wedge X_i$ is the antisymmetric part of $r_{\mathfrak{d}} = \xi^i \otimes X_i$ and $[\,,\,]_-, [\,,\,]_+$ are the commutator and anti-commutator, respectively. We denote a Heisenberg double by (D, Π_+). The reason for the $\pm$ in the notation should now be apparent since the difference between the Poisson structure of a Drinfeld double Π_- and the Poisson structure of a Heisenberg double Π_+ can be boiled down to a change in sign. Simplifying the Heisenberg Poisson structure Π_+ in equation 4.2 we get the equivalent formulas

$$\Pi_+ = \xi^{iL} \otimes X_i^L - X_i^R \otimes \xi^{iR} = \xi^{iR} \otimes X_i^R - X_i^L \otimes \xi^{iL}. \tag{4.4}$$

It is important to note that a Heisenberg double is not a Poisson-Lie group. Since $\Pi_+(e) = r_{\mathfrak{d}} - r_{\mathfrak{d}}^* \neq 0$ is non-degenerate, Π_+ is non-degenerate in a neighbourhood of the identity. Π_+ is an example of an **affine** Poisson structure. A Poisson structure Π on a group G is affine if

$$\Pi(gg') = (L_g)_* \Pi(g') + (R_{g'})_* \Pi(g) - (L_g)_* (R_{g'})_* \Pi(e)$$

for $g, g' \in G$ [29]. This condition is very similar to the multiplicativity condition of a Poisson structure, see (3.17) on page 26. In fact, we can characterize affine Poisson structures by their similarity to multiplicative ones. A Poisson structure Π on a group G is affine if and only if the new bivector field Π_L defined by

$$\Pi_L(g) = \Pi(g) - (L_g)_*\Pi(e)$$

is multiplicative (or, equivalently, if and only if Π_R is multiplicative where $\Pi_R(g) = \Pi(g) - (R_g)_*\Pi(e)$).

4.2 The Cotangent Bundle Relation

Recall in section 3.7 we saw that a cotangent bundle can be identified with a double group. This was done by recognizing that a Poisson-Lie group (G, Π) with trivial Poisson structure, i.e. $\Pi = 0$, has a corresponding double group D given by $D = G \ltimes \mathfrak{g}^*$. The factorizations of the elements of this group then corresponded to the trivializations of the cotangent bundle T^*G. Using this we extended our definition of body and space coordinates to general double groups. At the level of manifolds we then had a straightforward identification of T^*G with the double $D = G \ltimes \mathfrak{g}^*$. In that discussion we neglected to discuss the second important part of our construction, the Poisson structure. Cotangent bundles have a canonical symplectic structure arising from the Liouville form. This symplectic structure, which is non-degenerate by definition, then defines a canonical Poisson structure. The multiplicative Poisson structures of Poisson-Lie groups are degenerate about the identity and thus cannot be seen as a generalization of the canonical Poisson structure of a cotangent bundle.

We shall see now, presented in detail for the first time, how the affine structure of a Heisenberg double can be identified with the canonical Poisson structure of a cotangent bundle. In body and space coordinates the Poisson structure Π_+ of the Heisenberg double $(D = G \bowtie G^*, \Pi_+)$ can be written as

$$\Pi_+^B(g,u) = \Pi_G(g) + \Pi_{G^*}(u) + \sum_i (R_u)_*\xi^i \wedge (L_g)^* X_i \in \wedge^2 T(G \times G^*) \tag{4.5}$$

$$\Pi_+^S(v,h) = -\Pi_G(h) - \Pi_{G^*}(v) + \sum_i (L_v)_*\xi^i \wedge (R_h)^* X_i \in \wedge^2 T(G^* \times G) \tag{4.6}$$

where $g, h \in G, u, v \in G^*$, and Π_G, Π_{G^*} are as defined in (3.28), (3.29), respectively [6]. Supposing we choose coordinates x^i on the dual space $\mathfrak{g}^*$ so that any $\xi \in \mathfrak{g}^*$ can be written as $\xi = \sum_i x^i \xi^i$. Then on the double group $G \ltimes \mathfrak{g}^*$ we can identify the generators ξ^i of $\mathfrak{g}^*$

with $\frac{\partial}{\partial x^i}$. From the construction of the double we know $\Pi_G(g) = 0$ and

$$\Pi_{G^*}(\xi) = \sum_{ijk} x^i c^i_{jk} \frac{\partial}{\partial x^j} \otimes \frac{\partial}{\partial x^k}$$

is the linear Poisson structure on $\mathfrak{g}^*$ where c^i_{jk} are the structure constants of $\mathfrak{g}$. The dual $\mathfrak{g}^*$ is also abelian so $(R_u)_* \frac{\partial}{\partial x^i} = (L_u)_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}\big|_u$. Rewriting (4.5) and (4.6) with this knowledge we recover exactly (2.6) and (2.7), the canonical Poisson structure of the cotangent bundle in body and space coordinates.

Example 4.1. Recall example 3.18. There we constructed the double associated to the Poisson-Lie group $SO(3)$ with the trivial Poisson structure. Equipping this double with the Heisenberg Poisson structure the brackets on the group elements $g = (R, \mathbf{u}) \in ISO(3)$ can be computed easily using (4.1) to find

$$\{u_i, u_j\} = \epsilon^k_{ij} u_k, \quad \{u_i, R\} = RE_i, \quad \{R, R\} = 0.$$

This is in exact agreement with the Poisson brackets found on $T^*SO(3)$ in body coordinates, see example 2.6.

4.3 Symmetries and Momentum Maps

We saw that the symmetries of a symplectic manifold (M, ω) were captured by a symplectic group action Φ. This action respected the symplectic structure of (M, ω). Our notion of symmetry can now be generalized to include group actions where the group carries a non-trivial Poisson structure. Recall a Poisson action is an action $\Phi : G \times M \to M$ of a Poisson-Lie group (G, Π_G) on a Poisson manifold (M, Π_M) such that Φ is a Poisson map, i.e.

$$\{f, h\}_M \circ \Phi(g, p) = \{f \circ \Phi(\cdot, p), h \circ \Phi(\cdot, p)\}_G(g) + \{f \circ \Phi(g, \cdot), h \circ \Phi(g, \cdot)\}_M(p)$$

for $g \in G, p \in M$. If the Poisson-Lie group (G, Π_G) has trivial Poisson structure then Φ is simply the Poisson action of G on (M, Π_M), which is the obvious generalization of the symplectic action to Poisson manifolds.

