A Preservationist Approach to Relevant Logic

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The semantics I develop extend an approach to logic called preservationism. The preservationist approach to logic interprets non-classical consequence relations as preserving something other than truth. I specifically extend a preservationist approach, due to Bryson Brown, which interprets various paraconsistent consequence relations as preserving measures of ambiguity. Relevant logics are constructible by extending one of these logics with an implication connective. I develop a formal semantics which I show to be adequate for interesting relevant logics. I argue that the semantics I develop extend the preservationist approach to relevant logic by showing how the approach treats the implication connective. I conclude by arguing that some of the most pressing objections to the standard semantics for relevant logics do not apply to the ambiguity preservation account.

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Table of Contents

Chapter 1

Relevant Logic

1.1 Introduction

I will accomplish two overlapping goals. The first goal is to extend the preservationist project of Bryson Brown to relevant logic. So far, the preservationist approaches have been applied to paraconsistent and paracomplete logics. (In the second chapter I will explicate Brown's preservationist approach in detail, and will say more about what preservationism is in general.) An extension of the preservationist project which includes relevant logics would be beneficial to the preservationists, as this would mean that more logics are able to be captured by their approach. The second, and very similar, goal is to provide an additional motivation and interpretation for relevant logics in order to provide more support for relevant logics. More specifically, by extending a preservationist approach to relevant logic, I construct a semantics for a number of relevant logics. These novel semantics avoid many concerns had with the extant semantics. Of these concerns, which have been voiced both by logicians and philosophers alike, some of them claim that the extant semantics for relevant logic are merely model theoretic entities and not genuine semantics. The preservationist approach I will extend to relevant logic can be seen to solve these concerns. Naturally, then, the first two chapters will be dedicated to explicating relevant logic and Brown's preservationist approach. The third chapter will combine relevant logic with Brown's preservationist approach.

My aims in the first chapter are as follows. First, I will introduce relevant logic. I will show that there are some worthwhile motivations for using or accepting relevant logic. The relevant logic **B** and a number of its (relevant) extensions will introduced. These logics are related to one another model theoretically in a significant way, which will be shown when we have the logical machinery to do so. The concerns had with the semantics of relevant logic will be explicated in this first chapter. This will be accomplished by explicating a number of semantic approaches that have been taken, and then describing the various concerns had with these semantics. While the semantics for relevant logic are formally adequate, the concerns generally indicate that the semantics are not genuine semantics in some sense.

My aim in the second chapter is to explicate Brown's approach to preservationism. I will show how Brown's approach responds to a common objection to paraconsistent logics. Moreover, I will show that extending Brown's approach to relevant logic responds to a similar objection to relevant logics. In the third chapter I will combine relevant logics with Brown's approach by constructing a possible worlds semantics. I will argue that the constructed possible worlds semantics are able to be interpreted as an extension of Brown's preservationist project. The constructed models will not be able to model the logic **B**, but will model a number of **B**'s (relevant) extensions.

I claim that extending Brown's preservationist approach is able to address many of the concerns had with the semantics of relevant logic. In fact, extending Brown approach to construct a semantics for relevant logics creates a *genuine* semantics for relevant logic, as will be explained.

1.1.1 Notational Conventions

For the convenience of the reader, I will note some of the notational conventions I use. The first convention is merely for my convenience, but ought to cause no confusion with the reader. I will explicate a number of logical systems in what follows, and I use $\wedge$ for conjunction, $\vee$ for disjunction, $\neg$ for negation, and $\rightarrow$ for the conditional or implication connective in every one of these logics. Context should disambiguate when a conditional statement belongs to a specific logic, or is being used in a more general sense.

A *structure* is defined as follows. Where L is a language and '$|$' is a punctuation mark;

1. Every (well formed) formula of the language L is a structure.

2. If X and Y are structures, then $(X|Y)$ is also a structure.

The semicolon will the used as the standard punctuation mark of relevant logic. The structure $(X;Y)$ is the combination of the substructures X and Y. Outermost brackets will be omitted to aid readability.

Structural rules and their notation are used throughout the following. For our purposes, we may use Restall's definition of a structural rule as found in his introduction to substructural logics. Where X and X' and structures and A is a formula;

> A rule
> $$\frac{X \vdash A}{X' \vdash A}$$
> is a *structural rule* if it is closed under substitution for formulae. That is, given any instance of the rule, and any formula B appearing in either X or X' (or both), and given any structure Y you like, then the result of replacing every instance of B in X and X' by Y is still an [instance] of the rule. [25, p. 24]

As well, given any instance of the rule, uniformly replacing A in both sequents with another formula results in another instance of the rule. Note well that I will, as common in the literature, write a structural rule as $X \Leftarrow X'$, meaning that we may replace the structure X with the structure X'. An example of a structural rule is the commutativity of the semicolon. The structural rule of (weak) commutativity is $X; Y \Leftarrow Y; X$.

The terms *functional valuation* and *relational valuation* will be used throughout. The difference between a functional valuation and a relational valuation is that a functional valuation assigns sentences a single truth value, while the relational assigns sentences either no truth value, a single truth value, or many truth values. We write a functional valuation as $\nu(A) = \Phi$, where A is a sentence, and Φ is a single truth value. We interpret $\nu(A) = \Phi$, normally, as *'the truth value of A is Φ'*. On the other hand, a relational valuation is a written $(A)\rho\Phi$, where A is a sentence and Φ is a truth value. Using a relational valuation, it is possible that $(A)\rho\Phi$ and $(A)\rho\Psi$, or neither. We may interpret $(A)\rho\Phi$ as *'A is Φ'*. This notation is in accordance with fairly standard conventions in the literature.

Of course, we can translate relational valuations into functional valuations. Members of the powerset of relational truth values may be taken as the 'truth values' of a functional valuation. For instance, let the set of relational truth values be $\{\Phi, \Psi\}$. The truth values of the corresponding functional valuation are $\emptyset, \{\Psi\}, \{\Phi\}, \{\Psi, \Phi\}$. The relational valuation pair $(A)\rho\Phi$ and $(A)\rho\Psi$ would translate into functional valuations as $\nu(A) = \{\Phi, \Psi\}$. As well, if A bears the ρ relation to no truth value, we could translate this into functional valuations as $\nu(A) = \emptyset$. The notations may be freely interchanged unless otherwise noted. I will make more use of relational valuations for their notational convenience.

All other notational conventions will be explained as they arise.

1.2 Relevant logic: Implication

In this section I will introduce relevant logic. I will describe the motivations for relevant logic, the structural rules rejected by certain relevant logics, and the relevance conditions required by relevant logics. Note that relevant logic is not merely concerned with the relevance between antecedent structures and consequent structures, as its name might suggest. Richard Routley, Val Plumwood, Robert Meyer, and Ross Brady put the aims of relevant logic as follows:

> Implication, the main relation in this work, is fundamental in reasoning, particularly in deductive reasoning. Hence its central importance in philosophy, logic, and mathematics, where such notions as entailment and valid argument are central. [27, p. 1]

Anderson and Belnap have also noted that similar concerns motivated their two volumed work *Entailement*.

> Although there are many candidates for "logical connectives," such as conjunction, disjunction, negation, quantifiers, and for some writers even identity of individuals, we take the heart of logic to lie in the notion "if . . . then —". [1, p. 3]

Anderson and Belnap then discuss the so-called paradoxes of both material implication and strict implication.

Just as one might teach students in a first year course in symbolic (classical) logic, Anderson and Belnap have shown that when we take $P \to Q$ to be defined as $\neg P \vee Q$, then no *paradoxical* situation is observed [1, p. 3]. By defining the arrow as such, however, we are explicitly divorcing the arrow from the job of representing "if . . . then —". The meaning of $P \to Q$ in this case is the same as the meaning of $\neg P \vee Q$. When the so-called paradoxes are reinterpreted as disjunctive statements, then each 'paradox' is not paradoxical. For example, consider the sentence $P \to (Q \vee \neg Q)$. Interpreting the arrow as the material conditional, we get $\neg P \vee (Q \vee \neg Q)$. Since either Q or $\neg Q$ is true under classical assumptions, $\neg P \vee (Q \vee \neg Q)$ is true. Even those willing to reject the material conditional usually accept that $\neg P \vee (Q \vee \neg Q)$ is true, if they accept the law of excluded middle. There does not exist a so-called paradox when the material conditional is interpreted in disjunctive form. The paradoxes only arise when we interpret the arrow as implication.

Anderson and Belnap suggest that, in classical logic, the interdefinability of the $P \to Q$ with $\neg P \vee Q$ creates *some* notion of implication:

> Properly understood there are no "paradoxes" of implication. Of course this is a rather weak sense of "implication," and one may for certain purposes be interested in a stronger sense of the word. [1, p. 3]

The stronger sense of implication, they argue, requires a weaker logic. There are a number of inferences validated by logics stronger than the logic **R**[1] which Anderson and Belnap consider troubling, and large portions of *Entailment* [1] are dedicated to demonstrating why such inferences are troubling, and how we might construct logics without such inferences. The argument, then, is that implication should mean something stronger than a disjunction. That is, we ought not use the material conditional of classical logical as a syntactical representation of implication. Instead, a stronger connective is required to capture relevant logician's intuitions about implication.

Historically, the use of intentional connectives for implication was one of the first 'non-classical' approaches.[2] C.I. Lewis created an intensional arrow. By creating more places where propositions could be true or false, intensional connectives have truth conditions which reference other worlds.[3] Lewis defined the strict conditional with the tools of modal logic [15, 16]. However, there are still problematic validities to be found with strict implication. Consider the following:

1. $(P \wedge \neg P) \to Q$
2. $P \to (Q \to Q)$
3. $P \to (Q \vee \neg Q)$

[1]The logic **R** is a common relevant logic, which will be explicated below.

[2]Non-classical here indicating a deviation from the Frege-Russell tradition.

[3]If cannot be said of C.I. Lewis that he interpreted the model operators in terms of what is true "at possible worlds". The adoption of the possible worlds interpretation of model logic occurred only after Kripke's work in the 1950s and 1960s.

These three problems are 'paradoxes of strict implication'. They are paradoxes in the same way the paradoxes of classical implication are paradoxes. That is, given the meaning given by their formal definition, they are not paradoxes. However, if we take implication to be of the stronger kind which Anderson and Belnap argue for, then we are unable to interpret the strict conditional as implication. For instance, if we require implication to be such that $(Q \vee \neg Q)$ does not *follow from* (or is not implied by) P, as Anderson and Belnap argue, then strict implication does not formalize these intuitions.

What makes a logic a *relevant* logic is still under dispute. I will mention a few proposals for historical value. To aid further discussion and circumvent some disagreements about the essential properties of relevant logics, I will take the logic **B** and its extensions which reject both thinning[4] and explosion to be relevant logics. One criterion sometimes considered to be necessary for relevant logics is what has come to be called *Belnap's weak relevance criterion* [27, p. 3]. This criterion, named WR, is as follows:

> WR. That A implies B is a theorem, in symbols $\vdash A \to B$, only if A and B share a sentential variable. [27, p. 3]

This criterion allows a number of relatively weak logics (as compared to classical and intuitionistic logic), and disallows a number of inference rules. Notably the rule of explosion has to go. That is, we must reject $A, \neg A \vdash B$.

Routley takes the rejection of explosion, and thus disjunctive syllogism (hereafter DS), as a requirement of relevant logics [27, p. 5]. Routley points out that rejecting explosion commits us to rejecting other rules of inference which may appears less obviously incorrect. Consider the following argument;

$$\dfrac{\text{A} \qquad \dfrac{\neg A}{\neg A \vee B}\,(\vee\text{-I})}{\text{B}}\,(\text{DS})$$

If we wish to invalidate this argument (which is an instance of explosion), then we must either reject $\vee$-I or DS. Routley argues that, of these two rules, DS ought to be rejected.

To summarize, the motivations for relevant logic form one coherent picture. That is, relevant logicians desire logics with a conditional connective that affords a better interpretation in terms of implication. Whether or not the conditional of classical logic is

[4] Thinning will be defined below.

taken to be some form of implication, it is nonetheless the motivation of relevant logicians that such a conditional *ought not to be interpreted as implication or entailment*. Relevant logics have certain properties which make them more useful in the study of implication or entailment. There are no (or at least fewer) paradoxes of implication for relevant logics.[5] The motivation for relevant logic is not merely to avoid paradoxes. The motivation is to find a logic which better captures the meaning of implication and entailment. The so-called paradoxes are symptoms of the inadequacy of irrelevant logics, which are unable to capture implication as it should be taken to mean. Therefore, a logic which captures implication will happen to solve the so-called paradoxes. However, the paradoxes do not motivate relevant logic, despite the emphasis placed on the paradoxes in the first few pages in [27]. The paradoxes are informative; they indicate the divorce of the arrow connective and implication. Logic is the study of implication.

1.2.1 B: Axiomatization and Extensions

I will now provide axiomatizations for a number of relevant logics. As I will focus on the semantic side of relevant logic, I explicate the syntax in order to make clear which structural rules are accepted and rejected by different relevant logics. Relevant logics have been motivated by arguments rejecting certain structural rules. These structural rules correspond to semantic restrictions, as will be shown in the next section. Presumably the semantic restrictions will correspond (in a good semantics) to the arguments for and against the corresponding axioms. Therefore, we may also judge the semantics of relevant logics by how well the arguments for or against certain syntactical axioms are reflected in the semantics.

Though the names of the axioms and rules are for the most part consistent in the literature, I will use the naming conventions of Routley et al. found in [27] while constructing the axiomatization of **B**. There are other syntactical methods which produce these logics. For instance, there are consecution calculi, which more explicitly display the structural rules which a theorem or inference relies upon. A detailed treatment of the

[5] One might consider Curry's paradox a paradox of relevant logics with contraction. Very roughly, Curry's paradox is a conditional sentence which states something like "if this sentence is true, then P". If the sentence is true, then P is true by application of modus ponens. If the sentence is not true, then the sentence has a false antecedent. A false antecedent makes a material conditional true. Therefore P. Here I have used the material conditional to set up the paradox; however, much weaker logics are able to prove Curry's paradox. For a more detailed analysis, see [4, 29, 32].

consecution calculi can be found in [25]. The following are Hilbert style axioms for the relevant logic **B**:

(A1) $A \to A$

(A2) $(A \wedge B) \to A$

(A3) $(A \wedge B) \to B$

(A4) $((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$

(A5) $A \to (A \vee B)$

(A6) $B \to (A \vee B)$

(A7) $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$

(A8) $(A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$

(A9) $\neg\neg A \to A$

The rules which accompany these axioms are as follows:

(R1) $A, A \to B \vdash B$ (Modus Ponens)

(R2) $A, B \vdash A \wedge B$ (Adjunction)

(R3) $A \to B \vdash ((C \to A) \to (C \to B))$ (Prefixing)

(R4) $A \to B \vdash ((B \to C) \to (A \to C))$ (Suffixing)

(R5) $A \to \neg B \vdash B \to \neg A$ (Rule Contraposition)

The above nine axioms and five rules taken together form the logic **B**. The following non-exhaustive list of axioms may be appended to the logic B in order to produce other logics. I have chosen this small list of possible axioms to note the constructions of a number of popular relevant logics. Again, the naming scheme is consistent with Routley's in [27].

(B3) $(A \to B) \to ((B \to C) \to (A \to C))$

(B4) $(A \to B) \to ((C \to A) \to (C \to B))$

(B5) $(A \to (A \to B)) \to (A \to B)$

(B6) $A \to ((A \to B) \to B)$

(D4) $(A \to \neg B) \to (B \to \neg A)$

(BR1) If A is a theorem, so is $(A \to B) \to B$

Combinations of these axioms are used in the construction of the logics **T**, **R**, and **E**.[6]

T: = **B**+ D4 + B3 + B4 + B5.

R: = **T**+ B6.

E: = **T**+ BR1.[7]

Two more logics of interest are **RW** and **TW**. These logics and **R** and **T** respectively without the axiom B5. Known as absorption or contraction, B5 is worrisome to many relevant logicians. This is because Curry's paradox is derivable in logics with B5. Earlier I alluded to the fact that logics weaker than classical logic can prove Curry's paradox. Contraction, B5, is necessary to prove Curry's paradox, and is often rejected in order to solve the problem. Therefore, there have been a number of arguments which claim that we ought to accept or use an absorption-free (contraction-free) relevant logic [32, 29].

1.2.2 Relevant Logic: Thinning and Explosion

More pertinent to the motivations for relevant logic are the theorems which are rejected. First, I will provide a list of the theorems which are rejected by the logic **R**, but which are theorems of classical logic. Indeed, the theorems in question, in some presentations, are taken as axioms of classical logic. The map between relevant logics provided above will then be useful in terms of the arguments rejecting further theorems. Second, I will provide the arguments for the invalidity of these theorems in order to motivate accepting relevant logic.

[6] For the axiomatizations of other logics, two detailed axiom maps for relevant logics can be found in [19] and [27]. The former, being a relatively simple map, is found in Priest's chapter on relevant logics. In the later, a much more complex map, Routley introduces the axiomatizations for most relevant logics in the literature.

[7] The logic **E** is given a relatively simpler axiomatization in the first few pages of chapter IV in *Entailment* [1, p. 231–3].

I note two theorems which separate **R** from classical logic. The first is thinning, i.e. $A \to (B \to A)$. This theorem corresponds to the structural rule of weakening $(X \Leftarrow X; Y)$. The second is a property of negation usually called explosion, i.e. $A, \neg A \vdash B$. As mentioned above, the route used by Routley to block explosion involves blocking disjunctive syllogism.

The negation operator in **R** is what is called a *strict De Morgan negation* [25, p. 67]. A strict De Morgan negation does not satisfy explosion, but satisfies most other properties of classical negation. We construct a strict De Morgan negation from a De Morgan negation. Unlike many popular negations, a De Morgan negation satisfies all four De Morgan laws as well as double negation introduction and elimination rules [25, p. 65].[8] To obtain a strict De Morgan negation, we add the *semicolon negation rules* to a De Morgan negation, which links the negation to the arrow. Formally, it allows us to prove $A \to \neg B \vdash B \to \neg A$ [25, p. 67].[9]

Let us consider thinning first. The average student in a first year symbolic logic course might find thinning to be a peculiar theorem of classical logic. Thinning may appear to them, if not using structural rules, as $A \to (B \to A)$, or $A \vdash (B \to A)$. A translation of an instance of thinning may appear to be most confusing. Let A be a tautologous statement, or at least a theorem of whatever logic is chosen, say, $A := (P \to P)$. Let B be the contingent statement 'there has been a cat inside my house'. A translation, then, is as follows:

> From $(P \to P)$, we can prove that 'there has been a cat inside my house' implies that $(P \to P)$

This sentence appears to be a very odd claim for the semantics of our logic to validate. Below I will explicate Anderson and Belnap's argument against thinning, which relies upon the use of the word 'implies' in the above translation. On the other hand, the corresponding structural rule (weakening) might appear rather benign in comparison. $X \Leftarrow X; Y$ states that what is provable from one set (or piece of information, etc.) is provable from that set together with another set. However, Anderson and Belnap do not think *provable from* is correctly and accurately captured by a logic which accepts the structural rule of weakening.

[8] The De Morgan laws are $\neg(A \wedge B) \dashv\vdash (\neg A \vee \neg B)$ and $\neg(A \vee B) \dashv\vdash (\neg A \wedge \neg B)$.

[9] The details of the semicolon negation rules are not worth pausing over. What matters to this discussion is the the rule form of contraposition corresponds to the semicolon negation rules.

Anderson and Belnap claim that thinning commits a modal fallacy. To explicate their claim, I will familiarize the reader with some of the specialized terminology required. A familiar fallacy of modality is claimed to be committed when one derives a contingent proposition from a premise set containing only necessary propositions. For example consider an interpretation of the inference $(P \vee \neg P) \vdash P$, for a contingent P. An instance inference states that "from the necessary truth of *either my phone is in my pocket or it is not in my pocketer*, we can infer that *my phone is in my pocket*." This inference is obviously invalid. Disregarding the other oddities of the example, it seems fallacious that a contingent statement is *provable from* a necessary statement. Anderson and Belnap claim a similar fallacy exists for what they call *necessitive* propositions.

A necessitive proposition is a proposition A which is equivalent to another proposition $\Box B$ [1, p. 36]. Notably, in many cases we find that $\Box B$ is just $\Box A$, i.e. A is equivalent to $\Box A$. The exampled used by Anderson and Belnap is the sentences $(A \to A)$, where we have $A \to A$ is equivalent to $\Box(A \to A)$. The sentence $A \to A$ is a necessitive because $(A \to A)$ is equivalent to $\Box(A \to A)$. It is also noted that $(A \to A)$ is a necessary proposition. A necessary proposition is a proposition which is true at every possible world (place, model, etc.) $(A \to A)$ is a necessary proposition because "we have $\Box(A \to A)$" [1, p. 36].

The example given of a necessitive, but not necessary, proposition is the sentence $\Box A$, where A is a contingent proposition. $\Box A$ is not necessary, for *it is plain false*; however, it is necessitive because $\Box A$ is equivalent to $\Box\Box A$, in most modal logics [1, p. 36].[10]

An example of a necessary, non-necessitive proposition is $A \vee \neg A$. This example is used because Anderson and Belnap argue that there is no sentence B such that $A \vee \neg A$ is equivalent to $\Box B$. Their argument is as follows. We see that $A \to (A \vee B)$ even with a strong implication, because the antecedent makes the consequent true. When we take $B = \neg A$, then we get $A \to (A \vee \neg A)$. Still, we do not want to accept $A \to \Box(A \vee \neg A)$ under the stronger sense of implication [1, p. 245]. An instance of $A \to \Box(A \vee \neg A)$ is the sentence "The fact that my phone is in my pocket implies the additional fact that it is necessary that my phone is either in my pocket or not." The consequent of this natural language sentence is implies not by the antecedent, but my some logical, physical, or metaphysical law. Under the strong sense of implication, $A \to \Box(A \vee \neg A)$ must be rejected. We accept that $A \vee \neg A$ *follows from* logic, i.e. is necessary. $\Box(A \vee \neg A)$,

[10] The equivalence of $\Box A$ with $\Box\Box A$ is seen in modal logics wherein $\Box A$ is interpreted as A is true at every world in the model.

however, *follows from* the fact that $A \vee \neg A$ is necessary, and not from the contingent proposition A. Anderson and Belnap argue that the implication connective should make this distinction. When we make this distinction, we see that $A \vee \neg A$ and $\Box(A \vee \neg A)$ are not fully substitutable, and therefore not equivalent. Further the fact there there is no B such that $\Box B$ is equivalent to $(A \vee \neg A)$ follows from a distinction of what implies each. $(A \vee \neg A)$ (while true and necessarily so) is implied by A. For contingent A, no $\Box B$ sentence is implied by A, if we draw the distinction between being implied by logic and being implied by a specific sentence. The sentence $A \to \Box B$ (for any contingent A and for any B) is invalid on this distinction, and therefore A is not equivalent to $\Box B$.

Taking atomic sentences to be an example of non-necessary, non-necessitive propositions, we have examples of (1) necessary, necessitive propositions, (2), necessary, non-necessitive propositions, (3) non-necessary, necessitive propositions, and (4) non-necessary, non-necessitive propositions.

Important to Anderson and Belnap's rejection of thinning is the notion of a *pure non-necessitive*, which is "a proposition which does not entail a necessitive" [1, p. 38]. Anderson and Belnap argue for the existence of pure non-necessitives. That is, they argue that there is at least one p such that $p \not\to (A \to A)$ [1, p. 38]. They suggest the proposition 'Crater Lake is blue' is a pure non-necessitive. They argue that there is no non-necessitive which is entailed by the suggested proposition alone. As an example, they consider the sentence $p \to \Box\Diamond p$, which is valid in the logic **S5**. $\Box\Diamond p$ is a necessitive (in the light of **S5**), for $\Box\Diamond p \equiv_{S5} \Box\Box\Diamond p$. Anderson and Belnap claim that $p \to \Box\Diamond p$ is not universally valid.

> What we hold is that even if it *is* necessary that possibly Crater Lake is Blue [$\Box\Diamond p$], this putative fact does not follow logically from the proposition that Crater Lake is Blue. Even those who hold that it *is* necessarily possible rely on logical considerations ("it is no accident of nature") to buttress their claim — they don't go and look at the lake. [1, p. 38].

