Asynchronous Joint Source-Channel Communication: An Information-Theoretic Perspective

by

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I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Due to the increasing growth and demand for wireless communication services, new techniques and paradigms are required for the development of next generation systems and networks. As a first step to better differentiate between various options to develop future systems, one should consider fundamental theoretical problems and limitations in present systems and networks. Hence, some common ground between network information theory and mobile/wireless medium techniques should be explicitly addressed to better understand future generation trends.

Among practical limitations, a major challenge, which is inherent and due to the physics of many mobile/wireless setups, is the problem of asynchronism between different nodes and/or clients in a wireless network. Although analytically convenient, the assumption of full synchronization between the end terminals in a network is usually difficult to justify. Thus, finding fundamental limits for communication systems under different types of asynchronism is essential to tackle real world problems.

In this thesis, we study information theoretic limits that various multiuser wireless communication systems encounter under time or phase asynchronism between different nodes. In particular, we divide our research into two categories: phase asynchronous and time asynchronous systems.

In the first part of this thesis, we consider several multiuser networks with phase fading communication links, i.e., all of the channels introduce phase shifts to the transmitted signals. We assume that the phase shifts are unknown to the transmitters as a practical assumption which results in a phase asynchronism between transmitter sides and receiver sides. We refer to these communication systems as phase incoherent (PI) communication systems and study the problem of communicating arbitrarily correlated sources over them.

Specifically, we are interested in solving the general problem of joint source-channel coding over PI networks. To this end, we first present a lemma which is very useful in deriving necessary conditions for reliable communication of the sources over PI channels. Then, for each channel and under specific gain conditions, we derive sufficient conditions based on separate source and channel coding and show that the necessary and sufficient conditions match. Therefore, we are able to present and prove several separation theorems for channels under study under specific gain conditions.

In the second part of this thesis, we consider time asynchronism in networks. In particular, we consider a multiple access channel with a relay as a general setup to model many wireless networks in which the transmitters are time asynchronous in the sense that they cannot operate at the same exact time. Based on the realistic assumption of a time offset between the transmitters, we again consider the problem of communicating arbitrarily correlated sources over such a time-asynchronous multiple access relay channel (TA-MARC). We first derive a general necessary condition for reliable communication. Then, by the use of separate source and channel coding and under specific gain conditions, we show that the derived sufficient conditions match with the general necessary condition for reliable communications. Consequently, we present a separation theorem for this class of networks under specific gain conditions. We then specialize our results to a two-user interference channel with time asynchronism between the encoders.

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To

my parents,

Ensieh Jelodarian, and Hossein Ebrahimzadeh Saffar,

my wife,

Elham Zakeriashtiani,

my son,

Yousef Ebrahimzadeh Saffar,

and my brother

Alireza Ebrahimzadeh Saffar.

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Chapter 1

Introduction

Studying the fundamental limits of communications in wireless environments is essential to understand and design highly demanding future data communication systems. Although development of basic mathematical tools to meet this objective dates back to Shannon's pioneering paper [1], followed by a vast literature connecting wireless communications and information theory, there are still classes of problems which have emerged as subjects of further research. In particular, for multi-user systems, such as multiple-access, broadcast, interference and relaying systems, which form the main body of emerging applications, one faces more challenging problems due to the many components in the system.

Asynchronism or non-coherence between different nodes of a communication network is an inherent challenge to modern communication systems, usually due to propagation delays and/or other physical limitations. In particular, there are major factors in wireless systems, such as feedback delay, the bursty nature of some applications, and reaction delay which cause time or phase asynchronism between different nodes of a network [2]. However, in digital communication systems, synchronization based on time delay estimation and/or phase estimation of the transmitted signals at different nodes is usually necessary to adjust receiver sampling times and properly decode the messages [3]. Thus, synchronization circuits are essential elements of both single-user and multi-user communications system structures.

Synchronization between nodes of a communication network is a common assumption made to analyze and design such networks. In point-to-point wireless systems, using training sequences and/or feedback, it is possible in principle to achieve synchronization between the transmitter and the receiver. However, although analytically convenient, full synchronization is rarely a practical, or easily justified assumption. Furthermore, in multi-user systems, besides synchronization between transmitters and receivers, the large number of nodes and interference from other sources make synchronization much more difficult and in some cases theoretically infeasible [4]. As an example, in systems with different transmitters, the different transmitters must use their own locally generated clock. However, the initialization might be different for each clock and the frequencies at the local signal generators may not be perfectly matched [5]. Indeed, achieving time, phase or frequency synchronization in practical communication systems has been a major engineering issue and still remains an active area of research (see e.g., [2]). Thus, fundamental limits of communication in the presence of time and other types of asynchronism should be explicitly considered as a tool to better understand and tackle real-world challenges in the context of multiuser information theory.

The first studies of time asynchronism in point-to-point communications goes back to the 60's, [6], [7], where the receiver is not accurately aware of the time that the encoded symbols are transmitted. The recent work of [2], on the other hand, assumes a stronger form of time asynchronism, that is, the receiver knows neither the time at which transmission starts, nor the timing of the last information symbol. They propose a combined communication and synchronization scheme and discuss information-theoretical limits of the new method. Also, in multiuser communication settings, the problem of time asynchronism is addressed for example in [8], [9] for the particular case of the multiple access channels.

The problem of finding the capacity region of multiuser channels with no time synchronization between the encoders is considered in [8], [5], [10], and [11] from a channel coding perspective for the specific case of multiple access channels (MAC). In [12], a frame asynchronous MAC with memory is considered and it is shown that the capacity region can be drastically reduced in the presence of frame asynchronism. In [9], an asynchronous MAC is also considered, but with symbol asynchronism. All of these works constrain themselves to the study of channel coding only and disregard the source-channel communication of correlated sources over asynchronous channels. In this thesis, however, we are interested in both source coding and channel coding aspects for asynchronous communication networks.

Besides time asynchronism [2], which is present in most channels, other forms of asynchronism such as phase uncertainty are important in wireless systems. In fading channels, the channel state information (CSI) models amplitude attenuation and phase shifts (phase fading) introduced by the channels between the nodes. In many systems, it is difficult to know phase shifts at the transmitter side due to the delay and resource limits in feedback transmission. In particular, in highly mobile environments, fading in conjunction with feedback delay may result in out of date phase knowledge by the time it reaches the transmitters (see, e.g., [13]).

Although the issue of asynchronism has its own specific features, it can be analytically seen in the larger framework of *channel uncertainty*, that is, the communicating parties have to work under situations where the full knowledge of the law governing the channel (or channels in a multi-user setting) is not known to some or all of them [14]. In order to study this general problem from an information-theoretic point of view, the mathematical model of a *compound channel* (or state-dependent channel) has been introduced by different authors [15], [16], [17]. A compound channel is generally represented by a family of transition probabilities $p^{\boldsymbol{\theta}}_{Y|X}$, where the index $\boldsymbol{\theta} \in \Theta$ is the state of the channel and Θ represents the uncertainty of different parties about the exact channel's transition probability. A compound Gaussian multiple access channel (MAC), for example, is considered in [18] based on the lack of knowledge of the set of active users and their respective channels. A Gaussian MAC with unknown phase shifts is also a compound channel. Here, the lack of knowledge of the phase shifts at transmitters is known not to change the capacity region [19].

The problem of joint source-channel coding (JSCC) for multiuser networks is open in general. However, numerous results have been published on different aspects of the problem for specific channels and under specific assumptions such as phase or time asynchronism between the nodes. In [19], a sufficient condition for lossless communication of correlated sources over a discrete memoryless MAC is given. Although not always optimal, as shown in [20], the achievable scheme of [19] outperforms separate source-channel coding. In [21], however, the authors show that under phase fading, separation is optimal for the important case of a Gaussian MAC. Also, [22], [23] show the optimality of separate source-channel coding for several Gaussian networks with phase uncertainty among the nodes. Other authors have derived JSCC coding results for the broadcast channels [24], [25], interference relay channels [26], and other multiuser channels [27]. Furthermore, for lossy source-channel coding, a separation approach is shown in [28] to be optimal or approximately optimal for certain classes of sources and networks.

In this thesis, we consider the problem of JSCC for a range of Gaussian multiuser channels under phase or time uncertainty (asynchronism). In particular, in the first part of the research contributions, presented in Chapter 3, we address the problem of sending a pair of correlated sources over several phase asynchronous Gaussian multiuser channels, where by phase asynchronous we mean the channel-introduced phase shifts are not known to the transmitters. We consider lossless communication for both cognitive and non-cognitive phase asynchronous channels. For the case of an interference relay channel, however, we also study the problem for a lossy communication scheme.

In the second part of the research contributions, presented in Chapter 4, we consider the problem of sending K correlated sources over a multiple access relay channel with time asynchronism between the transmitters. In both parts of the research, we first derive general necessary conditions on reliable communications. Then, using separate source-channel coding and under specific channel gain conditions we show the same conditions to be sufficient for reliable communications. Therefore, we are able to prove several separate source-channel coding theorems for channels under study.

The rest of this thesis is organized as follows. In Chapter 2 we provide basic background on reliable communication systems and briefly review the literature on compound channels and channel uncertainty for both single and multiple user settings. Additionally, we review some existing mathematical models for abstracting asynchronism and channel uncertainty. In Chapter 3, we state and prove results for various classes of phase asynchronous multiuser channels, referred to as phase incoherent channels. Specifically, these are multiuser channels with channel phase shifts unknown to the transmitters. We show that if the phase shifts are unknown to the transmitters, they can perform no better than the scenario in which the information sources are independent, i.e., correlation between sources is not helpful. In Chapter 4, we introduce a general Gaussian multiple access relay network (MARC) where there is time asynchronism between the transmitters in the sense that they might have delays in time with respect to each other. We refer to this channel as a Gaussian time asynchronous MARC (TA-MARC). We derive a general necessary condition for reliable communication of K sources over the Gaussian TA-MARC. Then, under specific gain conditions, using separate source-channel coding, we derive the same conditions as sufficient conditions for reliable communications. Finally, the conclusion and future potential works are presented in Chapter 5.

Chapter 2

Literature Review

Communication over single-user channels as well as multiple-user channels has been traditionally divided into two separate parts: *source coding* and *channel coding*. Besides convenience of the two-stage method for both single-user and multi-user case, this separation is mainly justified by the fact that for the point-to-point single user systems, it is shown by Shannon [1] that the two-part method is as good as any other method of transmitting information over a noisy channel when the block length tends to infinity [29, Theorem 7.13.1]. A typical point-to-point communication system's schematic with separate source and channel coding is depicted in Figure 2.1. While the source coding block, consisting of source encoder and decoder, is responsible to compress the information coming out of the source and remove its redundancy, the channel coding part, consisting of channel encoder and decoder, adds extra redundancy to the data such that it can be robust against noise and interference introduced by the channel and other parties. Thus, to completely design a quality and/or cost efficient communication system, one needs to design both blocks properly. A brief description of these two fundamental blocks as well as their joint schemes follows in the sequel.

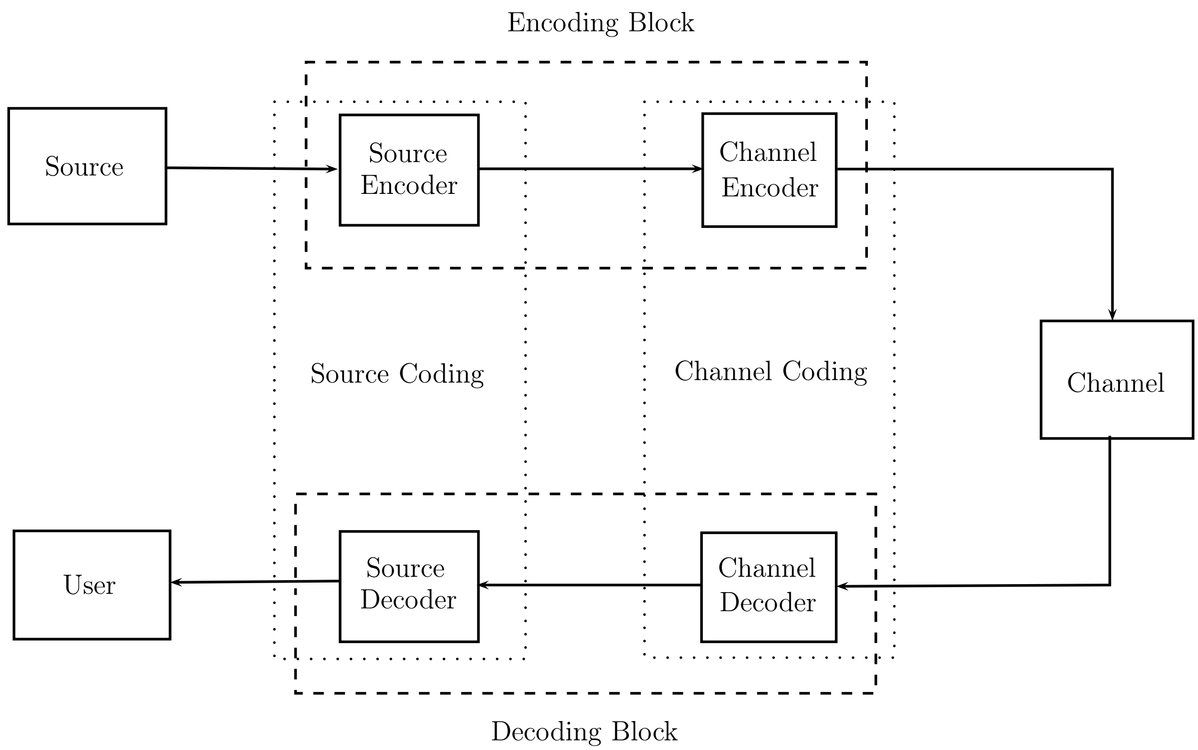

Figure 2.1: Point-to-point communication system model.

2.1 Source Coding

For a discrete memoryless source defined by independent and identically distributed (i.i.d) random variables $\{U_i\}_{i=1}^{\infty}$ with discrete alphabet $\mathcal{U}$ and probability distribution $\sim p_U(u)$, a source code of rate R is a source encoding mapping $f : \mathcal{U}^k \rightarrow \{1, 2, \cdots, 2^{nR}\}$ and a source decoding mapping $g : \{1, 2, \cdots, 2^{nR}\} \to \mathcal{U}^k$. The probability of error

$$P_e^n = P\left(g(f(\mathbf{U}^k)) \neq \mathbf{U}^k\right), \tag{2.1}$$

represents the probability that the source output $\mathbf{U}^k$ and the reconstructed signal $\hat{\mathbf{U}}^k$ are not the same, where the boldface letter $\mathbf{U}^k$ denotes a k-length vector. In his pioneered work [1], Shannon showed a source coding theorem, stating that the source code can represent the source with arbitrarily small probability of error as the block size $k \to \infty$ (losslessly), if the code rate satisfies

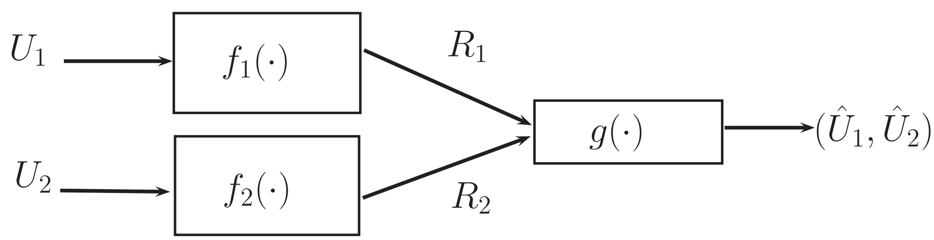

Figure 2.2: Slepian-Wolf (distributed source) coding

$$R > H(U), \tag{2.2}$$

where

$$H(U) = -\sum_{u \in \mathcal{U}} p_U(u) \log p_U(u), \tag{2.3}$$

is the entropy of the source. Thus, in order to reliably code a source U, a rate $R > H(U)$ is sufficient. Moreover, $R \geq H(U)$ is a necessary condition for reliable source coding.

Beyond the single-user case, constructing good source codes for multi-user systems, where several information sources are to be jointly compressed, is an important problem in source coding. An outstanding work in this regard is [30], where Slepian and Wolf formulated and proved a source coding theorem for two sources, referred to as *distributed* source coding. Namely, as shown in Figure 2.2, for two correlated sources $(U_1, U_2) \sim p_{U_1,U_2}(u_1, u_2)$ and two encoders f_1, f_2, with rates R_1, R_2 respectively, who wish to describe U_1, U_2 for a single decoder g, they proved the following theorem [29]:

Theorem 1. Slepian and Wolf*: A sequence of source codes of rates R_1, R_2 and block-length n can losslessly (i.e., with asymptotically vanishing probability of error) represent the distributed source (U_1, U_2) drawn i.i.d. $\sim p_{U_1,U_2}(u_1, u_2)$, if*

$$R_1 > H(U_1|U_2) \tag{2.4}$$

$$R_2 > H(U_2|U_1) \tag{2.5}$$

$$R_1 + R_2 > H(U_1, U_2). \tag{2.6}$$

Conversely, (2.4)-(2.6), with $>$ replaced by $\geq$, also describe a necessary condition for lossless source coding. □

The Slepian-Wolf theorem can also be generalized to the case of K correlated sources $U_1, \cdots, U_K$ [31]:

Theorem 2. *A sufficient condition for distributed lossless source coding of a set of K correlated sources $(U_1, \cdots, U_K)$ is given by the set of rate tuples $(R_1, \cdots, R_K)$ such that*

$$\sum_{j \in \mathcal{S}} R_j > H(U_{\mathcal{S}}|U_{\mathcal{S}^c}), \quad \forall \mathcal{S} \subseteq \{1, \cdots, K\}, \tag{2.7}$$

where $U_{\mathcal{S}} \triangleq \{U_i : i \in \mathcal{S}\}$. Moreover, (2.7) also represents necessary conditions for lossless source coding of $(U_1, \cdots, U_K)$, with $>$ replaced by $\geq$. □

2.2 Channel Coding

In his celebrated work [1], besides source coding, Shannon considered the problem of sending information over a probabilistically modeled communication channel. Specifically, for a memoryless channel with discrete input alphabet $\mathcal{X}$, discrete output alphabet $\mathcal{Y}$ and transition probability $P_{Y|X}(y|x)$, he showed that the maximum rate at which information can be reliably sent over the channel is

$$C = \max_{p_X} I(X;Y) = \max_{p_X} I(p_X; P_{Y|X}), \tag{2.8}$$

where

$$\begin{aligned} I(X;Y) &= I(p_X; P_{Y|X}) && (2.9) \\ &= \mathbb{E}_{X,Y}\left[\log \frac{P_{Y|X}}{p_Y}\right] && (2.10) \\ &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_X(x) P_{Y|X}(y|x) \log \frac{P_{Y|X}(y|x)}{p_Y(y)}, && (2.11) \end{aligned}$$

is referred to as the mutual information between random variables X and Y, p_X denotes the input distribution, and

$$p_Y(y) = \sum_{x'\in\mathcal{X}} P_{Y|X}(y|x') p_X(x'), \tag{2.12}$$

is the output distribution induced on y when the input distribution is $p_X(\cdot)$. The distribution p_X^* that maximizes the mutual information in (2.8) is called the *capacity achieving* or optimal input distribution.

Remark 1. *It can be shown [29] that $I(p_X; P_{Y|X})$ is a convex function of the conditional distribution $P_{Y|X}$ for a fixed input distribution p_X, while it is a concave function of p_X for a fixed $P_{Y|X}$.* □

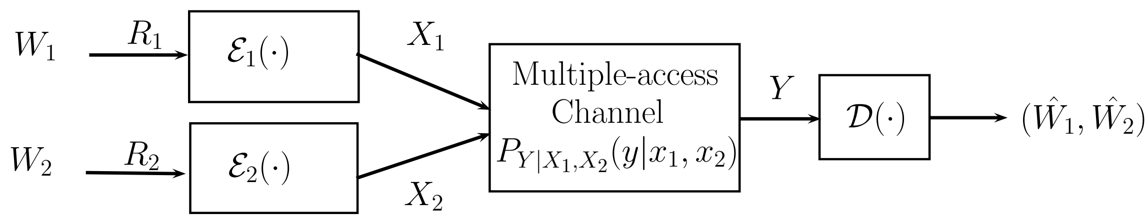

Figure 2.3: Two users, the discrete memoryless multiple-access channel and receiver

Channel coding for *multiuser* channels has also been widely studied [29] from an information-theoretic point of view. Among the important multi-user channels are the multiple-access channel (MAC), interference channel (IC), and broadcast channel (BC), where all of these can also employ relays to form more complicated channels. Herein, we briefly discuss a few of the important discrete memoryless multiuser channels as the main multiuser communication models addressed in this thesis.

A MAC is the channel coding counterpart of the Slepian-Wolf problem for a pair of correlated sources. A discrete memoryless MAC $(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{Y}, p_{Y|X_1,X_2}(y|x_1,x_2))$ consists of input alphabets $\mathcal{X}_1, \mathcal{X}_2$, output alphabet $\mathcal{Y}$, and transition probability law $p_{Y|X_1,X_2}(y|x_1,x_2)$, as depicted in Figure 2.3.

Definition 1. *A* $(2^{nR_1}, 2^{nR_2}, n)$ *code for the discrete memoryless MAC is a pair of encoders* $\mathcal{E}_1 : \{1,2,\cdots,2^{nR_1}\} \to \mathcal{X}_1^n$ *and* $\mathcal{E}_2 : \{1,2,\cdots,2^{nR_2}\} \to \mathcal{X}_2^n$ *and a decoder* $\mathcal{D} : \mathcal{Y}^n \to \{1,2,\cdots,2^{nR_2}\} \times \{1,2,\cdots,2^{nR_2}\}$. $\square$

Using a codebook, the users wish to send independent messages W_1, W_2 to a common receiver. The average probability of error for such a code under the assumption that message indices $W_1 \in \{1,2,\cdots,2^{nR_1}\}, W_2 \in \{1,2,\cdots,2^{nR_2}\}$ are drawn independently and according to a uniform distribution is given by

$$P_e^n = P\left(\mathcal{D}(\mathbf{Y}^n) \neq (W_1, W_2)\right) \tag{2.13}$$

$$= \frac{1}{2^{n(R_1+R_2)}} \sum_{w_1,w_2} P\left[\mathcal{D}(\mathbf{Y}^n) \neq (w_1, w_2) | (w_1, w_2) \text{ is sent}\right]. \tag{2.14}$$

Definition 2. *A pair (R_1, R_2) is said to be achievable if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes for which $P_n^e \to 0$, as $n \to 0$. Furthermore, the closure of set of achievable rate pairs (R_1, R_2) is called the capacity region of the MAC.* □

The capacity region of a 2-user MAC has been fully determined [32, 33] and stated as the following theorem:

Theorem 3. *The capacity region of a discrete memoryless multiple-access channel is given by the pairs (R_1, R_2) that satisfy*

$$R_1 \leq I(X_1; Y | X_2, Q), \tag{2.15}$$

$$R_2 \leq I(X_2; Y | X_1, Q), \tag{2.16}$$

$$R_1 + R_2 \leq I(X_1, X_2; Y | Q), \tag{2.17}$$

for some choice of the joint distribution $p(q)p(x_1|q)p(x_2|q)p(y|x_1, x_2)$, where the time-sharing random variable Q is chosen from a set $\mathcal{Q}$ with cardinality $|\mathcal{Q}| \leq 4$. □

The results for the 2-user MAC can also readily be extended to a K-user MAC. Here, we intend to send independent indices $W_1, \cdots, W_K$ over the channel to a common destination. Definitions of codes, achievability and capacity region are exactly the same as the 2-user case.

Theorem 4. *[31] The capacity region of the K-user MAC is the set of rate tuples $(R_1, \cdots, R_K)$ such that*

$$R_{\mathcal{S}} \leq I(X_{\mathcal{S}}; Y | X_{\mathcal{S}^c}, Q), \quad \forall \mathcal{S} \subseteq \{1, \cdots, K\}, \tag{2.18}$$

for some product distribution $p(q)\prod_{j=1}^{K}p_j(x_j|q)$ *with* $|\mathcal{Q}|\leq K$*, where* $R_{\mathcal{S}}=\sum_{j\in\mathcal{S}}R_j$*.* $\square$

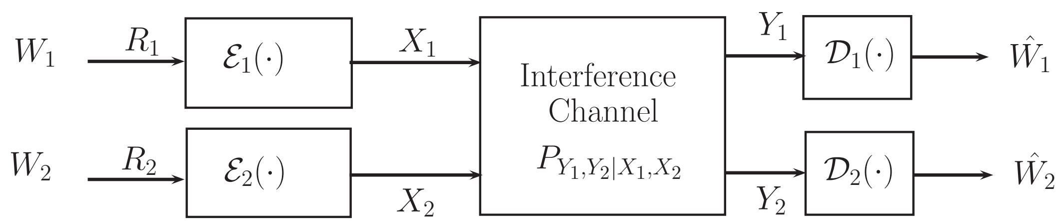

Figure 2.4: Two users, the discrete memoryless interference channel and receivers

In [34], Kramer and Wijngaarden introduced the multiple-access relay channel (MARC), where multiple sources wish to send their information to a common receiver with the help of a relay. Therein, the authors considered white Gaussian channels and by extending the coding technique of [35] which is based on block Markov encoding and successive decoding, derived an inner bound on the capacity region. Additionally they computed an outer bound for the MARC based on the cutset outer bound [29]. Later, inner and outer capacity bounds were derived for MARC and other relay networks in subsequent papers [36] and [37]. The bounds were especially computed for wireless channels with phase fading and it was shown that for some specific geometries the inner and outer bounds meet.

Another type of multiuser channel is the interference channel, where two users want to send their messages to two separate destinations respectively over a shared channel. The capacity of the interference channel is not in general known even in the 2-user case. Figure 2.4 depicts a 2-user discrete memoryless interference channel (DM-IC).

An important case in the analysis of the interference channel is when we are in the *strong interference* regime. A 2-user DM-IC is in the strong interference regime if

$$I(X_1;Y_1|X_2)\leq I(X_1;Y_2|X_2) \tag{2.19}$$

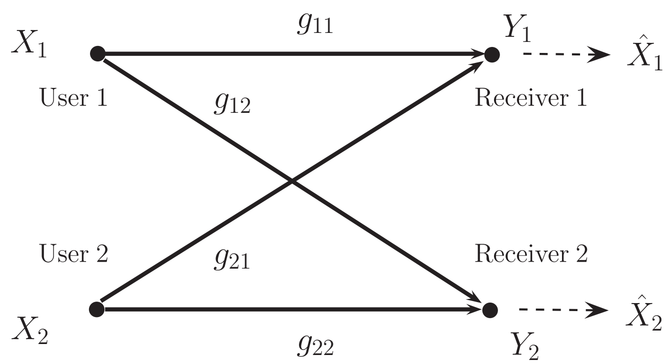

Figure 2.5: A 2-user Gaussian interference channel

$$I(X_2; Y_2|X_1) \leq I(X_2; Y_1|X_1), \tag{2.20}$$

for all $p(x_1)p(x_2)$.

In the wireless medium, the interference channel is modeled by a Gaussian interference channel depicted in Figure 2.5. The outputs of the Gaussian IC corresponding to the inputs X_1 and X_2 are

$$Y_1 = g_{11}X_1 + g_{21}X_2 + Z_1, \tag{2.21}$$

$$Y_2 = g_{12}X_1 + g_{22}X_2 + Z_2, \tag{2.22}$$

where g_{ij}, $i, j = 1, 2$ is the complex channel gain from the transmitter i to the receiver j, and Z_1, Z_2 are the noise signals.

The strong interference conditions (2.19)-(2.20) reduce to

$$|g_{12}| \geq |g_{11}|, \tag{2.23}$$

$$|g_{21}| \geq |g_{22}|. \tag{2.24}$$

Adding a relay to the interference channel results in an interference relay channel (IRC). For the IRC, an achievable region for the channel coding only problem is found in [38], while capacity results for ergodic phase fading and Rayleigh fading cases under strong and very strong interference conditions are reported in [39], [40].

The problem of finding the channel coding capacity of multiuser channels with no time synchronization between the encoders is considered in [8], [5], [10], and [11] from a channel coding perspective for multiple access channels (MAC). In [12], a frame asynchronous MAC with memory is considered and it is shown that the capacity region can be drastically reduced in the presence of frame asynchronism. In [9], an asynchronous MAC is also considered, but with symbol asynchronism. All of these works restrict themselves to the study of channel coding only and disregard the source-channel communication of correlated sources over an asynchronous MAC. In this thesis, we are interested in joint source-channel coding (JSCC) of a set of correlated sources over phase-asynchronous and time-asynchronous multiuser channels which can include relaying as well.

2.3 Joint Source-Channel Coding

Shannon's separation theorem establishes that there is no loss in terms of communication reliability by performing independent source and channel coding for a single point-to-point system. However, this result is correct under the implicit assumption of asymptotically large codewords lengths resulting in large system delays. Moreover for multi-user systems as well as some wireless applications in which the source and/or channel are non-stationary, the separation theorem might not hold [41]. Based on these observations, it is sometimes convenient to consider the design of source and channel codes *jointly*. Such schemes are generally called joint source channel coding [42]. An interesting work which relates the multi-user Slepian-Wolf and MAC schemes in a joint source channel coding framework is [19]. There, the authors show that for correlated sources, it is better to design channel codes based on the source outputs directly and thus, as opposed to classical MAC encoding, make the input distributions of the MAC correlated. Using this encoding scheme, they show that the achievable region can be enlarged, compared to that of the separate Slepian-Wolf and MAC coding. Later, by giving a counterexample, Dueck showed in [20] that the strategy of [19] is not, however, optimal, and can be improved upon.

The problem of JSCC for a network is open in general. Several works, however, have been published on this issue, e.g., for MAC [19], broadcast channels [24], [25], and other multiuser channels [26], [27]. In particular, [24] considers broadcasting a set of correlated sources by the means of some independent encoders to multiple receivers and shows that while joint source-channel coding at the encoding side is unnecessary, not using joint source-channel decoding at the decoding side is suboptimal. Also, [25] provides a complete solution for the JSCC problem of sending a pair of correlated Gaussian sources (also known as a bivariate Gaussian source) over a Gaussian broadcast channel, where each receiver is only interested in one component of the source. It is further shown in [25] that for the considered settings, the Gaussian scenario is the worst scenario among the sources and channel noises with the same covariances, in the sense that any distortion pair that is achievable in the Gaussian settings is also achievable for other sources and channel noises.

For lossy source-channel coding, a separation approach is shown in [28] to be optimal or approximately optimal for certain classes of sources and networks. In [19], on the other hand, a sufficient condition to losslessly send correlated sources over a MAC is given, along with a multi-letter expression for the outer bound. Although not always optimal, as shown in [20], the JSCC scheme of [19] outperforms separate source-channel coding, and thus separation is *not* optimal for *correlated* sources. In [21], [43], however, the authors show that performing *separate* source and channel coding for the important case of a Gaussian MAC with unknown phase shifts at transmitters (also referred to as phase incoherence), is optimal. Namely, in [21] and [43], F. Abi Abdallah et. al. showed the following separation theorem for multiple access channels with both non-ergodic and ergodic i.i.d. phase fading:

Theorem 5. *A necessary condition for reliable communication of the source pair* $(U_1^n, U_2^n) \sim \prod_i p(u_{1i}, u_{2i})$ *over a phase-faded MAC, with power constraints* P_1, P_2 *on the transmitters, and fading amplitudes* $g_1, g_2 \geq 0$*, is given by*

$$H(U_i|U_j) \leq \log(1 + g_i^2 P_i/N), \quad (i,j) \in S, \tag{2.25}$$

$$H(U_1, U_2) \leq \log(1 + (g_1^2 P_1 + g_2^2 P_2)/N), \tag{2.26}$$

where $S \triangleq \{(1,2),(2,1)\}$*, and* N *is the noise power.* (2.25)-(2.26) *also give a sufficient condition, with* $\leq$ *replaced by* $<$*.* □

Also, the recent work [44] addresses the same problem for an ergodic phase fading Gaussian multiple access relay channel (MARC) and proves a separation theorem under some channel coefficient conditions. For the achievability part, the authors use the results of [45], [37], and [36] based on a combination of *regular* Markov encoding at the transmitters and *backward* decoding at the receiver [46]. In particular, in order to derive the achievable region for discrete-memoryless MARC, the authors of [37] use codebooks of the same size which is referred to as regular Markov encoding. This is in contrast with block Markov encoding which was introduced by Cover and El-Gamal in [35] for the relay channel. There, the encoding is done using codebooks of different sizes and is referred to as *irregular* block Markov encoding.