Poisson actions like symplectic group actions define a corresponding vector field. A Poisson action Φ defines an infinitesimal Poisson action of the Lie bialgebra $(\mathfrak{g}, \delta)$ on (M, Π_M). It is a map from the Lie bialgebra to the space of vector fields $\mathfrak{X}(M)$

$$X \in \mathfrak{g} \mapsto \Phi_X(p) = \frac{d}{dt}\bigg|_{t=0} \Phi(\exp tX, p) \in \mathfrak{X}(M) \tag{4.7}$$

for all $p \in M$.

A **momentum map** corresponding to a (left) Poisson action $\Phi : G \times M \to M$ is a map $J : M \to G^*$ where G^* is the Poisson-Lie dual of G. J must satisfy

$$\Phi_X = \Pi_M(J^*(X^r), \cdot) \tag{4.8}$$

for all $X \in \mathfrak{g}$ where Φ_X is as defined in (4.7) and X^r is the right invariant 1-form on G^* defined by X. To clarify $J^*(X^r)$ is the image of X^r under the pullback of J, $J^* : T^*G^* \to T^*M$. This is sometimes referred to as the cotangent lift of J. Further, $X^r(u) = (R_{u^{-1}})^*X$ where $u \in G^*$ since $\langle X_i, \xi^j \rangle = \delta_i^j = (R_{u^{-1}})^*X_i)(\xi^{jR})(u)$ as $\xi^{jR}(u) = (R_u)_*\xi^j \in T_uG^*$.

In the special case where Φ is a Hamiltonian action then the above definition of momentum mapping recovers the traditional definition of a momentum map found on page 9. In this scenario the Poisson-Lie group (G, Π_G) has a trivial Poisson structure so that the dual $G^* = \mathfrak{g}^*$ is abelian and $X^r = X \in \mathfrak{g}$ is a constant 1-form on $\mathfrak{g}^*$. Then

$$J^*(X^r) = d\hat{J}(X)$$

where $\hat{J}(X)(p) = \langle J(p), X \rangle, p \in M$ and $J : M \to \mathfrak{g}^*$. So

$$\Phi_X = \Pi_M(d\hat{J}(X), \cdot) = \{\hat{J}(X), \cdot\}$$

which is exactly (2.5).

Momentum maps are not guaranteed to exist for all Poisson actions. However, we know that every Poisson action on a simply connected symplectic manifold has a momentum mapping [30].

Example 4.2. The left (and right) action of Poisson-Lie group on itself does not have a corresponding momentum map. By definition the multiplication map $\mu : G \times G \to G$ is a Poisson action. However, as the Poisson structure is trivial at the identity there is no corresponding momentum map. We can see this by the following, first we write the Poisson structure as,

$$\Pi_G(g) = \sum_i (L_g)_* \mathrm{pr}_{\mathfrak{g}}((Ad_{g^{-1}})_* \xi^i) \otimes (R_g)_* X_i.$$

The generator of the left action is a right invariant vector field. Thus a momentum map $J : G \to G^*$ must satisfy,

$$J^*(X_j^r)((L_g)_* \mathrm{pr}_{\mathfrak{g}}((Ad_{g^{-1}})_* \xi^i)) = \delta_j^i$$

for all $g \in G$. This is not possible at the identity.

There is a version of Noether's theorem that holds in this generalized framework [30]. It is the analogue of the conservation law presented in Theorem 1.

Theorem 4. *Let $\Phi : G \times M \to M$ be a Poisson action of a Poisson-Lie group G on a Poisson manifold M with momentum mapping $J : M \to G^*$. If $H \in \mathcal{F}(M)$ is invariant under the Poisson action Φ of G, i.e. $H(p) = H(\Phi_g(p))$ for all $p \in M, g \in G$ then J is an integral of the Hamiltonian vector field X_H.*

This can be proven similarly to Theorem 1. Using the invariance of H and differentiating we get,

$$0 = dH(\Phi_Y) = \Pi_M(J^*(Y^r), dH) = -J^*(Y^r)(X_H) = -Y^r(J_* X_H)$$

for all $Y \in \mathfrak{g}$. Hence $J_* X_H = 0$ and J is an integral of X_H. Thus J is conserved during time evolution.

There is a version of Euler's conservation law that holds as well in this context which we present now for the first time. First, we recognize that the left action σ of the Drinfeld double (D, Π_-) on the Heisenberg Double (D, Π_+) is a Poisson action. The left action

$$\begin{aligned} \sigma : (D, -\Pi_-) \times (D, \Pi_+) &\to (D, \Pi_+) \\ (b, a) &\mapsto ba \end{aligned}$$

satisfies

$$\{ba^{(1)}, ba^{(2)}\}_+ = (b \otimes b)\{a^{(1)}, a^{(2)}\}_+ - \{b^{(1)}, b^{(2)}\}_-(a \otimes a)$$

since

$$(ba \otimes ba)r_{\mathfrak{d}} - r_{\mathfrak{d}}^*(ba \otimes ba) = (b \otimes b)[(a \otimes a)r_{\mathfrak{d}} - r_{\mathfrak{d}}^*(a \otimes a)] - [(b \otimes b)r_{\mathfrak{d}} - r_{\mathfrak{d}}(b \otimes b)](a \otimes a)$$

and is thus Poisson [6].

