A Comprehensive Analysis of Lift-and-Project Methods for Combinatorial Optimization

by

Yu Hin Au

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In both mathematical research and real-life, we often encounter problems that can be framed as finding the best solution among a collection of discrete choices. Many of these problems, on which an exhaustive search in the solution space is impractical or even infeasible, belong to the area of combinatorial optimization, a lively branch of discrete mathematics that has seen tremendous development over the last half century. It uses tools in areas such as combinatorics, mathematical modelling and graph theory to tackle these problems, and has deep connections with related subjects such as theoretical computer science, operations research, and industrial engineering.

While elegant and efficient algorithms have been found for many problems in combinatorial optimization, the area is also filled with difficult problems that are unlikely to be solvable in polynomial time (assuming the widely believed conjecture $\mathcal{P} \neq \mathcal{NP}$). A common approach of tackling these hard problems is to formulate them as integer programs (which themselves are hard to solve), and then approximate their feasible regions using sets that are easier to describe and optimize over.

Two of the most prominent mathematical models that are used to obtain these approximations are linear programs (LPs) and semidefinite programs (SDPs). The study of these relaxations started to gain popularity during the 1960's for LPs and mid-1990's for SDPs, and in many cases have led to the invention of strong approximation algorithms for the underlying hard problems. On the other hand, sometimes the analysis of these relaxations can lead to the conclusion that a certain problem cannot be well approximated by a wide class of LPs or SDPs. These negative results can also be valuable, as they might provide insights into what makes the problem difficult, which can guide our future attempts of attacking the problem.

One mathematical framework that generates strong LP and SDP relaxations for integer programs is lift-and-project methods. Among many attractive features, an important advantage of this approach is that tighter relaxations can often be obtained without sacrificing polynomial-time solvability. Also, these procedures are able to generate relaxations systematically, without relying on problem-specific observations. Thus, they can be applied to improve any given relaxation.

In the past two decades, lift-and-project methods have garnered a lot of research attention. Many operators under this approach have been proposed, most notably those by Sherali and Adams; Lovász and Schrijver; Balas, Ceria and Cornuéjols; Lasserre; and Bienstock and Zuckerberg. These operators vary greatly both in strength and complexity, and their performances and limitations on many optimization problems have been extensively studied, with the exception of the Bienstock–Zuckerberg operator (and to a lesser degree, the Lasserre operator) in terms of limitations.

In this thesis, we aim to provide a comprehensive analysis of the existing lift and project operators, as well as many new variants of these operators that we propose in our work. Our new operators fill the spectrum of lift-and-project operators in a way which

makes all of them more transparent, easier to relate to each other, and easier to analyze. We provide new techniques to analyze the worst-case performances as well as relative strengths of these operators in a unified way. In particular, using the new techniques and a recent result of Mathieu and Sinclair, we prove that the polyhedral Bienstock–Zuckerberg operator requires at least $\sqrt{2n} - \frac{3}{2}$ iterations to compute the matching polytope of the $(2n+1)$-clique. We further prove that the operator requires approximately $\frac{n}{2}$ iterations to reach the stable set polytope of the n-clique, if we start with the fractional stable set polytope. Moreover, we obtained an example in which the Bienstock–Zuckerberg operator with positive semidefiniteness requires $\Omega(n^{1/4})$ iterations to compute the integer hull of a set contained in $[0,1]^n$. These examples provide the first known instances where the Bienstock–Zuckerberg operators require more than a constant number of iterations to return the integer hull of a given relaxation.

In addition to relating the performances of various lift-and-project methods and providing results for specific operators and problems, we provide some general techniques that can be useful in producing and verifying certificates for lift-and-project relaxations. These tools can significantly simply the task of obtaining hardness results for relaxations that have certain desirable properties.

Finally, we characterize some sets on which one of the strongest variants of the Sherali–Adams operator with positive semidefinite strengthenings does not perform better than Lovász and Schrijver's weakest polyhedral operator, providing examples where even imposing a very strong positive semidefiniteness constraint does not generate any additional cuts. We then prove that some of the worst-case instances for many known lift-and-project operators are also bad instances for this significantly strengthened version of the Sherali–Adams operator, as well as the Lasserre operator. We also discuss how the techniques we presented in our analysis can be applied to obtain the integrality gaps of convex relaxations.

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Chapter 1

Introduction

Optimization problems are abundant in everyday life, and we are often in situations where we are trying to make the best decision among a given set of feasible choices (e.g. finding the best route to travel from one city to another, given the schedule and prices of available flights; deciding on a location to build a certain facility to maximize its benefits). Thus, given a decision problem, it is natural to ask if it can be modelled as a mathematical problem, and whether this mathematical problem can be solved easily.

1.1 Linear programming

One of the most well-studied ways to model decision problems mathematically is via linear programming. A *linear program* (LP) is a problem of optimizing (i.e., either maximizing or minimizing) a linear function subject to linear equalities and/or inequalities. Thus, an LP can take the form:

$$\begin{array}{rrcl} \max & c^\top x & & \\ \text{subject to} & Ax & \leq & b, \end{array} \tag{1.1}$$

where A is an $m \times n$ real matrix, $b \in \mathbb{R}^m$, and $c \in \mathbb{R}^n$. Note that each LP has many different formulations that are equivalent to each other. For instance, the condition $Ax \leq b$ holds if and only if there exists a nonnegative vector $s \in \mathbb{R}^m$ such that $b = Ax + s$. Thus, the LP

$$\begin{array}{rrcl} \max & c^\top x & & \\ \text{subject to} & Ax + s & = & b \\ & s & \geq & 0, \end{array} \tag{1.2}$$

is equivalent to (1.1). In fact, it is not hard to show that *every* LP can be described in the form (1.1) for a suitable choice of the data A, b and c.

One of the most fascinating aspects of linear programming is its rich and robust duality theory. Given an LP in the form of (1.1), its *dual problem* is defined to be:

$$\begin{array}{rrcl} \min & b^\top y & & \\ \text{subject to} & A^\top y & = & c, \\ & y & \geq & 0. \end{array} \tag{1.3}$$

Note that the dual problem is also an LP. A LP is very closely related with its dual. For example, notice that, if x is feasible in (1.1) and y is feasible in (1.3), then

$$b^\top y \geq (Ax)^\top y = x^\top (Ay) = x^\top c = c^\top x,$$

and so the objective value of y in (1.3) is no less than that of x in (1.1). This is the *weak duality* relation between a primal-dual pair of LPs, which has many implications. First, if (P) is a maximization problem and (D) is its dual, then the objective value of any feasible solution in (D) is an upper bound of the optimal value of (P) (if it exists). On the other hand, if there are feasible solutions in (P) with arbitrarily high objective values, then the weak duality relation implies that (D) does not have any feasible solutions.

Furthermore, the following holds for every primal-dual pair of LPs, and is known as the *strong duality* relation.

Theorem 1. *If either an LP or its dual problem has an optimal solution, then both problems have optimal solutions. Moreover, the optimal values of the two problems must coincide.*

Another attractive property of LPs is that they can be solved efficiently, both theoretically and in practice. The first polynomial-time algorithm for solving LPs is the ellipsoid method, due to Khachiyan [Kha80]. While it has a significantly better worst-case run-time than previously known algorithms for LPs, the ellipsoid method does not work very well in practice. A few years later, Karmarkar [Kar84] invented an interior-point method, which can solve many large-scale LPs efficiently, and has been generalized to apply to other problems in convex optimization. The simplex algorithm due to Dantzig in 1947, is also proficient in solving LPs, as well as pure and mixed integer programs. Even though many variants of the simplex algorithm have been shown to have exponential run-time in the worst case (see, for instance, [KM72], for a well-known example), it is widely used and works remarkably well in practice.

Thus, while being able to model a wide variety of problems on their own, the simplicity and solvability of LPs also make them good candidates for providing "approximate models" to problems that are harder to solve directly, such as integer programs.

1.2 Integer programming

An *integer program* (IP) is an LP with the additional requirement that some or all of its variables can only take on integer values. Integer programming is a very powerful mathematical model both in theory and in practice, as it captures the discreteness that arises in many decision making processes, as well as in the study of mathematical objects such as sets and graphs. In particular, many problems that have a binary, yes/no nature in their decision variables can be naturally modelled by IPs in which every variable has to take on value 0 or 1. Thus, 0-1 integer programs make up an important subclass of IPs.

For example, consider the *stable set problem* of graphs. Given a graph $G = (V, E)$, we say that $S \subseteq V$ is a *stable set* if, for every edge $\{u, v\} \in E, |\{u, v\} \cap S| \leq 1$. That is, no two vertices in S are joined by an edge. Then the stable set problem is the problem of finding a stable set of maximum size in a given graph G. There are many real-life interpretations of this problem — for instance, we can take V to be a set of codewords we can transmit through a noisy channel, and we join two codewords by an edge if they could become indistinguishable after transmission. Then a solution to the stable set problem on this graph corresponds to a largest possible set of codewords in which no two will be mixed up after passing through the channel. Alternatively, one can take V as the set of possible locations in a city where a chain coffee shop can open up a new store, with two locations joined by an edge if they are deemed too close to each other, and that setting up shops at both locations can cause diminishing returns. Then the size of a largest stable set in this graph would be the maximum number of locations the chain can open without having the stores competing with each other.

We can easily formulate the stable set problem as an IP as follows. Define a 0-1 variable x_i for every vertex $i \in V$, and set $x_i = 1$ if and only if vertex i is included in our stable set, the solution to the stable set problem can be obtained by solving the IP:

$$\begin{array}{rrcll} \max & \sum_{i \in V} x_i & & & \\ \text{subject to} & x_i + x_j & \leq & 1 & \forall \{i, j\} \in E \\ & x_i & \in & \{0, 1\} & \forall i \in V. \end{array} \tag{1.4}$$

Despite its simple statement, the stable set problem is $\mathcal{NP}$-hard [Kar72]. In fact, it has been proven that the optimal value of the stable set problem on a general n-vertex graph cannot even be approximated to within a factor of $n^{1-\epsilon}$ for any $\epsilon > 0$ in polynomial time, assuming the widely believed conjecture $\mathcal{P} \neq \mathcal{NP}$ [Hås96]. Since any instance of the stable set problem can bereduced to an IP, we see that IPs in general are at least as hard to solve.

However, if we relax the integrality constraint of an IP, we obtain an LP, which can be solved efficiently. For some problems (such as bipartite matching and maximum flow), it turns out that the feasible regions of their LP relaxations always have integral extreme points, which can lead to polynomial-time algorithms for the underlying optimization problems. However, for hard problems such as the stable set problem, it usually does not work out that nicely. For example, if we relax the integrality constraint in (1.4), we obtain the LP:

$$\begin{array}{rrcll} \max & \sum_{i \in V} x_i & & & \\ \text{subject to} & x_i + x_j & \leq & 1 & \forall \{i, j\} \in E \\ & 0 \leq x_i & \leq & 1 & \forall i \in V. \end{array} \tag{1.5}$$

While the optimal value of (1.5) does provide an upper bound on the optimal value of (1.4), this bound can be extremely weak. For example, note that the solution obtained by setting $x_i = \frac{1}{2}$ for all $i \in V$ is feasible in (1.5), and thus the optimal value of (1.5) is at least $\frac{n}{2}$ for every graph G. However, the true optimal value can be much smaller — in the extreme case when $G = K_n$ (the complete graph on n vertices), the largest stable set in G only has size 1.

We can try to improve our LP relaxation by making observations about our problem at hand. For example, it is not hard to see that if $C \subseteq V$ is the vertex set of an odd cycle in G, then the inequality

$$\sum_{i \in C} x_i \leq \frac{|C| - 1}{2}$$

holds for all feasible solutions in (1.4), but is violated by the vector of all-halves. Thus, the following LP should be a tighter relaxation of (1.4) than (1.5):

$$\begin{array}{rrcll} \max & \sum_{i \in V} x_i & & & \\ \text{subject to} & x_i + x_j & \leq & 1 & \forall \{i,j\} \in E \\ & \sum_{i \in C} x_i & \leq & \frac{|C|-1}{2} & \forall \text{ odd cycles } C \\ & 0 \ \leq \ x_i & \leq & 1 & \forall i \in V. \end{array} \tag{1.6}$$

However, a couple issues arise: First, it is not clear that we could enumerate all odd cycles in a general graph G efficiently, and thus the "improvement" we obtain might be rendered meaningless if we have to sacrifice the polynomial-time solvability of the relaxation to get it. Secondly, this observation only applies to the stable set problem, and may not lend a clue to improving a relaxation of another optimization problem.

Therefore, it would be nice if, given an LP relaxation of an optimization problem, we can find means to "automatically" improve the relaxation without relying on specific observations on the problem. Moreover, we would hope that the improvement comes with only a minimal trade-off in the solvability of the relaxation. As we shall see, these goals (and more) can be achieved by lift-and-project methods.

1.3 Lift-and-project methods

Note that the feasible region of an LP relaxation of a 0-1 integer program is a polytope P contained in $[0, 1]^n$, the unit hypercube. Then we are interested in

$$P_I := \text{conv}(P \cap \{0, 1\}^n),$$

the *integer hull* of P. Then the desired optimal integer solutions of the underlying integer program must be one of the extreme points of P_I. While the set P_I itself is usually hard to describe, we want to strive for "tight" relaxations. For instance, if we can describe a set P' where $P_I \subseteq P' \subset P$, then optimizing over P' would yield a better approximation of our integer program than optimizing over P.

One way to obtain tight relaxations for 0-1 integer programs is via *lift-and-project* methods. Given a polytope $P \subseteq [0,1]^n$, a *lift-and-project operator* systematically generates a sequence of polyhedral or nonpolyhedral convex sets that converge to P_I. These operators all take the following approach: They lift P to a set in higher dimensions (usually a set of matrices), and use properties that we know are satisfied by points in P_I to derive inequalities that are valid for P_I but not P. Then the projection of this higher dimensional set to $\mathbb{R}^n$ provides a relaxation that is potentially tighter than P.

Lift-and-project methods have a special place in optimization as they lie at the intersection of combinatorial optimization and convex analysis (this goes back to work by Balas and others in the late 1960s and the early 1970s — see, for instance, [Bal98] and the references therein). Some of the most attractive features of these methods are:

- Convex relaxations of P_I obtained after $O(1)$ iterations of the procedure are *tractable* provided P is tractable. Here, tractable may mean either that the underlying linear optimization problem is polynomial-time solvable, say due to the existence of a polynomial-time weak separation oracle for P (the reader may refer to [MGS88] for a background on seperability and optimization); or, more strongly, that P has an explicitly given, polynomial size representation by linear inequalities (we will distinguish between these two versions of tractability, starting with the strength chart given in Figure 1.1). Moreover, all known lift-and-project operators converge to P_I in at most n iterations, for every input relaxation $P \subseteq [0,1]^n$.

- Notice that a projection of a set may have more facets than itself. Thus, the lifted (higher dimensional) representations for the relaxations used by these operators sometimes allow compact (polynomial size in the input) convex representations of exponentially many facets. We will see such an example in Section 2.1.

- Most of these methods allow easy addition of positive semidefiniteness constraints in the lifted space. This feature can make the relaxations much stronger in some cases, without sacrificing polynomial-time solvability (perhaps only approximately). Moreover, these semidefiniteness constraints can represent an uncountable family of defining linear inequalities, such as those of the theta body of a graph [Lov79]. We will formally introduce semidefinite programs in Chapter 2.

- Systematic generation of tighter and tighter relaxations converging to P_I makes the strongest of these methods good candidates for utilization in generating polynomial-time approximation algorithms for hard problems, or for proving large integrality gaps (hence providing a negative result about approximability in the underlying hierarchy of relaxations). Moreover, since these lift-and-project operators generate relaxations automatically and do not rely on specific observations on the underlying optimization problem, they can be applied to improve any relaxation.

In the last two decades, many lift-and-project operators have been proposed (see, for example, most notably those by Sherali and Adams [SA90]; Lovász and Schrijver [LS91];

Balas, Ceria and Cornuéjols [BCC93]; Lasserre [Las01]; and Bienstock and Zuckerberg [BZ04]. These operators vary greatly both in strength and complexity, and have been applied to various discrete optimization problems (see, for example, [SL96], [dKP02], [PVZ07] and [GL07]). Many families of facets of the stable set polytope of graphs are shown to be easily generated by these procedures [LS91, LT03]. Also studied are their performances on max-cut [Lau02], set covering [BZ04], k-constraint satisfiability problems [Sch08], knapsack [KMN11], sparsest cut [GTW13], directed Steiner tree [FKKK$^+$14], set partitioning [SL96], TSP relaxations [CD01, Che05, CGGS13], and matching [ST99, ABN04, MS09]. For general properties of these operators and some comparisons among them, see [GT01], [Lau03a] and [HT08].

Figure 1.1 provides a glimpse of the spectrum of these lift-and-project operators, as well as their strengths relative to each other. Note that some of them produce polyhedral relaxations, while a few others generate semidefinite relaxations. The complexity assumptions on their input relaxations are also distinguished. Also, each solid arrow in the chart denotes "is dominated by" (i.e., the operator that is at the head of an arrow is at least as strong as that at the tail). For instance, when applied to the same set P, the LS_0 operator yields a relaxation that is at least as tight as that obtained by applying the BCC operator.

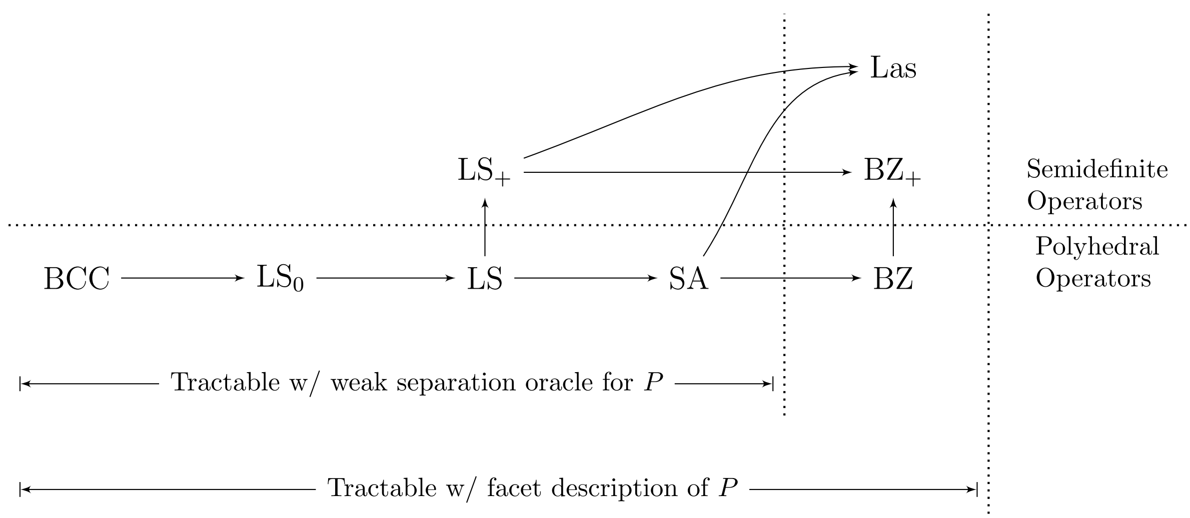

Figure 1.1: A strength chart of some previously known lift-and-project operators.

In our analysis, we have also defined many new operators. Two of them are strong, semidefinite versions of the Sherali–Adams operator that we call SA_+ and SA'_+. There are other weaker versions of these operators in the recent literature called *Sherali–Adams SDP* which have been previously studied, among others, by Chlamtac and Singh [CS08] and Benabbas et al. [BGM10, BM10, BCGM11, BGMT12], even though our versions are the strongest yet. We also propose operators that refine the Bienstock–Zuckerberg operators (BZ and BZ_+) and the Lasserre operator (Las). Note that BZ' and BZ'_+ are very strong operators that are not tractable, and we will mostly use them to establish inapproximability results. Also, while the Bienstock–Zuckerberg operators were defined to apply on any polyhedral relaxations contained in $[0,1]^n$, in this thesis we state and analyze restricted versions of these operators that only apply to lower-comprehensive polytopes, as we are mostly interested in applying these operators to polytopes that arise from set packing problems. Here, a set $P \subseteq [0,1]^n$ is *lower-comprehensive* if for every $x \in P$, every nonnegative y where $y \leq x$ belongs to P.

Figure 1.2 shows the dominance relationships between our new operators (boldfaced in the diagram) and the existing ones. These new operators fill the spectrum of polyhedral lift-and-project operators in a way which makes all of them more transparent, easier to relate to each other, and easier to analyze in a comprehensive way. These operators will be formally defined in Chapter 2 (with exception to $\mathrm{BZ}, \mathrm{BZ}_+$, BZ'' and BZ''_+, which are defined in the Appendix).

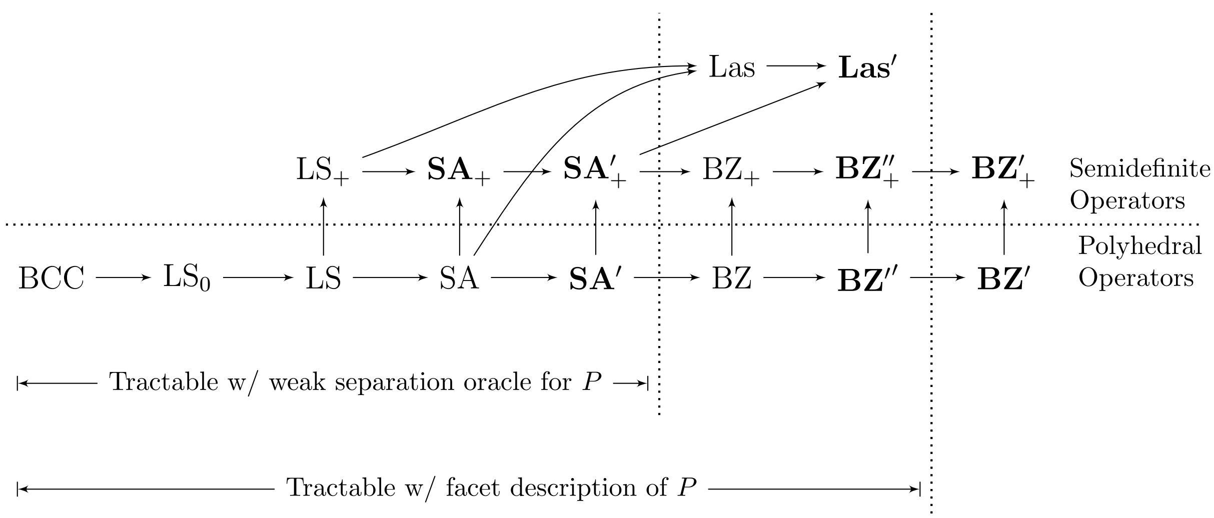

Figure 1.2: An updated strength chart with our new operators.

Observe that BCC is dominated by every other operator in Figure 1.2. Since BCC admits a very short and elegant proof that it returns P_I after n iterations for every $P \subseteq [0,1]^n$, it follows immediately that *every* operator in Figure 1.2 converges to P_I in at most n iterations. Moreover, if one can prove an upper-bound result for any operator Γ in Figure 1.2, then the same result applies to all operators in the diagram that can be reached from Γ by a directed path. Moreover, any lower-bound result on the BZ' operator implies the same result for all polyhedral lift-and-project operators in Figure 1.2. Likewise, to obtain a lower-bound result for all lift-and-project operators shown in the diagram, it suffices to show that the result holds for BZ'_+ and Las'. (For some bad instances for Las, see [Lau02] and [Che07], as well as Chapter 6 in this thesis.)

As seen in Figure 1.2, the strongest lift-and-project operators known to date are based on the operators SA, BZ and Las. We are interested in these strongest operators because they provide the strongest tractable relaxations obtained this way. On the other hand, if we want to prove that some combinatorial optimization problem is difficult to attack by lift-and-project methods, then we would hope to establish them on the strongest existing hierarchy for the strongest negative results. For example, some of the inapproximability results on vertex cover are based on the LS_+ operator [GMPT06, STT06], and some other integrality gap results are based on SA [CMM09].

Furthermore, it was shown in [CLRS13] that inapproximability results for the SA relaxations of approximate constraint satisfaction problems can be extended to lower-bound results on the extension complexity (i.e., the smallest number of variables needed to represent a given set as the projection of a tractable set in higher dimension) of the max-cut and max 3-sat polytopes. (The reader may refer to [Yan91] for the first major progress on the extension complexity of polytopes that arise from combinatorial optimization problems, and [Goe09, FMP$^+$12, Rot14] for some of the recent breakthroughs in this line of work.)

Therefore, by understanding the more powerful lift-and-project operators, we could either obtain better approximations for hard combinatorial optimization problems, or lay some of the groundwork for yet stronger inapproximability results. Moreover, we shall see that these analyses typically also lead to other crucial information about the underlying hierarchy of convex relaxations, such as their integrality gaps.

1.4 Results and organization of this thesis

With this thesis, we strive to provide a comprehensive analysis of the existing lift-and-project methods, with more focus on the strongest operators that give us the tightest relaxations, and their applications to relaxations that arise from combinatorial optimization problems. We study the most powerful cut-generating mechanisms of these operators, such as utilizing positive semidefiniteness constraints and lifting to higher dimensions to allow more variables in their formulations, and characterize situations where these conditions are helpful in delivering tight relaxations, or do not add extra strength over another operator that is less computationally costly. We prove numerous results that relate the performances of different lift-and-project operators, both in general and on relaxations with specific properties. Through the tools and techniques we provide, we aim to lay the groundwork for subsequent research of the lift-and-project approach, and help streamline and simplify the future analyses of these methods. Some of the results in this thesis were reported in an extended abstract and presented at IPCO 2011 [AT11].

Some of the highlights of the contributions of this thesis are:

- We define the notion of "admissible" lift-and-project operators, and identify variables used by an operator that are unhelpful in generating cuts. Under this framework, we prove that the BZ operator requires at least $\sqrt{2n} - \frac{3}{2}$ iterations to compute the matching polytope of the $(2n+1)$-clique, and approximately $\frac{n}{2}$ iterations to compute the stable set polytope of the n-clique. To the best of our knowledge, these results are the first results, since the invention of the BZ operator in 2004, which identify a class of combinatorial optimization problems on which BZ requires more than $O(1)$ iterations to reach the integer hull.

- We prove several results that can be seen as "approximate converses" of the dominance relationship among existing and new lift-and-project operators, each represented by dashed arrows in Figure 1.3. This shows that sometimes a weaker operator can be guaranteed to perform at least as well as a stronger one, by an appropriate increase of iterate number and/or certain assumptions on the given relaxation.

- We introduce the notion of "establishing" variables, and show that the presence of these variables in the formulation, together with a positive semidefiniteness constraint, can provide a guarantee on the overall performance of a lift-and-project operator. We then use these tools to show that SA'_+ requires at most $n - \left\lfloor \frac{\sqrt{2n+1}-1}{2} \right\rfloor$ iterations to compute the matching polytope of the $(2n+1)$-clique, while the BZ_+ operator requires no more than $\left\lceil \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rceil$ iterations.

- We characterize instances when SA_+ does not outperform polyhedral operators as weak as LS_0, generalizing on some of the results by Goemans and Tunçel in [GT01]. We also show that the notion of commutative maps for lift-and-project methods, which was proposed by Hong and Tunçel in [HT08] and proven to be useful in analyzing the Lovász–Schrijver operators, can be extended to handle stronger lift-and-project operators such as SA and SA_+.

- We provide a template for obtaining integrality gap results from lower-bound results for lift-and-project relaxations, and prove several integrality gap results on relaxations for the matching and stable set problem.

In Chapter 2, we introduce the existing lift-and-project methods, as well as many new operators. We will present multiple interpretations of these operators, such as viewing them as lifting to the face lattice of $[0,1]^n$ and the subset algebra of $\{0,1\}^n$. Some general properties of these operators will be discussed.

Next, we provide in Chapter 3 and 4 new techniques to analyze the worst-case performances, as well as relative strengths of these operators in a unified way. We show that, under certain conditions, the performances of SA' and BZ' are closely related to each other, and prove an analogous result relating the semidefinite operators SA'_+ and BZ'_+. Since SA' and SA'_+ inherit many properties from the well-studied SA operator, our findings provide another venue to understanding and analyzing BZ and BZ_+. We then utilize these tools we have established to prove the aforementioned inapproximability results on stable set and matching relaxations.

In Chapter 5, we turn to look at a few general tools that can help establish the membership of points in lift-and-project relaxations. Showing that a certain point lies in a lift-and-project relaxation usually requires constructing a matrix whose projection yields the desired point, and verifying that the matrix satisfies all conditions of the operator and indeed belongs to the lifted space. We shall see that the rows and columns of matrices we work with often have a lot of linear dependencies that are either enforced by a lift-and-project operator, or are present because the entries of our candidate matrix satisfy some nice set theoretical properties. These dependencies can simplify the task of verifying the membership conditions of these matrices. Next, since many semidefinite lift-and-project operators require the matrices in its lifted space to be positive semidefinite, we look into cases where there are natural connections between the eigenspaces of a candidate matrix and families of combinatorial objects, and present examples where these connections can give us valuable information about the eigenvectors and eigenvalues of our candidate matrix. We will also look at Hong and Tunçel's framework of proving lower-bound results using commutative maps for lift-and-project operators [HT08], and extend some of their results on the Lovász–Schrijver operators to stronger operators such as SA and SA_+. In particular, we will see that in the case when our given relaxation has a lot of symmetries, these maps can help reduce the task of checking the positive semidefiniteness of certificate matrices to verifying that on much smaller matrices. Examples where these tools applies (such as the relaxations for max-cut and matching) will be presented.

Then, in Chapter 6, we build on the work of Goemans and Tunçel in [GT01] and characterize some sets where applying SA_+ does not yield a tighter relaxation than applying LS_0, showing that sometimes even a very strong positive semidefiniteness constraint does not help generate any additional inequalities. We show that some well-known worst-case instances for LS_+ and SA are also worst-case instances for SA_+ and Las. In the process, we obtain a new example where the Las operator requires as many iterations as the dimension of the initial relaxation to return its integer hull. (The first of such results was obtained by Cheung in [Che07].) We also obtain what we believe to be the first example in which BZ_+ requires more than a constant number of iterations to return the integer hull of a set.

Finally, we illustrate in Chapter 7 how the analyses and the tools we provided may be used to prove integrality gaps for various classes of relaxations obtained from lift-and-project operators with some desirable invariance properties, and conclude with some possible future research directions in Chapter 8.

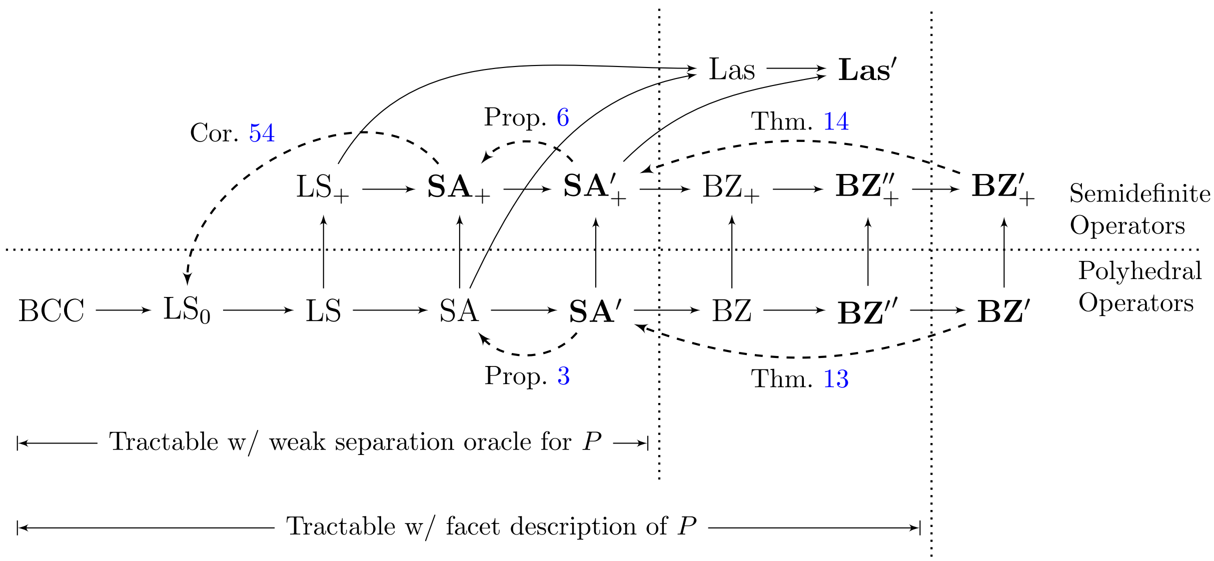

Figure 1.3: An illustration of several restricted reverse dominance results in this thesis.

Chapter 2

Preliminaries

In this section, we establish some notation and describe many lift-and-project operators that are known in the literature, as well as a few new ones that we devised. We start with the simpler (but weaker) procedures that produce polyhedral relaxations. Next, we give a brief introduction to semidefinite programming, and move on to discussing operators that utilize positive semidefiniteness constraints in their lifted spaces. We round up the chapter by defining slightly strengthened versions of operators by Bienstock and Zuckerberg.

2.1 The BCC operator and the Lovász–Schrijver operators

One of the most fundamental ideas behind the lift-and-project approach is convexification, which can be traced back to Balas' work on disjunctive cuts in the 1970s. For convenience, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$ herein. Observe that, given $P \subseteq [0,1]^n$, if we have mutually disjoint sets $Q_1, \ldots, Q_\ell \subseteq P$ such that their union, $\bigcup_{i=1}^{\ell} Q_i$, contains all integral points in P, then we can deduce that P_I is contained in $\operatorname{conv}\left(\bigcup_{i=1}^{\ell} Q_i\right)$, which therefore is a potentially tighter relaxation of P_I than P. Perhaps the simplest way to illustrate this idea is via the operator devised by Balas, Ceria and Cornuéjols [BCC93] which we call the BCC operator. Given $P \subseteq [0,1]^n$ and an index $i \in [n]$, define

$$\mathrm{BCC}_i(P) := \operatorname{conv}\left(\{x \in P : x_i \in \{0,1\}\}\right).$$

Moreover, we can apply BCC_i followed by BCC_j to a polytope P to make progress. In fact, it is well known that for every $P \subseteq [0,1]^n$,

$$\mathrm{BCC}_1(\mathrm{BCC}_2(\cdots(\mathrm{BCC}_n(P))\cdots)) = P_I.$$

This establishes that for every polytope P, one can obtain its integer hull with at most n applications of the BCC operator.

While iteratively applying BCC in all n indices is intractable (unless $\mathcal{P} = \mathcal{NP}$), applying them simultaneously to P and intersecting them is not. Furthermore, it is easy to see that P_I is contained in the intersection of these n sets. Thus,

$$\mathrm{LS}_0(P) := \bigcap_{i \in [n]} \mathrm{BCC}_i(P),$$

devised by Lovász and Schrijver [LS91], is a relaxation of P_I that is at least as tight as $\mathrm{BCC}_i(P)$ for all $i \in [n]$. Figure 2.1 illustrates how BCC and LS_0 operate in two dimensions.

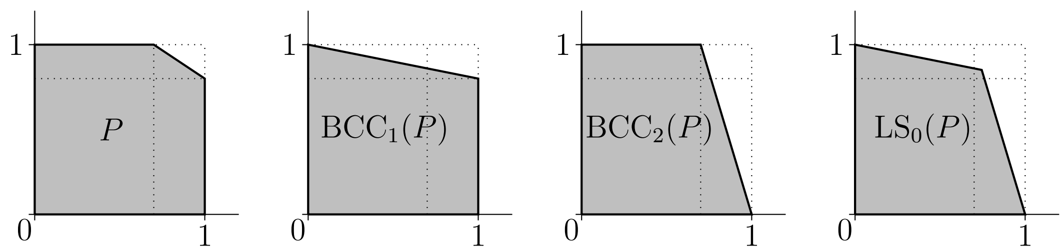

Figure 2.1: An illustration of BCC and LS_0 in two dimensions.