Anderson and Belnap claim that $\Box\Diamond p$ does not follow from p alone, but from p in conjunction with a necessitive.[11] This leads Anderson and Belnap to further claim that a pure non-necessitive cannot be expressed as a conjunction with a necessitive conjunct [1, p. 37].

[11] In other words, a proposition A is not equivalent (in terms of entailment) to $A \wedge B$, where B is a necessitive (or logical truth).

With this terminological distinction made, I am now able to explicate Anderson and Belnap's argument against thinning. Their argument against thinning is roughly that the consequent of thinning can be made false while the antecedent is true. In fact, this is the case whether or not the antecedent is both necessary and necessitive! The fallacy of modality lies in the consequent of thinning. It is easy to check that $(A \to A) \to (B \to (A \to A))$ is an instance of $A \to (B \to A)$, i.e. thinning. We note that $(A \to A)$ is necessary, necessitive, and made true by logic alone. To show that $(A \to A) \to (B \to (A \to A))$ is sometimes false, we must show that $(B \to (A \to A))$ is sometimes false. Anderson and Belnap do so by by showing that it commits a fallacy of modality.[12] The fallacy of modality in question is that a pure non-necessitive cannot imply or entail a necessitive [1, p. 37]. Taking B to be a pure non-necessitive, $(B \to (A \to A))$ commits the fallacy of modality, and has false instances.[13] We see, then, that thinning should fail because there is at least one significant instance of thinning which states a falsehood.

Parenthetically, I note that some fallacies of modality are contested [27, p. 15]. The fallacy of modality in question should nonetheless be treated as either a fallacy of modality, or at least something which is to be avoided. That is, however, if we are to accept the stronger sense of implication. So, if we are to accept the stronger sense of implication, then we should accept that these fallacies of modality are to be avoided. If we are to avoid these fallacies of modality, then we must reject thinning and explosion.

I now turn to explosion.

Arguments for the rejection of explosion may be found not only in the literature of relevant logic, but also the literature of paraconsistent logic. I will consider here only arguments for the rejection of explosion which are consistent both with the motivations concerning the study of implication explicated in section 1.2 and with the above arguments for the rejection of thinning. However, there are other arguments for the rejection of explosion. For examples of these arguments, see [18]. I aim to show one unified argument that motivates the rejection of both thinning and explosion.

[12] It just as easy to show that $A \to (B \to A)$ (for contingent A) commits a fallacy of modality when we take the instance where $A \to ((B \to B) \to A)$. For the consequent, $((B \to B) \to A)$, is a necessary sentence entailing a contingent sentence, which is a less contested fallacy of modality. That a necessary proposition entails a contingent proposition should be not dependent on a contingent truth. That is, observation (or whatever we use to determine what contingent propositions are true) should not determine what necessary truths logically entail.

[13] Mingle, written as $(A \to (A \to A))$, must also be rejected by this argument.

Let us suppose that the real world is such that $A \wedge \neg A$ never holds. Nonetheless, we should want a logic which is able to express proofs wherein $A \wedge \neg A$ is in the premise set, or provable from the premise set. If proofs lead from $A \wedge \neg A$ to triviality,[14] then we have an indication that our consequence relation, or conditional connective, does not capture implication in the stronger, relevant sense. We want a logic which captures implication or 'follows from' in which we may express what logically follows from a contradiction. As before, we can retain the assumption that $A \wedge \neg A$ never holds in the actual world. Nonetheless, we may want a logic capable of expressing what logically follows from the supposition that $A \wedge \neg A$ [27, p. 158]. Furthermore, the argument justifying the rejection of thinning may also be used to justify the rejection of explosion.

Consider an instance of $A \wedge \neg A$ where A is substituted for an atomic sentence (i.e. '$p \wedge \neg p$,' where p is an atomic sentence). $p \wedge \neg p$ is pure non-necessitive for neither p nor $\neg p$ is a necessitive or a conjunction. It is thus a fallacy of modality if $p \wedge \neg p$ implies a necessitive. Thus, our counterexample to explosion is that $p \wedge \neg p$ fails to imply $q \rightarrow q$. This counter-example to explosion is a counter-example for same reasons we wish to reject thinning. That is, it is an example of a pure non-necessitive implying a necessitive. Therefore, if we reject thinning for the reasons above, then we must also reject explosion. Although this is enough to justify the rejection of explosion under the larger motivation, one may be temped to accepted a weak version of explosion where a contradiction implies every non-necessitive propositions. This is not acceptable, for our motivations also warrant the rejection of explosion because B does not appear to follow from or be implied by $A \wedge \neg A$ by taking the stronger sense of implication. The stronger sense of implication does require relevance between antecedent and consequent, and this relevance appears to warrant the weak variable sharing property which immediately rules out explosion [27, p. 3–4].

I have now provided arguments for the rejection of both thinning and explosion with a single coherent motivation, i.e. taking implication as centrally important for logic. We now have motives for accepting relevant logics. However, these motives are met with concerns. Some philosophers have voiced concern, claiming that it is unclear what the connectives of relevant logic really mean. In the remainder of this chapter I will explicate these sorts of claims with respect to numerous semantic approaches. Each semantic approach to relevant logic is an attempt to explain what the connectives, and logic in general, mean. The general claim is that the semantic approaches to relevant logic are

[14] Triviality being the affirmation or truth of every sentences we are able to express in the language.

not genuine semantics, but merely formal tools. In the remaining chapters I will describe a genuine semantics and extend it to relevant logics in an attempt to address these concerns.

1.3 Semantics for Relevant Logic

There are a number of semantics for relevant logics. Each of the semantics appears to have its own philosophical problems. The primary philosophical problem for a number of these semantics is the ternary relation between worlds, as will be shown. I will explicate a number of semantics adequate for the relevant logic **B** and some of its extensions. The Routley$*$ semantics, the four valued semantics, and algebraic/operational semantics will be considered. The Routley$*$ semantics are a popular semantics in the literature and the four-valued semantics is parasitic on the Routley$*$ semantics. Algebraic semantics are discussed due to their usefulness.

1.3.1 Routley$*$

The first semantics I will describe is the Routley$*$ semantics. I follow Priest's explication of the Routley$*$ semantics for relevant logic found in [19]. The Routley$*$ semantics is a possible worlds semantics with two-valued worlds. The 'cost' of two valued worlds is an intensional negation: that is, in contrast to the truth conditions in the familiar possible worlds semantics for model logics, the truth conditions for intensional negations "require reference to worlds other than the world at which truth is being evaluated" [19, p. 151].

We begin by defining a Routley$*$ model. "A ternary ($*$) interpretation is a structure $\langle W, N, R, *, \nu \rangle$" [19, p. 189]. W is a set of worlds, points, states of information etc.; $N \subseteq W$ is a subset of worlds; R is a ternary relation on W ($R \subseteq W \times W \times W$); $*$ is a pairing function on worlds such that, for every world w, $w^{**} = w$. The set N of worlds is the set of *normal worlds*. A normal world is a world where the arrow behaves normally (as will be shown). Note that validities are sentences which are true *at every normal world* (unlike more familiar semantics, where validities are sentences true at every world). ν is a truth value assignment which assigns a truth value (1 or 0) to every atomic sentence at each world. The truth conditions for the extensional connectives are as follows [19, p. 151]:

$\nu_w(A \wedge B) = 1$ if $\nu_w(A) = 1$ and $\nu_w(B) = 1$; otherwise it is 0.

$\nu_w(A \vee B) = 1$ if $\nu_w(A) = 1$ or $\nu_w(B) = 1$; otherwise it is 0.

The remaining connectives, $\neg$ and $\rightarrow$, depend on other worlds for their corresponding truth conditions. First, the truth conditions for a negated sentence at a world rely on the world's $*$-pair [19, p. 151]:

$\nu_w(\neg A) = 1$ if $\nu_{w*}(A) = 1$; otherwise it is 0.

Finally there are the truth conditions for $\rightarrow$. I will state a general truth condition for the arrow, then add an extra condition which applies at normal worlds. The general condition is as follows:

$\nu_w(A \rightarrow B) = 1$ iff for all $x, y \in W$ such that $Rwxy$, if $\nu_x(A) = 1$, then $\nu_y(B) = 1$ [19, p. 189]

The normality condition is as follows [19, p. 189]:

For all $w \in W$, if $w \in N$, then $Rwxy$ iff $x = y$

The purpose of non-normal worlds is to create worlds where sentences like $P \rightarrow P$ may fail. Paraconsistent logics have models wherein worlds may model contradictions. That way the paraconsistent logic does not explode due to the consequence relation being satisfied trivially for premise sets containing contradictions.[15] Paracomplete logics dualize so that the denial of sentences like $P \vee \neg P$ may be modeled. Again, if we could not model the denial of such a sentence, then everything would trivially imply it. So, the purpose of non-normal worlds is to create places where the denial of $P \rightarrow P$ may be modeled, but which is not quantified over when determining validities so that $P \rightarrow P$ is a validity. The goal, then, like the paracomplete case, is to be able to model the denial of sentences like $P \rightarrow P$ so that they are not trivially implied by any sentence whatsoever.

The Routley$*$ semantics are also known as the Australian plan.

> On the Australian plan, we assign *exactly one* of 1 or 0 to each wff of a theory at that theory[16] [world, situation, etc...] (or whatever is taken to *realize* a

[15] A familiar and standard definition of a consequence relation might state something like 'if your premise set is modeled, then your conclusion set is also modeled'. In definitions like these, everything trivially follows from a premise set which is unable to be modeled.

[16] A 'wff of a theory' is a sentence in the language being modeled. They are modeled at worlds, which Routley also calls theories. A 'wff of a theory at that theory' means 'a sentence of a language at a model which models said language'.

> theory, ontically or neutrally or epistemically). But, where the theory is inconsistent or incomplete, there will be formula A such that both A, ¬A are assigned value 0, or else both assigned 1. [26, p. 132]

In other words, every sentence at every world is given only one truth value. That being said, take for example an inconsistent world where a sentence receives the truth value True and its negation receives the truth value True. At this world the sentence in question does not receive two truth values. The negation symbol, then, behaves quite differently than it does in classical logic.

We may extend the Routley∗ semantics to provide semantics for the extensions of **B**. One way to extend **B** is to place restraints on the accessibility relation [19, p. 194].[17] We will call this type of constraint a *relational constraint*. However, some relevant logics require another type of restraint, which is that the truths of some worlds bearing certain relations to other worlds will contain all the truths of the other world. We will call this type of restraint a *content constraint*. I will show below which constraints may be added to the Routley∗ semantics in order to produce semantics for **T**, **TW** and **R**.[18] It will be noted that similar types of constraints will be required for the four-valued semantics to be explicated in section 1.3.2.

Constraints placed on the accessibility relation are the first type of constraint I will discuss. Of the logics mentioned above, we are able to construct a semantics for the logic **TW** using only relational constraints. The relational constraints are conditions placed on the R relation itself. For a binary relation, relational constraints include reflexivity, symmetry, and anti-symmetry. Of course, since R is a three place relation, the constraints placed upon it will not be as familiar as the constraints placed upon the binary relation. In general, though, relational constraints do one of two things. Since the R relation is a set of 3-tuples, a relational constraint either forces R to contain certain 3-tuples, or to contain certain 3-tuples on the basis of the 3-tuples already in R.

To give a semantics for **TW** we need to add constraints which correspond to D4, B3, and B4. We find that the semantic conditions C8, C9, and C10[19] correspond to D4, B3, and B4 respectively [19, p. 194–6]. C11 corresponds to contraction, i.e. B5. Although B5

[17] Placing constraints on the accessibility relation is common practice in modal logics. For example, most normal extensions of the modal logic **K** are constructed by placing constraints on the binary relation of **K**. These constructions can be found, for instance, in [8].

[18] See [19] for a more exhaustive list of these two constraints, as I will not explicate as many below.

[19] The names of the constraints as given here as taken from [19].

is not needed to construct **TW**, it is a relational constraint which will be used to construct other logics. The constraints are as follows [19, p. 194–5]:

C8 If $Rabc$,then Rac^*b^*

C9 If there is an $x \in W$ such that $Rabx$ and $Rxcd$, then there is a $y \in W$ such that $Racy$ and $Rbyd$

C10 If there is an $x \in W$ such that $Rabx$ and $Rxcd$, then there is a $y \in W$ such that $Rbcy$ and $Rayd$

C11 If $Rabc$ then for some $x \in W, Rabx$ and $Rxbc$

The class of models for **TW** is a subclass of **B** models. It is the subclass of models where R satisfies C8, C9, and C10.

The second type of constraint requires a new formal symbol, which we will write as $\sqsubseteq$.

> $\sqsubseteq$ is a reflexive and transitive binary relation on worlds. Intuitively, $w_1 \sqsubseteq w_2$ means that everything true at w_1 is true at w_2. [19, p. 198]

Suppose we have $w \sqsubseteq w'$. For every atomic sentence P, the following hold:

1. if $\nu_w(P) = 1$ then $\nu_{w'}(P) = 1$
2. $w'^* \sqsubseteq w^*$
3. if $Rw'w_1w_2$ then ($w \in N$ and $w_1 \sqsubseteq w_2$) or ($w \notin N$ and Rww_1w_2) [19, p. 198]

The first condition ensures the intuitive reading mentioned by the quote just above. The second and third conditions "are sufficient to ensure that this condition holds for all sentences" [19, p. 198]. For instance, the second condition ensures that everything false at w' is also false at w, which leaves open the possibility that w' has more truths and fewer falsehoods.

With this novel machinery in place, I will now list some of the possible conditions which are now able to be formulated:

C12 If $Rabc$ then, for some x such that $a \sqsubseteq x$, $Rbxc$

C14 If $a \in N$, then $a^* \sqsubseteq a$; but if $a \in W - N$, then Raa^*a

Starting from the semantics for **TW** above, we may add C12 to construct the semantics for **RW**. From there, we may obtain semantics for **R** by adding C11. Semantics for **T** are obtained by adding both C11 and C14 to **TW**.[20]

1.3.2 American Plan

The American plan is an attempt to get back some of the classical intuitions about negation, but it does so at the cost of adding the truth values Both and Neither. For this reason the American plan semantics are sometimes referred to as the four-valued semantics. Originally, the American plan for the semantics of relevant logic was highly complicated, and required two ternary relations [27, p. 319]. With later developments, the American plan semantics (as well as the Routley$*$ semantics) were simplified in [21, 22]. The simplified four-valued semantics at that point could only provide semantics for the logic **BD** (which is weaker than **B**), and for the positive extensions of **B**. The paper [23] by Greg Restall provides a four valued semantics for **B** and some extensions.[21] I will first explicate the four-valued semantics for **B**, then show how to extend the four-valued semantics for extensions of **B**.

Restall proved that, "any four-valued model of **B** can be converted to a two-valued [Routley$*$] model, and conversely" [23, p. 149]. Although this is to say that the two semantic systems are formally equivalent, their equivalence is nevertheless due to the fact that the four-valued semantics is parasitic on the Routley* semantics. Specifically, the duality provided by the $*$ operation in Routley$*$ semantics is mimicked in order to provide sufficient truth conditions for negated conditionals. The American plan was motivated to escape the $*$ operation, and it has so far failed to escape the $*$ operation while at the same time providing a semantics for all of the extensions of **B** mentioned above. I will say more about this in section 1.4.

Part of the original motivation for the American plan is as follows:

> The American plan has us adhere resolutely to the classical view that $\neg$ A holds iff A does not hold, everywhere. But it allows that, in addition to being assigned a simple 1 or 0, a formula may be assigned both 1 and 0 or neither. [26, p. 132]

[20]We will note that C11 corresponds to the axiom B5, C12 to the axiom B6, and C14 to an axiom not previously listed.

[21]Edwin Mares extended this approach to provide the semantics for **R** in [17].

Restall shows that this is insufficient to produce a semantics for **B**. I will explicate (1) how Restall translated Routley$*$ models into American plan models, and (2) what additional requirement must be met by the American plan models in order to model **B**.

To understand how to translate the two-values $*$-models into four-valued models, we first need to define four-valued models. Then we will define the four-valued models which are *closed under duality*. Like the Australian plan, validities are those sentences true at every normal world. Thus, when we are using the semantics to produce counter-examples to corresponding invalidities, we need only consider one normal world in these models (i.e. world g in [23], and world 0 in the Routley$*$ semantics of [19]). In the literature there is often only one normal world considered, but there need not be.

In the American plan, the valuation rules for the extensional connectives $(\wedge, \vee, \neg)$ "can be made to *look just like* world-relativised classical evaluations" [26, p. 134]. Note here that my notation differs from both Routley and Restall: I use relational valuations. The truth conditions for negation, conjunction, and disjunction are as follows:[22]

$$\begin{aligned}
\textbf{Negation Rules: } & (\neg A)\rho_\alpha 1 \text{ iff } (A)\rho_\alpha 0 \\
& (\neg A)\rho_\alpha 0 \text{ iff } (A)\rho_\alpha 1 \\
\textbf{Conjunction Rules: } & (A \wedge B)\rho_\alpha 1 \text{ iff } (A)\rho_\alpha 1 \text{ and } (B)\rho_\alpha 1 \\
& (A \wedge B)\rho_\alpha 0 \text{ iff } (A)\rho_\alpha 0 \text{ or } (B)\rho_\alpha 0 \\
\textbf{Conjunction Rules: } & (A \vee B)\rho_\alpha 1 \text{ iff } (A)\rho_\alpha 1 \text{ or } (B)\rho_\alpha 1 \\
& (A \vee B)\rho_\alpha 0 \text{ iff } (A)\rho_\alpha 0 \text{ and } (B)\rho_\alpha 0
\end{aligned}$$

Restall proved that these truth conditions do not change when we are constructing four-valued models out of Routley$*$ models [23, p. 147].

To define truth conditions for the conditional, mimicking certain aspects of $*$ models is required. It is shown that "any $*$-interpretation that models **B** generates a four-valued interpretation on the same set of worlds, with exactly the same truths in each world" [23, p. 146]. The four-valued interpretation created takes a pair of worlds from the Routley$*$ interpretation, say w and w^*, and collapses them into one world. The collapse into one

[22] These truth and falsity conditions alone are sufficient to produce the logic **FDE**. The logic **FDE** is an conditional-free logic which in its usual interpretation has the truth values of True, False, Both, and Neither. **FDE** is the base of a number of relevant logics, including all the extensions of **B** to be considered. Intensional arrows are defined upon **FDE** to produce the relevant logics in question.

world is achieved by requiring the following two conditions are met [23, p. 146]:[23]

$$(A)\rho_w 1 \text{ if and only if } \nu_w(A) = 1,$$
$$(A)\rho_w 0 \text{ if and only if } \nu_{w*}(A) = 0$$

The truth conditions for negation, conjunction, and disjunction are as noted above for the four-valued semantics. The truth and falsity conditions for the conditional are significantly parasitic on Routley$*$ semantics. The truth condition is as follows [23, p. 147]:

$$(A \to B)\rho_w 1 \text{ iff } \forall(w', w'') \text{ such that } Rww'w'', \text{ if } (A)\rho_{w'}1, \text{ then } (B)\rho_{w''}1$$

For normal worlds, as it was in the Routley$*$ semantics, we require an additional constraint: if $a \in N$, then $Rabc$ if and only if $b = c$.

What is required for a suitable falsity conditions for the conditional connective of **B** is that the four-valued interpretations are closed under duality. An interpretation is *closed under duality* when "the dual of every world in the interpretation is also a world in the interpretation" [23, p. 148]. What is means is that for every world w in a model, there exists another world w° such that the following hold [23, p. 148];

1. Every sentence which receives the truth value True (False) receives the same truth value in the dual world.

2. Every sentence which receives the truth value Both (Neither) receives the truth value Neither (Both) in the dual world.

In other words, dual worlds are where sentences receiving the truth values Both and Neither trade truth values.

Having the dual operator in place, the falsity conditions are defined as follows [23, p. 148]:

$$(A)\rho_w 0 \text{ if and only if it is not the case that } (A)\rho_{w^\circ}1$$
$$(A \to B)\rho_w 0 \text{ if and only if it is not the case that } (A \to B)\rho_{w^\circ}1$$

We thus get sufficient falsity conditions for the conditional of **B**, but at the cost of adding a $*$ operator ($\circ$) into the four-valued semantics. Not only does the semantics require that

[23]Keeping the ρ valuations for four-valued interpretations, and the ν valuations for Routley$*$ interpretations.

the duals of every world are also in the model, but that the falsity conditions rely on the dual worlds.[24]

Nonetheless, with the $*$ operator explicitly in the four-valued semantics we get our desired result:

> The collection of four-valued interpretations closed under duality is sound and complete with respect to **B**. [23, p. 149]

We thus have a four-valued semantics for **B**. Extending **B** is quite similar to extending **B** in Routley$*$ semantics. A list of the axioms and their corresponding constraints, both on R and content inclusion, may be found one page 150 of [23]. Again, we just take the constraints for the corresponding axioms and we arrive at the semantics for the extensions of **B**.

For example, I will present the constraints which, when added to **B**, produce **TW**. We add what Restall calls D20, D3, and D4 [23, p. 143,150]:

D3 $\exists x(Rabx \wedge Rxcd) \Rightarrow \exists y(Racy \wedge Rbyd)$

D4 $\exists x(Rabx \wedge Rxcd) \Rightarrow \exists y(Rbcy \wedge Rayd)$

D20 $Rabc \Rightarrow Rac^*b^*$

We construct stronger logics by adding the appropriate constraints. The constraints are the same for the Australian plan and the American plan, so long as we take the $*$ operator in the constraints to be star-worlds in the Australian plan and dual worlds in the American plan.

1.3.3 Algebraic Semantics

I assume that most readers are familiar with some of the key concepts of algebraic logic. I will state a number of relevant definitions so that I may later refer back to them. An informative introduction to these concepts is Davey and Priestley's *Introduction to lattices and order* [10].

Definition 1. *A* partially ordered set *(or* poset*) $(S, \leq)$ is a set S ordered by a relation $\leq$ which is reflexive, transitive, and anti-symmetric.*

[24] Routley seems to have been aware of the link between the arrow and negation which is created in the four-valued semantics before and after the simplifications made [26, p. 133].

Definition 2. *Suppose X is a poset. If $Y \subseteq X$, a* meet *for Y is the greatest lower bound for Y in X, and a join for Y is the least upper bound. We write '$\cap$' for meet, and '$\cup$' for join.*[25]

Definition 3. *A poset is a* meet semi-lattice *if meet is idempotent, associative, and commutative — or, equivalently, that every non-empty finite subset of the poset has a meet. A poset is a* join semi-lattice *if join is idempotent, associative, and commutative — or, equivalently, that every non-empty finite subset of the poset has a join. A* lattice *is a poset which is both a meet semi-lattice and a join semi-lattice. Additionally, a lattice is* bounded *if it has a top element (usually called 1) and a bottom element (usually called 0).*

Definition 4. *A lattice is* distributive *if for all elements x, y, z in the lattice, $x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$.*

Definition 5. *The complement of an element x in a bounded lattice is another element y such that $x \cap y = 0$ and $x \cup y = 1$. A lattice is* complemented *if every element has a complement. It is well know that in any distributive lattice if an element has a complement, then it has only one (i.e., complements are unique). In such as lattice, we write x' as the complement of x.*

Definition 6. *A* Boolean algebra *is a distributive complemented lattice. It is common to write the Boolean algebra $\mathcal{B}$ as $\mathcal{B} = \langle B, \cap, \cup, ' \rangle$.*

Algebraic semantics is a common semantics for many systems of logic. For many logics there exists a natural class of corresponding algebras. The *operators* in these corresponding algebras quite naturally interpret the connectives present in the respective systems of logic. It turns our that the class of algebras which naturally interprets classical propositional logic is the class of Boolean algebras.

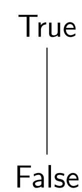
A *valuation* in a Boolean algebra $\mathcal{B} = \langle B, \cap, \cup, ' \rangle$ is a map from the atomic sentences of our language to the elements of the Boolean algebra. We write $\llbracket A \rrbracket$ for the element in the Boolean algebra the atomic sentence A is mapped. We then extend the valuation to every formula in our language as follows, where A and B are any sentence in the language:

$$\begin{aligned} \llbracket A \wedge B \rrbracket &= \llbracket A \rrbracket \cap \llbracket B \rrbracket \\ \llbracket A \vee B \rrbracket &= \llbracket A \rrbracket \cup \llbracket B \rrbracket \\ \llbracket \neg A \rrbracket &= \llbracket A \rrbracket' \end{aligned}$$

[25]This notation is to avoid confusion with the connectives of the logical systems investigated.