The left action $\mathcal{L} : (G, \Pi_G) \times (D, \Pi_+) \to (D, \Pi_+) : (g, a) \mapsto ga$ is also Poisson since $(G, -\Pi_G)$ is a Poisson Lie subgroup of (D, Π_-). The associated infinitesimal generator is

$$\mathcal{L}_X(a) = \left.\frac{d}{dt}\right|_{t=0} \exp tXa = X^R(a)$$

for $X \in \mathfrak{g}, a \in D$. A momentum map associated to this action is a map $J : D \to G^*$ satisfying (4.8), where G^* is the Poisson-Lie dual of (G, Π_G). Using the later half of (4.4)

$$\Pi_+(J^*(X^r), \cdot) = \sum_i [J^*(X^r)(\xi^{iR})]X_i^R - [J^*(X^r)(X_i^L)]\xi^{iL}$$

where we recall that $\xi \in \mathfrak{g}^* \subset \mathfrak{d}$ by construction of the double Lie bialgebra $(\mathfrak{d}, \delta(r_\mathfrak{d}))$. Thus J must satisfy

$$X_j^R = \sum_i [J^*(X_j^r)(\xi^{iR})]X_i^R - [J^*(X_j^r)(X_i^L)]\xi^{iL}$$

for all $X_j \in \mathfrak{g}$. Thus,

$$J^*(X_j^r)(X_i^L) = 0, \quad J^*(X_j^r)(\xi^{iR}) = \delta_j^i.$$

So in particular from the second equation, $J_*\xi^{iR} = \xi^{iR}|_{G^*} \in TG^*$. By definition

$$\xi^{iR}|_{G^*}(f)(w) = \frac{d}{dt}\bigg|_{t=0} f(\exp t\xi^i w), \quad J_*\xi^{iR}(f)(a) = \frac{d}{dt}\bigg|_{t=0}(f \circ J)(\exp t\xi^i a)$$

for all $f \in \mathcal{F}(G^*)$ and $w \in G^*, a \in D$. Supposing $J(a) = w \in G^*$, J must satisfy

$$\frac{d}{dt}\bigg|_{t=0} f(\exp t\xi J(a)) = \frac{d}{dt}\bigg|_{t=0} f(J(\exp t\xi a))$$

for all $f \in \mathcal{F}(G^*)$ and $\xi \in \mathfrak{g}^*$ where we consider the argument of J as an element in D. Hence,

$$J(a) = u^{-1}J(ua), \text{ for all } u \in G^*.$$

The map $J(a) = p_L^*(a)$ satisfies this equation. Further, this map satisfies the first equation, $J_*X_i^L = 0 \in TG^*$. We can see that

$$J_*X_i^L(f)(a) = \frac{d}{dt}\bigg|_{t=0} f(J(a\exp tX_i)) = \frac{d}{dt}\bigg|_{t=0} f(p_L^*(a\exp tX_i)) = 0$$

for all $f \in \mathcal{F}(G^*)$. Thus $J(a) = p_L^*(a)$ is a momentum map associated to the group action $\mathcal{L}$. In Table 4.1 we summarize this result in comparison to the Euler conservation law as presented on the cotangent bundle.

Taking advantage of the group structures that abound we can extend this to find the momentum maps of both the right and left actions of G and G^* on the Heisenberg double D. Continuing as in the previous case the left action $\mathcal{L}^* : (G^*, -\Pi_{G^*}) \times (D, \Pi_+) \to (D, \Pi_+) : (u, a) \mapsto ua$ is also Poisson since (G^*, Π_{G^*}) is a Poisson Lie subgroup of (D, Π_-). A momentum map associated to this action is a map $J : D \to G^{op}$ satisfying (4.8), where G^{op} is G with opposite product, i.e. $gh = h *^{op} g$. G^{op} is the Poisson-Lie dual of $(G^*, -\Pi_{G^*})$ since the Poisson structure is negative. Solving $\xi^{jR} = \Pi_+(J^*(\xi^{jr}), \cdot)$ we find that

$$J(a) = g *^{op} J(ga), \text{ for all } g \in G.$$

Table 4.1: Comparison of Euler's Conservation Law on the Heisenberg Double

Phase Space	Action	Momentum Map
T^*G	$L^{T*} : G \times T^*G \to T^*G$ $g' \times \alpha \mapsto (L_{g'^{-1}})^*\alpha$ Body: $L^B_{g'}(g, \mu) = (g'g, \mu)$ Space: $L^S_{g'}(\nu, h) = ((Ad_{g'^{-1}})^*\nu, g'h)$	$J : T^*_gG \mapsto \mathfrak{g}^*$ $J(\alpha) = (R_g)^*\alpha$ $J^B(g, \mu) = (Ad_{g^{-1}})^*\mu$ $J^S(\nu, h) = \nu$
(D, Π_+)	$\mathcal{L} : (G, \Pi_G) \times (D, \Pi_+) \to (D, \Pi_+)$ $g' \times a \mapsto g'a$ Body: $\mathcal{L}^B_{g'}(g, u) = (g'g, u)$ Space: $\mathcal{L}^S_{g'}(v, h) = (p^*_L(Ad_{g'}v), p_R(Ad_{g'}v)g'h)$	$J : D \mapsto G^*$ $J(a) = p^*_L(a)$ $J^B(g, u) = p^*_L(Ad_g u)$ $J^S(v, h) = v$

Thus the associated momentum map is $J(a) = [p_L(a)]^{-1} = p_R(a^{-1})$.

So far we have considered left actions as the underlying natural action of a group. We now give the definition of a momentum map for right actions. The momentum map of a right Poisson action $\Phi : M \times G \to M$ is a map $J : M \to G^*$ where G^* is the Poisson-Lie dual of G. J must satisfy

$$\Phi_X = -\Pi_M(J^*(X^l), \cdot) \tag{4.9}$$

for all $X \in \mathfrak{g}$ where Φ_X is as defined (4.7) and X^l is the left invariant 1-form on G^* defined by X. The difference between this definition and that of a left action is only a change in sign and a change from right invariance to left invariance of the 1-form.