Before we look into operators that are even stronger (and more sophisticated), it is helpful to understand the following alternative description of LS_0. Given $x \in [0,1]^n$, let $\hat{x}$ denote the vector $\begin{pmatrix} 1 \\ x \end{pmatrix}$ in $\mathbb{R}^{n+1}$, where the new coordinate is indexed by 0. Let e_i denote the i^{th} unit vector (of appropriate size), and for any square matrix M, let $\mathrm{diag}(M)$ denote the vector formed by the diagonal entries of M. Next, given $P \subseteq [0,1]^n$, define the cone

$$K(P) := \left\{ \begin{pmatrix} \lambda \\ \lambda x \end{pmatrix} \in \mathbb{R} \oplus \mathbb{R}^n : \lambda \geq 0, x \in P \right\}.$$

Then, it is shown in [LS91] that

$$\begin{aligned} \mathrm{LS}_0(P) \;\; = \;\; & \big\{ x \in \mathbb{R}^n : \exists Y \in \mathbb{R}^{(n+1)\times(n+1)}, Ye_i, Y(e_0 - e_i) \in K(P), \; \forall i \in [n], \\ & Ye_0 = Y^\top e_0 = \mathrm{diag}(Y) = \hat{x} \big\}. \end{aligned}$$

To see that $\mathrm{LS}_0(P) \supseteq P_I$ in this perspective, observe that given any integral vector $x \in P$, the matrix $Y := \hat{x}\hat{x}^\top$ is a matrix which "certifies" that $x \in \mathrm{LS}_0(P)$. Then $P_I \subseteq \mathrm{LS}_0(P)$ follows from the fact that the latter is obviously a convex set.

Next, observe that $\hat{x}\hat{x}^\top$ is symmetric for all $x \in \{0,1\}^n$. Thus, if we let $\mathbb{S}^n$ denote the set of $n \times n$ real, symmetric matrices, then

$$\mathrm{LS}(P) := \big\{ x \in \mathbb{R}^n : \exists Y \in \mathbb{S}^{n+1}, Ye_i, Y(e_0 - e_i) \in K(P), \; \forall i \in [n], Ye_0 = \mathrm{diag}(Y) = \hat{x} \big\}$$

also contains P_I by the same argument. By enforcing a symmetry constraint on the matrices in the lifted space (and still retaining all integral points in P), we see that $\mathrm{LS}(P)$ is a potentially tighter relaxation than $\mathrm{LS}_0(P)$.

We can also apply these operators iteratively to a polytope P to gain progressively tighter relaxations. Let $\mathrm{LS}_0^k(P)$ (resp. $\mathrm{LS}^k(P)$) denote the set obtained from applying LS_0 (resp. LS) to P iteratively for k times. It is apparent from their definitions that

$$\mathrm{LS}(P) \subseteq \mathrm{LS}_0(P) \subseteq \mathrm{BCC}_i(P),$$

for every $i \in [n]$. Hence, it follows that $\mathrm{LS}_0^n(P) = \mathrm{LS}^n(P) = P_I$, for every $P \subseteq [0,1]^n$.

Recall the stable set problem introduced in Section 1.2. What happens if we apply LS_0 and LS to the feasible region of the LP relaxation (1.5)? Turns out that, as shown in [LS91], the performance of LS_0 and LS coincide in this case, and we get exactly the feasible region of (1.6), for every graph G. Thus, while the feasible region of (1.6) may have exponentially many facets, this set can be expressed as the projection of a tractable set of dimension $O(|V|^2)$.

In general, it is easy to find examples where LS strictly performs better than LS_0, and that the additional condition in LS requiring the matrices in its lifted space to be symmetric do generate extra cuts [AT09]. Also, while applying LS_0 and LS once to the feasible region of (1.5) always return the same relaxation, there are graphs on which applying LS twice to the feasible region of (1.5) yields a strictly tighter relaxation than applying LS_0 twice. The reader may refer to [Au07, AT09] for details of these examples, as well as [LT03] for more properties of LS_0- and LS-relaxations of the stable set problem. In subsequent chapters, we will return to this problem and discuss more about the performances of the stronger lift-and-project operators on the stable set relaxations.

2.2 The face lattice interpretation and the Sherali–Adams operator

In the two aforementioned Lovász–Schrijver operators, the certificate matrices all have dimension $(n+1)$ by $(n+1)$. We next look into the potential of lifting the initial relaxation $P \subseteq [0,1]^n$ to sets of even higher dimensions.

Note that given a point $x \in \mathrm{LS}_0(P)$ (or $\mathrm{LS}(P)$), its certificate matrix Y and any index $i \in [n]$, we know that $Y[0,i] = Y[i,i]$ (by the condition $\mathrm{diag}(Y) = Y^\top e_0$) and $Ye_i \in K(P)$. Thus, we see that

$$Ye_i \in K(P \cap \{x \in \mathbb{R}^n : x_i = 1\}).$$

One can similarly derive that

$$Y(e_0 - e_i) \in K(P \cap \{x \in \mathbb{R}^n : x_i = 0\}).$$

Hence, we can interpret the column Ye_i as "representing" the set P intersected with the face of the unit hypercube that is consisted of the points of which the i-coordinate is 1. Likewise, $Y(e_0 - e_i)$ can be interpreted as corresponding to the points in P where x_i is 0.

2.2.1 Lifting to the face lattice of $[0,1]^n$

Lovász and Schrijver noted in [LS91] that this interpretation can be made more general. Let $\mathcal{F}$ denote $\{0,1\}^n$. Given a set of indices $S \subseteq [n]$ and $t \in \{0,1\}$, we define

$$S|_t := \{x \in \mathcal{F} : x_i = t,\ \forall i \in S\}.$$

In other words, $S|_t$ is the set of points in $\mathcal{F}$ whose coordinates in S are all equal to t. To reduce cluttering, we will write $i|_t$ instead of $\{i\}|_t$ when dealing with sets with just one index. Also, in the case when S is empty, we have $\emptyset|_1 = \emptyset|_0 = \mathcal{F}$.

Next, given an integer $\ell \in \{0,1,\ldots,n\}$, we define

$$\mathcal{A}_\ell := \{S|_1 \cap T|_0 : S, T \subseteq [n], S \cap T = \emptyset, |S| + |T| \leq \ell\}$$

and

$$\mathcal{A}_\ell^+ := \{S|_1 : S \subseteq [n], |S| \leq \ell\}.$$

Note that there is a natural one-to-one correspondence between sets in $\mathcal{A}_n$ and faces of the unit hypercube.

Now consider the following lift-and-project operator Ψ, defined as follows:

1. Let $\tilde{\Psi}(P)$ denote the set of matrices $Y \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_n}$ which satisfy all of the following conditions:
   - (Ψ1) $Y[\mathcal{F}, \mathcal{F}] = 1$.
   - (Ψ2) $Ye_\alpha \in K(P)$, for every $\alpha \in \mathcal{A}_n$.
   - (Ψ3) $$\sum_{S \subseteq [n]} Ye_{S|_1 \cap ([n] \setminus S)|_0} = Ye_{\mathcal{F}}.$$
   - (Ψ4) For all $\alpha \in \mathcal{A}_1^+, \beta \in \mathcal{A}_n$ such that $\alpha \cap \beta = \emptyset$, $Y[\alpha, \beta] = 0$.
   - (Ψ5) For all $\alpha_1, \alpha_2 \in \mathcal{A}_1^+, \beta_1, \beta_2 \in \mathcal{A}_n$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.
2. Define $$\Psi(P) := \left\{x \in \mathbb{R}^n : \exists Y \in \tilde{\Psi}(P), Ye_{\mathcal{F}} = \hat{x}\right\}.$$

Then we have the following:

Proposition 2. *For every $P \subseteq [0,1]^n$, $\Psi(P) = P_I$.*

Proof. First, we show that $P_I \subseteq \Psi(P)$. Given $x \in P \cap \mathcal{F}$, define $Y \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_n}$ such that

$$Y[\mathcal{F}, S|_1 \cap T|_0] = \left(\prod_{j \in S} x_j \right) \left(\prod_{j \in T} 1 - x_j \right),$$

and

$$Y[i|_1, S|_1 \cap T|_0] = \left(\prod_{j \in S \cup \{i\}} x_j \right) \left(\prod_{j \in T} 1 - x_j \right),$$

for every $S|_1 \cap T|_0 \in \mathcal{A}_n$, and for every $i \in [n]$. We let $Y[\mathcal{F}, \mathcal{F}]$, the empty product, to evaluate to 1. Note that Y is a 0-1 matrix.

Clearly, (Ψ1) holds, and (Ψ3), (Ψ4) and (Ψ5) are satisfied as well by construction. For (Ψ2), notice that given any $S \subseteq [n]$, $Ye_{S|_1 \cap ([n] \setminus S)|_0}$ is either the zero vector (which is in $K(P)$), or $Y[\mathcal{F}, S|_1 \cap ([n] \setminus S)|_0] = 1$, which implies that S is exactly the support of x, and $Ye_{S|_1 \cap ([n] \setminus S)|_0} = \hat{x} \in K(P)$ (since $x \in P$). Thus, $Y \in \tilde{\Psi}(P)$, and consequently $x \in \Psi(P)$ as $Ye_{\mathcal{F}} = \hat{x}$. Since $\Psi(P)$ is obviously convex, we see that $P_I \subseteq \Psi(P)$.

To show the reverse containment, let $x \in \Psi(P)$ and Y be its certificate matrix. Consider any $S \subseteq [n]$. Then note that by (Ψ4) and (Ψ5),

$$Y[i|_1, S|_1 \cap ([n] \setminus S)|_0] = \begin{cases} Y[\mathcal{F}, S|_1 \cap ([n] \setminus S)|_0] & \text{if } i \in S; \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$Ye_{S|_1 \cap ([n] \setminus S)|_0} = Y[\mathcal{F}, S|_1 \cap ([n] \setminus S)|_0] \begin{pmatrix} 1 \\ \chi^S \end{pmatrix},$$

for every $S \subseteq [n]$, where χ^S is the incidence vector of S. Then it follows from (Ψ1) and (Ψ3) that x is a convex combination of integral points in P, which means that $x \in P_I$. $\square$

Thus, we see that Ψ is an extremely powerful operator that always returns the integer hull of the given set P. However, the certificate matrices in $\tilde{\Psi}(P)$ have exponential size (in n), and explicitly constructing elements in a lifted space of such a high dimension could yield an intractable structure, which makes the underlying algorithm no better than simply enumerating the integral points in P.

Nevertheless, we can still take advantage of the framework we developed for Ψ, and try to obtain a tight relaxation by only working with polynomial-size submatrices of those in $\tilde{\Psi}(P)$, and imposing constraints that are relaxations (Ψ1)–(Ψ5), in hope of capturing some important inequalities that are valid for P_I but not P. In particular, we can devise a lift-and-project operator by looking at a matrix whose columns are indexed by a portion of sets in $\mathcal{A}_n$.

We next express the operators devised by Sherali and Adams [SA90] in this language.

2.2.2 The SA operator

For any fixed integer $k \in [n]$, the SA^k operator can be defined as follows:

1. Let $\tilde{\mathrm{SA}}^k(P)$ denote the set of matrices $Y \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_k}$ which satisfy all of the following conditions:

(SA 1) $Y[\mathcal{F}, \mathcal{F}] = 1$.

(SA 2) $Ye_\alpha \in K(P)$, for every $\alpha \in \mathcal{A}_k$.

(SA 3) For every $S|_1 \cap T|_0 \in \mathcal{A}_{k-1}$,

$$Ye_{S|_1 \cap T|_0 \cap j|_1} + Ye_{S|_1 \cap T|_0 \cap j|_0} = Ye_{S|_1 \cap T|_0}, \quad \forall j \in [n] \setminus (S \cup T).$$

(SA 4) For all $\alpha \in \mathcal{A}_1^+, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta = \emptyset$, $Y[\alpha, \beta] = 0$.

(SA 5) For all $\alpha_1, \alpha_2 \in \mathcal{A}_1^+, \beta_1, \beta_2 \in \mathcal{A}_k$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

2. Define

$$\mathrm{SA}^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{SA}}^k(P), Ye_{\mathcal{F}} = \hat{x} \right\}.$$

The SA^k operator was originally described by linearizing polynomial inequalities, as follows: given an inequality $\sum_{i=1}^n a_i x_i \leq a_0$ that is valid for P, disjoint subsets of indices $S, T \subseteq [n]$ such that $|S| + |T| \leq k$, SA^k generates the inequality

$$\left(\prod_{i \in S} x_i\right) \left(\prod_{i \in T} 1 - x_i\right) \left(\sum_{i=1}^n a_i x_i\right) \leq \left(\prod_{i \in S} x_i\right) \left(\prod_{i \in T} 1 - x_i\right) a_0, \tag{2.1}$$

and obtains a linear inequality by replacing x_i^j with x_i (for all $j \geq 2$) in all terms, and then by using a new variable to represent each remaining nontrivial product of monomials. In our definition of SA^k, the linearized inequality would be

$$\sum_{i=1}^n a_i Y[i|_1, S|_1 \cap T|_0] \leq a_0 Y[\mathcal{F}, S|_1 \cap T|_0],$$

which is enforced by (SA 2) on the column of Y indexed by the set $S|_1 \cap T|_0$. Also, for every set of indices $U \subseteq [n]$, the product of monomials $\prod_{i \in U} x_i$ could appear multiple times in the original formulation when we generate (2.1) using different S and T. Then SA^k identifies them all by the variable x_U in the linearized formulation. This requirement is enforced by (SA 5) in our definition.

It is not hard to see that $\mathrm{SA}^1(P) = \mathrm{LS}(P)$. In general, SA obtains extra strength over LS by lifting P to a set of matrices of higher dimension, and using some properties of sets in $\mathcal{A}_n$ to identify variables in the lifted space.

Also, note that SA^n is exactly Ψ, the operator we presented immediately before introducing SA. Thus, we see readily that SA^k is a "restricted" version of Ψ. This observation, together with Proposition 2, gives us another proof that SA converges to the integer hull of any set in at most n iterations.

2.2.3 The SA' operator

We now look into sharpening the condition (SA 4) to obtain a lift-and-project operator that is potentially stronger than SA. Recall that, in the proof of Proposition 2, each entry of the certificate matrix $Y \in \tilde{\Psi}(P)$ for an integral point x is a product of entries of x. Then if $x \in P \cap \mathcal{F}$, and $\alpha, \beta \in \mathcal{A}_n$ are sets such that $\alpha \cap \beta \cap P = \emptyset$, the entry $Y[\alpha, \beta]$ must be zero. Thus, imposing such a constraint still preserves all matrices in the lifted space which correspond to integral points in P.

As we will utilize this observation in some variants of the Bienstock–Zuckerberg operators and relate their performances to other operators (such as SA), it is worthwhile to investigate how this new condition impacts the performance of an operator. Given $P \subseteq [0,1]^n$, and integer $k \geq 1$, define

$$\mathrm{SA}'^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{SA}}'^k(P) : Y e_{\mathcal{F}} = \hat{x} \right\},$$

where $\tilde{\mathrm{SA}}'^k(P)$ is the set of matrices in $\tilde{\mathrm{SA}}^k(P)$ that satisfy

(SA′ 4) For all $\alpha \in \mathcal{A}_1^+, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta \cap P = \emptyset$, $Y[\alpha, \beta] = 0$.

Note that SA'^k yields a tractable algorithm when $k = O(1)$, as the condition (SA′ 4) can be verified efficiently (assuming P is tractable), and is only checked polynomially many times. Also, since (SA′ 4) is more restrictive than (SA 4), it is apparent that $\mathrm{SA}'^k(P) \subseteq \mathrm{SA}^k(P)$ in general. However, it turns out that in the case of SA, this extra condition would at most "save" one iteration.

Proposition 3. *For every $P \subseteq [0,1]^n$ and every $k \geq 1$,*

$$\mathrm{SA}^{k+1}(P) \subseteq \mathrm{SA}'^k(P).$$

Proof. Let $x \in \mathrm{SA}^{k+1}(P)$, and let $Y \in \tilde{\mathrm{SA}}^{k+1}(P)$ such that $Y e_{\mathcal{F}} = \hat{x}$. Define $Y' \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_k}$ such that $Y'[\alpha, \beta] = Y[\alpha, \beta]$, $\forall \alpha \in \mathcal{A}_1^+, \beta \in \mathcal{A}_k$ (i.e., Y' is a submatrix of Y). We show that $Y' \in \tilde{\mathrm{SA}}'^k(P)$. Since $Y' e_{\mathcal{F}} = Y e_{\mathcal{F}} = \hat{x}$, this would imply that $x \in \mathrm{SA}'^k(P)$.

First, since $Y \in \tilde{\mathrm{SA}}^{k+1}(P)$, it is obvious that $Y' \in \tilde{\mathrm{SA}}^k(P)$. Thus, we only need to show that Y' satisfies (SA′ 4). Given $\alpha \in \mathcal{A}_1^+, \beta \in \mathcal{A}_k$, suppose $\alpha = i|_1$ for some $i \in [n]$, and $\beta = S|_1 \cap T|_0$ for some $S, T \subseteq [n]$. Now $\alpha \cap \beta = (S \cup \{i\})|_1 \cap T|_0 \in \mathcal{A}_{k+1}$, and thus the entry $Y[\mathcal{F}, \alpha \cap \beta]$ exists.

Since $Y e_{\alpha \cap \beta} \in K(P)$ by (SA 2), $Y[\mathcal{F}, \alpha \cap \beta] > 0$ would imply that the point

$$\frac{1}{Y[\mathcal{F}, \alpha \cap \beta]} \left(Y[1|_1, \alpha \cap \beta], Y[2|_1, \alpha \cap \beta], \ldots, Y[n|_1, \alpha \cap \beta] \right)^\top$$

is in P, and thus $\alpha \cap \beta \cap P \neq \emptyset$. Hence, we see that (SA′ 4) holds, and our claim follows. $\square$

Proposition 3 establishes the dashed arrow from SA$'$ to SA in Figure 1.3, and assures that if one can provide a performance guarantee for SA$'$ on a polytope P, then the same can be said of the *weaker* SA operator by using one extra iteration. The meanings for the other four dashed arrows in Figure 1.3 are similar in nature — for some linear or quadratic function of the iterate number, the weaker operator performs at least as well as the stronger operator. However, they are much more involved than Proposition 3, and sometimes depend on the properties of the given set P. We will address them in detail in the subsequent chapters.

2.3 Utilizing positive semidefiniteness in lift-and-project operators

So far, every lift-and-project operator we have seen returns polyhedral tightened relaxations. Next, we will expand our discussion to operators that may not produce polyhedral relaxations. In particular, we will introduce several lift-and-project operators that utilize positive semidefiniteness, and look into the power and limitations of these additional constraints.

2.3.1 An extremely brief introduction to semidefinite programming

Before we do that, it is helpful to mention a few basic elements of semidefinite programming. Given two matrices $A, B \in \mathbb{S}^n$, we let

$$\langle A, B \rangle := \operatorname{tr}(A^\top B) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ij}$$

denote the standard inner product on $\mathbb{S}^n$. Also, we say that a matrix $A \in \mathbb{S}^n$ is *positive semidefinite* if $h^\top A h \geq 0$, $\forall h \in \mathbb{R}^n$. There are also several alternative ways to characterize positive semidefinite matrices:

Proposition 4. *Given a matrix $A \in \mathbb{S}^n$, the following are equivalent:*

1. *A is positive semidefinite;*
2. *All eigenvalues of A are nonnegative;*
3. *There exists a real matrix U such that $A = U^\top U$;*
4. *$\langle A, B \rangle \geq 0$ for every positive semidefinite matrix B;*
5. *There exists a collection of vectors $S \subseteq \mathbb{R}^n$ such that $A = \sum_{x \in S} x x^\top$.*

Let $\mathbb{S}^n_+ \subset \mathbb{S}^n$ denote the set of symmetric, positive semidefinite $n \times n$ matrices, and we indicate that a matrix A is positive semidefinite by writing $A \succeq 0$. Similarly, we define a matrix $A \in \mathbb{S}^n_+$ to be *positive definite* (denoted by $A \succ 0$), if $x^\top A x > 0$ for every nonzero vector $x \in \mathbb{R}^n$.

It is easy to see that, for all $x \in \mathbb{R}^n$, the matrix $xx^\top$ is positive semidefinite. Moreover, for all $A \in \mathbb{S}^n_+$ and nonnegative scalar $k \in \mathbb{R}$, the matrix kA is also positive semidefinite. Thus, the set of $n \times n$ positive semidefinite matrices form a cone in $\mathbb{R}^{n \times n}$. This, together with 5. of Proposition 4, implies that $\mathbb{S}^n_+$ is a convex cone contained in $\mathbb{R}^{n \times n}$, and the extreme rays of $\mathbb{S}^n_+$ are exactly the set of matrices of the form $xx^\top$ for some nonzero vector $x \in \mathbb{R}^n$.

As we have seen in the description of LS, the matrices in the lifted space that correspond to integer points are of the form $xx^\top$. Thus, $\mathbb{S}^n_+$ is a natural relaxation of the convex hull of these rank 1 matrices. This motivates the following model: Given matrices $C, A_1, \ldots, A_m \in \mathbb{S}^n$ and vector $b \in \mathbb{R}^m$, the mathematical program

$$\begin{array}{rrcll} \max & \langle C, X \rangle & & & \\ \text{subject to} & \langle A_i, X \rangle & = & b_i & \forall i \in [m] \\ & X & \succeq & 0, & \end{array} \tag{2.2}$$

is called a *semidefinite program* (SDP). SDPs are fundamental in the study of both continuous and discrete optimization, and have been used to obtain good approximation algorithms for many hard problems in combinatorial optimization (e.g. Goemans and Williamson's celebrated result on max-cut [GW95]). Semidefinite programming also boasts many applications in areas such as operations research, management sciences and quantum computing.

Also, SDP formulations are highly flexible. For instance, the formulation (2.2) can accommodate variations such as minimizations problems, inequality constraints, or even additional linear equations that we enforce on the entries of the matrix variable X. In particular, SDPs are a more general mathematical model than LPs. To see this, suppose we have the following LP:

$$\begin{array}{rrcll} \max & c^\top x & & & \\ \text{subject to} & \sum_{j=1}^n a_{ij} x_j & = & b_i & \forall i \in [m] \\ & x & \geq & 0. & \end{array} \tag{2.3}$$

Given a vector $x \in \mathbb{R}^n$, let $\operatorname{Diag}(x)$ denote the $n \times n$ matrix

$$\begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix}.$$

Then we define $C := \operatorname{Diag}(c)$ and $A_i := \operatorname{Diag}((a_{i1}, \ldots, a_{in})^\top)$ for all $i \in [m]$. We also define E_{ij} to be the $n \times n$ matrix whose ij entry is 1, and 0 everywhere else, for all

$i, j \in [n]$. Now consider the following:

$$\begin{array}{rrcl} \max & \langle C, X \rangle & & \\ \text{subject to} & \langle A_i, X \rangle & = & b_i \quad \forall i \in [m] \\ & \langle E_{ij}, X \rangle & = & 0 \quad \forall i, j \in [n], i < j \\ & X & \succeq & 0. \end{array} \tag{2.4}$$

Obviously, (2.4) is an SDP. Also, notice that every feasible solution X has to be a diagonal matrix, which is positive semidefinite if and only if all of its diagonal entries are nonnegative. Thus, we see that x is feasible in (2.3) if and only if $\mathrm{Diag}(x)$ is feasible in (2.4). Moreover, $c^\top x = \langle C, \mathrm{Diag}(x) \rangle$, and thus the two formulations are equivalent. Hence, every LP can be modelled as an SDP.

Another way to see that SDP generalizes LP is the following. Observe that the n-dimensional nonnegative orthant $\{x \in \mathbb{R}^n : x \geq 0\}$ can be expressed as

$$\underbrace{\mathbb{R}_+ \oplus \mathbb{R}_+ \oplus \cdots \oplus \mathbb{R}_+}_{n \text{ times}}.$$

Thus, the constraint $x \geq 0$ in the LP in the form of (2.3) can be interpreted as "x is in a convex cone that is a direct sum of n irreducible convex cones". Therefore, replacing each $\mathbb{R}_+$ by an irreducible cone $\mathcal{S}_+^{m_i}$ for some positive integer m_i, we arrive at the cone constraint

$$X \in \mathcal{S}_+^{m_1} \oplus \mathcal{S}_+^{m_2} \oplus \cdots \oplus \mathcal{S}_+^{m_n},$$

whose intersection with a polyhedron is a feasible region for an SDP.

Furthermore, semidefinite programming also has a nice duality theory, much of which resembles that of linear programming. Given an SDP in the form (2.2), its dual problem is

$$\begin{array}{rrcl} \min & b^\top y & & \\ \text{subject to} & \sum_{i=1}^m y_i A_i & \succeq & C. \end{array} \tag{2.5}$$

Note that (2.5) is itself a SDP, and can be put into the form of (2.2). Then the weak duality relation for LPs also hold for SDPs: if X, y are feasible in (2.2) and (2.5), respectively, then

$$b^\top y = \sum_{i=1}^m b_i y_i = \sum_{i=1}^m \langle A_i, X \rangle y_i = \left\langle \sum_{i=1}^m y_i A_i, X \right\rangle \geq \langle C, X \rangle .$$

However, unlike in LPs, it is possible for both an SDP and its dual to be feasible, but their optimal values not coincide. Given a maximization SDP (P) and its dual problem (D), their *duality gap* is defined to be the difference between the infimum of the objective values of feasible solutions in (D) and the supremum of the objective values of feasible solutions in (P). While this quantity can be nonzero for a general primal-dual pair of SDPs, it can be assured to be zero by the following condition: We say that an SDP satisfies the *Slater condition* if it has a strictly feasible solution (e.g. for (2.2), it would be a feasible X such that $X \succ 0$; for (2.5), it would be a vector y where $\sum_{i=1}^m A_i y_i \succ C$). Then we have the following:

Theorem 5. *Let (P) be an maximization SDP, and (D) be its dual.*

1. *If the objective value of (P) is bounded from above, and (P) satisfies Slater condition, then (P) and (D) have zero duality gap, and (D) has an optimal solution.*
2. *If the objective value of (D) is bounded from below, and (D) satisfies Slater condition, then (P) and (D) have zero duality gap, and (P) has an optimal solution.*

In particular, when both (P) and (D) satisfy the Slater condition, then they must have a zero duality gap, and there exist solutions in (P) and (D) that attain the optimal value. Thus, the strong duality relation, when present, assures us of many key properties of a primal-dual pair of SDPs that we can take for granted in the case of LPs.

Finally, assuming certain technical conditions, SDPs can be approximately solved to arbitrary precision, in polynomial time of the size of the input, by algorithms such as the ellipsoid method and interior-point methods. This is true even if we do not explicitly have the data $A_1, \ldots, A_m$ and b, but only a weak separation oracle for the feasible region of the SDP. The SDPs we will encounter in this thesis usually have feasible regions that are sets in the lifted space of an operator, and do satisfy these conditions. For instance, for most of the relaxations we look at, it is usually not hard to determine the affine hull of its lifted set from the conditions imposed by a lift-and-project operator, as well as our knowledge of some of the integer solutions of the underlying problem. For the full technicality of these details, as well as discussion on the theory and applications of SDPs in combinatorial optimization, the reader may refer to [Tun10].

Thus, we see that SDPs can be extremely useful in delivering tight, tractable relaxations of IPs that are difficult to tackle directly. Then, using duality theory (and other tools), we can uncover many nontrivial properties of the underlying optimization problems, such as tight bounds of their optimal values. As we shall see, lift-and-project operators that utilize positive semidefiniteness and return SDP relaxations can perform dramatically better than their polyhedral counterparts, in many cases.

2.3.2 The operators $\mathrm{LS}_+, \mathrm{SA}_+$ and SA'_+

Perhaps the most elementary operator that utilizes positive semidefiniteness is the LS_+ operator defined in [LS91]. Recall that one way to see why $P_I \subseteq \mathrm{LS}(P)$ in general is to observe that for every integral point $x \in P$, $\hat{x}\hat{x}^\top$ is a matrix that certifies x's membership in $\mathrm{LS}(P)$. Since $\hat{x}\hat{x}^\top$ is positive semidefinite for all x, it is easy to see that

$$\mathrm{LS}_+(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \mathbb{S}_+^{n+1}, Ye_i, Y(e_0 - e_i) \in K(P),\ \forall i \in [n], Ye_0 = \mathrm{diag}(Y) = \hat{x} \right\}$$

contains P_I as well. Also, by definition, $\mathrm{LS}_+(P) \subseteq \mathrm{LS}(P)$ for all $P \subseteq [0,1]^n$, and thus LS_+ potentially obtains a tighter relaxation than $\mathrm{LS}(P)$ in general.

It is also easy to see that for every set $P \subseteq [0,1]^n$, $\mathrm{LS}_+(P)$ can be expressed as the feasible region of an SDP, whose size is polynomial of the data describing P.

Next, we define two positive semidefinite variants of the SA operator that are even stronger than LS_+. Given a vector $y \in \mathbb{R}^{\mathcal{A}'}$ where $\mathcal{A}_1^+ \subseteq \mathcal{A}'$, we define

$$\hat{x}(y) := (y_{\mathcal{F}}, y_{1|_1}, \ldots, y_{n|_1})^\top.$$

Then, given any positive integer k, we define the operators SA_+^k and $\mathrm{SA}_+^{\prime k}$ as follows:

1. Let $\tilde{\mathrm{SA}}_+^k(P)$ be the set of matrices $Y \in \mathbb{S}_+^{\mathcal{A}_k}$ that satisfy all of the following conditions:

(SA$_+$ 1) $Y[\mathcal{F}, \mathcal{F}] = 1$.

(SA$_+$ 2) For every $\alpha \in \mathcal{A}_k$:

(i) $\hat{x}(Ye_\alpha) \in K(P)$;

(ii) $Ye_\alpha \geq 0$.

(SA$_+$ 3) For every $S|_1 \cap T|_0 \in \mathcal{A}_{k-1}$,

$$Ye_{S|_1 \cap T|_0 \cap j|_1} + Ye_{S|_1 \cap T|_0 \cap j|_0} = Ye_{S|_1 \cap T|_0}, \quad \forall j \in [n] \setminus (S \cup T).$$

(SA$_+$ 4) For all $\alpha, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta = \emptyset, Y[\alpha, \beta] = 0$.

(SA$_+$ 5) For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{A}_k$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2, Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

2. Let $\tilde{\mathrm{SA}}_+^{\prime k}(P)$ be the set of matrices $\tilde{\mathrm{SA}}_+^k(P)$ that also satisfy:

(SA$'_+$ 4) For all $\alpha, \beta \in \mathcal{A}_k$ such that $\alpha \cap \beta \cap P = \emptyset, Y[\alpha, \beta] = 0$.

3. Define

$$\mathrm{SA}_+^k(P) = \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{SA}}_+^k(P), \hat{x}(Ye_{\mathcal{F}}) = \hat{x} \right\},$$

and

$$\mathrm{SA}_+^{\prime k}(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{SA}}_+^{\prime k}(P), \hat{x}(Ye_{\mathcal{F}}) = \hat{x} \right\}.$$

The SA_+^k and $\mathrm{SA}_+^{\prime k}$ operators extend the lifted space of the SA^k operator (which are matrices of dimension $(n+1) \times O(n^k)$) to a set of square matrices, and impose an additional positive semidefiniteness constraint. What sets these two new operators apart is that $\mathrm{SA}_+^{\prime k}$ utilizes a (SA$'$ 4)-like condition to potentially obtain additional strength over SA_+^k. While we have seen in their polyhedral counterparts SA$'$ and SA that adding this additional constraint could decrease the number of iterations needed to reach the integer hull by at most one, the difference can be much more pronounced between SA_+ and SA'_+. Using the same observations in the proof of Proposition 3, we can prove the following:

Proposition 6. *For every $P \subseteq [0,1]^n$ and every $k \geq 1$,*

$$\mathrm{SA}_+^{2k}(P) \subseteq \mathrm{SA}_+^{\prime k}(P).$$

We shall see examples in Chapter 6 where SA'_+ requires $\Theta(n)$ less iterations than SA_+ to return the integer hull of a polytope.

Also, note that in (SA_+ 2) we have imposed that all certificate matrices in $\tilde{\mathrm{SA}}_+^k(P)$ (which contains $\tilde{\mathrm{SA}}_+'^k(P)$) have nonnegative entries, which obviously holds for matrices lifted from integral points. In contrast with (SA 2), the nonnegativity condition was not explicitly stated there as it is implied by the fact that $P \subseteq [0,1]^n$.

Next, we show that SA_+^k dominates the LS_+^k operator (i.e., k iterative applications of LS_+):

Proposition 7. *For every polytope $P \subseteq [0,1]^n$ and every integer $k \geq 1$,*

$$\mathrm{SA}_+^k(P) \subseteq \mathrm{LS}_+(\mathrm{SA}_+^{k-1}(P)).$$

Proof. Suppose $Y \in \tilde{\mathrm{SA}}_+^k(P)$ and $\hat{x}(Ye_{\mathcal{F}}) = \hat{x}$. Let Y' be the $(n+1)\times(n+1)$ symmetric minor of Y, with rows and columns indexed by elements in $\mathcal{A}_1^+$. To adapt to the notation for LS_+, we index the rows and columns of Y' by $0,1,\ldots,n$ (instead of $\mathcal{F}, 1|_1, \ldots, n|_1$). It is obvious that $Y' \in \mathbb{S}_+^{n+1}$, and $Y'e_0 = \operatorname{diag}(Y') = \hat{x}$. Thus, it suffices to show that $Y'e_i, Y'(e_0 - e_i) \in K(\mathrm{SA}_+^{k-1}(P)),\ \forall i \in [n]$.

We first show that $Y'e_1 \in K(\mathrm{SA}_+^{k-1}(P))$. If $(Y'e_1)_0 = 0$, then $Y'e_1$ is the zero vector and the claim is obviously true. Next, suppose $(Y'e_1)_0 > 0$. Define the matrix $Y'' \in \mathbb{S}^{\mathcal{A}_{k-1}}$, such that

$$Y''[\alpha,\beta] = \frac{1}{(Y'e_1)_0} Y[\alpha \cap 1|_1, \beta \cap 1|_1], \quad \forall \alpha,\beta \in \mathcal{A}_{k-1}.$$

Notice that Y'' is a positive scalar multiple of a symmetric minor of Y, and thus is positive semidefinite and nonnegative. Moreover, it satisfies (SA_+ 1) by construction, and inherits the properties (SA_+ 2)–(SA_+ 5) from Y. Thus, $Y'' \in \tilde{\mathrm{SA}}_+^{k-1}(P)$ and $\hat{x}(Y''e_{\mathcal{F}}) = \frac{1}{(Y'e_1)_0} Y'e_1 \in K(\mathrm{SA}_+^{k-1}(P))$. Hence, we obtain that $Y'e_1 \in K(\mathrm{SA}_+^{k-1}(P))$. Since we can use the same observations to show that $Y'e_i, Y'(e_0 - e_i) \in K(\mathrm{SA}_+^{k-1}(P))$ for all $i \in [n]$, our claim follows. ☐

Then it follows immediately from the definitions of $\mathrm{SA}_+, \mathrm{SA}'_+$ and Proposition 7 that

$$\mathrm{SA}_+'^k(P) \subseteq \mathrm{SA}_+^k(P) \subseteq \mathrm{LS}_+^k(P),$$

for every $k \geq 1$. The SA_+ and SA'_+ operators will be useful in simplifying our analysis and improving our understanding of the Bienstock–Zuckerberg operator enhanced with positive semidefiniteness.