We will call a formula A *valid* if every valuation on every Boolean algebra is such that $[\![A]\!] = 1$. We will say that $\Gamma \vdash \delta$ iff for every valuation on every Boolean algebra, $\bigvee\{\, [\![\gamma]\!] | \gamma \in \Gamma\} \leq [\![\delta]\!]$. The valid formulas defined this way are exactly the theorems of classical propositional logic, and the arguments made valid by this definition are exactly the inferences valid in classical propositional logic.

As should be expected, we are able to recover the truth table semantics of classical propositional logic. We may do so by considering an important Boolean algebra. Up to isomorphism, there is exactly one two element lattice, often called **2**. We represent the algebra **2** with a Hasse diagram as follows:



A valuation is a mapping from the atomic sentences of our language into the ordered set above. The mapping is extended to the connectives as above. Interpreting $[\![A]\!]$ as the truth value of A, the truth tables of classical logic fall out of this valuation.

The interpretation of the logical connectives of our language in terms of algebraic operators not only enables us to recover the truth tables, but also the inference rules corresponding to the connectives. If we interpret '$\leq$' as 'implies', then a number of common inference rules associated with conjunction and disjunction correspond to the following properties of Boolean algebras:

$$z \leq x \cap y \text{ iff } z \leq x \text{ and } z \leq y$$
$$x \cup y \leq z \text{ iff } x \leq z \text{ and } y \leq z$$

Indeed, the inferences rules these properties correspond to as exactly the introduction and elimination rules for disjunction and conjunction in natural deduction systems.

The arrow of classical propositional logic, the material conditional, is also translated quite simply into Boolean algebras. We may define the arrow with the following stipulation:

$$\text{(MC) } z \leq x \to y \text{ iff } z \cap x \leq y$$

The inference rules commonly known as modus ponens and conditional proof naturally correspond to this definition.

The conditional of relevant logic is not the material condition. The conditional of relevant logic, however, does satisfy a definition similar to that of (MC), but replacing the meet in (MC) with an operator we have not yet seen. This new connective is called *fusion*, and will be written as '$\circ$'. To interpret it, we have to add semi-group operations to our lattices.

Definition 7. *A* groupoid *is defined as "a collection of objects together with a binary operation on those objects" [25, p. 165]. We will write $\bullet$ as a groupoid operation.*

Definition 8. *A* semi-group *is a groupoid where the groupoid operation is associative. We say a groupoid is an* ordered groupoid *when the groupoid is ordered by an ordering relation which "*respects *the groupoid operation ... That is, if $a \leq a'$ and $b \leq b'$ then $a \bullet b \leq a' \bullet b'$" [25, p. 165].*

In a number of relevant logics, the Fusion operator, an operator often considered a sort of 'and' that differs from conjunction, corresponds to the groupoid operation in an ordered groupoid in the same way conjunction corresponds to meet [25, p. 165]. With this machinery, we may then define the conditional of relevant logic with the following definition:

$$\text{(RC) } z \leq x \to y \text{ iff } z \circ x \leq y$$

There are algebras with these properties. However, interpreting the connectives of a logic with the operations in these algebras is not as natural as it was for Boolean algebras and classical propositional logic.

In turns out that there are classes of algebra which naturally correspond to various relevant logics. For example, the logic **RM** is modeled by what are called Sugihara models. Note that **RM** is **R** with the addition of the axiom Mingle, i.e. $A \to (A \to A)$. I explicate Sugihara algebras, also known as RM_{2n+1} models, to show their complexity. Intuitively, the complexity of algebraic models may only increase as we consider the algebras for weaker logics. It will be noted in the next section how the complexity seen here is enough to dismiss algebraic models as a candidate for being combined with the preservationist approach explicated in the next chapter.

As explicated in by Restall, the propositions of our language will be the integers $-n$ to n, with the usual ordering:

$$\{-n, -(n-1), \ldots, -1, 0, 1, \ldots, n-1, n\} \text{ [25}, p.173]$$

Conjunction is interpreted at meet, Distinction as join. We interpret $\neg A$ as $-A$ [25, p. 173]. Fusion[26] and the arrow are interpreted as follows [25, p. 173]:

$$a \to b = \begin{cases} -a \vee b & \text{if } a \leq b \\ -a \wedge b & \text{if } a > b \end{cases} \qquad a \circ b = \begin{cases} a \wedge b & \text{if } a \leq -b \\ a \vee b & \text{if } a > -b \end{cases}$$

This produces a model for **RM**.

More generally, the model where we take the entire set of integers ($\mathrm{RM}_{\mathbb{Z}}$) "captures exactly the logic **RM** in the language $\wedge, \vee, \to, \circ, \neg, t$[27]" [25, p. 173]. Taking it that weaker logics include an increased number of algebraic models, we know that the complexity of algebraic models for relevant logics can be even more complex. The difficulties arise in the philosophical interpretation of the models.

1.4 Problems with the semantics

My aim in this section is to highlight a number of problems with the semantics introduced in the last section. I have not exhausted all of the possible problems with these semantics, but I have selected a number of problems based on (1) how seriously they appear to be taken in the literature by both relevant logicians and those more critical of relevant logic, and (2) how well they can be interpreted with the preservationist project of the next chapter after being reconstructed.

1.4.1 Routley$*$

Negation in Routley$*$ semantics is thought to be problematic by many philosophers, and, as we have seen in the motivation for the American plan, by other relevance logicians. It appears to lack a satisfactory philosophical interpretation. Soundness and completeness of a model theoretic approach does not means a philosophically satisfying semantics. If a model structure is sound and complete with respect to a logic, we write $\Gamma \vdash \delta$ iff $\Gamma \models \delta$. This means that for every correct proof there does not exists a countermodel, i.e. there is a semantic 'proof' or argument which demonstrates why you cannot model the conclusion if you have modeled the premise set. Additionally, it means that for every incorrect proof

[26] Fusion is a connective added in some presentations of relevant logics. I do not include the fusion connective in the relevant logics I consider, but one is definable from the arrow connective.

[27] t corresponds to *identity*, which is an element x such that, for any element y, $y = y \bullet x = x \bullet y$.

there does exist a corresponding countermodel. However, a sound and complete model theory does not mean that the model theory necessarily provides a philosophical interpretation of the logic. What is in need of explanation for the Routley$*$ models, at least, is the truth conditions for negations, i.e. $\nu_w(\neg A) = 1$ if $\nu_{w*}(A) = 1$.

A satisfactory philosophical interpretation or motivation for the Routley$*$ semantics and its negation should justify the properties of the $*$ operation. However, B.J. Copeland notes that "the characteristics of $*$ have been selected on a purely ad hoc basis" [9, p. 410]. The characteristics of the $*$ operation are only chosen, at least according to Copeland, in order to provide a semantics which does its formal job correctly.

> If the only constraint on $*$ is that the resulting theory should validate the right set of sentences, then we are indeed in the presence of merely formal model theory. [9, p. 410]

Copeland goes so far as to produce a philosophically uninteresting formal semantics using model theory. Copeland's uninteresting semantics is able to do a number of things the $*$ operation can do. There is, then, a significant philosophical difference between a formal, genuine semantics and a formal sound and complete model theory.

Philosophers and logicians have suggested various interpretations of the Routley$*$ semantics that answer objections including Copeland's. According to one such interpretation given by Restall, the $*$ worlds are given a philosophically satisfying interpretation in term of the compatibility of worlds with other worlds. Let us write A is true at the world x as $x \models A$. We write compatibility as xCy, which means the state (world) x is compatible with the state (world) y.[28]

Incompatibility is as follows:

> Consider what it is for $x \models \neg A$ and $y \models A$ to hold. Then x and y are incompatible, because according to x, A is false, while according to y, A is true. (This is more than the case where $x \not\models A$ and $y \models A$, for then x and y may still be compatible, for x may be incomplete 'about' A — x may neither support A nor $\neg A$.) [24, p. 61]

[28] Routley$*$ semantics include incomplete and inconsistent worlds. The compatibility relation is used to create/interpret $*$-worlds. Therefore inconsistent and incomplete worlds must be compatible with other worlds. That is, if we wish to model the more popular relevant logics.

Note that negation is naturally constrained by compatibility, and conversely. From here, a number of constraints on the C relation are considered which ultimately justifying the Routley$*$.

Restall's interpretation justifies $*$-worlds by supposing that for each world x, there exists a maximally compatible world. Let that world be called x^* [24, p. 63]. If we require that C is symmetrical, a seemingly natural assumption, and if we require that $x^{**} \leq x$, what appears to follow from our aforementioned desire to pick out a maximally compatible world, then we have produced the very $*$-function of relevant logic [24, p. 63]. I take the biggest addition to the justification of Routley$*$ semantics to be justification for the $*$ world as the maximally compatible world. If a world is inconsistent, then it is not maximally compatible with itself. Instead, an inconsistent world is maximally compatible with a world that is incomplete, because each the conjuncts of a contradiction at the inconsistent world must each not be true at the incomplete world.

However, there are still problems to be met. The constraints on the C relation which Restall considers are open to Copeland's criticism above. It is when all the conditions Restall considers are taken together that the Routley$*$ semantics are justified. Restall himself is not completely satisfied with the justification for these conditions:

> I am less certain of these than of the conditions we have seen so far. The discussion ahead is not intended to be 'the complete definitive story' about negation, but only one way that our account of negation can be developed. [24, p. 61]

We know that these conditions will give the correct models, but it is unclear whether or not this account of negation is philosophically satisfying. That is, it is unclear whether or not these models deserve to be called a (genuine) *semantics*. What we desire is a unified justification, both justifying peculiarities of the Routley$*$ semantics and the adoption of relevant logic. Restall's argument appears to justify (at least some of) the peculiarities of the Routley$*$ semantics, but it is unclear whether Restall's argument forms a coherent whole with the arguments for adoption of relevant logic.

To discuss the philosophical problems posed by the presence of non-normal worlds in our models, let us first reiterate the role of non-normal worlds. Specifically I emphasize two properties of non-normal worlds. The first, and often emphasized, is that these worlds are used as places where conditional statements may receive random and odd truth values so that sentences like $P \to P$ may fail to be modeled at some worlds. This ensures that not just any sentence implies sentences of the form $P \to P$. The second, is that theoremhood is never taken to quantify over the truths of non-normal worlds. If it did, then $A \to A$ would fail to be a necessary proposition and a validity.

The arrow connective relies on non-normal worlds in order to correctly show what logically follows from the failure of a necessitive arrow statement. The other connectives do well without non-normal worlds, as is demonstrated in both the Routley$*$ and the four-valued semantics for **FDE**. Non-normal worlds are required to treat the arrow connective as we treat the extensional connectives of **FDE**. Of the problems had with the Australian and American plans, non-normal worlds are one of the least challenging. **FDE** is rather weak and uninteresting without a implication connective. If we wish to add one, then we may want the same expressive power we gained with **FDE** in the first place, but applied to the arrow. That is, we may with to express implications where $A \to A$ fails, or its negation holds. Thus, I believe the question is not whether or not non-normal worlds make sense. There is a satisfactory philosophical interpretation of non-normal worlds. However, this interpretation relies on the justification for a logic powerful enough to express what is implied by the failure of, say, $A \to A$. The philosophical problems which arise given non-normal worlds are more interesting. In fact, the ternary relation enables the non-normal worlds to be used to provide the semantics for relevant logics.

There have been a number of attempts at an interpretation of the ternary relation. As should be expected, these attempts have been interpretations of the semantics in general. An inventory of some of the interpretations is given in [13], which includes the three interpretations given in [3]. I will briefly describe a few these possible interpretations.

After introducing the ternary relation in the autocommentary of the second edition of *In Contradiction*, Graham Priest summarizes both the use of the ternary relation and the difficulties which come with it:

> A natural question at this point is what, exactly, the ternary relation R means. Various suggestions concerning this have been made, though none of them is entirely satisfactory. But this is perhaps not so important. If w is a logically impossible world, then $\Rightarrow$ may behave in pretty much any way one likes. If $Rwxy$, then y just records whatever you can get from a conditional, $\alpha \Rightarrow \beta$, given the information α contained in x. [18]

Naturally, we would like to consider places where the consequent of thinning would fail. That is, we need a place where $p \to (A \to A)$ fails, while still being committed to

$(A \rightarrow A)$. It is important to keep in mind that theoremhood is determined by the truths at all and only normal worlds. So, introducing non-normal worlds appears to be apt. It affords us the ability to separate, in the semantics, p from $(A \rightarrow A)$. That is, it lets us consider worlds where p may be true while $(A \rightarrow A)$ is false.

Now, we want the truth and falsity of conditional statements at one world to depend on other worlds. That is, we want an intensional arrow.[29] However, we must now be careful in defining a relation between worlds. That is, we require that $(A \rightarrow A)$ is true at every normal world, and $p \rightarrow (A \rightarrow A)$ is not true at some normal world. Introducing a ternary relation and a normality condition, as described in section 1.3, lets us formally have normal worlds as which $(A \rightarrow A)$ is true and $p \rightarrow (A \rightarrow A)$ is false. However, the ternary relation lacks a philosophical interpretation. Why are we using it? Well, the ternary relation works; it does everything we want it to do formally. As Priest points out, we desire that the ternary relation lets normal worlds access non-normal worlds, and let non-normal worlds have conditionals which behave abnormally. The obstacle of trying to find an acceptable philosophical interpretation for the ternary relation is a difficult task. It is arguable that we have an acceptable philosophical interpretation for the familiar binary relation of possible worlds semantics. For example, some interpret Rxy as y is accessible to x, or as y is possible from the perspective of x. Even if one does accepts such an interpretation for the binary relation, the ternary relation is not given an adequate interpretation so easily.

Interpreting the ternary relation becomes even more difficult when the extra conditions are placed on the ternary relation to produce stronger relevant logics. For the binary relation, familiar properties such as reflexivity and symmetry have somewhat natural interpretations. Constraints on the ternary relation, such as ‘ If $Rabc$ then for some $x \in W, Rabx$ and $Rxbc$’, do not appear to have as natural an interpretation.

One interpretation of the ternary relation is that, for $Rxyz$, “one can view the points x, y, z as pieces of information, with R saying that z contains the combination of the information in x and y” [13, p. 536]. This interpretation developed out of Urquhart’s semantics for relevant logic found in [31]. Intuitively, this interpretation seems to motivate a logic stronger than **B** because in the semantics of **B** the truths (or falsehoods) of x and y are not necessarily contained in z (given $Rxyz$). This interpretation, however, could motivate the logic **B** with some work. In **B**, when $Rxyz$, the worlds x, y, and z can be completely independent of one another in terms of what is true and false at each

[29] This is motivated by the motivations for relevant logic found in the earlier in this chapter.

world. What happens when two pieces of information are combined in Urquhart's semantics is similar to (but not quite the same as) set theoretic union, and what it means for this piece of information to be contained at the world z is simply content inclusion. If combining pieces of information and containing pieces of information were given a weaker interpretation, then this approach may be capable of interpreting the logic **B**.

Another possible interpretation claims that, "One way to think of a conditional If A then B is as asserting an absence of counterexamples" [3, p. 600]. That is, there is no point (place, situation, world, etc.) where the antecedent is true and the consequent is not. The ternary relation, on this account, generalizes the point to pairs of points. A counterexample is, then, a pair of points where the antecedent is true at the first, and the consequent is not true at the second. Further restrictions may be placed on what counts as a counter-example point pair, and further development of this interpretation should be focused on giving a detailed account of point pairs.

These interpretations, and others, still have problems. Although they have problems, many of the extant interpretations are still being developed. For some interpretations, then, it is currently difficult to say whether or not it will be philosophically satisfying when fully developed. Nevertheless, as it stands there are problems with the extant interpretations.

1.4.2 American Plan

As I noted earlier, the American plan, in terms of the four-valued semantics given above, failed to achieving one of its goals. Specifically it failed to remove the $*$ operation in the semantics :

> The reason for the four-valued semantics is to get *away* from dualising operators, and to give negation a more pleasing modeling. [23, p. 150]

Parenthetically I note that other four-valued interpretations have been developed (to some extend). There exists a non-dualizing four-valued interpretation which is capable of modeling the logic **RW** (a.k.a. **C**) [23, p. 151]. However, it appears to be too limited, for it seems to only provide a semantics for **RW** [23]. Another option is suggested by Restall, but this option lacks any completeness proofs and "systems like **R** and **CK** cannot be modeled along these lines... [and] nothing weaker than **R** or **CK** can be modeled with this semantics" [23, p. 158]. Unlike the American plan, these interpretations are quite limited to specific relevant logics. Ultimately being limited as such could be to an interpretation's benefit, but *only* if the logic(s) the interpretation is limited to are desired. Early developments in the semantics of relevant logics showed a relationship between a large number of relevant logics. This relationship is similar to the relationship found in the possible worlds semantics between modal logics. The acceptance of possible worlds semantics was accelerated by the fact that possible worlds semantics helped show this relationship. The same could be hoped for any semantics of relevant logic which display the relationship between relevant logics.

The American plan was designed to solve the problem of the odd negation of the Routley$*$ semantics. Yet it requires part of the machinery in order to provide an interpretation for the basic relevant logic **B**.

> While this is saving the four-valued interpretation *by an explicit use of '$*$'*, which the four-valued interpretation is designed to avoid, *there does not seem to be any way of avoiding it*, if the truth conditions of entailment are to be kept as they are, as some kind of duality operator is the natural way to model rule-contraposition, which is the characteristic rule of **B**. [23, p. 149](Emphasis mine)

If it is unavoidable, then we should hope that it may be justified in the four-valued semantics independently. However, the four-valued interpretation does not seem to be able to provide independent reasons to suppose that the dual of a world will be in every model the world is in, and more importantly that the falsity conditions at that world refer to the dual world. This is the first problem of the American plan, and is a problem shared in slightly different words with the Routley$*$ semantics.

Non-normal worlds are still just as justified or unjustified as they are for the Routley$*$ semantics. There is no difference between the non-normal worlds of a $*$-interpretations and the non-normal worlds of a four-valued interpretation. We still require non-normal worlds in order to give a semantic interpretation of the invalidity of $(A \to A) \to (B \to (A \to A))$. That is, we use non normal worlds in order to provide worlds (or, situations, points, etc.) where $(A \to A)$ is not true. The simplified semantics given in Section 1.3.2 benefit over previously unsimplified semantics of the American plan, for they do not require two ternary relations. However, we appear to be stuck with at least one ternary relation. Therefore the American plan and Australian plan seems to have the same difficulties with non-normal worlds and the philosophical interpretation of the ternary relations. The difference, to the surprise of no one, is the treatment of negation.

1.4.3 Algebraic Semantics

Algebraic semantics appear natural to logics like classical logic and intuitionistic logic. For example, in classical logic we can map the sentences of the logics onto the a two-valued boolean algebra and understand it as mapping from sentences letters to true or to false. It is at times philosophically difficult to do so for relevant logics. There is no doubt that algebraic models for relevant logics are useful for many purposes, but they are somewhat philosophically unsatisfying. Consider the Sugihara algebras for the logic **RM**. The algebras for weaker logics only allow more complicated and complex algebras. I claim that the Sugihara models are not very philosophically satisfying as a semantics. What follows is that the algebraic semantics for relevant logics weaker than **RM** are, at best, equally unsatisfying.

Consider the Sugihara model for all of the integers. It is difficult to give a satisfying philosophical interpretation of the difference between the element -389 and the element -388. If we take each element of the algebra to be a different truth value, then we end up with a large number of truth values. There are some who advocate a 'degrees of truth' interpretation, which they claim is apt for modeling vagueness. I believe such an approach runs into serious objections.[30] Rather than explicating those objections, I will note that the difference between 389 and -389 is a more pressing difficulty, as we will see below.

This brings us the the interpretations of the logical connectives in terms of the algebra. We map atomics onto the elements of the algebra, but we find philosophical trouble with negation and implication. Implication is the main difficulty, for it does not map onto as simple an operation as the other connectives:

> There is no *operation* in intensional lattices that correspond to relevant implication. Relevant implication can be represented in intensional lattices only as a *relation*, so that an axiom of R like $(A \to B) \to ((B \to C) \to (A \to C))$ can only be imperfectly represented in intensional lattices. [1, p. 352]

I have shown earlier in this chapter that there is an operation which corresponds to

[30]To the interested reader I suggest the relevant Sorites paradox and fuzzy logic literature. For example, see [19, p.564–586].

implication in Sugihara models. This operation is as follows;

$$a \to b = \begin{cases} -a \vee b & \text{if } a \leq b \\ -a \wedge b & \text{if } a > b \end{cases}$$

This operation, however, is not a simple operation. In fact, it is not as easily given a philosophical interpretation as meet or join.

Negation is also troubling. We map $\neg n$ to $-n$. A good philosophical understanding of negation in this case would require a satisfying philosophical interpretation of the differences of the elements in the algebra. Again we require that each element of the algebra and the relations between them are given a satisfying philosophical interpretation. As noted above, one attempt to make sense of a large number of truth values is by considering them as 'degree of truth'. However, even if we can make sense of 'degrees of truth', it is not clear what a *negative* degree of truth might be. Negation is thought to have an intuitive interpretation in terms of positive degrees of truth. In such systems, $A \vee \neg A$ usually denotes the top element. In these cases, $\neg A$ is quite naturally interpreted as being true to the extent that A is less than completely true.[31] However, this intuitive interpretation is absent in the presence of negative and positive degrees of truth. The sequel sections attempt a philosophically satisfying semantics of relevant logic which do not require commitment to strange truth values. For my purposes, then, algebraic semantics will not be considered further.

[31] Consider fuzzy logic as an brief example. In some fuzzy logics, we assign a decimal between 0 and 1 (inclusively) as a sentence's truth value. The natural reading of negation is that, if the truth value of A was , say, 0.6, then the truth value of $\neg A$ would be 0.4.

Chapter 2

Preservationism

2.1 Introduction and Motivation: Preservation and Ambiguity

In the last chapter I presented relevant logic, its semantics, and a few of the common problems to be found with the semantics. In this chapter I will explicate Bryson Brown's preservationist approach to paraconsistent logic. By doing so, I will be providing an approach to logic and its semantics which I will extend to relevant logics in the next chapter. I choose to extend Brown's approach because the problems Brown's approach is intended to solve are also present in relevant logics; I will show that the extended approach solves these problems as well.

Brown defends paraconsistent logic from an objection given by B. H. Slater by developing a preservationist treatment of paraconsistent logic in which peculiar truth values are absent [5, p. 489]. Brown uses this approach to first develop **LP** [5], and then **FDE** [7, 6]. Slater's objection is similar to a number of the objections to relevant logic. So, by showing Slater's objection and how Brown defends paraconsistent logic against it, I will be able to discuss the analogs of Slater's objection and Brown's response in the case of relevant logics.[1] We are able to solve at least some of the problems with the semantics of relevant logic by extending Brown's approach to relevant logic. Therefore, in this chapter I plan to show Brown's approach and how it responds to Slater's objection.

[1]The analog of Slater's objection in the case of the semantics of relevant logic is the same paraconsistent behavior of negation.

Slater argues that the negation symbols in paraconsistent logic lack properties which actual negation has [5, p. 489]. The claim is that the negation symbol in paraconsistent logic cannot be correctly interpreted as the real negation. The objection given by Slater relies on the definitions of contraries and subcontraries. The formal definitions of both according to Graham Priest and Richard Routley, as Slater points out, are as follows. "Traditionally A and B are subcontraries if $A \vee B$ is a logical truth. A and B are contradictories if $A \vee B$ is a logical truth and $A \wedge B$ is logically false" [20, p. 165]. Slater worries that the negation symbol of paraconsistent logics does not give an account of contradictories which can be translated as the real negation. Slater notes how Priest's **LP** is supposed to be able to express that two propositions are contradictories [30, p. 451–2]. In **LP**, $A \vee \neg A$ is logically true for every proposition. That is, **LP**'s negation at least makes a proposition and its negation subcontraries. Priest believes that **LP**'s negation makes a proposition and its negation contradictories as well, as **LP** makes $A \wedge \neg A$ logically false for every proposition. In **LP**, $A \wedge \neg A$ is always logically false, but may sometimes be true as well. It is this fact which is used in Slater's objection.