The right action $\tau : (D, \Pi_+) \times (D, \Pi_-) \to (D, \Pi_+) : (a, c) \mapsto ac$ is a Poisson action since it satisfies

$$\{ac^{(1)}, ac^{(2)}\}_+ = (a \otimes a)\{c^{(1)}, c^{(2)}\}_- + \{a^{(1)}, a^{(2)}\}_+(c \otimes c).$$

Using this the following are right Poisson actions

$$\mathcal{R} : (D, \Pi_+)\times \to (D, \Pi_+) : (a, h) \mapsto ah$$
$$\mathcal{R}^* : (D, \Pi_+) \times (G^*, \Pi_{G^*}) \to (D, \Pi_+) : (a, v) \mapsto av$$

since $(G, -\Pi_G), (G^*, \Pi_{G^*})$ are Poisson-Lie subgroups of (D, Π_-). The momentum maps are

$$J : D \to G^{*op} : a \mapsto [p^*_R(a)]^{-1} = p^*_L(a^{-1})$$
$$J : D \to G : a \mapsto p_R(a)$$

Table 4.2: Euler's Conservation Laws on the Heisenberg Double

	Action	Momentum Map
	$\mathcal{L}:(G,\Pi_G)\times(D,\Pi_+)\to(D,\Pi_+)$	$J:D\mapsto G^*$
	$g'\times a\mapsto g'a$	$J(a)=p_L^*(a)$
Body:	$\mathcal{L}^B_{g'}(g,u)=(g'g,u)$	$J^B(g,u)=p_L^*(Ad_g u)$
Space:	$\mathcal{L}^S_{g'}(v,h)=(p_L^*(Ad_{g'}v),p_R(Ad_{g'}v)g'h)$	$J^S(v,h)=v$
	$\mathcal{L}^*:(G^*,-\Pi_{G^*})\times(D,\Pi_+)\to(D,\Pi_+)$	$J:D\mapsto G^{op}$
	$u'\times a\mapsto u'a$	$J(a)=p_R(a^{-1})$
Body:	$\mathcal{L}^{*B}_{u'}(g,u)=(p_L(Ad_{u'}g),p_R^*(Ad_{u'}g)u'u)$	$J^B(g,u)=g^{-1}$
Space:	$\mathcal{L}^{*S}_{u'}(v,h)=(u'v,h)$	$J^S(v,h)=p_L(Ad_v h)^{-1}$
	$\mathcal{R}:(D,\Pi_+)\times(G,-\Pi_G)\to(D,\Pi_+)$	$J:D\mapsto G^{*op}$
	$a\times g'\mapsto ag'$	$J(a)=p_L^*(a^{-1})$
Body:	$\mathcal{R}^B_{g'}(g,u)=(gg'p_L(Ad_{g'^{-1}}u),p_R^*(Ad_{g'^{-1}}u))$	$J^B(g,u)=u^{-1}$
Space:	$\mathcal{R}^S_{g'}(v,h)=(v,hg')$	$J^S(v,h)=p_R^*(Ad_{h^{-1}}v)^{-1}$
	$\mathcal{R}^*:(D,\Pi_+)\times(G^*,\Pi_{G^*})\to(D,\Pi_+)$	$J:D\mapsto G$
	$a\times u'\mapsto au'$	$J(a)=p_R(a)$
Body:	$\mathcal{R}^{*B}_{g'}(g,u)=(g,uu')$	$J^B(g,u)=p_R(Ad_{u^{-1}}g)$
Space:	$\mathcal{R}^{*S}_{g'}(v,h)=(vu'p_L^*(Ad_{u'^{-1}}h),p_R(Ad_{u'^{-1}}h))$	$J^S(v,h)=h$

associated to $\mathcal{R},\mathcal{R}^*$, respectively. We summarize this in Table 4.2.

So far we have examined the action of the Poisson-Lie subgroups on the Heisenberg double and seen how these actions and their associated momentum maps are related, particularly to the lifted left action on the cotangent bundle. We shall now see that the full symmetry structure of the Heisenberg double is the whole of the Drinfeld double. The momentum maps of the action of the Drinfeld double are presented here for the first time and have no known classical analogue.

The associated infinitesimal generators of the left and right Poisson actions, σ and τ respectively, of the Drinfeld double (D,Π_-) on the Heisenberg double (D,Π_+) are the right and left invariant vector fields of D, respectively. That is, for $X+\xi \in \mathfrak{d}, a \in D$

$$\sigma_{X+\xi}(a) = \frac{d}{dt}\bigg|_{t=0} \exp t(X+\xi)a = (X+\xi)^R(a),$$
$$\tau_{X+\xi}(a) = \frac{d}{dt}\bigg|_{t=0} a\exp t(X+\xi) = (X+\xi)^L(a).$$

A momentum map associated to these actions is a map $J : D \to D^*$ where $D^* = G^* \times G^{op}$ or $D^* = G^{*op} \times G$ is the Poisson-Lie dual of $(D, -\Pi_-)$ or (D, Π_-), respectively. The double Lie algebras of these are given by $\mathfrak{d}^* = \mathfrak{g}^* \times \mathfrak{g}^{op}$ and $\mathfrak{d}^* = \mathfrak{g}^{*op} \times \mathfrak{g}$, respectively.

Let us focus on the left action σ first. We let Y^i generate $\mathfrak{g}^*$ and γ_i generate $\mathfrak{g}^{op}$ in $\mathfrak{d}^* = \mathfrak{g}^* \times \mathfrak{g}^{op}$ so they satisfy the relations

$$[Y^i, Y^j] = f^{ij}_k Y^k, \quad [Y^i, \gamma_j] = 0, \quad [\gamma_i, \gamma_j] = -c^k_{ij}\gamma_k.$$

Further, the duality relations between these generators and those of $\mathfrak{d}$ read

$$\langle X_i, Y^j\rangle = \langle \xi^j, \gamma_i\rangle = \delta^j_i, \quad \langle X_i, \gamma_j\rangle = \langle \xi^i, Y^j\rangle = 0.$$

The Poison structure Π_+ of the Heisenberg double (D, Π_+) can be written

$$\Pi_+(a) = \sum_i (R_a)_*\xi^i \otimes (L_a)_*\mathrm{pr}_{\mathfrak{g}}((Ad_{a^{-1}})_*X_i) - (R_a)_*X_i \otimes (L_a)_*\mathrm{pr}_{\mathfrak{g}^*}((Ad_{a^{-1}})_*\xi^i)$$

for $a \in D$ [6]. A momentum map $J : D \to D^* = G^* \times G^{op}$ associated to σ must satisfy

$$X^R_j + \xi^{kR} = -\Pi_+(\cdot, J^*(X^r_j + \xi^{kr}))$$

from which we get

$$J^*(X^r_j + \xi^{kr})((L_a)_*\mathrm{pr}_{\mathfrak{g}}((Ad_{a^{-1}})_*X_i)) = -\delta^k_i$$
$$J^*(X^r_j + \xi^{kr})((L_a)_*\mathrm{pr}_{\mathfrak{g}^*}((Ad_{a^{-1}})_*\xi^i)) = \delta^i_j$$

for all $X_j \in \mathfrak{g}$, $\xi^k \in \mathfrak{g}^*$. Using the duality relations between $\mathfrak{d}$ and $\mathfrak{d}^*$ we see that the vector fields on D^* dual to X^r_j and ξ^{kr} are Y^{jR} and γ^R_k, respectively. Thus

$$J_*[(L_a)_*\mathrm{pr}_{\mathfrak{g}}((Ad_{a^{-1}})_*X_i)] = -\gamma^R_i, \quad J_*[(L_a)_*\mathrm{pr}_{\mathfrak{g}^*}((Ad_{a^{-1}})_*\xi^i)] = Y^{iR}.$$