2.4 The Lasserre operator

We turn our attention to Lasserre's operator defined in [Las01], denoted Las herein. While Las can be applied to semialgebraic sets, we restrict our discussion to their applications to polytopes contained in $[0,1]^n$. Also, our presentation of the operator is closer to that in [Lau03a]. Given $P := \{x \in [0,1]^n : Ax \leq b\}$, and an integer $k \in [n]$,

1. Let $\tilde{\mathrm{Las}}^k(P)$ denote the set of matrices $Y \in \mathbb{S}_+^{\mathcal{A}_{k+1}^+}$ that satisfy all of the following conditions:

(Las 1) $Y[\mathcal{F},\mathcal{F}] = 1$;

(Las 2) For every $i \in [m]$, define the matrix $Y^i \in \mathbb{S}^{\mathcal{A}_k^+}$ where

$$Y^i[S|_1, S'|_1] := b_i Y[S|_1, S'|_1] - \sum_{i=1}^{n} A[i,j] Y[(S \cup \{j\})|_1, (S' \cup \{j\})|_1],$$

and impose $Y^i \succeq 0$.

(Las 3) For every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{A}_k^+$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

2. Define

$$\mathrm{Las}^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{Las}}^k(P) : \hat{x}(Ye_{\mathcal{F}}) = \hat{x} \right\}.$$

We note that, unlike the previously mentioned operators, Las requires an explicit description of P in terms of valid inequalities. Also, similar to how we strengthened the original Sherali–Adams operator, we can define a refined version of Las as follows. Again, given $P := \{x \in [0,1]^n : Ax \leq b\}$, and an integer $k \in [n]$:

1. Let $\tilde{\mathrm{Las}}'^k(P)$ denote the set of nonnegative matrices $Y \in \mathbb{S}_+^{\mathcal{A}_{k+1}}$ that satisfy all of the following conditions:

(Las′ 1) $Y[\mathcal{F},\mathcal{F}] = 1$;

(Las′ 2) For every $i \in [m]$, define the matrix $Y^i \in \mathbb{S}^{\mathcal{A}_k}$ where

$$Y^i[\alpha, \beta] := b_i Y[\alpha, \beta] - \sum_{j=1}^{n} A[i,j] Y[\alpha \cap i|_1, \beta \cap i|_1].$$

Impose $Y^i \succeq 0$ and $Y^i \geq 0$.

(Las′ 3) For every $\alpha \in \mathcal{A}_k$,

$$Ye_{\alpha \cap j|_1} + Ye_{\alpha \cap j|_0} = Ye_\alpha, \quad \forall j \in [n].$$

(Las′ 4) For every $\alpha, \beta \in \mathcal{A}_{k+1}$ such that $\alpha \cap \beta \cap P = \emptyset, Y[\alpha, \beta] = 0$.

(Las′ 5) For all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{A}_k$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2, Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

2. Define

$$\mathrm{Las}'^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{Las}}'^k(P) : \hat{x}(Ye_{\mathcal{F}}) = \hat{x} \right\}.$$

Note that while $\tilde{\mathrm{SA}}'^k_+(P)$ and $\tilde{\mathrm{SA}}^k_+(P)$ have the same dimension, the matrices in $\tilde{\mathrm{Las}}'^k(P)$ have dimension $\mathcal{A}_{k+1} \times \mathcal{A}_{k+1}$, while those in $\tilde{\mathrm{Las}}^k(P)$ have size $\mathcal{A}^+_{k+1} \times \mathcal{A}^+_{k+1}$. Also, matrices in the lifted space of Las' are required to be nonnegative (which is not imposed by Las), and (Las' 4) is a condition that, like (SA' 4) and (SA'_+ 4), may force more variables in the lifted space to be zero. It is not hard to see that Las'^k dominates Las^k — if $x \in \mathrm{Las}'^k(P)$, then a symmetric minor of its certificate matrix would show that $x \in \mathrm{Las}^k(P)$. We next show that Las'^k also dominates SA'^k_+. Note that, from here on, we will sometimes use $v[i]$ to denote the i-entry of a vector v (instead of v_i).

Proposition 8. *For every polytopes $P := \{x \in [0,1]^n : Ax \leq b\}$ and for every integer $k \geq 1$, $\mathrm{Las}'^k(P) \subseteq \mathrm{SA}'^k_+(P)$.*

Proof. Let $x \in \mathrm{Las}'^k(P)$, and Y be its certificate matrix in $\tilde{\mathrm{Las}}'^k(P)$. Then Y is $\mathcal{A}_{k+1} \times \mathcal{A}_{k+1}$. Let Y' be the symmetric minor of Y whose rows and columns are indexed by $\mathcal{A}_k$. We show that $Y' \in \tilde{\mathrm{SA}}'^k_+(P)$.

First, (SA_+ 1), (SA_+ 2), (SA'_+ 4) and (SA_+ 5) follow from (Las' 1), (Las' 2), (Las' 4) and (Las' 5), respectively. Also, since $Y' \succeq 0$ and $Y \geq 0$, and Y' is a symmetric minor of Y, we obtain that Y' is also positive semidefinite and nonnegative. Thus, it only remains to verify that $\hat{x}(Ye_\alpha) \in K(P)$ for every $\alpha \in \mathcal{A}_k$. Observe that, for every inequality $\sum_{j=1}^n A[i,j]y \leq b[i]$, and $\alpha \in \mathcal{A}_k$,

$$\sum_{j=1}^n Y[j|_1, \alpha] \leq b[i]Y[\mathcal{F}, \alpha],$$

as $Y^i[\alpha, \alpha] \geq 0$. Thus, $\hat{x}(Ye_\alpha) \in K(P)$, and we are finished. □

In Chapter 6, we will provide examples in which Las' and SA'_+ strictly outperform their unprimed counterparts. Also, we remark that Gouveia, Parrilo and Thomas provided in [GPT10] an alternative description of the Las operator, where P_I is described as the variety of an ideal intersected with the solutions to a system of polynomial inequalities. For an example, given a graph $G = (V, E)$, if we let $H_1 := \{x \in \mathbb{R}^n : x_i(x_i - 1) = 0, \forall i \in V\}$ and $H_2 := \{x \in \mathbb{R}^n : 1 - x_i - x_j \geq 0, \forall \{i,j\} \in E\}$, then $STAB(G) = \mathrm{conv}(H_1 \cap H_2)$. Generalizing Lovász's theta body [Lov79], they proposed the notion of the theta body of an ideal, which they proved is equivalent to the Las operator, if Las was restricted to only using vanishing ideals of subsets of $\{0,1\}^n$ (without intersection with system of polynomial inequalities). For instance, the vanishing ideal of $STAB(G)$ is generated by the polynomials

$$\{x_i(x_i - 1) : i \in V\} \cup \{x_i x_j : \{i,j\} \in E\}.$$

One distinction of their relaxations is that they only depend on the desired set of integer points in P_I, instead of an initial, tractable relaxation. They showed that this more restricted version of the Las operator produces simpler but weaker relaxations than some of the other relaxations used commonly in the literature. We will look into these relaxations in more detail in Chapter 7.

2.5 The Bienstock–Zuckerberg operator and their variants

Finally, we look into the lift-and-project operators devised by Bienstock and Zuckerberg [BZ04].

Recall that the idea of convexification requires a collection of disjoint subsets of P whose union contains all integral points in P. So far, every operator that we have seen obtains these sets by intersecting P with faces of $[0,1]^n$. However, sometimes it is beneficial to allow more flexibility in choosing the way we partition the integral points in P. For example, consider

$$P := \left\{ x \in [0,1]^n : \sum_{i=1}^{n} x_i \leq n - \frac{1}{2} \right\}.$$

P is known to be a worst-case instance for many lift-and-project operators. For instance, we will see in Chapter 6 that $\mathrm{SA}_+^{n-1}(P)$, a relaxation obtained from using convexification with exponentially many sets that are all intersections of P and faces of $[0,1]^n$ as well as positive semidefiniteness, still strictly contains P_I. On the other hand, if we define

$$Q_j := \left\{ x \in P : \sum_{i=1}^{n} x_i = j \right\},$$

for every $j \in \{0, 1, \ldots, n\}$, then every integral point in P is contained in Q_j for some j, and

$$P_I = \operatorname{conv}\left(\bigcup_{i=0}^{n} Q_j \right).$$

We will see in the next section that the set $\operatorname{conv}\left(\bigcup_{i=0}^{n} Q_j\right)$ can be described as the projection of a set of dimension $O(n^2)$ that is tractable as long as P is.

Bienstock and Zuckerberg [BZ04] utilized this type of ideas and invented operators that use variables that were not exploited by the operators proposed earlier (namely, subsets of $\mathcal{F}$ that are not in $\mathcal{A}_n$), in conjunction with some new constraints. We will denote their operators by BZ and BZ_+, but we also present variants of them called $\mathrm{BZ}', \mathrm{BZ}'_+, \mathrm{BZ}''_+$ and BZ''_+. The operators BZ'' and BZ''_+ are slightly refined versions of BZ and BZ_+, respectively. They produce tractable relaxations for tractable relaxations, and will be defined and discussed in the Appendix, together with BZ and BZ_+. Our focus here will be on BZ' and BZ'_+, variants that are even stronger but we believe are simpler to present. While BZ' and BZ'_+ may require exponential run-time and space (in terms of the size of the data in the given relaxation), we will only use them to establish inapproximability results. Note that if one shows that, say, BZ' cannot return the integer hull of a relaxation in a certain number of iterations, the same can be said of the weaker operators such as BZ and BZ''.

Also, since we are mostly interested in applying these operators to polytopes that arise from set packing problems (such as the stable set and matching problems of graphs), we will state versions of these operators that only apply to lower-comprehensive polytopes. (Recall that a set $P \subseteq [0,1]^n$ is lower-comprehensive if for every $x \in P$, $y \in P$ for every y such that $0 \leq y \leq x$.) We will discuss this in more detail after stating the elements of their operators.

2.5.1 The subset algebra of $\mathcal{F}$

From here on, we let $\mathcal{A}$ denote $2^{\mathcal{F}}$, the power set of $\mathcal{F}$. For each $x \in \mathcal{F}$, we define the vector $x^{\mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$ where

$$x^{\mathcal{A}}[\alpha] = \begin{cases} 1 & \text{if } x \in \alpha; \\ 0 & \text{otherwise.} \end{cases}$$

That is, each coordinate of $\mathcal{A}$ corresponds to a subset of the vertices of the n-dimensional unit hypercube, and $x^{\mathcal{A}}[\alpha] = 1$ if and only if the point x is contained in the set α. It is not hard to see that for all $x \in \mathcal{F}$, we have $x^{\mathcal{A}}[\mathcal{F}] = 1$, and $x^{\mathcal{A}}[i|_1] = x_i, \forall i \in [n]$. Another important property of $x^{\mathcal{A}}$ is that, given disjoint subsets $\alpha_1, \alpha_2, \ldots, \alpha_k \subseteq \beta \subseteq \mathcal{F}$, we know that

$$x^{\mathcal{A}}[\alpha_1] + x^{\mathcal{A}}[\alpha_2] + \cdots + x^{\mathcal{A}}[\alpha_k] \leq x^{\mathcal{A}}[\beta], \tag{2.6}$$

and equality holds if $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ partitions β.

Thus, for any given $x \in \mathcal{F}$, if we define $Y^x_{\mathcal{A}} := x^{\mathcal{A}}(x^{\mathcal{A}})^\top$, then the entries of $Y^x_{\mathcal{A}}$ have considerable structure. Most notably, the following must hold:

Proposition 9. *For every $x \in \mathcal{F}$, the matrix $Y^x_{\mathcal{A}} = x^{\mathcal{A}}(x^{\mathcal{A}})^\top$ satisfies all of the following:*

(P1) $Y^x_{\mathcal{A}} e_{\mathcal{F}} = (Y^x_{\mathcal{A}})^\top e_{\mathcal{F}} = \operatorname{diag}(Y^x_{\mathcal{A}}) = x^{\mathcal{A}}$;

(P2) $Y^x_{\mathcal{A}} e_\alpha \in \left\{0, x^{\mathcal{A}}\right\}, \ \forall \alpha \in \mathcal{A}$;

(P3) $Y^x_{\mathcal{A}} \in \mathbb{S}^{\mathcal{A}}_+$;

(P4) $Y^x_{\mathcal{A}}[\alpha, \beta] = 1 \iff x \in \alpha \cap \beta$;

(P5) if $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, *then* $Y^x_{\mathcal{A}}[\alpha_1, \beta_1] = Y^x_{\mathcal{A}}[\alpha_2, \beta_2]$;

(P6) If y is a row or column vector of $Y^x_{\mathcal{A}}$, and $\alpha_1, \ldots, \alpha_k$ are disjoint subsets of $\beta \in \mathcal{A}$, then

$$\sum_{i=1}^k y[\alpha_i] \leq y[\beta].$$

Moreover, equality holds if $\alpha_1, \ldots, \alpha_k$ partition β.

As with matrices in the lifted space of the operator Ψ defined in Section 2.2.1, $Y^x_{\mathcal{A}}$ has exponential size, and utilizing all its entries in an operator can be computationally costly. However, we can again work with polynomial-size submatrices of $Y^x_{\mathcal{A}}$ to yield a tractable lift-and-project algorithm, and now we have much more flexibility in choosing variables as we can now use sets in $\mathcal{A}$ that are not in $\mathcal{A}_n$.

2.5.2 Obstructions, walls and tiers

Suppose we are given a polytope $P := \{x \in [0,1]^n : Ax \leq b\}$, where $A \in \mathbb{R}^{m \times n}$ has non-negative entires and $b \in \mathbb{R}^m$ is positive. The BZ$'$ operator can be viewed as a two-step process. The first step is *refinement*. Given a vector $v \in \mathbb{R}^n$, let

$$\operatorname{supp}(v) := \{i \in [n] : v_i \neq 0\}$$

denote the *support* of v. Also, for every $i \in [m]$, let A^i denote the i^{th} row of A. If $O \subseteq [n]$ satisfies

- $O \subseteq \operatorname{supp}(A^i)$;
- $\sum_{j \in O} A^i_j > b_i$; and
- $|O| \leq k+1$ or $|O| \geq |\operatorname{supp}(A^i)| - (k+1)$

for some $i \in [m]$, then we call O a *k-small obstruction*. Let $\mathcal{O}_k$ denote the collection of all k-small obstructions of P (or more precisely, of the system $Ax \leq b$). Notice that, for every obstruction $O \in \mathcal{O}_k$, and integral vector $x \in P$, the inequality $\sum_{i \in O} x_i \leq |O| - 1$ holds. Thus,

$$\mathcal{O}_k(P) := \left\{ x \in P : \sum_{i \in O} x_i \leq |O| - 1, \ \forall O \in \mathcal{O}_k \right\}$$

is a relaxation of P_I that is potentially tighter than P.

The second step of the BZ$'$ operator is *lifting*. Before we give the details of this step, we need another intermediate set of indices, called *walls*. For every $k \geq 1$, we define

$$\mathcal{W}_k := \left\{ \bigcup_{i,j \in [\ell], i \neq j} (O_i \cap O_j) : O_1, \ldots, O_\ell \in \mathcal{O}_k, \ell \leq k+1 \right\} \cup \{\{1\}, \ldots, \{n\}\}.$$

That is, each subset of up to $(k+1)$ k-small obstructions generate a wall, which is the set of elements that appear in at least two of the given obstructions. We also ensure that the singleton sets of indices are walls. Next, we define the collection of *tiers*

$$\mathcal{T}_k := \left\{ S \subseteq [n] : \exists W_{i_1}, \ldots, W_{i_k} \in \mathcal{W}_k, S \subseteq \bigcup_{j=1}^{k} W_{i_j} \right\}.$$

That is, we define a set of indices S to be a tier if there exist k walls whose union contains S. Note that every subset of $[n]$ of size up to k is a tier. Obstructions, walls and tiers play an integral role in the generation of BZ$'$-relaxations.

2.5.3 The BZ' and BZ'_+ operators

Given a set $U \subseteq [n]$ and a nonnegative integer r, we define

$$U|_{<r} := \left\{ x \in \mathcal{F} : \sum_{i \in U} x_i \leq r - 1 \right\}.$$

We shall see that the elements in $\mathcal{A}$ that are being generated by BZ' all take the form $S|_1 \cap T|_0 \cap U|_{<r}$, where S, T, U are disjoint sets of indices. Next, we describe the lifting step of BZ'^k and BZ'^k_+, for every integer $k \geq 1$:

1. Define $\mathcal{A}'$ to be the set consisting of the following. For each tier $S \in \mathcal{T}_k$, include:

$$(S \setminus T)|_1 \cap T|_0,$$

for all $T \subseteq S$ such that $|T| \leq k$; and

$$(S \setminus (T \cup U))|_1 \cap T|_0 \cap U|_{<|U|-(k-|T|)},$$

for every $T, U \subseteq S$ such that $U \cap T = \emptyset, |T| < k$ and $|U| + |T| > k$. We say these variables (indexed by the above sets) are associated with the tier S.

2. Let $\tilde{\mathrm{BZ}}'^k(P)$ denote the set of matrices $Y \in \mathbb{S}^{\mathcal{A}'}$ that satisfy all of the following conditions:

(BZ′ 1) $Y[\mathcal{F}, \mathcal{F}] = 1$.

(BZ′ 2) For every column x of the matrix Y,

(i) $0 \leq x_\alpha \leq x_\mathcal{F}$, for all $\alpha \in \mathcal{A}'$.

(ii) $\hat{x}(x) \in K(\mathcal{O}_k(P))$.

(iii) $x_{i|_1} + x_{i|_0} = x_\mathcal{F}$, for every $i \in [n]$.

(iv) For each $\alpha \in \mathcal{A}'$ in the form of $S|_1 \cap T|_0$ impose the inequalities

$$x_{i|_1} \geq x_\alpha, \quad \forall i \in S; \tag{2.7}$$

$$x_{i|_0} \geq x_\alpha, \quad \forall i \in T; \tag{2.8}$$

$$x_\alpha + x_{(S \cup \{i\})|_1 \cap (T \setminus \{i\})|_0} = x_{S|_1 \cap (T \setminus \{i\})|_0}, \quad \forall i \in T; \tag{2.9}$$

$$\sum_{i \in S} x_{i|_1} + \sum_{i \in T} x_{i|_0} - x_\alpha \leq (|S| + |T| - 1) x_\mathcal{F}. \tag{2.10}$$

(v) For each $\alpha \in \mathcal{A}'$ in the form $S|_1 \cap T|_0 \cap U|_{<r}$, impose the inequalities

$$x_{i|_1} \geq x_\alpha, \quad \forall i \in S; \tag{2.11}$$

$$x_{i|_0} \geq x_\alpha, \quad \forall i \in T; \tag{2.12}$$

$$\sum_{i \in U} x_{i|_0} \geq (|U| - (r-1)) x_\alpha; \tag{2.13}$$

$$x_\alpha = x_{S|_1 \cap T|_0} - \sum_{U' \subseteq U, |U'| \geq r} x_{(S \cup U')|_1 \cap (T \cup (U \setminus U'))|_0}. \tag{2.14}$$

(BZ′ 3) For all $\alpha, \beta \in \mathcal{A}'$ such that $\alpha \cap \beta \cap P = \emptyset$, $Y[\alpha, \beta] = 0$.

(BZ′ 4) For all $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathcal{A}'$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

3. Define
$$\mathrm{BZ}'^k(P) := \left\{x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{BZ}}'^k(P), \hat{x}(Ye_{\mathcal{F}}) = \hat{x}\right\}.$$
and
$$\mathrm{BZ}'^k_+(P) := \left\{x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{BZ}}'^k_+(P), \hat{x}(Ye_{\mathcal{F}}) = \hat{x}\right\},$$
where $\tilde{\mathrm{BZ}}'^k_+(P) := \tilde{\mathrm{BZ}}'^k(P) \cap \mathbb{S}^{\mathcal{A}'}_+$.

Similar to the case of SA^k, BZ'^k can be seen as creating columns that correspond to sets that partition $\mathcal{F}$. While SA^k only generates a partition for each subset of up to k indices, BZ'^k does so for every tier, which is a much broader collection of indices. For a tier S up to size k, it does the same as SA^k and generates $2^{|S|}$ columns corresponding to all possible negations of indices in S. However, for S of size greater than k, it generates a "k-deep" partition of S: a column for $(S \setminus T)|_1 \cap T|_0$ for each $T \subseteq S$ of size up to k, and a column for $S|_{<|S|-k}$. In fact, given a tier S and $T \subseteq S$ such that $|T| < k$, BZ'^k generates a $(k - |T|)$-deep partition of this set for each $U \subseteq S \setminus T$ such that $|U| + |T| > k$. First, the column for

$$(S \setminus (T \cup U'))|_1 \cap (T \cup U')|_0 = (S \setminus (T \cup U))|_1 \cap T|_0 \cap (U \setminus U')|_1 \cap U'|_0$$

is generated for all $U' \subseteq U$ of size $\leq k - |T|$. Then BZ'^k also generates

$$(S \setminus (T \cup U))|_1 \cap T|_0 \cap U|_{<|U|-(k-|T|)}$$

to capture the remainder of the partition.

Since each singleton index set is a wall, we see that every index set of size up to k is a tier. Thus, $\mathcal{A}'$ contains $\mathcal{A}_k$, and it is not hard to see that $\mathrm{BZ}'^k(P) \subseteq \mathrm{SA}'^k(\mathcal{O}_k(P))$ in general. Furthermore, notice that in BZ', we have generated exponentially many variables, whereas in the original BZ only polynomially many are selected. The role of walls is also much more important in selecting the variables in BZ, which we have intentionally suppressed in BZ' to make our presentation and analysis more transparent. Some of the details of the relationships between these modified operators and the original Bienstock–Zuckerberg operators are given in the Appendix.

One of the main results Bienstock and Zuckerberg achieved with the BZ^k operator is on set covering problems. Given an inequality $a^\top x \geq a_0$ such that $a \geq 0$ and $a_0 > 0$, its *pitch* is defined to be the smallest positive integer j such that

$$S \subseteq \mathrm{supp}(a), |S| \geq j \Rightarrow a^\top \chi^S \geq a_0.$$

Also, let $\bar{e}$ denote the all-ones vector of suitable size. Then they showed the following powerful result in [BZ04]:

Theorem 10. *Suppose $P := \{x \in [0,1]^n : Ax \geq \bar{e}\}$ where A is a $0,1$ matrix. Then for every $k \geq 1$, every valid inequality of P_I that has pitch at most $k+1$ is valid for $\mathrm{BZ}^k(P)$.*

Note that if all coefficients of an inequality are integral and at most k, then the pitch of the inequality is no more than k. An important property of the Bienstock–Zuckerberg operators is that its performance can vary upon different algebraic descriptions of the given set P, even if they geometrically describe the same set. For instance, adding a redundant inequality to the system $Ax \leq b$ could make many more sets qualify as k-small obstructions. This could increase the dimension of the lifted set as more walls and tiers are generated, and as a result strengthen the operator. We provide examples that illustrate this phenomenon after the definition of Bienstock and Zuckerberg's original operators in the Appendix.

Chapter 3

Lower-Bound Analysis

As we saw in Chapter 2, one way to gain additional strength in devising a lift-and-project operator is to lift to a space of higher dimension, and obtain a potentially tighter formulation by using more variables, albeit at a computational cost. In this Chapter, we provide conditions on sets and higher dimensional liftings which do not lead to strong cuts. As a result, we show in some cases, BZ'^k performs no better than SA'^ℓ for some suitably chosen k and ℓ. We also prove a similar result relating the performance of the operators BZ'^k_+ and SA'^ℓ_+. We then look into the consequences of these findings, and present some lower-bound results on the matching and stable set relaxations.

3.1 Identifying unhelpful variables in the lifted space

3.1.1 A general template

Recall that $\mathcal{F} = \{0,1\}^n$, and $\mathcal{A}$ is the power set of $\mathcal{F}$. A common theme among all lift-and-project operators we have looked at so far is that their lifted spaces can all be interpreted as sets of matrices whose columns and rows are indexed by elements in $\mathcal{A}$. Moreover, they all impose a constraint in the tune of "each column of the matrix belongs to a certain set linked to P" (e.g. conditions (SA 2) and (BZ′ 2)). This provides a natural way of partitioning the constraints of a lift-and-project operator into two categories: those that are present (and identical) for every matrix column, and the remaining constraints that cannot be captured this way.

We say that a lift-and-project operator Γ is *admissible under the pair* (f, g) if there exist functions f, g such that all of the following properties hold:

(I1) Given a convex set $P \subseteq [0,1]^n$, Γ lifts P to a set of matrices $\tilde{\Gamma}(P) \subseteq \mathbb{R}^{\mathcal{S} \times \mathcal{S}'}$, such that

$$\mathcal{A}_1^+ \subseteq \mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{A}.$$

(I2) f is a *column constraint function* that maps elements in $\mathcal{A}$ to subsets of $\mathbb{R}^{\mathcal{S}}$, and g is a *cross-column constraint function* that maps sets contained in $[0,1]^n$ to sets of matrices in $\mathbb{R}^{\mathcal{S}\times\mathcal{S}'}$, such that

$$\tilde{\Gamma}(P) = \{Y \in g(P) : Ye_{S'} \in f(S'),\ \forall S' \in \mathcal{S}'\}.$$

Furthermore, f has the property that, for every pair of disjoint sets $S, T \in \mathcal{S}'$:

1. $f(S) \cup f(T) \subseteq f(S \cup T)$;
2. $f(S) = f(T)$ if $S \cap P = T \cap P$.

(I3)

$$\Gamma(P) := \left\{x \in \mathbb{R}^n : \exists Y \in \tilde{\Gamma}(P), Y[\mathcal{F},\mathcal{F}] = 1, \hat{x}(Ye_{\mathcal{F}}) = \hat{x}\right\}.$$

Note that the notion of admissible operators is extremely broad. Every lift-and-project operator Γ is admissible under the pair (f,g), if we let $g(P) := \tilde{\Gamma}(P)$ and $f(S) := \mathbb{R}^{\mathcal{S}}$ for all $S \in \mathcal{A}$ (i.e., we define f to be trivial and "shove" all constraints of Γ under g). However, as mentioned above, the intention of this definition is that we try to capture as much of Γ as possible with f by using it to describe the constraints Γ places on every column of the matrices in the lifted space, and only include the remaining constraints in g. Thus, we want f to be maximal, and g to be minimal in this sense. For instance, we can show that SA^k is admissible under the pair (f,g) where $f(S) := K(P \cap \mathrm{conv}(S))$, $\forall S \in \mathcal{A}$, and let $g(P)$ to be the set of matrices in $\mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_k}$ that satisfy (SA 3), (SA 4) and (SA 5). All named operators mentioned in this manuscript can be shown to be admissible in this fashion — using f to describe that each matrix column has to be in some lifted set determined by P, and let g capture the remaining constraints.

For many known operators, these "other" constraints placed by g are relaxations of the set theoretical properties (P5) and (P6) of $Y_{\mathcal{A}}^x$ listed in Proposition 9. For instance, (SA 5) is in place to make sure the variables in the linearized polynomial inequalities that would be identified in the original description of SA^k would in fact have the same value in all matrices in $\tilde{\mathrm{SA}}^k(P)$. Likewise, (SA 3) and (SA 4) are also needed to capture the relationship between the variables that would be established naturally in the original description with polynomial inequalities.

Furthermore, sometimes using matrices to describe the lifted space and assigning set theoretical meanings to their columns and rows have advantages over using linearized polynomial inequalities directly. For instance, we again consider the set

$$P := \left\{x \in [0,1]^n : \sum_{i=1}^n x_i \leq n - \frac{1}{2}\right\}.$$

We have seen that if we define

$$Q_j := \left\{x \in P : \sum_{i=1}^n x_i = j\right\},$$

for every $j \in \{0,1,\ldots,n\}$, then $P_I = \operatorname{conv}\left(\bigcup_{j=0}^n Q_j\right)$. However, if we attempt to construct a formulation by linearizing polynomial inequalities as in the original description of SA, one would need to linearize

$$\sum_{\substack{S,T:S\cup T=[n],\\ S\cap T=\emptyset,|S|=j}} \left(\prod_{i\in S} x_i\right)\left(\prod_{i\in T} 1-x_i\right)\left(\sum_{i=1}^n a_i x_i\right) \leq \sum_{\substack{S,T:S\cup T=[n],\\ S\cap T=\emptyset,|S|=j}} \left(\prod_{i\in S} x_i\right)\left(\prod_{i\in T} 1-x_i\right) a_0$$

to capture the constraints for Q_j, for all facets $\sum_{i=1}^n a_i x_i \leq a_0$ of P. Of course, when $j \approx \frac{n}{2}$, the above constraint would have exponentially many terms.

However, we can obtain an efficient lifted formulation by doing the following: for each $j \in \{0,1,\ldots,n\}$, define $R_j \in \mathcal{A}$ where

$$R_j = \left\{x \in \mathcal{F} : \sum_{i=1}^n x_i = j\right\}.$$

Then $Q_j = \operatorname{conv}(P \cap R_j)$ for every j. Let $\mathcal{S} = \{\mathcal{F}, R_0, R_1, \ldots, R_n\}$. We now define Φ to be the lift-and-project operator as follows:

1. Let $\tilde{\Phi}(P)$ denote the set of matrices $Y \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{S}}$ such that
   (i) $Y[\mathcal{F},\mathcal{F}] = 1$.
   (ii) $Ye_{R_j} \in K(P \cap \operatorname{conv}(R_j)),\ \forall j \in \{0,\ldots,n\}$.
   (iii) $Ye_{\mathcal{F}} = \sum_{j=0}^n Ye_{R_j}$.
2. Define
$$\Phi(P) = \left\{x \in \mathbb{R}^n : \exists Y \in \tilde{\Phi}(P), Ye_{\mathcal{F}} = \hat{x}\right\}.$$

Then it is not hard to see that $\Phi(P) = \operatorname{conv}\left(\bigcup_{i=0}^n Q_j\right)$ for any set $P \subseteq [0,1]^n$. Note that we used constraint (iii) to enforce that the entries in the matrix behave consistently with their corresponding set theoretical meanings. In particular, since $R_0,\ldots,R_n$ partition $\mathcal{F}$, we require that the columns indexed by the sets $R_0,\ldots,R_n$ sum up to that representing $\mathcal{F}$.

Thus, the following notions are helpful when we attempt to analyze g more systematically. First, given $\mathcal{S},\mathcal{S}' \subseteq \mathcal{A}$, we say that $\mathcal{S}'$ *refines* $\mathcal{S}$ if for all $S \in \mathcal{S}$, there exist mutually disjoint sets in $\mathcal{S}'$ that partition S. Equivalently, given $\mathcal{S} \subseteq \mathcal{A}$, let $Y^x_{\mathcal{S}}$ denote the $\mathcal{A} \times \mathcal{S}$ submatrix of $Y^x_{\mathcal{A}}$ consisting of the columns indexed by sets in $\mathcal{S}$. Then $\mathcal{S}'$ refines $\mathcal{S}$ if and only if every column $Y^x_{\mathcal{S}}$ is contained in the cone generated by the column vectors of $Y^x_{\mathcal{S}'}$, for every $x \in \mathcal{F}$. For instance, $\mathcal{S}'$ refines $\mathcal{S}$ whenever $\mathcal{S} \subseteq \mathcal{S}'$. Another example is that $\mathcal{A}_k$ refines $\mathcal{A}_\ell$ whenever $k \geq \ell$. Note that the notion of refinement is transitive — if $\mathcal{S}'$ refines $\mathcal{S}$ and $\mathcal{S}''$ refines $\mathcal{S}'$, then $\mathcal{S}''$ refines $\mathcal{S}$.

Next, given $Y_1 \in \mathbb{R}^{\mathcal{S}_1 \times \mathcal{S}_1'}$ and $Y_2 \in \mathbb{R}^{\mathcal{S}_2 \times \mathcal{S}_2'}$ where $\mathcal{S}_1, \mathcal{S}_1', \mathcal{S}_2, \mathcal{S}_2' \subseteq \mathcal{A}$, we say that Y_1 and Y_2 are *consistent* if

$$\sum_{i=1}^{k} Y_1[S_{1i}, S_{1i}'] = \sum_{i=1}^{\ell} Y_2[S_{2i}, S_{2i}']$$

whenever $\{S_{1i} \cap S_{1i}' : i \in [k]\}$ and $\{S_{2i} \cap S_{2i}' : i \in [\ell]\}$ are both collections of mutually disjoint sets such that $\bigcup_{i=1}^{k} (S_{1i} \cap S_{1i}') = \bigcup_{i=1}^{\ell} (S_{2i} \cap S_{2i}')$. We will extend the above definition to determine whether two vectors are consistent with each other (viewing those vectors as $n \times 1$ matrices), and whether a matrix and a vector are consistent with each other.

We remark that our notion of consistency is closely related to some of similar notions by Zuckerberg [Zuc03]. Given $\mathcal{S} = \{S_1, \ldots, S_\ell\} \subseteq \mathcal{A}$, let $\mathcal{A}(\mathcal{S})$ denote the collection of subsets of $\mathcal{A}$ that can be obtained from taking unions of sets in the collection

$$\left\{ \left(\bigcap_{i \in I} S_i \right) \cap \left(\bigcap_{i \notin I} \mathcal{A} \setminus S_i \right) : I \subseteq [\ell] \right\}.$$

Equivalently, $\mathcal{A}(\mathcal{S})$ is the smallest collection of subsets of $\mathcal{A}$ that contains $\mathcal{S}$, and is closed under unions and intersections. Then, given a vector $y \in \mathbb{R}^{\mathcal{S}}$, Zuckerberg defines y to be $\mathcal{A}(\mathcal{S})$ *signed measure consistent* if there exists $z \in \mathbb{R}^{\mathcal{A}(\mathcal{S})}$ that is consistent (under our definition) with both y and itself. He also looked into a more restrictive notion of consistency, and defined a vector y to be $\mathcal{A}(\mathcal{S})$ *measure consistent* if there exists a nonnegative vector $z \in \mathbb{R}^{\mathcal{A}(\mathcal{S})}$ that witnesses y's signed measure consistency.

Next, given $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{A}$, we say that a matrix $Y \in \mathbb{R}^{\mathcal{S} \times \mathcal{S}'}$ is *overall measure consistent* (OMC) if it is consistent with itself. All matrices in the lifted spaces of all variants of $\text{SA}^k, \text{Las}^k$ and BZ^k satisfy (OMC), for all $k \geq 1$. One notable observation is that, in the case of vectors, if $\mathcal{S}'$ refines $\mathcal{S}$, and $x \in \mathbb{R}^{\mathcal{S}'}$ satisfies (OMC), then there is a unique $y \in \mathbb{R}^{\mathcal{S}}$ that is consistent with x.

Finally, we are ready to formally describe some variables that we will show are unhelpful in the lifted space under this framework. Given an admissible operator Γ and $P \subseteq [0,1]^n$, suppose $\tilde{\Gamma}(P) \subseteq \mathbb{R}^{\mathcal{S} \times \mathcal{S}'}$. We say that $S' \in \mathcal{S}'$ is *P-useless* if there is a collection $T = \{T_1, \ldots, T_k\} \subseteq \mathcal{S}'$ such that

1. $S' \in T$ and $\bigcup_{i=1}^{k} T_i \in \mathcal{S}' \setminus T$;
2. there exists $\ell \in [k]$ such that $P \cap \text{conv}(T_j) = \emptyset, \ \forall j \in [k], j \neq \ell$.

What does it mean for variables to be P-useless? For example, let $T = \{T_1, \ldots, T_k\} \subseteq \mathcal{S}'$ be a collection of variables such that $R := \bigcup_{i=1}^{k} T_i$ is itself a variable in $\mathcal{S}'$, and $R \notin T$. Further suppose there exists a unique $\ell \in [k]$ such that $P \cap \text{conv}(T_j) = \emptyset, \ \forall j \neq \ell$. This means that in this setting, each of the T_j has the set theoretical meaning of the empty set. Thus, we do not lose any points when projecting $\tilde{\Gamma}(P)$ to $\Gamma(P)$ if we assume that the matrix column indexed by T_j is the vector of all zeros. Moreover, since $R = \bigcup_{i=1}^{k} T_i$, we see that $R \cap P = T_\ell \cap P$, and so the variables R and T_ℓ can be interpreted as having the same set theoretical meaning in the formulation, and we can deem T_ℓ redundant. Therefore, in this case, $T_1, \ldots, T_k$ are all P-useless.