Slater does not think that **LP**'s negation symbol is capable of expressing the *fact* that a proposition and its negation are contradictories.

> For no change of language can alter the facts, only the mode of expression of them .... And one central fact is that *contradictories* cannot be true together — by definition [30, p. 453].

Slater's objection is similar to the just true (or, similarly, just false) problem. The just true problem follows from the that fact that even if one is to say that a statement is true, it may also be false under a paraconsistent logical framework [2, 18]. The problem is that we lack the expressive power to say that a sentence *is just true and not both true and false*. If $A \wedge \neg A$ is always false, then $A \wedge \neg(A \wedge \neg A)$ does not express that P is just true, for $A \wedge \neg A$ and $\neg A$ might both be true as well. The relation to Slater's objection is that Slater claims that the paraconsistent negation is incapable of expressing that a proposition negates another. We are incapable of expressing that the truth of a proposition forces its negation to be false *and just false*. Because of this lack of expressive power, the negation symbol of paraconsistent logics fails to capture what negation (supposedly) means.

The reason the ambiguity measure approach to preservationist logic taken by Bryson Brown solves the Slater's objection is that the so called 'true contradictions' and false ‘excluded middles’ can be treated as ambiguous while maintaining the consequence relation of glutty and gappy logics. So, while negation in the logic does not behave like Slater claims it ought to, the ambiguity based treatment of a classically inconsistent premise set can be modeled classically with a classical negation. If we have $P \wedge \neg P$ in our premise set, then we treat P ambiguously in order to produce a premise set able to be classically modeled. We are able to do so in a number of ways, and by quantifying over such ways Brown is able to construct consequence relations equivalent to **LP** and **FDE**. We are not, then, committed to the truth or some contradictions, but rather the ambiguity of the constituents of some contradictions. The approach treats some contradictions as cases of ambiguity. The contradictions in question, more specifically, are those which appear in the premise set of an inference. These contradictions are treated ambiguously in order make them at least true, in order to block explosion. There are many ways to disambiguate said contradictions. A possible motivation to create a consequence relation such as **LP** is as follows. Any one possible disambiguation or sentences in the premise set will necessitate certain inference being valid. Such inference are valid in virtue of accidental properties of the disambiguation. By quantifying over all such disambiguations we will show what follows by logic alone, and not by the accidental inferences made possible by any particular disambiguation. How Brown disambiguates and quantifies over the disambiguations will be shown in the next section.

Relevant logics are a subclass of paraconsistent logics, so we may be able to use Brown’s argument to defend the use of relevant logics from similar objections. A number of relevant logics can be constructed by extending the logic **FDE**. This is done by defining an intensional ‘arrow’ connective with specific properties. The relational semantics for relevant logics use worlds, points, situations, or information states which behave like models of **FDE** with respect to the extensional connectives. In the next chapter I will attempt to use Brown’s preservationist semantics to show that a preservationist account of the semantics of relevant logic is able to defend relevant logic against some of its objections. For example, I will attempt to explain the non-normal worlds needed for the relational semantics in terms of ambiguity measures, which will be explained in the next section. In this chapter I will explicate Bryson Brown’s preservationist approach to paraconsistent logic, focusing on his construction of preservationist consequence relation equivalent to **FDE**’s consequence relation.

2.2 Ambiguity-Measure Preserving Logics

Brown constructs consequence relations which preserves measures of ambiguity, then proves these consequence relations to be equivalent to a number of paraconsistent consequence relations. A measure of ambiguity is a measure of how inconsistent a set of sentences is. (For **FDE**, a dual notion of ambiguity measures will roughly capture how consistently deniable a set of sentences is.) Ambiguity, in this setting, allows us to classically model classically inconsistent sets of sentences by treating certain sentences as ambiguous. Informally, treating a sentence as ambiguous amounts to treating the instances of the sentence as different sentences. We can see that $P \wedge \neg P$ is able to be consistently modeled when we treat the instances of P as different sentences.

Ambiguity, then, is something very specific for Brown's preservationist approach. Ambiguous sentences are those we wish the instances of the sentence to be treated as different sentences. This may be done for different reasons, but Brown is concerned with treating sentences as ambiguous in order to classically model sentences which do not have classical models, i.e. the inconsistent and incomplete. More detail of Brown's formal treatment of ambiguity is provided below. After I explicate ambiguity's formal treatment, I will include an explication of what ambiguity means in these circumstances.

We start with what treating a sentence as ambiguous means. Formally, if we treat the sentence P as ambiguous, each and every instance of the P is replaced by one of the two new sentences P_t and P_f. Doing so produces a sort of disambiguation where P_t is taken to be a true sentence and P_f is taken to be a false sentence [7, p. 176]. The dual notion to be explicated below is similar. Thus treating a sentence as ambiguous is equivalent to treating the instances of the sentence as non-identical.

Brown's consequence relations preserve ambiguity measures, or levels of ambiguity. Ambiguity is used to make consistent models of inconsistent sets. By treating sentences as ambiguous we are able to model sentences such as $A \wedge \neg A$ consistently, as will be explained further below. With the formal notion of treating sentences as ambiguous to produce consistent images having been constructed, Brown defines the level of ambiguity of a set of sentences to be the smallest sufficient number of sentences letters we must treat ambiguously in order to produce a consistent image of the set:

> *Ambcon*: Ambcon(ξ, Γ) iff Γ can be made consistent by treating ξ sentences letters as ambiguous. [5, p. 495]

For example, Ambcon$(\xi, \{P, \neg P, Q\})$ can be made consistent by treating P as ambiguous, but can also be made consistent by treating both P and Q as ambiguous. The most interesting ξ for any set of propositions is the smallest of all ξ, for the smallest is the least amount of ambiguity needed to render the set classically consistent. Brown thus defines a set's level of ambiguity:

> 1 : The *level of ambiguity* ("lamb" for short) of a set Γ,
> 1 $(\Gamma) = \mathrm{Min}_{\xi} \mid \mathrm{Ambcon}(\xi, \Gamma)$. [5, p. 495]

When lamb $= 0$, the set is classically consistent. However, when lamb≥ 1, the set is classically inconsistent. With this distinction in place, we may make further distinctions which the classical treatment is unable to do. For example, let $\Gamma = \{A, \neg A\}$ and let $\Delta = \{A, \neg A, B, \neg B\}$. Γ's lamb $= 1$, while Δ's lamb $= 2$. This distinction is unable to be made within the classical setting. The distinction is also a useful distinction. First, this distinction allows us to construct consequence relations which preserve levels of ambiguity. Second, it allows us to measure how classically inconsistent a set is, which indicates roughly how much work is needed in order to make the set consistent.

As an example, let us consider scientific theories. The consequence relation does not explode while preserving levels, which allows us to reasoning non-trivially from inconsistent scientific theories (or a number of theories which are inconsistent when taken together). When an inconsistent theory appears in a premise set, an exploding consequence relation does not do us any good. We first note that inconsistent scientific theories are still used, and still useful.[2] It would be more useful to have a consequence relation which did not explode when inconsistent theories appear in the premise set. If we make an inference (using an exploding consequence relation) from any theory to a proposition which asserts something we deem by direct observation to be false, then either the theory is wrong and consistent, or it is inconsistent and the false conclusion follows from the inconsistency. Scientific theories are complicated. Note that some theories in general can be axiomatized in relatively simple and seemingly true axioms.[3] Therefore, finding inconsistencies in any given scientific theory may be difficult and impractical. What are we to do when we infer a false proposition from a theory using

[2]Graham Priest argues that we may, and do, rationally accept inconsistent scientific theories [18, p. 102]. Graham notes that Lakatos [14] and Feyerabend [12] contain numerous examples of inconsistent scientific theories.

[3]For example, take the axioms of Peano Arithmetic. They are seemingly true axioms which are inconsistent when taken together with classical logic.

classical logic? Either we go hunting for an inconsistency, or develop a brand new theory. Nonetheless, the theory may still be useful enough to be used in a subset of cases.

On the other hand, using a non-exploding consequence relation there are two conditions where an false proposition will be inferred. The first, as expected, is when the theory is incorrect. The second, and more interesting, is when the proposition and its negation are part of the theory. We are left again with two choices. Either we develop a new theory, or we fix this inconsistency. Indeed these choices are better looking, for the inconsistency in question should theoretically be easier to locate.[4] It will be easier to locate because the sentence inferred is guaranteed to be related importantly to the inconsistency; it is harder to locate in the classical logic case because any sentence may be inferred, and this includes the possibility of the inferred sentence having no relation to the inconsistency. Therefore, it is more useful to apply a consequence relation to scientific theories which does not explode.

The consequence relation which preserves level of ambiguity does not commit us to accepting that contradictions can be true in the sense of dialetheism, even though it is equivalent to the consequence relation of **LP**:

> We do not say that a classically inconsistent Γ is really consistent (or satisfiable) after all; we say only that preserving 1 [i.e. lamb] instead of preserving consistency leads to a new consequence relation which does not trivialize all inconsistent sets. Whether that consequence relation is interesting is another question. [5, p. 496]

Having constructed paraconsistent logics using an ambiguity measure preserving consequence relation, we should no longer think of entailment as the consequent being true in all models which satisfy the premise set. For if our models are classical, then the logic validates explosion, and if our models are **LP** models, then we have strange truth values to consider. Instead, Brown constructs a third option wherein the models are classical models, but where we are still able to model sentences such as $P \wedge \neg P$. Treating certain sentence letters an ambiguous allows us to classically model the premise set without requiring true contradictions.

I will now explicate Brown's preservationist interpretation of **LP** in order to demonstrate the metaphysical commitments of such an approach, and to show how such

[4] This is not to say that the inconsistency will be easy to locate, but that we may limit where look for the inconsistency.

an approach "focuses on the semantic side rather than on images of the premise set" [7, p. 179]. Brown uses what he calls *wildcard valuations*:

> *Wildcard valuations* allow inconsistent sets of sentences to be 'satisfied' by treating a set of 'wildcard' atoms in a way that allows ambiguity. [7, p. 179]

Note that the wildcards must be chosen carefully, as will be shown. A wildcard valuation begins with as classical assignment to a subset of the atoms. Let L be the language in question. Let W be a subset of the atomic sentences of L, and let the atoms of L be L_{At}. We begin with a two-valued truth assignment from the atoms not in W: $A_{At-W} : (L_{At} - W) \longrightarrow \{1, 0\}$. Our choice of wildcards is important, for the right selection of wildcards could invalidate too many inferences. In selecting wild cards, if we take the **LP** valuation assignment to be from sentences onto T, F, and Both, then the wildcards are taken to be the atomic sentences given the value Both. We see this in the wild card treatment extended to **FDE**, where " the atoms to which the Dunn valuation assigned T and F are assigned the values 1 and 0 respectively ... [and] the rest of the atoms are treated as wildcards" [7, p. 187]. Later we quantify over possible valuations given to wildcards, and not over possible wildcard choices. The careful selection of wildcards enables us to do so.

We are now able to give truth value assignments to the wildcards. "We assign 0 or 1 to each *instance* of an atom in W in each formula of L" [7, p. 179]. For example, if $p \in W$, then in the complex formula $p \wedge \neg p$ we may assign both instances of p the same value, or we may assign them different values. We will make use of these different possible assignments when constructing the consequence relation. The truth value assignment for complex sentences proceeds from here in the usual classical way, as all the atomics within the complex sentences now have a classical truth value. Each possible assignment to each instance of each atom of W produces a wildcard valuation, written as ${}^{W}V_{At-W}$ [7, p. 180]. "Let V_{At-W} be the set of all such valuations based on a given A_{At-W}" [7, p. 180]. That is, V_{At-W} is the set containing every possible combinations of true value assignments to the wildcard atoms, given a valuation on the non-wildcard atoms. The valuation of the non-wildcard atoms may be seen at the non-ambiguous propositions, while the wildcard atoms are ambiguous. Therefore V_{At-W} is the set containing all possible ways of disambiguating the wildcard atoms.

From here, "we quantify across V_{At-W} to obtain a more stable valuation based on all the wildcard valuations for each wildcard set W" [7, p. 180]. We call this valuation $\mathbf{V}_{At-W}$:

$\mathbf{V}_{At-W}(S) = 1$ if $\exists V \in V_{At-W} : V(S) = 1$
$\mathbf{V}_{At-W}(S) = 0$ else. [7, p. 180]

The analogous case in **LP** would be what are called designated values. If every valuation gives a sentence S the value 1, then it would given the value true in **LP**. Similarly, if every valuation gave S the value 0, then it would be false in **LP**. The only other case to consider is the case such that the totality of possible wildcard valuations assign the sentence S 1 in some cases and 0 in others. The final case would be the assignment of Both to the sentence in **LP**. Since the value Both is a designated value in **LP**, and because the consequence relation of **LP** preserves designated values, the following wildcard-based consequence relation is equivalent to **LP**:

$$\Gamma \vdash_W \alpha \Leftrightarrow \forall V_W[(\forall \gamma \in \Gamma, V_W(\gamma) = 1) \Rightarrow V_W(\alpha) = 1] \text{ [7, p. 180]}$$

With the analogy to designated values, we see that the above consequence relation is equivalent to **LP**'s consequence relation: if every formula on the left of the turnstyle is at least true (is either true is every wildcard valuation (True) or true in some of them (Both)), then the formula on the right side of the turnstyle is at least true.

Brown shows the preservation of ambiguity measures can result in **FDE**. That is, Brown creates a consequence relation which is equivalent to that of **FDE**. I explicate Brown's treatment of **FDE** by first introducing consistent images, the result of which is **LP**. Then the notion of consistently deniable images is introduced. With both notions, and with some minor tinkering, it is possible to construct a consequence relation equivalent to **FDE**.

"By treating certain sets of atomic sentences as ambiguous, we can produce consistent *images* of inconsistent premise sets" [7, p. 176]. A consistent image results from the doctoring of the original inconsistent premise set:

> A set of formulae, Γ', is a consistent image of Γ based on A (which we write $\text{ConIm}(\Gamma', \Gamma, A)$) iff A is a set of atoms, Γ' is consistent, and Γ results from the substitution, for each *occurence* of each member α of A in Γ, of one of a pair of new atoms α_f and α_t. [7, p. 176]

Brown defines the term *ambiguity set* as follows [7, p. 176]:

$$\text{Amb}(\Gamma) =_{def} \{A | \exists \Gamma' : \text{ConIm}(\Gamma', \Gamma, A) \wedge \forall A' \subset A, \neg \exists \Gamma' : \text{ConIm}(\Gamma', \Gamma, A')\}$$

This set is useful for a number of purposes. The cardinality of the set may be used as a (rough) measure of how inconsistent the set is [7, p. 176]. The more inconsistent the set is, the more sentence letters must be treated ambiguously in order to produce a consistent image of the inconsistent set.

Brown constructs a consequence relation by first defining acceptable extensions of inconsistent sets. An acceptable extension of a set should not require an ambiguity set which contains members not in the ambiguity set for the non-extended set. Allowing such extensions would ultimately result in a consequence relation which fails to block inferences which should be rejected. For example, we should reject $\{A, \neg A, B\} \vdash \neg B$, even on a dialetheist approach. The preservationist school of logic sometimes refers to the cardinality of the ambiguity set, or its equivalent, as the level of inconsistency of the set, and does not accept inferences with a higher level of inconsistency on the right side of the turnstyle [28]. Brown thus defines an acceptable extension of a set Γ to be a set Δ whose ambiguity set $\mathrm{Amb}(\Delta)$ is a subset of the ambiguity set Γ [7, p. 177]:

$$\mathrm{Accept}(\Delta, \Gamma) \Leftrightarrow \Gamma \subseteq \Delta \wedge \mathrm{Amb}(\Gamma \cup \Delta) \subseteq \mathrm{Amb}(\Gamma)$$

The consequence relation, as defined by Brown, preserves acceptable extensions. In other words, a formula follows from a set of premises if and only if all acceptable extensions of the premise set are also acceptable extensions of the set theoretic union of the premise set and the formula in question [7, p. 177]:

$$\Gamma \vdash_{Amb} \alpha \Leftrightarrow \forall \Delta : \mathrm{Accept}(\Delta, \Gamma) \rightarrow \mathrm{accept}(\Delta, \Gamma \cup \{\alpha\})$$

Brown proves that this constructed consequence relation turns out to be equivalent to **LP**. Brown proves this equivalence a number of ways in [5, 7].

Brown demonstrates how to construct a dual notion of ambiguity sets which apply to the right of turnstyle. Doing so blocks inferences from a set of contingent propositions to tautologous propositions.[5] However, this alone is not enough to construct **FDE**, as will be explained below. The dual notion of the ambiguity set, called $\mathrm{Amb}^*(\Delta)$ is "the set of minimal sets of sentence letters whose ambiguity if sufficient to project a consistently deniable image of Δ" [7, p. 181]. For example, if we commit ourselves to denying $A \vee \neg A$, the Amb^* set would be A. A consistent model which forces a denial of $A \vee \neg A$

[5] As well as rejecting explosion, blocking inferences from contingent premises to tautologous (trivial) conclusions illuminates the justification for using **FDE** as the starting point for many relevant logics.

must treat the proposition represented by the sentence letter A as ambiguous. That is, a consistently deniable image is a consistently deniable set of sentences which is the result of the substitution of certain sentence letters in the original inconsistently deniable set. This substitution process is similar to the substitution process for creating consistent images of inconsistent sets, as described above.

Acceptable extensions for the dual notion of ambiguity are also defined.

> Γ is an $\text{Amb}^*(\Delta)$-preserving extension of $\Delta \Leftrightarrow \text{Amb}^*(\Delta \cup \Gamma) \subseteq \text{Amb}^*(\Delta)$
>
> We write this as $\text{Accept}^*(\Gamma, \Delta)$, So a set Γ is acceptable as an extension of a commitment to *denying* Δ if and only if extending Δ with $\Gamma \ldots$ does not require any more ambiguity to produce a consistently deniable image than merely denying Δ does. [7, p. 181]

In other words, an acceptable extension is such that we do not have to add any sentence letters to the set Amb^* in order to maintain the consistently deniable image. For instance, the set $\{A \vee \neg A, B\}$ cannot be acceptably extended by the sentence $\neg B$, for the first set can be made consistently deniable by treating A ambiguously, while the extension would require treating both A and B ambiguously.

We again construct a consequence relation which preserves acceptable extensions. Note that sets of sentences will appear on the right, and singletons on the left[6] [7, p. 181]:

$$\gamma \vdash_{Amb*} \Delta \Leftrightarrow \forall \Delta : \text{Accept}^*(\Gamma, \Delta) \Rightarrow \text{Accept}^*(\Gamma \cup \{\gamma\}, \Gamma)^{7}$$

An example of an inference which this consequence relation invalidates is $B \vdash_{Amb*} \{A \vee \neg A\}$, which is, of course, also invalid in **FDE**. Note that an acceptable extension of $\{A \vee \neg A\}$ is $\neg B$, for the set $\{\neg B, A \vee \neg A\}$ can be made consistently deniable without adding to the ambiguity set. B is not an acceptable extension of $\{\neg B, A \vee \neg A\}$, for we must add B to the ambiguity set in order to preserve create a consistently deniable image. Thus, B is not an acceptable extension of an acceptable extension of $\{A \vee \neg A\}$.

[6] As explicated Brown produces the multiple conclusion logic known as $\mathbf{FDE}^+$. As a special case we get **FDE**, where the sets on the right are just singletons. Thus Brown shows how to construct both **FDE** and its multiple conclusion generalization.

[7] In Brown, [7], the second instance of γ is displayed as α. I believe this is simply a typographical error. I cite his translation of the quoted definition: "In English, a set Δ follows from a formula γ if and only if γ is an acceptable extension of every acceptable extension of Δ, a set we are committed to denying" [7, p. 181].

We construct a symmetrical consequence relation by putting the two consequence relations together while treating "sets of the left as closed under conjunction and sets of the right as closed under disjunction" [7, p. 181]:

$$\Gamma \vdash_{\text{Sym}} \Delta \Leftrightarrow \exists \delta \in \text{Cl}(\Delta, \vee) : \Gamma \vdash_{\text{Amb}} \delta \ \&$$
$$\exists \gamma \in \text{Cl}(\Gamma, \wedge) : \gamma \vdash_{\text{Amb}^*} \Delta$$

The resulting consequence relation is not quite **FDE**. In fact, the consequence relation is equivalent to $\mathbf{K}^4$, which is not to be confused with $\mathbf{K}_4$. The later is a logic with an arrow connective constructed using the Routley star method. $\mathbf{K}^4$ differs from **FDE** only when "classically trivial sets appear on both the left and the right [of the turnstyle]" [7, p. 182]. For example, let us consider $P \wedge \neg P \vdash_{\text{Sym}} Q \vee \neg Q$. Every acceptable Amb* extension of $Q \vee \neg Q$ can further be extended by $P \wedge \neg P$, for it is always consistently deniable without the need for extending the ambiguity set. The dual holds for the other direction.

However, a minor modification is sufficient to produce a consequence relation equivalent to **FDE**. If we allow the sentence letters treated ambiguously on each side of the turnstyle to be different, the resulting consequence relation is equivalent to the consequence relation of **FDE**

> The trick is to produce consistent images of premise sets and non-trivial images of conclusion sets simultaneously, while requiring that the sets of sentence letters used to project these images be *disjoint*. Then $\Gamma \vdash_{FDE} \Delta$ if and only if every such consistent image of Γ can be consistently extended by some member of each non-trivial image of Δ based on a disjoint set of sentence letters, or (now equivalently): $\Gamma \vdash_{FDE} \Delta$ if and only if every such non-trivial image of the conclusion set can be extended by some element of each non-contradictory image of the premise set while preserving its *consistent deniability*. [7, p. 182]

Consider the inference $P \wedge \neg P \vdash Q \vee \neg Q$ which is invalid in **FDE**. The above non-**FDE** criteria failed to invalidate this inference due to the fact that denying $P \wedge \neg P$ is always an acceptable extension of denying $Q \vee \neg Q$. However, when we simultaneously make consistent images, $P \wedge \neg P$ is treated ambiguously with P in the set $\text{Amb}(P \wedge \neg P)$. By making consistent images of the premise set and non-trivial images of the conclusion set simultaneously, we replace P and Q in such a way that the inference in question becomes the obviously invalid $P_t \wedge \neg P_f \vdash Q_f \vee \neg Q_t$

Brown proved that the resulting consequence relation is equivalent to that of **FDE**. He proved this equivalence by first constructing a game. A description of the game will show the commitments of Brown's approach in terms of truth values. It is important to see how this preservationist method addresses the objection to the strange truth values of Both and Neither. A direct consequence is that Brown's ambiguity measure preservation semantics block Slater's objections. To show this, Brown constructs a game which instantiates the process of finding acceptable extensions of acceptable extensions, the result of which proves to be equivalent to the truth tables of the Dunn semantics:

> The result shows that the *work* of a Dunn valuation can be done by a game that has nothing to do with peculiar truth values, because we can arrange the results of the game in tables isomorphic to Dunn's 4-valued tables for **FDE**. [7, p. 183]

Below I will show how this game, and Brown's project in general, approaches the strange truth values of many valued logics.

The game requires two players, which Brown aptly names Verum and Falsum [7, p. 183]. The game is played with a single formula. The formula is given a partial classical valuation, "which matches the values assigned to the atoms receiving the values T or F in the corresponding Dunn valuation" [7, p. 183].[8] The rest of the atoms are then given to the players, where the atoms assigned Both (Neither) in the Dunn valuation are given to Verum (Falsum). The game is played by having each player assign classical truth values to the instances of the atoms they received. The goal of Verum is to force the chosen formula to receive the value T. The goal of Falsum is to force the Formula to receive the truth value F. Note that no non-classical truth values are used by either player, or in the game set-up.

Only one player can win the game at a time. It is possible, in the game, to differentiate between situations where a Dunn valuation assigns Both (Neither) and True (False):

> If it's a won game for Verum, then the Dunn valuation assigns either True or Both to the formula.... Moreover, the Dunn valuation assigns the value True to the formula if and only if the game is won for Verum even if she and Falsum exchange their assigned letter instances. [7, p. 184]

[8] In the wildcard valuations, one must be careful when picking $At - W$. $At - W$ corresponds to the partial classical valuation in the game.