From these equations we get

$$J(a) = (e,h)\cdot J(ap_L(Ad_{a^{-1}}h)) \in G^* \times G^{op}$$
$$J(a) = (v,e)\cdot J(ap^*_L(Ad_{a^{-1}}v)) \in G^* \times G^{op}$$

where $h \in G, v \in G^*$. Combining the momentum maps of $\mathcal{L}, \mathcal{L}^*$ we can verify that $J(a) = (p_L^*(a), p_R(a^{-1}))$ satisfies these relations. Firstly, if $a = gu, g \in G, u \in G^*$ then

$$\begin{aligned} ap_L(Ad_{a^{-1}}h) &= hgp_L^*(g^{-1}h^{-1}gu) \\ ap_L^*(Ad_{a^{-1}}v) &= gp_L^*(g^{-1}vgu) \end{aligned}$$

where we have used the identity $p_L(a) = ap_L^*(a^{-1})$. Using our choice of factorization we have

$$\begin{aligned} p_L^*(hgp_L^*(g^{-1}h^{-1}gu)) &= p_L^*(gu), & p_R([hgp_L^*(g^{-1}h^{-1}gu)]^{-1}) &= g^{-1}h^{-1} \\ p_L^*(gp_L^*(g^{-1}vgu)) &= v^{-1}p_L^*(gu), & p_R([gp_L^*(g^{-1}vgu)]^{-1}) &= g^{-1}. \end{aligned}$$

and $J(gu) = (p_L^*(gu), g^{-1})$. Noting that we have the opposite product on G we see that

$$\begin{aligned} (e,h) \cdot (p_L^*(gu), g^{-1}h^{-1}) &= J(gu) \\ (v,e) \cdot (v^{-1}p_L^*(gu), g^{-1}) &= J(gu). \end{aligned}$$

Thus $J(a) = (p_L^*(a), p_R(a^{-1}))$ is a momentum map associated to the left action of the Drinfeld double $(D, -\Pi_-)$ on the Heisenberg double (D, Π_+).

In a similar manner we find that $J(a) = (p_L^*(a^{-1}), p_R(a)) \in G^{*op} \times G$ is a momentum map associated to τ, the right action of (D, Π_-) on (D, Π_+). For these calculations it is important to note that the Poison structure Π_+ can also be written as

$$\Pi_+(a) = \sum_i (L_a)_*\xi^i \otimes (R_a)_*\mathrm{pr}_{\mathfrak{g}}((Ad_a)_*X_i) - (L_a)_*X_i \otimes (R_a)_*\mathrm{pr}_{\mathfrak{g}^*}((Ad_a)_*\xi^i)$$

for $a \in D$ [6]. These results are summarized in Table 4.3.

Example 4.3 (The deformed rigid rotator)**.** From the examples 3.17, 3.19 we can see that the Heisenberg double $(SO(3,1) = SO(3) \bowtie AN(2), \Pi_+)$ is a deformation of $T^*SO(3)$. We now choose the Hamiltonian $H(a) = \frac{1}{2}Tr(aa^T) - 2, a \in SO(3,1)$ since $\frac{1}{2}Tr(aa^T) - 2 = \frac{1}{2}Tr(vv^T) - 2 \to \frac{1}{2}\mathbf{v}^T\mathbf{v}$ in the limit $\beta \to \infty$ which is exactly the Hamiltonian of the classical rotator (see example 2.8)[32]. Using Hamilton's equations we know that for any function $f \in \mathcal{F}(SO(3,1))$

$$\begin{aligned} \frac{d}{dt}f(a) &= \{H, f\}_+(a) \\ &= \frac{1}{2}\sum_i [x_i^L Tr(aa^T) - 4][X_i^L f(a)] - [X_i^R Tr(aa^T) - 4][x_i^R f(a)] \end{aligned}$$

Table 4.3: Actions of the Drinfeld Double on the Heisenberg Double

Action	Momentum Map
$\sigma : (D, -\Pi_-) \times (D, \Pi_+) \to (D, \Pi_+)$ $b \times a \mapsto ba$	$J : D \mapsto G^* \times G^{op}$ $J(a) = (p_L^*(a), p_R(a^{-1}))$
Body: $\sigma^B_{g'u'}(g, u) = (g' p_L(Ad_{u'} g), p_R^*(Ad_{u'} g) u' u)$ Space: $\sigma^S_{v'h'}(v, h) = (v' p_L^*(Ad_{h'} v), p_R(Ad_{h'} v) h' h)$	$J^B(g, u) = (p_L^*(Ad_g u), g^{-1})$ $J^S(v, h) = (v, p_L(Ad_v h)^{-1})$
$\tau : (D, \Pi_+) \times (D, \Pi_-) \to (D, \Pi_+)$ $a \times c \mapsto ac$	$J : D \mapsto G^{*op} \times G$ $J(a) = (p_L^*(a^{-1}), p_R(a))$
Body: $\sigma^B_{g'u'}(g, u) = (g g' p_L(Ad_{g'^{-1}} u), p_R^*(Ad_{g'^{-1}} u) u')$ Space: $\sigma^S_{v'h'}(v, h) = (v v' p_L^*(Ad_{v'^{-1}} h), p_R(Ad_{v'^{-1}} h) h')$	$J^B(g, u) = (u^{-1}, p_R(Ad_{u^{-1}} g))$ $J^S(v, h) = (p_R^*(Ad_{h^{-1}} v)^{-1}, h)$

where we have used the first half of (4.4) to capture the Poisson structure. In particular,

$$\frac{d}{dt}h = \{H, p_R\}_+(a) = \sum_i \frac{1}{2} Tr(h^T v^T v h(x_i + x_i^T)) h X_i$$

$$\frac{d}{dt}v = \{H, p_L^*\}_+(a) = \sum_i \frac{1}{2} Tr(v v^T (X_i + X_i^T)) x_i v = 0$$