With the notion of P-useless variables, we can show the following:

Proposition 11. *Let Γ_1, Γ_2 be two lift-and-project operators that are admissible under the pairs (f_1, g_1) and (f_2, g_2), respectively. Let $P \subseteq [0,1]^n$, and suppose $\tilde{\Gamma}_1(P) \in \mathbb{R}^{\mathcal{S}_1 \times \mathcal{S}_1'}$ and $\tilde{\Gamma}_2(P) \in \mathbb{R}^{\mathcal{S}_2 \times \mathcal{S}_2'}$. Also, let U be a set of P-useless variables in $\mathcal{S}_2'$. Further suppose that the following conditions hold:*

(i) Every matrix in $\tilde{\Gamma}_1(P)$ satisfies (OMC).

(ii) $\{S \cap S' : S \in \mathcal{S}_1, S' \in \mathcal{S}_1'\}$ refines $\{S \cap S' : S \in \mathcal{S}_2 \setminus U, S' \in \mathcal{S}_2' \setminus U\}$, and $\mathcal{S}_1'$ refines $\mathcal{S}_2' \setminus U$.

(iii) Let $Y \in \tilde{\Gamma}_1(P)$, and $S \in \mathcal{S}_2'$. If $y \in \mathbb{R}^{\mathcal{S}_2 \times \{S\}}$ is consistent with Y, then $y \in f_2(S)$.

(iv) If $Y_1 \in g_1(P)$ and $Y_2 \in \mathbb{R}^{\mathcal{S}_2 \times \mathcal{S}_2'}$ is consistent with Y_1, then $Y_2 \in g_2(P)$.

Then, $\Gamma_1(P) \subseteq \Gamma_2(P)$.

Intuitively, the above conditions are needed so that given a point $x \in \Gamma_1(P)$ and its certificate matrix $Y \in \tilde{\Gamma}(P)$, we know enough structure about the entries and set theoretic meanings of Y to construct a matrix in $\mathbb{R}^{(\mathcal{S}_2 \setminus U) \times (\mathcal{S}_2' \setminus U)}$ that is consistent with Y. Then using the fact that the variables in U are P-useless, we can extend this to a matrix in $\mathbb{R}^{\mathcal{S}_2 \times \mathcal{S}_2'}$ that certifies x's membership in $\Gamma_2(P)$. Also, for $y \in \mathbb{R}^{\mathcal{S}_2 \times \{S\}}$, we are referring to a vector with $|\mathcal{S}_2|$ entries that are indexed by elements of $\{T \cap S : T \in \mathcal{S}_2\}$. Since we will be talking about whether y is consistent with another vector or matrix, we will need to specify not only the entries of y, but also the sets in $\mathcal{A}$ these entries correspond to.

Now we are ready to prove Proposition 11.

Proof of Proposition 11. Suppose $x \in \Gamma_1(P)$. Let $Y \in \mathbb{R}^{\mathcal{S}_1 \times \mathcal{S}_1'}$ be a matrix in $\tilde{\Gamma}(P)$ such that $\hat{x}(Ye_{\mathcal{F}}) = \hat{x}$. First, we construct an intermediate matrix $Y' \in \mathbb{R}^{(\mathcal{S}_2 \setminus U) \times (\mathcal{S}_2' \setminus U)}$. For each $\alpha \in \mathcal{S}_2 \setminus U$ and $\beta \in \mathcal{S}_2' \setminus U$, we know (due to (ii)) that there exists a set of ordered pairs

$$I_{\alpha,\beta} \subseteq \{(S, S') : S \in \mathcal{S}_1, S' \in \mathcal{S}_1'\}$$

such that the collection $\{S \cap S' : (S, S') \in I_{\alpha,\beta}\}$ partitions $\alpha \cap \beta$. Next, we construct Y' such that

$$Y'[\alpha, \beta] := \sum_{(S,S') \in I_{\alpha,\beta}} Y[S, S'].$$

Note that by (OMC), the entry $Y'[\alpha, \beta]$ is invariant under the choice of $I_{\alpha,\beta}$. Also, since $\{(\mathcal{F}, \mathcal{F})\}$ is a valid candidate for $I_{\mathcal{F},\mathcal{F}}$, we see that $Y'[\mathcal{F}, \mathcal{F}] = Y[\mathcal{F}, \mathcal{F}] = 1$, and $\hat{x}(Y'e_{\mathcal{F}}) = \hat{x}(Ye_{\mathcal{F}}) = \hat{x}$.

Next, we construct $Y'' \in \tilde{\Gamma}_2(P)$ from Y'. Given $\alpha \in U$ such that $P \cap \operatorname{conv}(\alpha) \neq \emptyset$, there exists a set $h(\alpha) \in \mathcal{S}_2' \setminus U$ such that $\operatorname{conv}(\alpha) \cap P = \operatorname{conv}(h(\alpha) \cap P)$. Note that $h(\alpha)$ may not be unique, but any eligible choice would do.

Next, we define $V^1 \in \mathbb{R}^{(\mathcal{S}_2 \setminus U) \times \mathcal{S}_2}$ as follows:

$$V^1(e_\alpha) := \begin{cases} e_\alpha & \text{if } \alpha \in \mathcal{S}_2 \setminus U; \\ e_{h(\alpha)} & \text{if } \alpha \in U \text{ and } \operatorname{conv}(\alpha) \cap P \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we define $V^2 \in \mathbb{R}^{(\mathcal{S}_2' \setminus U) \times \mathcal{S}_2'}$ as follows:

$$V^2(e_\alpha) := \begin{cases} e_\alpha & \text{if } \alpha \in \mathcal{S}_2' \setminus U; \\ e_{h(\alpha)} & \text{if } \alpha \in U \text{ and } \operatorname{conv}(\alpha) \cap P \neq \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

We show that $Y'' := V^1 Y' (V^2)^\top \in \tilde{\Gamma}_2(P)$. Since our map from Y to Y'' preserves (OMC), Y'' is consistent with Y, and thus by (iv) it satisfies all constraints in g_2. Also, by (iii) it satisfies all column constraints in f_2 as well. Thus, $Y'' \in \tilde{\Gamma}_2(P)$. Since $\hat{x}(Y'' e_{\mathcal{F}}) = \hat{x}$, we are finished. $\square$

We note that, in some cases, we can relate the performance of two lift-and-project operators by assuming a condition slightly weaker than (OMC). Given a matrix $Y \in \mathbb{R}^{\mathcal{S} \times \mathcal{S}'}$, where $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{A}$, we say that it is *row and column measure consistent* (RCMC) if every column and row of Y satisfies (OMC). As is apparent in its definition, (RCMC) is less restrictive than (OMC). In fact, it is satisfied by all matrices in the lifted space of all named lift-and-project operators mentioned in this thesis. Then, we have the following result that is the (RCMC) counterpart of Proposition 11:

Proposition 12. *Let Γ_1, Γ_2 be lift-and-project operators that are admissible under the pairs (f_1, g_1) and (f_2, g_2), respectively. Also, let $P \subseteq [0,1]^n$, and suppose $\tilde{\Gamma}_1(P) \in \mathbb{R}^{\mathcal{S}_1 \times \mathcal{S}_1'}$ and $\tilde{\Gamma}_2(P) \in \mathbb{R}^{\mathcal{S}_2 \times \mathcal{S}_2'}$. Also, let U be a set of P-useless variables in $\mathcal{S}_2'$. Further suppose that all of the following conditions hold:*

- *(i) Every matrix in $\tilde{\Gamma}_1(P)$ satisfies (RCMC).*
- *(ii) $\mathcal{S}_1$ refines $\mathcal{S}_2 \setminus U$, and $\mathcal{S}_1'$ refines $\mathcal{S}_2' \setminus U$.*
- *(iii) Let $S \in \mathcal{S}_2'$. If $x \in \mathbb{R}^{\mathcal{S}_1 \times \{S\}}$ is contained in $f_1(S)$ and $y \in \mathbb{R}^{\mathcal{S}_2 \times \{S\}}$ is consistent with x, then $y \in f_2(S)$.*
- *(iv) If $Y_1 \in g_1(P)$ and $Y_2 \in \mathbb{R}^{\mathcal{S}_2 \times \mathcal{S}_2'}$ is consistent with Y_1, then $Y_2 \in g_2(P)$.*

Then, $\Gamma_1(P) \subseteq \Gamma_2(P)$.

Proof. The result can be shown by following the same outline as in the proof of Proposition 11. Suppose $x \in \Gamma_1(P)$ and $Y \in \mathbb{R}^{\mathcal{S}_1 \times \mathcal{S}_1'}$ is a certificate matrix for x. For each $\alpha \in \mathcal{S}_2 \setminus U$, define I_α to be a collection of sets in $\mathcal{S}_1$ that partitions α. Since $\mathcal{S}_1$ refines $\mathcal{S}_2 \setminus U$, such a collection must exist. Likewise, for all $\alpha \in \mathcal{S}_2' \setminus U$, we define I_α' to be a collection of sets in $\mathcal{S}_1'$ that partitions α.

Next, we define $Y' \in \mathbb{R}^{(\mathcal{S}_2 \setminus U) \times (\mathcal{S}_2' \setminus U)}$ such that

$$Y'[\alpha, \beta] := \sum_{S \in I_\alpha, S' \in I_\beta'} Y[S, S'].$$

Since Y satisfies (RCMC), $Y'[\alpha, \beta]$ is invariant under the choices of I_α and I_β'. From here on, we can define V_1, V_2 and $Y'' \in \mathbb{R}^{\mathcal{S}_2 \times \mathcal{S}_2'}$ as in the proof of Proposition 11, and apply the same reasoning therein to show that it is in $\tilde{\Gamma}_2(P)$. Now since $\hat{x}(Y'' e_{\mathcal{F}}) = \hat{x}(Y e_{\mathcal{F}}) = \hat{x}$, we conclude that $x \in \Gamma_2(P)$. $\square$

3.1.2 Relating $\mathrm{BZ}', \mathrm{BZ}'_+$ with $\mathrm{SA}', \mathrm{SA}'_+$

Next, we look into several applications of Proposition 11 and 12. First, they can be applied to relate the integrality gaps between relaxations. Given two operators Γ_1, Γ_2 and a set P such that $\Gamma_1(P) \subseteq \Gamma_2(P)$, it is apparent that the integrality gap of $\Gamma_1(P)$ is no more than that of $\Gamma_2(P)$ with respect to any chosen direction. We will formally define integrality gap and discuss these results in more depth in Chapter 7.

Next, we relate the performance of BZ' and SA' under some suitable conditions. First, we define a tier $S \in \mathcal{T}_k$ to be P-useless if all variables associated with S are P-useless. Then we have the following:

Theorem 13. *Suppose there exists $\ell \in [n]$ such that all tiers S generated by BZ'^k of size greater than ℓ are P-useless. Then*

$$\mathrm{BZ}'^k(P) \supseteq \mathrm{SA}'^{2\ell}(\mathcal{O}_k(P)).$$

Proof. Let $\Gamma_1 = \mathrm{SA}'^{2\ell}(\mathcal{O}_k(\cdot))$ and $\Gamma_2 = \mathrm{BZ}'^k(\cdot)$. We prove our assertion by checking all conditions listed in Proposition 11.

First of all, all matrices in the lifted space of $\mathrm{SA}'^{2\ell}$ satisfy (OMC). Next, since $\mathcal{S}_1 = \mathcal{A}_1^+$ and $\mathcal{S}_1' = \mathcal{A}_{2\ell}$, we see that $\{S \cap S' : S \in \mathcal{S}_1, S' \in \mathcal{S}_1'\}$ refines $\mathcal{A}_{2\ell}$. On the other hand, since every tier of size greater than ℓ is P-useless, we see that $\mathcal{A}_\ell$ refines both $\mathcal{S}_2 \setminus U$ and $\mathcal{S}_2' \setminus U$. Thus, $\mathcal{A}_{2\ell}$ refines $\{S \cap S' : S \in \mathcal{S}_2 \setminus U, S' \in \mathcal{S}_2' \setminus U\}$. Also, it is apparent that $\mathcal{S}_1' = \mathcal{A}_{2\ell}$ refines $\mathcal{S}_2' \setminus U$, so (ii) holds.

For (iii), we let $f_1(S) = K(\mathcal{O}_k(P) \cap \mathrm{conv}(S)),\ \forall S \in \mathcal{A}$, and

$$f_2(S) := \left\{ y \in \mathbb{R}^{\mathcal{S}_2'} : \hat{x}(y) \in K(\mathcal{O}_k(P) \cap \mathrm{conv}(S)), y \text{ satisfies } (\mathrm{BZ}'\,2) \right\}.$$

Note that all conditions in (BZ$'$ 2) are relaxations of constraints in (P5) and (P6) in Proposition 9, and thus are implied by (OMC). Let $Y \in \tilde{\mathrm{SA}}'^{2\ell}(\mathcal{O}_k(P))$, and Y'' be the matrix obtained from the construction in the proof of Proposition 11. Since Y satisfies (OMC) and Y'' is consistent with Y, the columns of Y'' must satisfy (BZ$'$ 2).

To check (iv), we see that $g_2(P)$ would be the set of matrices in the lifted space that satisfy (BZ$'$ 3) and (BZ$'$ 4). It is easy to see that (BZ$'$ 4) is implied by (OMC). For (BZ$'$ 3), suppose $S \in \mathcal{S}_2, S' \in \mathcal{S}_2'$, and $S \cap S' \cap \mathcal{O}_k(P) = \emptyset$. If $Y''[S, S'] \neq 0$, then we know that $P \cap \mathrm{conv}(S) \neq \emptyset$ and $P \cap \mathrm{conv}(S') \neq \emptyset$, by the construction of Y''. Thus, define $\alpha := S$ if $S \notin U$, and $\alpha := h(S)$ if $S \in U$. Likewise, define $\beta := S'$ if $S' \notin U$, and $\beta := h(S')$ if $S' \in U$. In all cases, we have now obtained $\alpha \in \mathcal{S}_2 \setminus U, \beta \in \mathcal{S}_2' \setminus U$ such that $Y''[\alpha, \beta] = Y''[S, S']$.

Since
$$Y''[\alpha, \beta] = Y'[\alpha, \beta] = \sum_{(T,T') \in I_{\alpha,\beta}} Y[T, T'],$$
we obtain $T \in \mathcal{A}_1^+, T' \in \mathcal{A}_k$ such that $Y[T, T'] \neq 0$. Then by (SA$'$ 4), $T \cap T' \cap P \neq \emptyset$. This implies that $S \cap S' \cap P \neq \emptyset$, and so (BZ$'$ 3) holds. □

We remark that, with a little more care and using the same observation as in the proof of Proposition 3, one can slightly sharpen Theorem 13 and show that $\mathrm{SA}^{2\ell}(\mathcal{O}_k(P)) \subseteq \mathrm{BZ}'^k(P)$ under these assumptions.

Next, we turn to relate the performances of BZ'_+ and SA'_+. Observe that in Proposition 12, in the special case of comparing two lift-and-project operators whose lifted spaces are both square matrices (i.e. $\mathcal{S}_1 = \mathcal{S}_1'$ and $\mathcal{S}_2 = \mathcal{S}_2'$), the construction of Y' and Y'' preserves positive semidefiniteness of Y. Thus, this framework can be applied even when g_1 and g_2 enforce positive semidefiniteness constraints in their respective lifted spaces. The following is an illustration of such an application:

Theorem 14. *Suppose there exists $\ell \in [n]$ such that all tiers S generated by BZ'^k_+ of size greater than ℓ are P-useless. Then*
$$\mathrm{BZ}'^k_+(P) \supseteq \mathrm{SA}'^\ell_+(\mathcal{O}_k(P)).$$

Proof. We prove our claim by verifying the conditions in Proposition 12. First, every matrix in the lifted space of SA'^ℓ_+ satisfies (OMC), which implies (RCMC). Next, since $\mathcal{S}_1 = \mathcal{S}_1' = \mathcal{A}_\ell$ and every tier of BZ'^k_+ that is not useless has size at most ℓ, we see that (ii) holds as well.

For (iii), note that we can let
$$f_1(S) = \left\{ y \in \mathbb{R}^{\mathcal{S}_1'} : \hat{x}(y) \in K(P \cap \mathrm{conv}(S)), y \text{ satisfies (OMC)} \right\},$$
and
$$f_2(S) = \left\{ y \in \mathbb{R}^{\mathcal{S}_2'} : \hat{x}(y) \in K(P \cap \mathrm{conv}(S)), y \text{ satisfies (BZ}' \text{ 2)} \right\}.$$

As mentioned before, all conditions in (BZ$'$ 2) are implied by (OMC) constraints and the fact that $\mathcal{A}_\ell$ refines $\mathcal{S}_2$. Thus, (iii) is satisfied.

For (iv), we see that $g_2(P)$ would be the set of matrices in $\mathbb{S}_+^{\mathcal{S}_2}$ that satisfy (BZ$'$ 3) and (BZ$'$ 4). It is easy to see that (BZ$'$ 4) is implied by (OMC). Also, (BZ$'$ 3) is implied by (SA$'_+$ 4). Thus, we are finished. □

3.2 Applications to matching and stable set relaxations

Next, we look into the lift-and-project ranks of a number of relaxations that arise from combinatorial optimization problems. Given a lift-and-project operator Γ and polytope P, the Γ-*rank of* P is defined to be the smallest integer k such that $\Gamma^k(P) = P_I$. The notion of rank gives us a measure of how close P is to P_I with respect to Γ. Moreover, it is useful when comparing the performances of different operators.

First, we look into the matching problem of graphs. Given a simple, undirected graph $G = (V, E)$, we define

$$MT(G) := \left\{ x \in [0,1]^E : \sum_{j:\{i,j\} \in E} x_{ij} \leq 1, \ \forall i \in V \right\}.$$

Then $MT(G)_I$ is the *matching polytope* of G, and is exactly the convex hull of the incidence vectors of the matchings of G.

While there exist efficient algorithms that solve the matching problem (e.g. Edmonds' seminal blossom algorithm [Edm65]), many lift-and-project operators have been shown to require exponential time to compute the matching polytope starting with $MT(G)$. In particular, $MT(K_{2n+1})$ is known to have LS$_+$-rank n [ST99] and BCC-rank n^2 [ABN04]. More recently, Mathieu and Sinclair [MS09] showed that the SA-rank of $MT(K_{2n+1})$ is $2n-1$. Using their result and Theorem 13, we can show that this polytope is also a bad instance for BZ$'$.

Theorem 15. *Let $G = K_{2n+1}$ for some integer $n \geq 1$. Then the* BZ$'$*-rank of $MT(G)$ is at least $\left\lceil \sqrt{2n} - \frac{3}{2} \right\rceil$.*

Proof. Let $P = MT(G)$. We first identify the tiers generated by BZ$'^k$ that are P-useless. Observe that a set $O \subseteq E$ is a k-small obstruction generated by BZ$'^k$ if there is a vertex that is incident with all edges in O, and that $|O| \leq k+1$ or $|O| \geq 2n-k$. Now suppose $W \in \mathcal{W}_k$ is a wall, and let $\{e_1, e_2, \ldots, e_p\}$ be a maximum cardinality matching contained in W. Notice that for $e_1 = \{u_1, v_1\}$ to be in W, it has to be contained in at least 2 obstructions, and each of these obstructions has to originate from the u_1- or v_1-constraint in the formulation of $MT(G)$. Now suppose $e_2 = \{u_2, v_2\}$. By the same logic, we deduce that the obstructions that allow e_2 to be in W have to be different from those that enabled e_1 to be in W. Since each wall is generated by at most $k+1$ obstructions, we see that $p \leq \frac{k+1}{2}$. Therefore, for every tier $S \in \mathcal{T}_k$ (which has to be contained in the union of k walls), the maximum cardinality matching contained in S has at most $\frac{k(k+1)}{2}$ edges.

Hence, if $|S| > \frac{k(k+1)}{2} + k$, then $S \setminus T$ is not a matching for any set $T \subseteq S$ of size up to k, which implies $(S \setminus T)|_1 \cap T|_0 \cap P = \emptyset$. Thus, the only variables α associated with S such that $\alpha \cap P \neq \emptyset$ take the form $\alpha = (S \setminus (T \cup U))|_1 \cap T|_0 \cap U|_{<|U|-(k-|T|)}$. In this case, it is not hard to see that $\alpha \cap P = (S \setminus (T \cup U))|_1 \cap T|_0 \cap P$. Since BZ'^k does generate the tier $S \setminus U$, we see that the variable $(S \setminus (T \cup U))|_1 \cap T|_0$ is present. Thus, all variables associated with S are P-useless. By Proposition 3 and the Mathieu–Sinclair result, we see that the SA$'$-rank of P is at least $2n-2$. Thus, by Theorem 13, for $\mathrm{BZ}'^k(P)$ to be equal to P_I, we need $2\left(\frac{k(k+1)}{2} + k\right) \geq 2n-2$, which is implied by $k \geq \sqrt{2n} - \frac{3}{2}$. $\square$

The best upper bound we know for the BZ$'$-rank of $MT(K_{2n+1})$ is $2n-1$ (due to Mathieu and Sinclair's result, and the fact that BZ'^k dominates SA^k). In contrast, we shall see in Chapter 4 that the BZ'_+-rank of $MT(K_{2n+1})$ is at most $\sqrt{2n}$, and utilizing positive semidefiniteness allows us to prove a much better upper bound in this case.

We next look at the stable set problem of graphs. Given a graph $G = (V, E)$, its *fractional stable set polytope* is defined to be

$$FRAC(G) := \left\{ x \in [0,1]^V : x_i + x_j \leq 1, \ \forall \{i,j\} \in E \right\}.$$

That is, $FRAC(G)$ is exactly the feasible region of the LP (1.5) mentioned in Section 1.2. Then the *stable set polytope* $STAB(G) := FRAC(G)_I$ is precisely the convex hull of incidence vectors of stable sets of G. Since there is a bijection between the set of matchings in G and the set of stable sets in its line graph $L(G)$, we obtain the next result readily from Theorem 15.

Corollary 16. *Let G be the line graph of K_{2n+1}. Then the* BZ$'$*-rank of $FRAC(G)$ is at least $\left\lceil \sqrt{2n} - \frac{3}{2} \right\rceil$.*

Proof. First, it is not hard to see that $MT(H) \subseteq FRAC(L(H))$, for every graph H. Also, it is apparent from the definition of BZ$'$ that $\mathcal{O}_k(P) \subseteq \mathcal{O}_k(P')$ implies $\mathrm{BZ}'^k(P) \subseteq \mathrm{BZ}'^k(P')$. Since the collection of k-small obstructions of $FRAC(G)$ is exactly the set of edges of G for all $k \geq 1$, we see that $FRAC(G) = \mathcal{O}_k(FRAC(G))$. Therefore,

$$\mathcal{O}_k(MT(K_{2n+1}) \subseteq MT(K_{2n+1}) \subseteq FRAC(G) = \mathcal{O}_k(FRAC(G)),$$

which implies that the BZ$'$-rank of $FRAC(G)$ is at least that of $MT(K_{2n+1})$. $\square$

Thus, we obtain from Corollary 16 a family of graphs on n vertices whose fractional stable set polytope has BZ$'$-rank $\Omega(n^{1/4})$.

We next turn to the complete graph $G := K_n$. Starting with the seminal paper of Lovász and Schrijver [LS91], it was noticed quite early that LS_0, LS and even SA perform poorly on $FRAC(G)$ when $G = K_n$. We show that this is also true for SA' and BZ'. First, we have the following:

Proposition 17. *For every graph G and every integer $k \geq 1$,*

$$\frac{1}{k+2}\bar{e} \in \mathrm{SA}'^k(FRAC(G)).$$

In particular, when $G = K_n$, the SA'-rank of $FRAC(G)$ is $n-2$, for every $n \geq 2$.

Proof. We define the matrix $Y \in \mathbb{R}^{\mathcal{A}_k \times \mathcal{A}_1^+}$, where

$$Y[\alpha, \beta] = \begin{cases} \frac{1}{k+2} & \text{if } \alpha \cap \beta = i|_1 \cap T|_0 \text{ for some } i \in [n], T \subseteq [n] \\ \frac{k+2-|T|}{k+2} & \text{if } \alpha \cap \beta = T|_0 \text{ for some } i \in [n], T \subseteq [n] \\ 0 & \text{otherwise.} \end{cases}$$

Next, we prove that $Y \in \tilde{\mathrm{SA}}'^k(FRAC(G))$. Since $\mathcal{F} = \emptyset|_0$, $Y[\mathcal{F}, \mathcal{F}] = 1$, and (SA 1) holds. Next, we see that for all $\alpha \in \mathcal{A}_k$, Ye_α is either $Y[\mathcal{F}, \alpha](e_0 + e_i)$ for some $i \in [n]$, or is less than or equal to $\begin{pmatrix} Y[\mathcal{F}, \alpha] \\ \frac{Y[\mathcal{F}, \alpha]}{2}\bar{e} \end{pmatrix}$. Thus, (SA 2) holds. (SA 3) and (SA 5) also hold by the construction of Y. Finally (SA' 4) is satisfied because whenever $Y[\alpha, \beta] > 0$, $\alpha \cap \beta$ contains at most one positive index, and thus has nonempty intersection with $FRAC(G)$. Thus, it follows that $\frac{1}{k-2}\bar{e} \in \mathrm{SA}'^k(FRAC(G))$.

For the second part of the claim, suppose $G = K_n$ for some $n \geq 2$. Since it is apparent that $\frac{1}{k-2}\bar{e} \notin STAB(G)$ whenever $k - 2 < n$, the SA'-rank of $FRAC(G)$ is at least $n-2$. On the other hand, it was shown in [LS91] that the LS_0-rank of $FRAC(G)$ is exactly $n-2$. Since SA' dominates LS_0, we see that the SA'-rank of $FRAC(G)$ is also exactly $n-2$. $\square$

Proposition 17 extends Lovász and Schrijver's result (Lemma 2.7 in [LS91]) which shows that $\frac{1}{k+2}\bar{e} \in LS^k(FRAC(G))$ for every $k \geq 1$. Now, we are ready to bound the BZ'-rank of $FRAC(K_n)$.

Theorem 18. *Suppose $G = K_n$ for some integer $n \geq 3$. Then the BZ'-rank of $FRAC(G)$ is between $\lceil \frac{n}{2} \rceil - 2$ and $\lceil \frac{n+1}{2} \rceil$. The same bounds apply to the BZ-rank.*

Proof. Let $P := FRAC(K_n)$. We first prove the lower bound, by showing that all tiers generated by BZ'^k of size greater than $k+1$ are P-useless. This, combined with Theorem 13, implies that $\mathrm{BZ}'^k(P) \supseteq \mathrm{SA}'^{2k+2}(\mathcal{O}_k(P))$.

Since the set of k-small obstructions of $FRAC(K_n)$ is exactly E for every $k \geq 1$, we see that $\mathcal{W}_k = \{W \subseteq [n] : |W| \leq k+1\}$ and $\mathcal{T}_k = \{S \subseteq [n] : |S| \leq k(k+1)\}$. Now if S is any tier of size at least $k+2$, we see that $(S \setminus T)|_1 \cap T|_0 \cap P = \emptyset$ for all $T \subseteq S$ such that $|T| \leq k$. This is because in such cases $|S \setminus T| \geq 2$, and there are no points in P which contain at least two 1s. Thus, the only variables α associated with S such that $\alpha \cap P \neq \emptyset$ take the form $(S \setminus (T \cup U))|_1 \cap T|_0 \cap U|_{<|U|-(k-|T|)}$. However, in this case we know that $S \setminus (T \cup U)$ has size 0 or 1, and thus $\alpha \cap P$ is equal to either $\mathcal{F} \cap P$ or $i|_1 \cap P$ for some $i \in [n]$. Therefore, all variables associated with S are P-useless, and so the tier S is P-useless.

Also, observe that $P = \mathcal{O}_k(P)$ for any $k \geq 1$, and from Proposition 17 we know that P has SA$'$-rank $n-2$. Thus, it follows that the BZ$'$-rank of P is at least $\lceil \frac{n}{2} \rceil - 2$. Moreover, since BZ$'$ dominates BZ$''$, it follows from Proposition 85 that $FRAC(G)$ has BZ$'$-rank at most $\lceil \frac{n+1}{2} \rceil$.

Finally, we turn to the BZ-rank of $FRAC(G)$. Again, $\mathcal{O}_k = E$ for all $k \geq 1$. Therefore, in this case the conditions (BZ 3) and (BZ$'$ 3) are equivalent. Since each vertex is incident with at least 2 edges, BZ does generate all the singleton sets as walls. Thus, the BZ- and BZ$'$-rank of $FRAC(G)$ must coincide. $\square$

Thus, we see that, as with all other popular polyhedral lift-and-project operators, BZ$'$ (which is already stronger than BZ) performs poorly on the fractional stable set polytope of complete graphs. On the other hand, Lovász and Schrijver [LS91] showed that $\text{LS}_+(FRAC(G))$ is contained in Lovász's theta body [Lov79] for every graph G. This implies that the LS_+-rank of $FRAC(G)$ is 1 for every perfect graph G (see [LS91, LT03] for more examples of graphs whose fractional stable set polytopes have low LS_+-rank). In particular, it follows that $FRAC(K_n)$ has rank 1 with respect to all lift-and-project operators we have discussed that utilize positive semidefiniteness. Also, Bienstock and Ozbay [BO04] showed that the SA-rank of $FRAC(G)$ is bounded above by the tree-width of G. Consequently, every operator that dominates SA can compute $STAB(G)$ in polynomial time for graphs G with known, bounded tree-widths. Interestingly, there is a family of graphs with tree-width 3 whose fractional stable set polytope is shown to have unbounded LS_0-rank [LT03]. These examples are all helpful in enhancing our understanding of the general behaviour of lift-and-project methods.

Chapter 4

Upper-Bound Analysis

In Chapter 3, we looked at tools that help establish lower-bound results, and relate the performances of different lift-and-project operators. In this Chapter, we look into some techniques that help with proving upper-bound results, and provide an application of these results on the matching relaxations.

4.1 Utilizing ℓ-establishing variables

Somewhat complementary to the notion of useless variables, here we look into instances where the presence of a certain set of variables in the lifted space provides a guarantee on the overall performance of the operator. Given $j \in \{0, 1, \ldots, n\}$, let $[n]_j$ denote the collection of subsets of $[n]$ of size j. Suppose $Y \in \mathbb{S}^{\mathcal{A}'}$ for some $\mathcal{A}' \subseteq \mathcal{A}$. We say that Y is *ℓ-established* if all of the following conditions hold:

(ℓ1) $Y[\mathcal{F}, \mathcal{F}] = 1$.

(ℓ2) $Y \succeq 0$.

(ℓ3) $\mathcal{A}_\ell^+ \subseteq \mathcal{A}'$.

(ℓ4) For all $\alpha, \beta, \alpha', \beta' \in \mathcal{A}_\ell^+$ such that $\alpha \cap \beta = \alpha' \cap \beta', Y[\alpha, \beta] = Y[\alpha', \beta']$.

(ℓ5) For all $\alpha, \beta \in \mathcal{A}_\ell^+, Y[\mathcal{F}, \beta] \geq Y[\alpha, \beta]$.

Notice that all matrices in $\tilde{\text{SA}}_+^{\ell}(P)$ (which contains $\tilde{\text{SA}}_+^{\prime\ell}(P)$) are ℓ-established, for all $P \subseteq [0,1]^n$. A matrix in $\tilde{\text{BZ}}_+^{\prime k}(P)$ is also ℓ-established if all subsets of size up to ℓ are generated as tiers (and thus every matrix in $\tilde{\text{BZ}}_+^{\prime k}(P)$ is at least k-established). Given such a matrix, we may define a vector y whose entries are indexed by sets in $\bigcup_{i=0}^{2\ell} [n]_i$, such that $y_S = Y[S'|_1, S''|_1]$, where S', S'' are subsets of $[n]$ of size at most ℓ such that

$S' \cup S'' = S$. Note that such choices of S', S'' must exist by ($\ell 3$), and by ($\ell 4$) the value of y_S is invariant under the choices of S' and S''.

Finally, we define $Z \in \mathbb{R}^{2\ell+1}$ such that

$$Z_i := \sum_{S \subseteq [n]_i} y_S, \quad \forall i \in \{0, 1, \ldots, 2\ell\}.$$

Note that Z_0 is always equal to 1 (by ($\ell 1$)), and $Z_1 = \sum_{i=1}^n Y[i|_1, \mathcal{F}]$. Also, observe that the entries of Z are related to each other. For example, if $\hat{x}(Ye_{\mathcal{F}})$ is an integral 0-1 vector, then by ($\ell 5$) we know that $y_S \leq 1$ for all S, and $y_S > 0$ only if $y_{\{i\}} = 1$, $\forall i \in S$. Thus, we can infer that

$$Z_j = \sum_{S \in [n]_j} y_S \leq \binom{Z_1}{j}, \quad \forall j \in [2\ell].$$

We next show that the positive semidefiniteness of Y also forces the Z_i's to relate to each other, somewhat similarly to the above. The following result would be more intuitive by noting that $\binom{p}{i+1} / \binom{p}{i} = \frac{p-i}{i+1}$.

Proposition 19. *Suppose $Y \in \mathbb{S}_+^{\mathcal{A}'}$ is ℓ-established, and y, Z are defined as above. If there exists an integer $p \geq \ell$ such that*

$$Z_{i+1} \leq \left(\frac{p-i}{i+1} \right) Z_i, \quad \forall i \in \{\ell, \ell+1, \ldots, 2\ell - 1\},$$

then $Z_i \leq \binom{p}{i}$, $\forall i \in [2\ell]$. In particular, $Z_1 \leq p$.

Proof. We first show that $Z_\ell \leq \binom{p}{\ell}$. Given $i \in [\ell]$, define the vector $v(i) \in \mathbb{R}^{\mathcal{A}'}$ such that

$$v(i)_\alpha := \begin{cases} \binom{p}{i} & \text{if } \alpha = \mathcal{F}; \\ -1 & \text{if } \alpha = S|_1 \text{ where } S \in [n]_i; \\ 0 & \text{otherwise.} \end{cases}$$

By the positive semidefiniteness of Y, we obtain

$$0 \leq v(\ell)^\top Y v(\ell) = \binom{p}{\ell}^2 - 2\binom{p}{\ell} Z_\ell + \sum_{S, S' \in [n]_\ell} Y[S|_1, S'|_1]. \tag{4.1}$$

Notice that for any $T \in [n]_{\ell+j}$, the number of sets $T', T'' \in [n]_\ell$ such that $T' \cup T'' = T$ is $\binom{\ell}{j}\binom{\ell+j}{\ell}$. Hence, this is the number of times the term y_T appears in $\sum_{S,S' \in [n]_\ell} Y[S|_1, S'|_1]$. We also know by assumption that for all $j \in [\ell]$,

$$\begin{aligned} Z_{\ell+j} &\leq \left(\frac{p-j-\ell+1}{j+\ell} \right) \left(\frac{p-j-\ell+2}{j+\ell-1} \right) \cdots \left(\frac{p-\ell}{\ell+1} \right) Z_\ell \\ &= \left(\frac{(p-\ell)!\ell!}{(p-\ell-j)!(j+\ell)!} \right) Z_\ell. \end{aligned} \tag{4.2}$$

Thus,

$$\begin{aligned}
\sum_{S,S'\in[n]_\ell} Y[S|_1, S'|_1] &= \sum_{j=0}^{\ell} \sum_{S\in[n]_{\ell+j}} \binom{\ell+j}{\ell}\binom{\ell}{j} y_S \\
&= \sum_{j=0}^{\ell} \binom{\ell+j}{\ell}\binom{\ell}{j} Z_{\ell+j} \\
&\le \sum_{j=0}^{\ell} \binom{\ell}{j}\left(\frac{(\ell+j)!}{j!\ell!}\right)\left(\frac{(p-\ell)!\ell!}{(p-\ell-j)!(\ell+j)!}\right) Z_\ell \\
&= \binom{p}{\ell} Z_\ell.
\end{aligned}$$

Therefore, we conclude from (4.1) that $0 \le \binom{p}{\ell}^2 - \binom{p}{\ell} Z_\ell$, which implies that $Z_\ell \le \binom{p}{\ell}$. Together with (4.2), this implies that $Z_{\ell+j} \le \binom{p}{\ell+j}$, $\forall j \in \{0, 1, \ldots, \ell\}$.