2.4 Compound Channels

Shannon's work does not address the more realistic situations in which the transition probability (or channel law) $P_{Y|X}$ may depend on external parameters, such as channel gains, phase, etc. The probability distribution in these cases is usually denoted by $P^{\theta}_{Y|X}$, where θ is a parameter. Hence, the rule governing transmission in such situations is not unique or even fixed in time, but it is chosen from a class of distributions. The oldest work addressing coding for a class of channels is [15]. Later, Wolfowitz [17] and Csiszar [16] formulated the same problem and named such channels as *compound* channels. A formal definition of the compound channel is as follows.

Definition 3. *As shown in Figure 2.6, a class or family of discrete alphabet channels* $(\mathcal{X}, \mathcal{Y}, P^{\theta}_{Y|X})$ *is defined by discrete input and output sets* $\mathcal{X}, \mathcal{Y}$*, and set of probability distributions* $P^{\theta}_{Y|X}$*, indexed by* $\theta \in \Theta$*, where* Θ *represents the index set.* θ *is also known as channel* state *or* parameter*. In each time period, the communication is performed over a specific channel from the class.* $\square$

Intuitively, the capacity of such a channel can not be greater than the infimum of the mutual informations of the channels belonging to the class. Since the capacity achieving input distribution is not the same for each channel, one would intuitively think that the input distribution should be chosen such that the infimum of all capacities is maximized resulting in

$$C = \max_{p_X} \inf_{\theta} I(p_X; P^{\theta}_{Y|X}), \tag{2.27}$$

where $I(p_X; P^{\theta}_{Y|X})$ denotes actual mutual information of a particular channel in the set. Indeed, this is correct as will be elaborated in the following discussions.

As opposed to time-varying channels, if the channel over which the communication is performed is fixed for a block-length n, then the class can be denoted by block representation $(\boldsymbol{x} \in \mathcal{X}^n, \boldsymbol{y} \in \mathcal{Y}^n, P^{\theta}_{\mathbf{Y}|\mathbf{X}}(\boldsymbol{y}|\boldsymbol{x}))$. For a discrete memoryless class of channels, for example, we have

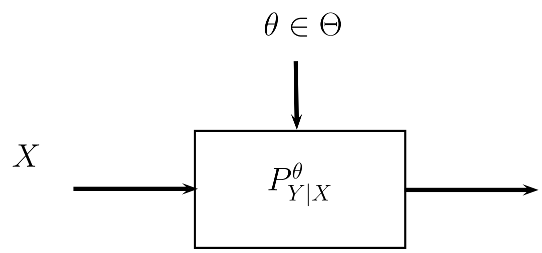

Figure 2.6: Compound channel

$$P^{\theta}_{Y|X}(\boldsymbol{y}|\boldsymbol{x}) = \prod_{i=1}^{n} P^{\theta}_{Y|X}(y_i|x_i). \tag{2.28}$$

For convenience, we shall simply denote such a class of channels by $P^{\theta}_{Y|X}$.

A block code $(\mathcal{E}, \mathcal{D})$ of length n for a class $P^{\theta}_{Y|X}$ of channels, consists of

1. an encoding function

$$\mathcal{E} : \{1, 2, \cdots, 2^k\} \to \mathcal{X}^n, \tag{2.29}$$

and

2. a decoding function

$$\mathcal{D} : \mathcal{Y}^n \to \{1, 2, \cdots, 2^k\}, \tag{2.30}$$

that divides $\mathcal{Y}^n$ into 2^k disjoint decision subsets $\mathcal{B}_i, i = 1, 2, \cdots, k$.

The *rate* of such a code is

$$R = \frac{k}{n}. \tag{2.31}$$

The probability of error for a specific message $i \in \{1, 2, \cdots, k\}$ when a code $(\mathcal{E}, \mathcal{D})$ and a specific channel $\theta \in \Theta$ are used can be written as

$$P_e^\theta(i) = P_{Y|X}^\theta(\mathcal{B}_i^c |\ i \text{ sent}) \tag{2.32}$$

where a set A^c denotes the complement of A. Therefore the average probability of error over the message set is given by

$$\overline{P}_e^\theta = \frac{1}{2^k} \sum_{i=1}^{2^k} P_e^\theta(i), \tag{2.33}$$

and the maximum probability of error is

$$P_{e,max}^\theta = \max_{i \in \{1,2,\cdots,k\}} P_e^\theta(i). \tag{2.34}$$

The definitions of achievable rates and capacity for a class of channels $P_{Y|X}^\theta$ which are more involved than that of Shannon' channel model are as follows:

Definition 4. *[14] A rate R is said to be ϵ-achievable, $0 < \epsilon < 1$, on the compound channel $P_{Y|X}^\theta$, for maximum (resp. average) probability of error if: for every $\delta > 0$, one can find an integer N such that for all $n > N$, there exists a block code $(\mathcal{E}, \mathcal{D})$ of length n with rate*

$$R - \delta < \frac{k}{n} \tag{2.35}$$

and maximum (resp. average) probability of error satisfying

$$\sup_{\theta \in \Theta} P_{e,max}^\theta \leq \epsilon, \tag{2.36}$$

$$(\text{resp.} \quad \sup_{\theta \in \Theta} \overline{P}_e^\theta \leq \epsilon.) \tag{2.37}$$

□

Definition 5. *A rate R is achievable for maximum (resp. average) probability of error if it is ϵ-achievable for every $0 < \epsilon < 1$.* $\square$

Indeed, it can be seen that if a rate R is achievable for the class of channels, it will be achievable for all of the channels $\{P^{\theta}_{Y|X}\}_{\theta\in\Theta}$ in the class.

Definition 6. *The ϵ-capacity of a compound channel for maximum (resp. average) probability of error is the supremum of all the ϵ-achievable rates as determined by (2.35) and (2.36) (resp. (2.37)) and is denoted by C^m_ϵ (resp. C^a_ϵ)* $\square$

Definition 7. *The capacity of a class of channels for maximum (resp. average) probability of error is the supremum of all achievable rates for maximum (resp. average) probability of error and is denoted by C^m (resp. C^a).* $\square$

Clearly, in general $C^m_\epsilon < C^a_\epsilon$ and $C^m < C^a$. If the capacities are the same for both maximum and average probability of error criteria, they are simply denoted by C [14]. Also note that the capacities C^m and C^a can be defined as the limits of the corresponding ϵ-capacities as $\epsilon \to 0$. Namely,

$$\lim_{\epsilon\to 0} C^m_\epsilon = C^m, \qquad \lim_{\epsilon\to 0} C^a_\epsilon = C^a \tag{2.38}$$

The capacity expression of the compound discrete memoryless channel (DMC) of (2.28) is determined in [15, Theorem 1] and subsequent works [47], [16], and [17]. It is assumed that both encoder and decoder are ignorant of the channel law ruling the communication, they are only aware of the class Θ to which the law belongs. Obviously, the compound capacity cannot be larger than any of the individual capacities of the channels belonging to the class. However, this bound is not tight as the input probability mass functions (pmf) that achieve capacity for any of the channels in the class may be different. Nevertheless, in [17], it is shown that the capacity of the compound channel is positive if and only if (iff) the infimum of the capacities in the class Θ is positive. We state the capacity of the compound channel as the following theorem:

Theorem 6. *The capacity of the compound DMC $P^{\theta}_{Y|X}$ of (2.28) for both maximum and average probability of error criteria is given by*

$$C = \sup_{p_X} \inf_{\theta} I_{\theta}(X;Y) = \sup_{p_X} \inf_{\theta} I(p_X; P^{\theta}_{Y|X}), \tag{2.39}$$

where the subscript θ in $I_{\theta}(X;Y)$ demonstrates explicitly that a specific parameter θ is used.

$\square$

Remark 2. *[17], [47] The knowledge of the parameter θ at the receiver side does not increase the capacity. However, if the channel state information is known to the encoder, the capacity is increased in general and is equal to the infimum of the capacities of the channels in class. This is true even though the decoder is unaware of the channel state information.* $\square$

2.4.1 Channels with Uncertainty

Compound channel modeling can be seen as a technique of formulating the more general problem of communication under *uncertainty*. Indeed, compound channels are subsets of channels with uncertainty. In real situations, it is very common that the codebook or decoder should be chosen without having full knowledge of the probabilistic law governing the channel. By defining a class of channels and confining the knowledge of different nodes to the class, as opposed to the actual channel used in the course of communication, the concept of uncertainty can be modeled. For instance, in [48], it is assumed that the encoder knows only the class of channels over which the communication is done. It is also assumed there is no feedback available to the transmitter and thus the codewords should be fixed before transmission begins. Therein, like Blackwell. et. al. [15], they assumed that the receiver does not know the channel either and consequently, the decoding rule should not depend on the actual channel used in communication.

For studying the broad issue of channel uncertainty, one can propose other models, as well as generalizing the notion of compound channels to more complicated settings, while still adhering to the concept of a *family* of channels [14]. A more severe case of uncertainty than the conventional compound channel defined in Definition 3, for instance, arises when the channel parameter is not fixed for a whole block, but changes arbitrarily from symbol to symbol. This channel, firstly introduced by [49] and sometimes referred to as an arbitrarily varying channel (AVC), can be modeled by defining the set $\Sigma = \Theta^{\infty}$, called state space, and for the discrete memoryless case, setting

$$P^{\boldsymbol{\theta}}_{Y|X}\left(\boldsymbol{y}|\boldsymbol{x}\right) = \prod_{i=1}^{n} P^{\theta_i}_{Y|X}\left(y_i|x_i\right), \tag{2.40}$$

where the transmission block length is n.

The following example of the discrete memoryless AVC is given in [49].

Example 7. *Let $\mathcal{X} = \Theta = \{0,1\}$ and $\mathcal{Y} = \{0,1,2\}$ and define*

$$P^{\theta}_{Y|X}(y|x) = \begin{cases} 1, & \text{if } y = x + \theta \\ 0, & \text{otherwise.} \end{cases} \tag{2.41}$$

Since all of the transition probabilities are either 0 or 1, such an AVC is referred to as a deterministic *AVC. Note also that it can be equivalently described by the equation*

$$y_i = x_i + \theta_i. \tag{2.42}$$

□

In the AVC problem, it is usually assumed that the state vector $\boldsymbol{\theta}$ is not known to the either the transmitter or receiver. There are various types of problems pertaining to determining the capacity of AVCs, depending on whether the maximum or average error probability criteria and deterministic or *randomized* coding [14, 49] strategies are chosen. The cases also differ based on the degree of the mutual knowledge that encoders and external sources that form state sequences have of each other.

A key communication technique used in different papers for the AVC is *randomized coding*. Randomized coding introduces a common source of randomness to both encoder and decoder. Thus, the input and output signals of the channel can be further dependent on the outcome of a random experiment. Therefore, the encoding and decoding strategies, and consequently sufficient tools to prove the corresponding coding theorems, are enriched by having a probabilistic approach to code design. Note that a randomized code constitutes a communication technique which should not be confused with random-coding argument as a proof technique. In particular, random coding is often used to establish the existence of a *deterministic* code which yields good performance on a known channel, without actually constructing the code. To this end, a probability mass function is introduced on an ensemble of codes and the average performance over such an ensemble is computed. Then, by arguing that the average performance is good, it is concluded that there must exist at least one code in the ensemble with good performance. Randomized coding, however, uses *stochastic* functions as encoders and decoders. The formal definition of randomized codes is now stated.

Definition 8. *A randomized code (e, d) is a random variable taking values in the set of all codes $(\mathcal{E}, \mathcal{D})$ of block length n defined by (2.29), (2.30), with the same message set $\mathcal{M} = \{1, 2, \cdots 2^k\}$. The pmf of the RV (e, d) may depend on the knowledge of the class Θ but should not depend on a specific state $\theta \in \Theta$ ruling a particular transmission or on the* *chosen message to be sent over the channel.* □

Different error probabilities for randomized codes are defined in a way similar to those defined for the deterministic codes in (2.32), (2.33), and (2.34). Analogous notions of ϵ-capacity and capacity are also defined for the randomized codes. The average and maximum probabilities of error, however, lead to the same results for the randomized codes.

Based on this, in the original work of Blackwell. et. al. [49], a coding theorem is proved for *correlated random* codes, that is, codes and decoders are chosen by a random experiment known to both encoder and decoder. Although a powerful tool, it is important however to note that within the randomized coding framework, there should be a further part in the system to inform the encoder and decoder of the random experiment's outcome. Thus, in [48], for example, deterministic codes with maximum error probability were applied to the AVC. In order to state the key result on the capacity of the AVC, we give the following definition [14].

Definition 9. *For the AVC of* (2.40)*, let ψ be a pmf on Θ and define by $W^{\psi}_{Y|X}$ the averaged channel transition probability with respect to ψ given by*

$$W^{\psi}_{Y|X}(y|x) = \sum_{\theta \in \Theta} P^{\theta}_{Y|X}(y|x)\psi(\theta). \tag{2.43}$$

The capacity of the AVC (2.40) for randomized codes is analogous to that of the compound DMC. In particular, one should find the input distribution that maximizes the minimum of the mutual information over averaged distributions $W^{\psi}_{Y|X}$. This result is stated as the following theorem [49], [50]

Theorem 8. *The randomized code capacity of the AVC is given by*

$$C = \max_{p_X} \min_{\psi} I(p_X; W^{\psi}_{Y|X}) = \min_{\psi} \max_{p_X} I(p_X; W^{\psi}_{Y|X}). \tag{2.44}$$

□

In some of the applications involving channel uncertainty, some of the (possibly unknown) channel parameters are fixed during the course of transmission while certain other parameters change arbitrarily from symbol to symbol. To model such a situation, one can introduce a *hybrid* of compound and AVC channels. Namely, let the state space $\Sigma = \Theta^\infty \times \Phi$, where $\{\theta_i\}_{i=1}^\infty \in \Theta^\infty$ represent the varying parameter while $\phi \in \Phi$ represent a fixed parameter. A hybrid discrete memoryless channel (DMC) can be described by

$$P_{Y|X}^{\boldsymbol{\theta},\phi}\left(\boldsymbol{y}|\boldsymbol{x}\right) = \prod_{i=1}^{n} P_{Y|X}^{\theta_i,\phi}\left(y_i|x_i\right). \tag{2.45}$$

Also, as another example, in [51], the class of channels involves linear dispersive channels with unknown dispersive filter, as opposed to the discrete memory channels (DMC) considered in [15].

In [52], [53], [54], and [55], variations of finite-state channels with memory are considered to model the effect of fading in mobile wireless communications. Also, in [56], a universal decoder is proposed which is based on family of channels under consideration, and not on the individual channel over which the communication takes place. Therein, the authors show that their proposed decoder can perform as well as a maximum-likelihood decoder chosen for the actual channel in use, in terms of error exponent.

In [57], communication over a channel $P_{Y|X,S}$ with *side information* S was introduced. They assumed the side information is non-causally known to the transmitter. In the more recent work of [58], Mitran et. al. generalized the notion of channels with side information to the compound channels. Namely, they considered the problem in which a sender wishes to communicate its message over a channel $P_{Y|X,S}^{\theta}$, where the non-causal side information S is only known to the encoder and the channel parameter θ is only known to the decoder. Furthermore, although the input and output alphabets in compound channels are usually assumed to be drawn from finite sets, they generalized the discrete-time compound channels

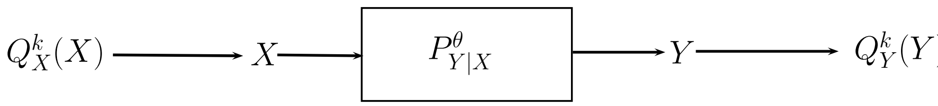

Figure 2.7: Input, output and their quantized versions of a compound channel

to the discrete-time continuous alphabet case, using a quantization/dithering argument. In particular, they first generate a quantized random variable $Q_X^k(X)$ and think of the original input of the channel X as $Q_X^k(X)$ plus a random *dither* to the quantization cells. The output Y of the channel is also quantized to form $Q_Y^k(Y)$ and thus an equivalent discrete alphabet channel from $Q_X^k(X)$ to $Q_Y^k(Y)$ is formed (see Figure 2.7). Using this model, they relate continuous and discrete alphabet channels and show that by letting the resolution of the quantizers go to infinity, the mutual information of the discrete alphabet channel converges to that of the continuous alphabet channel:

$$I(X;Y) = \lim_{k\to\infty} I(Q_X^k(X); Q_Y^k(Y)). \tag{2.46}$$

This result is shown to be very useful to derive certain results for the continuous alphabet channels, inspired by their discrete alphabet counterparts.

They also derived capacity upper and lower bounds for both discrete and continuous alphabet compound channels with knowledge of side information, such as interference, at the transmitter.

2.4.2 Multiuser Channels

Although *compoundness* in *multi-user* systems was marginally addressed in [17] and [15], an early work to formulate multi-user communication under uncertainty was [59]. There, an extension of the single-user AVC model for arbitrarily varying multiple-access channels is proposed, in which the channel state can change in each time slot. Thus, as opposed to (2.28), the arbitrarily varying multiple access channel (AVMAC) can be described by

$$P^{\boldsymbol{\theta}}_{Y|X}\left(\boldsymbol{y}|\boldsymbol{x}_1,\boldsymbol{x}_2\right)=\prod_{i=1}^{n}P^{\theta_i}_{Y|X}\left(y_i|x_{1i},x_{2i}\right), \tag{2.47}$$

where $\boldsymbol{\theta}$ is the sequence of states ruling the transmission.

Using a randomized code, based on designing a random experiment outcome which should be known to both encoder and decoder, Jahn [59] obtains a deterministic code with a vanishing average probability of error and thus determines the average error capacity region of an AVMAC. Moreover, in [59], the author formalizes the arbitrarily varying *broadcast* channel and derives an achievable region for that.

As far as multi-user settings are concerned, compound channels and communications under uncertainty have received much more interest recently.

For example, in [18], the authors consider Gaussian compound multiple-access channel with K transmitters and a common receiver. In each block, only k transmitters are active and accordingly they assume the following additive Gaussian noise channel

$$y=\sum_{i=1}^{k}g_ix_i+n, \tag{2.48}$$

where x_i's are the transmitters input signals and y is the received signal. The state of the channel is described by the vector $\boldsymbol{\theta}=(g_1,g_2,\cdots,g_K)\in\Theta$ where the state space of the MAC is a subset of $\Theta\subseteq(\mathbb{R}^+)^K$. Both the channel gains and number of active nodes are unknown to all parties. Without giving a rigorous proof, it is stated in [18] that the capacity region of such a compound multiple access channel is given by the intersection of the capacities of the family. Namely, they give the intuitive result that

$$C = \bigcap_{\boldsymbol{\theta} \in \Theta} C_{\boldsymbol{\theta}}, \tag{2.49}$$

based on the fact that since a Gaussian input distribution simultaneously achieves all capacities of the class channels (i.e., for each $\boldsymbol{\theta}$), the intersection of the class capacities is the ultimate capacity.

Furthermore, using polling, channel estimation and feedback techniques, they propose a protocol which achieves within $\log(K)$ bits/s/Hz of the capacity of a K-user Gaussian multiple access system with perfect feedback.

The recent work of [60], also addresses the compound multiple-access channel where the encoders can partially cooperate to send their own private messages and a common message to a single destination. The encoders are free to exchange information not only about their messages, but also with respect to the channel state information.

2.5 Asynchronism

In practice, the transmitter and receiver systems can never be perfectly synchronized because of propagation delay. However, using the knowledge of time or phase differences between nodes, synchronization subsystems are designed to provide the desired synchronism. If the knowledge of time or phase is not available at designated nodes, then the asynchronism can be analyzed as a communication under uncertainty problem.

The problem of *phase* asynchronism can be formulated as a special case of compound channels. Namely, the phase shift terms $e^{j\theta}$ will play the role of the channel's parameter, where θ denotes the phase introduced by the channel. In this thesis, we have modeled multiple access networks with phase uncertainty at the transmitters side, using the notion of compound channels (refer to Chapter 3 for details).

On the other hand, there have been several works to model and analyze the problem of *time* asynchronism in communication systems. One of the earlier works to address time asynchronism is [6]. Therein, block coding is used to encode the stream of messages and the blocks are sent successively through the channel. The receiver obtains the channel outputs only from some random specific time slot on and tries to find the boundary of the next codeword to decode the subsequent messages. Another well-known model for time asynchronism is introduced in the Dobrushin's paper [7] which introduces so called insertion, deletion and substitution (IDS) channel. Abstracting the timing error and irregularity in communication medium and transceiver hardware, in the IDS channel model, each symbol of the sent codeword is received in the form of different length sequences of symbols, probably manipulated by channel's insertion, deletion and substitution of symbols. The receiver, however, as opposed to [6], knows the time at which transmission begins.

Besides studying and designing the synchronization subsystem as a separate part from transceiver and coding subsystems, on which the vast majority of the literature focuses, joint design of synchronization and communication parts is addressed in the recent work of Tchamkerten et. al. [2]. Indeed, they proposes a universal framework to discuss asynchronism in point-to-point communication systems. In this work, it is assumed that the receiver knows neither the start time nor the duration of the transmission. In their system model, transmitter commences to send the message codeword at a randomly chosen point in time and within a prescribed window. The length of the window scales exponentially with the length of codeword and the scaling parameter is referred to as asynchronism exponent. The receiver is only aware of the transmission window and not the transmission time. The communication rate under discussion is defined as the ratio of the message size and the time between beginning of transmission and decoding decision. An interesting result addressed in this work is that it is possible to achieve reliable communication at all rates less than the capacity of the synchronized channel (i.e., classical capacity). This is established by the use of sequential decoding for a specific group of codes. Also, they have determined the largest asynchronism exponent under which one can reliably communicate over the channel, regardless of the rate.

For the multiple user channels, and in particular the important class of multiple access channels, [8] and [9] have shown interesting results. In [9], a continuous-time MAC is considered under symbol time asynchronism between transmitters. The issue of symbol synchronism arises in continuous-time channels where a codeword $(c_1, c_2, \cdots, c_n)$ modulates a finite-duration waveform which in general has the form

$$\sum_{i=1}^{n} s(t - iT; c_i), \tag{2.50}$$

where the waveforms $s(t; c_i)$ are the waveforms assigned to codeword symbols and vanish outside the interval $[0, T]$. Thus, to correctly decode the codewords, sampling synchronization at the decoder becomes essential. This is indeed the case in most of the standard digital communication systems. In [9], Verdú considers this issue for a Gaussian MAC where two users channels have time delays of $\tau_1, \tau_2 \in [0, T)$ respectively. Also, each user modulates linearly a fixed *signature* waveform $s_k(t)$, i.e., $s_k(t; c_i) = c_i s_k(t)$, where $k = 1, 2$ is the user index. If the users use the same signature waveforms and $\tau_1 = \tau_2$, it is easy to see that the channel is equivalent to the standard Gaussian MAC with known capacity region. Interestingly, it is shown in [9] that this result still holds if the users are not symbol synchronous, that is $\tau_1 \neq \tau_2$, as long as transmitters use the same signature waveforms. Namely, in this setting, if the transmitters are assigned the same signature waveforms, asynchronism will not degrade the capacity region of white Gaussian MAC. However, if the assigned waveforms are different, the capacity region is changed. In order to derive the results, an equivalent channel model is obtained with discrete-time outputs and the lack of symbol asynchronism has been modeled by finite channel memory. To this end, the results of [12] on the capacity of the discrete-multiple access channels with finite memory is exploited.

In [8], the authors consider a MAC with no common time base between encoders. There, the encoders transmit with an unknown offset with respect to each other, and the offset is bounded by a maximum value $\mathsf{d}_{\mathsf{max}}(n)$ that is a function of coding block length n. Using a time-sharing argument, it is shown that the capacity region is the same as the capacity of the ordinary MAC as long as $\mathsf{d}_{\mathsf{max}}(n)/n \to 0$. On the other hand, [5] considers a *totally asynchronous* MAC in which the coding blocks of different users can potentially have no overlap at all, and thus potentially have several block lengths of shifts between themselves (denoted by random variables Δ_i). Moreover, the encoders have different clocks that are referenced with respect to a standard clock, and the offsets between the start of code blocks for the standard clock and the clock at transmitter i are denoted by random variables D_i. For such a scenario, in [5], it is shown that the capacity region differs from that of the synchronous MAC only by the lack of the convex hull operation. In [61], Poltyrev also considers a model with arbitrary delays, known to the receiver (as opposed to [5]). Among other related works is the recent paper [10] that finds a single letter capacity region for the case of a 3 sender MAC, 2 of which are synchronized with each other and both asynchronous with respect to the third one.

Chapter 3

Phase Asynchronous Systems

3.1 Introduction

In this chapter[1], we consider the problem of joint source-channel coding (JSCC) for a range of compound Gaussian multiuser channels with phase uncertainty. We refer to such channels as phase incoherent (PI) channels. We assume that the phase shifts over channels under consideration are stationary non-ergodic phase fading processes which are chosen randomly and fixed over the block length. The phase information $\boldsymbol{\theta}$ (as the channel parameter) is assumed to be unknown to the transmitters and known at the receiver side(s) as a practical assumption. However, since it is straight forward for the receivers to estimate the channel side information, in this chapter, we assume that each receiver knows its own channels' phases.

In Section 3.2, we first present the preliminaries and definitions along with a lemma

[1]The results of this chapter (except for Sections 3.5 and 3.6) are published in the IEEE Transactions on Communications, vol. 62, no. 8, pp. 2996–3003, Aug. 2014. [22]

that is useful in the derivation of all the converse results in this chapter.

In Sections 3.3-3.4, we consider the problem of JSCC for a series of phase incoherent multiuser channels. We examine both cognitive and non-cognitive models. In particular, we consider phase incoherent multiple access relay channels (PI-MARC), in which one of the encoders is helped by the other one, either causally or non-causally. These can model, for example, a cognitive radio communication scenario in which the cognitive user is aware of the primary user's message. We refer to such networks as phase-incoherent causal/non-causal cognitive MARCs or PI-CC-MARC/PI-NC-MARCs for short. Furthermore, we also consider a phase-incoherent interference relay channel (PI-IRC) under asymmetric gain conditions. For the IRC, an achievable region for the channel coding only problem is found in [38], while capacity results for ergodic phase fading and Rayleigh fading cases under strong and very strong interference conditions are reported in [39], [40].

Furthermore, in this chapter, in Section 3.5, we consider two classes of phase incoherent cognitive Gaussian interference channels and study the *lossless* communication of primary and secondary sources. We assume the sources to be correlated as may be in practical situations where the primary and secondary transmit information acquired from the same environment. In particular, sufficient and necessary conditions for reliable lossless communication of two correlated sources over classes of phase asynchronous cognitive interference channels are derived. Namely, we consider interference channels in which one of the encoders, i.e., the secondary or cognitive user, is causally or non-causally aware of the other's message. We show that, for both classes of causal and noncausal cooperation, under strong interference conditions, separate source and channel coding is optimal for reliable communication of both users. Also, we derive necessary and sufficient conditions for reliable communication of the cognitive radio transmission while the primary is able to maintain the same information rate it could reliably send in the absence of the secondary.

To the best of our knowledge, this is the first work to find fundamental limits of lossless reliable communication for cognitive interference channels.

For all of the channels considered, we derive both necessary and sufficient conditions for reliable communication. Furthermore, we show that if the phase shifts are unknown to the transmitters, then the optimal performance of JSCC is no better than the scenario in which the information sources are first source coded and then channel coded separately. In particular, correlation between the sources does not change the necessary and sufficient conditions for reliable communication, as opposed to cases where the transmitters have knowledge of the phase shifts and could potentially use beamforming, for example, to joint source-channel code the data and achieve higher rates.

Finally, in Section 3.6, we study the lossy communication of a bivariate Gaussian source over a phase incoherent interference relay channel (PI-IRC). We derive inner and outer bounds for the achievable distortion region, where the inner bound is derived under specific strong interference conditions as well as strong gain conditions between transmitters and the relay. When the sources are correlated, we find an approximate achievable distortion region in the high SNR regime. In case of independent sources, the bounds are tight and by explicitly providing the achievable distortion region, we show that a separation theorem results for the PI-IRC under strong interference conditions. By removing the relay, the result also specializes to the lossy source-channel communications of independent sources over an interference channel.

3.2 Preliminaries and a Useful Lemma

Consider two finite alphabet sources $\{(U_{1i}, U_{2i})\}$ with correlated outputs that are jointly drawn according to a distribution $P[U_{1i} = u_1, U_{2i} = u_2] = p(u_1, u_2)$. The sources are memoryless, i.e., (U_{1i}, U_{2i})'s are i.i.d. Both of the sources are to be transmitted to the corresponding destinations through complex-valued discrete-time memoryless non-ergodic phase fading Gaussian channel models. Channels are parameterized by the phase shifts that are introduced by different paths of the network which are not known to the transmitters. The vector $\boldsymbol{\theta}$ is a vector with one element from $[0, 2\pi)$ for each channel, and denotes the non-ergodic phase fading parameters. For simplicity, throughout the chapter, we assume that transmitter node with index $i \in \{1, 2, r\}$ has power constraint P_i and the noise power at all corresponding receiving nodes is N.

Throughout the chapter, we use the notations $X_{ik}, Y_k, Y_{ik} \in \mathbb{C}$, $Z_k, Z_{ik} \sim \mathcal{CN}(0, N)$ in order to show the input and output symbols of corresponding channels along with circularly symmetric complex Gaussian noise signals respectively. Also note that the first subscript $i \in \{1, 2, r\}$ denotes the node index while the second subscript $k \in \{1, 2, \cdots, n\}$ denotes the time index, with the exception of Y_k, Z_k, which have only a time index. Without confusion X_i^n denotes the length-n vector $(X_{i1}, \cdots, X_{in})$. Additionally, we denote the path gains from node i to node j by $g_{ij}e^{\theta_{ij}}$, where θ_{ij} shows the phase shift introduced by the path, unknown at i, and g_{ij} represents the amplitude gain that is known to the transmitter node i, and can model e.g., line of sight path gains. Finally, let $S = \{(1, 2), (2, 1)\}$.

3.2.1 Useful Lemma

Let $X^m = (X_1, X_2, \cdots, X_m)$, be a vector of random variables with joint distribution p_{X^m} and $\max_i \mathbb{E}\|X_i\|^2 < \infty$. Also let the scalar RV $V \triangleq \sum_{i=1}^m g_i e^{j\theta_i} X_i + Z$, where $g_i e^{j\theta_i}$ are arbitrary complex coefficients and $Z \sim \mathcal{CN}(0, N)$.

We now state and prove the following lemma which asserts that the minimum over $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_m)$ of the mutual information between X^m and V is maximized when X^m

is a zero-mean Gaussian vector with independent elements, i.e., RVs $X_1, X_2, \cdots, X_m$ are independent Gaussians with zero mean.

Lemma 1. *Let* $\mathcal{P} = \{p_{X^m} : \mathbb{E}\|X_i\|^2 \leq P_i, \forall i\}$*. Then,*

$$\max_{p_{X^m} \in \mathcal{P}} \min_{\boldsymbol{\theta}} I(X^m; V) = \log(1 + \sum_{i=1}^{m} g_i^2 P_i / N),$$

i.e., when $\boldsymbol{\theta}$ *is chosen adversarially, the best* X^m *is a zero-mean Gaussian vector with independent elements and* $\mathrm{Var}(X_i) = P_i, \forall i.$ □

Proof. By letting $\mathbb{E}\|X_i\|^2 = \sigma_i^2$, $\mathbb{E}(X_i X_j) = \rho_{ij}\sigma_i\sigma_j$, it can be easily seen that the RV V has a fixed variance σ_V^2 which is equal to

$$\begin{aligned}
\sigma_V^2 &= (\sum_{i=1}^{m} g_i^2 \sigma_i^2) + N + 2\sum_{i<j} g_i g_j \sigma_i \sigma_j \ \Re\big\{\rho_{ij} e^{j(\theta_i - \theta_j)}\big\} - [\mathbb{E}(V)]^2 \\
&\leq (\sum_{i=1}^{m} g_i^2 \sigma_i^2) + N + 2\sum_{i<j} g_i g_j \sigma_i \sigma_j \ \Re\big\{\rho_{ij} e^{j(\theta_i - \theta_j)}\big\} \qquad (3.1) \\
&\triangleq \tilde{\sigma}_V^2.
\end{aligned}$$

As for a given variance, the Gaussian distribution maximizes the differential entropy [29, Thm. 8.4.1], we have

$$I(X^m; V) \leq \log(2\pi e \tilde{\sigma}_V^2) - h(Z). \qquad (3.2)$$

Next, note that $\min_{\boldsymbol{\theta}} \tilde{\sigma}_V^2$ is maximized when $\rho_{ij} = 0, \forall i, j$. It can be seen from (3.1) that if $\rho_{ij} \neq 0$, the parameters $\theta_1, \theta_2, \cdots, \theta_m$ can be chosen such that the term $2\sum_{i<j} g_i g_j \sigma_i \sigma_j \ \Re\big\{\rho_{ij} e^{j(\theta_i - \theta_j)}\big\}$ is strictly negative. Thus

$$\min_{\boldsymbol{\theta}} I(X^m; V) \leq \log(\frac{\sum_{i=1}^{m} g_i^2 \sigma_i^2}{N} + 1)$$

$$\leq \log\left(\frac{\sum_{i=1}^{m} g_i^2 P_i}{N} + 1\right), \tag{3.3}$$

where (3.3) follows since $\sigma_i^2 \leq P_i$. Finally, note that the bound in (3.3) is achieved by zero-mean independent Gaussians and the lemma is proved. □

Remark 3. *For the ergodic setting, where $\boldsymbol{\theta}$ is i.i.d. from channel use to channel use, uniformly distributed over $[0, 2\pi)^m$, and the* averaged *mutual information over $\boldsymbol{\theta}$ is to be maximized, a similar result is given in [45, Thm. 8]. Specifically,*

$$\max_{p_{X^m}} \mathbb{E}_{\boldsymbol{\theta}} I(X^m; V) = \log(1+\sum_{i=1}^{m} g_i^2 P_i/N). \qquad \square$$

Note that Lemma 1 applies to the phase incoherent scenario as opposed to an ergodic phase fading scheme. Also, its statement and proof are different than [45, Thm. 8].