Now in the limit $\beta \to \infty$ we have $\sum_i \frac{1}{2} Tr(h^T v^T v h(x_i + x_i^T)) h X_i \to v_i J_i h$. And thus we can recover the dynamics of the standard rigid rotator. In the deformed case, $V_i = \frac{1}{2} Tr(h^T v^T v h(x_i + x_i^T)) h X_i$ plays the role of the angular momenta. This Hamiltonian is invariant under the left and right actions of $SO(3)$ thus v and $p_R^*(Ad_{h^{-1}} v)^{-1}$ are conserved, see Table 4.2. This action is not invariant under the left or right action of whole of $SO(3, 1)$. Nonetheless we see that the lack of invariance of the Hamiltonian under the left and right action of $AN(2)$ tells us that dynamics breaks the $AN(2)$ Poisson-Lie symmetry of the space.

Chapter 5

Conclusion

We have seen that Heisenberg doubles provide a well-defined framework in which to deform the momentum of classical phase spaces. Beginning with a new notion of symmetry, where we replaced symplectic group actions with Poisson group actions, we were able to enlarge the set of phase spaces under consideration. The addition of a non-trivial Poisson structure on the symmetry group, controlled by a single deformation parameter, ultimately resulted in the non-abelian group structure of momentum space. We could say that the curvature of momentum was a result of the new symmetries or that the deformation of momentum led to the new symmetries. Either way, these symmetries are hoped to contain clues informing us about the structure of spacetime in the quantum gravity regime. The Heisenberg double gives us a well-defined laboratory in which to investigate some of the first order effects of quantum gravity.

With a direct route to classical mechanics, Heisenberg doubles provide a bridge between classical mechanics and quantum groups. They allow us to introduce a cosmological constant and quantum gravity scale by hand. We saw that Heisenberg doubles encompass the cotangent bundles of Lie groups and that many notions defined in the classical setting carry over to the new deformed picture. These include, choices of coordinates, momentum maps and conservation laws. Further we saw that the Heisenberg double has a larger symmetry structure given by the Drinfeld double. The momentum maps of these symmetries was presented here for the first time.

Snyder first demonstrated that the curvature of momentum space implies the non-commutativity of spacetime [39]. With non-commutativity hinted at by many approaches to quantum gravity and given that quantum groups are a powerful and consistent approach to defining non-commutative spacetimes, quantum groups serve as an important tool to investigate models of quantum gravity. Poisson-Lie groups, the classical counterpart of quantum groups, are then another tool to shed light on the mysteries of quantum gravity. In this thesis we have illustrated a method to take the phase space of a classical system and introduce some quantum gravitational effects by the introduction of a scale. What this method will tell us about our space-time, is yet unknown. Components of this approach have begun to be utilized to investigate (2+1) dimensional gravity theories [14, 9, 12].

Of particular interest for future work is the deformed particle with spin in 4 dimensions. In example 3.19 we have deformed a particle with spin in 2 dimensions. A deformation of a particle with spin in the context of Poisson-Lie symmetries and the Heisenberg double has not yet been done. The undeformed phase space of a particle with spin is given by $T^*ISO(3,1)$ [25, 28]. To begin, it is known that there are at least 21 possible coboundary Lie bialgebras on $\mathfrak{iso}(3,1)$ [41] of which only one corresponds to the common κ-Minkowski space. This provides us with many deformation possibilities to investigation. Following the notation of [41] the basis of $\mathfrak{iso}(3,1)$ is given by $H, X_\pm, JH, JX_\pm, e_0, \ldots, e_3$ with relations

$$\begin{gathered}
[H, X_+] = X_+, \quad [H, X_-] = -X_-, \quad [X_+, X_-] = 2H, \\
[JA, JB] = -[A, B], \quad [JA, B] = J[A, B] \\
[H, e_0] = e_3, \quad [H, e_3] = e_0, \quad [JH, e_1] = -e_2, \quad [JH, e_2] = e_1 \\
[X_\pm, e_0] = e_1, \quad [X_\pm, e_1] = e_0 \pm e_3, \quad [X_\pm, e_3] = \mp e_1, \\
[JX_\pm, e_0] = \mp e_2, \quad [JX_\pm, e_2] = \mp e_0 - e_3, \quad [JX_\pm, e_3] = e_2
\end{gathered}$$

and all other brackets zero, where $A, B \in \{H, X_\pm\}$. The $H, X_\pm, JH, JX_\pm$ generate $\mathfrak{so}(3,1)$ and the $e_0, \ldots, e_3$ generate $\mathbb{R}^4$. If we choose the r-matrix

$$r = \gamma JH \wedge H - 2e_0 \wedge e_3 + \tilde{\alpha} e_1 \wedge e_2$$

where $\gamma, \alpha, \tilde{\alpha} \in \mathbb{R}$ (this corresponds to $N = 1$ of [41]) then using the Sklyanin bracket the dual Lie algebra $\mathfrak{iso}(3,1)^*$ with generators $H^*, X^*_\pm, JH^*, JX^*_\pm, e^*_0, \ldots, e^*_3$ has brackets given by

$$\begin{gathered}
[H^*, X^*_\pm] = \mp\gamma JX^*_\pm, \quad [H^*, JX^*_\pm] = \pm X^*_\pm, \quad [H^*, e^*_1] = \gamma e^*_2, \quad [H^*, e^*_2] = -\gamma e_1, \\
[JH^*, X^*_\pm] = \mp\gamma JX^*_\pm, \quad [JH^*, JX^*_\pm] = \mp X^*_\pm, \quad [JH^*, e^*_1] = \gamma e^*_2, [H^*, e^*_2] = -\gamma e_1, \\
[e^*_0, e^*_1] = \tilde{\alpha}(JX^*_+ - JX^*_-) + 2\alpha(X^*_+ - X^*_-), \\
[e^*_0, e^*_2] = -2\alpha(JX^*_+ + JX^*_-) + \tilde{\alpha}(X^*_+ + X^*_-), \\
[e^*_1, e^*_3] = -\tilde{\alpha}(JX^*_+ + JX^*_-) - 2\alpha(X^*_+ + X^*_-), \\
[e^*_2, e^*_3] = 2\alpha(JX^*_+ - JX^*_-) - \tilde{\alpha}(X^*_+ - X^*_-).
\end{gathered}$$

This is 10 dimensional solvable Lie algebra. The next stage, ripe for work, would be to define the constraints on the corresponding dual group such that they recover the mass-shell and spin constraints of a classical particle in the limit $\gamma, \alpha, \tilde{\alpha} \to 0$. These deformed constraints would then inform us of some first-order corrections in terms of the deformation parameter to the behaviour of a particle with spin. This is hoped to be carried out in future work.