It remains to show that $Z_i \le \binom{p}{i}$, $\forall i \in [\ell-1]$. To do that, it suffices to show that $Z_i \le \binom{p}{i}$ can be deduced from assuming $Z_{i+j} \le \binom{p}{i+j}$, $\forall j \in [i]$. Then applying the argument recursively would yield the result for all i. Observe that

$$\begin{aligned}
\sum_{S,S'\in[n]_i} Y[S|_1, S'|_1] &= \sum_{j=0}^{i} \sum_{S\in[n]_{i+j}} \binom{i+j}{i}\binom{i}{j} y_S \\
&\le Z_i + \sum_{j=1}^{i} \binom{i+j}{i}\binom{i}{j}\binom{p}{i+j} \\
&= Z_i - \binom{p}{i} + \binom{p}{i}^2.
\end{aligned}$$

Hence,

$$0 \le v(i)^\top Y v(i) \le \binom{p}{i}^2 - 2\binom{p}{i} Z_i + \left(Z_i - \binom{p}{i} + \binom{p}{i}^2\right) = 2\left(\binom{p}{i} - 1\right)\left(\binom{p}{i} - Z_i\right),$$

and we conclude that $Z_i \le \binom{p}{i}$. □

An immediate but noteworthy implication of Proposition 19 is the following:

Corollary 20. *Suppose $Y \in \mathbb{S}^{\mathcal{A}'}$ is ℓ-established, and y, Z are defined as before. If $Z_i = 0$, $\forall i > \ell$, then $Z_1 \le \ell$.*

Proof. Since $Y \succeq 0$,

$$0 \le v(\ell)^\top Y v(\ell) = 1 - 2Z_\ell + \sum_{S,S'\in[n]_\ell} Y[S|_1, S'|_1].$$

Since $Z_i = 0,\ \forall i > \ell, Y[S|_1, S'|_1] > 0$ only if $S = S'$. Therefore,

$$\sum_{S,S' \in [n]_\ell} Y[S|_1, S'|_1] = \sum_{S \in [n]_\ell} Y[S|_1, S|_1] = Z_\ell,$$

and we deduce that $Z_\ell \leq 1$. Then we can apply Proposition 19 and deduce that $Z_i \leq \binom{\ell}{i},\ \forall i \in [2\ell]$. In particular, $Z_1 \leq \ell$. $\square$

4.2 Applications to matching relaxations

We now employ the upper-bound proving techniques developed earlier and the notion of ℓ-established matrices to prove the following result on the matching polytope of graphs.

Theorem 21. *The* SA'_+*-rank of* $MT(K_{2n+1})$ *is at most* $n - \left\lfloor \frac{\sqrt{2n+1}-1}{2} \right\rfloor$.

Proof. Suppose $G = K_{2n+1}$ and let $P = MT(G)$. Let $Y \in \tilde{\mathrm{SA}}'^k_+(P)$. Since Y is k-established, it suffices to show that $Z_{i+1} \leq \binom{n-i}{i+1} Z_i$ for all integer $i \in \{k, k+1, \ldots, 2k-1\}$ whenever $k \geq n - \left\lfloor \frac{\sqrt{2n+1}-1}{2} \right\rfloor$. Then it follows from Proposition 19 that $Z_1 \leq n$, which implies $\sum_{e \in E} x_e \leq n$ is valid for $\mathrm{SA}'^k_+(P)$.

By the fact that the maximum cardinality matching in G has size n and the condition (SA'_+ 4), $Z_i = 0,\ \forall i > n$. Thus, it suffices to verify the above claim for the case when $k \leq i \leq n-1$. Let S be a matching of size k that saturates the vertices $\{2n-2k+2, \ldots, 2n+1\}$, let T be a matching of size $i-k$ that saturates vertices $\{2n-2i+2, \ldots, 2n-2k+1\}$ and let E' be the set of edges that do not saturate any vertices in S or T. Also, for each $U \subseteq E'$, we define the vector $f_U \in \mathbb{R}^{|E'|+1}$ (indexed by $\{0\} \cup E'$) such that

$$(f_U)_i := \begin{cases} Y[(T \cup U)|_1 \cap (E' \setminus U)|_0, S|_1] & \text{if } i = 0 \text{ or if } i \in U; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $k \geq n - \frac{\sqrt{2n+1}-1}{2}$ implies $k \geq \binom{2n+1-2k}{2} \geq |E'| + |T|$. Therefore, the above entries in Y do exist, and the vectors f_U are well-defined. Also, applying (SA 3) iteratively on each index in E' gives

$$\sum_{U \subseteq E'} f_U = (Y[T|_1, S|_1], Y[(T \cup \{e_1\})|_1, S|_1], \ldots, Y[(T \cup \{e_{|E'|}\})|_1, S|_1])^\top, \tag{4.3}$$

where $e_1, \ldots, e_{|E'|}$ are the edges in E'.

Moreover, observe that $f_U = \begin{pmatrix} (f_U)_0 \\ (f_U)_0 \chi^U \end{pmatrix}$ for all $U \subseteq E'$, and by (SA'_+ 4) we know that $(f_U)_0 > 0$ only if $U \cup T \cup S$ is a matching of G, which implies that U is a matching contained in E'. Since E' spans $2n-2i+1$ vertices, such a U must have size at most $n-i$. Thus, for each f_U such that $(f_U)_0 > 0$, we know that $\sum_{i \in E'} (f_U)_i \leq (n-i)(f_U)_0$. Therefore, by (4.3),

$$\binom{2n-2i+1}{2} \frac{Z_{i+1}}{|\mathcal{M}_{n,i+1}|} = \sum_{i \in E'} Y[(T \cup \{e_i\})|_1, S|_1] \leq (n-i) Y[T|_1, S|_1] = (n-i) \frac{Z_i}{|\mathcal{M}_{n,i}|},$$

where we used $\mathcal{M}_{n,i}$ to denote the set of all matchings of size i in K_n. Notice that

$$|\mathcal{M}_{n,i}| = \frac{1}{i!} \binom{n}{2} \binom{n-2}{2} \cdots \binom{n-2i+2}{2} = \frac{n!}{2^i i! (n-2i)!}.$$

Thus, we obtain that

$$Z_{i+1} \leq \frac{(n-i)|\mathcal{M}_{n,i+1}|}{\binom{2n-2i+1}{2} |\mathcal{M}_{n,i}|} Z_i = \frac{n-i}{i+1} Z_i.$$

This concludes the proof, as we see that the facets of $MT(G)_I$ corresponding to smaller odd cliques in G are also generated by $\mathrm{SA}_+^{\prime k}$. $\square$

Recall that the LS_+-rank of $MT(K_{2n+1})$ is exactly n, as shown in [ST99]. Thus, the techniques we proposed prove that SA'_+ performs strictly better on this family of polytopes.

Next, we show that the notion of ℓ-established matrices can also be applied to provide an upper bound on the BZ'_+-rank of $MT(K_{2n+1})$.

Theorem 22. *The* BZ'_+*-rank of* $MT(K_{2n+1})$ *is at most* $\left\lceil \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rceil$.

Proof. Let $P = MT(K_{2n+1})$. First, we show that every subset $W \subseteq E$ of size up to $\left\lfloor \frac{k+1}{2} \right\rfloor$ is a wall generated by $\mathrm{BZ}_+^{\prime k}$. Given any edge $\{i,j\} \in W$, take a vertex $v \notin \{i,j\}$. Then $\{\{i,v\}, \{i,j\}\}$ and $\{\{j,v\}, \{i,j\}\}$ are both k-small obstructions for any $k \geq 1$, and their intersection contains $\{i,j\}$. If we do this for every edge in W, then we see that there is a set of at most $2|W| \leq k+1$ obstructions that generate W as a wall.

Therefore, every set S of size up to $k \left\lfloor \frac{k+1}{2} \right\rfloor$ is a tier, and the variable $S|_1$ is generated. Since $k \geq \left\lceil \sqrt{2n + \frac{1}{4}} - \frac{1}{2} \right\rceil$ implies $k \left\lfloor \frac{k+1}{2} \right\rfloor \geq n$, we see that any $Y \in \tilde{\mathrm{BZ}}_+^{\prime k}(P)$ is n-established. By (BZ' 3), $Y[S|_1, S'|_1] > 0$ only if $S \cup S'$ is a matching, which implies $Z_i = 0$, $\forall i > n$. Thus, we can apply Corollary 20 and deduce that $Z_1 \leq n$. Therefore, $\sum_{e \in E} x_e \leq n$ is valid for $\mathrm{BZ}_+^{\prime k}(P)$.

Again, since the facets of $MT(G)_I$ corresponding to smaller odd cliques in G are also generated by $\mathrm{BZ}_+^{\prime k}$, we are finished. $\square$

Note that the above upper bound also applies to the slightly weaker BZ_+ operator.

Chapter 5

Tools for Constructing and Verifying Certificate Matrices

In Chapter 3, we presented tools that can help establishing a lower-bound result by relating the performances of two lift-and-project operators. We saw that these tools can reduce the task of constructing a certificate matrix for a complicated operator (e.g. BZ' or BZ'_+) to constructing one for a simpler operator (e.g. SA' or SA'_+). We now proceed to look at ideas that can further simplify the construction and verification of certificate matrices, using ideas such as linear dependence, symmetries, convexity and connections with combinatorial objects.

First, we shall see that many matrices in the lifted space of a lift-and-project operators have very low rank compared to their dimensions. This is because many operators enforce linear dependencies on the matrices in their lifted spaces to try to capture the relationships between the sets in $\mathcal{A}$ that are represented by the rows and columns of the matrices. Also, sometimes dependencies can also arise when the entries of the matrices that represent larger subsets of $\mathcal{F}$ align with those for the smaller subsets in a special way. In these cases, verifying the membership of the matrix in the lifted space can be reduced to verifying some of the conditions of a symmetric minor that has the same rank as the original matrix.

Also, since many lift-and-project operators impose positive-semidefiniteness constraints on the matrices in their lifted spaces, we provide some techniques that can simplify this task. In this context, we will first look into the case when there is a connection between the eigenspaces of a candidate matrix and sets of combinatorial objects, which may provide valuable information about the eigenvectors and eigenvalues (and their multiplicities) of our matrix. We will then look into some maps that commute with lift-and-project operators. Hong and Tunçel provided in [HT08] a template of proving a lower-bound result for the Lovász–Schrijver operators by using such maps, and we show that some of their work can be extended to the stronger SA-based operators. We will also look at the special case when the map is a permutation of the n coordinates, and use them to show that when the given initial relaxation has a lot of symmetries, we might be able to conclude that there are certificate matrices in the lifted space with very few distinct entries. This in turn can allow us to verify the positive semidefiniteness of a very large matrix by only doing that on a much smaller matrix.

The idea of using symmetry and convexity to reduce the number of parameters involved in a problem instance have been widely exploited in both computational work and theoretical research. This at least goes back to Lovász's seminal work on the theta function in [Lov79] and related findings by Schrijver in [Sch79]. Also during the 1970s, Godsil used similar ideas in his work in algebraic graph theory (see [CG97] for a more recent survey). More recently, these ideas have also been proven useful in reducing SDP instances [GP04, dKPS07], bounding the crossing number of graphs [dKMP+06], and obtaining SDP relaxations for polynomial optimization problems [MWT13].

In this chapter, we will focus on how these ideas can help simplify the construction and verification of certificate matrices. Notice that the conditions imposed by the named lift-and-project operators can be loosely partitioned into the following three categories:

1. P**-related constraints:** Enforcing some vector formed by a subset of matrix entries to be contained in some lifted version of the initial relaxation P (e.g. the conditions $Ye_i, Y(e_0 - e_i) \in K(P)$ for the Lovász–Schrijver operators, conditions (SA 2), and (BZ$'$ 2)(ii)).

2. **Set theoretical constraints:** Establishing dependencies between matrix entries to ensure some consistency between the behaviour of these entries with their intended set theoretical values (e.g. the condition $Y = Y^\top$ in LS; (SA 3), (SA 4) and (SA 5); (BZ$'$ 2) (iii)-(v), (BZ$'$ 3) and (BZ$'$ 4).)

3. **Positive Semidefiniteness:** Requiring that a matrix (or a certain submatrix) is positive semidefinite.

Depending on what we know at hand (about the problem, the lift-and-project operator, related known lower-bound results, etc.), we can take different approaches to constructing a certificate matrix, such that one or two of the above set of conditions are guaranteed to hold. Then it only suffices to check the remaining (perhaps more challenging) constraints to establish the desired lower-bound result.

We now look at several such approaches, highlighting how each could simplify the construction and verification of certificate matrices. We will also see how many of these ideas apply to specific lift-and-project relaxations, such as the Lasserre relaxations of the max-cut problem, and the SA_+'-relaxations of the matching problem.

5.1 Reducing certificate matrices using linear dependencies

Recall that given $x \in \{0,1\}^n$, $Y^x_{\mathcal{A}}$ is the matrix whose rows and columns are indexed by sets in $\mathcal{A}$, where $Y^x_{\mathcal{A}}[S,T] = 1$ if and only if $x \in S \cap T$. Notice that the columns of $Y^x_{\mathcal{A}}$ are not linearly independent. For example,

$$Y^x_{\mathcal{A}} e_\alpha = Y^x_{\mathcal{A}} e_{\alpha \cap i|_1} + Y^x_{\mathcal{A}} e_{\alpha \cap i|_0},$$

for all $\alpha \in \mathcal{A}$ and $i \in [n]$. The above equation captures the trivial fact that for every point $y \in \alpha$, either $y \in \alpha \cap i|_1$ or $y \in \alpha \cap i|_0$. Recall that many existing lift-and-project operators can be interpreted as working with submatrices of $Y^x_{\mathcal{A}}$, and many of them do enforce this type of linear dependence in the lifted space to make the matrix variables behave as consistently with their set theoretical meanings as possible (e.g. the condition (SA 3)). Knowing this, when we try to construct a certificate matrix in the lifted space of an operator, we really only have to construct enough of it such that the rest of the matrix would be determined by the linear dependence conditions imposed by the operator. Even when these dependence conditions are not enforced by the given operator, we can use them as guidance to help us construct such matrices.

Let us formalize the above observations. First, recall that given $\mathcal{S} \subseteq \mathcal{A}$, we let $Y^x_{\mathcal{S}}$ denote the $\mathcal{A} \times \mathcal{S}$ submatrix of $Y^x_{\mathcal{A}}$ consisting of the columns indexed by sets in $\mathcal{S}$. Then given $\mathcal{S}, \mathcal{S}' \subseteq \mathcal{A}$, we say that $\mathcal{S}$ *generates* $\mathcal{S}'$ if the column space of $Y^x_{\mathcal{S}}$ contains that of $Y^x_{\mathcal{S}'}$, for all $x \in \mathcal{F}$.

It is pretty easy to see that if $\mathcal{S}$ refines $\mathcal{S}'$ (recall that $\mathcal{S}$ refines $\mathcal{S}'$ if every set in $\mathcal{S}'$ can be expressed as a disjoint union of sets in $\mathcal{S}$), then $\mathcal{S}$ generates $\mathcal{S}'$. The converse to this is not true — an important observation to construct counterexamples is that $\mathcal{A}^+_k$ generates $\mathcal{A}_k$, for all $k \in [n]$. To see this, let $\alpha = S|_1 \cap T|_0 \in \mathcal{A}_k$. Then it is not hard to check that

$$Y^x_{\mathcal{A}} e_\alpha = \sum_{U \subseteq T} (-1)^{|U|} Y^x_{\mathcal{A}} e_{(S \cup U)|_1},$$

for every $x \in \mathcal{F}$.

Using the above observation, we can give the following alternative definition of the SA^k operator.

Proposition 23. *For any fixed integer $k \in [n]$ and $P \subseteq [0,1]^n$, define the lift-and-project operator Γ^k as follows:*

1. *Let $\tilde{\Gamma}^k(P)$ denote the set of matrices $Y \in \mathbb{R}^{\mathcal{A}^+_1 \times \mathcal{A}^+_k}$ which satisfy all of the following conditions:*

   ($\Gamma 1$) $Y[\mathcal{F}, \mathcal{F}] = 1$.

(Γ2) *For all disjoint subsets $S, T \subseteq [n]$ where $|S| + |T| \leq k$,*

$$\sum_{U \subseteq T} (-1)^{|U|} Y e_{(S \cup U)|_1} \in K(P).$$

(Γ3) *For all $\alpha_1, \alpha_2 \in \mathcal{A}_1^+, \beta_1, \beta_2 \in \mathcal{A}_k^+$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.*

2. *Define*

$$\Gamma^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\Gamma}^k(P), Y e_{\mathcal{F}} = \hat{x} \right\}.$$

Then $\Gamma^k(P) = \mathrm{SA}^k(P)$ for every $k \in [n]$ and $P \subseteq [0,1]^n$.

Proof. First, we show that $\mathrm{SA}^k(P) \subseteq \Gamma^k(P)$. Given $x \in \mathrm{SA}^k(P)$, let Y' be its certificate matrix in $\tilde{\mathrm{SA}}^k(P)$, and Y be the submatrix of Y consisting of only the columns indexed by sets in $\mathcal{A}_k^+$. Then we see that (Γ1) is implied by (SA 1), (Γ2) follows from (SA 2) and (SA 3), and (Γ3) follows from (SA 5). Since $\hat{x} = Y e_{\mathcal{F}}$, $x \in \Gamma^k(P)$.

Next, we prove that $\Gamma^k(P) \subseteq \mathrm{SA}^k(P)$. Given $x \in \Gamma^k(P)$, let $Y \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_k^+}$ be its certificate matrix in $\tilde{\Gamma}(P)$. We define the matrix $Y' \in \mathbb{R}^{\mathcal{A}_1^+ \times \mathcal{A}_k}$ where

$$Y' e_{S|_1 \cap T|_0} = \sum_{U \subseteq T} (-1)^{|U|} Y e_{(S \cup U)|_1},$$

for all disjoint $S, T \subseteq [n]$ such that $|S| + |T| \leq k$. Then we see that $Y' \in \tilde{SA}^k(P)$, as (SA 1) follows from (Γ1), (SA 2) and (SA 3) follow from (Γ2). Also, both (SA 4) and (SA 5) are implied by (Γ3). Since $\hat{x} = Y' e_{\mathcal{F}}$, we obtain that $x \in \mathrm{SA}^k(P)$. □

Similarly, since $\mathcal{A}_k \setminus \mathcal{A}_{k-1}$ refines (and therefore generates) $\mathcal{A}_k$, the definition of SA^k can also be rewritten with the lifted space being a set of matrices whose columns are indexed by $(\mathcal{A}_k \setminus \mathcal{A}_{k-1})$.

More generally, suppose $P \subseteq [0,1]^n$, Γ is an admissible lift-and-project operator, and $\tilde{\Gamma}(P) \subseteq \mathbb{R}^{\mathcal{S} \times \mathcal{T}}$ where $\mathcal{S}, \mathcal{T} \subseteq \mathcal{A}$. Furthermore, suppose $\mathcal{S}'$ generates $\mathcal{S}$ and $\mathcal{T}'$ generates $\mathcal{T}$. Then there exists $U \in \mathbb{R}^{\mathcal{S}' \times \mathcal{S}}$ such that $Y^x_{\mathcal{S}'} U = Y^x_{\mathcal{S}}$ for all $x \in \mathcal{F}$. Similarly, there exists $V \in \mathbb{R}^{\mathcal{T}' \times \mathcal{T}}$ such that $Y^x_{\mathcal{T}'} V = Y^x_{\mathcal{T}}$ for all $x \in \mathcal{F}$. If all matrices in $Y \in \tilde{\Gamma}(P)$ satisfy (RCMC), then we know that we can write $Y = U^\top Y' V$ for some $Y' \in \mathbb{R}^{\mathcal{S}' \times \mathcal{T}'}$. In many cases, Y' might be easier to obtain than directly computing Y (e.g. when $\mathcal{S}', \mathcal{T}'$ are subsets of $\mathcal{S}, \mathcal{T}$ respectively, or when their set theoretical structures relate better to our problem at hand). Moreover, it is apparent that extending Y' to Y preserves conditions such as (OMC) and (RCMC). Thus, some of the conditions of Γ might only need to be verified on Y' instead of Y, which can simplify our construction of certificate matrices.

In particular, when $\tilde{\Gamma}(P)$ is a set of square matrices whose rows and columns are indexed by the same sets in $\mathcal{A}$ (such as in the cases when $\Gamma \in \left\{ \mathrm{SA}_+, \mathrm{SA}'_+, \mathrm{BZ}'_+ \right\}$), the above construction preserves positive semidefiniteness. Thus, instead of checking $Y \succeq 0$, it suffices to check $Y' \succeq 0$.

Sometimes we can further reduce the task of checking $Y' \succeq 0$ by the following simple observation:

Proposition 24. *Suppose $Y \in \mathbb{S}^n$ can be written as $\begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}$, where A and C are square matrices. If there exists a matrix L such that $AL = B$, then $Y \succeq 0$ if and only if $A \succeq 0$ and $C - L^\top AL \succeq 0$.*

Proof. Since $AL = B$, $B^\top = L^\top A$. Then observe that

$$\begin{pmatrix} I & -L \\ 0 & I \end{pmatrix}^\top Y \begin{pmatrix} I & -L \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & C - L^\top AL \end{pmatrix}.$$

Since $\begin{pmatrix} I & -L \\ 0 & I \end{pmatrix}$ is invertible, we see that $Y' \succeq 0$ if and only if $A \succeq 0$ and $C - L^\top AL \succeq 0$. □

Note that in the case when A is positive definite, Proposition 24 is an application of the Schur complement. Moreover, if the sets indexing the rows and columns of Y' — and the entries of Y' themselves — have considerable structure and symmetry, it might be possible to find a matrix L that has some combinatorial meaning, which can help explain and establish the positive semidefiniteness of A and $C - L^\top AL$. We shall discuss this further in the next section.

5.2 Verifying positive semidefiniteness when weights align

In many cases, the last hurdle of verifying if a matrix is a certificate matrix is establishing that it is in fact positive semidefinite. This could happen, for instance, when we use tools outlined in previous section to construct a candidate matrix that already satisfies P-related and set theoretical constraints. Also, if a lower-bound result for a polyhedral operator (e.g. LS or BZ$'$) has already been established, and in addition one can verify that the certificate matrix is also positive semidefinite, then the lower-bound result can be established on an even stronger operator (e.g LS$_+$ or BZ$'_+$). Here, we look into some cases when we know more about the candidate certificate matrices, and provide some tools that can simplify the task of verifying their positive semidefiniteness.

5.2.1 The "last block" approach

Suppose we would like to show that a matrix Y is positive semidefinite. Then the following simple observation could be helpful:

Proposition 25. *Let Y be a symmetric matrix, and Y' be a symmetric minor of Y. If $\operatorname{rank}(Y') = \operatorname{rank}(Y)$, then $Y \succeq 0 \iff Y' \succeq 0$.*

Proof. First, the forward implication is obvious, as Y' is a symmetric minor of Y. For the other direction, suppose Y is $n \times n$, and assume without loss of generality that Y' is the $m \times m$ leading symmetric minor of Y. Then we can write $Y = \begin{pmatrix} Y' & U \\ U^\top & W \end{pmatrix}$ for some matrices U, W. Now since $\operatorname{rank}(Y') = \operatorname{rank}(Y)$, every column in U can be expressed as a linear combination of columns in Y', and so there exists an $m \times (n-m)$ matrix L where $Y'L = U$. By the same linear dependence argument, we obtain that such a matrix L must also satisfy $U^\top L = W$. Thus, we obtain that $W = U^\top L = (LY')^\top L = L^\top Y' L$, and we have the desired result by Proposition 24. We can also directly compute:

$$Y = \begin{pmatrix} Y' & Y'L \\ L^\top Y' & L^\top Y' L \end{pmatrix} = \begin{pmatrix} I \\ L^\top \end{pmatrix} Y' \begin{pmatrix} I & L \end{pmatrix}.$$

Thus, $Y' \succeq 0 \Rightarrow Y \succeq 0$. □

While Proposition 25 is very elementary, it has quite a few interesting applications in the certificate matrices for lift-and-project relaxations. One particular way of utilizing it is the following: Let $\{B_0, B_1, \ldots, B_k\}$ be a partition of the rows (and thus also columns) of Y. Then we can rewrite Y in the block matrix form:

$$Y = \begin{pmatrix} Y_{0,0} & Y_{0,1} & \cdots & Y_{0,k} \\ Y_{1,0} & Y_{1,1} & \cdots & Y_{1,k} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{k,0} & Y_{k,1} & \cdots & Y_{k,k} \end{pmatrix}, \tag{5.1}$$

where $Y_{i,j} \in \mathbb{R}^{B_i \times B_j}$, for all $i, j \in \{0, \ldots, k\}$. Note that if Y is a matrix in the in the lifted space for operators such as SA_+ and Las, then there is a natural way to partition of the rows and columns of a certificate matrix (e.g. B_i could be $A_i \setminus A_{i-1}$ in the case of SA_+, and $A_i^+ \setminus A_{i-1}^+$ in case of Las). Nonetheless, the following observations apply to whichever way we partition the rows and columns of Y'.

Proposition 26. *Let Y be a matrix whose rows and columns are partitioned into sets $B_0, \ldots, B_k$ as in (5.1). Suppose $\operatorname{rank}(Y) = \operatorname{rank}(Y_{k,k})$. Then there exists matrices $L_0, L_1, \ldots, L_k$ such that*

$$Y_{i,j} = L_i Y_{k,k} L_j,$$

for all $i, j \leq k$. In particular, $Y_{k,k} \succeq 0 \iff Y' \succeq 0$.

Proposition 26 is specialization of Proposition 25, and can be proved by essentially the same argument. Thus, we see that when $\text{rank}(Y) = \text{rank}(Y_{k,k})$, the $Y_{i,j}$ matrices are all connected to each other by linear dependencies in a special way, and that the task of establishing $Y \succeq 0$ is reduced to showing the "last" symmetric minor block of Y is positive semidefinite. While the assumption in Proposition 26 might seem very restrictive, such relationships between entries of the matrix can be very insightful. For example, in the case when $B_i = \mathcal{A}_i^+ \setminus \mathcal{A}_{i-1}^+$, one can interpret the above property as entries in Y being "weights" assigned to certain "slices" of $\mathcal{F}$. Then the presence of these L_i matrices can be interpreted as the fact that the entries of Y corresponding to larger subsets can be expressed as linear functions of the weights of finer, smaller subsets.

For an example of a certificate matrix with this property, we turn to the max-cut problem and some of its known lower-bound results. The max-cut problem is the problem of, given a graph $G = (V, E)$, find a set of vertices $U \subseteq V$ such that the size of the cut induced by U,

$$\delta(U) := \{\{i,j\} \in E : |\{i,j\}| \cap U = 1\}$$

is maximized. The max-cut problem can be formulated as the following nonlinear integer program:

$$\begin{array}{ll} \max & \displaystyle\sum_{\{i,j\} \in E} w_{i,j}(x_i + x_j - 2x_i x_j) \\ \text{subject to} & \qquad x \in \{0,1\}^V. \end{array} \tag{5.2}$$

Note that for all $x \in \{0,1\}^V$,

$$x_i + x_j - 2x_i x_j = x_i^2 + x_j^2 - 2x_i x_j = (x_i - x_j)^2,$$

which evaluates to 1 if x_i and x_j differ, and 0 otherwise. Thus, $\bar{x}$ is an optimal solution of (5.2) if and only if $U := \{i : \bar{x}_i = 1\}$ produces a maximum cut in the graph G.

The following series of semidefinite relaxations of max-cut have been studied by researchers including Laurent [Lau03b] and Georgiou [Geo10], and have often been referred to as the "Lasserre relaxations" of the max-cut problem. While these relaxations are not obtained by applying the Las operator as defined in Chapter 2 to an initial relaxation, it was proposed by Lasserre in [Las00]. To state these relaxations, it is helpful to introduce the notion of moment matrices. Let $y \in \mathbb{R}^{\mathcal{A}_{2k}^+}$ for some integer k. Then for every integer $\ell \leq k$, we define the matrix $M_\ell(y) \in \mathbb{R}^{\mathcal{A}_\ell^+ \times \mathcal{A}_\ell^+}$ where $M_\ell(y)[\alpha, \beta] = y[\alpha \cap \beta]$ for all $\alpha, \beta \in \mathcal{A}_\ell^k$. Then for every $k \geq 1$, we define the semidefinite program:

$$\begin{array}{ll} \max & \displaystyle\sum_{\{i,j\} \in E} w_{i,j}(y[i|_1] + y[j|_1] - 2y[\{i,j\}|_1]) \\ \text{subject to} & \quad M_k(y) \succeq 0, y \in \mathbb{R}^{\mathcal{A}_{2k}^+}, y[\mathcal{F}] = 1. \end{array} \tag{5.3}$$

Note that given a set of vertices U, we may define $y \in \mathbb{R}^{\mathcal{A}_{2k}^+}$ where

$$y[S|_1] = \begin{cases} 1 & \text{if } S \subseteq U; \\ 0 & \text{otherwise.} \end{cases}$$

Then $M_k(y)[\alpha, \beta] = y[\alpha]y[\beta]$ for every $\alpha, \beta \in \mathcal{A}_k^+$, and thus is positive semidefinite since $y \geq 0$. Moreover, the objective value given by this y is exactly that of the total weight of the cut induced by U. Thus, we see that (5.3) indeed gives a family of relaxations of the max-cut problem.

Consider $G := K_{2n+1}$. The maximum cut in G has size $n(n+1)$. Laurent [Lau03b] showed (in a slightly different language, as she started with a $[-1, 1]$-relaxation as opposed to a $[0, 1]$-relaxation) that the k^{th} relaxation in (5.3) has optimal value greater than $n(n+1)$ if and only if $k \leq n$. Subsequently, Georgiou [Geo10] provided an alternative proof to Laurent's result, while showing the following:

Theorem 27 (Theorem 9.3.2 in [Geo10])**.** *Given $G = K_{2n+1}$, construct $y \in \mathbb{R}^{\mathcal{A}_{2n}^+}$ where*

$$y[S|_1] := \frac{\binom{n+1/2}{|S|}}{\binom{2n+1}{|S|}}.$$

Then $M_n(y) \succeq 0$.

This shows that, for all integers $k \leq n$, the optimal value of the k^{th} relaxation of (5.3) is at least

$$\sum_{\{i,j\} \in E(G)} \left(\frac{1}{2} + \frac{1}{2} - 2 \left(\frac{2n-1}{8n} \right) \right) = \frac{(2n+1)^2}{4} > n(n+1).$$

Consequently, the relaxation (5.3) is not exact for all $k \leq n$. One of the key observations Georgiou used in establishing Theorem 27 is the following: let $y \in \mathbb{R}^{\mathcal{A}_{2n}^+}$ be the vector as defined in Theorem 27, and $Y = M_n(y)$. Georgiou showed that, if we define $B_0 := \{\mathcal{F}\}$ and $B_i := \mathcal{A}_i^+ \setminus \mathcal{A}_{i-1}^+$ for all $i \in [n]$ and rewrite Y in block matrix form as in (5.1), then the rank of $Y_{n,n}$ is equal to the rank of Y. This implies that, to show $Y \succeq 0$, it suffices to show that $Y_{n,n} \succeq 0$.

The certificate matrix Laurent used in her proof of the result in [Lau03b] was different, but she used similar steps to establish the positive semidefiniteness of her matrix. More precisely, let $Y \in \mathbb{R}^{\mathcal{A}_n^+ \times \mathcal{A}_n^+}$ denote the certificate matrix used in her proof. Laurent proved that the principle minor of Y with rows and columns indexed by the sets in $\mathcal{A}_n^+ \setminus \mathcal{A}_{n-1}^+$ is positive definite, and that the nullspace of Y has dimension at least $|\mathcal{A}_{n-1}^+|$. Then it follows that $Y \succeq 0$.

We now provide another example of this phenomenon, showing how the weight-aligning property in a matrix could simplify the task for verifying certificate matrices in $\tilde{\text{SA}}_+^{\prime k}(MT(G))$. Recall that $\mathcal{M}_{n,k}$ denotes the set of matchings of K_n of size k, and

$$|\mathcal{M}_{n,k}| = \frac{1}{k!} \binom{n}{2} \binom{n-2}{2} \cdots \binom{n-2k+2}{2} = \frac{n!}{2^k k!(n-2k)!} = (2k-1)!! \binom{n}{2k}.$$

Note that we have used $d!!$ to denote the *double factorial* of d, which can be defined recursively as follows: $(-1)!! = 0!! = 1$, and for all $d \geq 1, d!! := d \times (d-2)!!$. Next, given integers n, k, ℓ such that $n \geq \max\{2k, 2\ell\}$, we define $Y_{n,k,\ell}$ to be the $\mathcal{M}_{n,k} \times \mathcal{M}_{n,\ell}$ matrix where

$$Y_{n,k,\ell}[S,T] := \begin{cases} (n-1-2(|S \cup T|))!! & \text{if } S \cup T \text{ is a matching;} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following:

Theorem 28. *If $Y_{2n+1,k,k} \succeq 0$, then $\frac{1}{2n}\bar{e} \in \mathrm{SA}_+^{\prime k}(MT(K_{2n+1}))$.*

Observe that $Y_{2n+1,k,k}$ is much smaller than matrices in $\tilde{\mathrm{SA}}_+^{\prime k}(MT(K_{2n+1}))$. For instance, $Y_{13,3,3}$ is 25740×25740, while matrices in $\tilde{\mathrm{SA}}_+^{\prime 3}(MT(K_{13}))$ are 620777×620777.

Before we prove Theorem 28, we establish several combinatorial identities that relate the entries of $Y_{n,k,k}$:

Lemma 29. *Given integers n, i, j where $i + j \leq \frac{n}{2}$,*

$$f_{n,i,j} := \sum_{S \in \mathcal{M}_{n,i}, S' \in \mathcal{M}_{n,j}} Y_{n,i,j}[S,S'] = (n-1)!!\binom{n/2}{i}\binom{n/2}{j}.$$

Proof. We first prove the claim for the case when $i = 0$. Notice that $Y_{n,0,j}$ is just a row vector with $|\mathcal{M}_{n,j}|$ entries, each being $(n-2j-1)!!$ (since $j \leq \frac{n}{2}$).Therefore,

$$f_{n,0,j} = (2j-1)!!\binom{n}{2j}(n-2j-1)!! = \frac{(2j-1)!!n!(n-2j-1)!!}{(2j)!(n-2j)!} = (n-1)!!\binom{n/2}{j}.$$

Next, we compute $f_{n,i,j}$. It is easy to see that $Y_{n,i,j} = (Y_{n,j,i})^\top$. Hence, $f_{n,i,j} = f_{n,j,i}$, and we may assume without loss of generality that $i \geq j$.