Similar conditions hold with regards to Falsum and the truth values Neither and False.

For each game there are two ways for each player to win. The first way the winner is determined by the partial classical valuation, and is thus determined no matter how the remaining atoms are distributed among the players [7, p. 184]. The second way a player may win depends on which atoms the player receives of the distributed atoms [7, p. 184]. The second type of winning is only determined by playing the game two times for the formula in question. That is, once with one distribution of the ambiguous atoms, and once with the distributed atoms exchanged. In effect the game draws out the differences between the Dunn valuation of True (False) and Both (Neither), as explained above.

Brown proves that each player can win a game in one of two ways. The significant of this result allows us to construct truth tables which exhaust the possible outcomes for complex sentences, as the atomic sentences are trivially covered. The resulting truth tables are equivalent to the truth tables for the Dunn four-valued semantics [7, p. 186]. The game therefore "does the work of a Dunn valuation" [7, p. 186]. In other words, the game-based truth tables may be used for the semantics of **FDE** just as well as the Dunn valuation truth tables.

2.3 Upshots

Brown's ambiguity measure preservation method allows us to construct a consequence relation equivalent to **FDE** in a way that blocks objections to paraconsistent and paracomplete logics. It first blocks Slater's objection, and similar objections, that the negation of such logics is not the real negation. Second, and more generally, Brown's treatment of **FDE** does not require being committed to non-classical truth values. The game explicated above, as shown by the constructed truth tables, is the basis for a semantic consequence relation which quantifies over the possible ways to treat a set of atomic sentences as ambiguous. In this way, non-classical truth values are avoided:

> This trick produces yet another way of applying ambiguity *to replace strange truth values* — the rules of the game allow Verum and Falsum to treat ambiguously the atoms assigned to them, as they attempt to produce an assignment that makes the target formula True of False, but each player uses the leeway that ambiguity grants her in a particular way. [7, p. 186] (Emphasis mine)

Thus, we may reinterpret **FDE** models to purge them of the truth values Both and Neither. We treat the atomic sentences receiving the values Both and Neither in the Dunn valuations as ambiguous. In doing so we are able to classically model the premise set (or the denial of the conclusion set) by treating different instances of the ambiguous atomic sentences as different sentences.

There are a number of benefits to adopting Brown's approach. The first is that Slater's objection, and other similar objections, are blocked. What negation really means, at least according to Slater, is what it continues to mean under Brown's approach. If we find that a a formula and its negation are not full contraries but merely subcontraries, then may treat certain propositional parameters present in the formula as ambiguous. The result of such a treatment is that the negation symbol will behave as Slater wishes it to behave. For example, let P and $\neg P$ be subcontrary, but not fully contrary in **FDE**. Under Brown's approach, we treat P ambiguously such that in our model we now have two sentences, P_t and $\neg P_f$, which are subcontrary, but not contrary. However, any sentences and its negation within our new models are full contraries, and thus our new models are immune to Slater's objection. (A similar objection might be dual to Slater's, involving false excluded middles. Brown's treatment of **FDE** provides a response to such as objection as easily as it responds to Slater's.) Moreover, the models created by the ambiguity approach are models which do not use truth values beyond Just True and Just False. The upshot here is that the consequence relations of logics which reject explosion (such as **LP**) can be created using just these two classical truth values. Additionally we may construct the logic **FDE**, which is the base for many relevant logics. Notably, **FDE** is known to have the weak variable sharing property, i.e. the weak relevance criterion.

Chapter 3

A Preservationist Approach to Relevant Logic

3.1 Introduction

In this chapter I will combine Brown's preservationist approach presented in the last chapter with model theoretic semantics for relevant logics. The ultimate goal is to extend Brown's approach to the conditional of relevant logics. I will also argue for the appropriateness of relevant logic for extending Brown's approach to the conditional.

In section 3.2, I will construct models which will model the logic K_4. I will prove that the models I construct are equivalent to a model theoretic semantics which is known to be sound and complete for K_4. I will prove this equivalence by constructing a translation scheme. To then show that the models I have constructed qualify as genuine semantics for the logic K_4, I will argue that an extension of Brown's preservationist semantics is aptly represented by the model theoretic structure I construct.

In section 3.3, I will modify the models I have constructed with the addition of non-normal worlds in order to create a model structure sound and complete with the logic N_4. I will show that the truth conditions for the conditional at non-normal worlds are not given a more satisfying philosophical interpretation, and are not capable of representing Brown's approach as I explicated it for the models of K_4.

In section **B**, I will construct two new types of models. The first is is a model theory which extends Brown's approach and is capable of modeling the logic **B** and its extensions. However, despite being equivalent to the American plan semantics, these models do not represent an extension of Brown's preservationist approach to the arrow connective. The second type of models I construct do adequately extend Brown's approach to the arrow connective. However, as will be shown, the logic **B** is not modeled by these models. I will show that a relevant logic is modeled by these models, and that these models are able to be extended to model other relevant logics. Thus, there exists relevant logics which are modeled by an extension of Brown's preservationist approach.

3.1.1 $\mathbf{K}_4$: A Non-Relevant Arrow

Arguably the logic **FDE** is rather uninteresting, as it does not have any connective which may be considered as a candidate for being an implication connective. The first arrow we will consider extending **FDE** with will produce the logic $\mathbf{K}_4$. The logic $\mathbf{K}_4$ is not a relevant logic, for $\vdash_{\mathbf{K}4} (P \to (Q \to Q))$ [19, p. 167]. Unlike the material conditional, the conditional of $\mathbf{K}_4$ is an *intensional* arrow, so I will be able to discuss the intensional arrow in the light of Brown's preservationist project without the distractions more complicated models produce. Additionally, the models I construct here provide the basis for the more complicated models needed for relevant logics.

I now reproduce models which are already known to be adequate for $\mathbf{K}_4$ [19, p. 180–2]. I will later show that these models are equivalent to the models I will construct. Priest's models for $\mathbf{K}_4$ are order pairs, $\langle W, \rho \rangle$ such that W is a set of worlds, and ρ is a relational-valuation of propositions at each world. The truth and falsity conditions for the connectives in Priest's explication of $\mathbf{K}_4$ are as follows;

$$
\begin{aligned}
(\neg A)\rho_w 1 &\text{ iff } (A)\rho_w 0 \\
(\neg A)\rho_w 0 &\text{ iff } (A)\rho_w 1 \\
(A \wedge B)\rho_w 1 &\text{ iff } A\rho_w 1 \text{ and } B\rho_w 1 \\
(A \wedge B)\rho_w 0 &\text{ iff } A\rho_w 0 \text{ or } B\rho_w 0 \\
(A \vee B)\rho_w 1 &\text{ iff } A\rho_w 1 \text{ or } B\rho_w 1 \\
(A \vee B)\rho_w 0 &\text{ iff } A\rho_w 0 \text{ and } B\rho_w 0 \\
(A \to B)\rho_w 1 &\text{ iff for all } w' \in W \text{ such that } A\rho_{w'}1, B\rho_{w'}1 \\
(A \to B)\rho_w 0 &\text{ iff for some } w' \in W, A\rho_{w'}1 \text{ and } B\rho_{w'}0 \ [19, p.164]
\end{aligned}
$$

Note that models of $\mathbf{K}_4$ are collections of worlds, where each world is effectively a model for **FDE**, to which we have added an $\to$ connective.

3.2 K4

In this section I will construct models that work as a formal semantics for $\mathbf{K}_4$. The constructed models will be apt to represent the preservationist project as explicated in the previous chapter, as I will argue in section 3.2.1. I will also produce a translation schema to translate between the newly constructed models and Priest's models for $\mathbf{K}_4$ as shown in section 3.1.1. I will refrain from extensive philosophical discussion until section 3.2.1. The constructed models are roughly based off of Chellas' presentation of modal logic in [8].[1]

I begin by defining two set of atomic propositions. The first set is written and ordered as follows;

$$\{\mathbb{P}_1, \mathbb{P}_{-1}, \mathbb{P}_2, \mathbb{P}_{-2}, \mathbb{P}_3, \mathbb{P}_{-3}, \dots\}.$$

This first set of atomic propositions will provide the basis for our models, and the truth and falsity of the members second set will depend on this first set. We will call this first set of the atomic propositions *type 1* atomic propositions. The second set matches the atomic sentences of the syntax being modeled, and will be written as follows;

$$\{\alpha_1, \alpha_2, \alpha_3, \dots\}.$$

The second set of atomic propositions will be called *type 2* atomic propositions. With the truth and falsity of the members of the second set depending on the truth and falsity of the first set (in ways to be defined), there will be two related levels of description for each world. That is, worlds may be described in terms of the first set of atomic sentences, but may also be described in terms of the atomic sentences of the second set. I will further explicate these levels and their philosophical importance below.

Definition 9. *A* ***model*** *M is a structure $\langle W, P, Am, f, R\rangle$. W is a set of worlds. Worlds shall be written as w_i. R is an accessibility relation such that $R = W \times W$. P is a function with the domain $\{1, -1, 2, -2, 3, -3, \dots\}\dots$ such that for each number n, P_n is a subset of W;*

$$P : \{1, -1, 2, -2, 3, -3, \dots\} \longrightarrow \wp(W)\ [8, p.35]^{2}$$

[1] Chellas' initial models for modal logic are $\langle W, P, R\rangle$ [8, p. 35]. I extend these models to suit my purposes.

We use this P function to pair each atomic proposition $\mathbb{P}_n$ with a set of worlds P_n. This set of worlds is to be thought of as the set of worlds where the atomic sentence $\mathbb{P}_n$ is true. As in Chellas [8], we write the sentence '$\mathbb{P}_n$ is true at world w_i in the model M' as $\models^M_{w_i} \mathbb{P}$.

To formally ensure that P_n is the set of worlds where the atomic sentence $\mathbb{P}_n$ is true, the require that

$$\models^M_{w_i} \mathbb{P}_n \text{ iff } w_i \in P_n.$$

Note that $\langle W, P, R\rangle$ is a model structure for classical modal logic, and for that reason the following hold [8, p. 35]:

1. $\models^M_{w_i} \neg A$ iff $\not\models^M_{w_i} A$
2. $\models^M_{w_i} (A \wedge B)$ iff $\models^M_{w_i} A$ and $\models^M_{w_i} B$
3. $\models^M_{w_i} (A \vee B)$ iff $\models^M_{w_i} A$ or $\models^M_{w_i} B$

The function f pairs type 2 atomic sentences with the type 1 atomic sentences. f is a function which takes members of $\{\alpha_1, \alpha_2, \alpha_3 \ldots\}$ as arguments and returns ordered pairs of type 1 atomic sentences such that:

$$\begin{gathered} f(\alpha_1) = \langle \mathbb{P}_1, \mathbb{P}_{-1}\rangle \\ f(\alpha_2) = \langle \mathbb{P}_2, \mathbb{P}_{-2}\rangle \\ \vdots \end{gathered}$$

A philosophical interpretation of f will be given below.

Lastly, Am, is the set of all ordered pairs, $\langle w_i, \alpha_n\rangle$, such that exactly one of $\mathbb{P}_n$ and $\mathbb{P}_{-n}$ is true at w_i in the model. Each ordered pair, $\langle w_i, \alpha_n\rangle$, is to be interpreted as 'α_n is ambiguous at w_i'. We write $Am(w_i)$ as a shorthand for the set of atomic propositions ambiguous at w_i;

$$Am(w_i) = \{\alpha_n : \langle w_i, \alpha_n\rangle \in Am \text{ and } \alpha_n \in \{\alpha_1, \alpha_2, \alpha_3 \ldots\}\}$$

Brown, as shown in the last chapter, defines treating a sentence ambiguously as treating the instances of the sentence as one of two new sentences. Of the two new sentences, one is true and one is false. This is used to model things which cannot be modeled by classical logic alone. When a type 2 atomic sentence is ambiguous in my models, the two corresponding type 1 atomic sentences are such that one is true and the other false. Roughly, then, the type 1 atomic sentence corresponding to a type 2 atomic sentence may be interpreted as the instances of the type 2 sentence. When the instances all receive the same truth value, there is no ambiguity. This motivates the following truth and falsity conditions.

Definition 10. *For members of* $\{\alpha_1, \alpha_2, \alpha_3 \ldots\}$, *the following hold;*

1. $\models^M_w \alpha_n$ iff $f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $\models^M_w \mathbb{P}_n$.
2. $\models^M_w \neg\alpha_n$ iff $f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $\not\models^M_w \mathbb{P}_{-n}$.
3. $\models^M_w (A \wedge B)$ iff $\models^M_w A$ and $\models^M_w B$
4. $\models^M_w \neg(A \wedge B)$ iff $\models^M_w \neg A$ or $\models^M_w \neg B$
5. $\models^M_w (A \vee B)$ iff $\models^M_w A$ or $\models^M_w B$
6. $\models^M_w \neg(A \vee B)$ iff $\models^M_w \neg A$ and $\models^M_w \neg B$

Note that these conditions reduce to implication-free classical logic at each world when no sentence is ambiguous, i.e. when Am is just the empty set.[3]

The above truth and falsity conditions for disjunction and conjunction are, not surprisingly, the same as the truth and falsity conditions for **FDE**, though indexed to worlds. The main formal difference between my models and Priest's models for $\mathbf{K}_4$ is that my models have two levels of description. The literals[4] in the above truth and falsity conditions are dependent on the truth and falsity of the type 1 atomic sentences, which, as I have observed, behave decidedly classical. This formal difference affords a difference in philosophical interpretation. More specifically, my models are capable of formalizing Brown's preservationist project as explicated in the previous chapter. I will argue for this in section 3.2.1.

The arrow connective is related to the consequence relation, and is defined as follows:

[3] Implication-free logic being a logic without the symbol '$\rightarrow$', though one may easily be defined at these worlds by extending the logic with an arrow, and defining $A \rightarrow B$ as $\neg A \vee B$.

[4] Literals being atomic sentences and their negations.

7. $\models_w^M (A \to B)$ iff for all $w' \in W$ such that $\models_{w'}^M A$, $\models_{w'}^M B$.[5]

8. $\models_w^M \neg(A \to B)$ iff there exists a $w' \in W$ such that $\models_{w'}^M A$ and $\models_{w'}^M \neg B$.

In the next section I will show that, although my models are formally different from Priest's, my models nevertheless are sound and complete with respect to $\mathbf{K}_4$. I do this by providing a translation schema between the newly constructed models and Priest models.

Example 1. To get a feel for how these models work, let us construct a world in a model which at which $P \wedge \neg P$ but not Q. Suppose $P = \alpha_n$ and $Q = \alpha_k$ and $\alpha_n \neq \alpha_k$. Let $W = \{w\}$, and let $f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $f(\alpha_k) = \langle \mathbb{P}_k, \mathbb{P}_{-k} \rangle$. As we may assign to each type 1 proposition any subset of W, let $w \in P_n$, $w \notin P_{-n}$, $w \notin P_k$, and $w \notin P_{-k}$. From the truth conditions for type 1 atomic sentences, $\models_w \mathbb{P}_n$ and $\not\models_w \mathbb{P}_{-n}$. By the truth conditions for type 2 atomic sentences, $\models_w \alpha_n$ and $\models_w \neg\alpha_n$, and thus $\models_w (\alpha_n \wedge \neg\alpha_n)$ — i.e. $\models_w P \wedge \neg P$. Additionally, by the truth conditions for type 1 atomic sentences, $\not\models_w \mathbb{P}_k$ and $\not\models_w \mathbb{P}_{-k}$. By the truth conditions for type 2 atomic sentences, $\not\models_w \alpha_k$ — i.e. $\not\models_w Q$. Thus we have both $\models_w (P \wedge \neg P)$, and $\not\models_w Q$. This model is, then, a counterexample to explosion. That is, $(P \wedge \neg P) \not\models Q$.

Example 2. We may also construct a countermodel to $Q \vdash P \vee \neg P$. Suppose again that $P = \alpha_n$ and $Q = \alpha_k$ and $\alpha_n \neq \alpha_k$. Let $W = \{w\}$, and let $f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $f(\alpha_k) = \langle \mathbb{P}_k, \mathbb{P}_{-k} \rangle$. As we may assign to each type 1 proposition any subset of W, let $w \notin P_n$, $w \in P_{-n}$, $w \in P_k$, and $w \in P_{-k}$. By the truth conditions for type 1 atomic sentences, $\models_w \mathbb{P}_k$ and $\models_w \mathbb{P}_{-k}$. By the truth conditions for type 2 atomic sentences, $\models_w \alpha_k$ — i.e. $\models_w Q$. Additionally, by the truth conditions for type 1 atomic sentences, we have $\not\models_w \mathbb{P}_n$ and $\models_w \mathbb{P}_{-n}$. By the truth conditions for type 2 atomic sentences, $\not\models_w \alpha_n$ and $\not\models_w \neg\alpha_n$, and thus $\not\models_w (\alpha_n \vee \neg\alpha_n)$ — i.e. $\not\models_w P \vee \neg P$. Thus we have both $\models_w Q$, and $\not\models_w P \vee P$, as required.

Translation Schema

In this section I will prove that my models are models for the logic $\mathbf{K}_4$ by providing a translation scheme between the models of $\mathbf{K}_4$ as constructed by Priest in [19] and the models I have constructed. That is, given an arbitrary world of either type of model, there exists a world in the other type of model which has the same truths and falsehoods.

[5] Because $R = W \times W$, we omit the clause wRw' in these definitions. The definition written fully would be $\models_w^M (A \to B)$ iff for all $w' \in W$ such that wRw' and $\models_{w'}^M A$, $\models_{w'}^M B$.

I will define two functions, g and h, which will be maps from one type of model to the other. g will take Priest models as its arguments, and h will take *Ferenz Models* as its arguments.[6]

Definition 11. *The function g takes as argument a Priest model M' and returns Ferenz models $g(M')$. Let $M' = \langle W', \rho \rangle$ be our Priest model. The Ferenz model $g(M') = \langle W, P, Am, f, R \rangle$ is such that:*

$$
\begin{aligned}
W &= W' \\
R &= W \times W \\
f(\alpha_n) &= \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle \\
\textit{If } n \textit{ is positive, } P_n &= \{w_i | (\alpha_n)\rho_{w_i} 1\} \\
\textit{If } n \textit{ is negative, } P_n &= \{w_i | \textit{ it is not the case that } (\alpha_n)\rho_{w_i} 0\} \\
\langle w_i, \alpha_n \rangle \in Am & \textit{ iff both } (\alpha_n)\rho_{w_i} 0 \textit{ and } (\alpha_n)\rho_{w_i} 1 \textit{ or neither}
\end{aligned}
$$

Theorem 1.

For every world x in M', the corresponding world w in the Ferenz model $g(M')$ is such that;

1. $(\alpha_n)\rho_x 1$ *if and only if* $\models_w \alpha_n$.
2. $(\alpha_n)\rho_x 0$ *if and only if* $\models_w \neg\alpha_n$.

The proof is quite simple. For the first case, note that $\models_w \alpha_n$ is the case if and only if $f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $\models_w \mathbb{P}_n$, which is the case if and only if $w \in P_n$, which by definition is the case if and only if $(\alpha_n)\rho_x 1$. The second case is quite similar.

Definition 12. *The function h takes as argument a Ferenz models M returns Priest models $h(M)$. Let $M = \langle W, P, Am, f, R \rangle$ be our Ferenz model. The Priest model $h(M) = \langle W', \rho \rangle$ is such that:*

$$
\begin{aligned}
W' &= W \\
(\alpha_n)\rho_{w_i} 1 & \textit{ iff } w_i \in P_n \\
(\alpha_n)\rho_{w_i} 0 & \textit{ iff } w_i \notin P_{-n}
\end{aligned}
$$

[6] We name the newly constructed models Ferenz models, or, more accurately, *Ferenz models for* K_4.

Theorem 2.

For every world w in M, the corresponding world x in the Priest model $h(M)$ is such that;

1. $\models_w \alpha_n$ *if and only if* $(\alpha_n)\rho_x 1$.

2. $\models_w \neg\alpha_n$ *if and only if* $(\alpha_n)\rho_x 0$.

The proof is quite similar to the proof of Theorem 1

Corollary 1. *For every Priest model M', $g(M')$ is indeed a Ferenz model, and for every Ferenz model M, $h(M)$ is indeed a Priest model.*

Theorem 3. *For any x in M', the corresponding world w in $g(M')$ is such that $(A)\rho_x 1$ if and only if $\models_w A$.*

The proof is straightforward and found in the appendix.

Theorem 4. *For any w in M, the corresponding world x in $h(M)$ is such that $\models_w A$ if and only if $(A)\rho_x 1$.*

The proof is straightforward and found in the appendix.

Theorem 5. *For any Priest model M', $M' \models A$[7] if and only if $g(M') \models A$.*

Proof. Left to right: assume $M' \models A$. For reductio, suppose that $g(M') \not\models A$. From this supposition, it follows that there exists a world w in the Ferenz model $g(M')$ such that $\not\models_w^{g(M')} A$. From this we are able to prove that there exists a world x in M' such that $\not\models_x^{M'} A$ by means Theorem 3. The induction is straightforward, and will be omitted. From $\not\models_x^{M'} A$ we get $M' \not\models A$, which contradicts our original assumption. Therefore $g(M') \not\models A$.

Right to Left: assume $g(M') \models A$. For reduction, suppose that $M' \not\models A$. From this supposition, it follows that there exists a world x in the Priest model M' such that $\not\models_x^{M'} A$. From this we are able to prove that there exists a world w in $g(M')$ such that $\not\models_w^{g(M')} A$ by means of an Theorem 3. Again, this induction is straightforward and will be omitted. From $\not\models_w^{g(M')} A$ we get $g(M') \not\models A$, which contradicts our original assumption. Therefore $M' \models A$, as required. □

Theorem 6. *For any Ferenz model M, $M \models A$ if and only if $h(M) \models A$.*

[7] $M' \models A$ means that A is true at every world in M'.

Proof. The proof is similar to the proof of Theorem 5, and will be omitted. □

It follows from the above theorems that any inference has a countermodel in the class of Priest models if and only if it has a countermodel in the class of Ferenz models.

Theorem 7. *The functions g and h are such that, given any Ferenz model M, $g(h(M)) = M$, and given any Priest model M', $h(g(M')) = M'$.*

Proof. If it were the case that $g(h(M)) \neq M$, then either the number of worlds in the model $g(h(M))$ is different from the number of worlds in the model M, or there exists a world in the model M such that its corresponding world in $g(h(M))$ has a different set of truths. We can easily see that the number of worlds remains constant. We know that every world w_i in M has a corresponding world x_i in $h(M)$ at which the truth assignment to the literals[8] is the same. That is, $\models_{w_i} \alpha_n$ iff $(\alpha_n)\rho_{x_i}1$ and $\models_{w_i} \neg\alpha_n$ iff $(\alpha_n)\rho_{x_i}0$. Furthermore, we know that every world x_i in $h(M)$ has a corresponding world w_i in $g(h(M))$ at which the truth assignment to the literals is the same. That is, $(\alpha_n)\rho_{x_i}1$ iff $\models_{w_i} \alpha_n$ and $(\alpha_n)\rho_{x_i}0$ iff $\models_{w_i} \neg\alpha_n$. Thus, every world w_i in M has a corresponding world w_k in $g(h(M))$ such that $\models_{w_i} \alpha_n$ iff $\models_{w_k} \alpha_n$ and $\models_{w_i} \neg\alpha_n$ iff $\models_{w_k} \neg\alpha_n$. It follows from Theorem 3 and 4 that $\models_{w_i} A$ iff $\models_{w_k} A$ for every A. Thus $g(h(M)) = M$.

The proof that $h(g(M')) = M'$ is similar. □

3.2.1 Philosophical discussion

Let us briefly examine how we are able to model **FDE** using these models. We restrict the models to those only containing one world which we shall call w in each model, and we remove the arrow connective and the R relation. For Am, we may use $Am(w)$. The set $Am(w)$, then, is the set of sentences which are treated ambiguously in any given case.

Recall from the last chapter the sets Amb and Amb*. Amb was a set consisting of atomic sentences treated ambiguously in order to classically model a inconsistency, and Amb* was a set consisting of atomic sentences treated ambiguously in order to classically model incompleteness. We model inconsistency in the constructed models in a similar way. Given what these sets do, a sentence α_n is in Brown's Amb if and only if the sentence is in $Am(w)$, $f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $\models_w \mathbb{P}_n$.[9] Furthermore, a sentence α_k is in Brown's Amb* if and only if $\alpha_k \in Am(w)$, $f(\alpha_k) = \langle \mathbb{P}_k, \mathbb{P}_{-k} \rangle$ and $\models_w \mathbb{P}_{-k}$. Finally, the

[8] That is, the type 2 atomics in the Ferenz models and the atomics in the Priest models.