APPENDICES

Appendix A

Lie Algebra Cohomology

Lie bialgebras by construction are related with a Chevalley-Eilenberg Lie algebra cohomology. The classic cohomolgy question, 'when is a closed form exact?' is important in the distinction of special cocycles that are called coboundary and correspond to r-matrices. The following is a summary of material found in [40, 27, 17].

Let $\mathfrak{g}$ be a Lie algebra with bracket $[\ ,\ ,]$ and V a vector space. A linear map $\rho : \mathfrak{g} \to End(V)$ is called a **representation** of $\mathfrak{g}$ on V if

$$\rho([X,Y]) = [\rho(X), \rho(Y)]$$

for all $X, Y \in \mathfrak{g}$, where $End(V)$ is the space of linear endomorphisms of V. We say that $\mathfrak{g}$ **acts on** V and denote $\rho(X)(v)$ as $X.v$ for $X \in \mathfrak{g}, v \in V$. We call a representation **trivial** if $V = \mathbb{R}$ and sometimes write $\rho = 0$.

Every finite dimensional Lie algebra $\mathfrak{g}$ has a **generalized adjoint representation**. For fixed $k \in \mathbb{N}$ it is a representation of $\mathfrak{g}$ on $\otimes^k \mathfrak{g} = \mathfrak{g} \otimes \cdots \otimes \mathfrak{g}$ (p times). Given a basis X_i of $\mathfrak{g}$ an element $Z \in \otimes^k \mathfrak{g}$ can be written as $Z = Z^{i_1 \cdots i_k} X_{i_1} \otimes \cdots X_{i_k}$ and the representation $\rho : \mathfrak{g} \to End(\otimes^k \mathfrak{g})$ is then

$$Y.Z = ad_Y^{(k)}(Z) = Z^{i_1 \cdots i_k} \sum_{n=1}^{k} X_{i_1} \otimes \cdots \otimes ad_Y(X_{i_n}) \otimes \cdots \otimes X_{i_k}$$

for $Y \in \mathfrak{g}$ where $ad_X(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$. In particular, for $k = 2$,

$$ad_Y^{(2)}(Z) = Z^{ij}(ad_Y X_i \otimes X_j + X_i \otimes ad_Y X_j) = Z^{ij}([Y, X_i] \otimes X_j + X_i \otimes [Y, X_j]).$$

If we denote the identity map from $\mathfrak{g}$ to $\mathfrak{g}$ by 1 then we may write

$$ad_Y^{(2)}(Z) = (ad_Y \otimes 1 + 1 \otimes ad_Y)(Z) = [Y \otimes 1 + 1 \otimes Y, Z].$$

This expression can be further decomposed in terms of the structure constants of $\mathfrak{g}$ and the components of Z.

For integers $k \in \mathbb{N}$ a **k-cochain** of $\mathfrak{g}$ with values in V is a skew-symmetric k-linear mapping from $\mathfrak{g}$ to V, where V is the vector space of a representation of $\mathfrak{g}$. The **coboundary operator** δ is a map that promotes k-cochains to $(k+1)$-cochains. Let u be an arbitrary k-cochain then the **coboundary** $\delta(u)$ of u is given by

$$\delta(u)(Y_0, \ldots, Y_k) = \sum_{i=0}^{k} (-1)^i Y_i.(u(Y_0, \ldots, \hat{Y}_i, \ldots, Y_k)) \\ + \sum_{i<j} (-1)^{i+j} u([Y_i, Y_j], Y_0, \ldots, \hat{Y}_i, \ldots, \hat{Y}_j, \ldots, Y_k)$$

for $Y_0, \ldots, Y_k \in \mathfrak{g}$ where $\hat{Y}_i$ indicates the element Y_i is omitted. We note that this equation resembles the formula for the explicit computation of the exterior derivative of a differential k-form. If u is a k-form on the Lie group G and the Y_is are vector fields on G, then u takes values in $\mathcal{F}(G)$ and if we replace $\rho(Y_i)$ by the action of the vector field Y_i on $\mathcal{F}(G)$ then the definiton of δ is just the definition of the exterior derivative d. If we consider the generalized adjoint action then for $k = 0$, u is simply a constant element of $\mathfrak{g}$ and $\delta(u)(X) = [X, u], X \in \mathfrak{g}$ and for $k = 1$, u is a linear map from $\mathfrak{g}$ to itself and

$$\delta(u)(X, Y) = [X, u(Y)] - [Y, u(X)] - u([X, Y]), \quad X, Y \in \mathfrak{g}.$$

A k-cochain is called a **k-cocycle** if $\delta(u) = 0$ or **k-coboundary** if $u = \delta(v)$ for some $(k-1)$-cochain v. In particular, a 0-cocycle is an invariant element of $\mathfrak{g}$, i.e. $[X, u] = 0$ for all $X \in \mathfrak{g}$ and a 1-cocycle satisfies $u([X, Y]) = [X, u(Y)] - [Y, u(X)]$. Since $\delta(\delta(u)) = 0$ always, any k-coboundary is a k-cocycle. The converse is not true.

For a representation ρ of $\mathfrak{g}$ on V the quotient of the vector space of k-cocycles $Z^k_\rho(\mathfrak{g}, V)$ by the vector space of k-coboundaries $B^k_\rho(\mathfrak{g}, V)$

$$H^k_\rho(\mathfrak{g}, V) = Z^k_\rho(\mathfrak{g}, V)/B^k_\rho(\mathfrak{g}, V)$$

is called the **k-th cohomology space** of $\mathfrak{g}$ on V corresponding to the representation ρ.