3.3 Phase Incoherent Multiple Access Relay Channels

In this section, we consider the problem of joint source-channel coding over a series of PI-MARCs in which the encoders can cooperate in a unidirectional manner, i.e., one of the encoders can be a cognitive radio. Namely, we first study a MARC in which one of the encoders has knowledge of the other encoder's message causally (PI-CC-MARC). We also consider an ordinary MARC, and a PI-MARC with non-causal cognitive cooperation between the encoders (PI-NC-MARC), as special cases of PI-CC-MARC.

We first derive a set of necessary conditions for reliable communication over a PI-CC-MARC which can be considered as a unified outer bound for PI-CC-MARC and other MARCs under consideration. The outer bound will be presented in the form of a lemma from which the converse results for PI-MARC and PI-NC-MARC as special cases of PI-CC-MARC can be deduced. Afterwards, using separate source-channel coding and under specific gain conditions, we derive achievability constraints that match with the unified outer bound for each of the three considered channels. Therefore, we prove separation theorems under specific gain conditions along with necessary and sufficient conditions for reliable communication for each channel.

3.3.1 PI-CC-MARC

We now consider sending sources U_1, U_2 over a PI-CC-MARC which is depicted in Figure 3.1, and denoted by $(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_r, \mathcal{Y}_1 \times \mathcal{Y}_r \times \mathcal{Y}, p_{\boldsymbol{\theta}}(y_1, y_r, y|x_1, x_2, x_r))$. The received signal at the destination is given by

$$Y_i = g_1 e^{j\theta_1} X_{1i} + g_2 e^{j\theta_2} X_{2i} + g_r e^{j\theta_r} X_{ri} + Z_i, \tag{3.4}$$

and the signal received at the relay can be written as

$$Y_{ri} = g_{1r} e^{j\theta_{1r}} X_{1i} + g_{2r} e^{j\theta_{2r}} X_{2i} + Z_{ri} \tag{3.5}$$

where Z_{ri} is independent of Z_i.

As it can be seen from Figure 3.1, the encoder X_1 receives a noisy phase faded version of X_2 through the link from node 2 to node 1. Indeed, the first transmitter works as a relay for the other one while communicating its own information. Here, the relationship

$$Y_{1i} = g_{21} e^{j\theta_{21}} X_{2i} + Z_{1i}, \tag{3.6}$$

with Z_{1i} being independent of Z_i, Z_{ri}, describes the link from node 2 to node 1. The parameter $\boldsymbol{\theta}$ for the PI-CC-MARC is the vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_r, \theta_{1r}, \theta_{2r}, \theta_{12}) \in [0, 2\pi)^6$. X_{1i} is a function of the source signal U_1^n and its past received signals $Y_1^{(i-1)}$.

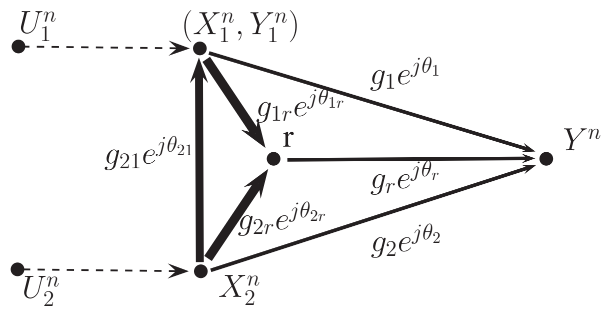

Figure 3.1: Correlated sources and phase incoherent causal cognitive multiple access relay channel (PI-CC-MARC).

In the sequel, we state and prove a separation theorem for the PI-CC-MARC under specific gain conditions. In particular, we first define the settings and afterwards derive an outer bound for the PI-CC-MARC in the form of a lemma. The outer bound also applies to the cases of PI-MARC and PI-NC-MARC, thus it is named as the unified outer bound.

Definition 10. Joint source-channel code*: A joint source-channel code of length n for a PI-CC-MARC with correlated sources is defined by*

1. *Two sets of encoding functions $\{\mathcal{E}_{ji}\}_{i=1}^n, j = 1, 2$. Furthermore, we define relay encoding functions by $x_{ri} = f_i(y_{r1}, y_{r2}, \cdots, y_{r(i-1)}), i = 1, 2, \cdots, n$. The sets of codewords are denoted by the codebook $\mathcal{C} = \{(\{\mathcal{E}_{1i}(u_1^n, y_1^{i-1})\}_{i=1}^n, \{\mathcal{E}_{2i}(u_2^n)\}_{i=1}^n) : u_1^n \in \mathcal{U}_1^n, u_2^n \in \mathcal{U}_2^n\}$.*

2. *Power constraint P_1, P_2 and P_r at the transmitters, i.e.,*

$$\mathbb{E}\ (\frac{1}{n}\sum_{i=1}^{n} \|X_{ji}\|^2) \leq P_j,\ \ j = 1, 2, r, \tag{3.7}$$

where $\mathbb{E}$ is the expectation operation over the distribution induced by U_1^n, U_2^n.

3. A decoding function $g^n_{\boldsymbol{\theta}} : \mathcal{Y}^n \to \mathcal{U}_1^n \times \mathcal{U}_2^n$.

$\square$

Upon reception of the received vector Y^n, the receiver decodes $(\hat{U}_1^n, \hat{U}_2^n) = g_{\boldsymbol{\theta}}(Y^n)$ as the transmitted source outputs. The probability of an erroneous decoding depends on $\boldsymbol{\theta}$ and is denoted by $P_e^n(\boldsymbol{\theta})$.

Definition 11. *We say the source $\{U_{1i}, U_{2i}\}_{i=1}^n$ of i.i.d. discrete random variables with joint probability mass function $p(u_1, u_2)$* can be reliably sent *over the PI-CC-MARC, if there exists a sequence of encoding functions $\mathcal{E}_n \triangleq \{\{\mathcal{E}_{1i}\}_{i=1}^n, \{\mathcal{E}_{2i}\}_{i=1}^n, f_1, f_2, \cdots, f_n\}$ and decoders $g^n_{\boldsymbol{\theta}}$ such that the source sequences U_1^n and U_2^n can be estimated with asymptotically small probability of error (uniformly over all parameters $\boldsymbol{\theta}$) at the receiver side from the received sequence Y^n, i.e., $\sup_{\boldsymbol{\theta}} P_e^n(\boldsymbol{\theta}) \longrightarrow 0, \quad \text{as} \quad n \to \infty$.* $\square$

Note that although our definitions concern the case where source vectors of length n are mapped into channel vectors of the same length (n), i.e., *bandwidth matched* case, all of the results extend easily to the mismatched case as well.

Now, we present a converse result for the PI-CC-MARC which can be considered as a unified outer bound for all of the MARCs considered in this section.

Lemma 2. Unified converse*: Let $\mathcal{E}_n$, and $g^n_{\boldsymbol{\theta}}$ be a sequence in n of encoders and decoders for the PI-CC-MARC for which $\sup_{\boldsymbol{\theta}} P_e^n(\boldsymbol{\theta}) \longrightarrow 0$, as $n \to \infty$. Then*

$$H(U_1|U_2) \leq \log(1 + (g_1^2 P_1 + g_r^2 P_r)/N), \tag{3.8}$$

$$H(U_1, U_2) \leq \log(1 + (g_1^2 P_1 + g_2^2 P_2 + g_r^2 P_r)/N). \tag{3.9}$$

$\square$

Proof. First, fix a PI-CC-MARC with given parameter $\boldsymbol{\theta}$, a codebook $\mathcal{C}$, and induced *empirical* distribution $p_{\boldsymbol{\theta}}(u_1^n, u_2^n, x_1^n, x_2^n, x_r^n, y_1^n, y_r^n, y_r^n)$ by the codebook. Since for this fixed choice of $\boldsymbol{\theta}$, $P_e^n(\boldsymbol{\theta}) \to 0$, from Fano's inequality, we have

$$\frac{1}{n}H(U_1^n, U_2^n|Y^n, \boldsymbol{\theta}) \le \frac{1}{n}P_e^n(\boldsymbol{\theta}) \log \|\mathcal{U}_1^n \times \mathcal{U}_2^n\| + \frac{1}{n} \triangleq \epsilon_n(\boldsymbol{\theta}), \tag{3.10}$$

and $\epsilon_n(\boldsymbol{\theta}) \to 0$, where convergence is uniform in $\boldsymbol{\theta}$. Defining $\sup_{\boldsymbol{\theta}} \epsilon_n(\boldsymbol{\theta}) = \epsilon_n$ and following the similar steps as in [19, Section 4], we have

$$\begin{aligned}
H(U_1|U_2) &= \frac{1}{n}H(U_1^n|U_2^n) \\
&\overset{\text{(a)}}{=} \frac{1}{n}H(U_1^n|U_2^n, X_2^n, \boldsymbol{\theta}) \\
&= \frac{1}{n}I(U_1^n; Y^n|X_2^n, U_2^n, \boldsymbol{\theta}) + \frac{1}{n}H(U_1^n|Y^n, X_2^n, U_2^n, \boldsymbol{\theta}) \\
&\overset{\text{(b)}}{\le} \frac{1}{n}I(U_1^n; Y^n|U_2^n, X_2^n, \boldsymbol{\theta}) + \epsilon_n \\
&\overset{\text{(c)}}{\le} \frac{1}{n}I(X_1^n; Y^n|X_2^n, U_2^n, \boldsymbol{\theta}) + \epsilon_n & (3.11) \\
&\le \frac{1}{n}I(X_1^n, X_r^n; Y^n|X_2^n, U_2^n, \boldsymbol{\theta}) + \epsilon_n, & (3.12)
\end{aligned}$$

where (a) follows from the fact that X_2^n is only a function of U_2^n, and (b) follows from (3.10), and (c) follows from data processing inequality. Similarly, it can be shown that

$$\begin{aligned}
H(U_1, U_2) &= \frac{1}{n}H(U_1^n, U_2^n), & (3.13) \\
&= \frac{1}{n}I(U_1^n, U_2^n; Y^n|\boldsymbol{\theta}) + \frac{1}{n}H(U_1^n, U_2^n|Y^n, \boldsymbol{\theta}) \\
&\le \frac{1}{n}I(X_1^n, X_2^n, X_r^n; Y^n|\boldsymbol{\theta}) + \epsilon_n. & (3.14)
\end{aligned}$$

We now further upper bound (3.12), (3.14). First, we expand Y^n in (3.12) to upper bound $H(U_1|U_2)$ as follows:

$$H(U_1|U_2) \le \frac{1}{n}\, I(X_1^n, X_r^n; Y^n|X_2^n, U_2^n, \boldsymbol{\theta}) + \epsilon_n$$

$$
\begin{aligned}
&= \frac{1}{n}\ I(X_1^n, X_r^n; g_1 e^{j\theta_1} X_1^n + g_2 e^{j\theta_2} X_2^n + g_r e^{j\theta_r} X_r^n + Z^n | U_2^n, X_2^n) + \epsilon_n \\
&= \frac{1}{n} I(X_1^n, X_r^n; g_1 e^{j\theta_1} X_1^n + g_r e^{j\theta_r} X_r^n + Z^n | U_2^n, X_2^n) + \epsilon_n \\
&= \frac{1}{n}\left[h(g_1 e^{j\theta_1} X_1^n + g_r e^{j\theta_r} X_r^n + Z^n | U_2^n, X_2^n) - h(Z^n)\right] + \epsilon_n \\
&\leq \frac{1}{n}\left[h(g_1 e^{j\theta_1} X_1^n + g_r e^{j\theta_r} X_r^n + Z^n) - h(Z^n)\right] + \epsilon_n \\
&= \frac{1}{n}\ I(X_1^n, X_r^n; g_1 e^{j\theta_1} X_1^n + g_r e^{j\theta_r} X_r^n + Z^n) + \epsilon_n \\
&\leq \frac{1}{n}\sum_{i=1}^{n} I(X_{1i}, X_{ri}; g_1 e^{j\theta_1} X_{1i} + g_r e^{j\theta_r} X_{ri} + Z_i) + \epsilon_n \\
&\overset{\text{(a)}}{=} I(X_1, X_r; g_1 e^{j\theta_1} X_1 + g_r e^{j\theta_r} X_r + Z | W) + \epsilon_n \\
&= \left[h(g_1 e^{j\theta_1} X_1 + g_r e^{j\theta_r} X_r + Z | W) - h(Z)\right] + \epsilon_n \\
&\leq \left[h(g_1 e^{j\theta_1} X_1 + g_r e^{j\theta_r} X_r + Z) - h(Z)\right] + \epsilon_n \\
&= I(X_1, X_r; g_1 e^{j\theta_1} X_1 + g_r e^{j\theta_r} X_r + Z) + \epsilon_n, \qquad (3.15)
\end{aligned}
$$

where (a) follows by defining new random variables

$$X_j = X_{jW},\ j \in \{1, 2, r\}, \qquad (3.16)$$

$$Z = Z_W, \qquad (3.17)$$

$$W \sim \text{Uniform}\{1, 2, \cdots, n\}. \qquad (3.18)$$

From (4.5), the input signals X_1, X_r satisfy the power constraints

$$\mathbb{E}|X_j|^2 = \mathbb{E}\ (\frac{1}{n}\sum_{i=1}^{n} \|X_{ji}\|^2) \leq P_j,\ j = 1, r, \qquad (3.19)$$

and $Z \sim \mathcal{CN}(0, N)$.

Moreover, following similar steps from (3.14), we have

$$H(U_1, U_2) = \frac{1}{n} H(U_1^n, U_2^n)$$

$$
\begin{aligned}
&= \frac{1}{n} I(U_1^n, U_2^n; Y^n|\boldsymbol{\theta}) + \frac{1}{n} H(U_1^n, U_2^n|Y^n, \boldsymbol{\theta}) \\
&\leq \frac{1}{n} I(U_1^n, U_2^n; Y^n|\boldsymbol{\theta}) + \epsilon_n \\
&\leq \frac{1}{n} I(X_1^n, X_2^n; Y^n|\boldsymbol{\theta}) + \epsilon_n \\
&\leq \frac{1}{n} I(X_1^n, X_2^n, X_r^n; Y^n|\boldsymbol{\theta}) + \epsilon_n \\
&= \frac{1}{n}\, I(X_1^n, X_2^n, X_r^n; g_1 e^{j\theta_1} X_1^n + g_2 e^{j\theta_2} X_2^n + g_r e^{j\theta_r} X_r^n + Z^n) + \epsilon_n \\
&\leq \ \frac{1}{n} \sum_{i=1}^{n} I(X_{1i}, X_{2i}, X_{ri}; g_1 e^{j\theta_1} X_{1i} + g_2 e^{j\theta_2} X_{2i} + g_r e^{j\theta_r} X_{ri} + Z_i) + \epsilon_n \\
&\leq \ I(X_1, X_2, X_r; g_1 e^{j\theta_1} X_1 + g_2 e^{j\theta_2} X_2 + g_r e^{j\theta_r} X_r + Z) + \epsilon_n, \qquad (3.20)
\end{aligned}
$$

where the last step follows with the same RVs as in (3.130)-(3.132).

The constraints defined by (3.15) and (3.20) is an outer bound on the capacity region. But since it applies for a fixed $\boldsymbol{\theta}$, it is also true for all choices of $\boldsymbol{\theta}$. The proof of the lemma is completed by taking the intersection of the upper bounds over all values of $\boldsymbol{\theta}$, letting $n \to \infty$, and noting by Lemma 1 that the resulting bounds are simultaneously maximized by choosing X_1, X_2, X_r to be zero-mean complex independent Gaussians. □

Remark 4. *Note that to prove Lemma 2 as the converse part for PI-CC-MARC, we do not need the receiver to know the CSI $\boldsymbol{\theta}$. This is indeed true for other separation theorems of the chapter as well.* □

Now, by combining Lemma 2 as the converse part and an achievbility argument as follows, we state and prove the following separation theorem for the PI-CC-MARC.

Theorem 9. *Consider a PI-CC-MARC, and the source pair $(U_1^n, U_2^n) \sim \prod_i p(u_{1i}, u_{2i})$. Furthermore, assume the gain conditions*

$$
g_{1r}^2 P_1 \geq g_1^2 P_1 + g_r^2 P_r, \qquad (3.21)
$$

Encoder	Block 1	Block 2	Block B	Block $B+1$
1	$x_1^n(1, W_{11}, 1)$	$x_1^n(W_{11}, W_{12}, W_{21})$	$x_1^n(W_{1(B-1)}, W_{1B}, W_{2(B-1)})$	$x_1^n(W_{1B}, 1, W_{2B})$
2	$x_2^n(1, W_{21})$	$x_2^n(W_{21}, W_{22})$	$x_2^n(W_{2(B-1)}, W_{2B})$	$x_2^n(W_{2B}, 1)$
r	$x_r^n(1, 1)$	$x_r^n(W_{11}, W_{21})$	$x_r^n(W_{1(B-1)}, W_{2(B-1)})$	$x_r^n(W_{1B}, W_{2B})$

Table 3.1: Block Markov encoding scheme for the PI-CC-MARC.

$$g_{2r}^2 P_1 \geq g_1^2 P_1 + g_2^2 P_1 + g_r^2 P_r, \tag{3.22}$$

$$1 + \frac{g_{21}^2 P_2}{N} \geq 2^{-H(U_1|U_2)} \left(1 + (g_1^2 P_1 + g_2^2 P_2 + g_r^2 P_r)/N\right). \tag{3.23}$$

Then, necessary conditions for reliable communication of the correlated sources (U_1^n, U_2^n) *over such a channel are given by* (3.8) *and* (3.9)*. Conversely,* (3.8) *and* (3.9) *also describe sufficient conditions with* $\leq$ *replaced by* $<$*.* □

Proof. The achievability argument is as follows:

Achievability

For the achievability part, we use separate source and channel coding. We need to show that given (3.8) and (3.9), we can first losslessly source code the sources to indices $W_1 \in [1, 2^{nR_1}], W_2 \in [1, 2^{nR_2}]$ and then send W_1, W_2 over the channel with arbitrarily small error probability.

Source Coding: Using Slepian-Wolf coding [30], for asymptotically lossless representation of the source (U_1^n, U_2^n), we should have the rates (R_1, R_2) satisfying

$$R_1 > H(U_1|U_2), \tag{3.24}$$

$$R_2 > H(U_2|U_1), \tag{3.25}$$

$$R_1 + R_2 > H(U_1, U_2). \tag{3.26}$$

The source codes are represented by indices W_1, W_2 which are then channel coded before being transmitted.

Channel Coding: The channel coding argument for a PI-CC-MARC is based on block Markov coding with backward decoding as shown in Table 3.1. First fix a distribution $p(x_1)p(x_2)p(x_r)$ and construct random codewords x_1^n, x_2^n, x_r^n based on the corresponding distributions. Namely, the first encoder generates $2^{n(2R_1+R_2)}$ random codewords $x_1^n(W_1, W_1', W_2)$ according to the distribution $\prod_{i=1}^n p(x_{1i})$, the second encoder generates $2^{n(2R_2)}$ random codewords $x_2^n(W_2, W_2')$ according to the distribution $\prod_{i=1}^n p(x_{2i})$, and the relay encoder generates $2^{n(R_1+R_2)}$ random codewords $x_r^n(W_1, W_2)$ according to the distribution $\prod_{i=1}^n p(x_{ri})$. The message W_i of each encoder is then divided to B blocks $W_{i1}, W_{i2}, \cdots, W_{iB}$ of 2^{nR_i} bits each, $i = 1, 2$. The codewords are transmitted in $B+1$ blocks based on the block Markov encoding scheme depicted in Table 3.1. Since U_2^n is not perfectly and non-causally known to the first encoder, node 1 needs to first decode W_{2t} after block t from its received signal over the link between the encoders. In the t-th block, the first transmitter sends the codeword $x_1^n(W_{1(t-1)}, W_{1t}, W_{2(t-1)})$, while the second transmitter uses codeword $x_2^n(W_{2(t-1)}, W_{2t})$ and the relay sends the codeword $x_r^n(W_{1(t-1)}, W_{2(t-1)})$. We let $B \to \infty$ to approach the original rates R_1, R_2.

At the end of each block b, the relay decodes W_{1b}, W_{2b}, referred to as forward decoding [35]. Indeed, at the end of the first block, the relay decodes W_{11}, W_{21} from the received signal $Y_r^n(W_{11}, W_{21})$. In the second block, nodes 1 and 2 transmit $x_1^n(W_{11}, W_{12}, W_{21})$ and $x_2^n(W_{21}, W_{22})$, respectively. The relay decodes W_{12}, W_{22}, using the knowledge of W_{11}, W_{21}, and this is continued until the last block.

The decoding at the destination, however, is performed based on backward decoding [36], [62], i.e., starting from the last block back to the former ones. As depicted in Table 3.1, at the end of block $B+1$, the receiver can decode W_{1B}, W_{2B}. Afterwards, by using the knowledge of W_{1B}, W_{2B}, the receiver goes one block backwards and decodes $W_{1(B-1)}, W_{2(B-1)}$. This process is continued until the receiver decodes all of the messages. Thus, in order to guarantee correct decoding at the relay and correct backward decoding at the destination when $n \to \infty$, using standard random coding arguments, the following conditions should be satisfied:

$$R_1 < I(X_1; Y_r | X_2, X_r, \boldsymbol{\theta}), \tag{3.27}$$

$$R_2 < I(X_2; Y_r | X_1, X_r, \boldsymbol{\theta}), \tag{3.28}$$

$$R_1 + R_2 < I(X_1, X_2; Y_r | X_r, \boldsymbol{\theta}), \tag{3.29}$$

for decoding at the relay and

$$R_1 < I(X_1, X_r; Y | X_2, \boldsymbol{\theta}), \tag{3.30}$$

$$R_1 + R_2 < I(X_1, X_2, X_r; Y | \boldsymbol{\theta}). \tag{3.31}$$

for decoding at the destination respectively.

Additionally, to reliably decode the second encoder's message at the first encoder (which plays the role of a relay), we need to satisfy the condition

$$R_2 < I(X_2; Y_1 | X_1, X_r, \boldsymbol{\theta}). \tag{3.32}$$

Computing these conditions for independent Gaussian inputs and using conditions (3.21) and (3.22), we find the following achievable region for channel coding:

$$R_1 < \log(1 + (g_1^2 P_1 + g_r^2 P_r)/N), \tag{3.33}$$

$$R_2 < \log(1 + g_{21}^2 P_2/N), \tag{3.34}$$

$$R_1 + R_2 < \log(1 + (g_1^2 P_1 + g_2^2 P_2 + g_r^2 P_r)/N). \tag{3.35}$$

In order to make the inner bounds of (3.33)-(3.35) coincide with the outer bounds (3.8), (3.9), we need to have

$$\log(1 + (g_1^2 P_1 + g_2^2 P_2 + g_r^2 P_r)/N) - R_1 < \log(1 + g_{21}^2 P_2/N),$$

so that we can drop (3.34) from the achievability constraints. But by choosing $R_1 = H(U_1|U_2) + \epsilon$, with $\epsilon > 0$ arbitrary, condition (3.23) makes (3.34) dominated by (3.35) for the Gaussian input distributions. Therefore, since $\epsilon > 0$ is arbitrary, one can easily verify that given (3.8) and (3.9) with $\leq$ replaced by $<$, along with the conditions (3.21)-(3.23), source and channel codes of rates R_1, R_2 can be found such that (3.24)-(3.26), and (3.27)-(3.32) simultaneously hold. □

We now present the following corollary for a phase incoherent causal cognitive MAC (PI-CC-MAC) which can be constructed from a corresponding MARC by eliminating the relay.

Corollary 1. Reliable communication over a PI-CC-MAC*: Necessary conditions for reliable communication of the sources* (U_1, U_2) *over the causal PI-CC-MAC with power constraints* P_1, P_2 *on transmitters, fading amplitudes* $g_1, g_2 > 0$*, and source pair* $(U_1^n, U_2^n) \sim \prod_i p(u_{1i}, u_{2i})$*, is given by*

$$H(U_1|U_2) \leq \log(1 + g_1^2 P_1/N), \tag{3.36}$$

$$H(U_1, U_2) \leq \log(1 + (g_1^2 P_1 + g_2^2 P_2)/N), \tag{3.37}$$

provided

$$1+\frac{g_{21}^2P_2}{N}\geq 2^{-H(U_1|U_2)}\left(1+\frac{g_1^2P_1+g_2^2P_2}{N}\right). \tag{3.38}$$

Given (3.38), sufficient conditions for reliable communications are also given by (3.48) and (3.49), with $\leq$ replaced by $<$. □

Proof. The PI-CC-MAC is equivalent to a PI-CC-MARC where the relay has power constraint $P_r = 0$. As the relay is thus silent, we may assume without loss that g_{1r}, g_{2r} are arbitrarily large. The conditions (3.21)-(3.22) of Theorem 9 with (3.23) being changed to (3.38) are then trivially satisfied. □

3.3.2 PI-MARC

If the cognitive link in the PI-CC-MARC is silent ($g_{21} = 0$), we will have an ordinary PI-MARC.

Corollary 2. *Consider a PI-MARC with the gain conditions*

$$g_{ir}^2P_i\geq g_i^2P_i+g_r^2P_r,\ \ i=1,2. \tag{3.39}$$

Then, a necessary condition for reliably sending the source pair $(U_1^n, U_2^n) \sim \prod_i p(u_{1i}, u_{2i})$, over such a PI-MARC, is given by

$$H(U_i|U_j)\leq\log(1+(g_i^2P_i+g_r^2P_r)/N),\ \ (i,j)\in S, \tag{3.40}$$

$$H(U_1,U_2)\leq\log(1+(g_1^2P_1+g_2^2P_2+g_r^2P_r)/N). \tag{3.41}$$

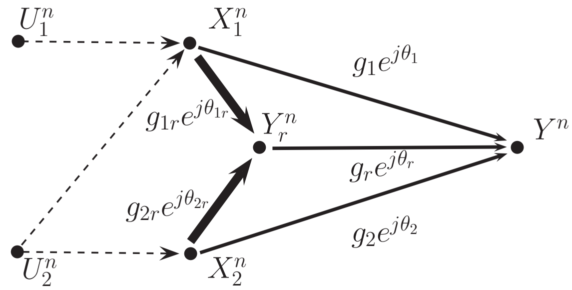

Figure 3.2: Correlated sources and phase incoherent non-causal cognitive multiple access relay channel (PI-NC-MARC).

Moreover, (3.40)-(3.41) *also describes sufficient conditions for reliable communications with* $\leq$ *replaced by* $<$. *The sufficient conditions are achieved by separate source-channel coding.*

□

Proof. The converse proof is a direct result of Lemma 2 by intersecting the outer bounds for PI-CC-MARCs including either directions of the cognitive link. The achievability proof is similar to that of [44, Thm. 4]. □

3.3.3 PI-NC-MARC

A PI-NC-MARC is an extension of a PI-MARC, in which the first encoder (cognitive encoder) has non-causal access to the second source U_2. Fig. 3.2 depicts a PI-NC-MARC. Like PI-CC-MARC, the input/output relationships of the channel for the receiver and the relay are given by (3.4) and (3.5).

In the sequel, we present and prove necessary and sufficient conditions for the reliable communications of a pair of arbitrarily correlated sources (U_1, U_2) over a PI-NC-MARC, in the form of a separation theorem.

Theorem 10. *Consider a PI-NC-MARC with the gain conditions*

$$g_{1r}^2 P_1 \geq g_1^2 P_1 + g_r^2 P_r, \tag{3.42}$$

$$g_{1r}^2 P_1 + g_{2r}^2 P_2 \geq g_1^2 P_1 + g_2^2 P_2 + g_r^2 P_r. \tag{3.43}$$

Necessary conditions for reliably sending a source pair $(U_1^n, U_2^n) \sim \prod_i p(u_i, v_i)$*, over such a PI-NC-MARC are given by* (3.8)-(3.9)*. Furthermore, eqs.* (3.8)-(3.9) *also give the sufficient conditions for reliable communications over such PI-NC-MARC with* $\leq$ *replaced by* $<$*.* □

Proof. The converse proof is a direct result of Corollary 1 (and in turn Lemma 2), since PI-NC-MARC can be considered as a PI-MARC with the source pair $((U_1^n, U_2^n), U_2^n)$. The achievability part is again obtained by a separate source and channel coding approach.

Achievability

We now establish the same region (described by (3.8)-(3.9)) as achievable for the PI-NC-MARC. To derive the achievable region, we perform separate source-channel coding. Again, as for the PI-CC-MARC, the source coding is performed by Slepian-Wolf coding and the channel coding argument is based on regular block Markov encoding in conjunction with backward decoding [37]. Both source coding and channel coding schemes are explained as follows.

Source Coding: Recall that the first encoder has non-causal access to the second source U_2^n. From Slepian-Wolf coding [30], for asymptotically lossless representation of the source $((U_1^n, U_2^n), U_2^n)$, we should have the rates (R_1, R_2) satisfying

$$R_1 > H(U_1|U_2),$$

Encoder	Block 1	Block 2	Block B	Block $B+1$
1	$x_1^n(1, W_{11}, W_{21}, 1)$	$x_1^n(W_{11}, W_{12}, W_{22}, W_{21})$	$x_1^n(W_{1(B-1)}, W_{1B}, W_{2B}, W_{2(B-1)})$	$x_1^n(W_{1B}, 1, 1, W_{2B})$
2	$x_2^n(1, W_{21})$	$x_2^n(W_{21}, W_{22})$	$x_2^n(W_{2(B-1)}, W_{2B})$	$x_2^n(W_{2B}, 1)$
r	$x_r^n(1, 1)$	$x_r^n(W_{11}, W_{21})$	$x_r^n(W_{1(B-1)}, W_{2(B-1)})$	$x_r^n(W_{1B}, W_{2B})$

Table 3.2: Block Markov encoding scheme for a NC-MARC.

$$R_1 + R_2 > H(U_1, U_2).$$

Channel Coding: Similar to the discussion presented in Section 3.3.1 for the PI-CC-MARC, an achievable region for a discrete memoryless NC-MARC with 2 users is given based on the block Markov coding scheme shown in Table 3.2 combined with backward decoding. Note that the results readily extend to a PI-NC-MARC with phase vector $\boldsymbol{\theta}$ known to the receiver side.

First, fix a distribution $p(x_1)p(x_2)p(x_r)$ and construct random codewords x_1^n, x_2^n, x_r^n based on the corresponding distributions. The message W_i of each encoder is divided to B blocks $W_{i1}, W_{i2}, \cdots, W_{iB}$ of 2^{nR_i} bits each, $i = 1, 2$. The codewords are transmitted in $B+1$ blocks based on the block Markov encoding scheme depicted in Table 3.2. Using its non-causal knowledge of the second source, transmitter 1 sends the information using the codeword $x_1^n(W_{1(t-1)}, W_{1t}, W_{2t}, W_{2(t-1)})$, while transmitter 2 uses codeword $x_2^n(W_{2(t-1)}, W_{2t})$ and the relay sends the codeword $x_r^n(W_{1(t-1)}, W_{2(t-1)})$. We let $B \to \infty$ to approach the original rates R_1, R_2.