Focus on any matching $S' \in \mathcal{M}_j$. The number of matchings $S \in \mathcal{M}_i$ such that $S \cup S'$ is a matching and $|S \cap S'| = p$ is $\binom{j}{p}|\mathcal{M}_{n-2j,i-p}|$. Thus, these matchings contribute

$$\binom{j}{p}|\mathcal{M}_{n-2j,i-p}|(n-2(i+j-p)-1)!! = (n-2j-1)!!\binom{j}{p}\binom{(n-2j)/2}{i-p}$$

to $f_{n,i,j}$ (assuming $i + j \leq \frac{n}{2}$). Also, by the symmetry of K_n and our definition of $Y_{n,i,j}$, $\sum_{S \in \mathcal{M}_i} Y_{n,i,j}[S,S']$ is the same for each $S' \in \mathcal{M}_j$. Therefore,

$$\begin{aligned} f_{n,i,j} &= |\mathcal{M}_{n,j}|\sum_{p=0}^{j}(n-2j-1)!!\binom{j}{p}\binom{(n-2j)/2}{i-p} \\ &= f_{n,0,j}\sum_{p=0}^{j}\binom{j}{p}\binom{(n-2j)/2}{i-p} \\ &= f_{n,0,j}\left([x^p](1+x)^j\right)\left([x^{i-p}](1+x)^{\frac{n-2j}{2}}\right) \\ &= f_{n,0,j}[x^i](1+x)^{\frac{n}{2}} \\ &= (n-1)!!\binom{n/2}{j}\binom{n/2}{i}. \end{aligned}$$

Note that we have used $[x^k]f(x)$ to denote the coefficient of x^k in $f(x)$. This completes the proof of our claim. $\square$

Next, we introduce another family of matrices that are closely related to $Y_{n,k,k}$ matrices. Given positive integers n, k, ℓ such that $2k \leq 2\ell \leq n$, we define the matrix $L_{n,k,\ell}$ to be the $\mathcal{M}_{n,k} \times \mathcal{M}_{n,\ell}$ matrix where

$$L_{n,k,\ell}[S,T] := \begin{cases} 1 & \text{if } S \subseteq T; \\ 0 & \text{otherwise.} \end{cases}$$

The next result shows a key property of L matrices in relating Y matrices.:

Lemma 30. *Given integers i, j, k, n such that $i, j \leq k \leq n/2$,*

$$Y_{n,i,j} = \binom{(n-2i)/2}{k-i}^{-1} \binom{(n-2j)/2}{k-j}^{-1} L_{n,i,k} Y_{n,k,k} L_{n,j,k}^{\top}. \tag{5.4}$$

Proof. For convenience, let Y' denote the right hand side of (5.4). Given $S \in \mathcal{M}_i, S' \in \mathcal{M}_j$, we see that

$$\begin{aligned} Y'[S,S'] &= \binom{(n-2i)/2}{k-i}^{-1} \binom{(n-2j)/2}{k-j}^{-1} \langle Y_{n,k,k}, (L_{n,i,k}e_S)(L_{n,j,k}e_{S'})^{\top} \rangle \\ &= \binom{(n-2i)/2}{k-i}^{-1} \binom{(n-2j)/2}{k-j}^{-1} \sum_{\substack{T,T' \in \mathcal{M}_k, \\ T \supseteq S, T' \supseteq S'}} Y_{n,k,k}[T,T']. \end{aligned}$$

Notice that if $S \cup S'$ is not a matching, $T \supseteq S$ and $T' \supseteq S'$, then $T \cup T'$ is not a matching either, and we know that $Y_{n,k,k}[T,T'] = 0$. Therefore, if $S \cup S'$ is not a matching, $Y'[S,S'] = 0$.

Now suppose $S \cup S'$ is a matching. We want to sum over the entries $Y_{n,k,k}[T,T']$ such that $T \supseteq S$ and $T' \supseteq S'$. Notice that for sets $W \subseteq S \setminus S', W' \subseteq S' \setminus S$,

$$\begin{aligned} & \sum_{\substack{T \in \mathcal{M}_k, T \supseteq S, T \cap (S' \setminus S) = W' \\ T' \in \mathcal{M}_k, T' \supseteq S', T' \cap (S' \setminus S) = W}} Y_{n,k,k}[T,T'] \\ = \;& (n - 2|S \cup S'| - 1)!! f_{n-2|S \cup S'|, k-|S|-|W'|, k-|S'|-|W|}. \end{aligned}$$

Then we have

$$\begin{aligned}
&\sum_{\substack{W\subseteq(S\setminus S'),\\ W'\subseteq(S'\setminus S)}} f_{n-2|S\cup S'|,k-|S|-|W'|,k-|S'|-|W|}\\
=\ & \sum_{i=0}^{|S\setminus S'|}\sum_{j=0}^{|S'\setminus S|}\binom{|S\setminus S'|}{i}\binom{|S'\setminus S|}{j} f_{n-2|S\cup S'|,k-|S|-j,k-|S'|-i}\\
=\ & \sum_{i=0}^{|S\setminus S'|}\sum_{j=0}^{|S'\setminus S|}\binom{|S\setminus S'|}{i}\binom{|S'\setminus S|}{j}\binom{(n-2|S\cup S'|)/2}{k-|S|-j}\binom{(n-2|S\cup S'|)/2}{k-|S'|-i}\\
=\ & \left(\sum_{i=0}^{|S\setminus S'|}\binom{|S\setminus S'|}{i}\binom{(n-2|S\cup S'|)/2}{k-|S'|-i}\right)\left(\sum_{j=0}^{|S'\setminus S|}\binom{|S'\setminus S|}{j}\binom{(n-2|S\cup S'|)/2}{k-|S|-j}\right)\\
=\ & \binom{|S\setminus S'|+(n-2|S\cup S'|)/2}{k-|S'|}\binom{|S'\setminus S|+(n-2|S\cup S'|)/2}{k-|S|}\\
=\ & \binom{(n-2j)/2}{k-j}\binom{(n-2i)/2}{k-i}.
\end{aligned}$$

Therefore, $Y'[S,S'] = (n-2|S\cup S'|-1)!!$, which coincides with $Y_{n,i,j}[S,S']$. Hence, we are finished. $\square$

We are now ready to prove Theorem 28.

Proof of Theorem 28. For convenience, we let $G := K_{2n+1}$ and $P = MT(G)$. We construct $Y \in \mathbb{S}^{\mathcal{A}_k}$ such that $\hat{x}(Ye_{\mathcal{F}}) = \frac{1}{2n}\bar{e}$, and show that $Y \in \tilde{\mathrm{SA}}_+^{\prime k}(P)$. Define the vector $y \in \mathbb{R}^{\mathcal{A}_{2k}}$ as follows. Given $S \subseteq E(G)$ and $|S| \le k$, we set

$$y_{S|_1} := \begin{cases} \frac{(2n-2|S|)!!}{(2n)!!} & \text{if } S \text{ is a matching;}\\ 0 & \text{otherwise.}\end{cases}$$

Next, given disjoint S,T, we define

$$y_{S|_1\cap T|_0} := \sum_{U\subseteq T}(-1)^{|U|} y_{(S\cup U)|_1}.$$

Notice that by the above construction, the entries of y satisfy

$$y_{S|_1\cap T|_0} = y_{S|_1\cap T|_0\cap j|_1} + y_{S|_1\cap T|_0\cap j|_0},$$

for any disjoint S,T such that $|S|+|T| < k$, and $j \notin S\cup T$. Thus y satisfies (OMC). Moreover, y is nonnegative. Given disjoint sets of edges S,T, if S is not a matching, then

$y_{(S\cup U)|_1} = 0$ for every U, and thus $y_{S|_1 \cap T|_0} = 0$. Now suppose S is in fact a matching. Then

$$\begin{aligned}
y_{S|_1\cap T|_0} &= \sum_{U\subseteq T} (-1)^{|U|} y_{(S\cup U)|_1} \\
&= y_{S|_1} + \sum_{i=1}^{|T|} \sum_{U\subseteq T, |U|=i} (-1)^{|U|} y_{(S\cup U)|_1} \\
&\geq \frac{1}{2n!!}\left((2n-2|S|)!! - \sum_{i:1\leq i\leq |T|, i \text{ odd}} \binom{|T|}{i} (2n-2|S|-2i)!! \right) \\
&\geq \frac{(2n-2|S|)!!}{2n!!}\left(1 - \sum_{i:1\leq i\leq |T|, i \text{ odd}} \frac{1}{2^i}\right) \\
&\geq 0.
\end{aligned}$$

Next, we define the matrix $Y \in \mathbb{S}^{\mathcal{A}_k}$ such that

$$Y[S|_1 \cap T|_0, S'|_1 \cap T'|_0] := \begin{cases} y_{(S\cup S')|_1 \cap (T\cup T')|_0} & \text{if } (S\cup S')\cap(T\cup T') = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that ($\text{SA}_+ 1$), ($\text{SA}_+ 3$), ($\text{SA}'_+ 4$) and ($\text{SA}_+ 5$) all hold by the construction of Y. Also, it was shown in [MS09] that $\hat{x}Ye_\alpha \in K(P)$ for all $\alpha \in \mathcal{A}_k$. Since $y \geq 0$, Y has nonnegative entries, and so ($\text{SA}_+ 2$) holds as well. Thus, it only remains to verify that Y is positive semidefinite.

We now utilize the linear dependence of the rows and columns of Y to reduce it to a smaller matrix. Consider a column in Y indexed by $S|_1 \cap T|_0$ for some disjoint S, T. By the construction of y and Y,

$$Ye_{S|_1\cap T|_0} = \sum_{U\subseteq T} (-1)^{|U|} Ye_{(S\cup U)|_1}.$$

Furthermore, notice that $Ye_{(S\cup U)|_1}$ is the zero vector if $S \cup U$ is not a matching. Thus, every column of Y is a linear combination of columns corresponding $S|_1$, where S is a matching.

Hence, if we let $\mathcal{A}' \subseteq \mathcal{A}_k$ be this collection of sets, and Y' be the symmetric minor of Y consisting of the columns and rows from $\mathcal{A}'$, then by the symmetry of Y and the above claim, there exists a matrix $U \in \mathbb{R}^{\mathcal{A}'\times\mathcal{A}_k}$ such that $Y = U^\top Y'U$. Thus, to verify that $Y \succeq 0$, it suffices to show that $Y' \succeq 0$.

Notice that there is a natural one-to-one correspondence between $\mathcal{A}'$ and $\bigcup_{i=0}^n \mathcal{M}_{2n+1,i}$. In fact, Y' can be expressed as the block matrix

$$\frac{1}{(2n)!!}\begin{pmatrix} Y_{2n+1,0,0} & Y_{2n+1,0,1} & \cdots & Y_{2n+1,0,k} \\ Y_{2n+1,1,0} & Y_{2n+1,1,1} & \cdots & Y_{2n+1,1,k} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{2n+1,k,0} & Y_{2n+1,k,1} & \cdots & Y_{2n+1,k,k} \end{pmatrix}.$$

Now by Lemma 30 and Proposition 26, the positive semidefiniteness of $Y' \succeq 0$ follows from that of $Y_{2n+1,k,k}$. Thus, it follows that $\frac{1}{2n}\bar{e} \in \mathrm{SA}'^{k}_{+}(P)$. □

We will revisit these $Y_{2n+1,k,k}$ matrices in Section 5.4, where we present other tools that can further simplify the task of establishing their positive semidefiniteness.

5.2.2 The inductive approach

Note that Proposition 26 can be seen as a special instance of Proposition 24, where we used $Y_{k,k}$ as the "A" block, and $C - L^\top AL$ happens to be the zero matrix. On the other hand, if the blocks of Y' are related by L_i matrices as described in Proposition 26, we can also establish $Y' \succeq 0$ in an inductive way as follows:

Proposition 31. *Let Y' be a matrix whose rows and columns are partitioned into sets $B_0, \dots, B_k$ as in (5.1). Suppose, for every $i \leq k-1$, there exists a $B_i \times B_{i+1}$ matrix $\tilde{L}_i$ where $Y_{i,i}\tilde{L}_i = Y_{i,i+1}$. Further assume that $Y_{0,0} \succeq 0$, and*

$$Y_{i+1,i+1} - \tilde{L}_i^\top Y_{i,i}\tilde{L}_i \succeq 0,$$

for all $i \in \{0, 1, \dots, k-1\}$. Then $Y' \succeq 0$.

Proof. By Proposition 26, it suffices to show that $Y_{k,k} \succeq 0$, which we do by induction on k. The base case holds by assumption, as $Y_{0,0} \succeq 0$. Assuming $Y_{i,i} \succeq 0$ for all $i \in \{0, \dots, k-1\}$ and the existence of $\tilde{L}_{k-1}$, we can write

$$Y_{k,k} = (Y_{k,k} - \tilde{L}_{k-1}^\top Y_{k-1,k-1}\tilde{L}_{k-1}) + \tilde{L}_{k-1}^\top Y_{k-1,k-1}\tilde{L}_{k-1}.$$

The both summands on the right hand side are positive semidefinite (first one by assumption and second one by the inductive hypothesis). Thus, it follows that $Y_{k,k} \succeq 0$, and consequently $Y' \succeq 0$. □

Of course, in general if we know that $Y_{i,i} \succeq 0$, and *any* $B_i \times B_{i+1}$ matrix M where $Y_{i+1,i+1} - M^\top Y_{i,i} M \succeq 0$, then $Y_{i+1,i+1} \succeq 0$ follows. However, we believe that using the $\tilde{L}_i$ matrices described in Proposition 31 to establish the positive semidefiniteness of $Y_{k,k}$ can have special combinatorial meaning — at each inductive step, we are essentially "peeling off" the information in $Y_{i,i}$ that is "inherited" from $Y_{i-1,i-1}$. More importantly, if $Y_{k,k}$ is in fact positive semidefinite, then we are assured that such $\tilde{L}_i$ matrices exist.

Proposition 32. *Suppose the assumption in Proposition 26 holds, and $Y_{k,k} \succeq 0$. Then for all $i \in \{0, \dots, k-1\}$, there exists a $B_i \times B_{i+1}$ matrix $\tilde{L}_i$ where $Y_{i,i}\tilde{L}_i = Y_{i,i+1}$.*

Proof. The claim holds if and only if the column space of $Y_{i,i+1}$ is contained in that of $Y_{i,i}$, which is true if and only if the left nullspace of $Y_{i,i}$ is contained in that of $Y_{i,i+1}$. Thus, it suffices to show that whenever $Y_{i,i}^\top x = Y_{i,i}x = 0$, then $Y_{i,i+1}^\top x = Y_{i+1,i}x = 0$.

First, $Y_{k,k} \succeq 0$ implies that $L_{i+1,k} Y_{k,k} L_{i+1,k}^\top = Y_{i+1,i+1} \succeq 0$. Then we can write $Y_{i+1,i+1} = U^\top U$ for some matrix U. Now,

$$\begin{aligned} Y_{i,i}x = 0 \ &\Rightarrow \ x^\top Y_{i,i} x = 0 \\ &\Rightarrow \ x^\top L_{i,i+1} Y_{i+1,i+1} L_{i,i+1}^\top x = 0 \\ &\Rightarrow \ (U L_{i,i+1}^\top x)^\top (U L_{i,i+1}^\top x) = 0 \\ &\Rightarrow \ U L_{i,i+1}^\top x = 0 \\ &\Rightarrow \ U^\top U L_{i,i+1}^\top x = 0 \\ &\Rightarrow \ Y_{i+1,i} x = 0. \end{aligned}$$

Hence, we are finished. □

Thus, we see that Proposition 31 provides a template to inductively establish the positive semidefiniteness of our certificate matrix. Essentially, we want to show that

$$\begin{pmatrix} LAL^\top & AL^\top \\ LA & A \end{pmatrix} \succeq 0, \tag{5.5}$$

while knowing that $LAL^\top \succeq 0$ and there exists $\tilde{L}$ such that $LA = LAL^\top \tilde{L}$. We have seen that the task of showing (5.5) can be reduced to showing $A - \tilde{L}^\top LAL^\top \tilde{L} \succeq 0$. Next, we consider situations when the latter may be easier to verify than just directly showing $A \succeq 0$. One way of showing that a matrix is positive semidefinite is by finding all of its eigenvectors and eigenvalues. Here, we show that sometimes $A - \tilde{L}^\top LAL^\top \tilde{L}$ does have a simpler eigenspace structure than A, in the presence of the above $L, \tilde{L}$ matrices.

Proposition 33. *Suppose $A \in \mathbb{S}^n$, and $L, \tilde{L}$ are $m \times n$ matrices such that $LAL^\top \tilde{L} = LA$. Let $B := \tilde{L}^\top LAL^\top \tilde{L}$. If x is an eigenvector of A with eigenvalue λ, then*

$$Bx = \begin{cases} \lambda x & \text{if } x \text{ is in the rowspace of } L; \\ 0 & \text{if } x \text{ is in the nullspace of } L. \end{cases}$$

Proof. First, note that $B = \tilde{L}^\top LA$. Thus, if $Lx = 0$, then

$$Bx = (\tilde{L}^\top LA)x = \lambda \tilde{L}^\top Lx = 0.$$

On the other hand, if x is in the rowspace of L, then $\exists y \in \mathbb{R}^m, x = L^\top y$. Therefore,

$$Bx = (\tilde{L}^\top LA)(L^\top y) = (LA)^\top y = AL^\top y = \lambda x.$$

Thus, our claim follows. □

Proposition 33 tells us that, if no eigenspace of A has nontrivial intersection with both the rowspace and nullspace of L (which are orthogonal complements of each other in $\mathbb{R}^n$), then the eigenspaces of B and $A - B$ are aligned (i.e. x is an eigenvector of one if and only if it is also an eigenvector of the other). Moreover, B is obviously positive semidefinite by the fact that $LAL^\top$ is positive semidefinite, and no eigenvector of A can have nonzero eigenvalue for both B and $A - B$. Thus, B essentially maximizes the eigenspace information in A we can obtain from the fact that $LAL^\top \succeq 0$. Now if we could find all eigenvectors and eigenvalues of $A - B$, then we can write

$$A = B + \sum \lambda v v^\top$$

where v's are the normalized eigenvectors of $A - B$ (which we know are in the nullspace of L), and λ are their eigenvalues.

Another way to look at the above observation is the following:

Proposition 34. *Suppose $A \in \mathbb{S}^n$, and $L, \tilde{L}$ are $m \times n$ matrices such that $LAL^\top \tilde{L} = LA$. If*

$$A \in \left(\left\{ \tilde{L}^\top M \tilde{L} : M \in \mathbb{S}_+^m \right\} + \left\{ v v^\top : v \in \operatorname{Null}(L) \right\} \right),$$

then $A \succeq 0$.

While the proof to Proposition 34 is elementary (if A is contained in the Minkowski sum of two sets of positive semidefinite matrices, it is certainly positive semidefinite itself), it translates the task of proving $A \succeq 0$ into a membership problem, where various linear algebraic and geometric tools could be applied as the set in question is a convex cone. Also, depending on what we know about the matrices A, L and $\tilde{L}$, the set in Proposition 34 can be loosened of tightened to allow easier verification of the containment of A.

5.3 Connecting eigenspaces of certificate matrices with combinatorial objects

While there are many different ways one can prove that a matrix is positive semidefinite, sometimes finding all of its eigenvalues is the most straightforward approach. In this section, we present several examples where elegant connections can be made between the eigenspaces of certificate matrices and certain families of combinatorial objects. These connections can provide deeper insights into the underlying optimization problem at hand, and perhaps even give rise to new combinatorial identities and spark new threads of research.

For example, in establishing the LS_+-rank of $MT(K_{2n+1})$, Stephen and Tunçel [ST99] proved the following:

Proposition 35 (Lemma 4.2 in [ST99]). *Let $G = (V, E)$ be the complete graph on $2n+1$ vertices where $n \geq 1$. Define $Y \in \mathbb{R}^{(\{0\} \cup E) \times (\{0\} \cup E)}$ where*

- $Y[0, 0] = 1$;

- *$Y[i,0] = Y[i,i] = Y[0,i] = \frac{1}{2n}$ for all $i \in E$;*
- *$Y[i,j] = \frac{1}{2n(2n-2)}$ for all $i,j \in E$ such that $\{i,j\}$ is a matching of size 2;*
- *$Y[i,j] = 0$ otherwise.*

Then $Y \succeq 0$.

Stephen and Tunçel proved Proposition 35 by listing all eigenvalues and their multiplicities of Y. Some of the eigenvectors of Y can be described as follows:

- Let C be any even cycle of length $2k$ in G, and let $\{i_1, i_2, \ldots, i_{2k}\}$ be the edges on the cycle in order. Define $x \in \mathbb{R}^{E \cup \{0\}}$ such that

  $$x[j] = \begin{cases} 1 & \text{if } j \in \{i_1, i_3, \ldots, i_{2k-1}\}; \\ -1 & \text{if } j \in \{i_2, i_4, \ldots, i_{2k}\}; \\ 0 & \text{otherwise.} \end{cases}$$

  Then x is an eigenvector of Y with eigenvalue $\frac{2n-1}{2n(2n-2)}$. In particular, let H be a fixed set of edges that make up a cycle of G that passes through all $2n+1$ vertices. Then for every $j \in E \setminus H$, $H \cup \{j\}$ contains a unique even cycle. There are

  $$|E| - |H| = \binom{2n+1}{2} - (2n+1) = (2n+1)(n-1)$$

  choices of the edge j, and the eigenvectors obtained from these $(2n+1)(n-1)$ even cycles are linearly independent, since $x_j \neq 0$ for exactly one of these vectors for every $j \in E \setminus H$.

- Let v_1, v_2 be two fixed vertices in G. Define $x \in \mathbb{R}^{E \cup \{0\}}$ such that

  $$x[j] = \begin{cases} 1 & \text{if } j \in E \text{ is incident with } v_1 \text{ but not } v_2 \\ -1 & \text{if } j \in E \text{ is incident with } v_2 \text{ but not } v_1; \\ 0 & \text{otherwise.} \end{cases}$$

  Then x is an eigenvector of Y with eigenvalue 0. Now it is not hard to see that, if we fix v_1 and let v_2 vary over $V \setminus \{v_1\}$, we obtain $2n$ linearly independent eigenvectors.

Notice that

$$(2n+1)(n-1) + 2n = \binom{2n+1}{2} - 2,$$

and hence all but two of the $|E|+1$ eigenvalues of Y are accounted by the two families of eigenvectors described above, and their multiplicity arguments are very transparent when described combinatorially with even cycles and complete bipartite graphs.

Next, we turn to the matrix studied by Georgiou in his result on max-cut in [Geo10], as defined in Proposition 27. Again, let $y \in \mathbb{R}^{\mathcal{A}_{2n}^+}$ where

$$y[S|_1] := \frac{\binom{n+1/2}{|S|}}{\binom{2n+1}{|S|}}.$$

Let Y be the symmetric minor of $M_n(y)$ whose rows and columns are indexed by elements of $\mathcal{A}_n^+$. Recall that, as observed by Georgiou in [Geo10], Y has the same rank as $M_n(y)$. Thus, the task of proving that $M_n(y) \succeq 0$ can be reduced to establishing the positive semidefiniteness of Y, which is a $\binom{2n+1}{n} \times \binom{2n+1}{n}$-matrix.

Georgiou then pointed out that Y can be expressed as a linear combination of the adjacency matrices of a generalization of Kneser graphs: Given integers n, r, s where $n \geq r \geq s$, define $G_{r,s}^n$ to be the graph whose vertices are the subsets of $[n]$ of size r, and we join two vertices S, T by an edge if and only if $|S \cap T| = s$. In the case of $s = 0$, we obtain the traditional Kneser graphs, whose eigenvectors and eigenvalues were known (e.g. see [Lov79] and [Kar99]). Georgiou then showed that the eigenvectors for Kneser graphs are also eigenvectors for the generalized version, and went on to prove that the eigenvalues of Y are nonnegative.

We now return to the matching problem. Recall the matrices $Y_{n,k,k}$, as defined immediately before Theorem 28. For the rest of this section, we will focus on these matrices and describe some of its eigenvectors, which seem to have a lot of connections with integer partitions and Young diagrams. Somewhat surprisingly, many of these connections seem to only rely on the sparsity pattern of $Y_{n,k,k}$, and not on the value of its entries. To illustrate this, let $m \in \mathbb{R}^{k+1}$ be a vector whose entries are indexed by $\{k, k+1, \ldots, 2k\}$. We define $Y_{n,k,k}(m) \in \mathbb{R}^{\mathcal{M}_{n,k} \times \mathcal{M}_{n,k}}$ to be the matrix where

$$Y_{n,k,k}(m)[S,T] := \begin{cases} m_i & \text{if } S \cup T \text{ is a matching of size } i; \\ 0 & \text{otherwise.} \end{cases} \tag{5.6}$$

Thus, $Y_{n,k,k}$ is merely $Y_{n,k,k}(m)$ with

$$m = \left((n-2k-1)!!, (n-2k-3)!!, \ldots, (n-4k-1)!!\right)^\top.$$

Before we describe some eigenvectors of $Y_{n,k,k}(m)$, we need some elementary facts about integer partitions and Young tableaux. There are many other instances in the literature where various algebraic objects based on matchings in graphs were associated with Young tableaux (for a recent paper involving 3-matchings, see [GMP14]). Given a nonnegative integer k, a *partition* of k is a sequence of nonincreasing, positive integers that sum up to k. For instance, the five distinct partitions of 4 are

$$(4), (3,1), (2,2), (2,1,1), (1,1,1,1).$$

Moreover, each partition λ can be visualized by a *Young diagram*, an array with rows aligned to the left such that row i has λ_i boxes. For example, the five partitions of 4

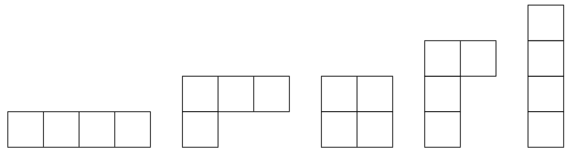

Figure 5.1: Young diagrams for all partitions of size 4.

above can be represented by the five Young diagrams in Figure 5.1, respectively. We now construct a family of vectors that are eigenvectors of $Y_{n,k,k}$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of size k, define $\lambda_0 := \lambda_1$. Consider a set of vertices

$$S := \{v_{ij} : 0 \leq i \leq \ell : 1 \leq j \leq 2\lambda_i\}. \tag{5.7}$$

Thus, S is a set of $2k + 2\lambda_1$ vertices. We can imagine S as being laid out in a grid as suggested by the indexing of its elements (i.e. we say $v_{ij}, v_{i'j'} \in S$ are in the same row if $i = i'$, and in the same column if $j = j'$, and so on). Note that S has row lengths (from top to bottom) $2\lambda_1, 2\lambda_1, 2\lambda_2, \ldots, 2\lambda_\ell$. For example, Figure 5.2 gives a visualization of this layout of vertices for $\lambda = (3, 3, 2)$.

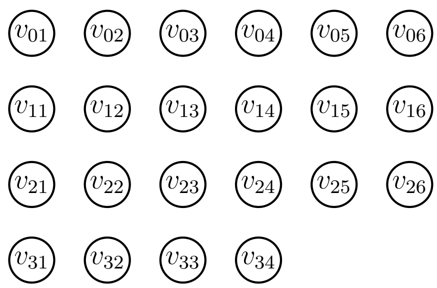

Figure 5.2: Laying out vertices of S in a grid for $\lambda = (3, 3, 2)$.

Next, we define $\mathcal{T}(\lambda)$ to be set of λ-*permutations*, which are Young diagrams of shape $(2\lambda_0, 2\lambda_1, \ldots, 2\lambda_\ell)$, such that each column of size q is filled with entries from 0 to $q-1$, each appearing exactly once. For instance, Figure 5.3 shows a λ-permutation for $\lambda = (3, 3, 2)$.

1	1	1	3	0	1
0	3	3	2	1	0
3	2	0	1	2	2
2	0	2	0		

Figure 5.3: Example of a $(3, 3, 2)$-permutation.

As the name suggests, for each λ-permutation $T \in \mathcal{T}(\lambda)$, we look at each column of T as a permutation on $\{0, 1, \ldots, q-1\}$. Given such a permutation π, we let $f(\pi)$ denote the minimum number of two-element-swaps to transform π into the identity permutation. For instance, if we let $\pi_1, \ldots, \pi_6$ denote the permutations corresponding to the 6 columns of the λ-permutation in Figure 5.3, then $f(\pi_i) = 2, 2, 3, 2, 0, 1$ for $i \in \{1, 2, 3, 4, 5, 6\}$, respectively. We slightly abuse notation and define $f(T) := \sum_{i=1}^{\lambda_1} f(\pi_i)$ a λ-partition T. Next, for each $T \in \mathcal{T}(\lambda)$, we define $T(S)$ to be the rearrangement of vertices in S such that the i^{th} row of vertices in $T(S)$ is $\left\{ v_{T(i,j),j} : j \in [2\lambda_i] \right\}$ for all $i \in \{0, 1, \ldots, \ell\}$. For instance, for the set S in Figure 5.2 and T in Figure 5.3, the set $T(S)$ would be arranged as follows:

$$
\begin{array}{cccccc}
v_{11} & v_{12} & v_{13} & v_{34} & v_{05} & v_{16} \\
v_{01} & v_{32} & v_{33} & v_{24} & v_{15} & v_{06} \\
v_{31} & v_{22} & v_{03} & v_{14} & v_{25} & v_{26} \\
v_{21} & v_{02} & v_{23} & v_{04} & &
\end{array}
$$

Figure 5.4: A rearrangement of S under a λ-permutation.

Given a λ-permutation T, we say that a matching $M \in \mathcal{M}_{n,k}$ is *aligned* with T if M saturates every vertex in $T(S)$ except those in the top (zeroth) row, and every edge in M joins two vertices on the same row in $T(S)$. Notice that if M is aligned with T, then it saturates exactly $2\lambda_1 + \cdots + 2\lambda_\ell = 2k$ vertices, and thus $|M| = k$.

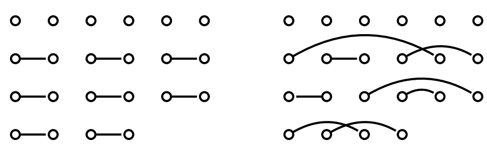

Figure 5.5: Visualizing aligning matchings.

Next, given a set of vertices S and a λ-permutation T, we define the vector $u^S \in \mathbb{R}^{\mathcal{M}_{n,k}}$ such that

$$
u^{T(S)}[M] := \begin{cases} 1 & \text{if } M \text{ is aligned with } T; \\ 0 & \text{otherwise.} \end{cases}
$$

Then we define the vector

$$
z^S := \sum_{T \in \mathcal{T}(\lambda)} (-1)^{f(T)} u^{T(S)}. \tag{5.8}
$$

It is also not hard to see that if $M \in \mathcal{M}_{n,k}$ is aligned with both $T(S)$ and $T'(S)$, then $(-1)^{f(T)} = (-1)^{f(T')}$. Thus, we see that $|z^S[M]|$ gives the number of λ-partitions that are aligned with M. Moreover, we can extend the parity function f and define M to be even (resp. odd) if $(-1)^{f(T)}$ is equal to 1 (resp. -1) for some $T \in \mathcal{T}(\lambda)$ that is aligned with M. Note that we do not define the parity of a matching M where $z^S[M] = 0$.

For an example, let $\lambda := (3,1)$ and S is a fixed set of 14 vertices, as arranged in Figure 5.6. Consider the matchings M_1 and M_2.

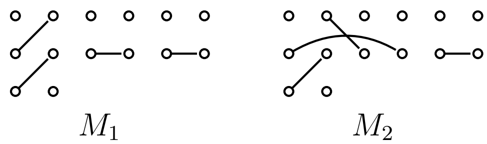

Figure 5.6: A set of vertices corresponding to $\lambda = (3,1)$ and two relevant matchings.

Then $z^S[M_1] = 2$ as M_1 is aligned with the two λ-permutations in Figure 5.7, all of which satisfies $(-1)^{f(T)} = 1$. On the other hand, $z^S[M_2] = 1$ as it is only aligned with the first λ-permutation in Figure 5.7.

0	2	0	0	0	0
1	0	1	1	1	1
2	1				

0	2	0	0	0	0
2	1	1	1	1	1
1	0				

Figure 5.7: λ-permutations related to matchings in Figure 5.6.

We next show that, for some special partitions λ, z^S gives an eigenvector of $Y_{n,k,k}(m)$:

Theorem 36. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of size k such that either $\ell = 1$ or $\lambda_2 = 1$. Then for every set of $2k + 2\lambda_1 \leq n$ vertices S as labelled as in (5.7), z^S is an eigenvector of $Y_{n,k,k}(e_{k+i})$ with eigenvalue*

$$\mu := \binom{k-\ell+1}{i}(2i-1)!! + (\ell-1)\binom{k-\ell}{i-1}(2i-3)!!,$$

for all $i = \{0, \ldots, k\}$.

Proof. First, given a set of vertices S, we say that a matching $M \in \mathcal{M}_{n,k}$ is S*-relevant* if for every column of vertices in S of size q, M saturates exactly $q-1$ of the vertices. Notice that $z^S[M] \neq 0$ implies that S is M-relevant.

We first show that, if M is not S relevant, then $(Y_{n,k,k}(e_i)z^S)[M] = 0$. Since M is not S relevant, there must exist some column i of S where M leaves both vertices v_{ij_1}, v_{ij_2} unsaturated. Now consider

$$Y_{n,k,k}(e_i)z^S[M] = \sum_{M' \in \mathcal{M}_{n,k} : M \cup M' \in \mathcal{M}_{n,k+i}} z^S[M']. \tag{5.9}$$

Observe that every M' that contributes to (5.9) must be S-relevant, and thus must saturate at least one of v_{ij_1}, v_{ij_2}. If it saturates exactly one of them, then assume without loss of generality that it saturates v_{ij_1} but not v_{ij_2}. Let $\{u, v_{ij_1}\}$ be the edge in M' that saturates v_{ij_1},and define the matching

$$M'' := (M' \cup \{\{u, v_{ij_2}\}\}) \setminus \{\{u, v_{ij_1}\}\}.$$

Then, it is not hard to see that $M \cup M'' \in \mathcal{M}_{n,k+i}$, and $z^S[M'] = -z^S[M'']$. Thus, the contributions of different matchings cancel out, and the sum in (5.9) vanishes. Next, suppose M' saturates both v_{ij_1}, v_{ij_2}. Then M' must contain edges $\{u_1, v_{ij_1}\}, \{u_2, v_{ij_2}\}$. In this case, define

$$M'' := (M' \cup \{\{u_1, v_{ij_2}\}, \{u_2, v_{ij_1}\}\}) \setminus \{\{u_1, v_{ij_1}\}, \{u_2, v_{ij_2}\}\}.$$

Then again, $M \cup M'' \in \mathcal{M}_{n,k+i}$ and $z^S[M'] = -z^S[M'']$. Hence, the sum in (5.9) again evaluates to zero.

Next, we focus on M that are S-relevant. Given the partition λ that either has at most one part of size greater than 1, we partition S into S_1, S_2, where

$$\begin{aligned} S_1 &:= \{v_{ij} : j \leq 2\}; \\ S_2 &:= \{v_{ij} : 0 \leq i \leq 1, 3 \leq j \leq \lambda_1\}. \end{aligned}$$

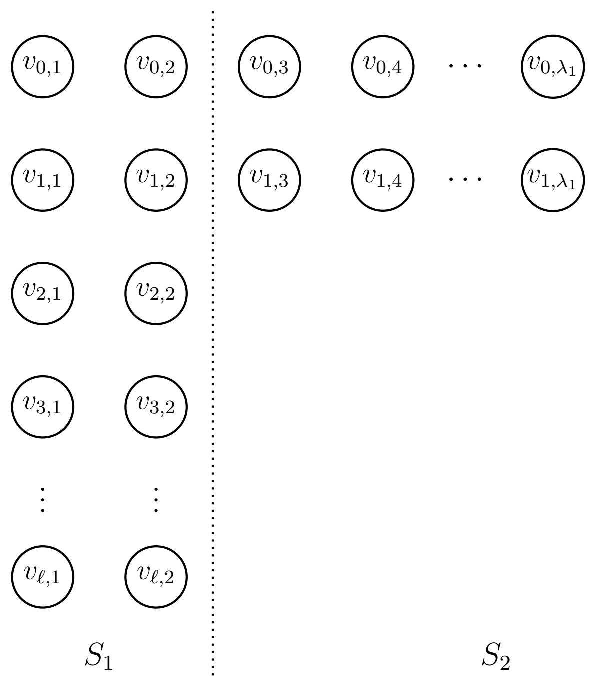

Figure 5.8: Illustrating the partition of S into S_1, S_2.