[9] This ensures that $\models_w \alpha_n$ and $\models_w \neg\alpha_n$.

sentences in neither Amb nor Amb* are not in $Am(w)$ in the newly constructed models. In order words, the ambiguous sentences are mapped onto pairs of sentences, each of which has a different truth value: the non-ambiguous sentences are mapped onto pairs of sentences, each of which has the same truth value. The benefit of the two types of atomic sentences, and thus the two level of description, is emphasized by this point. The non-ambiguous sentences are described using the second type of atomic sentences. The ambiguous sentences are described using the first level of description, in order to classically model inconsistent and incomplete sets of sentences. That is, the atomic sentences which are treated ambiguously are separated into true and false instances. The treatment of ambiguous atomic sentences in these models is similar to their treatment in Brown's approach.

Equivalent to Brown's construction of the consequence relation, we say that $\Gamma \vdash_{FDE} \delta$ if and only if every model in which all the members of Γ are true, so is δ. Dually, $\Gamma \vdash_{FDE} \delta$ if and only if every model in which δ is not true, at least one of the members of Γ is also not true.

Example 4. Consider (in **FDE**) the invalid inference $P \wedge \neg P \vdash Q \vee \neg Q$. I will construct a countermodel using the newly constructed models. Let M be a model with only one world, w, and suppose $P = \alpha_n$, $Q = \alpha_k$, and $\alpha_n \neq \alpha_k$. Let $f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $f(\alpha_k) = \langle \mathbb{P}_k, \mathbb{P}_{-k} \rangle$. As any classical truth assignment to type 1 atomic sentences is possible, let $\models_w \mathbb{P}_n$, $\not\models_w \mathbb{P}_{-n}$, $\not\models_w \mathbb{P}_k$, and $\models_w \mathbb{P}_{-k}$. It immediately follows by the conditions for membership that $P, Q \in Am(w)$.[10] Given the truth and falsity conditions for the second type of atomic sentences, we see that $\models_w \alpha_n$ and $\models_w \neg\alpha_n$, and therefore $\models_w (\alpha_n \wedge \neg\alpha_n)$. Thus $\models_w (P \wedge \neg P)$. What is left to show is that this world does not model $Q \vee \neg Q$.

By the truth and falsity conditions of the second type of atomic sentences, $\not\models_w \alpha_k$, and $\not\models_w \neg\alpha_k$, and therefore $\not\models_w (\alpha_k \vee \neg\alpha_k)$ — indeed $\not\models_w (Q \vee \neg Q)$. Thus the world w is indeed a world where $\models_w (P \wedge \neg P)$ and $\not\models_w (Q \vee \neg Q)$ as desired. We therefore have a countermodel to the inference $P \wedge \neg P \vdash_{FDE} Q \vee \neg Q$. Furthermore, the dual definition of the consequence relation will give us the same result. That is, the world w is a model which models the denial of $(Q \vee \neg Q)$, but does not model the denial of $(P \wedge \neg P)$

The f function requires ordered pairs because of its philosophical interpretation.

[10] In fact, P satisfies the conditions given earlier in this section for membership in Brown's Amb, and Q in Amb*.

Brown's formal method of treating an atomic sentence as ambiguous is to separate the sentence into two sentences. One of these new sentences is treated as true, the other as false. Members of Amb and members of Amb* are similar in that their members are treated as pairs of new sentences. How Amb and Amb* differ is how the truth value assignments to members of the pairs of new sentences.

For members of Amb, the truth value assignments on the corresponding pairs of new sentences assign True to one member of the pair and False to the other, and do so in order to model inconsistent sets. For members of Amb*, the truth value assignments on the corresponding pairs of new sentences assign True to one member of the pair and False to the other, and do so in order to model incompleteness. What the f function does is allow us to assign pairs of type 1 atomic sentences to each type 2 atomic sentence. How those sentences receive truth values is dependent on the individual models, but the f function serves a vital role in formalizing Brown's preservationist approach in model theory. The ordering of the pairs the f function maps onto allows us to distinguish between members of Amb and Amb* within the model theoretic approach, which lets us interpret Brown's approach directly into the models I have constructed.

The models I have constructed for $\mathbf{K}_4$ differ by the addition of a intensional arrow connective. To accommodate the intensional connective, the models are generalized to contain more than one world. I will show that if Brown were to extend his project to include logics with a conditional connective, then an intensional arrow much like the arrow of $\mathbf{K}_4$, (or indeed of $\mathbf{N}_4$ or $\mathbf{B}$) would be suitable.

One feature of many logical systems is that the *deduction theorem* is provable for them. Formally, we may write the deduction theorem as follows;

$$\Gamma \cup \{A\} \vdash B \text{ if and only if } \Gamma \vdash A \to B$$

In the above statement, A and B are formula, and Γ is a set of formula. We may, however, distinguish this deduction theorem from a semantic deduction theorem, which may be stated as follows;

$$\Gamma \cup \{A\} \models B \text{ if and only if } \Gamma \models A \to B$$

These statements are equivalent if our semantics is sound and complete with respect to the syntax, but I will respect this distinction in what is to follow.

The Deduction Theorem emphasizes the relationship between the conditional connective and the consequence relation. In fact, the following statement of the deduction theorem in *Entailment* Vol 1 shows that the relevant logicians are motivated by creating an arrow which has a specific relationship with the provability;

> THEOREM. $A \to B$ is a theorem of $\mathbf{R}_{\to}$ just in case there is a proof of B *from* the hypothesis A. [1, p. 20]

Note carefully two things. The first is that the arrow encodes that there exists a proof, i.e. that $A \vdash B$. The second is that 'from' is emphasized in the above quote. How 'from' is cashed out by Anderson and Belnap is the construction of a new turnstyle (well, many) which aim to better encode their relevant intuitions of 'provable from'.

The semantic deduction theorem should also share this relationship. That is, replacing proof with semantic entailment, we wish that the arrow encodes that the antecedent semantically entails the consequent. For B to be semantically entailed by A in the semantics for $\mathbf{K}_4$, it must be the case that every world at which A is modeled, B is also modeled. In other words, every way we can make the antecedent true, makes the consequent true.

Looking at the definition of an arrow in the models for $\mathbf{K}_4$, we find that the arrow indeed has the desired relationship with semantic entailment. That is, $A \to B$ if and only if every world at which A is modeled, B is also modeled. For logics where the antecedent structure of the turnstyle or double turnstyle is closed under conjunction, and the consequent structure is close under disjunction, every inference may be translated into a conditional sentence.

The models I have constructed capture the essence of Brown's preservationist approach and extend it to the logic $\mathbf{K}_4$. These models allow us to note clearly a few key points. The first is that negation is fully classical for those sentence not treated ambiguously, i.e. those sentences which are not in Am. The sentence in Am are not the sentences of the base level of our models. When modeling a sentence in Am, it is intuitive to think that we are modeling pairs of sentences which behave classically. The models I have constructed clearly show that atomic sentences treated ambiguously may be replaced with pairs of sentences. These pairs are such that one member is true and the other is false. These pairs represent the true and false instances of the ambiguous sentences, as required by Brown's approach. Given this, and given that the arrow constructed is suitable (for now), the models I have constructed for $\mathbf{K}_4$ are a coherent extension of Brown's preservationist project explicated in the last chapter.

Furthermore, the models I have constructed need not be accepted only by those who accept the logic $\mathbf{K}_4$ or **FDE**. In terms of metaphysical commitments, my models do not require being committed to the possibility of glutty or gappy worlds. The inconsistent and incomplete worlds that appear in most extant model theories of K_4 are only described as such when mapping to pairs of sentences in my models, while the worlds themselves are complete and consistent.[11] We can then make sense of the logic K_4 by means of the models I have constructed, and with completely classical commitments. Another question is whether or not this K_4 is useful.

The usefulness of K_4 could be instrumental. Many scientific theory and set of beliefs are likely to be inconsistent and incomplete — at least, that is, under closure of the classical consequence relation.[12] Instrumentally we cannot use classical logic to reason from these theories or belief sets. If we did use classical logic when reasoning from these sets, and if the inferences of classical logic are to be considered rational, then by rational inference alone our belief sets and scientific theories include every sentence when closed under rational inference. However, even if belief is not closed under rational inference, it seems reasonable to think that being closed under rational inference should not trivialize your beliefs. Rational inference, then, is better captured by a paraconsistent logic which does not trivialize inconsistent belief sets or scientific theories.

Instrumentally, then, we ought to use a logic which is capable of describing inconsistent and incomplete theories without trivializing. Such a logic would allow us to use every inference valid in the logic without having to worry that applying the logic will lead to triviality. That is, we should not have to worry about the inconsistency or incompleteness of a set while reasoning from it, and we should not have to worry about whether or not a sentence follows *just* because of inconsistency or incompleteness. Nonetheless, it is worthwhile in either case to aim for complete and consistent premise sets.

I want to note a distinction which is important to all the non-classical logics mentioned here. There is a difference between the sentence $P \wedge \neg P$ being in our premise set, and our logic proving $P \wedge \neg P$ from consistency (the null set or a consistent set). In the second case, the connective $\wedge$ or the connective $\neg$ must mean something other than what they mean in classical logic. In this case we would no longer be reasoning from our premise set

[11] For those still worried about the commitments of mapping to pairs, we need only map the members of Am to pairs of sentences. The formalism is much smoother when we map every sentence to pairs of sentences.

[12] Though some belief sets are explicitly inconsistent, as some may "believe that the Russell set is both a member of itself and not a member of itself " [18, p. 96]. Other belief sets might only be contradictory under a closure relation, such as the Peano axioms with closed under classical logic [11, p. 486].

in any reasonable sense of the word. Inferring a contradiction from a consistent premise set does not preserve truth, nor is it helpful. This second case is much more radical than the the first case. In the first case, we are trying to reason *from* a contradiction. It seems perfectly reasonable to make use of a premise set from which $P \wedge \neg P$ is derivable. For example, people often reason from inconsistent theories or belief sets.

The distinction being made is the distinction between inference and assumption for an inference. Suppose $P \wedge \neg P$ for some P. Should it follow that our inference relation is no longer useful? I think not. We ought to have a useful inference relation which is not trivialized by the appearance of $P \wedge \neg P$ in the premise set. That being said, we want an inference relation which is still useful when $P \wedge \neg P$ appears in the premise set. Being able to infer nothing at all is still not useful. The logic $\mathbf{K}_4$ retains the meanings of all the connectives (including negation when using the models I have constructed with Brown's motivation). Thus $\mathbf{K}_4$ is useful, even if only instrumentally for someone with the metaphysical commitments of classical logic.

$\mathbf{K}_4$ is one way to extend the logic **FDE** with the addition of an arrow. In fact, the arrow is almost a relevant arrow. The validities of $\mathbf{K}_4$ which exclude $\mathbf{K}_4$ from being considered a relevant logic in agreement with the earlier discussion in chapter 1 (e.g., $\vdash P \to (Q \to Q)$) follow from the properties of the arrow alone, and not the other connectives. What we must change to construct a relevant logic is the behavior of the arrow. In the next section I will motivate changing the consequence relation in order to make the arrow relevant. One logic which has a relevant arrow is $\mathbf{N}_4$, for which I will also construct adequate models based on the models constructed for $\mathbf{K}_4$.

3.3 N4

3.3.1 Introductions and Motivations

In $\mathbf{K}_4$, $P \to (Q \to Q)$ is a valid formula, and therefore the logic $\mathbf{K}_4$ is not a relevant logic. We want a logic in which $\models (Q \to Q)$, but $\not\models P \to (Q \to Q)$. Syntactically, the solution is relatively easy.[13] The common semantic (model theoretic) solution is the introduction of worlds at which $(Q \to Q)$ fails. These new worlds are named non-normal worlds, and it is specified that $\models A$ is valid if and only if A is modeled at every *normal* world. This last condition ensures that $\models (Q \to Q)$, while at the same time $\not\models P \to (Q \to Q)$.

[13] In a Hilbert axiom system, we change our set of axioms.

One way to motivate the inclusion of non-normal worlds in our models is that the non-normal worlds explicitly separate the truth and falsity a conditional formula from the truth and falsity of its antecedent and consequent. This motivation aligns itself with the preservationist approach of Brown. Take the inference $\models Q \rightarrow (P \rightarrow P)$. This is always true in the logic K_4, for there is no way to model the denial of $(P \rightarrow P)$. (Brown notes a similar situation with classical logic and the inference $Q \models (P \vee \neg P)$.) I expand upon this motivation below. There I will argue that this motivation is a coherent extension of Brown's project, and indeed one which captures the motivations of relevant logic as well. First, however, I will construct models which extend the models of the last section with non-normal worlds. With this machinery in place, I will discuss the motivation for the inclusion of non-normal worlds.

I will ultimately extend the models of K_4 by the inclusion of non-normal worlds in two ways. First to model the logic N_4, then to model the logic B. Each extension adds non-normal worlds to the models of K_4.

Non-Normal Worlds: Relevant Logic and Model Theory

Given the motivations for the rejection of thinning and explosion found in the first chapter, we would like a semantics in which $\not\models Q \rightarrow (P \rightarrow P)$ and $\not\models \neg Q \rightarrow (P \rightarrow P)$. With a model theoretic semantics in place, our requirements amount to requiring that $P \rightarrow P$ is not modeled at every world which models Q (or $\neg Q$). What we require, then, is worlds within our models where Q is true and $P \rightarrow P$ is not true. This is precisely the work a non-normal world does in our model. So, if we are committed to a model theoretic approach, and if we are committed to invalidating $Q \rightarrow (P \rightarrow P)$, then we must have worlds within our model which model Q and not $P \rightarrow P$.

There are, however, further requirements we wish to impose on our semantics. We require that $\models (P \rightarrow P)$ be the case. The distinction which appears to be made in the literature between $\models (P \rightarrow P)$ and $\models Q \rightarrow (P \rightarrow P)$ is as follows.[14] The first inference, $\models (P \rightarrow P)$, states that the formula $(P \rightarrow P)$ *follows from* logic alone. The second inference states that $(P \rightarrow P)$ *follows from* the formula Q. One reason we may be motivated to distinguish between these inferences is the benefit we gain in terms of expressive power. Consider the following example of Priest's;

[14] The distinction uses the refined notion *following from* developed in [1], and the distinction made between normal and non-normal (logically impossible) worlds by Priest in chapter 9 of [19].

$q \to q$ is an instance of the law of identity. Yet the following conditional would hardly seem to be true: if every instance of the law of identity failed, then, if cows were black, cows would be black. If every instance of the law failed, then it would precisely *not* be the case that if cows were black, they would be black. [19, p. 167]

This example shows at least one type of situation where the distinction is important. If we wish to be able to express that the sentence referred to by Priest in the above quote is indeed false, then we require non-normal worlds in our models. Furthermore, we ought to be able to express the truth and falsity of such sentences. The truth and falsity of such sentences can be informative, and can be used in the study of implication. Denying ourselves the ability to express such truths and falsehoods throws away the baby with the bathwater.

For example, consider sentences of the form 'if logic L_n was the case, then ... '.[15] There seems to be true and false sentences of this form. The expressive power needed to express these sentences in a logic requires a distinction between sentences of the form $\models (P \to P)$ and sentences of the form $\models Q \to (P \to P)$. $\models (P \to P)$ states that $(P \to P)$ follows from the logic it is being expressed within. On the other hand, $\models Q \to (P \to P)$ may very well be false, say, when the formula Q states that a certain logic, in which the law of identity is false, is the case. A logic which is capable of representing this sort of reasoning is required to have the expressive power to make the distinction between the $\models (P \to P)$ and $\models Q \to (P \to P)$. Model theoretically this is achieved by the introduction of non-normal worlds to our models.

3.3.2 N_4: Models

Definition 13. *A **model** M for N_4 is a structure $\langle W, N, P, Am, f, R\rangle$. W, Am, f, and R are defined as they were for K_4. N is a subset of W. The members of N are normal worlds. P assigns sets of worlds to atomic sentences of the first type, as before. We may again think of the set of worlds assigned to a proposition as the set of worlds where that proposition is true. However, P also assigns sets of non-normal worlds to conditional sentences and negations of conditional sentences. That is, P assigns a set of non-normal worlds to each sentence of the form $A \to B$ and to each sentence of the form $\neg(A \to B)$. What this expanded P will amount to will be explained below.*

[15] In fact, Priest discusses this type of sentence as well [19, p. 171].

We will write $P_{(A\to B)}$ as the set of non-normal worlds which P assigns to $(A \to B)$, and we write $P_{\neg(A\to B)}$ as the set of non-normal worlds which P assigns to $\neg(A \to B)$. The truth and falsity conditions for the first type of atomic sentence remain unchanged. The truth and falsity conditions for second type of atomic sentence, and complex sentences built from the second type of atomic sentence, remain the same with the exception of the conditional sentence and their negations. For normal worlds, the truth and falsity conditions for arrow remain the same. That is, the conditions for normal worlds w are as follows;

1. $\models_w^M (A \to B)$ iff for all $w' \in W$ such that $\models_{w'}^M A$, $\models_{w'}^M B$.
2. $\models_w^M \neg(A \to B)$ iff there exists a $w' \in W$ such that $\models_{w'}^M A$ and $\models_{w'}^M \neg B$.

For non-normal worlds the truth and falsity conditions for conditional sentences as their negations are as follows;

1. $\models_w^M (A \to B)$ iff $w \in P_{(A\to B)}$.
2. $\models_w^M \neg(A \to B)$ iff $w \in P_{\neg(A\to B)}$.

Note that these conditions ensure that the truth and falsity conditions of conditional sentences and their negations at non-normal worlds are neither truth functional nor intensional. Rather, they are assigned truth values at those worlds just like the first type of atomic sentence are assigned truth values.

Validity is defined at follows. Note that the definitions for validity are modified from the definitions given by Priest for non-normal modal logics [19, p. 65]. For every non-empty set Γ, validity is defined as follows;

$$\begin{gathered}\Gamma \models \delta \text{ iff for every model and every } w \in N: \\ \text{if } \models_w B \text{ for all } B \in \Gamma, \text{ then } \models_w \delta.\end{gathered}$$

When Γ is the empty set, validity is defines as;

$$\models \delta \text{ iff for every model and every } w \in N, \models_w \delta.$$

To get a feel for how these models work, and to check that $P \to P$ is indeed valid and $Q \to (P \to P)$ is invalid, I construct the following examples.

EXAMPLE 5. $(P \to P)$ is an validity in the constructed models. I show this by showing that $(P \to P)$ is true at every normal world in every model. Let $w \in N$, and let

$f(P) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$. We know that $\models_w (P \to P)$ if and only if every $w' \in W$ such that $\models_{w'} P$, $\models_{w'} P$. It is obvious that every $w' \in W$ such that $\models_{w'} P$ is also such that $\models_{w'} P$. Thus $\models_w (P \to P)$. Furthermore, because w was an arbitrary normal world, and we only used the fact that it was normal, the result hold for every normal world in every model. Therefore $(P \to P)$ is a validity.

EXAMPLE 6. Consider $Q \to (P \to P)$. To show this sentence is invalid, I construct a countermodel. Let M be a model which contains two worlds w_1 and w_2. Let $N = \{w_1\}$. Let $f(P) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $f(Q) = \langle \mathbb{P}_k, \mathbb{P}_{-k} \rangle$ for $k \neq n$. Let $w_2 \notin P_{P \to P}$, and let $P_k = \{w_1, w_2\}$. I will show that $\not\models_{w1} Q \to (P \to P)$, and thus $\not\models Q \to (P \to P)$. Give the truth conditions, we have $\models_{w1} Q$ and $\models_{w2} Q$. For $\models_{w1} Q \to (P \to P)$ to be the case, every world w' such that $\models_{w'} Q$, $\models_{w'} (P \to P)$. We see that at $\models_{w2} Q$ and $\not\models_{w2} (P \to P)$. We then have $\not\models_{w1} Q \to (P \to P)$, and thus $Q \to (P \to P)$ is invalid , as desired.

3.3.3 Philosophical discussions

One may be tempted to use the set Am in determining the truth values of conditional sentences at non-normal worlds. After all, the set Am is used to explain the non-classical behavior of the other connectives. However, this temptation is misguided for the current models of the logic $\mathbf{N}_4$. For, as I will now show, the failure of some instances of the law of identity at non-normal worlds cannot be the result of ambiguity. (It is no surprise that the truth and falsity of conditional sentences at non-normal worlds cannot be extensional, even when based upon the ambiguity at that world.)

Theorem 8. *For the current models of the logic $\mathbf{N}_4$, the truth and falsity of conditional sentences at any given non-normal world are not extensionally determined by the f function, the set Am and the truth and falsity of the atomic sentences at that world.*

Proof. Let M be a model of $\mathbf{N}_4$. Let M contain two worlds, w_1 and w_2 such that $N = \{w_1\}$. Let Am be the empty set. That is, let no type 2 atomic sentence be ambiguous at either world. Because w_2 is a non-normal world, the truth and falsity of conditional sentences (and their negations) are assigned by the P function. Since any assignment is possible, let our model be such that $\not\models_{w2} P \to P$ and $\models_{w2} \neg(P \to P)$. Because no sentence is ambiguous at w_2, $P \to P$ being just false at w_2 cannot be because of ambiguous atomic sentences at w_2. $\square$

A natural next step is to define truth and falsity conditions for conditional sentences at non-normal worlds which are intensional. That is, we define the truth and falsity conditions such that they rely on the truth and falsity (and ambiguity) of sentences at other worlds. I am unsure how to construct a semantics for $\mathbf{N}_4$ using this method, or if it is even possible. Interaction between ambiguity and the failure of $P \to P$ at non-normal worlds is desirable if Brown's preservationist project is to be extended to relevant logics. Without some interaction, Brown's approach only helps the interpretations of extensional connectives. I have shown using the logic $\mathbf{K}_4$ that Brown's approach is capable of being extended to intensional connectives. Modifying the models I have constructed so that ambiguity plays a role in the truth conditions for conditional sentences at non-normal worlds, and doing so in such a way that the models still model the logic $\mathbf{N}_4$, would provide a way to extend Brown's approach to the logic $\mathbf{N}_4$ in a more satisfactory way. However, I will not do so here. I leave this as a future project. However, in the next section I will attempt to construct models for the logic $\mathbf{B}$ where the truth conditions for conditionals at non-normal worlds are intensional.

To deserve to be considered an extension of Brown's project, the truth and falsity of conditionals at non-normal worlds must be due to some sort of ambiguity. Indeed, consider the sentence $P \to P$. If we treat P ambiguously, then, as in Brown's project, we have $P_t \to P_f$. Treating P ambiguously in this case amounts to assigning different truth values to different instances of P in the formula (or inference). To adequately extend Brown's preservationist project, this type of ambiguity must be the reason $P \to P$ fails at some non-normal worlds. The goal, then, is to be able to classically model a denial of $P \to P$ by treating P ambiguously. I say classically model, but the term may be misleading in this case because the conditional connective is intensional. The move to intensionality was argued for in section 3.2.1, and is also motivated by the inability to formalize the desired interaction between conditionals, non-normal worlds, and ambiguity using extensionality alone.

The above proof shows, at least for $\mathbf{N}_4$, that this ambiguity is not ambiguity *at the non-normal world* which $P \to P$ is being evaluated. Thus, $P \to P$ will not fail at a non-normal world because of ambiguity at that world, but because of ambiguity at other worlds, if we are to adequately extend Brown. We therefore need an accessibility relation which we can be used in defining truth and falsity conditions at non-normal worlds in order that the truth and falsity of conditionals at non-normal worlds interacts with ambiguity. Ideally, sentences which we cannot model at normal worlds, such as the denial of $P \to P$, will be the sentences which require ambiguity to model at non-normal worlds.

In the next section I will construct models in which the truth (and falsity) of conditionals at non-normal worlds is intensional. This will provide the machinery needed to construct truth and falsity conditions for conditionals at non-normal worlds which rely on ambiguity, as far as ambiguity has been formalized so far.