At the end of each block b, the relay performs forward decoding to reconstruct W_{1b}, W_{2b}. In particular, at the end of the first block, the relay decodes W_{11}, W_{21} from the received signal $Y_r^n(W_{1b}, W_{2b})$. In the second block, nodes 1 and 2 transmit $x_1^n(W_{11}, W_{12}, W_{22}, W_{21})$ and $x_2^n(W_{21}, W_{22})$, respectively. The relay decodes W_{12}, W_{22}, using the knowledge of W_{11}, W_{21}, and this is continued until the last block. Using random coding arguments and forward decoding from the first block, for reliable decoding of messages $W_{1(b-1)}, W_{2(b-1)}$ at the relay after the bth block, when $n \to \infty$, it is sufficient to have

$$R_1 < I(X_1; Y_r | X_2, X_r, \boldsymbol{\theta}), \tag{3.44}$$

$$R_1 + R_2 < I(X_1, X_2; Y_r | X_r, \boldsymbol{\theta}). \tag{3.45}$$

The destination performs backward decoding to sequentially reconstruct

$$(W_{1B}, W_{2B}), (W_{1(B-1)}, W_{2(B-1)}), \cdots, (W_{11}, W_{21}),$$

as shown in Table 3.2. Thus, by applying regular block Markov encoding and backward decoding based on the configuration shown in Table 3.2, one finds that the destination can decode the messages reliably if $n \to \infty$ and

$$R_1 < I(X_1, X_r; Y | X_2, \boldsymbol{\theta}), \tag{3.46}$$

$$R_1 + R_2 < I(X_1, X_2, X_r; Y | \boldsymbol{\theta}). \tag{3.47}$$

The achievability part is complete by first choosing X_1, X_2, and X_r as independent Gaussians and observing that under conditions (3.42) and (3.43), (3.46) and (3.47) are tighter bounds than (3.140) and (3.141).

□

As a result of Theorem 10, in the following corollary, we state a separation theorem for the PI-NC-MAC, i.e., a PI-MAC with non-causal cooperation between encoders.

Corollary 3. Reliable Communication over a PI-NC-MAC*: Necessary conditions for reliable communication of the source* (U_1, U_2) *over a PI-NC-MAC with power constraints* P_1, P_2

on transmitters, fading amplitudes $g_1, g_2 > 0$*, and source pair* $(U_1^n, U_2^n) \sim \prod_i p(u_{1i}, u_{2i})$*, are given by*

$$H(U_1|U_2) \leq \log(1 + g_1^2 P_1/N), \tag{3.48}$$

$$H(U_1, U_2) \leq \log(1 + (g_1^2 P_1 + g_2^2 P_2)/N). \tag{3.49}$$

Sufficient conditions for reliable communication are also given by (3.48)-(3.49)*, with* $\leq$ *replaced by* $<$*.* $\square$

Proof. The PI-NC-MAC is equivalent to a PI-NC-MARC where the relay has power constraint $P_r = 0$. As the relay is thus silent, we may assume without loss that g_{1r}, g_{2r} are arbitrarily large, and the conditions (3.42) and (3.43) are trivially satisfied. $\square$

Remark 5. *Note that although the PI-NC-MARC can be considered as a special case of the PI-MARC, Theorem 10 is not a special case of the Corollary 2 for the source pair* $((U_1^n, U_2^n), U_2^n)$*. Specifically, because of the channel coding configuration used for the PI-NC-MARC, the gain conditions* (3.42)-(3.43) *of Theorem 10 are weaker than the gain conditions* (3.39) *of Corollary 2.*

3.4 Interference Relay Channel

The network model we consider in this section is an interference channel with two transmitters and a relay referred to as *interference relay channel* (IRC). We then study the interference channel (IC) as a special case of the IRC.

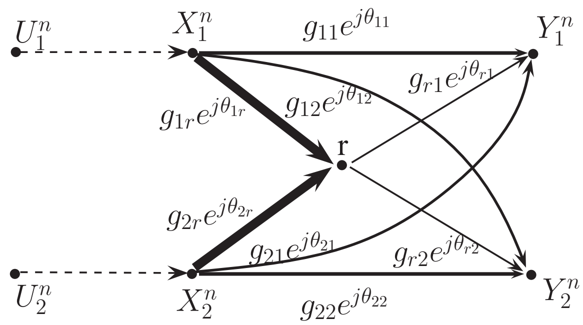

Figure 3.3: Correlated sources and phase incoherent interference relay channel (PI-IRC).

3.4.1 PI-IRC

The PI-IRC $(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_r, \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_r, p_{\boldsymbol{\theta}}(y_1, y_2, y_r | x_1, x_2, x_r))$ is depicted in Fig. 3.3, and is described by relationships

$$Y_{1i} = g_{11}e^{j\theta_{11}}X_{1i} + g_{21}e^{j\theta_{21}}X_{2i} + g_{r1}e^{j\theta_{r1}}X_{ri} + Z_{1i},$$
$$Y_{2i} = g_{12}e^{j\theta_{12}}X_{1i} + g_{22}e^{j\theta_{22}}X_{2i} + g_{r2}e^{j\theta_{r2}}X_{ri} + Z_{2i},$$
$$Y_{ri} = g_{1r}e^{j\theta_{1r}}X_{1i} + g_{2r}e^{j\theta_{2r}}X_{2i} + Z_{ri},$$

where parameter $\boldsymbol{\theta} = (\theta_{11}, \theta_{21}, \theta_{r1}, \theta_{12}, \theta_{22}, \theta_{r2}, \theta_{1r}, \theta_{2r}) \in [0, 2\pi)^8$ represents the phase shifts introduced by the channel to inputs X_1, X_2 and X_r, respectively.

In the sequel, we prove a separation theorem, along with the necessary and sufficient conditions of the reliable communications for the PI-IRC, under some non-trivial constraints on the channel gains which can be considered as an *asymmetric interference* situation for the IRC. The definitions of joint source-channel codes and reliable communication for the PI-IRC are similar to the ones given in Section 3.3.1 except for the fact that there are two decoders $g^n_{1\boldsymbol{\theta}}, g^n_{2\boldsymbol{\theta}}$ and two error probability functions $P^n_{e1}(\boldsymbol{\theta}), P^n_{e2}(\boldsymbol{\theta})$ in this setup.

We first state the separation theorem and consequently give the proofs of the converse and achievability parts.

Theorem 11. *Consider a PI-IRC with the gain conditions*

$$g_{11} = \alpha g_{12},\ g_{r1} = \alpha g_{r2},\ \alpha < 1, \tag{3.50}$$

$$g_{1r}^2 P_1 \geq g_{11}^2 P_1 + g_{r1}^2 P_r, \tag{3.51}$$

$$g_{2r}^2 P_2 \geq g_{22}^2 P_2 + g_{r2}^2 P_r, \tag{3.52}$$

$$\alpha^2 g_{r2}^2 P_r \geq (1-\alpha^2)\ g_{12}^2 P_1, \tag{3.53}$$

$$g_{21}^2 P_2 \geq \left(1-\alpha^2\right) g_{12}^2 P_1 + \left(1-\alpha^2\right) g_{r2}^2 P_r + g_{22}^2 P_2. \tag{3.54}$$

Then, necessary conditions for reliably sending a source pair $(U_1^n, U_2^n) \sim \prod_i p(u_{1i}, u_{2i})$*, over such PI-IRC are given by*

$$H(U_i|U_j) \leq \log(1 + (g_{ii}^2 P_1 + g_{ri}^2 P_r)/N),\ (i,j) \in S \tag{3.55}$$

$$H(U_1, U_2) \leq \log(1 + (g_{12}^2 P_1 + g_{22}^2 P_2 + g_{r2}^2 P_r)/N), \tag{3.56}$$

Moreover, a sufficient condition is also given by (3.55)-(3.56)*, with* $\leq$ *replaced by* $<$*.* □

Note that the gain conditions described by (3.106)- (3.54) imply asymmetric strong interference gains in the PI-IRC, as g_{21} is large while g_{12} can be relatively small. Indeed, these gain conditions can model, for example, physical proximity between the transmitters and the opposite receivers, specifically between the second transmitter and the first receiver. The proof of Theorem 11 is discussed in the two following subsections.

Converse

Lemma 3. *Let $\mathcal{E}_n$ be a sequence in n of encoders, and $g^n_{1\boldsymbol{\theta}}, g^n_{2\boldsymbol{\theta}}$ be sequences in n of decoders for the PI-IRC for which $\sup_{\boldsymbol{\theta}} P^n_{e1}(\boldsymbol{\theta}), P^n_{e2}(\boldsymbol{\theta}) \longrightarrow 0$, as $n \to \infty$, then we have*

$$H(U_i|U_j) \le \min_{\boldsymbol{\theta}\in\boldsymbol{\Phi}_c} I(X_i, X_r; g_{ii}e^{j\theta_{ii}}X_i + g_{ri}e^{j\theta_{ri}}X_r + Z),\ (i,j) \in \{(1,2),(2,1)\}, \tag{3.57}$$

$$H(U_1, U_2) \le \min_{\boldsymbol{\theta}\in\boldsymbol{\Phi}_c} I(X_1, X_2, X_r; g_{12}e^{j\theta_{12}}X_1 + g_{22}e^{j\theta_{22}}X_2 + g_{r2}e^{j\theta_{r2}}X_r + Z), \tag{3.58}$$

for some joint *distribution p_{X_1,X_2,X_r} such that $\mathbb{E}|X_1|^2 \le P_1, \mathbb{E}|X_2|^2 \le P_2, \mathbb{E}|X_r|^2 \le P_r$, where $\boldsymbol{\Phi}_c \triangleq \{\boldsymbol{\theta} : \theta_{11} = \theta_{12}, \theta_{r1} = \theta_{r2}\}$.* $\square$

Proof. First, fix a PI-IRC with given parameter $\boldsymbol{\theta} \in \boldsymbol{\Phi}_c$, a codebook $\mathcal{C}$, and induced *empirical* distribution $p_{\boldsymbol{\theta}}(u^n_1, u^n_2, x^n_1, x^n_2, x^n_r, y^n_1, y^n_2, y^n_r)$. Since for this fixed choice of $\boldsymbol{\theta}$, $P^n_{e1}(\boldsymbol{\theta}), P^n_{e2}(\boldsymbol{\theta}) \to 0$, from Fano's inequality, we have

$$\frac{1}{n}H(U^n_i|Y^n_i, \boldsymbol{\theta}) \le \frac{1}{n}P^n_{ei}(\boldsymbol{\theta}) \log \|\mathcal{U}^n_i\| + \frac{1}{n} \triangleq \epsilon_{in}(\boldsymbol{\theta}),$$

and $\epsilon_{in}(\boldsymbol{\theta}) \to 0, i = 1, 2$, where convergence is uniform in $\boldsymbol{\theta}$. Defining $\sup_{\boldsymbol{\theta}} \epsilon_{in}(\boldsymbol{\theta}) = \epsilon_{in}, i = 1, 2$ and following similar steps as those resulting in (3.12), we have

$$H(U_i|U_j) \le \frac{1}{n}I(X^n_i, X^n_r; Y^n_i|U^n_j, X^n_j, \boldsymbol{\theta}) + \epsilon_{in}, \tag{3.59}$$

for $(i,j) \in S$. Noting that (3.59) holds for every choice of $\boldsymbol{\theta}$, as in Section 3.3.1, we can upper bound (3.59) and derive (3.57). Next, to derive (3.58), we define a random vector $\tilde{Z}^n{}_1 \sim \mathcal{CN}(0, (1-\alpha^2)N\mathbf{I})$ with $\mathbf{I}$ the $n \times n$ identity matrix, and bound $H(U_1, U_2)$ as follows:

$$\begin{aligned} H(U_1, U_2) &= \frac{1}{n}H(U^n_1, U^n_2) \qquad (3.60)\\ &= \frac{1}{n}H(U^n_2) + \frac{1}{n}H(U^n_1|U^n_2, X^n_2)\\ &\le \frac{1}{n}H(U^n_2) + \frac{1}{n}H(U^n_1|X^n_2) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n} I(U_2^n; Y_2^n | \boldsymbol{\theta}) + \frac{1}{n} I(U_1^n; Y_1^n | X_2^n, \boldsymbol{\theta}) + \frac{1}{n} H(U_2^n | Y_2^n, \boldsymbol{\theta}) + \frac{1}{n} H(U_1^n | X_2^n, Y_1^n, \boldsymbol{\theta}) \\
&\leq \frac{1}{n} I(X_2^n; Y_2^n | \boldsymbol{\theta}) + \frac{1}{n} I(X_1^n, X_r^n; Y_1^n | X_2^n, \boldsymbol{\theta}) + \epsilon_{1n} + \epsilon_{2n} \\
&= \frac{1}{n} I(X_2^n; Y_2^n | \boldsymbol{\theta}) + \frac{1}{n} I(X_1^n, X_r^n; g_{11} e^{j\theta_{11}} X_1^n + g_{r1} e^{j\theta_{r1}} X_r^n + Z_1^n | X_2^n) + \epsilon_{1n} + \epsilon_{2n}
\end{aligned} \tag{3.61}
$$

$$
\begin{aligned}
&\overset{(a)}{=} \frac{1}{n} I(X_2^n; Y_2^n | \boldsymbol{\theta}) + \frac{1}{n} I(X_1^n, X_r^n; g_{11} e^{j\theta_{11}} X_1^n + g_{r1} e^{j\theta_{r1}} X_r^n + \alpha Z_1^n + \tilde{Z}_1^n | X_2^n) \\
&\quad + \epsilon_{1n} + \epsilon_{2n} \\
&\overset{(b)}{=} \frac{1}{n} I(X_2^n; Y_2^n | \boldsymbol{\theta}) + \frac{1}{n} I(X_1^n, X_r^n; g_{11} e^{j\theta_{11}} X_1^n + g_{r1} e^{j\theta_{r1}} X_r^n + \alpha Z_2^n + \tilde{Z}_1^n | X_2^n) \\
&\quad + \epsilon_{1n} + \epsilon_{2n} \\
&\overset{(c)}{=} \frac{1}{n} I(X_2^n; Y_2^n | \boldsymbol{\theta}) + \frac{1}{n} I(X_1^n, X_r^n; \alpha g_{12} e^{j\theta_{12}} X_1^n + \alpha g_{r2} e^{j\theta_{r2}} X_r^n + \alpha Z_2^n + \tilde{Z}_1^n | X_2^n) \\
&\quad + \epsilon_{1n} + \epsilon_{2n} \\
&\overset{(d)}{\leq} \frac{1}{n} I(X_2^n; Y_2^n | \boldsymbol{\theta}) + \frac{1}{n} I(X_1^n, X_r^n; \alpha g_{12} e^{j\theta_{12}} X_1^n + \alpha g_{r2} e^{j\theta_{r2}} X_r^n + \alpha Z_2^n | X_2^n) \\
&\quad + \epsilon_{1n} + \epsilon_{2n} \\
&\overset{(e)}{=} \frac{1}{n} I(X_2^n; Y_2^n | \boldsymbol{\theta}) + \frac{1}{n} I(X_1^n, X_r^n; Y_2^n | X_2^n, \boldsymbol{\theta}) + \epsilon_{1n} + \epsilon_{2n} \\
&= \frac{1}{n} I(X_1^n, X_2^n, X_r^n; Y_2^n | \boldsymbol{\theta}) + \epsilon_{1n} + \epsilon_{2n},
\end{aligned} \tag{3.62}
$$

where (a), (b) follows from the fact that by preserving the noise marginal distribution, the mutual information does not change. The noise term Z_1^n in (3.61) is thus divided into two independent terms $\alpha Z_1^n + \tilde{Z}_1^n$, and then Z_1^n is replaced by Z_2^n. Also, (c) follows from (3.106) and the fact that in $\boldsymbol{\Phi}_c$, $\theta_{11} = \theta_{12}$ and $\theta_{r1} = \theta_{r2}$, (d) follows since reducing the noise may only increase the mutual information, and (e) follows from the fact that linear transformation does not change mutual information.

To derive (3.58), we further upper bound (3.62) by a time sharing argument similar to the one resulted in (3.15), the fact that the upper bound is true for all $\boldsymbol{\theta} \in \boldsymbol{\Phi}_c$, and letting $n \to \infty$. $\square$

Using Lemma 1, we maximize the upper bounds of Lemma 3 with independent Gaussians and the proof of the converse part is complete.

Achievability

The achievability part is again proved by separate source-channel coding:

Source Coding: Using Slepian-Wolf coding, the source (U_1^n, U_2^n) is source coded, requiring the rates (R_1, R_2) to satisfy (3.24)-(3.26).

Channel Coding: Using block Markov coding in conjunction with backward decoding at the receivers (note: both receivers decode all messages) and forward decoding at the relay, we derive the following achievable region for a compound IRC with 2 transmitters and a relay r [38]:

$$R_i < \min\big\{I(X_i; Y_r | X_j, X_r, \boldsymbol{\theta}), I(X_i, X_r; Y_i | X_j, \boldsymbol{\theta}), I(X_i, X_r; Y_j | X_j, \boldsymbol{\theta})\big\}, \quad (i,j) \in \{(1,2),(2,1)\}, \tag{3.63}$$

$$R_1 + R_2 < \min\big\{I(X_1, X_2; Y_r | X_r, \boldsymbol{\theta}), I(X_1, X_2, X_r; Y_1 | \boldsymbol{\theta}), I(X_1, X_2, X_r; Y_2 | \boldsymbol{\theta})\big\}, \tag{3.64}$$

for some input distribution $p(x_1)p(x_2)p(x_r)$.

Computing the mutual informations in (3.63)-(3.64) for independent Gaussians $X_1 \sim \mathcal{CN}(0, P_1)$, $X_2 \sim \mathcal{CN}(0, P_2)$, $X_r \sim \mathcal{CN}(0, P_r)$, we find by (3.104)-(3.53) and (3.54) that

$$I(X_i; Y_r | X_j, X_r, \boldsymbol{\theta}) \geq I(X_i, X_r; Y_i | X_j, \boldsymbol{\theta}),$$
$$I(X_i, X_r; Y_j | X_j, \boldsymbol{\theta}) \geq I(X_i, X_r; Y_i | X_j, \boldsymbol{\theta}),$$

respectively for $(i, j) \in \{(1,2),(2,1)\}$. Also, the conditions (3.51)-(3.53) together result in

$$I(X_1, X_2; Y_r | X_r, \boldsymbol{\theta}) \geq I(X_1, X_2, X_r; Y_2 | \boldsymbol{\theta}),$$

while the condition (3.54) makes

$$I(X_1, X_2, X_r; Y_1 | \boldsymbol{\theta}) \geq I(X_1, X_2, X_r; Y_2 | \boldsymbol{\theta}).$$

Hence, due to (3.51)-(3.54), the larger terms will drop off from the constraints (3.63)-(3.64) and we may rewrite the sufficient conditions as

$$R_i \leq \log(1 + (g_{ii}^2 P_i + g_{ri}^2 P_r)/N), \quad i = 1, 2,$$
$$R_1 + R_2 \leq \log(1 + (g_{12}^2 P_1 + g_{22}^2 P_2 + g_{r2}^2 P_r)/N).$$

Thus, combining the source coding and channel coding, the achievable region is the same as the outer bound and the proof of Theorem 11 is complete.

3.4.2 PI-IC

We now consider an interference channel as a special case of an interference relay channel. For the gain conditions of strong interference, we have the following source-channel coding theorem for the PI-IC:

Corollary 4. *Necessary conditions for reliably sending arbitrarily correlated sources* (U_1, U_2) *over a PI-IC with strong interference conditions*

$$g_{11} \leq g_{12}, \tag{3.65}$$

$$g_{22} \leq g_{21}, \tag{3.66}$$

are given by

$$H(U_i|U_j) \leq \log(1 + g_{ii}^2 P_i/N), \ (i,j) \in S \tag{3.67}$$

$$H(U_1, U_2) \leq \min\{\log(1 + (g_{11}^2 P_1 + g_{21}^2 P_2)/N), \log(1 + (g_{12}^2 P_1 + g_{22}^2 P_2)/N)\}. \tag{3.68}$$

The same conditions (4.50)-(4.51) with $\leq$ replaced by $<$ describe sufficient conditions for reliable communication. □

Proof. We note that by using the strong interference conditions, in the converse, one can argue that both of the receivers can decode both of the sequences U_1^n, U_2^n (see [63] for details). Thus, U_1^n, U_2^n can both be decoded from both Y_1^n, Y_2^n. Hence, we have the intersection of two PI-MACs and the result follows from the Theorem presented in the introduction [21]. □

As a result, joint source-channel coding is not necessary under non-ergodic phase incoherence for the networks and channel gains studied in this section, and separate source-channel coding can achieve optimal performance.

3.5 Cognitive Interference Channels (CIC)

In this section[2], we consider phase incoherent cognitive Gaussian interference channels and study the lossless communication of correlated primary and secondary sources over them.

In this section, we are interested in finding conditions under which reliable communication (in the information theoretic sense) can be accomplished for

[2]The results of this section were presented at the 2012 IEEE Global Communications Conference (Globecom) [23].

1. both the primary and secondary users,
2. for the secondary user while it causes no degradation for the primary, i.e., it can still establish reliable communication using the same conditions and procedures as it did when the secondary was absent

In both cases, we find necessary and sufficient conditions to reliably send a pair of correlated sources over phase incoherent cognitive interference channels under strong interference conditions. We prove separation theorems for both phase incoherent non-causal cognitive interference channel (PI-NCIC) and causal cognitive interference channel (PI-CCIC). Furthermore, we show that by performing source coding and channel coding separately, one can asymptotically achieve the best possible performance. To the best of our knowledge, this is the first work to address lossless joint source-channel coding for cognitive interference channels.

3.5.1 Non-causal Cognitive Interference Channel (NCIC)

In this subsection, we consider a two-user interference channel with strong interference, where one of the transmitters knows the message of the other non-causally. In particular, the message set of the primary is fully available to the secondary encoder. Both of the transmitters wish to send their own messages reliably to their respective receivers. Furthermore, we assume that the phase shifts are not known to the encoders making the channel a phase asynchronous one. We refer to such a network as a phase incoherent non-causal cognitive interference channel (PI-NCIC). The setup is depicted in Fig. 3.4.

A continuous alphabet, discrete-time memoryless interference channel (IC) with phase fading is denoted by $(\mathcal{X}_1 \times \mathcal{X}_2, \mathcal{Y}_1 \times \mathcal{Y}_2, p_{\boldsymbol{\theta}_1,\boldsymbol{\theta}_2}(y_1, y_2|x_1, x_2))$ and its probabilistic character-

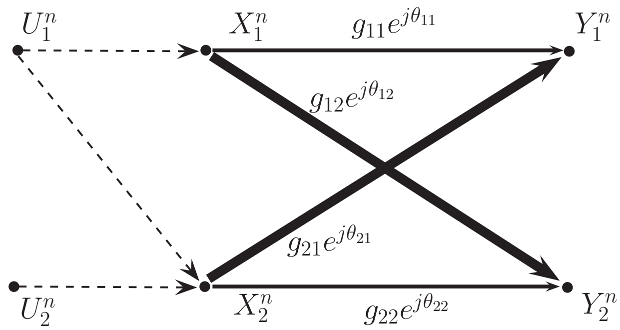

Figure 3.4: Correlated sources and phase incoherent noncausal cognitive interference channel

ization is described by the relationship

$$Y_{1i} = g_{11}e^{j\theta_{11}}X_{1i} + g_{21}e^{j\theta_{21}}X_{2i} + Z_{1i}, \tag{3.69}$$

$$Y_{2i} = g_{12}e^{j\theta_{12}}X_{1i} + g_{22}e^{j\theta_{22}}X_{2i} + Z_{2i}, \tag{3.70}$$

where $X_{1i}, X_{2i}, Y_i \in \mathbb{C}$, $Z_i \sim \mathcal{CN}(0, N)$ is circularly symmetric complex Gaussian noise, $g_{11}, g_{12}, g_{21}, g_{22}$ are non-ergodic complex channel gains, and parameters $\boldsymbol{\theta}_1 \triangleq (\theta_{11}, \theta_{21}) \in [0, 2\pi)^2$, $\boldsymbol{\theta}_2 \triangleq (\theta_{12}, \theta_{22}) \in [0, 2\pi)^2$ represent the phase shifts introduced by the channel to inputs X_1 and X_2, respectively. Also, we denote by $\boldsymbol{\theta}$ the vector $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$.

Definition 12. Code*: A block joint source-channel code of length n for the PI-NCIC with correlated sources is defined by*

1. *Two encoding functions*

$$f_1^n : \mathcal{U}_1^n \to \mathcal{X}_1^n$$

$$f_2^n : \mathcal{U}_1^n \times \mathcal{U}_2^n \to \mathcal{X}_2^n,$$

that map the source outputs to the codewords. The sets of codewords are denoted by the codebook $\mathcal{C} = \{(f_1^n(u_1^n), f_2^n(u_1^n, u_2^n)) : u_1^n \in \mathcal{U}_1^n, u_2^n \in \mathcal{U}_2^n\}$.

2. *Power constraint* P_1 *and* P_2 *on the codewords.*

3. *Two decoding functions*

$$g_1^n(\cdot|\boldsymbol{\theta}_1) : \mathcal{Y}_1^n \to \mathcal{U}_1^n, \quad g_2^n(\cdot|\boldsymbol{\theta}_2) : \mathcal{Y}_2^n \to \mathcal{U}_2^n. \tag{3.71}$$

The estimated vectors $g_1^n(Y_1^n|\boldsymbol{\theta}_1), g_2^n(Y_2^n|\boldsymbol{\theta}_2)$ *are denoted by* $\hat{U}_1^n, \hat{U}_2^n$ *respectively.*

□

Upon reception of the received vectors Y_1^n, Y_2^n, using knowledge of the channel parameter vectors $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$, the receivers find $\hat{U}_1^n = g_1^n(Y_1^n|\boldsymbol{\theta}_1)$, and $\hat{U}_2^n = g_2^n(Y_2^n|\boldsymbol{\theta}_2)$ as the transmitted source outputs respectively. Thus, the probability of an erroneous decoding is given by

$$\begin{aligned} P_{e_1}^n(\boldsymbol{\theta}_1) &= P\{U_1^n \neq \hat{U}_1^n|\boldsymbol{\theta}_1 = (\theta_{11}, \theta_{21})\} \\ &= \sum_{u_1^n \in \mathcal{U}_1^n} p(u_1^n) P\{\hat{U}_1^n \neq u_1^n \, |u_1^n, \boldsymbol{\theta}_1\}. \end{aligned} \tag{3.72}$$

$$\begin{aligned} P_{e_2}^n(\boldsymbol{\theta}_2) &= P\{U_2^n \neq \hat{U}_2^n|\boldsymbol{\theta}_2 = (\theta_{21}, \theta_{22})\} \\ &= \sum_{u_2^n \in \mathcal{U}_2^n} p(u_2^n) P\{\hat{U}_2^n \neq u_2^n \, |u_2^n, \boldsymbol{\theta}_2\}. \end{aligned} \tag{3.73}$$

Reliable communication for both users

Definition 13. *We say the source* $\{U_{1i}, U_{2i}\}_{i=1}^n$ *of i.i.d. discrete random variables with joint probability mass function* $p(u_1, u_2)$ can be reliably sent *(or is achievable) over the*

PI-NCIC, if there exists a sequence of block codes $\{f_1^n(U_1^n), f_2^n(U_1^n, U_2^n)\}$ *and decoders* $g_1^n(\cdot|\boldsymbol{\theta}_1), g_2^n(\cdot|\boldsymbol{\theta}_2)$ *such that the output sequences* U_1^n *and* U_1^n *of the source can be estimated with arbitrarily asymptotically small probability of error over all parameters* $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ *at the receiver side from the received sequences* Y_1^n, Y_1^n*, i.e.,*

$$\left[\sup_{\boldsymbol{\theta}_i} P_{ei}^n(\boldsymbol{\theta}_i)\right] \longrightarrow 0, \text{ as } n \to \infty, \quad i = 1, 2. \tag{3.74}$$

□

Herein, we find necessary and sufficient conditions under which both of the users can reliably communicate their own messages to their respective decoders in the sense of Definition 13.

Remark 6. *The following theorem and other theorems in the section can be considered as joint source-channel coding separation theorems for phase asynchronous cognitive interference channels as all of them prove the separation approach to achieve the optimal performance.*

Theorem 12. Reliable Communication over a PI-NCIC*: Consider a phase incoherent cognitive interference channel with non-causal unidirectional cooperation between the encoders and with power constraints* P_1, P_2 *on transmitters, and fading amplitudes* $g_{11}, g_{12}, g_{21}, g_{22} > 0$ *between the transmitters and the receivers. Moreover, assume the strong interference gain conditions*

$$g_{22} \le g_{21}, \tag{3.75}$$

$$g_{11} \le g_{12}, \tag{3.76}$$

hold.

A necessary condition for sending a source pair $(U_1^n, U_1^n) \sim \prod_i p(u_{1i}, u_{2i})$*, over such a PI-NCIC is given by*

$$H(U_2|U_1) \leq \log(1 + g_{22}^2 P_2/N), \tag{3.77}$$

$$\begin{aligned} H(U_1, U_2) \leq \min\{&\log(1 + (g_{12}^2 P_1 + g_{22}^2 P_2)/N), \\ &\log(1 + (g_{11}^2 P_1 + g_{21}^2 P_2)/N)\}. \end{aligned} \tag{3.78}$$

Furthermore, eqs. (3.77)-(3.78) *also give the sufficient conditions for reliable communications over such PI-NCIC with* $\leq$ *replaced by* $<$*.* □

Proof. See Appendix 3.A.1. □

Cognitive Reliable Communication

In this section, we consider the scenario in which the cognitive user wishes to reliably communicate its information while the primary can reliably communicate with its receiver whenever $H(U_1) \leq \log(1 + g_{11}^2 P_1/N)$. In particular we give the following definition for *cognitive reliable communication.*

Definition 14. *We say the cognitive source* $\{U_{2i}\}_{i=1}^n$ can be reliably sent *over the PI-NCIC, if there exists a sequence of block codes* $\{f_1^n(U_1^n), f_2^n(U_1^n, U_2^n)\}$ *and decoders* $g_1^n(\cdot|\boldsymbol{\theta}_1), g_2^n(\cdot|\boldsymbol{\theta}_2)$ *such that*

1. *the output sequence* U_2^n *can be estimated with arbitrarily asymptotically small probability of error over all parameters* $\boldsymbol{\theta}_2$ *at the secondary receiver side*
2. *if*

$$H(U_1) \leq \log(1 + g_{11}^2 P_1/N), \tag{3.79}$$

then the primary can reliably communicate with its decoder in the sense of Definition 13.

Theorem 13. Cognitive reliable communication over a PI-NCIC*: Consider a PI-NCIC with power constraints P_1, P_2 on transmitters, and strong interference gain conditions*

$$g_{22} \leq g_{21}, \tag{3.80}$$

$$g_{11} \leq g_{12}, \tag{3.81}$$

as well as the additional condition

$$\min\{1 + (g_{11}^2 P_1 + g_{21}^2 P_2)/N, 1 + (g_{12}^2 P_1 + g_{22}^2 P_2)/N\} \geq 2^{H(U_2|U_1)}(1 + g_{11}^2 P_1/N). \tag{3.82}$$

A necessary condition for reliable cognitive communication of U_2^n over such a PI-NCIC is given by

$$H(U_2|U_1) \leq \log(1 + g_{22}^2 P_2/N), \tag{3.83}$$

$$\begin{aligned} H(U_1, U_2) \leq \min\{&\log(1 + (g_{12}^2 P_1 + g_{22}^2 P_2)/N), \\ &\log(1 + (g_{11}^2 P_1 + g_{21}^2 P_2)/N)\}. \end{aligned} \tag{3.84}$$

Furthermore, eqs. (3.83)-(3.84) *also give sufficient conditions for cognitive reliable communications with $\leq$ replaced by $<$.* □

Proof. See Appendix 3.A.2 □

The condition (3.82) depends on the entropy content of the sources U_1, U_2. We now specialize Theorem 13 to the following corollary, where we only have two symmetric SNR-dependant gain conditions.