An important result that allows us to classify the Poisson structures of Poisson-Lie groups is the following lemma:

Theorem 5. *(Whitehead's Lemma [17]) If* $\mathfrak{g}$ *is semi-simple then* $H_0^2(\mathfrak{g}, \mathbb{R}) = 0$.

Since r-matrices are coboundary this result tells us that all Poisson-Lie structures of semi-simple Lie groups have an associated r-matrix.

Appendix B

The Schouten Bracket

The Schouten bracket is a generalization of the commutator of vector fields discovered by Jan Schouten in 1940 and further developed by his student Albert Nijenhuis. It plays a role in finding the r-matrices of coboundary Lie groups. The following is a summary of definitions found in [40, 29, 27].

Let M be a manifold of dimension n. We denote the space of of k-times contravariant skew-symmetric tensor fields on M by $V_k(M)$. In particular, $V_0(M) = \mathcal{F}(M)$ and $V_1(M) = \mathfrak{X}(M)$. The **multivector field algebra** is the direct sum

$$V(M) := \bigoplus_{k=0}^{n} V_k(M)$$

equipped with the exterior product $\wedge$. Multivectors in a particular subspace $V_k(M)$ are **homogoneous**. The **degree** of a homogeneous non-zero element $A \in V_k(M)$ is defined as k and denoted $|A| = k$. Elements in $V_k(M)$ of the form $X_1 \wedge \cdots \wedge X_k$ where $X_i \in \mathfrak{X}(M)$ are called **simple**.

The **Schouten bracket** of multivector fields is an $\mathbb{R}$-bilinear map on multivector fields. On homogeneous simple elements of $V(M)$ it is defined as

$$[|X_1\wedge\cdots\wedge X_n, Y_1\wedge\cdots\wedge Y_m|] = \sum_{i=1}^{n}\sum_{j=1}^{n}(-1)^{i+j}[X_i, Y_j]\wedge X_1\wedge\cdots\wedge\hat{X}_i\wedge\cdots\wedge X_n\wedge Y_1\wedge\cdots\wedge\hat{Y}_j\wedge\cdots\wedge Y_m$$

where $\hat{X}_i$ denotes omission of the vector field in the product. It is the unique map satisfying:

1. it is a biderivation of degree -1, i.e. for all $A, B, C \in V(M)$

- it is bilinear
- $|[|A,B|]| = |A| + |B| - 1$
- $[|A, B \wedge C|] = [|A,B|] \wedge C + (-1)^{(|A|+1)|B|} B \wedge [|A,C|]$

2. for all $f, g \in \mathcal{F}(M)$ and $X, Y \in \mathfrak{X}(M)$

   - $[|f,g|] = 0$
   - $[|X,f|] = Xf$
   - $[|X,Y|] = [X,Y]$ (the usual Lie bracket for vector fields)

3. $[|A,B|] = (-1)^{|A||B|}[|B,A|]$

The Schouten bracket also satisfies the graded Jacobi identity

$$(-1)^{|A||C|}[|[|A,B|],C|] + (-1)^{|A||B|}[|[|B,C|],A|] + (-1)^{|C||B|}[|[|C,A|],B|] = 0.$$

For an explicit expression of the Schouten bracket in coordinates see [40].

Every bivector $\Pi \in V_2(M)$ that satisfies $[|\Pi,\Pi|] = 0$ defines a Poisson bracket on M. And conversely, the Poisson bivector of every a Poisson manifold (M,Π) satisfies $[|\Pi,\Pi|] = 0$. This ensures that the Poisson structure satisfies the Jacobi identity. At the infinitesimal level of a coboundary Poisson-Lie group (G, Π_G) it is required that the r-matrix of the corresponding Lie bialgebra $(\mathfrak{g}, \delta(r))$ defines a Lie bracket $[\ ,\ ]^r$ on $\mathfrak{g}^*$. In particular, the Jacobi identity must be satisfied by that bracket. This condition has an alternate formulation involving the Schouten bracket.

Working with the generalized adjoint representation of $\mathfrak{g}$, the **algebraic Schouten bracket** of an element $r \in \wedge^2\mathfrak{g}$ is an element in $\wedge^3\mathfrak{g}$ defined as

$$[|r,r|](\xi,\eta,\zeta) := \circlearrowleft_{\xi,\eta,\zeta} -2\langle \zeta, [\underline{r}(\xi), \underline{r}(\eta)]\rangle$$

where $\underline{r} : \mathfrak{g}^* \to \mathfrak{g}$ is defined by $\langle \zeta, \underline{r}(\xi)\rangle = \underline{r}(\xi)(\zeta) = r(\zeta,\eta)$. This expression is equivalent to (3.22) found on page 35, see [27] for details. If we write our elements in terms of a chosen basis then

$$[|r,r|](\xi,\eta,\zeta) = -2r^{ij}r^{kl}c^m_{jl}(\circlearrowleft_{i,k,m} \xi_i\eta_k\zeta_m)$$

where c^k_{ij} are the structure constants of $\mathfrak{g}$.

We can view $[|r,r|]$ as a 0-cochain of $\mathfrak{g}$ with values in $\otimes^3\mathfrak{g}$ and so for $X \in \mathfrak{g}$ we have

$$\delta([|r,r|])(X) = ad^{(3)}_X[|r,r|]$$

where δ is the coboundary operator and $ad_X^{(3)} = (ad_X \otimes 1 \otimes 1 + 1 \otimes ad_X \otimes 1 + 1 \otimes 1 \otimes ad_X)$. It can be proven that the ad-invariance of $[|r, r|]$ (i.e. $\delta([|r, r|]) = 0$) is equivalent to $[\ ,\ ]^r$ satisfying the Jacobi identity. In terms of a chosen basis this ad-invariance can be expressed as

$$\circlearrowleft_{i,j,k} (r^{pm} r^{il} c^j_{ml} c^k_{np} + r^{pl} r^{km} c^j_{ml} c^i_{np} + r^{km} r^{il} c^p_{ml} c^j_{np}) = 0.$$

Thus the algebraic Schouten bracket gives us a condition at the level of r-matrices to ensure we have Poisson-Lie structure at the group level, just as the Schouten bracket of multivector fields gives us a condition to guarantee a Poisson structure on a manifold.

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