3.4 B

3.4.1 B: Models

Here I will extend the models constructed for K_4 with the addition of non-normal worlds and a ternary relation. This allows us to define intensional truth and falsity conditions for conditionals at non-normal worlds. I aim to produce models similar to the American Plan models as constructed in [23].[16] The models constructed will be models for the logic **B**.

Definition 14. *A model M for* **B** *is a structure $\langle W, N, P, Am, f, R\rangle$. $W, Am, P,$ and f are defined as they were for K_4. $N \subseteq W$. The set N is the set of normal worlds. R is a ternary accessibility relation between worlds, i.e. $R \subseteq W \times W \times W$.*

The models are further constrained by the requirement that each model is *closed under duality*. Restall has shown that this is required for the American Plan to model the logic **B**. A model M' is closed under duality when, for each world $w \in M'$, there exists a world w^* such that [23, p. 148];

$$\begin{array}{l} \models^{M'}_{w} A \text{ and } \not\models^{M'}_{w} \neg A \text{ iff } \models^{M'}_{w^*} A \text{ and } \not\models^{M'}_{w^*} \neg A \\ \not\models^{M'}_{w} A \text{ and } \models^{M'}_{w} \neg A \text{ iff } \not\models^{M'}_{w^*} A \text{ and } \models^{M'}_{w^*} \neg A \\ \not\models^{M'}_{w} A \text{ and } \not\models^{M'}_{w} \neg A \text{ iff } \models^{M'}_{w^*} A \text{ and } \models^{M'}_{w^*} \neg A \\ \models^{M'}_{w} A \text{ and } \models^{M'}_{w} \neg A \text{ iff } \not\models^{M'}_{w^*} A \text{ and } \not\models^{M'}_{w^*} \neg A \end{array}$$

That is, in the terms of four-valued logic, every world has a star world where every just True and just False sentence is the same, but sentences receiving the truth value Both (Neither) at a world receive the truth value Neither (Both) at its star world.

For type 2 atomic sentences, the truth and falsity conditions for the extensional connectives $(\wedge, \vee, \neg)$ remain the same. The truth condition for the conditional relies on the ternary relation, and the falsity condition relies on star (dual) worlds.

[16] See Chapter 1 for the essential details.

Definition 15. *At every $w \in W$, for members of $\{\alpha_1, \alpha_2, \alpha_3 \dots\}$, the following hold;*

1. $\models^M_w \alpha_n$ iff $f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $\models^M_w \mathbb{P}_n$.
2. $\models^M_w \neg\alpha_n$ iff $f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$ and $\not\models^M_w \mathbb{P}_{-n}$.
3. $\models^M_w (A \wedge B)$ iff $\models^M_w A$ and $\models^M_w B$
4. $\models^M_w \neg(A \wedge B)$ iff $\models^M_w \neg A$ or $\models^M_w \neg B$
5. $\models^M_w (A \vee B)$ iff $\models^M_w A$ or $\models^M_w B$
6. $\models^M_w \neg(A \vee B)$ iff $\models^M_w \neg A$ and $\models^M_w \neg B$
7. $\models^M_w (A \to B)$ iff for every x, y such that $Rwxy$: if $\models^M_x A$, then $\models^M_y B$
8. $\models^M_w \neg(A \to B)$ iff $\not\models^m_{w^*} (A \to B)$

We must make the further stipulation that, for any normal world x, $Rxyz$ iff $y = z$ [19, p. 189]. The truth condition for normal worlds, then, is as follows;

7N. $\models^M_w (A \to B)$ iff for every w' such that $\models^M_{w'} A$, $\models^M_{w'} B$

The condition 7N is the condition added for conditionals in the logic $\mathbf{K}_4$, i.e. a strict conditional. A conditional is true at a normal world if and only if every way to model the antecedent also models the consequent. Validity is defined as it was for the logic $\mathbf{N}_4$. That is, validity is defined over normal worlds.

Example 6. Here I construct a model in order to illustrate how the falsity conditions for conditionals work. The model to be constructed also demonstrates the use of the ternary relation. The model, however, is perhaps not the most illustrative example in terms of the ternary relation. Let M be a model with two worlds. let the first world be called w and let the second world be the star world of w, w^*. Let $N = \{w\}$. R shall contain all ordered set required by the normality of W. Additionally, let Rw^*ww (i.e. $\langle w^*, w, w \rangle$). Let $\models_w A$, $\models_w B$, $\models_w \neg B$, and $\models_{w^*} A$. Note carefully two things that follow. The first is that $\not\models_w (A \to B)$, for there exists a world in the model, w^* such that $\models_{w^*} A$ and $\not\models_{w^*} B$. The second relies on the fact that $\models_{w^*} (A \to B)$. This fact is easy to check and relies on the set chosen for R. The second interesting property, then, is that the falsity conditions for the conditional imply that $\not\models_w \neg(A \to B)$.

B: Translation Scheme

To prove that the models constructed are sound and complete for the logic **B**, I will provide a translation scheme between between the models constructed and the models of the American Plan for the logic **B**. The American Plan models are closed under duality, as explicated in [23]. The essentials of these models are explicated above in chapter 1. However, I will use slightly different notation than I did in Chapter 1. In Chapter 1 I used relational valuations for the American plan semantics. Here I use $\models_x A$ to stand for $(A)\rho_x 1$, and I use $\models_x \neg A$ to stand for $(A)\rho_x 0$. The truth conditions for the connectives are capable of being expressed in this notation as well.[17] I will differentiate the different types of models by using $w, w', w'' \ldots$ to stand for worlds of Ferenz models, and using $x, y, z, x', \ldots$ to stand for worlds of the American plan models. Again, I will define two functions, g and h, which will be maps from one type of model the the other. g will take American Plan models as arguments, and h will take Ferenz Models as arguments.

Definition 16. *The function g takes as argument an American plan model M' and returns a Ferenz models $g(M')$. Let $M' = \langle W', N', R', \rho\rangle$ be our American plan model closed under duality. The ferenz model $g(M') = \langle W, N, P, Am, f, R\rangle$ is such that:*

$$W = W'$$
$$R = R'$$
$$N = N'$$
$$f(\alpha_n) = \langle \mathbb{P}_n, \mathbb{P}_{-n}\rangle$$
$$\textit{If } n \textit{ is positive, } P_n = \{w_i | (\alpha_n)\rho_{w_i} 1\}$$
$$\textit{If } n \textit{ is negative, } P_n = \{w_i | \textit{ it is not the case that } (\alpha_n)\rho_{w_i} 0\}$$
$$\langle w_i, \alpha_n\rangle \in Am \textit{ iff both } (\alpha_n)\rho_{w_i} 0 \textit{ and } (\alpha_n)\rho_{w_i} 1 \textit{ or neither}$$

Theorem 9. *For every world x in M', the corresponding world w in the Ferenz model $g(M')$ such that:*

1. $\models_x \alpha_n$ *if and only if* $\models_w \alpha_n$
2. $\models_x \neg\alpha_n$ *if and only if* $\models_w \neg\alpha_n$

[17] Consider the following example. On page 24 I defined the truth condition for conjunctions as "$(A \wedge B)\rho_x 1$ iff $(A)\rho_x 1$ and $(B)\rho_x 1$". This becomes "$\models_x (A \wedge B)$ iff $\models_x A$ and $\models_x B$" with no loss (or gain) of information.

3. *$Rxyz$ is in M' iff $Rww'w''$ is in $g(M')$, where the worlds w, w', w'' correspond respectively to x, y, z.*

4. *If the world corresponding to x is w, then the world corresponding to x^* is w^*.*

The proof is quite simple. The first two cases are covered by the proof of Theorem 1. The third and fourth cases follow trivially from the definition of W, N, and R, in $g(M')$.

Definition 17. *The function h takes as argument a Ferenz model M and returns an American plan model $h(M)$. Let $M = \langle W, N, P, Am, f, R\rangle$ be our Ferenz model. The American plan model $h(M) = \langle W', N', R', \rho\rangle$ is such that:*

$$W' = w$$
$$R' = R$$
$$N' = N$$
$$(\alpha_n)\rho_{w_i}1 \text{ iff } w_i \in P_n$$
$$(\alpha_n)\rho_{w_i}0 \text{ iff } w_i \notin P_{-n}$$

Theorem 10. *For every world w in M, the corresponding world x in the American plan model $h(M)$ is such that:*

1. *$\models_w \alpha_n$ if and only if $\models_x \alpha_n$*

2. *$\models_w \neg\alpha_n$ if and only if $\models_x \neg\alpha_n$*

3. *$Rww'w''$ is in M iff $Rxyz$ is in $h(M)$, where the worlds x, y, z correspond respectively to w, w', w''.*

4. *If the world corresponding to w is x, then the world corresponding to w^* is x^*.*

The proof is simple, and similar to the proof of Theorem 9

Corollary 2. *For every American plan model M', $g(M')$ is indeed a Ferenz model, and for every Ferenz model M, $h(M)$ is indeed an American plan model.*

Theorem 11. *For any x in M', the corresponding world w in $g(M')$ is such that $\models_x A$ if and only if $\models_w A$.*

The proof is straightforward and found in the appendix.

Theorem 12. *For any w in M, the corresponding world x in $h(M)$ is such that $\models_w A$ if and only if $\models_x A$.*

The proof is straightforward and found in the appendix.

Theorem 13. *For any American plan model M', $M' \models A$ if and only if $g(M') \models A$.*

Proof. Left to right: assume $M' \models A$. For reductio, suppose that $g(M') \not\models A$. From this supposition, it follows that there exists a *normal* world w in the Ferenz model $g(M')$ such that $\not\models_w^{g(M')} A$. From this, we are able to prove that there exists a normal world x in M' such that $\not\models_x^{M'} A$ by the application of Theorem 11.[18] From $\not\models_x^{M'} A$, we get $M' \not\models A$, which contradicts our original assumption. Therefore $g(M') \models A$, as required.

Right to left: assume $g(M') \models A$. For reductio, suppose that $M' \not\models A$. From this supposition, it follows that the exists a *normal* world x in the American Plan model M' such that $\not\models_x^{M'}$. From this, we are able to prove that there exists a normal world w in $g(M')$ such that $\not\models_w^{g(M')} A$ by application of Theorem 12. From $\not\models_w^{g(M')} A$, we get $g(M') \not\models A$, which contradicts our original assumption. Therefore $M' \models A$, as required. $\square$

Theorem 14. *For any Ferenz model M, $M \models A$ if and only if $h(M) \models A$.*

Proof. The proof is similar to the proof of Theorem 13. $\square$

It follows from the above theorems that a sentence has a countermodel in the class of American plan models if and only if it has a countermodel in the class of Ferenz models.

Theorem 15. *The functions g and h are such that, given any Ferenz model M, $g(h(M)) = M$, and given any American plan Model M', $h(g(M')) = M'$.*

[18] We take the base case of the induction to be the truth value assignment to the literals in the same in each type of model. This is ensured by the definition of the function g. The truth conditions for the each connective are same same in each model. Further, the R relation is such that $Rww'w''$ is in $g(M')$ if and only if $Rxyz$ is in M' and w, w', w'' are the worlds corresponding respectively to x, y, z. Therefore, because each model has the same number of worlds with the same truth value assignment to the literals at corresponding worlds, the induction is very straightforward.

Proof. If it were the case that $g(h(M)) \neq M$, then either the number of worlds in the model $g(h(M))$ is different from the number of worlds in the model M, or there exists a world in the model M such that its corresponding world in $g(h(M))$ has a different set of truths. We can easily see by the definitions of g and h that the number of worlds remains constant.

We know that every world w_i in M has a corresponding world x_i in $h(M)$ at which (1) the truth value assignment to the literals is the same, (2) $w_i \in N$ iff $x_i \in N$, and (3) $Rww'w''$ is in M iff $Rxyz$ is in $h(M)$, where the worlds x, y, z correspond respectively to w, w', w''. We know, then, that $\models_{w_i} \alpha_n$ iff $\models_{x_i} \alpha_n$ and $\models_{w_i} \neg\alpha_n$ iff $\models_{x_i} \neg\alpha_n$. Furthermore, we know that every world x_i in $h(M)$ has a corresponding world w_k in $g(h(M))$ at which (1) the truth assignment to the literals is the same, (2) $x_i \in N$ iff $w_k \in N$, and (3) $Rww'w''$ is in M iff $Rxyz$ is in $h(M)$, where the worlds x, y, z correspond respectively to w, w', w''. We know, then, that $\models_{x_i} \alpha_n$ iff $\models_{w_k} \alpha_n$ and $\models_{x_i} \neg\alpha_n$ iff $\models_{w_k} \neg\alpha_n$.

Thus, every world w_i in M has a corresponding world w_k in $g(h(M))$ such that (1) $\models_{w_i} \alpha_n$ iff $\models_{w_k} \alpha_n$ and $\models_{w_i} \neg\alpha_n$ iff $\models_{w_k} \neg\alpha_n$, (2) $w_i \in N$ iff $w_k \in N$, and (3) $Rww'w''$ is in M iff $Rw_kw'_kw''_k$ is in $g(h(M))$, where the worlds w_k, w'_k, w''_k correspond respectively to w, w', w''. A simple application of Theorems 11 and 12 show that $\models_{w_i} A$ iff $\models_{w_k} A$ for every A. Thus $g(h(M)) = M$. ☐

3.4.2 Ambiguity and the Conditional.

In this section I discuss the models as currently constructed, and I show that these models lack the desired interaction between ambiguity and the truth and falsity of conditionals.

The desired interaction between ambiguity and the conditional will allows us to model the denial of sentences like $A \to A$. If A is an atomic sentence, then the denial of $A \to A$ would be modeled by treating A ambiguously so that the first instance of A in the formula received the truth value True while the second instance received the truth value False. Further, I have shown that the the truth and falsity conditions must be intensional, so we cannot get away with treating A ambiguous at the world where $A \to A$ fails. Ambiguity (ideally) will be used to deny sentences like $Q \to (P \to P)$, which will be treated as $Q \to (P_t \to P_f)$. Ambiguity should also be used, for instance, in inferences with a premise of the form $\neg(P \to P)$ in order to non-vacuously model the premise set. The premise, treated ambiguously, would then be $\neg(P_t \to P_f)$.

In the earlier discussion of the models of $\mathbf{K}_4$, we saw how we can interpret the models in terms of ambiguity. The interpretations for the models of $\mathbf{K}_4$ used the f function to separate true and false instances of type 2 atomic sentence by treating the type 2 atomic sentence as two type 1 atomic sentences. To formalize the interpreted formula $A_t \to A_f$, the truth and falsity conditions for conditionals at non-normal worlds have to non-trivially include the f function and the truth and falsity of the associated type 1 atomic sentences. However, the models for **B** as constructed above fail to have this desired interaction. To see this, consider the following model.

Let M be a model with four worlds. Let us call the worlds w, w_1, w_2, and w_3. Let $N = \{w\}$. Let $f(Q) = \langle \mathbb{P}_k, \mathbb{P}_{-k} \rangle$ and let $f(P) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$. In addition to the members of R required by the normality of world w, let $\langle w_1, w_2, w_3 \rangle \in R$, and let R contain no other ordered 3-tuple. Let $\models_{w_1} \mathbb{P}_k$, and thus $\models_{w_1} Q$. Lastly, let $\models_{w_2} \mathbb{P}_n$ and $\not\models_{w_3} \mathbb{P}_{-n}$. [19]

I will evaluate $Q \to (P \to P)$ at w in order to show that the desired interaction between ambiguity and $(P \to P)$ failing at the non-normal world w_1 is absent. If our desires interaction were to be found, then $Q \to (\mathbb{P}_n \to \mathbb{P}_{-n})$ would fail for the very reason $Q \to (P \to P)$ fails in the American Plan models.

$\models_w Q \to (\mathbb{P}_n \to \mathbb{P}_{-n})$ if and only if for every world w' such that $\models_{w'} Q$, $\models_{w'} (\mathbb{P}_n \to \mathbb{P}_{-n})$. Therefore I show that there is one world, namely w_1 such $\models_{w_1} Q$ and $\not\models_{w_1} (\mathbb{P}_n \to \mathbb{P}_{-n})$. We see that $\models_{w_1} Q$, by our assumptions. By the truth conditions for conditionals $\not\models_{w_1} (\mathbb{P}_n \to \mathbb{P}_{-n})$, because Rw_1, w_2, w_3, $\models_{w_2} \mathbb{P}_n$, and $\models_{w_2} \mathbb{P}_{-n}$.

Now that I have shown why $Q \to (P \to P)$ fails in set of models satisfying my assumptions, I will show that the desired interaction between ambiguity and the truth and falsity of conditionals is not found in these models. Under the assumptions specifying the set of models I am considering, the worlds w_2 and w_3 might not have any sentences being treated ambiguously. That is, $Am(w_2)$ and $Am(w_3)$ may very well be empty sets. In fact, we may further specify that no atomic sentence at any world in the model is to be treated ambiguously. The sentence $Q \to (P \to P)$ still fails, and fails because of the ternary relation, as shown above, and *not* because of ambiguity.

The present model theory does not qualify as an extension of Brown's semantics. While the extensional connectives do formalize an extension of Brown's semantics, the intensional arrow of these models does not. The question remains, however, of whether or not we can construct a semantics for relevant logics in which ambiguity has a suitable

[19] Note that I have no specified a single model, for, for instance, I have not specified whether the truth value of $\mathbb{P}_k$ is the same as the truth value of $\mathbb{P}_{-k}$ at the world w_1. I have, however, specified a number of models which share the same property. That is, $Q \to (P \to P)$ fails at the normal world w.

role. Of the many possible approaches which may be fruitful in this project, the approach I pursue replaces the ternary relation with a binary relation.

Consider the model specified above. If we require that $w_2 = w_3$ in this model, and if we replace Rw_1, w_2, w_3 with the binary Rw_1, w_2, then (for $Q \to (\mathbb{P}_n \to \mathbb{P}_{-n})$ to fail) $P \in Am(w_2)$ and $\models_{w_2} \mathbb{P}_n$. In this model, for the sentence $Q \to (P \to P)$, the desired interaction is found when we further require that $w_2 = w_3$. For this to be useful, we must generalize this new construction to all models. One way to do so would to to require that that ternary relation is replaced a binary relation, and that every ternary relation $Rxyz$ in the previously constructed models is replaced with the binary relation Rxy.[20] We want $A \to A$ to fail at a non-normal world because it is R related to a world where A is ambiguous. The is a truth condition for conditionals which allows us to do this, but only when A is a type 2 atomic sentence. Where A is an atomic sentence, w is a non-normal world, and $f(A) = \langle \mathbb{P}_n, \mathbb{P}_{-n} \rangle$, the following definition formalizes the desired interaction between the conditional and ambiguity;[21]

$$\models_w A \to B \text{ iff for every } w' \text{ such that} Rww' : \text{ if } \models_{w'} \mathbb{P}_n, \text{ then } \models_{w'} \mathbb{P}_{-n}$$

This definition only works when A is a type 2 atomic sentence, for the f function only takes type 2 atomic sentences as arguments.

Nonetheless the definition is able to be extended so that A can be any formula, but at the cost of making even more use of the $*$-worlds. Let A be an atomic sentence of type 2, and let w be a world. If A is ambiguous at w, either $A \wedge \neg A$ is modeled at w or $A \vee \neg A$ is not modeled at w. Consider the world w^*. For instance, when $A \wedge \neg A$ is modeled at w, it is not modeled at w^*. So if $A \to A$ is to fail because A is ambiguous at some world, then a method of determining whether or not A is ambiguous at that world is crucial. It turns out, in fact, that we can determine whether or not A is ambiguous at a world by observing the truths and falsehoods at the $*$-world of the world in question. We may define the truth condition for conditionals as the following;

$$\models_w A \to B \text{ iff for every } w' \text{ such that} Rww' : \text{ if } \models_{w'} A, \text{ then } \models_{w'^*} B$$

This definition ensures that, when $(A \to A)$ fails at non-normal worlds, $(A \to A)$ fails because of ambiguity. More specifically, $(A \to A)$ fails at a world w, when A is

[20] In the model we are generalizing, we required that $w_2 = w_3$. This works for this specific model, but not all models when generalizing. All that we require for the generalization is that the ternary relation is replaced with binary relation.

[21] Though, the logic produced by such a definition is unknown.

ambiguous at a world w' such that Rww'. By treating A as ambiguous, we treat the first instance of A and the second instance of A in $A \to A$ as different sentences. We produce a counter-model, then, when we are able to treat the first instance of A as a true sentence, and the second instance as a false sentence.

3.4.3 B: The extension DW

I have proposed a truth condition for conditionals at non-normal worlds. This truth condition formalizes the desired interaction between ambiguity and the truth values of conditionals at non-normal worlds. While the logic created may be philosophically interesting because of how the logic can be interpreted in terms of ambiguity, another question is whether or not these constructed models are models for the logic **B**. Again, the new truth condition for conditionals at non-normal worlds is as follows:

$$\models_w A \to B \text{ iff for every } w' \text{ such that} Rww' : \text{ if } \models_{w'} A, \text{ then } \models_{w'^*} B$$

It is worth noting parenthetically that the essential use of $*$-worlds in this truth condition ensures that these models cannot model the positive fragments of relevant logics.

Theorem 16. *The models I have constructed with the new truth condition for conditionals at non-normal worlds are sound with respect to the logic* **B**.

Proof. The proof is by cases. I prove that there does not exist a counter-model to each axiom and each rule.

A1 $\models A \to A$ iff for ever w such that $\models_w A$, $\models_w A$. Ever world w is such that if $\models_w A$, then $\models_w A$, as required.

A2 $\models (A \wedge B) \to A$ iff for ever w such that $\models_w (A \wedge B)$, $\models_w A$. Assume for an arbitrary world w' that $\models_{w'} (A \wedge B)$, then $\models_{w'} A$ by the truth conditions, as required.

A3 Trivially similar to A2

A4 $\models ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$ iff for ever w such that $\models_w ((A \to B) \wedge (A \to C))$, $\models_w (A \to (B \wedge C))$.

Assume w' is a normal world. $\models_{w'} ((A \to B) \wedge (A \to C))$ if and only if $\models_{w'} (A \to B)$ and $\models_{w'} (A \to C)$. From the truth conditions, every world w'' at which $\models_{w''} A$, $\models_{w''} B$ and $\models_{w''} C$, and thus $\models_{w''} (B \wedge C)$. By the truth conditions, $\models_{w'} (A \to (B \wedge C))$, as required.

On the other hand, assume w' is a non-normal world. $\models_{w'} ((A \to B) \wedge (A \to C))$ if and only if $\models_{w'} (A \to B)$ and $\models_{w'} (A \to C)$. From the fact that $w'^{**} = w$, given the new truth truth condition, for every world w'' such that $Rw'w''$ if $\models_{w''} A$, then $\models_{w''^*} B$ and $\models_{w''^*} C$ and thus $\models_{w''^*} (B \wedge C)$. By the truth conditions for non-normal worlds, $\models_{w'} (A \to (B \wedge C))$, as required.

A5 $\models A \to A \vee B$ iff for ever w such that $\models_w A$, $\models_w (A \vee B)$. Assume for an arbitrary world w' that $\models_{w'} A$, then by the truth condition for disjunction $\models_{w'} (A \vee B)$, as required.

A6 Trivially similar to A5

A7 Not as trivially similar to A4 as A6 is to A5, but easy to verify there cannot exist any counter-model to A7 for similar reasons.

A8 $\models (A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C))$ iff for ever w such that $\models_w (A \wedge (B \vee C))$, $\models_w ((A \wedge B) \vee (A \wedge C))$. Assume w is an arbitrary world such that $\models_w (A \wedge (B \vee C))$. By the truth conditions, $\models_w A$ and $\models_w (B \vee C)$. As such, $\models_w B$ or $\models_w C$. Assume $\models_w B$. Then $\models_w (A \wedge B)$, and therefore $\models_w ((A \wedge B) \vee (A \wedge C))$, as required. A similar subproof exists for the assumption that $\models_w C$. Therefore it must be the case that $\models_w ((A \wedge B) \vee (A \wedge C))$, as required.

A9 $\models \neg\neg A \to A$ iff for ever w such that $\models_w \neg\neg A$, $\models_w A$. Assume for an arbitrary world at $\models_w \neg\neg A$, then $\not\models_{w^*} \neg A$ by the requirements of $*$-worlds. Again, by the properties of $*$-worlds, $\models_w A$, as required.