Corollary 5. Cognitive reliable communication over a PI-NCIC*: Assume the gain conditions*

$$g_{21}^2 \geq g_{22}^2(1 + g_{11}^2 P_2/N) \tag{3.85}$$

$$g_{12}^2 \geq g_{11}^2(1 + g_{22}^2 P_1/N) \tag{3.86}$$

hold for a PI-NCIC. Then a necessary condition for reliable cognitive communication of U_1^n over such a PI-NCIC is given by (3.83)*, and* (3.84)*. Furthermore, eqs.* (3.83) *and* (3.84) *also give the sufficient conditions for reliable communications over such PI-NCIC with $\leq$ replaced by $<$.* $\square$

Proof. It is straightforward to see that (3.85), (3.86) imply the conditions (3.80), (3.81) of Theorem 13. Since in the converse part of the proof of Theorem 13, we only used the strong interference conditions (3.80), (3.81), the necessity part of Corollary 5 is established. For the sufficiency part, notice that (3.85), (3.86), along with the constraint (3.83), result in the required additional condition (3.82). $\square$

3.5.2 Causal Cognitive Interference Channel (CCIC)

Now, we consider an interference channel with causal unidirectional cooperation between the encoders. As opposed to the noncausal case, the secondary encoder does not have noncausal knowledge of the primary's message. However, there is a unidirectional communication link from the primary to the secondary which can be used by the secondary in a causal manner. The setup is depicted in Fig. 3.5.

All the definitions and preliminaries are the same as those presented in Section 3.5.1. Additionally, the causal channel between the encoders is described by the relationship

$$Y_s = g_c e^{j\theta_c} X_1 + Z_s. \tag{3.87}$$

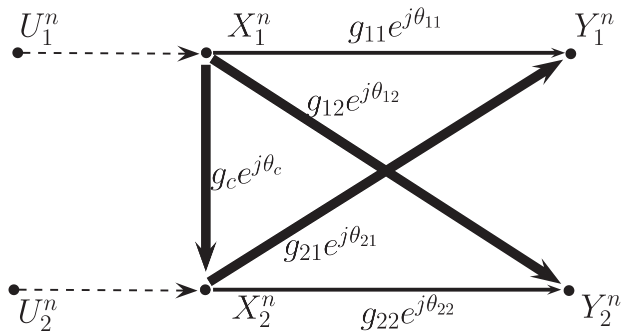

Figure 3.5: Correlated sources and phase incoherent causal cognitive interference channel

Reliable communication for both users

Theorem 14. Reliable communication over a PI-CCIC*: Consider a PI-CCIC with power constraints P_1, P_2 on transmitters, path gains $g_{11}, g_{12}, g_{21}, g_{22} > 0$ between transmitters and the receivers and $g_c > 0$ between the two transmitters. Moreover, assume the gain conditions*

$$\begin{aligned}
&g_{21} \geq g_{22},\\
&g_{12} \geq g_{11}\\
1+\frac{g_c^2P_2}{N} &\geq 2^{-H(U_2|U_1)}\min\Big\{\left(1+\frac{g_{11}^2P_1+g_{21}^2P_2}{N}\right),\\
&\left(1+\frac{g_{12}^2P_1+g_{22}^2P_2}{N}\right)\Big\}.
\end{aligned} \tag{3.88}$$

A necessary condition for reliably sending a source pair $(U_1^n, U_1^n) \sim \prod_i p(u_{1i}, u_{2i})$, over such a PI-CCIC is given by

$$H(U_2|U_1) \leq \log(1+g_{22}^2P_2/N), \tag{3.89}$$

$$H(U_1, U_2) \leq \min\{\log(1+(g_{12}^2P_1+g_{22}^2P_2)/N),$$

$$\log(1 + (g_{11}^2 P_1 + g_{21}^2 P_2)/N)\}. \tag{3.90}$$

Furthermore, eqs. (3.89)-(3.90) *also give the sufficient conditions for reliable communications with* $\leq$ *replaced by* $<$. □

Proof. See Appendix 3.A.3. □

Cognitive reliable communication

Theorem 15. Cognitive reliable communication over a PI-CCIC*: Consider a PI-CCIC and assume the gain conditions given by* (3.80)*,* (3.81)*, and* (3.82) *hold. A necessary condition for reliable cognitive communication of* U_1^n *over such a PI-CCIC is given by* (3.89)*, and* (3.90)*. Furthermore, eqs.* (3.89)-(3.90) *also give the sufficient conditions for cognitive reliable communications over such a PI-CCIC with* $\leq$ *replaced by* $<$. □

Proof. See Appendix 3.A.4. □

Corollary 6. Cognitive reliable communication over a PI-CCIC*: Consider a PI-CCIC. Assume the gain conditions* (3.85)*,* (3.86) *hold. Then, a necessary condition for reliable cognitive communication of* U_2^n *over such PI-CCIC is given by* (3.89)-(3.90)*. Furthermore, eqs.* (3.89)-(3.90) *also give the sufficient conditions for cognitive reliable communications with* $\leq$ *replaced by* $<$. □

3.6 Lossy Communication over a Phase-Incoherent Interference Relay Channel

In this section[3], we are interested in lossy joint source-channel coding for phase asynchronous wireless networks involving essential elements of wireless communications, i.e, interference and cooperation. For lossy source-channel coding over interference networks, a separation approach is shown in [64] to be optimal or approximately optimal to communicate *independent* sources. Their results, however, are based on building channel codes on top of previously existing joint source-channel codes and thus the distortion region (or inner/outer bounds on it) is not found.

Herein, we consider the problem of sending a pair of correlated Gaussian sources over a phase-fading Gaussian interference relay channel which manifests both interference and cooperation in wireless communications. The transmitters encode the continuous sources and send them over the channel while satisfying certain power constraints. We assume that the phase shifts over channels under consideration are random but fixed over the block length. Again, as a practical assumption, we assume that the phases are not known to the transmitters and the relay while the channel state information (CSI) is available to decoders. We thus refer to the channel under consideration as a phase incoherent interference relay channel (PI-IRC). At the receivers, the sources are intended to be reconstructed with the best possible minimum square error distortions. The contributions of this section are as follows:

- We first find a rectangular outer bound to the distortion region which is represented by constraints on D_1 and D_2.

[3]The results of this section were presented at the 2012 IEEE International Symposium on Information Theory (ISIT) [26].

- Under specific strong interference gain conditions, and under the extra condition of strong gains from transmitters to the relay, we find an inner bound to the distortion region represented by constraints on D_1, D_2, and $D_1 D_2$. For a fixed correlation coefficient between the sources, we show that in the high SNR regime, the constraints on D_1 and D_2 of the inner bound coincide with those of the outer bound whereas the third constraint of the inner bound shrinks a portion of the optimal achievable distortion region proportional to $\frac{1}{1-\rho^2}$ where ρ is the correlation coefficient.

- In the case of independent sources (again under strong interference conditions), we show that the inner bound exactly matches the outer bound and consequently fully characterize the achievable distortion region. Namely, we find the optimal distortion region and determine the optimality of separate source and channel coding for the phase incoherent case, as opposed to cases where the transmitters have knowledge of the phase shifts and could potentially achieve higher rates using beamforming, for example.

- Similar inner and outer bounds can be found for an interference channel with an arbitrary number of relays under phase asynchronism. Our results can also be specialized to an interference channel by omitting the relay, i.e., an inner and outer bound to the distortion region as well as the optimal joint source channel coding distortion region for independent sources are also found. For the case of no relay (interference channel), the results hold for both phase coherent and incoherent scenarios.

Although we assume non-ergodic phase shifts throughout the section, as in [21], [45, Thm. 2], our results also apply to the case where the phases change i.i.d. from symbol to symbol. Also, in this section, we focus on the strong interference regime, leaving other interference conditions as considerable future works.

3.6.1 Problem Statement

Consider a memoryless bivariate Gaussian source consisting of two zero-mean correlated Gaussian outputs (U_1, U_2) with covariance matrix

$$C_{U1,U2} = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

where $\rho \in [-1, 1]$. Both of the sources are to be transmitted to the corresponding destinations through a continuous alphabet and discrete-time memoryless non-ergodic Gaussian interference relay channel shown in Fig. 3.3. The channel is parameterized by the phase shifts that are introduced by different paths of the network and are, as a realistic assumption for wireless networks, not known to the transmitters. The vector $\boldsymbol{\theta}$ denotes the phase fading parameters. Encoders wish to use codes that are robust for all $\boldsymbol{\theta}$. In our model, the receiver is fully aware of $\boldsymbol{\theta}$. For simplicity, throughout the section, we assume that transmitter node with index $i \in \{1, 2, r\}$ has power constraint P_i and the noise power at all corresponding receiving nodes is N.

The PI-IRC $(\mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_r, \mathcal{Y}_1 \times \mathcal{Y}_2 \times \mathcal{Y}_r, p_{\boldsymbol{\theta}}(y_1, y_2, y_r | x_1, x_2, x_r))$ and all of its related definitions are presented in Section 3.4.

Definition 15. Joint source-channel code*: A joint source-channel code of length n for the PI-IRC with correlated sources is defined by*

1. *Two encoding functions*

$$f_1^n : \mathbb{R}^n \to \mathcal{X}_1^n,$$
$$f_2^n : \mathbb{R}^n \to \mathcal{X}_2^n,$$

that map the source outputs to the codewords. Furthermore, we define relay encoding functions by

$$x_{ri} = f_i(y_{r1}, y_{r2}, \cdots, y_{r(i-1)}), \quad i = 1, 2, \cdots, n.$$

2. *Power constraint P_1, P_2 and P_r at the transmitters.*

3. *Two decoding functions*

$$g^n_{\boldsymbol{\theta},1} : \mathcal{Y}^n_1 \to \mathbb{R}^n, g^n_{\boldsymbol{\theta},2} : \mathcal{Y}^n_2 \to \mathbb{R}^n. \tag{3.91}$$

The estimated vectors $g^n_{\boldsymbol{\theta},1}(Y^n_1), g^n_{\boldsymbol{\theta},2}(Y^n_2)$ are denoted by $\hat{U}^n_1, \hat{U}^n_2$ respectively.

Definition 16. *A distortion pair (D_1, D_2) is said to be* achievable *if there exists a sequence of encoding functions satisfying the corresponding power constraints and decoding functions, such that the average minimum squared error (MSE) resulting from functions satisfy*

$$\limsup_{n\to\infty} \ \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} (U_{ji} - \hat{U}_{ji})^2)\right] \ \le D_j, \ j = 1, 2.$$

3.6.2 Inner and Outer Bounds on the Distortion Region

Outer bound

Theorem 16. *Let*

$$\begin{aligned} A = \max\Big\{ & \frac{1}{[1 + (g_{11}^2 P_1 + g_{21}^2 P_2 + g_{r1}^2 P_r)/N]^2}, \\ & \frac{1-\rho^2}{[1 + (g_{11}^2 P_1 + g_{r1}^2 P_r)/N]^2}\Big\}, \\ B = \max\Big\{ & \frac{1}{[1 + (g_{12}^2 P_1 + g_{22}^2 P_2 + g_{r2}^2 P_r)/N]^2}, \end{aligned} \tag{3.92}$$

$$\frac{1-\rho^2}{[1+(g_{22}^2P_2+g_{r2}^2P_r)/N]^2}\Big\}. \tag{3.93}$$

A necessary condition for the pair (D_1, D_2) *to be achievable is given by*

$$D_1 \geq A, \quad D_2 \geq B.$$

Proof. Let $\{f_1^n(u_1^n), f_2^n(u_2^n)\}$, and $g_{1\boldsymbol{\theta}}^n, g_{2\boldsymbol{\theta}}^n$ be sequences in n of codebooks and decoders for the PI-IRC for which (D_1, D_2) is achievable. Fix a PI-IRC with given parameter $\boldsymbol{\theta}$, a codebook $\mathcal{C}$, and induced *empirical* distribution $p(u_1^n, u_2^n, x_1^n, x_2^n, x_r^n)p_{\boldsymbol{\theta}}(y_1^n, y_2^n, y_r^n|x_1^n, x_2^n, x_r^n)$. Then we have

$$\begin{aligned}
I(U_1^n, X_r^n; Y_1^n|X_2^n, \boldsymbol{\theta}) &= h(Y_1^n|X_2^n, \boldsymbol{\theta}) - h(Y_1^n|X_2^n, U_1^n, X_r^n, \boldsymbol{\theta}) \\
&\leq h(g_{11}X_1^n e^{j\theta_{11}} + g_{r1}X_r^n e^{j\theta_{r1}} + Z_1^n) - h(Z_1^n) \\
&\leq \sum_{i=1}^{n} h(g_{11}X_{1i}e^{j\theta_{11}} + g_{r1}X_{ri}e^{j\theta_{r1}} + Z_{1i}) - h(Z_1^n) \\
&\overset{\text{(a)}}{=} n\ h(g_{11}X_{1W}e^{j\theta_{11}} + g_{r1}X_{rW}e^{j\theta_{r1}} + Z_{1W}|W) - h(Z_1^n) \\
&\overset{\text{(b)}}{\leq} n\ h(g_{11}X_1e^{j\theta_{11}} + g_{r1}X_re^{j\theta_{r1}} + Z_1) - n\ h(Z_1)
\end{aligned} \tag{3.94}$$

where (a) and (b) follow by defining new random variables

$$\begin{aligned}
&W \sim \text{Uniform}\{1, 2, \cdots, n\}, \\
&X_j = X_{jW},\ j \in \{1, r\}, \\
&Z_1 = Z_{1W}.
\end{aligned} \tag{3.95}$$

Note that from (4.5), the input signals X_1, X_r satisfy the power constraints

$$\mathbb{E}|X_j|^2 = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\|X_{ji}\|^2\right] \leq P_j,\ j = 1, r,$$

and $Z_1 \sim \mathcal{CN}(0, N)$.

Since (3.94) can be derived for all values of $\boldsymbol{\theta}$, we have

$$\begin{aligned}\min_{\boldsymbol{\theta}} I(U_1^n, X_r^n; Y_1^n | X_2^n, \boldsymbol{\theta}) &\leq n \max_{p_{X_1,X_r}} \min_{\boldsymbol{\theta}} h(g_{11} X_1 e^{j\theta_{11}} + g_{r1} X_r e^{j\theta_{r1}} + Z_1) - n\ h(Z_1) \\ &\overset{(c)}{\leq} n \log\bigl(1 + (g_{11}^2 P_1 + g_{r1}^2 P_r)/N\bigr). \end{aligned} \tag{3.96}$$

where p_{X_1,X_r} is the joint distribution of (X_1, X_r) and (c) follows directly from Lemma 1.

On the other hand,

$$\begin{aligned} I(U_1^n, X_r^n; Y_1^n | X_2^n, \boldsymbol{\theta}) &\geq I(U_1^n; Y_1^n | X_2^n, \boldsymbol{\theta}) \\ &= h\bigl(U_1^n | X_2^n\bigr) - h(U_1^n | X_2^n, Y_1^n, \boldsymbol{\theta}) \\ &\geq h\bigl(U_1^n | U_2^n, X_2^n\bigr) - h\bigl(U_1^n - \hat{U_1^n}\bigr) \\ &= h\bigl(U_1^n | U_2^n\bigr) - h\bigl(U_1^n - \hat{U_1^n}\bigr). \end{aligned} \tag{3.97}$$

But (3.97) is true for all values of $\boldsymbol{\theta}$. Hence, we have

$$\liminf_{n\to\infty}\ \min_{\boldsymbol{\theta}} \frac{1}{n} I(U_1^n, X_r^n; Y_1^n | X_2^n, \boldsymbol{\theta}) \overset{(d)}{\geq} \frac{1}{2} \log \frac{(1-\rho^2)}{D_1}, \tag{3.98}$$

where (d) is a straight forward result of (3.97) following from the achievability assumption on D_1. Thus combining the lower and upper bounds (3.96), (3.98) on

$$\liminf_{n\to\infty} \min_{\boldsymbol{\theta}} I(U_1^n, X_r^n; Y_1^n | X_2^n, \boldsymbol{\theta}),$$

we have

$$\log\left(1 + \frac{g_{11}^2 P_1 + g_{r1} P_r}{N}\right) \geq \frac{1}{2} \log \frac{(1-\rho^2)}{D_1}.$$

Similarly, we derive the same inequality for D_2 and therefore two of the inequalities of Theorem 16 are established.

Now, by similar arguments and reusing Lemma 1, we derive lower and upper bounds on

$$\liminf_{n\to\infty} \min_{\boldsymbol{\theta}} \frac{1}{n} I(X_1^n, X_2^n, X_r^n; Y_1^n|\boldsymbol{\theta}),$$

and

$$\liminf_{n\to\infty} \min_{\boldsymbol{\theta}} \frac{1}{n} I(X_1^n, X_2^n, X_r^n; Y_2^n|\boldsymbol{\theta}),$$

as follows:

$$\min_{\boldsymbol{\theta}} \frac{1}{n} I(X_1^n, X_2^n, X_r^n; Y_1^n|\boldsymbol{\theta}) \leq \log\bigl(1 + (g_{11}^2 P_1 + g_{21}^2 P_2 + g_{r1}^2 P_r)/N\bigr), \tag{3.99}$$

and

$$\begin{aligned} I(X_1^n, X_2^n, X_r^n; Y_1^n) &\geq I(U_1^n; \hat{U_1^n}) \qquad (3.100)\\ &\geq h(U_1^n) - h(U_1^n - \hat{U_1^n}), \end{aligned}$$

which results in

$$\liminf_{n\to\infty} \min_{\boldsymbol{\theta}} \frac{1}{n} I(X_1^n, X_2^n, X_r^n; Y_1^n) \geq \frac{1}{2} \log \frac{1}{D_1}. \tag{3.101}$$

Combining (3.99) and (3.101), we have

$$\log\left(1 + \frac{g_{11}^2 P_1 + g_{21}^2 P_2 + g_{r1}^2 P_r}{N}\right) \geq \frac{1}{2} \log \frac{1}{D_1}.$$

The proof of the converse is complete by noting that a similar bound on D_2 can be found by following similar steps for $I(X_1^n, X_2^n, X_r^n; Y_2^n|\boldsymbol{\theta})$. □

Inner bound

The following theorem describes an inner bound on the distortion region for specific gain conditions that can be thought of as counterparts to the *very strong* interference conditions for the interference channel. In particular, by eliminating the role of the relay, i.e., setting $g_{r1} = g_{r2} = 0$, the conditions will reduce to the very strong interference conditions for the Gaussian interference channel.

Theorem 17. *Suppose the strong interference gain conditions*

$$g_{12}^2 \geq g_{11}^2(1 + g_{22}^2 P_2/N + g_{r2}^2 P_r/N) + g_{r1}^2 \frac{P_r}{P_1}(1 + g_{22}^2 \frac{P_2}{N} + g_{r2}^2 P_r/N), \tag{3.102}$$

$$g_{21}^2 \geq g_{22}^2(1 + g_{11}^2 P_1/N + g_{r1}^2 P_r/N) + g_{r2}^2 \frac{P_r}{P_2}(1 + g_{22}^2 \frac{P_1}{N} + g_{r2}^2 P_r/N), \tag{3.103}$$

as well as the encoders to relay strong gain conditions

$$g_{11}^2 P_1 + g_{r1}^2 P_r \leq g_{1r}^2 P_1, \tag{3.104}$$

$$g_{22}^2 P_2 + g_{r2}^2 P_r \leq g_{2r}^2 P_2, \tag{3.105}$$

$$\min_{i \in \{1,2\}} \left\{ g_{1i}^2 P_1 + g_{ri}^2 P_r + g_{2i}^2 P_2 \right\} \leq g_{1r}^2 P_1 + g_{2r}^2 P_2. \tag{3.106}$$

hold. An achievable distortion region for source-channel communication of (U_1, U_2) over the PI-IRC is given by

$$D_1 \geq A + \frac{\rho^2}{(1-\rho^2)^2} \cdot A \cdot B, \tag{3.107}$$

$$D_2 \geq B + \frac{\rho^2}{(1-\rho^2)^2} \cdot A \cdot B, \tag{3.108}$$

$$D_1 D_2 \geq \frac{A \cdot B}{(1-\rho^2)} + \rho^2 \frac{A^2 \cdot B^2}{(1-\rho^2)^4}. \tag{3.109}$$

Proof. To establish the achievability argument, we follow a separate source-channel coding scheme based on lossy distributed source coding and reliable channel coding for the PI-IRC where both of the receivers are forced to estimate both of the sources with respective distortions D_1, D_2. The source coding indices and channel coding rates are denoted by ω_{1i}, ω_{2i}, and R_1, R_2 respectively.

Source Coding: An inner region $\mathcal{R}_{in}(D_1, D_2)$ on the rate region $\mathcal{R}(D_1, D_2)$ of distributed source coding of two Gaussian sources for a common decoder is given in [65], [66]. The inner region can be reexpressed as the following achievable distortion region

$$\begin{aligned}\mathcal{D}_{in}(D_1, D_2) = \Bigg\{&(D_1, D_2) : \\ &D_1 \geq (1-\rho^2)2^{-2R_1} + \rho^2 2^{-2(R_1+R_2)}, \\ &D_2 \geq (1-\rho^2)2^{-2R_2} + \rho^2 2^{-2(R_1+R_2)}, \\ \frac{2D_1D_2}{1+\sqrt{1+\frac{4\rho^2 D_1 D_2}{(1-\rho^2)^2}}} &\geq (1-\rho^2)2^{-2(R_1+R_2)}\Bigg\}.\end{aligned} \tag{3.110}$$

Channel Coding: Using block Markov coding in conjunction with backward decoding at the receivers and forward decoding at the relay, as shown in Table 3.3, we can derive the following sufficient conditions to reliably decode all messages at both decoders for a compound IRC [22]:

$$R_1 < \min\Big\{I(X_1; Y_r|X_2, X_r, \boldsymbol{\theta}), I(X_1, X_r; Y_1|X_2, \boldsymbol{\theta}), I(X_1, X_r; Y_2|X_2, \boldsymbol{\theta})\Big\}, \tag{3.111}$$

$$R_2 < \min\Big\{I(X_2; Y_r|X_1, X_r, \boldsymbol{\theta}), I(X_2, X_r; Y_1|X_1, \boldsymbol{\theta}), I(X_2, X_r; Y_2|X_1, \boldsymbol{\theta})\Big\}, \tag{3.112}$$

$$R_1 + R_2 < \min\Big\{I(X_1, X_2; Y_r|X_r, \boldsymbol{\theta}), I(X_1, X_2, X_r; Y_1|\boldsymbol{\theta}), I(X_1, X_2, X_r; Y_2|\boldsymbol{\theta})\Big\}, \tag{3.113}$$

for some input distribution $p(x_1)p(x_2)p(x_r)$.

For independent Gaussians $X_1 \sim \mathcal{CN}(0, P_1)$, $X_2 \sim \mathcal{CN}(0, P_2)$, $X_r \sim \mathcal{CN}(0, P_r)$, the conditions (3.104)-(3.106) make the first terms of (3.111)-(3.113) larger than the others and hence we can drop them from the constraints. Also due to the strong interference conditions of (3.102) and (3.103), we can drop the mutual information terms

Encoder	Block 1	Block 2	Block B	Block $B+1$
1	$x_1^n(1,\omega_{11})$	$x_1^n(\omega_{11},\omega_{12})$	$x_1^n(\omega_{1(B-1)},\omega_{1B})$	$x_1^n(\omega_{1B},1)$
2	$x_2^n(1,\omega_{21})$	$x_2^n(\omega_{21},\omega_{22})$	$x_2^n(\omega_{2(B-1)},\omega_{2B})$	$x_2^n(\omega_{2B},1)$
r	$x_r^n(1,1)$	$x_r^n(\omega_{11},\omega_{21})$	$x_r^n(\omega_{1(B-1)},\omega_{2(B-1)})$	$x_r^n(\omega_{1B},\omega_{2B})$

Table 3.3: Block Markov encoding scheme for an IRC.

$I(X_1, X_r; Y_2|X_2, \boldsymbol{\theta}), I(X_2, X_r; Y_1|X_1, \boldsymbol{\theta})$ from (3.111) and (3.112). Hence, for such independent Gaussians X_1, X_2, X_r, the sufficient conditions reduce to

$$R_1 \leq \log(1 + (g_{11}^2 P_1 + g_{r1}^2 P_r)/N), \tag{3.114}$$

$$R_2 \leq \log(1 + (g_{22}^2 P_2 + g_{r2}^2 P_r)/N), \tag{3.115}$$

$$R_1 + R_2 \leq \min\Big\{ \log(1 + (g_{12}^2 P_1 + g_{22}^2 P_2 + g_{r2}^2 P_r)/N), \log(1 + (g_{11}^2 P_1 + g_{21}^2 P_2 + g_{r1}^2 P_r)/N)\Big\}. \tag{3.116}$$

By choosing R_1 and R_2 as

$$R_1 = \min\big\{ \log(1 + (g_{11}^2 P_1 + g_{r1}^2 P_r)/N), \log\big[\sqrt{1-\rho^2}(1 + (g_{11}^2 P_1 + g_{r1}^2 P_r + g_{21}^2 P_2)/N)\big]\big\} \tag{3.117}$$

$$R_2 = \min\big\{ \log(1 + (g_{22}^2 P_2 + g_{r2}^2 P_r)/N), \log\big[\sqrt{1-\rho^2}(1 + (g_{12}^2 P_1 + g_{r2}^2 P_r + g_{22}^2 P_2)/N)\big]\big\}, \tag{3.118}$$

and by the strong interference conditions (3.102) and (3.103), the constraints (3.114)-(3.116) are satisfied and we are guaranteed to have reliable decodings of both the first and second channel encoders codewords. Furthermore, by replacing R_1, R_2, as chosen in (3.117) and (3.118), in (3.110), one obtains the achievable distortion region given by (3.107)-(3.109). $\square$

Remark 7. *A similar outer bound and inner bound can be derived for a phase incoherent interference channel with multiple relays.*

Remark 8. *Note that if there are no relays in the network, for deriving the outer bound in Section 3.6.2, Lemma 1 is not needed since there is only one phase in* (3.94) *then. Therefore, as opposed to the case of an interference relay channel, for an interference channel, the results carry on to the scenario in which the encoders are aware of the phase shifts as well. Namely, by removing relay dependant terms and redefining A and B in* (3.92) *and* (3.93)*, Theorem 16 applies to a general Gaussian interference channel. Theorem 17 can also be specialized to an interference channel with equations* (3.102)*,* (3.103) *replaced by* very strong *interference conditions*

$$g_{12}^2 \geq g_{11}^2(1 + g_{22}^2 P_2/N), \tag{3.119}$$

$$g_{21}^2 \geq g_{22}^2(1 + g_{11}^2 P_1/N). \tag{3.120}$$

Approximate inner bound

In the moderate to high SNR regime, i.e., when the noise power N is relatively small, the second terms in the right hand sides of (3.107)-(3.109) will be negligible compared to the first terms. The inner bound can thus be approximately (or exactly in the limit) described by $\{D_1 \geq A, D_2 \geq B, D_1 D_2 \geq \frac{AB}{1-\rho^2}\}$. Therefore, the constraints on D_1 and D_2 coincide in both the inner region and outer region. The third constraint on $D_1 D_2$ makes the inner region restricted by a curve and results in a ρ-dependant gap between the regions. This can be inferred as approximate optimality of separation for values of the correlation coefficient that have small magnitude. A typical example is depicted in Fig. 3.6, where the approximate inner bound practically matches the actual inner bound. We also see that for $\rho = 0$, the regions exactly coincide for all SNR regimes.

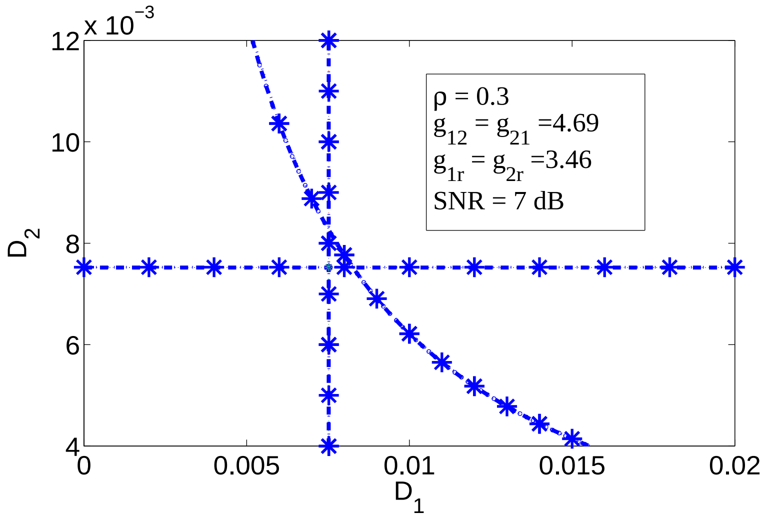

Figure 3.6: An achievable distortion region with $\rho = 0.3$, $P_1 = P_2 = P_r = 5, N = 1$. All channel gains (except for g_{12}, g_{21}, g_{1r}, g_{2r}) are 1. The exact inner bound is sketched with starred lines whereas dashed lines depict the approximate inner bound.

3.6.3 Optimal Distortion Regions

For the special case of $\rho = 0$, the inner bound of Theorem 17 will coincide with the outer bound given in Theorem 16. Therefore, we can fully characterize the optimal distortion regions for the case of independent Gaussians and state the following separation theorem as a corollary.

Corollary 7. *Provided the gain conditions (3.102)-(3.106) are met, the set of all achievable distortion pairs (D_1, D_2) for a PI-IRC with $\rho = 0$ is given by*

$$D_1 \geq \frac{1}{\left[1 + (g_{11}^2 P_1 + g_{r1}^2 P_r)/N\right]^2},$$
$$D_2 \geq \frac{1}{\left[1 + (g_{22}^2 P_1 + g_{r2}^2 P_r)/N\right]^2}.$$

Furthermore, in order to achieve this distortion region, it is sufficient to perform lossy source coding and channel coding separately.

A similar corollary can be stated for an interference channel:

Corollary 8. *Provided the gain conditions* (3.119), *and* (3.120) *are met, the set of all achievable distortion pairs* (D_1, D_2) *for an interference channel, with a pair of independent Gaussian sources, is given by*

$$D_1 \geq \frac{1}{\left[1 + g_{11}^2 P_1/N\right]^2}, \quad D_2 \geq \frac{1}{\left[1 + g_{22}^2 P_1/N\right]^2}.$$

Furthermore, in order to achieve this distortion region, it is enough to perform lossy source coding and channel coding separately.

3.A Appendices

3.A.1 Proof of Theorem 12

The proof of the theorem is divided into two parts: achievability and converse. The achievability part is obtained by a separate source and channel coding approach. The source coding part involves Slepian-Wolf coding followed by channel coding based on random coding arguments. In the channel coding part, the secondary receiver (Y_2) decodes both of the source coded indices making a multiple access channel from the encoders to the second receiver. The converse and achievability parts of Theorem 12 are discussed and proved in the sequel.

Converse

We derive an outer bound on the entropy content of U_1, U_2 for the PI-NCIC under strong interference gain conditions (3.75), (3.76) and prove the converse part of Theorem 12. The proof also applies similarly to the converse part of separation Theorem 14 for a PI-CCIC.