If M is S-relevant, then either $|z^S[M]| = \ell!$ (if it has no edges crossing S_1 and S_2) or $|z^S[M]| = (\ell - 1)!$ (if it has two edges crossing S_1 and S_2). We say that M is of type 0 if

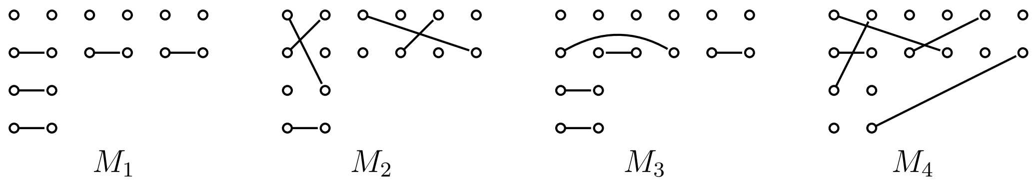

Figure 5.9: Examples of type 0 matchings (M_1, M_2) and type 1 matchings (M_3, M_4).

it has no edge crossing S_1, S_2, and type 1 otherwise. For instance, in Figure 5.9, M_1, M_2 are type 0 matchings, while M_3, M_4 are type 1.

Now it only remains to evaluate (5.9) for each of these two cases. Suppose M is of type 0, and $z^S[M] = \ell!$. Then M must contain ℓ edges that join vertices in S_1, and $k-\ell$ edges that join vertices between S_2. Now let us exhaust all the possible matchings M', where M' is S-relevant and $M \cup M' \in \mathcal{M}_{n,k+i}$. There are three cases:

1. M' contains all ℓ S_1-edges from M. In that case, M' must be of type 0, and contains exactly i S_2 edges that are not in M. There are $\binom{k-\ell}{i}|\mathcal{M}_{2i,i}|$ to select these edges, and thus these matchings contribute a total of $\binom{k-\ell}{i}|\mathcal{M}_{2i,i}|\ell!$ to (5.9).

2. M' contains $\ell-1$ S_1-edges from M and is of type 0. The total contribution in this case is $\ell\binom{k-\ell}{i-1}|\mathcal{M}_{2i-2,i-1}|\ell!$.

3. M' contains $\ell-1$ S_1-edges from M, and is of type 1. The total contribution is $\ell\binom{k-\ell}{i-1}(2i-2)(2i-3)|\mathcal{M}_{2i-4,i-2}|(\ell-1)!$.

Thus, in this case, (5.9) evaluates to

$$
\begin{aligned}
& \binom{k-\ell}{i}|\mathcal{M}_{2i,i}|\ell! + \ell\binom{k-\ell}{i-1}|\mathcal{M}_{2i-2,i-1}|\ell! + \ell\binom{k-\ell}{i-1}(2i-2)(2i-3)|\mathcal{M}_{2i-4,i-2}|(\ell-1)! \\
= & \left((2i-1)\binom{k-\ell}{i} + (\ell+2i-2)\binom{k-\ell}{i-1}\right)(2i-3)!!\ell! \\
= & \left((2i-1)\left(\binom{k-\ell}{i} + \binom{k-\ell}{i-1}\right) + (\ell-1)\binom{k-\ell}{i-1}\right)(2i-3)!!\ell! \\
= & \left((2i-1)\binom{k-\ell+1}{i} + (\ell-1)\binom{k-\ell}{i-1}\right)(2i-3)!!\ell! \\
= & \mu z^S[M].
\end{aligned}
$$

Note that, in all cases, if $M, M' \in \mathcal{M}_{n,k}$ are S-relevant, and $M \cup M'$ is also a matching, then M, M' must have the same parity. Next, we turn to the case when $z^S[M] = (\ell-1)!$, and let f_1, f_2 be the two edges in M that cross S_1 and S_2. Then M must contain $(\ell-1)$ S_1-edges and $(k-\ell-1)$ S_2-edges. Here are all the possible cases where M' is also S-relevant, and $M \cup M' \in \mathcal{M}_{n,k+i}$:

1. M' contains all $(\ell-1)$ S_1-edges in M. There are a total of $\binom{k-\ell+1}{i}|\mathcal{M}_{2i,i}|$ such matchings, and $\binom{k-\ell-1}{i-2}|\mathcal{M}_{2i-2,i-1}|$ of them are of type 0, with the remainder being type 1. Thus, the total contribution in this case is
$$\left(\binom{k-\ell+1}{i}|\mathcal{M}_{2i,i}| - \binom{k-\ell-1}{i-2}|\mathcal{M}_{2i-2,i-1}|\right)(\ell-1)! + \binom{k-\ell-1}{i-2}|\mathcal{M}_{2i-2,i-1}|\ell!.$$

2. M' contains only $(\ell-2)$ S_1-edges in M. Then it must contain both f_1 and f_2, and must be of type 1. The total contribution here is $(\ell-1)\binom{k-\ell-1}{i-1}|\mathcal{M}_{2i-2,i-1}|(\ell-1)!$.

Adding them up, we get
$$\begin{aligned}
& \left(\binom{k-\ell+1}{i}|\mathcal{M}_{2i,i}| - \binom{k-\ell-1}{i-2}|\mathcal{M}_{2i-2,i-1}|\right)(\ell-1)! + \\
& \binom{k-\ell-1}{i-2}|\mathcal{M}_{2i-2,i-1}|\ell! + (\ell-1)\binom{k-\ell-1}{i-1}|\mathcal{M}_{2i-2,i-1}|(\ell-1)! \\
= \; & \left(\binom{k-\ell+1}{i}(2i-1) + (\ell-1)\binom{k-\ell-1}{i-2} + (\ell-1)\binom{k-\ell-1}{i-1}\right)(2i-3)!!(\ell-1)! \\
= \; & \left((2i-1)\binom{k-\ell+1}{i} + (\ell-1)\binom{k-\ell}{i-1}\right)(2i-3)!!(\ell-1)! \\
= \; & \mu z^S[M].
\end{aligned}$$

Thus, we obtain in both cases that (5.9) evaluates to $\mu z^S[M]$. The cases for $z^S[M] = -\ell!$ and $-(\ell-1)!$ can be shown similarly. Hence, we conclude that z^S is an eigenvector of $Y_{n,k,k}(e_{k+i})$ with eigenvalue μ. $\square$

Since $Y_{n,k,k}(m) = \sum_{i=k}^{2k} m_i Y_{n,k,k}(e_i)$, it follows that z^S is an eigenvector of $Y_{n,k,k}(m)$ for all $m \in \mathbb{R}^{k+1}$, for the partitions described above.

In the next sections, when we explore additional techniques that will further reduce the task of proving $Y_{n,k,k} \succeq 0$, we will point out another connection between matchings and integer partitions (Proposition 50). We will also see another example of relating eigenvectors and eigenvalues to combinatorial objects when we compute the eigenvalues of $ZZ^\top$ (where Z is the zeta matrix of a set) in Chapter 6.

5.4 Commutative maps and reductions using symmetries

We have seen that it can be helpful to partition the rows and columns of a certificate matrix into "slices" (as in (5.1)), and that sometimes their eigenvectors can be related to families of combinatorial objects. Moreover, such structures and connections lend themselves rather naturally to inductive arguments. For instance, we saw in Proposition 31 how one could verify the positive semidefiniteness of a matrix within this setup by induction.

In [HT08], Hong and Tunçel outlined a framework of establishing lower-bound results for the Lovász–Schrijver operators. They looked at maps that are commutative with these operators, and showed that, using these maps, the certificate matrix of a smaller instance of a problem can often be lifted to form one for a larger instance. This type of technique can be extremely useful when one tries to verify a family of certificate matrices inductively. Herein, we show that some of their results can be extended to stronger lift-and-project operators (such as SA and SA_+), and look at how these maps can simplify the construction and verification of certificate matrices for sets that have a lot of symmetries.

5.4.1 Maps that commute with operators based on SA

First, we define a lift-and-project operator Γ to be *union-commutative* if

$$\mathrm{conv}(\Gamma(P_1) \cup \Gamma(P_2)) \subseteq \Gamma(\mathrm{conv}(P_1 \cup P_2)),$$

for all $P_1, P_2 \subseteq [0,1]^n$. It was shown in [HT08] (in a slightly different language) that the three Lovász–Schrijver operators $\mathrm{LS}_0, \mathrm{LS}$ and LS_+ are all union-commutative. Here, we show that this property is also shared by many other operators:

Proposition 37. *The operators* $\mathrm{SA}^k, \mathrm{SA}'^k, \mathrm{SA}_+^k$ *and* $\mathrm{SA}_+'^k$ *are all union-commutative, for every* $k \geq 1$.

Proof. We only show the proof of the result for SA'^k, as the arguments for the other operators rely on the same observations. Let $x \in \mathrm{conv}(\mathrm{SA}'^k(P_1) + \mathrm{SA}'^k(P_2))$. Then, there exists $Y = \lambda Y_1 + (1-\lambda)Y_2$, where $Ye_{\mathcal{F}} = \hat{x}, Y_i \in \widetilde{\mathrm{SA}}'^k(P_i), i \in \{1,2\}$, and $\lambda \in [0,1]$. Now

$$Y[\mathcal{F},\mathcal{F}] = \lambda Y_1[\mathcal{F},\mathcal{F}] + (1-\lambda)Y_2[\mathcal{F},\mathcal{F}] = \lambda + (1-\lambda) = 1,$$

so (SA 1) holds. Next, it is not hard to see that for every $\alpha \in \mathcal{A}_k$, $Y_1 e_\alpha \in K(P_1)$ and $Y_2 e_\alpha \in K(P_2)$ imply that $Ye_\alpha \in K(\mathrm{conv}(P_1 \cup P_2))$, so (SA 2) is satisfied as well. Since taking convex combination of matrices preserves (OMC), (SA 3) and (SA 5) also hold. Finally, for (SA′ 4), if $Y[\alpha,\beta] \neq 0$, then at least one of $Y_1[\alpha,\beta], Y_2[\alpha,\beta]$ is nonzero, which means that $\alpha \cap \beta \cap (P_1 \cup P_2) \neq \emptyset$. Hence, $\alpha \cap \beta \cap \mathrm{conv}(P_1 \cup P_2) \neq \emptyset$. □

Note that given polytopes $P_1, P_2 \subseteq [0,1]^n$, a point $x \in \mathrm{conv}(P_1 \cup P_2)$ if and only if the system

$$\begin{array}{rcl} \hat{x} & = & y+z \\ y & \in & K(P_1) \\ z & \in & K(P_2) \end{array}$$

is feasible. This shows that if we have a separation oracle for P_1, P_2, then $\mathrm{conv}(P_1 \cup P_2)$ is also tractable. In particular, $\Gamma(\mathrm{conv}(P_1 \cup P_2))$ is then also tractable if Γ only requires a weak separation oracle for the input relaxation. However, it is not apparent that we can efficiently generate a facet description of $\text{conv}(P_1 \cup P_2)$ from facet descriptions of P_1 and P_2. Thus, for operators that depend on the algebraic description of the relaxation (such as Las and the Bienstock–Zuckerberg variants), it is not clear if the linear optimization problem over $\Gamma(\text{conv}(P_1 \cup P_2))$ is polynomial-time solvable.

Next, given a function $M : \mathbb{R}^n \to \mathbb{R}^p$, we extend the point-to-point mapping M to a set-to-set mapping by defining

$$M(P) := \{M(x) : x \in P\}$$

for all $P \subseteq \mathbb{R}^n$. Then we say that a lift-and-project operator Γ is *M-commutative* if

$$M(\Gamma(P)) \subseteq \Gamma(M(P)),$$

for every set $P \subseteq [0,1]^n$. Here, we show that SA_+^k and $\text{SA}_+'^k$ are commutative with a special family of maps.

Proposition 38. *Suppose $M : \mathbb{R}^n \to \mathbb{R}^p$ is a function such that, for all $i \in [p]$, one of the following applies:*

1. $(M(x))_i = 0$ *for all* $x \in \mathbb{R}^n$*;*
2. $(M(x))_i = 1$ *for all* $x \in \mathbb{R}^n$*;*
3. *there exists* $j \in [n]$ *such that* $(M(x))_i = x_j$ *for all* $x \in \mathbb{R}^n$*;*
4. *there exists* $j \in [n]$ *such that* $(M(x))_i = 1 - x_j$ *for all* $x \in \mathbb{R}^n$*.*

Then SA_+^k and $\text{SA}_+'^k$ are both M-commutative, for every $k \geq 1$.

Proof. We only prove the claim for SA_+^k, as the same argument applies for $\text{SA}_+'^k$. Herein, let $\mathcal{A}_{n,k}$ denote $\{S|_1 \cap T|_0 : S, T \subseteq [n], |S| + |T| \leq k\}$, and define $\mathcal{A}_{p,k}$ similarly. For every $i \in [p]$, we define

$$g(i|_1) := \begin{cases} \mathcal{F} & \text{if } (M(x))_i = 1 \text{ for all } x \in \mathbb{R}^n; \\ j|_1 & \text{if } (M(x))_i = x_j \text{ for all } x \in \mathbb{R}^n; \\ j|_0 & \text{if } (M(x))_i = 1 - x_j \text{ for all } x \in \mathbb{R}^n; \\ \emptyset & \text{if } (M(x))_i = 0 \text{ for all } x \in \mathbb{R}^n. \end{cases}$$

We also define $g(i|_0) := \mathcal{F} \setminus g(i|_1)$ for every $i \in [p]$. Furthermore, given $\alpha = S|_1 \cap T|_0 \in \mathcal{A}_{p,k}$, define

$$g(\alpha) := \left(\bigcap_{j \in S} g(j|_1) \right) \cap \left(\bigcap_{j \in T} g(j|_0) \right).$$

Note that this definition is consistent with that of $g(i|_1)$ and $g(i|_0)$. Also observe that $g(\alpha) \in \mathcal{A}_{n,k}$, for all $\alpha \in \mathcal{A}_{p,k}$.

Next, define $\tilde{M} \in \mathbb{R}^{\mathcal{A}_{p,k} \times \mathcal{A}_{n,k}}$, where $\tilde{M}[\alpha, \beta] = 1$ if $g(\beta) = \alpha$, and 0 otherwise. We claim that if $Y \in \tilde{\mathrm{SA}}_{+}^{k}(P)$, then $\tilde{M} Y \tilde{M}^{\top} \in \tilde{\mathrm{SA}}_{+}^{k}(M(P))$. This is due to the construction of $\tilde{M}$, as $\tilde{M} Y \tilde{M}^{\top}[\alpha, \beta] = Y[g(\alpha), g(\beta)]$ for every $\alpha, \beta \in \mathcal{A}_{p,k}$. Thus, $\tilde{M} Y \tilde{M}^{\top}$ inherits the conditions $(\mathrm{SA}_+ 1)$ – $(\mathrm{SA}_+ 5)$ from Y. Also, $Y \succeq 0 \Rightarrow \tilde{M} Y \tilde{M}^{\top} \succeq 0$. Since $\tilde{M} Y \tilde{M}^{\top}$ is a certificate for $M(x)$. we see that SA_{+}^{k} is M-commutative. $\square$

These mappings M can be described as taking a set in $[0,1]^n$ and producing a set in $[0,1]^p$ where each of the p coordinates is either obtained by embedding (i.e. $(M(x))_i = 0$ or 1), duplicating an existing coordinate (i.e. $(M(x))_i = x_j$) or flipping an existing coordinate (i.e. $(M(x))_i = 1 - x_j$). Also, one can use similar observations to show that such mappings are also commutative with operators such as SA^k and Las^k. A broader family of maps that contains those we just described were shown in [HT08] to be commutative with all three Lovász–Schrijver operators.

Hong and Tunçel remarked that these commutative maps lend themselves readily to be used in inductive arguments. Several such examples were provided in [HT08], and we expand on one of them here. Consider the graph K_{2n+3}, and label its edges from 1 to $\binom{2n+3}{2}$ by their lexicographic order. That is, the edges are labelled in order

$$\{1,2\}, \{1,3\}, \ldots, \{1, 2n+3\}, \{2,3\}, \ldots, \{n-1, n\}.$$

For each edge $f \in E(K_{2n+3})$, we let $K_{2n+3} \setminus f$ denote the graph with f and the two vertices it is incident with removed. We then define $M_f : \mathbb{R}^{E(K_{2n+1})} \to \mathbb{R}^{E(K_{2n+3})}$ where

$$(M_f(x))_i = \begin{cases} 1 & \text{if } i = f; \\ 0 & \text{if } i \neq f, \text{ but they share a common vertex;} \\ x_j & \text{if } i \text{ is the } j^{\text{th}} \text{ edge in } K_{2n+3} \setminus f. \end{cases}$$

In establishing their result on the LS_+-rank of the matching polytope of complete graphs in [ST99], Stephen and Tunçel proved the following:

Proposition 39 (Lemma 4.1 in [ST99]). *If* $\frac{1}{2n}\bar{e} \in \mathrm{LS}_{+}^{k}(MT(K_{2n+1}))$, *then* $\frac{1}{2n+2}\bar{e} \in \mathrm{LS}_{+}^{k+1}(MT(K_{2n+3}))$.

The sketch of their proof is the following: Since the map M_f is LS_+-commutative,

$$M_f\left(\frac{1}{2n}\bar{e}\right) \in \mathrm{LS}_{+}^{k}(M_f(MT(K_{2n+1})) \subseteq \mathrm{LS}_{+}^{k}(MT(K_{2n+3})),$$

for every edge f. Note that the last containment follows because $M_f(MT(K_{2n+1}) \subseteq MT(K_{2n+3})$, and LS_{+}^{k} preserves containment. Then it turns out that the matrix $Y \in \mathbb{R}^{(\{0\} \cup E(K_{2n+3})) \times (\{0\} \cup E(K_{2n+3}))}$ where

$$Ye_f = \begin{cases} \begin{pmatrix} 1 \\ \frac{1}{2n+2}\bar{e} \end{pmatrix} & \text{if } f = 0; \\ \frac{1}{2n+2}\hat{x}\left(M_f(\frac{1}{2n}\bar{e})\right) & \text{for all } f \in E(K_{2n+3}), \end{cases}$$

is a certificate for $\frac{1}{2n+2}\bar{e} \in \text{LS}_+^{k+1}(MT(K_{2n+3}))$. Note that one main ingredients that make this type of inductive proofs relatively simple is the fact that $\text{LS}_+^k(P) = \text{LS}_+(\text{LS}_+^{k-1}(P))$ for every set $P \subseteq [0,1]^n$ and integer $k \geq 1$ (same applies for the operators LS_0, LS). Thus, such a template cannot be readily extended for proving lower-bound results for, say, SA_+, since $\text{SA}_+^k(P)$ might be strictly contained in $\text{SA}_+(\text{SA}_+^{k-1}(P))$ in general. Another hurdle is that while a certificate matrix for a point in $\text{LS}_+^k(P)$ has dimension $(n+1) \times (n+1)$ for every $k \geq 1$, the certificate matrices for $\text{SA}_+^k(P)$ grows in size rather rapidly as k increases.

Nonetheless, we can still use the fact that the map M_f is SA_+-commutative to show that the SA_+-rank of $MT(K_{2n+1})$ is indeed nondecreasing with respect to n.

Proposition 40. *If $\frac{1}{2n}\bar{e} \in \text{SA}_+'^k(MT(K_{2n+1}))$, then $\frac{1}{2n+2}\bar{e} \in \text{SA}_+'^k(MT(K_{2n+3}))$.*

Proof. First, by Proposition 38, we obtain that $\text{SA}_+'^k$ is M_f-commutative, and thus

$$M_f\left(\frac{1}{2n}\bar{e}\right) \in \text{SA}_+'^k(M_f(MT(K_{2n+1})) \subseteq \text{SA}_+'^k(MT(K_{2n+3})).$$

Thus, by the convexity of $\text{SA}_+'^k(MT(K_{2n+3}))$, the point

$$\binom{2n+3}{2}^{-1} \sum_{f=1}^{\binom{2n+3}{2}} M_f\left(\frac{1}{2n}\bar{e}\right) = \frac{1}{2n+2}\bar{e}$$

is contained in $\text{SA}_+'^k(MT(K_{2n+3}))$. □

Proposition 40 immediately implies the following about $Y_{n,k,k}$ matrices:

Corollary 41. *If $Y_{2n+1,k,k} \succeq 0$, then $Y_{2p+1,k,k} \succeq 0$ for all integers $p > n$.*

Proof. By Theorem 28, $Y_{2n+1,k,k} \succeq 0$ implies that $\frac{1}{2n}\bar{e} \in \text{SA}_+'^k(MT(K_{2n+1}))$, which in turn implies that $\frac{1}{2n+2}\bar{e} \in \text{SA}_+'^k(MT(K_{2n+3}))$ by Proposition 40. Then we can construct a certificate matrix for $\frac{1}{2n+2}\bar{e}$ in $\tilde{\text{SA}}_+'^k(MT(K_{2n+3}))$ by taking the certificate matrices for $M_f(\frac{1}{2n}\bar{e})$ (as constructed in the proof of Proposition 38), and taking the average of these matrices. That would result in the matrix

$$Y = \begin{pmatrix} Y_{2n+3,0,0} & Y_{2n+3,0,1} & \cdots & Y_{2n+3,0,k} \\ Y_{2n+3,1,0} & Y_{2n+3,1,1} & \cdots & Y_{2n+3,1,k} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{2n+3,k,0} & Y_{2n+3,k,1} & \cdots & Y_{2n+3,k,k} \end{pmatrix}.$$

Since $Y \succeq 0$, its symmetric minor $Y_{2n+3,k,k}$ must also be positive semidefinite. Proceeding with this argument iteratively, and we obtain that $Y_{2p+1,k,k} \succeq 0$ for all integers $p > n$. □

5.4.2 Reducing matrices using permutation-commutative maps

Notice that Proposition 40 had a rather simple proof largely because we were working with a certificate matrix with very few distinct parameters, and that the underlying relaxation $MT(K_n)$ is highly symmetric. We now look into a particular type of commutative maps, and illustrate how the symmetries present in the given relaxation P can allow us assume that, if there exists a point in a lift-and-project relaxation that violates a valid inequality of the integer hull, then there exists a point that inherits all symmetries of P. We will further see that often times we can further assume that the certificate matrix of such a point also possesses these symmetries.

Given a compact, convex set $P \subseteq [0,1]^n$ and an $n \times n$ matrix Q, we define

$$Q(P) := \{Qx : x \in P\}.$$

The following elementary result shows that all matrices Q which satisfy $Q(P) = P$ must have a very simple structure.

Lemma 42. *Suppose $P \subseteq [0,1]^n$ is a compact, convex set that contains the unit simplex. If a matrix Q satisfies $Q(P) = P$, then Q is a permutation matrix.*

Proof. First, we show that each column of Q have exactly one nonzero entry. Since both P and $Q(P)$ are fully dimensional sets, Q must be invertible, and it is not hard to see that $Q^{-1}(P) = P$.

Now fix $i \in [n]$ and consider $Q\left(\frac{1}{2}e_i\right)$. We know $Q\left(\frac{1}{2}e_i\right) \neq 0$ (since Q is invertible). Next, consider $S \subset P$ to be the line segment joining 0 and $\frac{1}{2}e_i$. Since S is contained in exactly one edge in P, by linearity of the mapping, so must $Q(S)$. Moreover, since $Q(0) = 0$, we know that $Q(S)$ must also be an edge containing 0, and thus we deduce that $Q\left(\frac{1}{2}e_i\right) = ke_j$ for some scalar $k \in [0,1]$ and $j \in [n]$.

Next, we argue that $k = \frac{1}{2}$. If $k > \frac{1}{2}$, then $Qe_i \notin [0,1]^n$, contradicting $Q(P) = P$. On the other hand, if $k < \frac{1}{2}$, then $Qe_i = 2ke_j < e_j$, and is not an extreme point of P while e_i is one, a contradiction. Thus, we see that for every $i \in [n]$, there exists $j \in [n]$ such that $Qe_i = e_j$.

Again, by the linearity of the mapping, every row of Q cannot contain more than one nonzero entry (otherwise there exist $i, j, k \in [n]$ such that $Qe_i = Qe_j = e_k$). Thus, we conclude that Q must be a permutation matrix. □

Thus, we see that in such cases, there exists an integer N such that $Q^N = I$, the identity matrix (e.g. $N = n!$ would do). From now on, we say that a lift-and-project operator Γ is *permutation-commutative* if it is Q-commutative for all permutation matrices Q. Note that mappings defined by permutation matrices fall under those described in Proposition 38. Thus, it follows that many operators, including all Lovász–Schrijver and Sherali–Adams variants that we discussed, are permutation-commutative.

When trying to establish a lower-bound result for a lift-and-project operator Γ applied to P, we are often trying to construct a certificate matrix for a point $x \in \Gamma(P)$ that violates some inequality in the form of $a^\top y \leq \beta$ that is valid for P_I. We now look at how the symmetries in the initial relaxation P, and in the coefficients of a, can allow us to construct a violating point $x \in \Gamma(P)$ that also has a lot of symmetries.

Proposition 43. *Let $P \subseteq [0,1]^n$ be a compact, convex set, and Γ be a permutation-commutative lift-and-project operator such that $\Gamma(P)$ is also convex. Then the inequality $a^\top y \leq \beta$ is not valid for $\Gamma(P)$ if and only if there exists $x \in \Gamma(P)$ such that $\alpha^\top x > \beta$, and $x_i = x_j$ whenever there exists a permutation matrix Q such that $Q(P) = P, Qa = a$ and $Qe_i = e_j$.*

Proof. The "if" portion of the claim is clear. For the "only if" part, take any point $x \in \Gamma(P)$ that violates $a^\top y \leq \beta$, and a permutation matrix Q such that $Q(P) = P$ and $Qa = a$. Then we see that

$$Qx \in Q(\Gamma(P)) \subseteq \Gamma(Q(P)) = \Gamma(P),$$

since Γ is Q-commutative. Thus, it follows that $Q^i x \in \Gamma(P)$ for all $i \geq 1$. Moreover, let N be an integer such that $Q^N = I$. Then we see that

$$a^\top (Qx) = (Q^\top a)^\top x = (Q^{N-1} a)^\top x = a^\top x,$$

where we used the fact that $Q^\top = Q^{-1}$ for all permutation matrices Q, and applied $Qa = a$ iteratively for $N-1$ times.

Therefore, the above observations imply that if we let $z := \frac{1}{N}\sum_{i=0}^{N-1} Q^i x$, then $z \in \Gamma(P)$ by convexity, and

$$a^\top z = \frac{1}{N}\sum_{i=0}^{N-1} a^\top (Q^i x) = a^\top x > \beta.$$

Moreover, by the construction of z,

$$Qz = Q\left(\frac{1}{N}\sum_{i=0}^{N-1} Q^i x\right) = \frac{1}{N}\sum_{i=1}^{N} Q^i x = z.$$

Thus, $z_i = z_j$ whenever $Qe_i = e_j$, and our claim follows. □

Thus, we know that if a certain inequality $a^\top y \leq \beta$ is violated by some point x, then it has to be violated by a point that inherits all symmetries of P and the vector a. Notice that, in many optimization problems, we are interested in computing the largest or smallest cardinality of a set among a given collection (e.g. the stable set problem, matching problem, and the max-cut problem). Thus, we are often optimizing in the direction of $\bar{e}$. Moreover, we have seen that many hardness results have been achieved by highly symmetric combinatorial objects (e.g. the complete graph), which correspond to polytopes that have a lot of symmetries. Thus, we define a set $P \subseteq [0,1]^n$ to be *symmetric* if for all $i, j \in [n]$, there exists a permutation matrix Q such that $Q(P) = P$ and $Qe_i = e_j$. Then the following special instance of Proposition 43 is particularly noteworthy.

Corollary 44. *Let $P \subseteq [0,1]^n$ be a compact, convex set that is symmetric. If a lift-and-project operator Γ is permutation-commutative and $\Gamma(P)$ is convex, then the inequality $\bar{e}^\top y \leq \beta$ is valid for $\Gamma(P)$ if and only if*

$$\max\left\{k : k\bar{e} \in \Gamma(P)\right\} \leq \frac{\beta}{n}.$$

Hence, not only can the above observations simplify lower-bound analyses by introducing symmetries and guiding us to find points that may have "nice" certificate matrices, it can also reduce the task of proving an upper-bound result from showing $P_I \subseteq \Gamma(P)$ to verifying if a specific point is on the boundary of $\Gamma(P)$.

Moreover, in addition to using the presence of Q to construct points in $\Gamma(P)$ that have a lot of symmetries, sometimes we can show that these same symmetries can be assumed upon the matrices in $\tilde{\Gamma}(P)$.

Corollary 45. *Let $P \subseteq [0,1]^n$ be a compact, convex set, and Q be a permutation matrix such that $Q(P) = P$. Then the inequality $\bar{e}^\top y \leq \ell$ is not valid for $\mathrm{SA}_+^k(P)$ if and only if there exists $Y \in \tilde{\mathrm{SA}}_+^k(P)$ such that*

- $\sum_{i=1}^n Y[i|_1, \mathcal{F}] > \ell$, *and*
- $Y[S_1|_1 \cap T_1|_0, S_1' \cap T_1'|_0] = Y[S_2|_1 \cap T_2|_0, S_2'|_1 \cap T_2'|_0]$ *whenever* $Q\chi^{S_1 \cup S_1'} = \chi^{S_2 \cup S_2'}$ *and* $Q\chi^{T_1 \cup T_1'} = \chi^{T_2 \cup T_2'}$.

The same assertion also holds for the operators $\mathrm{SA}^k, \mathrm{SA}'^k$ and SA'^k_+, for every $k \geq 1$.

Proof. The proof closely follows that of Proposition 43. First, let $x \in \mathrm{SA}_+^k(P)$ and Y be its certificate matrix. Since SA_+^k is Q-commutative, we can follow the proof of Proposition 38 to construct a matrix that certifies $Qx \in \mathrm{SA}_+^k(P)$. Let us slightly abuse notation and denote this matrix by $Q(Y)$. We can likewise construct $Q^i(Y)$ that certifies $Q^i x \in \mathrm{SA}_+^k(P)$ for all $i \geq 1$.

Again, let N be an integer such that $Q^N = I$. Then it is not hard to see that

$$\frac{1}{N}\sum_{i=0}^{N-1} Q^i(Y)$$

is in $\tilde{\mathrm{SA}}_+^k(P)$, by the convexity of the set $\tilde{\mathrm{SA}}_+^k(P)$, and this "averaged out" matrix would fulfill the symmetry assertions. The same argument applies to the other operators. $\square$

In the case when Γ is a lift-and-project operator whose performance can vary upon the algebraic descriptions of P, simply assuming $Q(P) = P$ is not enough. However, similar results can be achieved with a slightly stronger assumption. For instance, we illustrate such a result for the Lasserre operator:

Proposition 46. *Let $P = \{y : Ay \leq b\} \subseteq [0,1]^n$ be a polytope, and Q be a permutation matrix such that $AQ = A$. Then the inequality $\bar{e}^\top y \leq \ell$ is not valid for $\text{Las}^k(P)$ if and only if there exists $Y \in \tilde{\text{Las}}^k(P)$ such that*

- $\sum_{i=1}^n Y[i|_1, \mathcal{F}] > \ell$, *and*
- $Y[S_1|_1, S_1'|_1] = Y[S_2|_1, S_2'|_1]$ *whenever* $Q\chi^{S_1 \cup S_1'} = \chi^{S_2 \cup S_2'}$.

The same assertion also holds for Las'^k, *for every* $k \geq 1$.

Thus, we see that the symmetries of P can allow us to construct symmetric points and certificate matrices in its lift-and-project relaxations. Next, we look to further exploit these symmetries to simply the verification of these certificate matrices. In particular, we show how these ideas can be utilized to establish the positive semidefiniteness of a certificate matrix by doing that on another matrix that is potentially much smaller. First, we have the following elementary result:

Proposition 47. *Suppose $Y \in \mathbb{S}^n$, and Q is a permutation matrix such that $QY = Y$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_k \subseteq [n]$ be the disjoint cycles of the permutation on $[n]$ that correspond to Q, and x be an eigenvector of Y with eigenvalue λ. Define the vector z where*

$$z_j = \frac{\sum_{j \in \mathcal{C}_i} x_j}{|\mathcal{C}_i|}, \ \forall i \in \mathcal{C}_i, \ \forall i \in [k].$$

Then $Yz = \lambda z$.

Proof. Observe that

$$Y(Qx) = (YQ)x = (Q^\top Y^\top)^\top x = (Q^{-1}Y)^\top x = Yx = QYx = \lambda Qx.$$

Again, $Q^\top = Q^{-1}$ for all permutation matrices Q. Thus, we see that $Q^i x$ is an eigenvector of Y with eigenvalue λ, for all $i \geq 1$. If $Q^N = I$ for some integer N, then $z := \sum_{i=0}^{N-1} Q^i x$ satisfies $Yz = \lambda z$. Also, it is not hard to see that $z_i = z_j$ whenever $Qe_i = e_j$ (i.e. when i, j belong to the same cycle). Thus, our claim follows. □

Therefore, when such a permutation matrix exists, we can assume that for every eigenvalue of Y, we can obtain eigenvectors whose entries are identical within each orbit of Q. The only problem is that z above could be the zero vector, and one has to exercise caution in choosing x and Q to not "lose" these eigenvectors in the process.

One way to do that is the following: We say that an $n \times n$ matrix Y is *coordinate-transitive* if, for all $i, j \in [n]$, there exists a permutation matrix Q such that $QY = Y$ and $Qe_i = e_j$. Given such a matrix, we can define an equivalence relation on $[n]$, and say that $i \sim j$ if there exists a permutation matrix Q where $QY = Y, Qe_1 = e_1$ and $Qe_i = e_j$. Let $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ be the resulting equivalence classes that partition $[n]$. Then we have the following:

Proposition 48. *Suppose Y is a coordinate-transitive matrix, and the equivalence classes $\mathcal{C}_1, \ldots, \mathcal{C}_\ell$ are defined as above. Define the $n \times \ell$ matrix L where*

$$L[i,j] = \begin{cases} 1 & \text{if } i \in \mathcal{C}_j; \\ 0 & \text{otherwise.} \end{cases}$$

Then λ is an eigenvalue of Y if and only if it is also an eigenvalue of

$$Y' := (L^\top L)^{-1} L^\top Y L.$$

Proof. Let λ be an eigenvalue of Y, with a corresponding eigenvector x. Since Y is coordinate-transitive, we may assume that there is such an x where $x_1 \neq 0$. Then using permutation matrices Q where $QY = Y$ and $Qe_1 = e_1$, we can construct from Proposition 47 a vector z whose entries are uniform among each $\mathcal{C}_i$. Also, since $z_1 = x_1$, z is nonzero and thus indeed an eigenvector of Y with eigenvalue λ.