R1 $A, A \to B \models B$

Assume for all w that $\models_w A$ and assume for all w that if $\models_w A$, then $\models_w B$. By these assumptions, every world w is such that $\models_w B$, as required.

R2 $A, B, \models A \wedge B$

Assume for all w that $\models_w A$ and $\models_w B$. By the truth conditions, $\models_w A \wedge B$, as required.

R3 $A \to B \models ((C \to A) \to (C \to B))$

Assume for all w that if $\models_w A$, then $\models_w B$. Let w' be an arbitrary world at which $\models_{w'} C \to C$.

If w' is a normal world, then for every w'' such that $\models_{w''} C$, $\models_{w''} A$. Let x be a world such that $\models_x C$. Then, by our assumption, $\models_x A$. By the first assumption, $\models_x B$, as required.

On the other hand, if w' is a non-normal world, then for every w'' such that $Rw'w''$, if $\models_{w''} C$, then $\models_{w''*} A$. Let x be an arbitrary world such that $Rw'x$ and $\models_x C$. By our assumptions, $\models_{x*} A$ and thus $\models_{x*} B$. It must be the case, as required, that $\models_{w'} (C \to B)$, for for every w'' such that $Rw'w''$, if $\models_{w''} C$, then $\models_{w''*} B$.

R4 This proof is trivially similar to the proof of R3

R5 $(A \to \neg B) \models (B \to \neg A)$

Assume for all w that if $\models_w A$, then $\models_w \neg B$. Then assume that $\models_{w'} B$ for an arbitrary world w'. Then, assume that $\not\models_{w'} \neg A$. By the properties or $*$-worlds, $\models_{w'*} A$. Then, by our original assumption, $\models_{w'*} \neg B$. But then, by the properties of $*$-worlds, $\not\models_{w'} B$, which contradictions our original assumption about the world w'. Thus, for any world w', if $\models_{w'} B$, then $\models_{w'} \neg A$, as required.

$\square$

The logic determined by the class of models I have defined is at least as strong as the logic **B**. Moreover, the logic determined by these models is also at least as strong as the logic **DW**, which is **B** plus the axiom D4: $(A \to \neg B) \to (B \to \neg A)$:

D4 $\models (A \to \neg B) \to (B \to \neg A)$ iff for every w such that $\models (A \to \neg B)$, $\models (B \to \neg A)$.

Assume w is an arbitrary non-normal world at which $\models_w (A \to \neg B)$. Suppose (for Reductio) that $\not\models_w (B \to \neg A)$, then there exists world w' such that Rww', $\models_{w'} B$, and $\not\models_{w'*} \neg A$. By the requirements of $*$-worlds, we know that $\models_{w'} A$ and $\not\models_{w'*} \neg B$. By our assumptions, $\models_{w'*} \neg B$, a contradiction. Therefore $\models_w (B \to \neg A)$ from the assumption that $\models_w (A \to \neg B)$, as required.

On the other hand, assume w is an arbitrary normal world at which $\models_w (A \to \neg B)$. Suppose that $\not\models_w (B \to \neg A)$. Then there exists a world w' at which $\models_{w'} B$ and

$\not\models_{w'} \neg A$. Since the models are closed under duality, there exists a the dual world w'^* at which $\models_{w'^*} A$. By our assumptions, $\models_{w'^*} \neg B$. Finally, by duality $\not\models_{w'} B$, a contradiction. Thus $\models_w (B \to \neg A)$ from the assumption that $\models_w (A \to \neg B)$.

These models determine a logic which is at least as strong as **B** and reject both thinning and explosion.[22] By the criteria I adopted in the first chapter, the logic these models exactly model is a relevant logic. Indeed the logic may be the logic **DW**, but I currently lack a completeness proof.

3.4.4 B: Ambiguity and Preservationism

I have constructed a semantics for a relevant logic which is an extension of the ambiguity-measure preservationist semantics developed by Brown. That is, the models do the work of Brown's ambiguity-measure preserving approach, and the models extend the approach to the conditional.

To see how the models may be interpreted in terms of ambiguity-measure preservation, I show how to interpret the models and the work they do. Suppose that Q and P are type 2 atomic sentences. (The result generalizes to all sentences build from type 2 atomic sentences.) In the models constructed, $Q \to (P \to P)$ is not a validity because we can create a model in which there is a world w such that $\models_w Q$ and $\not\models_w (P \to P)$. We are able to to do so because the world w is R-related to a world where P is ambiguous. By treating the first instance of P as the true type 1 atomic sentence, and the second instance as the false type 1 atomic sentence, we get $Q \to (P_t \to P_f)$.

Taking these lessons outside of the model theoretic realm, we see that $Q \to (P \to P)$ is not a validity because we can consistently deny the consequent without being forced to deny the antecedent. We can consistently deny the consequent by treating P as ambiguous. Furthermore, being committed to denying $P \to P$ does not mean being rationally required by logic to deny Q.

Non-normal worlds are afforded a somewhat novel interpretation in terms of ambiguity-measure preservation. The denial of $P \to P$ in these new models is just like the denial of $P \vee \neg P$ in Brown. The difference is that the sentence $P \to P$ is intensional, and its denial its not achieved by merely changing the truths and falsehoods, but by also changing accessibility relations. The worlds which consistently deny $P \vee \neg P$ can do so simple by treating P as ambiguous at that world. However, consistently denying $P \to P$

[22]This fact is trivial to check.

by treating it as $P_t \to P_f$ is not merely achieved by treating P ambiguously at the world where $P \to P$ is being evaluated. Consistently denying $P \to P$ is achieved by treating the sentence as $P_t \to P_f$ at a world which is R-related to another world where P_t is true and P_f is not.

The arrow encodes semantic entailment by showing that every world R-related is such that, if the antecedent is modeled, then the consequent must be modeled as well. So, the first property a non-normal world must have is a restricted R relation. That is, a non-normal world cannot have access to every world. Second, by treating $P \to P$ as $P_t \to P_f$, we can have non-normal worlds have access to worlds at which P_t and not P_f. These non-normal worlds represent commitments to denying sentences such as $P \to P$. That is, they represent places from which not every world which models P_t also models P_f.

In Brown's preservationism, as explained in the previous chapter, $P \models P$ is the case (for paraconsistent and paracomplete logics) because every way to model P, also models P. In this case, we do not need ambiguity. $P \to P$ encodes something similar. For a sentence like $Q \to (P \to P)$, denying the consequent requires the use of ambiguity. So a non-normal world is not that different from a paraconsistent world or a paracomplete world. First, paracomplete, paraconsistent, and non-normal worlds can be given the ambiguity-measure preservation account, which is psuedo-classical in its commitments. Second, they represent commitments to denying or accepting sentences which cannot be denied or accepted classically. The difference, I reiterate, is that what you are denying at a non-normal world necessarily changes not only the truths and falsehoods at the non-normal world, but also the relations the non-normal worlds has with other worlds and the truths and falsehoods at these related worlds. Note that this interpretation can even be given to the non-normal worlds of the American plan semantics.

Brown suggests another interpretation of his ambiguity-measure preservationism which claims that he is preserving the classical consequence relation [7, p. 188].

> $\Gamma \vdash_{FDE} \Delta$ iff every image of the premise and conclusion sets, $I(\Gamma)$, $I^*(\Delta)$ obtained by treating disjoint sets of sentence letters as ambiguous is such that $I(\Gamma) \vdash I^*(\Delta)$.
>
> This suggests a new preservationist strategy for producing new consequence relations from old. We can say that the new consequence relation holds when and only when the old relation holds in all of a range of cases *anchored to* the original premise and conclusion sets. This strategy eliminates

> or reduces trivialization by ensuring that the *range* of cases considered includes some non-trivial ones, even when the instance forming our 'anchor' is trivial. [7, p. 188]

I claim to have produced a relevant consequence relation which preserves the classical consequence relation when treating sentences as ambiguous. The models constructed using the new truth condition model a relevant logic, but do so by treating some atomic sentences as ambiguous. In the semantics for classical logic, we say $A \models B$ if and only if every model of A is also a model of B. The arrow encodes similar information. The models of classical logic have become worlds in the new semantics, with the exception of the arrow. The truth and falsity conditions for the conditional, once dependent on possible models, are now dependent on the possible (and impossible) worlds. Consider again the sentence $Q \to (P \to P)$. It is impossible to classically deny $P \to P$, but we do not want $Q \to (P \to P)$ to be a validity trivially. Thus, we use $P \to P$ as an 'anchor', so to speak, and we let P or a subset of the atomic sentences in P be ambiguous in such a way as to *classically deny* $P \to P$. The sentence then becomes $Q \to (P_t \to P_f)$, where $P_t \neq P_f$. This new sentence is not a theorem of classical logic. We can then see that the classical consequence relation is preserved in this treatment of relevant logic.

Future Projects

Because of the essential use of $*$ worlds in the new truth condition for conditionals at non-normal worlds, there are difficulties to overcome in order to modify the models I have constructed to model some extensions of **B**. The list which enumerates the axioms to be added to **B** with the corresponding modifications to the models (content constraints and relational constraints) must be modified. Some of these correspondences will remain unaffected. For example, the axiom $A \to (B \to B)$ corresponds to the content inclusion constraints $Rxyz \Rightarrow y \sqsubseteq z$ [23, p. 150]. It is easy to check that this correspondence remains unaffected.

However, the relation constraint which corresponds to the axiom $A \to ((A \to B) \to B)$ must be modified, if it is possible to do so. Note that this axiom is required to model the logic **R**. Under the American Plan semantics, the relation constraint was $Rxyz \Rightarrow Ryxz$ [23, p. 143]. This restraint becomes too strong in the new models. Because the new truth condition refers to three worlds in the exact same way, we might (instrumentally) think of the new relation as $Rww'w'^*$. If the condition

$Rxyz \Rightarrow Ryxz$ were to hold, then $w = w'$, which is a lot stronger than required. Perhaps, then, there is no corresponding relational or content constraint(s) for the axiom $A \to ((A \to B) \to B)$ in the new models. Whether or not there is a corresponding constraint which may be added is left as a future project. The construction of the list which enumerates the axioms used in common extensions of **B** with their corresponding model theoretic constraints is also left as a future project.

In addition to this project, there should be other formal semantics for relevant logic which may also be interpreted as an extension of Brown's ambiguity-measure preservation approach. One promising semantics is explicated briefly in [3] and section 10.7 in [19]. This approach used the ceteris peribus enthymemes of conditional logic as explicated in [19], but uses the two-valued $*$-worlds of the Australian Plan. Again, we will also see that non-normal worlds require restrictions onto what worlds are accessible from them.

We introduce binary relations that are indexed to formula. We write this as $R_{|A|}xy$, where x, y are worlds and A is a formula. $R_{|A|}xy$ shall be interpreted as x is A-related to y. For normal worlds, intuitively, $R_{|A|}xy$ means that A is true at y. However, for non-normal worlds we do not require that A is true at y when $R_{|A|}xy$; this is what lets sentences of the form $P \to P$ fail at non-normal worlds. We note that "$|A|_M$ is the set of points in the model M at which A holds" [3, p. 606]. One possible way to develop a connection between the truth condition for conditionals at non-normal worlds and ambiguity would be to state that, from a non-normal world, it is ambiguous where the sentence A is true. For instance, when $P \to P$ becomes $P_t \to P_f$, worlds in $|P_t|$ need not all make P_f true as well.

By holding our mouths just right and adding the right restrictions on these models,[23] these models should model relevant logics. Treating $P \to P$ as $P_t \to P_f$ lets us extend Brown's ambiguity-measure preservation approach to relevant logic in such a way that we may treat conditionals in the same way we treat conjunctions and disjunctions, at least in terms of ambiguity. These models may be promising in terms of ambiguity because if we interpret R_Axy as saying that A is true at y, then when we treat A as ambiguous, it then ambiguous where A holds. Or, at least, treating A as ambiguous allows us to separate A into two sentences. Thus there will be worlds where A does not hold, but which are R_A related. This approach appears to be promising in terms of ambiguity-measure preservation, but developing this into a complete semantics is beyond the scope of this thesis and well be left as a future project.

[23] For a detailed account of what restrictions must be places, we [3, p. 605–8] and [19, p. 209–11].

3.4.5 Conclusion

I have achieved a few things in the these 3 chapters. The first is that I have given another motivation for relevant logic. As explicated above, we would like a logic which does not trivialize when $P \to P$ is in the consequent of a conditional. We want to be able to express what one must be committed to by denying sentences like $P \to P$. This motivation is a consistent extension of Brown's preservationist approach.

The second is that I have extended Brown's preservationist approach to at least one relevant logic. That is, there exist relevant logics which may be given an ambiguity-measure preservation semantics. If the desired interaction between antecedents and consequents is not essential, then I have created models which extend Brown's approach to every relevant logic which is both an extension of **B** and capable of being modeled by the American plan semantics. The desired interaction appears essential to providing a genuine semantics and avoiding many of the concerns described in Chapter 1. The models I have constructed with the desired interaction are models for at least one relevant logic. This means that there is at least one relevant logic which is capable of being interpreted in terms of ambiguity-measure preservation. Nonetheless, without the desired interaction, the extension of Brown's approach still provides novel interpretations for the extensional connectives. For example, consider non-normal worlds. As seen above, by extending Brown's approach I have aided in the justification of relevant logic. By extending Brown's approach, non-normal worlds and the properties they have are justified by what it means to be committed to the denial of sentences like $P \to P$.

There are a number of benefits gained by achieving the above goals. The first is that relevant logic has more motivating material. This is good for relevant logicians. The more motivating material for relevant logic, the more relevant logic seems interesting, and, hopefully, the more inclined others are to use it. Anderson and Belnap state that;

> It seems to be generally conceded that formal systems are natural and substantial if they can be looked at from several points of view. We tend to think of systems as artificial or *ad hoc* if most of their formal properties arise from some *one* notational system in terms of which they are described. [1, p. 50]

Starting with the ambiguity-measure preserving approach for **FDE**, I have extended this *point of view* to relevant logic. (And I have done two in two separate ways.) The formal properties of the relevant logics and their semantics are motivated by the ambiguity-measure preserving approach. I will explain below that this motivation for relevant logic allows one to hold the metaphysical commitments of classical logic firm as one ventures into relevant logic. Even the classical logician, then, I claim, should start seeing relevant logic as more 'natural and substantial', and even more so from the logics with semantics that have the desired interaction between the conditional and ambiguity.

Another benefit is that one is now able to accept at least one relevant logic while keeping classical metaphysical commitments,[24] if one is so inclined. There are three ways someone with the metaphysical commitments of classical logic might object to relevant logic. The first two are the paracompleteness and paraconsistency of relevant logic. Brown's ambiguity-measure preserving account of **FDE** presented in Chapter 2 should convince one that the metaphysical commitments of classical logicians can be preserved, even in the logic **FDE**. One may use the logic **FDE**, but still accept that a sentences like $P \vee \neg P$ or an equivalent is necessary and true. One can accept that any old sentence Q does not and should not imply $P \vee \neg P$, while still accepting $P \vee \neg P$. The case is similar for contradiction. One can accept that $P \wedge \neg P$ is false and necessarily so, but still accept that a paraconsistent logic should be used. Brown's approach affords us a response to Slater's objection which is unavailable to previous semantic accounts. Furthermore, when this approach is extended to relevant logics, we are afforded a similar response to an account of Slater's objection which has been extended to the negation of relevant logics. This is the case whether we use the new truth condition for conditionals at non-normal worlds or not. With the old truth condition for conditionals at non-normal worlds, Brown's approach may be used for the extensional connectives to show that a contradiction is never modeled (i.e. true). Instead a new sentence *anchored to the contradiction* is modeled. This new sentence is related to the contradiction, for it is the result of treating certain atomic sentences within the contradiction as ambiguous.

The third objection to relevant logic is the seemingly odd behavior of the arrow. However, the metaphysical commitments of classical logicians can be preserved while accepting the conditional of relevant logic. The key to this, again, is ambiguity. Modeling the denial of necessary necessitives such as $P \to P$ is *only* done by treating certain atomic formula within P as ambiguous. That is, however, at least for the models with the desired interaction between ambiguity and the conditionals. In these models the arrow connective behaves 'classically'.[25] The arrow connective may be interpreted as it is

[24]The metaphysical commitments of classical logic being that the world is consistent and complete.
[25]Classical in the sense of a modal interpretation, where the models of classical logic are worlds.

in the semantics of classical logic. That is, $A \rightarrow B$ if and only if every way of modeling A models B. This is, perhaps, the most important benefit of the models I have constructed; even the (seemingly odd) behavior of the arrow connective in relevant logic can be interpreted classically, given the formal treatment of ambiguity. Thus the negation, conjunction, disjunction, and arrow connectives of relevant logic can be given a fully classical interpretation, provided we interpret them in terms of ambiguity.

That being said, one need not accept the metaphysical commitments of classical logic in order to accept or use relevant logic. I have not shown that any specific metaphysical commitments are required to accept relevant logic. All I have shown here is that the metaphysical commitments of classical logic are compatible with relevant logic under an ambiguity-measure preservation interpretation. Moreover, I have shown that all of the connectives[26] can be given a classical interpretation when we include a formal notion of ambiguity in our models. So not only have we gained a novel and significant interpretation of relevant logic, but we have also gained additional motivation to reason using relevant logic in areas which are reasonably thought to be both complete and consistent.

[26]That is, all of the connectives of relevant logic which are also connectives in the usual interpretation of classical logic.

APPENDICES

Appendix A

Proofs of (some) Theorems

A.1 K4

Proof for theorem 3: For a reminded, the theorem states that for any x in M', the corresponding world w in $g(M')$ is such that $(A)\rho_x 1$ if and only if $\models_w A$.

Proof. The proof is a straightforward induction on the construction of A. The base case is the truth value assignment to the literals. The base case is ensured by the first stipulation in Theorem 1. The induction hypothesis is that for every x' in M', the corresponding world w' in $g(M')$ is such that $(B)\rho_{x'}1$ if and only if $\models_{w'} B$ and $(C)\rho_{x'}1$ if and only if $\models_{w'} C$.

I will show one extensional case and one intensional case. The first case is the extensional case. Let A be of the form $(B \wedge C)$. Assume that $(A)\rho_x 1$. Then $(B)\rho_x 1$ and $(C)\rho_x 1$. By the induction hypothesis, $\models_w B$ and $\models_w C$. Therefore $\models_w A$, as required. On the other hand assume that it is not the case that $(A)\rho_x 1$. Then it is also not the case that both $(B)\rho_x 1$ and $(C)\rho_x 1$. By the induction hypothesis, not both $\models_w B$ and $\models_w C$. Therefore $\not\models_w A$, as required.

Let A be of the form $B \rightarrow C$. Suppose that $(A)\rho_x 1$. Then for every $x' \in W$ such that $(B)\rho_{x'}1$, $(C)\rho_{x'}1$. By the induction hypothesis, and the fact that the number of worlds does not change with the application of g, every world $w' \in W$ is such that if $\models_{w'} B$, then $\models_{w'} C$. Therefore $\models_w A$, as required. On the other hand suppose it is not the case that $(A)\rho_x 1$. Then there is a world x' such that $(B)\rho_{x'}1$ and not $(C)\rho_{x'}1$. By the induction hypothesis, there exists a world w' in $g(M')$ such that $\models_{w'} B$ and $\not\models_{w'} C$. Therefore

$\not\models_w A$, as required. □

Proof for theorem 4: For a reminded, the theorem states that for any w in M, the corresponding world x in $h(M)$ is such that $\models_w A$ if and only if $(A)\rho_x 1$.

Proof. The proof is by induction on the construction of A. The base case is the truth value assignment to the literals. The base case is ensured by the first stipulation in Theorem 2. The induction hypothesis is that for every world w' in M, the corresponding world x' in $h(M)$ is such that $\models_{w'} B$ if and only if $(B)\rho_{x'}1$ and $\models_{w'} C$ if and only if $(C)\rho_{x'}1$.

I will again show one extensional case and one intensional case. The first case is the extensional case. Let A be of the world $(B \wedge C)$. Suppose that $\models_w A$. Then $\models_w B$ and $\models_w C$. By the induction hypothesis, $(B)\rho_x 1$ and $(C)\rho_x 1$. Therefore $(A)\rho_x 1$, as required. On the other hand, suppose that $\not\models_w A$. Then either $\models_w B$ or $\models_w C$. By the induction hypothesis, either not $(B)\rho_x 1$ or not $(C)\rho_x 1$. Therefore it is not the case that $(A)\rho_x 1$, as required.

Let A be of the form $B \rightarrow C$. Suppose that $\models_w A$. Then for every world w' in M, if $\models_{w'} B$, then $\models_{w'} C$By the induction hypothesis, and the fact that the number of worlds does not change with the application of h, every world x' in $h(M)$ is such that if $(B)\rho_{x'}1$, then $(C)\rho_{x'}1$. Therefore $(A)\rho_x 1$, as required. On the other hand suppose that $\not\models_w A$. Then there exists a world w' in M such that $\models_{w'} B$ and $\not\models_{w'} C$. By the induction hypothesis, there exists a world x' in $h(M)$ such that $(B)\rho_{x'}1$ and not $(C)\rho_{x'}1$. Therefore it is not the case that $(A)\rho_x 1$, as required. □

A.2 B

The notational conventions of this section are as they were in section 3.4.1, where the theorems being proved were originally stated.

Proof for theorem 11: For a reminder, the theorem states that for any x in M', the corresponding world w in $g(M')$ is such that $\models_x A$ if and only if $\models_w A$.

Proof. The proof is by induction on the construction of A. The base case is the truth value assignment to the literals. The base case is ensured by Theorem 9. The induction hypothesis is that for every x' in M', the corresponding world w' in $g(M')$ is such that $\models_{x'} B$ if and only if $\models_{w'} B$ and $\models_{x'} C$ if and only if $\models_{w'} C$.

The extensional cases are the same as they were for the proof of Theorem 3. The intensional cases are also straightforward. I will show one intensional case as an example. Let A be of the form $B \to C$. Suppose that $\models_x A$. Then, for every y, z such that $Rxyz$, if $\models_y B$, then $\models_z C$. It follows from Theorem 9, that $Rww'w''$ iff w' and w'' correspond to y and z respectively and $Rxyz$. By the induction hypothesis, for every w', w'' such that $Rww'w''$, if $\models_{w'} B$, then $\models_{w''} C$. Therefore $\models_w A$, as required.

On the other hand, suppose that $\not\models_x A$. Then there exists worlds y, z such that $Rxyz$, $\models_y B$, and $\not\models_z C$. It follows from Theorem 9 that $Rww'w''$ iff w' and w'' correspond to y and z respectively and $Rxyz$. By the induction hypothesis, there exist worlds w', w'' such that $Rww'w''$, $\models_{w'} B$, and $\not\models_{w''} C$. Therefore $\not\models_w A$, as required.

□

Proof for theorem 12: For a reminder, the theorem states that for any w in M, the corresponding world x in $h(M)$ is such that $\models_w A$ if and only if $\models_x A$.

Proof. The proof is by induction on the construction of A. The base case is the truth value assignment to the literals. The base case is ensured by Theorem 10.The induction hypothesis is that for every world w' in M, the corresponding world x' in $h(M)$ is such that $\models_{w'} B$ if and only if $\models_{x'} B$ and $\models_{w'} C$ if and only if $\models_{x'} C$.

The extensional cases are the same as they were for the proof of Theorem 4. The intensional cases are also straightforward. I will show one intensional case as an example. Let A be of the form $B \to C$. Suppose that $\models_w A$. Then, for every w', w'' such that $Rww'w''$, if $\models_{w'} B$, then $\models_{w''} C$. It follows from Theorem 10 that $Rxyz$ iff y and z correspond to w' and w'' respectively and $Rww'w''$. By the induction hypothesis, for every y, z such that $Rxyz$, if $\models_y B$, then $\models_z C$. Therefore $\models_x A$, as required.

On the other hand, suppose that $\not\models_w A$. Then there exists worlds w', w'' such that $Rww'w''$, $\models_{w'} B$, and $\not\models_{w''} C$. It follows from Theorem 10 that $Rxyz$ iff y and z correspond to w' and w'' respectively and $Rww'w''$. By the induction hypothesis, there exist worlds y, z such that $Rxyz$, $\models_y B$, and $\not\models_z C$. Therefore $\not\models_x A$, as required.

□

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