Lemma 4. Converse*: Let* f_1^n, f_2^n*, and* $g_1^n(\cdot|\boldsymbol{\theta}_1)$*,* $g_2^n(\cdot|\boldsymbol{\theta}_2)$ *be a sequence in* n *of encoders and decoders for the PI-NCIC for which* $\sup_{\boldsymbol{\theta}_i} P_{ei}^n(\boldsymbol{\theta}_i) \longrightarrow 0$*, as* $n \to \infty$ *for* $i = 1, 2$*. Then*

$$H(U_2|U_1) \leq \min_{\boldsymbol{\theta}_2} I(X_2; g_{22}e^{j\theta_{22}}X_2 + Z_2), \tag{3.121}$$

$$H(U_1, U_2) \leq \min_{\boldsymbol{\theta}_1} I(X_1, X_2; g_{11}e^{j\theta_{11}}X_1 + g_{21}e^{j\theta_{21}}X_2 + Z_1), \tag{3.122}$$

$$H(U_1, U_2) \leq \min_{\boldsymbol{\theta}_2} I(X_1, X_2; g_{12}e^{j\theta_{12}}X_1 + g_{22}e^{j\theta_{22}}X_2 + Z_2), \tag{3.123}$$

for some joint *distribution* p_{X_1,X_2} *such that* $\mathbb{E}|X_1|^2 \leq P_1, \mathbb{E}|X_2|^2 \leq P_2$*, with* $Z_1, Z_2 \sim \mathcal{CN}(0, N)$*.* $\square$

Proof. First, fix a PI-NCIC with given parameters $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$, a codebook $\mathcal{C}$, and induced *empirical* distribution $p_{\boldsymbol{\theta}}(u_1^n, u_2^n, x_1^n, x_2^n, y_1^n, y_2^n)$ by the codebook. Note that by using the strong interference conditions of (3.75) and (3.76), availability of phase parameters $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$ at both receivers, and following the fact that there are encoders and decoders such that each receiver can reliably decode its *own* source sequence, one can argue that both of the receivers can decode both of the sequences U_1^n, U_2^n (see [63] for details). Therefore, from Fano's inequality, we have

$$\frac{1}{n} H(U_i^n | Y_i^n, \boldsymbol{\theta}_i) \leq \frac{1}{n} P_{ei}^n(\boldsymbol{\theta}_i) \log \|\mathcal{U}_i^n\| + \frac{1}{n} \triangleq \epsilon_{in}(\boldsymbol{\theta}_i), \tag{3.124}$$

and $\epsilon_{in}(\boldsymbol{\theta}_i) \to 0$, where convergence is uniform in $\boldsymbol{\theta}_i$ by (4.6) for $i = 1, 2$. Defining $\sup_{\boldsymbol{\theta}_i} \epsilon_{in}(\boldsymbol{\theta}_i) = \epsilon_{in}$ and following the similar steps as in [19, Section 4], we have

$$\begin{aligned}
H(U_2|U_1) &= \frac{1}{n} H(U_2^n | U_1^n) \\
&\overset{\text{(a)}}{=} \frac{1}{n} H(U_2^n | U_1^n, X_1^n) \\
&= \frac{1}{n} I(U_2^n; Y_2^n | U_1^n, X_1^n, \boldsymbol{\theta}_2) + \frac{1}{n} H(U_1^n | U_1^n, Y_2^n, X_1^n, \boldsymbol{\theta}_2) \\
&= \frac{1}{n} I(U_2^n; Y_2^n | U_1^n, X_1^n, \boldsymbol{\theta}_2) + \frac{1}{n} H(U_1^n | Y_2^n, \boldsymbol{\theta}_2) \\
&\overset{\text{(b)}}{\leq} \frac{1}{n} I(U_2^n; Y_2^n | U_1^n, X_1^n, \boldsymbol{\theta}_2) + \epsilon_{2n} \\
&\overset{\text{(c)}}{\leq} \frac{1}{n} I(X_2^n; Y_2^n | U_1^n, X_1^n, \boldsymbol{\theta}_2) + \epsilon_{2n},
\end{aligned} \tag{3.125}$$

where (a) follows from the fact that X_1^n is only a function of U_1^n, (b) follows from (3.124), and (c) follows from data processing inequality. It can also be shown that

$$\begin{aligned}
H(U_1, U_2) &= \frac{1}{n} I(U_1^n, U_2^n; Y_1^n | \boldsymbol{\theta}_1) + \frac{1}{n} H(U_1^n, U_2^n | Y_1^n, \boldsymbol{\theta}_1) \\
&= \frac{1}{n} I(U_1^n, U_2^n; Y_1^n | \boldsymbol{\theta}_1) + \frac{1}{n} H(U_2^n | Y_1^n, U_1^n, \boldsymbol{\theta}_1) + \frac{1}{n} H(U_1^n | Y_1^n, \boldsymbol{\theta}_1) \\
&\leq \frac{1}{n} I(X_1^n, X_2^n; Y_1^n | \boldsymbol{\theta}_1) + \frac{1}{n} H(U_2^n | Y_1^n, U_1^n, \boldsymbol{\theta}_1) + \epsilon_{1n},
\end{aligned} \tag{3.126}$$

and similarly,

$$H(U_1, U_2) \leq \frac{1}{n} I(X_1^n, X_2^n; Y_2^n | \boldsymbol{\theta}_2) + \frac{1}{n} H(U_1^n | Y_2^n, U_2^n, \boldsymbol{\theta}_2) + \epsilon_{2n}. \tag{3.127}$$

We now define the region $C_n(\boldsymbol{\theta})$ as

$$\begin{aligned} C_n(\boldsymbol{\theta}) = \Bigg\{ &(R_1, R_2) : R_1 < \frac{1}{n}\ I(X_2^n; Y_2^n | U_1^n, X_1^n, \boldsymbol{\theta}_2) + \epsilon_{2n}, \\ &R_2 < \frac{1}{n}\ I(X_1^n, X_2^n; Y_1^n | \boldsymbol{\theta}_1) + \frac{1}{n} H(U_1^n | Y_1^n, U_2^n, \boldsymbol{\theta}_1) + \epsilon_{1n}, \\ &R_2 < \frac{1}{n}\ I(X_1^n, X_2^n; Y_2^n | \boldsymbol{\theta}_2) + \frac{1}{n} H(U_1^n | Y_2^n, U_1^n, \boldsymbol{\theta}_2) + \epsilon_{2n} \Bigg\}, \end{aligned} \tag{3.128}$$

for the empirical distribution induced by the nth codebook

$$\prod_{i=1}^{n} p(u_{1i}, u_{2i})\ p(x_1^n | u_1^n) p(x_2^n | u_1^n, u_2^n) \prod_{i=1}^{n} p_{\boldsymbol{\theta}}(y_{1i}, y_{2i} | x_{1i}, x_{2i}).$$

Hence, the outer bounds (3.125) and (3.126) can be equivalently described by $C_n(\boldsymbol{\theta})$:

$$(H(U_2|U_1), H(U_1, U_2)) \in C_n(\boldsymbol{\theta}).$$

We then note that the outer bound $C_n(\boldsymbol{\theta})$ on $(H(U_2|U_1), H(U_1, U_2))$ applies for all $\boldsymbol{\theta}$ and thus can be tightened by taking the intersection over $\boldsymbol{\theta}$ and letting $n \to \infty$.

First, we expand Y_2^n in the right hand side of (3.125) to upper bound $H(U_2|U_1)$ as follows:

$$\begin{aligned} H(U_2|U_1) &\leq \frac{1}{n}\ I(X_2^n; Y_2^n | U_1^n, X_1^n, \boldsymbol{\theta}_2) + \epsilon_{2n} \\ &= \frac{1}{n}\ I(X_2^n; g_{12} e^{j\theta_{12}} X_1^n + g_{22} e^{j\theta_{22}} X_2^n + Z_2^n | U_1^n, X_1^n) + \epsilon_{2n} \\ &= \frac{1}{n}\ I(X_2^n; g_{22} e^{j\theta_{22}} X_2^n + Z_2^n | U_1^n, X_1^n) + \epsilon_{2n} \\ &= \frac{1}{n}\ \left[h(g_{22} e^{j\theta_{22}} X_2^n + Z_2^n | U_1^n, X_1^n) - h(Z_2^n) \right] + \epsilon_{2n} \end{aligned}$$

$$\leq \frac{1}{n}\left[h(g_{22}e^{j\theta_{22}}X_2^n + Z_2^n) - h(Z_2^n)\right] + \epsilon_{2n}$$
$$\leq \frac{1}{n}\sum_{i=1}^{n} h(g_{22}e^{j\theta_{22}}X_{2i} + Z_{2i}) - h(Z_{2i}) + \epsilon_{2n}$$
$$= \left[h(g_{22}e^{j\theta_{22}}X_2 + Z_2|W) - h(Z_2)\right] + \epsilon_{2n}$$
$$\leq \left[h(g_{22}e^{j\theta_{22}}X_2 + Z_2) - h(Z_2)\right] + \epsilon_{2n}$$
$$= \ I(X_2; g_{22}e^{j\theta_{22}}X_2 + Z_2) + \epsilon_{2n}, \tag{3.129}$$

where (a) follows by defining new random variables

$$X_j = X_{jW},\ j \in \{1,2\}, \tag{3.130}$$
$$Z_j = Z_W,\ j \in \{1,2\} \tag{3.131}$$
$$W \sim \text{Uniform}\{1,2,\cdots,n\}. \tag{3.132}$$

From (4.5), the input signal X_2 satisfies the power constraint

$$\mathbb{E}|X_2|^2 = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\|X_{2i}\|^2\right] \leq P_2, \tag{3.133}$$

and $Z_2 \sim \mathcal{CN}(0,N)$. Since (3.129) is true for all $\boldsymbol{\theta}_2$, by minimizing over all $\boldsymbol{\theta}_2$, (3.121) follows.

Afterwards, we follow similar steps for (3.126), (3.127). The terms $\frac{1}{n}H(U_2^n|Y_1^n, U_2^n, \boldsymbol{\theta}_1)$, and $\frac{1}{n}H(U_1^n|Y_2^n, U_2^n, \boldsymbol{\theta}_2)$ can be upper bounded by ϵ_{1n}, and ϵ_{2n} respectively and thus be vanished when $n \to \infty$. This is due to the fact that although only $\boldsymbol{\theta}_i$ is available at Y_i^n, but for the case of $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$, Y_1^n (Y_2^n) can reliably decode U_2^n (U_1^n). However, taking minimum over all $\boldsymbol{\theta}$ subject to the constraint $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$, does not change the results as it can only loosen the upper bounds. Hence, we have

$$H(U_1,U_2) \leq I(X_1,X_2; g_{11}e^{j\theta_{11}}X_1 + g_{21}e^{j\theta_{21}}X_2 + Z_1) + 2\epsilon_{1n}, \tag{3.134}$$

$$H(U_1, U_2) \leq I(X_1, X_2; g_{12}e^{j\theta_{12}}X_1 + g_{22}e^{j\theta_{22}}X_2 + Z_2) + 2\epsilon_{2n}. \tag{3.135}$$

The constraints defined by (3.129), (3.134), (3.135) form an outer bound on $C_n(\boldsymbol{\theta})$. But since it applies for a fixed $\boldsymbol{\theta}$, it is also true for all choices of $\boldsymbol{\theta}$. By taking infimum over all $\boldsymbol{\theta}$ and letting $n \to \infty$, the proof of Lemma 4 is complete.

□

To prove the converse part of Theorem 12, we note by Lemma 1 that each of the bounds of Lemma 4 are simultaneously maximized by independent Gaussians. The proof of the converse is complete.

Remark 9. *Note that to prove the converse part of the Theorem 12, we do not need the receivers to know the CSI $\boldsymbol{\theta}$. This is indeed true for the converse parts of all theorems in this chapter.* □

Achievability

Source Coding: Recall that the secondary encoder has non-causal access to the primary source U_1^n. From Slepian-Wolf coding [30], for asymptotically lossless representation of the "source pair" $((U_2^n, U_1^n), U_1^n)$, we should have the rates (R_1, R_2) satisfying

$$R_2 > H(U_2|U_1), \tag{3.136}$$

$$R_1 + R_2 > H(U_2, U_1). \tag{3.137}$$

The source codes are represented by indices W_1, W_2 which are then channel coded before being transmitted.

Channel Coding: We use random coding arguments to establish an achievable region for the CIC. First fix a distribution $p(x_1)p(x_2)$ and construct random codewords x_1^n, x_2^n

based on the corresponding distributions. In particular, the encoding and decoding parts are as follows.

Encoding: The primary encoder generates 2^{nR_1} random codewords $x_1^n(W_1)$ according to the distribution $\prod_{i=1}^{n} p(x_{1i})$. Also, using the knowledge of W_1, the secondary encoder generates $2^{n(R_1+R_2)}$ random codewords $x_2^n(W_1, W_2)$ according to the distribution $\prod_{i=1}^{n} p(x_{2i})$.

Decoding: We assume $\boldsymbol{\theta}_1$, and $\boldsymbol{\theta}_2$ are known to the first and second decoders respectively.

At the end of the block, the first decoder decodes $\hat{W}_1, \hat{W}_2$ by finding codewords

$$X_1^n(\hat{W}_1), X_2^n(\hat{W}_1, \hat{W}_2)$$

such that

$$(Y_1^n, X_1^n(\hat{W}_1), X_2^n(\hat{W}_1, \hat{W}_2))$$

is jointly typical. By classical random coding arguments and error probability computations, one needs to have

$$R_1 < I(X_1, X_2; Y_1|\boldsymbol{\theta}_1), \tag{3.138}$$

$$R_1 + R_2 < I(X_1, X_2; Y_1|\boldsymbol{\theta}_1) \tag{3.139}$$

in order to reliably decode W_1. Note that W_2 may not be decoded correctly at the primary decoder.

By applying a similar jointly typical decoding procedure, the secondary claims $\hat{\hat{W}}_1, \hat{\hat{W}}_2$ as decoded messages. To decode W_2 correctly, we need the constraints

$$R_2 < I(X_2; Y_2|X_1, \boldsymbol{\theta}_2), \tag{3.140}$$

$$R_1 + R_2 < I(X_2, X_1; Y_2|\boldsymbol{\theta}_2), \tag{3.141}$$

which imply that both the messages should be decoded reliably at the secondary decoder, making a MAC from the encoders to the second decoder.

Combining all of the constraints, for any distribution $p(x_1)p(x_2)$, we can describe the corresponding achievable region by

$$R_2 < I(X_2; Y_2 | X_1, \boldsymbol{\theta}_2), \tag{3.142}$$

$$R_1 + R_2 < I(X_2, X_1; Y_2 | \boldsymbol{\theta}_2), \tag{3.143}$$

$$R_1 + R_2 < I(X_2, X_1; Y_1 | \boldsymbol{\theta}_1). \tag{3.144}$$

The achievability part of Theorem 12 is complete by first choosing X_1, and X_2, as independent Gaussians.

3.A.2 Proof of Theorem 13

Converse

The necessity part of Theorem 13 is a direct part of Theorem 12. Namely, if there exist sequences of secondary encoders and decoders such that the cognitive reliable communication is established in the sense of Definition 14, then under condition (3.79), there exist sequences of primary encoders and decoders for reliable communication for the primary user. Therefore, by the converse part of Theorem 12, and under strong interference conditions (3.80), (3.81), eqs. (3.83) and (3.84) hold.

Achievability

We follow the same separation approach, source coding, and channel coding procedures as in the achievability proof of Theorem 12 in Section 3.A.1. Given eqs. (3.83) and (3.84),

Encoder	Block 1	Block 2	Block B	Block $B+1$
1	$x_1^n(1, W_{11})$	$x_1^n(W_{11}, W_{12})$	$x_1^n(W_{1(B-1)}, W_{1B})$	$x_1^n(W_{1B}, 1)$
2	$x_2^n(1, W_{21}, 1)$	$x_2^n(W_{21}, W_{22}, W_{11})$	$x_2^n(W_{2(B-1)}, W_{2B}, W_{1(B-1)})$	$x_2^n(W_{2B}, 1, W_{1B})$

Table 3.4: Block Markov encoding scheme for the PI-CCIC.

by choosing $R_2 = H(U_2|U_1)$, and $R_2 + R_1 = H(U_1, U_2)$, the source coding part is complete from (3.136) and (3.137). For the channel coding part, by noting that all rates $H(U_1) = R_1$ satisfying the condition (3.79) should be achieved at the primary, one can see that (3.82) assures that the choices of R_1, R_2 satisfy the constraints (3.142)–(3.144).

3.A.3 Proof of Theorem 14

Converse

The converse is the same as the converse part of Theorem 12 presented in Section 3.A.1.

Achievability

Again, the separation approach is adopted. The source coding part is the same as Section 3.A.1. For the channel coding part, the secondary plays the role of a full-duplex relay and causally decodes the primary message. In particular, by the use of block Markov coding along with backward decoding [22], [45], the secondary can causally acquire knowledge about the message of the primary. The block Markov coding scheme performed in B blocks is shown in Table 3.4. By letting $B \to \infty$ and simple random coding arguments, the achievable rates can be shown to be given by

$$R_2 < I(X_2; Y_2|X_1, \boldsymbol{\theta}_2), \tag{3.145}$$

$$R_1 < I(X_1; Y_s | X_2, \theta_c), \tag{3.146}$$

$$R_1 + R_2 < I(X_2, X_1; Y_2 | \boldsymbol{\theta}_2), \tag{3.147}$$

$$R_1 + R_2 < I(X_2, X_1; Y_1 | \boldsymbol{\theta}_1), \tag{3.148}$$

for some distribution $p(x_1)p(x_2)$. By choosing Gaussian X_1, X_2 and applying the condition (3.88), both source coding and channel coding constraints are satisfied.

3.A.4 Proof of Theorem 15

Converse

The necessity part of Theorem 15 is a direct result of Theorem 14.

Achievability

The proof is similar to the proof of the achievability part of Theorem 13 in Section 3.A.2. The only difference is that we follow the same channel coding procedures as in the achievability proof of Theorem 14 in Section 3.A.3. In particular, the channel coding part is based on the block Markov coding with backward decoding presented in Section 3.A.3.

Chapter 4

Time Asynchronous Systems

4.1 Introduction

In this chapter[1], we study the communication of K correlated sources over a time-asynchronous MARC (TA-MARC) where the encoders cannot synchronize the starting times of their codewords. Rather, they transmit with unknown positive time delays $d_1, d_2, \cdots, d_{K+1} \geq 0$ with respect to a time reference. The time shifts are also bounded by $d_\ell \leq \mathsf{d}_{\mathsf{max}}(n)$, $\ell = 1, \cdots, K+1$, where n is the codeword block length. Moreover, we assume that the offsets $d_1, d_2, \cdots, d_{K+1}$ are unknown to the transmitters as a practical assumption since they are not controlled by the transmitters. We further assume that the maximum possible offset $\mathsf{d}_{\mathsf{max}}(n) \to \infty$ as $n \to \infty$ while $\mathsf{d}_{\mathsf{max}}(n)/n \to 0$.

In [67], we have considered a two user time asynchronous Gaussian MAC with a pair of correlated sources. There, we have derived necessary and sufficient conditions for reliable communication and consequently derived a separation theorem for the problem. This

[1]The results of this chapter are submitted to the IEEE Transactions on Information Theory.

chapter extends the work of [67] to a more general setup with K nodes and a relay. In particular, in this chapter, we first derive general necessary conditions for reliable communication and then derive matching sufficient conditions under specific gain conditions. Furthermore, we have defined a robust notion of reliable communication in which the system should have vanishing error probabilities for all possible values of offsets $d_1, \cdots, d_{K+1}$ between transmitters.

It is shown that separate source-channel coding is optimal and the encoders can first perform source coding and then perform channel coding with *independent* inputs, with no loss. Specifically, the encoders have no way to exploit the correlation of the sources to increase the capacity region. If they plan to correlate the transmitted codewords, since they are not aware of of the offset values $d_1, \cdots, d_{K+1}$, they need to correlate them for all possible offsets (otherwise for some choices of offsets, they do not achieve any beamforming gain). But making the transmitted codewords cross correlated under all possible values of offsets imply that the code letters in a single transmitted codeword should also be correlated in time. However, the correlated codewords in time carry little information and as the transmitters encode their corresponding sources, the transmitted codewords can in turn communicate little information about the sources. Hence, there is a tradeoff between the gain in information reliably sent about the sources and the information loss due to the correlation in time needed to accomplish this. As our analysis effectively shows, the tradeoff is optimized when there is no attempt at correlating the transmissions, i.e., there is no attempt at beamforming, and thus separate source-channel coding is optimal.

For other multiuser networks such as interference channels and broadcast channels, JSCC capacity results under phase asynchronism can be found in [23], [26]. Also, the recent work [68] considers the point-to-point state-dependent and cognitive multiple access channels with time asynchronous side information.

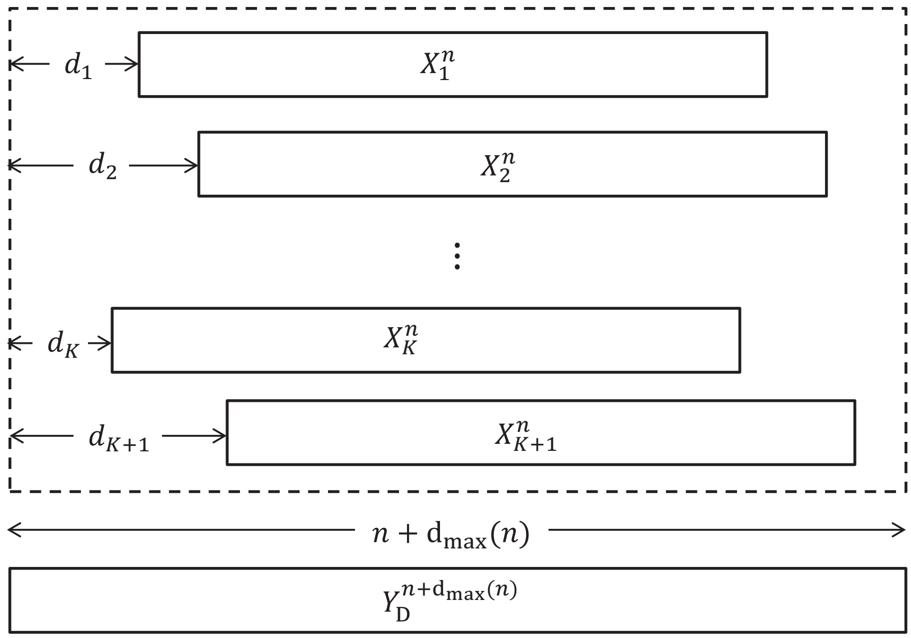

Figure 4.1: Gaussian time asynchronous multiple access relay channel (TA-MARC), with delays $d_1, \cdots, d_{K+1}$.

4.2 Problem Statement and a Useful Lemma

Notations: In what follows, we denote random variables by upper case letters, e.g., X, their realizations by lower case letters, e.g., x, and their alphabet by calligraphic letters, e.g., $\mathcal{X}$. For integers $0 \leq a \leq b$, Y_a^b denotes the $b-a+1$-tuple $(Y[a], \cdots, Y[b])$, and Y^b is a shorthand for Y_0^{b-1}. Without confusion, X_ℓ^n denotes the length-n MARC input codeword $(X_\ell[0], \cdots, X_\ell[n-1])$ of the ℓth transmitter, and based on this, we also denote $(X_\ell[a], \cdots, X_\ell[b])$ by $X_{\ell,a}^b$. The n-length discrete Fourier transforms (DFT) of the n-length codeword X_ℓ^n is denoted by $\hat{X}_\ell^n = \text{DFT}(X_\ell^n)$. Furthermore, let $[1, K] \triangleq \{1, \cdots, K\}$, for $\forall K \in \mathbb{N}$.

Consider K finite alphabet sources $\{(U_1[i], U_2[i], \cdots, U_K[i])\}_{i=0}^{\infty}$ as correlated random variables drawn according to a distribution $p(u_1, u_2, \cdots, u_K)$. The sources are memoryless, i.e., $(U_1[i], U_2[i], \cdots, U_K[i])$'s are independent and identically distributed (i.i.d) for $i = 1, 2, \cdots$. The indices $1, \cdots, K$ represent the transmitter nodes and the index $K+1$ represents the relay transmitter. All of the sources are to be transmitted to a destination by the help of a relay through a continuous alphabet, discrete-time memoryless multiple-access relay channel (MARC) with time asynchronism between different transmitters and the relay. Specifically, as depicted in Fig. 4.1, the encoders use different time references and thus we assume that the encoders start transmitting with offsets of

$$0 \leq d_\ell \leq \mathsf{d}_{\mathsf{max}}(n), \quad \ell = 1, \cdots, K+1, \tag{4.1}$$

symbols with respect to a fixed time reference, where d_{K+1} is the offset for the relay transmitter with respect to the time reference.

Hence, the probabilistic characterization of the time-asynchronous Gaussian MARC, referred to as a Gaussian TA-MARC and denoted by $\mathcal{M}([1, K+1])$ throughout the section, is described by the relationships

$$Y_{\mathsf{D}}[i] = \sum_{\ell=1}^{K+1} g_{\ell\mathsf{D}} X_\ell[i - d_\ell] + Z_{\mathsf{D}}[i], \quad i = 0, 1, \cdots, n + \mathsf{d}_{\mathsf{max}}(n) - 1, \tag{4.2}$$

as the ith entry of the received vector $Y_{\mathsf{D}}^{n+\mathsf{d}_{\mathsf{max}}(n)}$ at the destination (D), and

$$Y_{\mathsf{R}}[i] = \sum_{\ell=1}^{K} g_{\ell\mathsf{R}} X_\ell[i - d_\ell] + Z_{\mathsf{R}}[i], \quad i = 0, 1, \cdots, n + \mathsf{d}_{\mathsf{max}}(n) - 1, \tag{4.3}$$

as the ith entry of the received vector $Y_{\mathsf{R}}^{n+\mathsf{d}_{\mathsf{max}}(n)}$ at the relay (R), where

- $g_{\ell\mathsf{D}}, \ell = 1, \cdots, K+1$, are complex gains from transmission nodes as well as the relay (when $\ell = K+1$) to the destination, and $g_{\ell\mathsf{R}}, \ell = 1, \cdots, K$, are complex gains from the transmission nodes to the relay,
- $X_\ell[i - d_\ell], \ell = 1, \cdots, K+1$, are the delayed channel inputs such that $X_\ell[i - d_\ell] = 0$ if $(i - d_\ell) \notin \{0, 1, \cdots, n-1\}$ and $X_\ell[i - d_\ell] \in \mathbb{C}$ otherwise,
- $Z_{\mathsf{D}}[i], Z_{\mathsf{R}}[i] \sim \mathcal{CN}(0, N)$ are circularly symmetric complex Gaussian noises at the destination and relay, respectively.

Fig. 4.1 depicts the delayed codewords of the encoders, and the formation of the received codeword for the TA-MARC.

We now define a joint source-channel code and the notion of reliable communication for a Gaussian TA-MARC in the sequel.

Definition 17. *A block joint source-channel code of length n for the Gaussian TA-MARC with the block of correlated source outputs*

$$\{(U_1[i], U_2[i], \cdots, U_K[i])\}_{i=0}^{n-1}$$

is defined by

1. *A set of encoding functions with the bandwidth mismatch factor of unity*[2], *i.e.,*

$$f_\ell^n : \mathcal{U}_\ell^n \to \mathbb{C}^n, \quad \ell = 1, 2, \cdots, K,$$

that map the source outputs to the codewords, and the relay encoding function

$$x_{(K+1)}^{i+1} = f_{(K+1)}^{i+1}(y_{\mathsf{R}}[0], y_{\mathsf{R}}[1], \cdots, y_{\mathsf{R}}[i]), \quad i = 0, 2, \cdots, n-2. \tag{4.4}$$

The sets of encoding functions are denoted by the codebook $\mathcal{C}^n = \left\{ f_1^n, \cdots, f_K^n, \{f_{(K+1)}^{i+1}\}_{i=0}^{n-2} \right\}$.

2. *Power constraints* P_ℓ, $\ell = 1, \cdots, K+1$, *on the codeword vectors* X_ℓ^n, *i.e.,*

$$\mathbb{E}\left[\frac{1}{n}\sum_{i=0}^{n-1} |X_\ell[i]|^2\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=0}^{n-1} |\hat{X}_\ell[i]|^2\right] \leq P_\ell, \tag{4.5}$$

for $\ell = 1, \cdots, K+1$ *where we recall that* $\hat{X}_\ell^n = DFT\{X_\ell^n\}$, *and* $\mathbb{E}[\cdot]$ *represents the expectation operator.*

3. *A decoding function* $g^n(y_{\mathsf{D}}^{n+\mathsf{d}_{\max}} | d_1^{K+1}) : \mathbb{C}^{n+\mathsf{d}_{\max}} \times [0, \mathsf{d}_{\max}]^{K+1} \to \mathcal{U}_1^n \times \cdots \times \mathcal{U}_K^n$.

Definition 18. *We say the source* $\{(U_1[i], U_2[i], \cdots, U_K[i])\}_{i=0}^{n-1}$ *of i.i.d. discrete random variables with joint probability mass function* $p(u_1, u_2, \cdots, u_K)$ can be reliably sent *over a Gaussian TA-MARC, if there exists a sequence of codebooks* $\mathcal{C}^n$ *and decoders* g^n *in* n *such that the output sequences* $U_1^n, U_2^n, \cdots, U_K^n$ *of the source can be estimated from* $Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}(n)}$ *with arbitrarily asymptotically small probability of error uniformly over* all *choices of delays* $0 \leq d_\ell \leq \mathsf{d}_{\max}(n)$, $\ell = 1, \cdots, K+1$, *i.e.,*

$$\sup_{0 \leq d_1, \cdots, d_{K+1} \leq \mathsf{d}_{\max}(n)} P_e^n(d_1^{K+1}) \longrightarrow 0, \quad \text{as} \quad n \to \infty, \tag{4.6}$$

[2] The assumption of unity mismatch factor is without loss of generality and for simplicity of exposition. Extension to the more general setting with different mismatch factors can be achieved simply by modifying the final result with a constant factor (cf. Remark 11).

where

$$P_e^n(d_1^{K+1}) \triangleq P[g(Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}(n)}|d_1^{K+1}) \neq (U_1^n, U_2^n, \cdots, U_K^n)|d_1^{K+1}], \tag{4.7}$$

is the error probability for a given set of offsets d_1^{K+1}. □

We now present a key lemma that plays an important role in the derivation of our results. In order to state the lemma, we first need to define the notions of a *sliced* MARC and a *sliced cyclic* MARC as follows:

Definition 19. *Let* $\mathcal{S} \subseteq [1, K+1]$ *be a subset of transmitter node indices. A Gaussian sliced MARC* $\mathcal{M}(\mathcal{S})$ *corresponding to the Gaussian TA-MARC* $\mathcal{M}([1, K+1])$ *defined by* (4.2)-(4.3)*, is a MARC in which only the codewords of the encoders with indices in* $\mathcal{S}$ *contribute to the destination's received signal, while the received signal at the relay is the same as that of the original Gaussian TA-MARC* $\mathcal{M}([1, K+1])$*.*

In particular, for the Gaussian sliced MARC $\mathcal{M}(\mathcal{S})$*, the received signals at the destination and the relay at the* i*th time index, denoted by* $Y_{\mathsf{D}(\mathcal{S})}[i]$ *and* $Y_{\mathsf{R}(\mathcal{S})}[i]$ *respectively, are given by*

$$Y_{\mathsf{D}(\mathcal{S})}[i] = \sum_{\ell \in \mathcal{S}} g_{\ell \mathsf{D}} X_\ell[i - d_\ell] + Z_{\mathsf{D}}[i], \quad i = 0, \cdots, n + \mathsf{d}_{\max} - 1, \tag{4.8}$$

and

$$Y_{\mathsf{R}(\mathcal{S})}[i] = Y_{\mathsf{R}}[i], \quad i = 0, \cdots, n + \mathsf{d}_{\max} - 1. \tag{4.9}$$

Definition 20. *A sliced cyclic MARC* $\widetilde{\mathcal{M}}(\mathcal{S})$*, corresponding to the sliced TA-MARC* $\mathcal{M}(\mathcal{S})$ *defined by* (4.8)-(4.9)*, is a sliced TA-MARC in which the codewords are cyclicly shifted around the* n*th time index to form new received signals at the destination only. Specifically,*

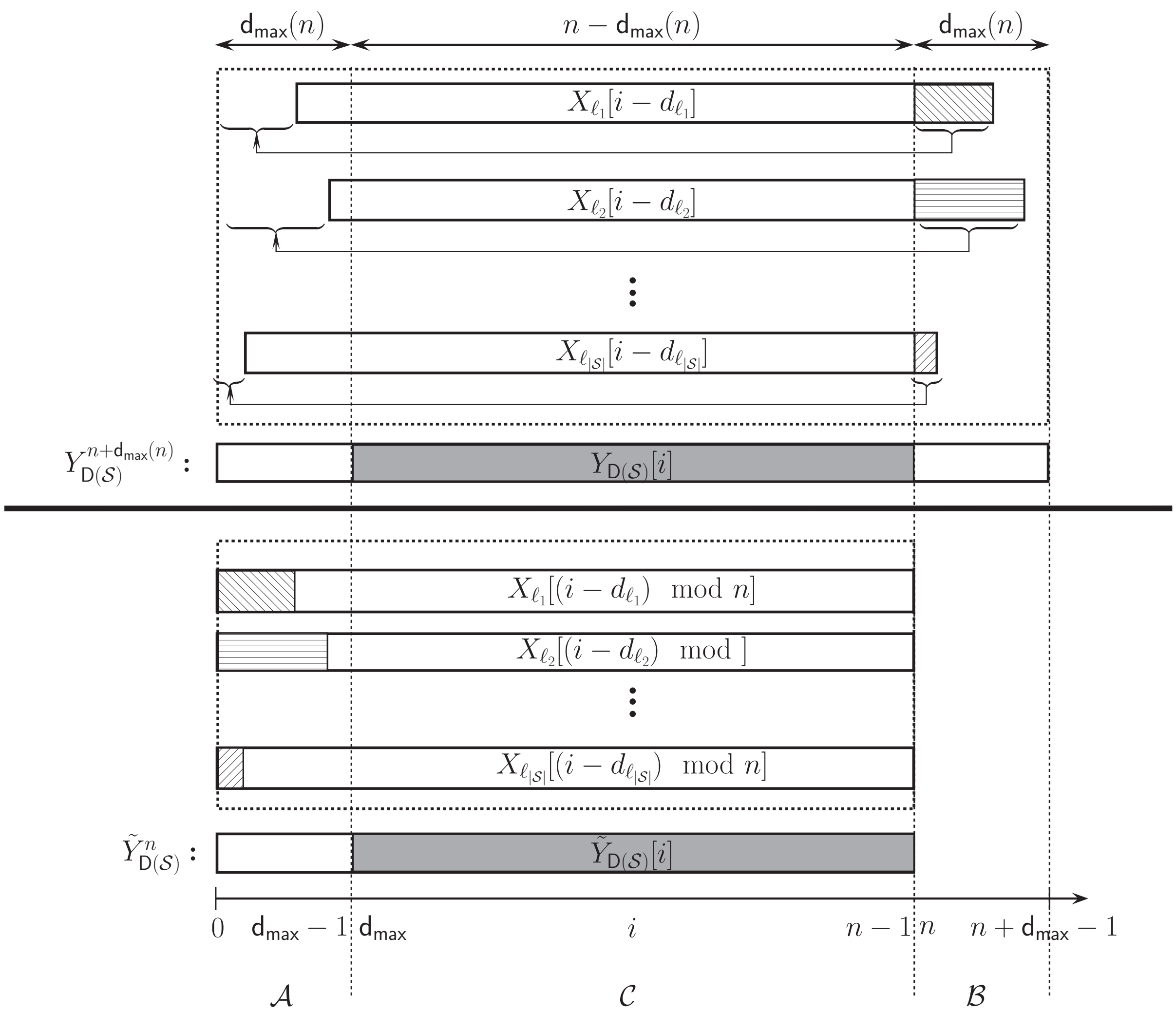

Figure 4.2: Codewords of a Gaussian sliced TA-MARC $\mathcal{M}(\mathcal{S})$ (top) and the corresponding sliced cyclic MARC $\tilde{\mathcal{M}}(\mathcal{S})$ (bottom).

the corresponding outputs of the sliced cyclic MARC $\widetilde{\mathcal{M}}(\mathcal{S})$ *at the destination and the relay at the ith time index, denoted by* $\tilde{Y}_{\mathsf{D}(\mathcal{S})}[i]$ *and* $\tilde{Y}_{\mathsf{R}(\mathcal{S})}[i]$ *respectively, can be written as*

$$\tilde{Y}_{\mathsf{D}(\mathcal{S})}[i] = \sum_{\ell \in \mathcal{S}} g_{\ell \mathsf{D}} X_\ell[(i - d_\ell) \mod n] + Z_{\mathsf{D}}[i], \quad i = 0, \cdots, n-1, \tag{4.10}$$

and

$$\begin{aligned} \tilde{Y}_{\mathsf{R}(\mathcal{S})}[i] &= \sum_{\ell=1}^{K} g_{\ell \mathsf{R}} X_\ell[i - d_\ell] + Z_{\mathsf{R}}[i], \quad i = 0, \cdots, n-1 \\ &= Y_{\mathsf{R}}[i]. \end{aligned} \tag{4.11}$$

In particular, as shown in Fig. 4.2, the tail of the codewords are cyclicly shifted to the beginning of the block, where the start point of the block is aligned with the first time instant. The destination's output $\tilde{Y}^n_{\mathsf{D}(\mathcal{S})}$ *of the sliced cyclic MARC is the n-tuple that results by adding the shifted versions of the codewords* $X^n_\ell, \ell \in \mathcal{S}$. *As indicated in Fig. 4.2, we divide the entire time interval* $[0, n + \mathsf{d}_{\max} - 1]$ *into three subintervals* $\mathcal{A}, \mathcal{B}$, *and* $\mathcal{C}$ *where*

- $\mathcal{A}$ *is the sub-interval representing the left tail of the received codeword, i.e.,* $[0, \mathsf{d}_{\max} - 1]$,
- $\mathcal{B}$ *represents the right tail, i.e.,* $[n, n + \mathsf{d}_{\max} - 1]$,
- $\mathcal{C}$ *represents a common part between the sliced TA-MARC and sliced cyclic MARC, i.e.,* $[\mathsf{d}_{\max}, n - 1]$.

Remark 10. *In both sliced TA-MARC and sliced cyclic MARC, the observation* $Y_{\mathsf{R}}^{n+\mathsf{d}_{\max}}$ *of the relay remains unchanged. Therefore, the generated channel input at the relay* X^n_{K+1} *is the same as the original TA-MARC due to (4.4) when the same relay encoding functions are used.*

The following lemma implies that, for every choice of $\mathcal{S} \subseteq [1, K+1]$, the mutual information rate between the inputs and the destination's output in the Gaussian sliced TA-MARC $\mathcal{M}(\mathcal{S})$ and the sliced cyclic MARC $\widetilde{\mathcal{M}}(\mathcal{S})$ are asymptotically the same, i.e., their difference asymptotically vanishes. This fact will be useful in the analysis of the problem in Section 4.3, where we can replace a sliced TA-MARC with the corresponding sliced cyclic MARC.

Before stating and proving the useful lemma, we define the following notations:

$$Y_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] \triangleq \{Y_{\mathsf{D}(\mathcal{S})}[i] : i \in \mathcal{A}\}, \tag{4.12}$$

$$\tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] \triangleq \{\tilde{Y}_{\mathsf{D}(\mathcal{S})}[i] : i \in \mathcal{A}\}, \tag{4.13}$$

$$X_{\mathcal{S}}^n \triangleq \{X_\ell^n : \ell \in \mathcal{S}\}, \tag{4.14}$$

$$\vec{X}_{\mathcal{S}}[\mathcal{A}] \triangleq \{X_\ell[i - d_\ell] : \ell \in \mathcal{S}, i \in \mathcal{A}\}, \tag{4.15}$$

$$\tilde{\vec{X}}_{\mathcal{S}}[\mathcal{A}] \triangleq \{X_\ell[i - d_\ell \bmod n] : \ell \in \mathcal{S}, i \in \mathcal{A}\}, \tag{4.16}$$

where $\mathcal{S} \subseteq [1, K+1]$ is an arbitrary subset of transmitter nodes indices. Similarly, we can define $Y_{\mathsf{D}(\mathcal{S})}[\mathcal{B}]$, $Y_{\mathsf{D}(\mathcal{S})}[\mathcal{C}]$, $\tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{B}], \cdots$, by replacing $\mathcal{A}$ with $\mathcal{B}$ or $\mathcal{C}$ in the above definitions.

Lemma 5. *For a Gaussian sliced TA-MARC $\mathcal{M}(\mathcal{S})$, and the corresponding sliced cyclic MARC $\widetilde{\mathcal{M}}(\mathcal{S})$,*

$$\frac{1}{n}\left| I(X_{\mathcal{S}}^n; Y_{\mathsf{D}(\mathcal{S})}^{n+\mathsf{d}_{\mathsf{max}}} | d_1^{K+1}) - I(X_{\mathcal{S}}^n; \tilde{Y}_{\mathsf{D}(\mathcal{S})}^n | d_1^{K+1}) \right| \leq \epsilon_n, \quad \forall\ d_1^{K+1} \in [0, \mathsf{d}_{\mathsf{max}}(n)]^{K+1}, \tag{4.17}$$

for all $\mathcal{S} \subseteq [1, K+1]$, where ϵ_n is independent of d_1^{K+1} and $\epsilon_n \to 0$, as $n \to \infty$. □

Proof. Noting that the mutual information between subsets of two random vectors is a lower bound on the mutual information between the original random vectors, we first lower bound the original mutual information $I(X_{\mathcal{S}}^n; Y_{\mathsf{D}(\mathcal{S})}^{n+\mathsf{d}_{\mathsf{max}}} | d_1^{K+1})$:

$$I(\vec{X}_{\mathcal{S}}[\mathcal{C}]; Y_{\mathsf{D}(\mathcal{S})}[\mathcal{C}] | d_1^{K+1}) \leq I(X_{\mathcal{S}}^n; Y_{\mathsf{D}(\mathcal{S})}^{n+\mathsf{d}_{\mathsf{max}}} | d_1^{K+1}). \tag{4.18}$$

Then, by splitting the entropy terms over the intervals $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ as depicted in Fig. 4.2, we upper bound the same mutual information term $I(X^n_{\mathcal{S}}; Y^{n+\mathsf{d}_{\max}}_{\mathsf{D}(\mathcal{S})} | d_1^{K+1})$ as follows:

$$\begin{aligned}
I(X^n_{\mathcal{S}}; Y^{n+\mathsf{d}_{\max}}_{\mathsf{D}(\mathcal{S})} | d_1^{K+1}) &= h(Y^{n+\mathsf{d}_{\max}}_{\mathsf{D}(\mathcal{S})} | d_1^{K+1}) - h(Y^{n+\mathsf{d}_{\max}}_{\mathsf{D}(\mathcal{S})} | X^n_{\mathcal{S}}, d_1^{K+1}) \\
&\leq h(Y_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) + h(Y_{\mathsf{D}(\mathcal{S})}[\mathcal{B}] | d_1^{K+1}) + h(Y_{\mathsf{D}(\mathcal{S})}[\mathcal{C}] | d_1^{K+1}) \\
&\quad - \sum_{i=0}^{n+\mathsf{d}_{\max}-1} h(Z_{\mathsf{D}}[i]) \\
&= I(\vec{X}_{\mathcal{S}}[\mathcal{A}]; Y_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) + I(\vec{X}_{\mathcal{S}}[\mathcal{B}]; Y_{\mathsf{D}(\mathcal{S})}[\mathcal{B}] | d_1^{K+1}) \\
&\quad + I(\vec{X}_{\mathcal{S}}[\mathcal{C}]; Y_{\mathsf{D}(\mathcal{S})}[\mathcal{C}] | d_1^{K+1}).
\end{aligned} \tag{4.19}$$

Also, the mutual information term $I(X^n_{\mathcal{S}}; \tilde{Y}^n_{\mathsf{D}(\mathcal{S})} | d_1^{K+1})$ which is associated to the cyclic MARC can be similarly lower bounded as

$$I(\tilde{\vec{X}}_{\mathcal{S}}[\mathcal{C}]; \tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{C}] | d_1^{K+1}) \leq I(X^n_{\mathcal{S}}; \tilde{Y}^n_{\mathsf{D}(\mathcal{S})} | d_1^{K+1}), \tag{4.20}$$

and upper bounded as

$$\begin{aligned}
I(X^n_{\mathcal{S}}; \tilde{Y}_{\mathsf{D}(\mathcal{S})} | d_1^{K+1}) &= h(\tilde{Y}_{\mathsf{D}(\mathcal{S})} | d_1^{K+1}) - h(\tilde{Y}_{\mathsf{D}(\mathcal{S})} | X^n_{\mathcal{S}}, d_1^{K+1}) \\
&\leq h(\tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) + h(\tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{C}] | d_1^{K+1}) - \sum_{i=0}^{n-1} h(Z_{\mathsf{D}}[i]) \\
&= I(\tilde{\vec{X}}_{\mathcal{S}}[\mathcal{A}]; \tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) + I(\tilde{\vec{X}}_{\mathcal{S}}[\mathcal{C}]; \tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{C}] | d_1^{K+1}) \\
&= I(\tilde{\vec{X}}_{\mathcal{S}}[\mathcal{A}]; \tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) + I(\vec{X}_{\mathcal{S}}[\mathcal{C}]; Y_{\mathsf{D}(\mathcal{S})}[\mathcal{C}] | d_1^{K+1}),
\end{aligned} \tag{4.21}$$

where in the last step, we used the fact that for any $\mathcal{S} \subseteq [1, K+1]$, $\tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{C}] = Y_{\mathsf{D}(\mathcal{S})}[\mathcal{C}]$ and $\tilde{\vec{X}}_{\mathcal{S}}[\mathcal{C}] = \vec{X}_{\mathcal{S}}[\mathcal{C}]$, as there is no cyclic foldover for $i \in \mathcal{C}$.

Hence, combining (4.18)-(4.19), and (4.20)-(4.21), we can now bound the difference between the mutual information terms as

$$\frac{1}{n} \left| I(X^n_{\mathcal{S}}; Y^{n+\mathsf{d}_{\max}}_{\mathsf{D}(\mathcal{S})} | d_1^{K+1}) - I(X^n_{\mathcal{S}}; \tilde{Y}^n_{\mathsf{D}(\mathcal{S})} | d_1^{K+1}) \right|$$

$$\leq \frac{1}{n} I(\vec{X}_{\mathcal{S}}[\mathcal{A}]; Y_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) + \frac{1}{n} I(\vec{X}_{\mathcal{S}}[\mathcal{B}]; Y_{\mathsf{D}(\mathcal{S})}[\mathcal{B}] | d_1^{K+1}) + \frac{1}{n} I(\tilde{\vec{X}}_{\mathcal{S}}[\mathcal{A}]; \tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}). \tag{4.22}$$

But all of the terms in the right hand side of (4.22) can also be bounded as follows. Consider the first term:

$$\begin{aligned}
\frac{1}{n} I(\vec{X}_{\mathcal{S}}[\mathcal{A}]; Y_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) &= \frac{1}{n} \left[h(Y_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) - h(Z_{\mathsf{D}}[\mathcal{A}]) \right] \\
&\leq \frac{1}{n} \sum_{i \in \mathcal{A}} \left[h(Y_{\mathsf{D}(\mathcal{S})}[i] | d_1^{K+1}) - h(Z_{\mathsf{D}}[i]) \right] \\
&\leq \frac{1}{n} \sum_{i \in \mathcal{A}} \left[h\left(\sum_{\ell \in \mathcal{S}} g_{\ell\mathsf{D}} X_\ell[i - d_\ell] + Z_{\mathsf{D}}[i] \right) - h(Z_{\mathsf{D}}[i]) \right] \\
&\overset{(a)}{\leq} \frac{1}{n} \sum_{i \in \mathcal{A}} \log\left(1 + \frac{\mathbb{E} \left| \sum_{\ell \in \mathcal{S}} g_{\ell\mathsf{D}} X_\ell[i - d_\ell] \right|^2}{N} \right) \\
&\overset{(b)}{\leq} \frac{1}{n} \sum_{i \in \mathcal{A}} \log\left(1 + \frac{\sum_{\ell \in \mathcal{S}} |g_{\ell\mathsf{D}}|^2 \cdot \sum_{\ell \in \mathcal{S}} \mathbb{E}|X_\ell[i - d_\ell]|^2}{N} \right) \\
&\overset{(c)}{\leq} \frac{|\mathcal{A}|}{n} \log\left(1 + \frac{\sum_{i \in \mathcal{A}} \left[\sum_{\ell \in \mathcal{S}} |g_{\ell\mathsf{D}}|^2 \cdot \sum_{\ell \in \mathcal{S}} \mathbb{E}|X_\ell[i - d_\ell]|^2 \right]}{|\mathcal{A}| N} \right) \\
&\overset{(d)}{=} \frac{\mathsf{d}_{\max}}{n} \log\left(1 + \frac{\sum_{\ell \in \mathcal{S}} |g_{\ell\mathsf{D}}|^2 \cdot \sum_{\ell \in \mathcal{S}} \mathbb{E} \left[\sum_{i \in \mathcal{A}} |X_\ell[i - d_\ell]|^2 \right]}{\mathsf{d}_{\max} N} \right) \\
&\leq \frac{\mathsf{d}_{\max}}{n} \log\left(1 + \frac{\sum_{\ell \in \mathcal{S}} |g_{\ell\mathsf{D}}|^2 \cdot \sum_{\ell \in \mathcal{S}} \mathbb{E} \sum_{i=0}^{n-1} |X_{\ell i}|^2}{\mathsf{d}_{\max} N} \right) \\
&\overset{(e)}{\leq} \frac{\mathsf{d}_{\max}}{n} \log\left(1 + \frac{n}{\mathsf{d}_{\max}} \frac{\sum_{\ell \in \mathcal{S}} |g_{\ell\mathsf{D}}|^2 \cdot \sum_{\ell \in \mathcal{S}} P_\ell}{N} \right) \\
&\triangleq \gamma\left(\frac{\mathsf{d}_{\max}}{n} \right),
\end{aligned} \tag{4.23}$$

where (a) follows by the fact that Gaussian distribution maximizes the differential entropy [29, Thm. 8.4.1], (b) follows from the Cauchy-Schwartz inequality:

$$\left| \sum_{\ell \in \mathcal{S}} g_{\ell\mathsf{D}} X_\ell[i - d_\ell] \right|^2 \leq \left(\sum_{\ell \in \mathcal{S}} |g_{\ell\mathsf{D}}|^2 \right) \left(\sum_{\ell \in \mathcal{S}} |X_\ell[i - d_\ell]|^2 \right), \tag{4.24}$$

(c) follows from concavity of the log function, (d) follows from the fact that $|\mathcal{A}| = \mathsf{d}_{\mathsf{max}}$, and (e) follows from the power constraint in (4.5).

Similarly, for the second term in the right hand side of (4.22), it can be shown that

$$\frac{1}{n} I(\vec{X}_{\mathcal{S}}[\mathcal{B}]; Y_{\mathsf{D}(\mathcal{S})}[\mathcal{B}] | d_1^{K+1}) \leq \gamma\left(\frac{\mathsf{d}_{\mathsf{max}}}{n}\right). \tag{4.25}$$

Following similar steps that resulted in (4.23), we now upper bound the third term in the right hand side of (4.22) as follows

$$\begin{aligned}
\frac{1}{n} I(\tilde{\vec{X}}_{\mathcal{S}}[\mathcal{A}]; \tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) &= \frac{1}{n}\left[h(\tilde{Y}_{\mathsf{D}(\mathcal{S})}[\mathcal{A}] | d_1^{K+1}) - h(Z_{\mathsf{D}}[\mathcal{A}])\right] \\
&\leq \frac{1}{n}\sum_{i\in\mathcal{A}}\left[h(\tilde{Y}_{\mathsf{D}}[i] | d_1^{K+1}) - h(Z_{\mathsf{D}}[i])\right] \\
&= \frac{1}{n}\sum_{i\in\mathcal{A}}\left[h\left(\sum_{\ell\in\mathcal{S}} g_{\ell\mathsf{D}} X_\ell[(i-d_\ell) \mod n] + Z_{\mathsf{D}}[i] \middle| d_1^{K+1}\right) - h(Z_{\mathsf{D}}[i])\right] \\
&\leq \frac{1}{n}\sum_{i\in\mathcal{A}} \log\left(1 + \frac{\mathbb{E}\left|\sum_{\ell\in\mathcal{S}} g_{\ell\mathsf{D}} X_\ell[(i-d_\ell) \mod n]\right|^2}{N}\right) \\
&\leq \frac{\mathsf{d}_{\mathsf{max}}}{n}\log\left(1 + \frac{n}{\mathsf{d}_{\mathsf{max}}}\frac{\sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2 \cdot \sum_{\ell\in\mathcal{S}} P_\ell}{N}\right) \\
&= \gamma\left(\frac{\mathsf{d}_{\mathsf{max}}}{n}\right).
\end{aligned} \tag{4.26}$$

Based on (4.23), (4.25), and (4.26), the absolute difference between the mutual informations in (4.17) is upper bounded by $3\gamma(\mathsf{d}_{\mathsf{max}}/n)$. One can see that $3\gamma\left(\mathsf{d}_{\mathsf{max}}(n)/n\right) \to 0$ as $n \to \infty$, since for any $a > 0$, $z_n \log(1 + a/z_n) \to 0$ as $z_n \to 0$, and the lemma is proved by taking $z_n = \mathsf{d}_{\mathsf{max}}(n)/n$ and $a = \sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2 \sum_{\ell\in\mathcal{S}} P_\ell/N$.

$\square$

4.3 Converse

Lemma 6. *Consider a Gaussian TA-MARC with power constraints* $P_1, P_2, \cdots, P_K$ *on the transmitters, and the power constraint* P_{K+1} *on the relay, and the set of encoders' offsets* d_1^{K+1}*. Moreover, assume that the set of offsets* d_1^{K+1} *are known to the receiver,* $\mathsf{d}_{\max}(n) \to \infty$*, and* $\mathsf{d}_{\max}(n)/n \to 0$ *as* $n \to \infty$*. Then, a necessary condition for reliably communicating a source tuple* $(U_1^n, U_2^n, \cdots, U_K^n) \sim \prod_{i=0}^{n-1} p(u_1[i], u_2[i], \cdots, u_K[i])$*, over such a Gaussian TA-MARC, in the sense of Definition 18, is given by*

$$H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) \leq \log\left(1 + \frac{\sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2 P_\ell}{N}\right), \quad \forall \mathcal{S} \subseteq [1, K+1] \tag{4.27}$$

where $\mathcal{S}$ *includes the relay, i.e.,* $\{K+1\} \in \mathcal{S}$*, where by definition* $U_{K+1} \triangleq \emptyset$*, and* $\mathcal{S}^c \triangleq [1, K+1]/\{\mathcal{S}\}$*.* □

Remark 11. *The result of (4.27) can be readily extended to the case of mapping blocks of source outputs of the average length of* m *to channel inputs of the average length of* n*. In particular, for the average bandwidth mismatch factor of* $\kappa \triangleq n/m$*, the converse result in (4.27), to be proved as an achievability result in Section 4.4 as well, can be generalized to*

$$H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) \leq \kappa \log\left(1 + \frac{\sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2 P_\ell}{N}\right), \quad \forall \mathcal{S} \subseteq [1, K+1]. \tag{4.28}$$

Since considering a general mismatch factor $\kappa > 0$ *obscures the proof, in the following we only present the proof for the case of* $\kappa = 1$*.*

Proof. First, fix a TA-MARC with given offset vector d_1^{K+1}, a codebook $\mathcal{C}^n$, and induced *empirical* distribution

$$p(u_1^n, \cdots, u_K^n, x_1^n, \cdots, x_{K+1}^n, y_{\mathsf{R}}^{n+\mathsf{d}_{\max}}, y_{\mathsf{D}}^{n+\mathsf{d}_{\max}} | d_1^{K+1}).$$

Since for this fixed choice of the offset vector d_1^{K+1}, $P_e^n(d_1^{K+1}) \to 0$, from Fano's inequality, we have

$$\frac{1}{n}H(U_1^n, U_2^n, \cdots, U_K^n | Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}}, d_1^{K+1}) \le \frac{1}{n} P_e^n(d_1^{K+1}) \log \|\mathcal{U}_1^n \times \mathcal{U}_2^n \times \cdots \times \mathcal{U}_K^n\| + \frac{1}{n} \triangleq \delta_n, \tag{4.29}$$

and $\delta_n \to 0$, where convergence is uniform in d_1^{K+1} by (4.6).

Now, we can upper bound $H(U_{\mathcal{S}}|U_{\mathcal{S}^c})$ as follows:

$$\begin{aligned}
H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) &= \frac{1}{n}H(U_{\mathcal{S}}^n|U_{\mathcal{S}^c}^n, d_1^{K+1}) \\
&\overset{\text{(a)}}{=} \frac{1}{n}H(U_{\mathcal{S}}^n|U_{\mathcal{S}^c}^n, X_{\mathcal{S}^c}^n, d_1^{K+1}) \\
&= \frac{1}{n}I(U_{\mathcal{S}}^n; Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}}|U_{\mathcal{S}^c}^n, X_{\mathcal{S}^c}^n, d_1^{K+1}) + \frac{1}{n}H(U_{\mathcal{S}}^n|Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}}, U_{\mathcal{S}^c}^n, X_{\mathcal{S}^c}^n, d_1^{K+1}) \\
&\overset{\text{(b)}}{\le} \frac{1}{n}I(X_{\mathcal{S}}^n; Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}}|U_{\mathcal{S}^c}^n, X_{\mathcal{S}^c}^n, d_1^{K+1}) + \delta_n \\
&\overset{\text{(c)}}{=} \frac{1}{n}h(Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}}|U_{\mathcal{S}^c}^n, X_{\mathcal{S}^c}^n, d_1^{K+1}) - \frac{1}{n}h(Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}}|U_{\mathcal{S}^c}^n, X_{[1,K+1]}^n, d_1^{K+1}) + \delta_n \\
&\overset{\text{(d)}}{\le} \frac{1}{n}h(Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}}|X_{\mathcal{S}^c}^n, d_1^{K+1}) - \frac{1}{n}h(Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}}|U_{\mathcal{S}^c}^n, X_{[1,K+1]}^n, d_1^{K+1}) + \delta_n \\
&= \frac{1}{n}h(\{\sum_{\ell=1}^{K+1} g_{\ell\mathsf{D}} X_\ell[i-d_\ell] + Z_{\mathsf{D}}[i]\}_{i=0}^{n+\mathsf{d}_{\max}-1}|X_{\mathcal{S}^c}^n, d_1^{K+1}) - \frac{1}{n}h(Z_{\mathsf{D}}^{n+\mathsf{d}_{\max}}) + \delta_n \\
&= \frac{1}{n}h(\{\sum_{\ell\in\mathcal{S}} g_{\ell\mathsf{D}} X_\ell[i-d_\ell] + Z_{\mathsf{D}}[i]\}_{i=0}^{n+\mathsf{d}_{\max}-1}|X_{\mathcal{S}^c}^n, d_1^{K+1}) - \frac{1}{n}h(Z_{\mathsf{D}}^{n+\mathsf{d}_{\max}}) + \delta_n \\
&\le \frac{1}{n}h(Y_{\mathsf{D}(\mathcal{S})}^{n+\mathsf{d}_{\max}}|d_1^{K+1}) - \frac{1}{n}h(Z_{\mathsf{D}}^{n+\mathsf{d}_{\max}}) + \delta_n \\
&= \frac{1}{n}I(X_{\mathcal{S}}^n; Y_{\mathsf{D}(\mathcal{S})}^{n+\mathsf{d}_{\max}}|d_1^{K+1}) + \delta_n
\end{aligned} \tag{4.30}$$

where in (a) we used the fact that $X_{\mathcal{S}^c}^n$ is a function of only $U_{\mathcal{S}^c}^n$, in (b) we used the data processing inequality and (4.29), in (c) we used $X_{[1,K+1]}^n$ based on the definition in (4.14), and lastly in (d) we made use of the fact that conditioning does not increase the entropy.

But (4.30) represents the mutual information at the destination's output of the Gaussian sliced TA-MARC $\mathcal{M}(\mathcal{S})$ corresponding to the original Gaussian TA-MARC. Thus, using Lemma 5, we can now further upper bound the mutual information term in (4.30) by the corresponding mutual information term in the corresponding sliced cyclic MARC and derive

$$H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) \leq \frac{1}{n} I(X_{\mathcal{S}}^n; \tilde{Y}_{\mathsf{D}(\mathcal{S})}^n | d_1^{K+1}) + \epsilon_n + \delta_n. \tag{4.31}$$

Now, let $D_\ell, \ell = 1, \cdots, K+1$, be a sequence of random variables that are each uniformly distributed on the set $\{0, 1, \cdots, \mathsf{d}_{\mathsf{max}}(n)\}$ and independent of $\{U_\ell^n\}_{\ell=1}^{K+1}$, $\{Z_{\mathsf{D}}[i]\}_{i=0}^{n-1}$, and $\{Z_{\mathsf{R}}[i]\}_{i=0}^{n-1}$. Since (4.31) is true for every choice of $d_1^{K+1} \in \{0, 1, \cdots, \mathsf{d}_{\mathsf{max}}(n)\}^{K+1}$, $H(U_{\mathcal{S}}|U_{\mathcal{S}^c})$ can also be upper bounded by the average over d_1^{K+1} of $I(X_{\mathcal{S}}^n; \tilde{Y}_{\mathsf{D}(\mathcal{S})}{}^n | d_1^{K+1})$. Hence,

$$\begin{aligned} H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) &\leq I(X_{\mathcal{S}}^n; \tilde{Y}_{\mathsf{D}(\mathcal{S})}^n | D_1^{K+1}) + \epsilon_n + \delta_n \\ &\overset{\text{(a)}}{=} I(X_{\mathcal{S}}^n; \hat{\tilde{Y}}_{\mathsf{D}(\mathcal{S})}^n | D_1^{K+1}) + \epsilon_n + \delta_n, \end{aligned} \tag{4.32}$$

where $\hat{\tilde{Y}}_{\mathsf{D}(\mathcal{S})}^n = \mathrm{DFT}(\tilde{Y}^n_{\mathsf{D}(\mathcal{S})})$, and (a) follows from the fact that the DFT is a bijection.

Expanding $I(X_{\mathcal{S}}^n; \hat{\tilde{Y}}_{\mathsf{D}(\mathcal{S})}^n | D_1^{K+1})$ in the right hand side of (4.32),

$$\begin{aligned} H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) &\leq \frac{1}{n}[h(\hat{\tilde{Y}}_{\mathsf{D}(\mathcal{S})}^n | D_1^{K+1}) - h(\hat{\tilde{Y}}_{\mathsf{D}(\mathcal{S})}^n | X_{\mathcal{S}}^n, D_1^{K+1})] + \epsilon_n + \delta_n \\ &\leq \frac{1}{n}[h(\hat{\tilde{Y}}_{\mathsf{D}(\mathcal{S})}^n) - h(\hat{Z}_{\mathsf{D}}^n)] + \epsilon_n + \delta_n, \end{aligned}$$

where $\hat{Z}_{\mathsf{D}}^n = \mathrm{DFT}(Z_{\mathsf{D}}^n)$ has i.i.d. entries with $\hat{Z}_{\mathsf{D}}[i] \sim \mathcal{CN}(0, N)$. Recall $\hat{X}_\ell^n = \mathrm{DFT}(X_\ell^n)$. Then,

$$h(\hat{\tilde{Y}}_{\mathsf{D}(\mathcal{S})}^n) = h\left(\sum_{\ell \in \mathcal{S}} e^{-j\boldsymbol{\theta}(D_\ell)} \odot g_{\ell\mathsf{D}} \hat{X}_\ell^n + \hat{Z}_{\mathsf{D}}^n\right)$$

$$\leq \sum_{i=0}^{n-1} h\left(\sum_{\ell\in\mathcal{S}} e^{\frac{-j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}}\hat{X}_\ell[i] + \hat{Z}_{\mathsf{D}}[i]\right),$$

where $e^{-j\boldsymbol{\theta}(D)} \triangleq (e^{\frac{-j2\pi i D}{n}})_{i=0}^{n-1}$ is an n-length vector, and $\odot$ denotes element-wise vector multiplication. Thus,

$$H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) \leq \frac{1}{n}\sum_{i=0}^{n-1}\left[h\left(\sum_{\ell\in\mathcal{S}} e^{\frac{-j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}}\hat{X}_\ell[i] + \hat{Z}_{\mathsf{D}}[i]\right) - h(\hat{Z}_{\mathsf{D}}[i])\right] + \epsilon_n + \delta_n$$

$$\leq \frac{1}{n}\sum_{i=0}^{n-1}\log\left(1+\frac{\mathbb{E}\left|\sum_{\ell\in\mathcal{S}} e^{\frac{-j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}}\hat{X}_\ell[i]\right|^2}{N}\right) + \epsilon_n + \delta_n. \tag{4.33}$$

We now divide the sum in (4.33) into three terms for $0 \leq i \leq \alpha(n)-1$, $\alpha(n) \leq i \leq n-\alpha(n)-1$, and $n-\alpha(n) \leq i \leq n-1$, where $\alpha(n) : \mathbb{N} \to \mathbb{N}$ is a function such that

$$\frac{\alpha(n)}{n} \to 0, \quad \frac{\alpha(n)\mathsf{d}_{\mathsf{max}}(n)}{n} \to \infty. \tag{4.34}$$

An example of such an $\alpha(n)$ is the function $\alpha(n) = \lceil \frac{n}{\mathsf{d}_{\mathsf{max}}(n)} \log \mathsf{d}_{\mathsf{max}}(n)\rceil$. Consequently, we first upper bound the tail terms and afterwards the main term in the sequel.

For the terms in $0 \leq i \leq \alpha(n)-1$, we have

$$\frac{1}{n}\sum_{i=0}^{\alpha(n)-1}\log\left(1+\frac{\mathbb{E}\left|\sum_{\ell\in\mathcal{S}} e^{\frac{-j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}}\hat{X}_\ell[i]\right|^2}{N}\right)$$

$$\stackrel{\text{(a)}}{\leq} \frac{1}{n}\sum_{i=0}^{\alpha(n)-1}\log\left(1+\frac{\sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2\cdot\sum_{\ell\in\mathcal{S}}\mathbb{E}|\hat{X}_\ell[i]|^2}{N}\right)$$

$$\stackrel{\text{(b)}}{\leq} \frac{\alpha(n)}{n}\log\left(1+\frac{\sum_{i=0}^{\alpha(n)-1}\left[\sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2\cdot\sum_{\ell\in\mathcal{S}}\mathbb{E}|\hat{X}_\ell[i]|^2\right]}{\alpha(n)N}\right)$$

$$\stackrel{\text{(c)}}{\leq} \frac{\alpha(n)}{n}\log\left(1+\frac{n}{\alpha(n)}\frac{\sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2\cdot\sum_{\ell\in\mathcal{S}}P_\ell}{N}\right)$$

$$\triangleq \lambda_n, \tag{4.35}$$

where (a) follows by the Cauchy-Schwartz inequality (cf. (4.24)), (b) follows by the concavity of the log function and (c) follows by the power constraints (4.5). Also, for $n - \alpha(n) \leq i \leq n-1$, a similar upper bound can be derived by the symmetry of the problem as follows

$$\frac{1}{n}\sum_{i=n-\alpha(n)}^{n-1} \log\left(1 + \frac{\mathbb{E}\left|\sum_{\ell\in\mathcal{S}} e^{\frac{-j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}}\hat{X}_\ell[i]\right|^2}{N}\right) \leq \lambda_n. \tag{4.36}$$

To bound the third component of (4.33) for $\alpha(n) \leq i \leq n-\alpha(n)-1$, we first obtain that

$$\mathbb{E}\left|\sum_{\ell\in\mathcal{S}} e^{\frac{-j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}}\hat{X}_\ell[i]\right|^2 = \sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2\mathbb{E}|\hat{X}_\ell[i]|^2 + \sum_{\substack{(\ell,\ell')\in\mathcal{S}^2\\ \ell<\ell'}} 2\Re\mathbb{E}\left\{e^{\frac{-j2\pi i(D_\ell - D_{\ell'})}{n}} g_{\ell\mathsf{D}}g^*_{\ell'\mathsf{D}}\hat{X}_\ell[i]\hat{X}^*_{\ell'}[i]\right\}, \tag{4.37}$$

where $\Re(z)$ is the real part of $z\in\mathbb{C}$. Now, the following two cases can occur

i) $\ell < \ell' < K+1$: In this case, both $\hat{X}_\ell[i]$ and $\hat{X}^*_{\ell'}[i]$ are independent of D_ℓ and $D_{\ell'}$.

ii) $\ell < \ell' = K+1$: In this case, $\hat{X}_\ell[i]$ and $\hat{X}^*_{\ell'}[i]$ are independent of $D_{\ell'}$. However, $\hat{X}^*_{\ell'}[i]$, that corresponds to the channel input of the relay, is a function of $\{Y_{\mathsf{R}}[0], Y_{\mathsf{R}}[1], \cdots, Y_{\mathsf{R}}[i-1]\}$ and is thus correlated with delays of all transmitters, *i.e.*, $D_\ell, \ell = 1,2,\cdots,K$, due to (4.3).

In either scenario, we can proceed from (4.37) by separating $e^{\frac{-j2\pi i D_{\ell'}}{n}}$ from the remaining terms inside the expectation. Specifically,

$$\mathbb{E}\left|\sum_{\ell\in\mathcal{S}} e^{\frac{-j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}}\hat{X}_\ell[i]\right|^2$$

$$
\begin{aligned}
&= \sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2\mathbb{E}|\hat{X}_\ell[i]|^2 + \sum_{\substack{(\ell,\ell')\in\mathcal{S}^2\\ \ell<\ell'}} 2\Re\left(\mathbb{E}\left\{e^{\frac{-j2\pi i D_{\ell'}}{n}}\right\}\mathbb{E}\left\{e^{\frac{j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}} g^*_{\ell'\mathsf{D}}\hat{X}_\ell[i]\hat{X}^*_{\ell'}[i]\right\}\right)\\
&\le \sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2\mathbb{E}|\hat{X}_\ell[i]|^2 + \sum_{\substack{(\ell,\ell')\in\mathcal{S}^2\\ \ell<\ell'}} 2\left|\mathbb{E}\left\{e^{\frac{-j2\pi i D_{\ell'}}{n}}\right\}\mathbb{E}\left\{e^{\frac{j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}} g^*_{\ell'\mathsf{D}}\hat{X}_\ell[i]\hat{X}^*_{\ell'}[i]\right\}\right|\\
&= \sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2\mathbb{E}|\hat{X}_\ell[i]|^2 + \sum_{\substack{(\ell,\ell')\in\mathcal{S}^2\\ \ell<\ell'}} 2|g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|\left|\mathbb{E}\left\{e^{\frac{-j2\pi i D_{\ell'}}{n}}\right\}\right|\left|\mathbb{E}\left\{e^{\frac{j2\pi i D_\ell}{n}}\hat{X}_\ell[i]\hat{X}^*_{\ell'}[i]\right\}\right|\\
&\overset{(a)}{\le} \sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2\mathbb{E}|\hat{X}_\ell[i]|^2 + \frac{1}{\mathsf{d}_{\max}(n)|\sin(\frac{\pi i}{n})|}\sum_{\substack{(\ell,\ell')\in\mathcal{S}^2\\ \ell<\ell'}} |g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|\left(\mathbb{E}|\hat{X}_\ell[i]|^2+\mathbb{E}|\hat{X}_{\ell'}[i]|^2\right)\\
&\overset{(b)}{\le} \sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2\mathbb{E}|\hat{X}_\ell[i]|^2 + \frac{1}{\mathsf{d}_{\max}(n)|\sin(\frac{\pi\alpha(n)}{n})|}\sum_{\substack{(\ell,\ell')\in\mathcal{S}^2\\ \ell<\ell'}} |g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|\left(\mathbb{E}|\hat{X}_\ell[i]|^2+\mathbb{E}|\hat{X}_{\ell'}[i]|^2\right),
\end{aligned}
\tag{4.38}
$$

where the derivation of (a) is presented in Appendix 4.A, and (b) follows from the inequality

$$
\sin(\frac{\pi\alpha(n)}{n}) \le \sin(\frac{\pi i}{n}),\ \ \text{for all } i\in[\alpha(n), n-\alpha(n)-1]. \tag{4.39}
$$

By summing (4.38) over $\alpha(n)\le i\le n-\alpha(n)-1$, we further obtain

$$
\begin{aligned}
&\sum_{i=\alpha(n)}^{n-\alpha(n)-1}\mathbb{E}\left|\sum_{\ell\in\mathcal{S}} e^{\frac{-j2\pi i D_\ell}{n}} g_{\ell\mathsf{D}}\hat{X}_\ell[i]\right|^2\\
&\le \sum_{i=\alpha(n)}^{n-\alpha(n)-1}\sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2\mathbb{E}|\hat{X}_\ell[i]|^2\\
&\quad+\frac{1}{\mathsf{d}_{\max}(n)|\sin(\frac{\pi\alpha(n)}{n})|}\sum_{i=\alpha(n)}^{n-\alpha(n)-1}\sum_{\substack{(\ell,\ell')\in\mathcal{S}^2\\ \ell<\ell'}} |g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|\left(\mathbb{E}|\hat{X}_\ell[i]|^2+\mathbb{E}|\hat{X}_{\ell'}[i]|^2\right)\\
&\overset{(a)}{\le} \sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2 nP_\ell + \frac{1}{\mathsf{d}_{\max}(n)|\sin(\frac{\pi\alpha(n)}{n})|}\sum_{\substack{(\ell,\ell')\in\mathcal{S}^2\\ \ell<\ell'}} |g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|(nP_\ell+nP_{\ell'})
\end{aligned}
$$

$$= n\left[\sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2P_\ell + \frac{\zeta(\mathcal{S})}{\mathsf{d}_{\max}(n)|\sin(\frac{\pi\alpha(n)}{n})|}\right], \tag{4.40}$$

where (a) is due to the power constraint in (4.5), and

$$\zeta(\mathcal{S}) \triangleq \sum_{\substack{(\ell,\ell')\in\mathcal{S}^2\\ \ell<\ell'}}|g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|(P_\ell+P_{\ell'}). \tag{4.41}$$

Based on the result in (4.40), we upper bound the third component of (4.33) as below

$$\begin{aligned}
&\frac{1}{n}\sum_{i=\alpha(n)}^{n-\alpha(n)-1}\log\left(1+\frac{\mathbb{E}\left|\sum_{\ell\in\mathcal{S}}e^{\frac{-j2\pi iD_\ell}{n}}g_{\ell\mathsf{D}}\hat{X}_\ell[i]\right|^2}{N}\right)\\
&\overset{(a)}{\le}\frac{n-2\alpha(n)}{n}\log\left(1+\frac{\sum_{i=\alpha(n)}^{n-\alpha(n)-1}\left[\mathbb{E}\left|\sum_{\ell\in\mathcal{S}}e^{\frac{-j2\pi iD_\ell}{n}}g_{\ell\mathsf{D}}\hat{X}_\ell[i]\right|^2\right]}{N(n-2\alpha(n))}\right)\\
&\overset{(b)}{\le}\frac{n-2\alpha(n)}{n}\log\left(1+\frac{n}{n-2\alpha(n)}\frac{\sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2P_\ell+\frac{\zeta(\mathcal{S})}{\mathsf{d}_{\max}(n)|\sin(\frac{\pi\alpha(n)}{n})|}}{N}\right),
\end{aligned} \tag{4.42}$$

where (a) follows by the concavity of the log function, and (b) follows from (4.40).

Now, by combining (4.33), (4.35), (4.36), and (4.42) we derive

$$H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) \le \frac{n-2\alpha(n)}{n}\log\left(1+\frac{n}{n-2\alpha(n)}\frac{\sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2P_\ell+\frac{\zeta(\mathcal{S})}{\mathsf{d}_{\max}(n)|\sin(\frac{\pi\alpha(n)}{n})|}}{N}\right)+2\lambda_n+\epsilon_n+\delta_n. \tag{4.43}$$

To obtain the asymptotic bound, we recall that that due to the choice of $\alpha(n)$ in (4.34),

$$\begin{aligned}
\frac{n-2\alpha(n)}{n} &\to 1,\\
\sin\left(\frac{\pi\alpha(n)}{n}\right)/\frac{\pi\alpha(n)}{n} &\to 1,
\end{aligned}$$

$$\frac{1}{\mathsf{d}_{\mathsf{max}}(n)|\sin(\frac{\pi\alpha(n)}{n})|} \to \frac{n}{\pi \mathsf{d}_{\mathsf{max}}(n)\alpha(n)} \to 0,$$

as $n \to \infty$. Therefore, it can be easily verified from (4.43) that since $\zeta(\mathcal{S}) < \infty$, and $\lambda_n, \delta_n, \epsilon_n \to 0$ as $n \to \infty$,

$$H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) \leq \log\left(1 + \frac{\sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2 P_\ell}{N}\right), \tag{4.44}$$

where we recall that the subset $\mathcal{S} \subseteq [1, K+1]$ includes the relay, i.e., $\{K+1\} \in \mathcal{S}$. $\square$

4.4 Achievability

We now focus on demonstrating the achievability of the region that was proved to be an outer bound on the capacity region in Lemma 6 and thus conclude that the region is indeed the capacity region. To establish the achievability argument, we follow a *tandem* (separate) source-channel coding scheme. Thus, the communication process will be divided into two parts: source coding and channel coding. In the sequel, we simply state the results for each of both source and channel coding, and finally by combining them prove the achievability lemma.

Source Coding: From Slepian-Wolf coding [30], for the correlated source $(U_1^n, U_2^n, \cdots, U_K^n)$, if we have K n-length sequences of source codes with rates $(R_1, R_2, \cdots, R_K)$, for asymptotically lossless representation of the source, we should have

$$H(U_{\mathcal{S}}|U_{\mathcal{S}^c}) < \sum_{\ell\in\mathcal{S}} R_\ell, \quad \forall \mathcal{S} \subseteq [1, K+1] : \{K+1\} \in \mathcal{S}, \tag{4.45}$$

where by definition $R_{K+1} \triangleq 0$, and $U_{K+1} \triangleq \emptyset$.

Channel Coding: Next, for fixed source codes with rates $(R_1, R_2, \cdots, R_K)$, we make channel codes for the TA-MARC separately such that the channel codes can be reliably

Encoder	Block 1	Block 2	$\cdots$	Block B	Block $B+1$
1	$x_1^n(1, W_{11})$	$x_1^n(W_{11}, W_{12})$	$\cdots$	$x_1^n(W_{1(B-1)}, W_{1B})$	$x_1^n(W_{1B}, 1)$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
K	$x_K^n(1, W_{K1})$	$x_K^n(W_{K1}, W_{K2})$	$\cdots$	$x_K^n(W_{K(B-1)}, W_{KB})$	$x_K^n(W_{KB}, 1)$
$K+1$	$x_{K+1}^n(1, \cdots, 1)$	$x_{K+1}^n(W_{11}, \cdots, W_{K1})$	$\cdots$	$x_{K+1}^n(W_{1(B-1)}, \cdots, W_{K(B-1)})$	$x_{K+1}^n(W_{1B}, \cdots, W_{KB})$

Table 4.1: Block Markov encoding scheme for the Gaussian TA-MARC.

decoded at the receiver side. In particular, we use the block Markov coding scheme used in [37] on top of the coding strategy used in [8], in order to make reliable channel codes. Indeed, we directly apply the decoding technique of [8] to a series of block Markov codes which results in an achievable rate region equivalent to the intersection of two MACs with encoders as the transmitters with indices $1, \cdots, K$, and decoders as the relay and destination. In the sequel, we briefly give some details of the block Markov coding scheme and the coding strategy for the delayed codewords.

- *Block Markov coding*: Table I shows the block Markov coding configuration used to transmit the codewords of the encoders of the Gaussian TA-MARC. First fix a distribution $p(x_1) \cdots p(x_{K+1})$ and construct random codewords $x_1^n, \cdots, x_{K+1}^n$ based on the corresponding distributions. The message W_i of each encoder is divided to B blocks $W_{i1}, W_{i2}, \cdots, W_{iB}$ of 2^{nR_i} bits each, $i = 1, \cdots, K$. The codewords are transmitted in $B+1$ blocks based on the block Markov encoding scheme depicted in Table I. After each block, the relay makes a MAC decoding and uses the decoded messages $W_{1(i-1)}, \cdots, W_{K(i-1)}$ to send the codewords in the next block. Also, the decoding at the destination is performed at the end of the last block and in a backward block-by-block manner, also known as *backward decoding* [37]. We let $B \to \infty$ to approach the original rates $R_1, \cdots, R_K$.

- *Coding strategy of [8]*: The encoders transmit their codewords as shown in Table I and in B blocks, albeit with delays $d_1, \cdots, d_{K+1}$. Note that if the MARC was synchronous, one would obtain the achievable rate region resulting from the intersection of two MACs. However, using a simply generalized version of the coding strategy used in [8], it can be seen that the same region is achievable for the time asynchronous case. In particular, at the end of the ith block, the relay decoder inspects the received vector $Y_{\mathsf{R}}^{n+\mathsf{d}_{\max}(n)}$ for the presence of codewords $x_1^n(W_{1i}), \cdots, x_K^n(W_{Ki})$, embedded in it with arbitrarily shifts. Likewise, at the end of the last block, the destination decoder inspects the received vector $Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}(n)}$ to first decode $W_{1B}, \cdots, W_{KB}$ and consequently decode the previous messages in a backward manner. In all of these decoding cases, like [8], we look for the codewords under all possible shifts up to the maximum delay $\mathsf{d}_{\max}$ such that the shifted codewords and the $(n + \mathsf{d}_{\max})$-length received vector are jointly typical. Therefore, the decoders at the relay and destination need to look for $\mathsf{d}_{\max}(n)^K$ combination of codewords and find the one that is jointly typical with $Y_{\mathsf{R}}^{n+\mathsf{d}_{\max}(n)}$ or $Y_{\mathsf{D}}^{n+\mathsf{d}_{\max}(n)}$. Following similar error analysis as in [8], now for a K user system with K delays, and due to the assumption that $\mathsf{d}_{\max}(n)/n \to 0$, it can be seen that the standard synchronous K user MAC capacity constraints are derived in order to achieve asymptotically vanishing probability of error.

Hence, for reliable communication of the source indices over the Gaussian TA-MARC, the following sets of inequalities that represents MAC decoding at the relay and destination should be satisfied:

$$R_{\mathcal{S}} < I(X_{\mathcal{S}}; Y_{\mathsf{R}} | X_{\mathcal{S}^c}), \quad \forall \mathcal{S} \subseteq [1, K], \tag{4.46}$$

and

$$R_{\mathcal{S}} < I(X_{\mathcal{S}}; Y_{\mathsf{D}} | X_{\mathcal{S}^c}), \quad \forall \mathcal{S} \subseteq [1, K+1] : \{K+1\} \in \mathcal{S}, \tag{4.47}$$

for an input distribution $p(x_1)\cdots p(x_{K+1})$.

By choosing Gaussian input distributions, the constraints in (4.46)-(4.47) will be reduced to logarithmic rate functions. It is then straight forward to see that under the gain conditions

$$\sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{R}}|^2 P_\ell \geq \sum_{\ell\in\mathcal{S}} |g_{\ell\mathsf{D}}|^2 P_\ell, \quad \forall\mathcal{S}\subseteq[1,K+1]:\{K+1\}\in\mathcal{S} \tag{4.48}$$

where $g_{(K+1)\mathsf{R}} \triangleq 0$, the destination decoding constraints (4.47) will dominate (4.46), and we can thus derive the following conditions on $R_1,\cdots,R_K$, as sufficient conditions for reliable communication of source coded indices over a Gaussian TA-MARC:

$$\sum_{\ell\in\mathcal{S}} R_\ell < \log\left(1+\frac{\sum_{\ell\in\mathcal{S}}|g_{\ell\mathsf{D}}|^2 P_\ell}{N}\right), \quad \forall\mathcal{S}\subseteq[1,K+1]:\{K+1\}\in\mathcal{S}. \tag{4.49}$$

Lemma 7. *A sufficient condition for reliable communication of the source* $(U_1^n,\cdots,U_K^n)$ *over the TA-MARC defined by* (4.2)-(4.3) *is given by* (4.27)*, with* $\leq$ *replaced by* $<$.

Proof. From (4.27), it can be seen that there exist choices of $R_1,\cdots,R_2$ such that the Slepian-Wolf conditions (4.45) and the channel coding conditions (4.49) are simultaneously satisfied. Since error probabilities of both the source coding part and channel coding part vanish asymptotically, then the error probability of the combined tandem scheme also vanishes asymptotically and the proof of the lemma is complete. $\square$

4.5 Separation Theorems

Based on the converse and achievability results presented in Sections 4.3 and 4.4, we can now combine the results and state the following separation theorem for a Gaussian TA-MARC

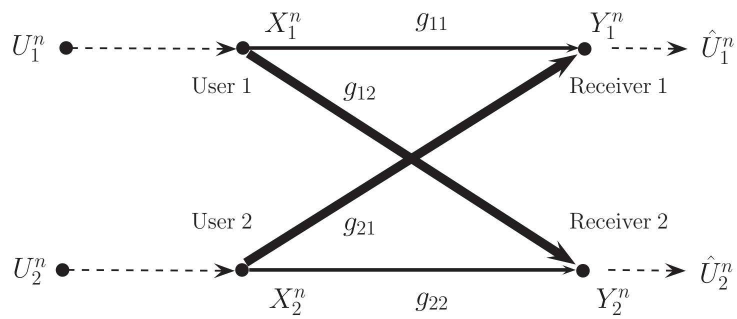

Figure 4.3: Gaussian Time-Asynchronous Interference Channel (TA-IC) with Strong Interference Gains.

Theorem 18. Reliable Communication over a Gaussian TA-MARC*: Consider a Gaussian TA-MARC with the gain conditions (4.48). Then, necessary conditions for reliably sending a source* $(U_1^n, \cdots, U_K^n) \sim \prod_i p(u_{1i}, \cdots, u_{Ki})$*, over such a TA-MARC are given by (4.27). Furthermore, (4.27), with* $\leq$ *replaced by* $<$*, also gives a sufficient condition for reliable communications over such a TA-MARC and can be achieved by separate source-channel coding.* □

Theorem 18 can be easily specialized to a MAC if we impose $P_{K+1} = 0$ and eliminate the role of the relay. Thus, the result of [67] for a 2-user TA-MAC is a direct consequence of Theorem 18. As a result, we can also state the following corollary for a Gaussian time asynchronous interference channel (TA-IC) with strong interference conditions depicted in Fig. 4.3. The result of the corollary is based on the fact that in the strong interference regime, the Gaussian interference channel can be reduced to the intersection of two Gaussian MACs with no loss. Namely, if each receiver can correctly decode its own channel input sequence, in the strong interference regime, it can also correctly decode the other channel input sequence (see [63] for details). In the context of JSCC, we note that by using the strong interference conditions and the one-to-one mappings between source and channel sequences, one can argue that both of the receivers can recover both source sequences U_1^n, U_2^n provided there are encoders and decoders such that each receiver can reliably decode its *own* source sequence. Specifically, if the first receiver can decode U_1^n and the second receiver can decode U_2^n, this in turn enables each receiver to reconstruct the channel input X_1^n from U_1^n or X_2^n from U_2^n, then X_2^n from X_1^n or X_1^n from X_2^n, and finally the source sequence U_2^n from X_2^n or U_1^n from X_1^n. Therefore, under the strong interference regime, the JSCC capacity region (i.e., a set of necessary and sufficient conditions for reliable communications) is described by the intersection of JSCC capacity regions of two MACs.

Corollary 9. *Necessary conditions for reliably sending arbitrarily correlated sources* (U_1, U_2) *over a TA-IC with strong interference conditions* $|g_{11}| \leq |g_{12}|, |g_{22}| \leq |g_{21}|$ *are given by*

$$H(U_i|U_j) \leq \log(1 + |g_{ii}|^2 P_i/N), \quad (i,j) \in \{(1,2),(2,1)\} \tag{4.50}$$

$$H(U_1, U_2) \leq \min\{\log(1 + (|g_{11}|^2 P_1 + |g_{21}|^2 P_2)/N), \log(1 + (|g_{12}|^2 P_1 + |g_{22}|^2 P_2)/N)\}, \tag{4.51}$$

where $g_{ij}, i, j \in \{1, 2\}$ *represents the complex gain from node* i *to the receiver* j *in a two user interference channel. The same conditions* (4.50)-(4.51) *with* $\leq$ *replaced by* $<$ *describe sufficient conditions for reliable communication.* □

4.A Appendix

Since $D_{\ell'}$ has a uniform distribution over $\{0, 1, \cdots, \mathsf{d_{max}}\}$ we have

$$\left|\mathbb{E}\left\{e^{\frac{-j2\pi i D_{\ell'}}{n}}\right\}\right| = \left|\sum_{d=0}^{\mathsf{d_{max}}} \frac{1}{\mathsf{d_{max}}+1} e^{\frac{-j2\pi i d}{n}}\right| \tag{4.52}$$

$$= \left|\frac{1}{\mathsf{d_{max}}+1} \frac{e^{\frac{-j2\pi i(\mathsf{d_{max}}+1)}{n}} - 1}{e^{\frac{-j2\pi i}{n}} - 1}\right| \tag{4.53}$$

$$= \left|\frac{1}{\mathsf{d_{max}}+1} \frac{\sin(\frac{\pi i(\mathsf{d_{max}}+1)}{n})}{\sin(\frac{\pi i}{n})}\right| \tag{4.54}$$

$$\leq \frac{1}{\mathsf{d_{max}}|\sin(\frac{\pi i}{n})|}. \tag{4.55}$$

Thus, we obtain the following inequality

$$\sum_{\substack{(\ell,\ell')\in\mathcal{S}^2 \\ \ell<\ell'}} 2|g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|\left|\mathbb{E}\left\{e^{\frac{-j2\pi i D_{\ell'}}{n}}\right\}\right|\left|\mathbb{E}\left\{e^{\frac{j2\pi i D_{\ell}}{n}} \hat{X}_{\ell}[i]\hat{X}^*_{\ell'}[i]\right\}\right|$$

$$\leq \frac{1}{\mathsf{d_{max}}|\sin(\frac{\pi i}{n})|} \sum_{\substack{(\ell,\ell')\in\mathcal{S}^2 \\ \ell<\ell'}} 2|g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|\mathbb{E}\left\{\left|e^{\frac{j2\pi i D_{\ell}}{n}} \hat{X}_{\ell}[i]\hat{X}^*_{\ell'}[i]\right|\right\} \tag{4.56}$$

$$= \frac{1}{\mathsf{d_{max}}|\sin(\frac{\pi i}{n})|} \sum_{\substack{(\ell,\ell')\in\mathcal{S}^2 \\ \ell<\ell'}} 2|g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|\mathbb{E}\left\{\left|\hat{X}_{\ell}[i]\right|\left|\hat{X}^*_{\ell'}[i]\right|\right\} \tag{4.57}$$

$$\overset{\text{(a)}}{\leq} \frac{1}{\mathsf{d_{max}}|\sin(\frac{\pi i}{n})|} \sum_{\substack{(\ell,\ell')\in\mathcal{S}^2 \\ \ell<\ell'}} |g_{\ell\mathsf{D}}||g_{\ell'\mathsf{D}}|(\mathbb{E}|\hat{X}_{\ell}[i]|^2 + \mathbb{E}|\hat{X}_{\ell'}[i]|^2), \tag{4.58}$$

where (a) follows by the geometric inequality $2\sqrt{ab} \leq a + b$ with $a = |\hat{X}_{\ell}[i]|^2$ and $b = |\hat{X}_{\ell'}[i]|^2 = |\hat{X}^*_{\ell'}[i]|^2$.

Chapter 5

Conclusion and Future Work

5.1 Conclusion

In this thesis, we established necessary and sufficient conditions for reliable communication of correlated sources over several multiuser channels including phase and time asynchronism between different nodes of the network. Namely, we divided our results into two general parts: phase asynchronous and time asynchronous networks. For the phase asynchronous systems, the main assumption was unknown phases of the communication links at the encoders while for the time asynchronous systems, we assumed that the transmitters cannot exactly synchronize the timing of their transmissions.

In general, for all of these systems, we first derived necessary conditions for reliable communications for *all* possible channel gains and then we derived sufficient conditions for *specific* gain conditions. Noting the coincidence of necessary and sufficient conditions under the specific gain conditions, we then stated and proved that under the gain conditions, a separation approach is optimal for all of these scenarios and one can achieve the optimal performance by first Slepian-Wolf coding the correlated sources and then performing channel coding with independent input distributions. Indeed, we showed that under the specific gain conditions, the correlation between the sources cannot enlarge the achievable region compared to separate source-channel coding schemes.

Specifically, in Chapter 3, Sections 3.1-3.4, the problem of sending arbitrarily correlated sources over variations of the phase asynchronous multiple access relay channel and the interference relay channel with non-ergodic phase fading was considered. Namely, in light of Lemma 1, outer bounds on the source entropy content $(H(U_1|U_2), H(U_2|U_1), H(U_1, U_2))$ were first derived under phase uncertainty at the encoders. The outer bounds were then shown to match the achievable regions obtained by separate source-channel coding under some restrictions on the channel gains. We also conjecture that the optimality of separation is true not only for the specific gain conditions we state, but also for all possible values of path gains. In particular, as implied by Lemma 1, for a general network, independent channel inputs avoid the reduction in received power caused by adversarial choices of unknown phases. Therefore, regardless of the channel gains, this suggests that separate source-channel coding is likely optimal for the considered networks. Hence, we conjecture that separation is optimal for unrestricted forms of the phase incoherent Gaussian phase asynchronous channels discussed in Sections 3.1-3.4. The approach we used here to prove the separation theorems which is based on computing necessary and sufficient conditions for reliable communication, however, may not be viable to prove the conjecture. Indeed, if the approach taken here was useful to derive optimal joint source-channel coding regions of the considered networks for all channel gains, it would imply that we could also compute their corresponding phase-fading channel coding capacity. However, finding these channel coding capacities are long standing open problems in network information theory.

In Chapter 3, Section 3.5, we extended the results to the case of Gaussian cognitive interference channels. In particular, we derived necessary and sufficient conditions for reliable communication of primary and secondary over classes of phase asynchronous cognitive interference channels. Furthermore, we also derived necessary and sufficient conditions for reliable communication of secondary while it causes no degradation to the primary, i.e., it can still establish reliable communication using the same conditions and procedures as it did when the secondary was absent. Our results hold under strong interference conditions. In particular, we proved separation theorems for both cases of noncausal and causal unidirectional cooperation between the encoders. This is the first work to address the problem of joint source-channel coding for cognitive interference channels in the lossless setup, to the best of our knowledge.

In Chapter 3, Section 3.6, we derived a general outer bound on the distortion region, for sending a bivariate Gaussian source over an IRC under phase uncertainty at the transmitters. Using a separation approach, we then derived an inner bound for the distortion region under specific SNR-dependant gain conditions which mainly represent strong interferences between the transmitters and the unwanted receivers. Next, an approximation to the inner bound in the high SNR regime was found. Under the specified gain conditions and phase uncertainty at the transmitters, we consequently characterized the full achievable distortion region for independent sources and proved a separation theorem. By removing the relay, our results were specialized to communication of independent Gaussians over an interference channel with SNR-dependant strong interference gains.

Finally, in Chapter 4, the problem of sending arbitrarily correlated sources over a time asynchronous multiple-access relay channel with maximum offset between encoders $\mathsf{d}_{\mathsf{max}}(n) \to \infty$, as $n \to \infty$, was considered. Necessary and sufficient conditions for reliable communication were presented under the assumption of $\mathsf{d}_{\mathsf{max}}(n)/n \to 0$. Namely, a general outer bound on the capacity region (i.e., a necessary condition on the reliable communications) was first derived and then was shown to match the separate source-channel coding achievable region under specific gain conditions. Therefore, under the gain conditions, separation was shown to be optimal and as a result, joint source-channel coding is not necessary under time asynchronism.

5.2 Future Directions

In the following, we list a few possible future directions we plan to pursue. While some of those are extensions of the *time* asynchronism problem to other settings such as cognitive networks or discrete alphabets, others may facilitate new approaches to phase asynchronism.

5.2.1 Extension of the Time Asynchronous Settings to the Cognitive Networks

As we studied the problem of JSCC for phase asynchronous cognitive multiple access channels, one possible research direction is to examine this problem when we have time asynchronism in the network. In Chapter 4, all of the results were derived under the assumption that there is no cooperation/cognition between the encoders. Incorporating the notion of cooperation/cognition into the problem already studied can be interesting as it better models emerging systems where cognitive radios and cooperative schemes are to play key roles.

5.2.2 Time Asynchronous Networks with Discrete Alphabets

In Chapter 4, we considered the JSCC problem for a time asynchronous MARC with *complex-valued* input and output symbols. Our proof technic relies on the continuous alphabet and additive nature of the channel, e.g., by computing the DFT of a vector and using the fact that the DFT conserves the vector's power. In a general setup, where the input and output symbols are chosen from a discrete alphabet and their relationship is characterized by a conditional probability mass function only, a new technic to derive necessary and sufficient conditions for reliable source-channel communications would be needed. Therefore, studying a more general setting with arbitrary channel alphabet is an interesting further step to address the problem of JSCC for asynchronous networks.

5.2.3 Phase Asynchronous Side Information

State dependant channels with known side information at the encoder were first introduced by Shannon. In [69], Shannon found the capacity of a state-dependant channel with state causally known to the encoder. Later, Gel'fand and Pinsker [57] found a single-letter capacity expression for channels with non-causal knowledge of the i.i.d. state at the encoder and Costa [70] extended their results to an additive continuous setup referred to as dirty paper coding. Subsequently, other chain of works were published to address the problem of state-dependant channels with side information available at the encoder for different settings, such as [71] for cognitive radios, [72] for multiple access channels with channel side information, and [73] for multiple-input multiple-output broadcast channels.

The usual underlying assumption in the analysis of these channels in the literature is that the side information signal and the transmitted or received signal are fully synchronized. An exception is [74] where phase faded side information with unknown fading coefficient at the transmitter is considered. Since the synchronized side information is not a practical assumption in many real-world problems, considering state-dependant channels with some kind of asynchronous side information is an interesting research direction in network information theory. When the synchronization assumption, either in time, phase or frequency, does not hold, most of the existing results for the aforementioned channels are questionable and new models and analysis should be considered. Hence, it is clear that many different problems can be defined in this context for various setups and topologies. In the following, we briefly introduce one of these possible problems which can be studied as a future research direction.

The problem of lossy source coding with side information at the decoder is first addressed by Wyner and Ziv in [75]. As a future direction, we propose to study the case where the side information undergoes a random phase shift where the phase in not known to the decoder side. It is interesting to analyze the upper bound and lower bound on the distortion and examine whether the lack of phase knowledge renders the side information useless. It appears that both the converse and achievability parts of the coding theorem need to be revisited.

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