Define $\bar{z} \in \mathbb{R}^\ell$ where $\bar{z}_i = z_j$, where $j \in \mathcal{C}_i$, for all $i \in [\ell]$. (Again, since all entries of z corresponding to $\mathcal{C}_i$ are identical, we can choose any $j \in \mathcal{C}_i$.) Note that $z = L\bar{z}$. Now,

$$\begin{aligned} Y'\bar{z} &= (L^\top L)^{-1} L^\top Y L \bar{z} \\ &= (L^\top L)^{-1} L^\top Y z \\ &= \lambda (L^\top L)^{-1} L^\top z \\ &= \lambda (L^\top L)^{-1} L^\top L \bar{z} \\ &= \lambda \bar{z}. \end{aligned}$$

Next, we show that $Y'\bar{z} = \lambda\bar{z}$ implies $Yz = \lambda z$. Notice $L^\top L$ is the $\ell \times \ell$ diagonal matrix where $L^\top L[i,i] = |\mathcal{C}_i|$ for all $i \in [\ell]$. Thus, $(L^\top L)^{-1}$ must exist, and is the diagonal matrix whose i^{th} diagonal entry is $\frac{1}{|\mathcal{C}_i|}$. Hence, $Y'[i,j] = \frac{1}{|\mathcal{C}_i|} \sum_{p \in \mathcal{C}_i, q \in \mathcal{C}_j} Y[p,q]$, and from that it is easy to see that $LY' = YL$. Hence,

$$Yz = YL\bar{z} = LY'\bar{z} = \lambda L\bar{z} = \lambda z,$$

and our claim follows. □

Thus, if we need to verify if Y is positive semidefinite, it is equivalent to verify that for Y', a matrix that is potentially much smaller. In fact, the same can be said about several other $\ell \times \ell$ matrices:

Corollary 49. *Let Y be a coordinate-transitive matrix, and L as defined in Proposition 48. Then the following are equivalent:*

1. $Y \succeq 0$;
2. $(L^\top L)^{-1} L^\top Y L \succeq 0$;
3. $L^\top Y L \succeq 0$;

4. $(L^\top L)^{-1/2} L^\top Y L (L^\top L)^{-1/2} \succeq 0.$

While the proof to Corollary 49 is elementary, we remark that each of the above matrices boasts different advantages:

- 2. and 4. have the same eigenvalues as Y, while 3. may not;
- 3. and 4. are symmetric but 2. is generally not;
- 2. and 3. have rational entries (given Y has rational entries) and 4. may not.

Thus, depending on individual circumstances, some of these matrices might be more fruitful to work with than the others.

Next, we utilize Proposition 48 to compute eigenvalues for $Y_{n,k,k}(m)$ (the matrix whose rows and columns are indexed by matchings of size k in K_n, defined in (5.6)). First, it is apparent that $Y_{n,k,k}(m)$ is coordinate-transitive. Thus, let M be a fixed matching of sized k, and consider all automorphisms of K_n that fixes M. More precisely, $P : [n] \to [n]$ is an isomorphism that fixes $M = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_k, v_k\}\}$ if

$$\{\{P(u_1), P(v_1)\}, \{P(u_2), P(v_2)\}, \ldots, \{P(u_k), P(v_k)\}\} = M.$$

Note that the sets above are unordered, and $\{u_i, v_i\}$ is not necessarily equal to $\{P(u_i), P(v_i)\}$ for any particular $i \in [k]$. Hence, among the $n!$ automorphisms of K_n, there are exactly $(2k)!!$ that fix M. Next, we define the equivalence classes $\mathcal{C}_1, \ldots \mathcal{C}_\ell$ as in Proposition 48 and the matrix L, and look into the reduced matrix

$$Y'_{n,k,k}(m) = (L^\top L)^{-1} L^\top Y_{n,k,k}(m) L.$$

We assume throughout this section that $n \geq 2k$. In such cases, the number of equivalence classes in $\mathcal{M}_{n,k}$ is only dependent on k. Recall that, given a polynomial or power series $f(x)$, $[x^k]f(x)$ denotes the coefficient of x^k in $f(x)$. The following result gives one way of counting these equivalence classes:

Proposition 50. *Let $M \in \mathcal{M}_{n,k}$ be a fixed matching, and we define the equivalence relation on $\mathcal{M}_{n,k}$ such that $S_1 \sim S_2$ if there exists an automorphism on K_n that fixes M and maps S_1 to S_2. Then the total number of equivalence classes under this relation is*

$$\alpha_k := [x^k] \frac{1}{2} \left(\prod_{j \geq 1} \frac{1}{(1-x^j)^4} + \prod_{j \geq 1} \frac{1}{(1-x^j y)(1-x^j y^{-1})(1-x^j)^2} \right).$$

Proof. First, we construct a bijection between the equivalence classes, and the set of partitions of k with four types of parts $\{p^+, p, p', p^- : p \geq 1\}$, in which the number of parts of the kind p^+ is no more than that of the kind p^-. For instance, when $k=2$, there are 10 such partitions:

$$(2^-),(1^-,1^-),,(1^-,1),(1^-,1'),(1^-,1^+),(2),(1,1),(1,1'),(2'),(1',1').$$

Given an equivalence class $\mathcal{C}_i$, take any matching $S \in \mathcal{C}_i$, and consider the components in $S \cup M$. For each component that contains p edges from S (where $p \geq 1$), we assign it to a part as follows:

- p^- if the component is a path of length $2p-1$;
- p if the component is a cycle of length $2p$ where $p \geq 2$, or when the component consists of one edge belonging to both M and S;
- p' if the component is a path of length $2p$;
- p^+ if the component is a path of length $2p+1$.

If we do that for each component that contains some edges from S, we obtain parts that add up to k. Moreover, at most k of the edges from M would be accounted for. Thus, since each p^--component contains $p-1$ edges from M, and each p^+-component contains $p+1$ edges from M, the partition of k we obtain must contain as least as many p^- parts as p^+ parts. Figure 5.10 illustrates the bijection between the 10 partitions and the 10 nonisomorphic cases for $S \cup M$ when $k=2$.

By the definition of our equivalence relation and the construction of the partition, we see that $S_1 \sim S_2$ if and only if $S_1 \cup M$ and $S_2 \cup M$ correspond to the same partition. Also, the construction is reversible — given such a partition, we can construct all components of $S \cup M$, and recover the equivalence class that contains S.

Finally, we count the number of such partitions. The total number of partitions of k with four kinds of parts is $[x^k]\prod_{j\geq 1}\frac{1}{(1-x^j)^4}$. Now there is a natural bijection between partitions with more p^+ parts than p^- parts and vice versa. Thus, the number of partitions we want is the total number of partitions of k with four kinds of different parts, plus such partitions with the same number of p^+ and p^- parts, all divided by two. Note that since $[x^k y^\ell](1-x^j y)^{-1}$ is the number of partitions of k with ℓ parts, $[x^k]\left((1-x^j y)(1-x^j y^{-1})\right)^{-1}$ counts the number of partitions of k with two kinds of parts, with the same number of parts of each kind. This finishes our proof. $\square$

Computations show that the first few values of α_k are 3 , 10 , 27 , 69 , 161 , 361 and 767. Thus, we see that it suffices to verify the positive semidefiniteness of an $\alpha_k \times \alpha_k$ matrix to prove that $Y_{n,k,k}(m) \succeq 0$.

Next, we look into these reduced matrices of $Y_{n,k,k}(m)$ for small k. For $k=1$, we have $\alpha_1 = 3$. If we let $M = \{\{u,v\}\}$, then the three equivalence classes (and their corresponding partitions) are:

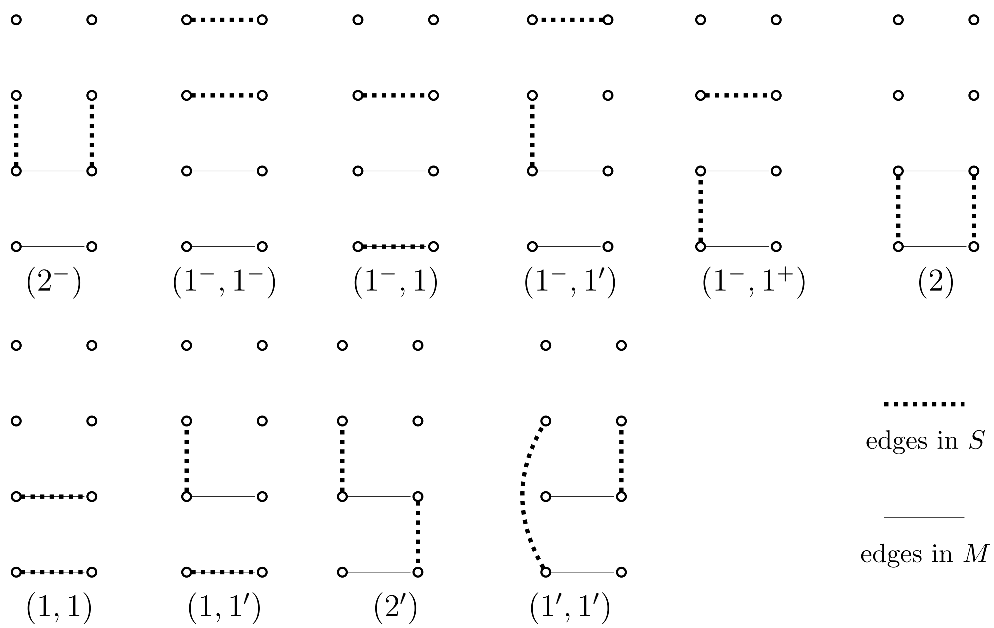

Figure 5.10: The bijection between integer partitions and nonisomorphic unions of two matchings.

(1) $\mathcal{C}_1$, which consists of the edge $\{u, v\}$;

(1′) $\mathcal{C}_2$, the edges that are incident with u or v, but not both;

(1⁻) $\mathcal{C}_3$, the edges that are neither incident with u nor v.

Thus, while $Y_{n,1,1}(m)$ is an $\binom{n}{2} \times \binom{n}{2}$ matrix, $Y'_{n,1,1}(m)$ is 3×3, regardless of n. For example,

$$Y'_{5,1,1}(m) = \begin{pmatrix} m_1 & 0 & 3m_2 \\ 0 & m_1 + 2m_2 & m_2 \\ m_2 & 2m_2 & m_1 \end{pmatrix},$$

where the rows and columns are ordered according to the listing of the equivalence classes above (e.g. $Y'_{5,1,1}(m)[1,3] = 3m_2$ since $\frac{1}{|\mathcal{C}_1|}\sum_{p\in\mathcal{C}_1, q\in\mathcal{C}_3} Y_{5,1,1}[p,q] = 3m_2$).

Then the eigenvalues of Y' are m_1+m_2, m_1-2m_2 and m_1-3m_2, each with multiplicity one. Thus, by Proposition 48, these are also the eigenvalues of $Y_{5,1,1}(m)$.

For $k = 2$, there are $\alpha_2 = 10$ equivalence classes. We list their corresponding partitions in order:

$$(1,1), (2), (2^-), (1', 1'), (1^-, 1^-), (1, 1'), (1^-, 1), (2'), (1^-, 1^+), (1^-, 1^-).$$

Then one can compute that $Y'_{9,2,2}(m)$ is the matrix

$$\begin{pmatrix}
b_2 & 0 & 0 & 0 & 15b_4 & 0 & 20b_3 & 0 & 0 & 0 \\
0 & b_2 & 0 & 0 & 15b_4 & 0 & 0 & 0 & 20b_3 & 0 \\
0 & 0 & b_2+6b_4 & 12b_3 & 0 & 2b_3 & 3b_4 & 0 & 0 & 6b_3+6b_4 \\
0 & 0 & 6b_3 & b_2+6b_3+6b_4 & 0 & 0 & 0 & 2b_3 & 3b_4 & 6b_3+6b_4 \\
b_4 & 2b_4 & 0 & 0 & b_2 & 4b_4 & 4b_3 & 8b_4 & 8b_3 & 8b_3 \\
0 & 0 & 4b_3 & 0 & 3b_4 & b_2+4b_3 & 6b_3 & 0 & 0 & 6b_3+12b_4 \\
b_3 & 0 & 6b_4 & 0 & 3b_3 & 6b_3 & b_2+4b_3+3b_4 & 0 & 0 & 6b_3+6b_4 \\
0 & 0 & 0 & 4b_3 & 3b_4 & 0 & 0 & b_2+4b_3 & 6b_3 & 6b_3+12b_4 \\
0 & b_3 & 0 & 6b_4 & 3b_3 & 0 & 0 & 6b_3 & b_2+4b_3+3b_4 & 6b_3+6b_4 \\
0 & 0 & 2b_3+2b_4 & 4b_3+4b_4 & b_3 & b_3+2b_4 & b_3+b_4 & 2b_3+4b_4 & 2b_3+2b_4 & b_2+7b_3
\end{pmatrix}.$$

The eigenvalues of $Y'_{9,2,2}(m)$ are:

Eigenvalue	Multiplicity	Corresponding eigenvalue of $Y_{9,2,2}$
$m_2 + 20m_3 + 15m_4$	1	63
$m_2 + 2m_3 + \frac{9m_4+\sqrt{336m_3^2-168m_3m_4+81m_4^2}}{2}$	2	33
$m_2 + 2m_3 + 3m_4$	1	15
$m_2 + 2m_3$	1	12
$m_2 - 4m_3$	1	0
$m_2 - m_3 - 6m_4$	1	0
$m_2 + 2m_3 - 12m_4$	1	0
$m_2 + 2m_3 + \frac{9m_4-\sqrt{336m_3^2-168m_3m_4+81m_4^2}}{2}$	2	0
Total =	10	

Table 5.1: Eigenvalue breakdown of the reduced $Y_{9,2,2}(m)$.

Thus, we see that $Y_{9,2,2} \succeq 0$. Together with Corollary 41, we obtain that $Y_{n,2,2} \succeq 0$ for all odd integers $n \geq 9$.

For $k = 3, \alpha_3 = 27$. While we were not able to compute the eigenvalues of the matrix $Y'_{13,3,3}(m)$ for a general vector m, the eigenvalues of $Y'_{13,3,3}$ were relatively easy to obtain:

For $k = 4$ there are $\alpha_4 = 69$ equivalence classes, and the eigenvalue breakdown of $Y'_{17,4,4}$ is:

Note that $Y_{17,4,4}$ is a 2552550×2552550 matrix, and Proposition 48 allows us to completely recover its eigenvalues by looking into a mere 69×69 matrix. Thus, it follows that $Y_{n,3,3} \succeq 0$ for all odd $n \geq 13$, and $Y_{n,4,4} \succeq 0$ for all odd $n \geq 17$. We will further discuss these findings about $Y_{n,k,k}$ in Chapter 8.

Eigenvalue	Multiplicity
1287	1
657	2
287	2
272	2
105	1
80	1
72	1
0	17
Total =	27

Table 5.2: Eigenvalue breakdown of the reduced $Y_{13,3,3}$.

Eigenvalue	Multiplicity
36465	1
18447	2
8241	3
8052	2
3087	2
2872	3
2760	2
945	1
700	1
640	1
616	1
576	1
0	49
Total =	69

Table 5.3: Eigenvalue breakdown of the reduced $Y_{17,4,4}$.

Chapter 6

When Positive Semidefiniteness Does Not Help

In earlier chapters, we looked at tools such as Theorem 11 and 12 that can relate the performances of two lift-and-project operators. In the way they are derived, the comparison is usually applied to two operators that are either both polyhedral or both semidefinite, and showing that those operators do not gain any strength by lifting a given set to a higher dimension (e.g. Theorem 13 and 14). In this chapter, we are more interested in comparing two operators from the two different realms, such as SA^k and SA_+^k. While we already know that SA_+^k dominates SA^k in general, it is worthwhile to ask the following: Under what conditions does SA_+^k *not* perform strictly better than SA^k? In such cases, it would be silly to apply SA_+^k, verify a complex set of conditions, only to yield the exact same relaxation as that obtained by applying the much simpler SA^k.

6.1 When SA_+^k does not outperform LS_0^k

In [GT01], Goemans and Tunçel studied the question of when LS_+ does not outperform LS_0, and they proved the following:

Theorem 51. *Suppose $P \subseteq [0,1]^n$. Given $x \in P$, let*

$$S(x) := \{i \in [n] : 0 < x_i < 1\}.$$

If the points obtained from x by

1. *increasing x_i to 1, and*
2. *decreasing x_i to 0*

are both in P, $\forall i \in S(x)$, then $x \in \mathrm{LS}_+(P)$.

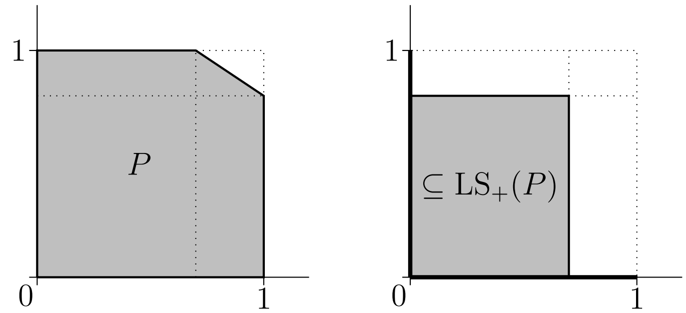

Figure 6.1: An illustration of Theorem 51.

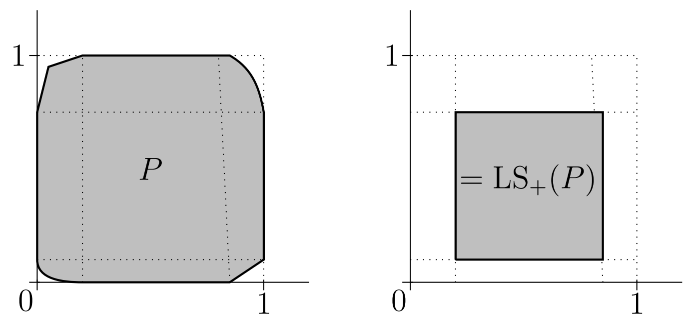

Figure 6.2: An illustration of Corollary 52.

For an example, if P is the set on the left hand side of Figure 6.1, then Theorem 51 implies that the grey area on the right hand side of Figure 6.1, as well as all points on the two axes, are contained in $\mathrm{LS}_+(P)$.

Theorem 51 then immediately implies the following:

Corollary 52. *Suppose* $P \subseteq [0,1]^n$*, and*

$$\{x \in P : x_i = 1\} = \{x \in P : x_i = 0\} + \{e_i\}$$

for all $i \in [n]$*, then*

$$\mathrm{LS}_+(P) = \mathrm{LS}_0(P) = \{x : x - x_j e_j \in P, \ \forall j \in [n]\}\,.$$

For instance, Figure 6.2 provides an illustration of Corollary 52. In this example, the points described in Theorem 51 accounts for all points in the set $\mathrm{LS}_+(P)$.

Herein, we extend Theorem 51 and Corollary 52, and provide some conditions under which SA_+^k does not outperform LS_0^k. Given $x \in [0,1]^n$ and two disjoint sets of indices $I, J \subseteq [n]$, we define the vector $x_J^I \in [0,1]^n$ where

$$x_J^I[i] = \begin{cases} 1 & \text{if } i \in I; \\ 0 & \text{if } i \in J; \\ x[i] & \text{otherwise.} \end{cases}$$

In other words, x_J^I is the vector obtained from x by setting all entries indexed by elements in I to 1, and all entries indexed by elements in J to 0. Then we have the following.

Theorem 53. *Let $P \subseteq [0,1]^n$ and $x \in P$. If $x_J^I \in P$ for all $I, J \subseteq S(x)$ such that $|I| + |J| \leq k$, then $x \in \mathrm{SA}_+^k(P)$.*

Proof. We prove our claim by constructing a matrix in $\mathbb{R}^{\mathcal{A}_k \times \mathcal{A}_k}$ that certifies $x \in \mathrm{SA}_+^k(P)$. Recall that

$$\mathcal{A}_k = \{S|_1 \cap T|_0 : S, T \subseteq [n], S \cap T = \emptyset, |S| + |T| \leq k\},$$

and

$$\mathcal{A}_k^+ = \{S|_1 : |S| \leq k\}.$$

For each $I \subseteq [n], |I| \leq k$, define $y^{(I)} \in \mathcal{A}_k^+$ such that

$$y^{(I)}[S|_1] = \begin{cases} \prod_{i \in S \setminus I} x_i & \text{if } I \subseteq S; \\ 0 & \text{otherwise.} \end{cases}$$

Note that in the case of $y^{(I)}[I|_1]$, the empty product is defined to evaluate to 1. Next, we define $Y \in \mathbb{R}^{\mathcal{A}_k^+ \times \mathcal{A}_k^+}$ as

$$Y := \sum_{S \subseteq [n], |S| \leq k} \left(\prod_{i \in S} x_i(1 - x_i) \right) y^{(S)} (y^{(S)})^\top.$$

Note that $Y \succeq 0$. Now given $S, T \subseteq [n], |S|, |T| \leq k$, observe that

$$\begin{aligned} Y[S|_1, T|_1] &= \sum_{U \subseteq S \cap T} \left(\prod_{i \in U} x_i(1 - x_i) \right) \left(\prod_{i \in S \setminus U} x_i \right) \left(\prod_{i \in T \setminus U} x_i \right) \\ &= \left(\prod_{i \in S \cup T} x_i \right) \left(\sum_{U \subseteq S \cap T} \left(\prod_{i \in U} (1 - x_i) \right) \left(\prod_{i \in (S \cap T) \setminus U} x_i \right) \right) \\ &= \prod_{i \in S \cup T} x_i. \end{aligned}$$

Next, define $U \in \mathbb{R}^{\mathcal{A}_k \times \mathcal{A}_k^+}$ such that

$$U^\top (e_{S|_1 \cap T|_0}) = \sum_{U : S \subseteq U \subseteq (S \cup T)} (-1)^{|U \setminus S|} e_{U|_1},$$

for all disjoint $S, T \subseteq [n]$ such that $|S| + |T| \leq k$. We claim that $Y' := UYU^\top \in \tilde{\mathrm{SA}}_+^k(P)$.

First, notice that $Y'[\mathcal{F}, \mathcal{F}] = Y[\mathcal{F}, \mathcal{F}] = 1$, so ($\mathrm{SA}_+$ 1) holds. Next, given $\alpha = S|_1 \cap T|_0 \in \mathcal{A}_k$,

$$\hat{x}(Y' e_\alpha) = \begin{pmatrix} Y'[\mathcal{F}, \alpha] \\ Y'[\mathcal{F}, \alpha] x_T^S \end{pmatrix} = \begin{pmatrix} Y'[\mathcal{F}, \alpha] \\ Y'[\mathcal{F}, \alpha] x_{T \cap S(x)}^{S \cap S(x)} \end{pmatrix} \in K(P),$$

by the assumption that $x_J^I \in P$ for all $I, J \subseteq S(x)$ where $|I| + |J| \leq k$. Also, we see that given $\alpha, \beta \in \mathcal{A}_k$ where $\alpha = S|_1 \cap T_0, \beta = S'|_1 \cap T'|_0$,

$$Y'[\alpha, \beta] = \left(\prod_{i \in S \cup S'} x_i \right) \left(\prod_{i \in T \cup T'} (1 - x_i) \right),$$

and hence is nonnegative. Thus, (SA$_+$ 2) is satisfied as well. (SA$_+$ 3)−(SA$_+$ 5) are ensured by the construction of U. Also, since $Y \succeq 0$, $Y' \succeq 0$ as well.

Finally, since $\hat{x}(Y' e_{\mathcal{F}}) = \hat{x}$, it follows that $x \in \mathrm{SA}_+^k(P)$. □

From the above, we are able to characterize some convex sets for which SA_+^k does not produce a tighter relaxation than an operator as weak as LS_0^k.

Corollary 54. *Suppose $P \subseteq [0,1]^n$ is a convex set such that, for all $x \in P$ and for all $I, J, I', J' \subseteq [n]$ such that $I \cup J = I' \cup J'$ and $|I| + |J| = k$,*

$$x_J^I \in P \iff x_{J'}^{I'} \in P.$$

Then

$$\mathrm{SA}_+^k(P) = \mathrm{LS}_0^k(P) = \bigcap_{I \subseteq [n], |I| = k} \left\{ x : x_\emptyset^I \in P \right\}.$$

The two results above generalize Theorem 51 and Corollary 52, respectively.

6.2 Some bad instances for $\mathrm{SA}_+, \mathrm{Las}$ and BZ'_+

In this section, we look at two polytopes that have been shown to be bad instances for many known lift-and-project operators, and analyze their ranks with respect to some of the strongest operators. For the first example, consider the set

$$P(\alpha) := \left\{ x \in [0,1]^n : \sum_{i=1}^n x_i \leq n - \alpha \right\}.$$

Using Theorem 53, we have the following for the SA$_+$-rank of $P(\alpha)$:

Proposition 55. *For every $n \geq 2$, if $\alpha \in (0, n)$ is not an integer and $k < \frac{n(\lceil \alpha \rceil - \alpha)}{\lceil \alpha \rceil}$, then the* SA$_+$*-rank of $P(\alpha)$ is at least $k + 1$.*

Proof. First observe that for every α where $0 < \alpha < n$,

$$P(\alpha)_I = \left\{ x \in [0,1]^n : \sum_{i=1}^n x_i \leq n - \lceil \alpha \rceil \right\}.$$

Next, note that

$$k < \frac{n(\lceil\alpha\rceil - \alpha)}{\lceil\alpha\rceil} \iff (n-k)\left(\frac{n-\lceil\alpha\rceil}{n}\right) + k < n - \alpha$$

Thus, there exists $\ell \in \mathbb{R}$ such that $(n-k)\ell + k < n - \alpha$ and $\ell > \frac{n-\lceil\alpha\rceil}{n}$. Now consider the point $x := \ell\bar{e}$. Since $\ell > \frac{n-\lceil\alpha\rceil}{n}$, $x \notin P(\alpha)_I$. However, for every disjoint sets of indices $I, J \subseteq [n]$ where $|I| + |J| \leq k$, we have

$$\sum_{i=1}^{n} x_J^I[i] \leq (n-k)\ell + k < n - \alpha,$$

by the choice of ℓ. Thus, $x_J^I \in P(\alpha)$ for any such choices of I, J. Note that the first inequality above follows from the fact that x_J^I is maximized by choosing any I with $|I| = k$ and $J = \emptyset$. Thus, it follows from Theorem 53 that $x \in \mathrm{SA}_+^k(P(\alpha))$. This proves that $\mathrm{SA}_+^k(P(\alpha)) \neq P(\alpha)_I$, and hence the SA_+-rank of $P(\alpha)$ is at least $k+1$. □

Using Proposition 55 and some results shown earlier in this thesis, we obtain a lower-bound result on the BZ'_+-rank of $P(\alpha)$, establishing what we believe to be the first example in which BZ'_+ (and, as a result, BZ_+) requires more than a constant number of iterations to return the integer hull of a set.

Proposition 56. *Suppose an integer $n \geq 19$ is not a perfect square. Then there exists $\alpha \in (\lfloor\sqrt{n}\rfloor, \lceil\sqrt{n}\rceil)$ such that the BZ'_+-rank of $P(\alpha)$ is at least $\left\lfloor \sqrt{\frac{\sqrt{n}-1}{2}} \right\rfloor + 1$.*

Proof. For convenience, let $k := \left\lfloor \sqrt{\frac{\sqrt{n}-1}{2}} \right\rfloor$. Notice that for all $n \geq 19$, $k + 2 < \lfloor\sqrt{n}\rfloor$, and so BZ'^k_+ does not generate any k-small obstructions of $P(\alpha)$ for all $\alpha > \lfloor\sqrt{n}\rfloor$. Thus, $\mathcal{O}_k(P(\alpha)) = P(\alpha)$, and the walls generated by BZ'^k_+ is exactly the subsets of $[n]$ of size at most k in this case. Hence, every tier has size at most k^2. Then it follows from Theorem 14 that, for all $\alpha > \lfloor\sqrt{n}\rfloor$, $\mathrm{BZ}'^k_+(P(\alpha)) \supseteq \mathrm{SA}'^{k^2}_+(P(\alpha))$. Then Proposition 6 in turn implies that $\mathrm{SA}'^{k^2}_+(P(\alpha)) \supseteq \mathrm{SA}^{2k^2}_+(P(\alpha))$.

Now choose $\epsilon > 0$ small enough such that

$$\sqrt{n} - 1 < \frac{n(\lceil\sqrt{n}\rceil - (\lfloor\sqrt{n}\rfloor + \epsilon)}{\lceil\sqrt{n}\rceil},$$

and let $\alpha := \lfloor\sqrt{n}\rfloor + \epsilon$. From Proposition 55, as long as $2k^2 \leq \sqrt{n} - 1$, $\mathrm{SA}^{2k^2}_+(P(\alpha)) \neq P(\alpha)_I$. Thus, the BZ'_+-rank of $P(\alpha)$ is at least $\left\lfloor \sqrt{\frac{\sqrt{n}-1}{2}} \right\rfloor + 1$. □

Note that one particular consequence of Proposition 55 is that $P(\alpha)$ has SA_+-rank n for all $\alpha \in (0, \frac{1}{n})$. We next show that the SA_+-rank is in fact n for all $\alpha \in (0, 1)$:

Proposition 57. *For every $n \geq 2$, the* SA_+*-rank of $P(\alpha)$ is n for all $\alpha \in (0,1)$.*

Proof. We prove our claim by showing that $\left(1 - \frac{\alpha}{\alpha n+1-\alpha}\right)\bar{e} \in \mathrm{SA}_+^{n-1}(P(\alpha)) \setminus P(\alpha)_I$. First,

$$n\left(1 - \frac{\alpha}{\alpha n + 1 - \alpha}\right) = n - \frac{\alpha n}{\alpha n + 1 - \alpha} > n - 1,$$

and so $\left(1 - \frac{\alpha}{\alpha n+1-\alpha}\right)\bar{e} \notin P(\alpha)_I$. We next show that this vector is in $\mathrm{SA}_+^{n-1}(P(\alpha))$. Let $\mathcal{A}^- := \{S|_0 : S \subseteq [n]\}$. It is clear that $\mathcal{A}^-$ generates $\mathcal{A}$ (for the same reason $\mathcal{A}_k^+$ generates $\mathcal{A}_k$, as explained in Section 5.1). Define $y \in \mathbb{R}^{\mathcal{A}^-}$ where

$$y[S|_0] = \begin{cases} 1 & \text{if } S = \emptyset; \\ \frac{\alpha}{\alpha n+1-\alpha} & \text{if } |S| = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Now let $Y \in \mathbb{R}^{\mathcal{A}_{n-1} \times \mathcal{A}_{n-1}}$ be the unique matrix that is consistent with y. We claim that $Y \in \tilde{\mathrm{SA}}_+^{n-1}(P(\alpha))$. First, ($\mathrm{SA}_+$ 1) holds as $Y[\mathcal{F},\mathcal{F}] = y[\emptyset|_0] = 1$. It is also not hard to see that $Y \geq 0$, as every entry in Y is either 0 or $1 - k\frac{\alpha}{\alpha n+1-\alpha}$ for some integer $k \in \{0,\ldots,n\}$. Next, we check that $\hat{x}(Ye_\beta) \in K(P(\alpha))$ for all $\beta \in \mathcal{A}_{n-1}$. Given $\beta = S|_1 \cap T|_0$, $\hat{x}(Ye_\beta)$ is the zero vector whenever $|T| \geq 2$, and is the vector $\frac{\alpha}{\alpha n+1-\alpha}(\bar{e} - e_i)$ whenever $T = \{i\}$ for some $i \in [n]$.

Finally, suppose $\beta = S|_1$ for some $S \subseteq [n]$. Then

$$\hat{x}(Ye_\beta)[i] = \begin{cases} 1 - \frac{k\alpha}{\alpha n+1-\alpha} & \text{if } i = 0 \text{ or } i \in S; \\ 1 - \frac{(k+1)\alpha}{\alpha n+1-\alpha} & \text{if } i \in [n] \setminus S. \end{cases}$$

Now

$$\begin{aligned} & k\left(1 - \frac{k\alpha}{\alpha n + 1 - \alpha}\right) + (n-k)\left(1 - \frac{(k+1)\alpha}{\alpha n + 1 - \alpha}\right) \\ = \;& n\left(1 - \frac{k\alpha}{\alpha n + 1 - \alpha}\right) - \alpha\left(\frac{n-k}{\alpha n + 1 - \alpha}\right) \\ \leq \;& (n - \alpha)\left(1 - \frac{k\alpha}{\alpha n + 1 - \alpha}\right). \end{aligned}$$

Thus, $\hat{x}(Ye_\beta) \in K(P)$ in this case as well. Next, (SA_+ 3), (SA_+ 4) and (SA_+ 5) hold since Y satisfies (OMC) by construction. Finally, to see that $Y \succeq 0$, let Y' be the symmetric minor of Y indexed by rows and columns from $\mathcal{A}^-$. Then $Y' \succeq 0$ as it is diagonally dominant. Also, since $\mathcal{A}^-$ generates $\mathcal{A}_{n-1}$, there exists matrix L such that $Y = LY'L^\top$. Hence, we conclude that $Y \succeq 0$ as well. This completes our proof. $\square$

Note that Proposition 57 slightly strengthens a result of Cheung's in [Che07], who used similar techniques to show that the SA-rank and LS_+-rank of $P(\alpha)$ are both n for all $\alpha \in (0,1)$. Interestingly, Cheung also showed the following in the same paper:

Theorem 58. *For every integer $n \geq 4$,*

1. *The* Las*-rank of $P(\alpha)$ is at most $n-1$ for all $\alpha \geq \frac{1}{n}$;*
2. *There exists $\alpha \in \left(0, \frac{1}{n}\right)$ such that the* Las*-rank of $P(\alpha)$ is n.*

Thus, Las seems to be the only lift-and-project operator (among those we consider in this thesis) whose performance is sensitive to the perturbation of the parameter α.

Also, we remark that the proof of Proposition 85 can be adapted to show that the SA'_+-rank and Las'-rank of any polytope contained in $[0,1]^n$ is at most $\left\lceil \frac{n+1}{2} \right\rceil$. Since we can choose α to be small enough such that $P(\alpha)$ has Las-rank n (by Theorem 58), we obtain a family of examples whose Las-rank and Las'-rank differ by roughly $\frac{n}{2}$. We also note that the BZ-rank of $P(\alpha)$ is 1 for every $\alpha \in (0,1)$. This is because the set $[n]$ is a k-small obstruction for every $k \geq 1$, and so $\sum_{i=1}^n x_i \leq n-1$ is valid for $\mathcal{O}_k(P(\alpha))$, and the refinement step in BZ already suffices in generating the integer hull of $P(\alpha)$.

We next turn our attention to another well-studied example. Let $\alpha > 0$, and define the set

$$Q(\alpha) := \left\{ x \in [0,1]^n : \sum_{i \in S} (1 - x_i) + \sum_{i \notin S} x_i \geq \alpha, \ \forall S \subseteq [n] \right\}.$$

Observe that, for every $S \subseteq [n]$, its incidence vector χ^S violates the inequality corresponding to S in the description of $Q(\alpha)$. Thus, we see that $Q(\alpha)_I = \emptyset$. Also, $Q(\frac{1}{2})$ has been shown to be a worst-case instance for many known lift-and-project methods. We show that it is no different for SA_+.

Corollary 59. *For every $n \geq 2$ and $\alpha \in \left(0, \frac{1}{2}\right]$, $\mathrm{SA}_+^k(Q(\alpha)) = Q\left(\frac{k+\alpha}{2}\right)$ for every $k \in [n]$. In particular, the SA_+-rank of $Q(\alpha)$ is n.*

Proof. Since $Q(\alpha)$ fulfills the description in Corollary 54, it follows that

$$\mathrm{SA}_+^k(Q(\alpha)) = \bigcap_{I \subseteq [n], |I| = k} \left\{ x : x_\emptyset^I \in Q(\alpha) \right\} = Q\left(\frac{k+\alpha}{2}\right).$$

Since $Q(\alpha)_I = \emptyset$, and $\frac{1}{2}\bar{e} \in Q\left(\frac{k+\alpha}{2}\right)$ for all $k \leq n-1$, it follows that the SA_+-rank of $Q(\alpha)$ is n. □

Since $\mathrm{SA}_+^{2k}(P) \subseteq \mathrm{SA}_+'^{k}(P)$ in general (Proposition 6), the SA'_+-rank of $Q(\alpha)$ is at least $\left\lceil \frac{n}{2} \right\rceil$ for every $\alpha \in (0, \frac{1}{2}]$. Also, as mentioned above, the SA'_+-rank of any polytope contained in $[0,1]^n$ is at most $\left\lceil \frac{n+1}{2} \right\rceil$. Thus, we see that in this case, SA'_+ requires roughly $\frac{n}{2}$ fewer rounds than SA_+ to show that $Q(\alpha)$ has an empty integer hull.

It was shown in [BZ04] that $\mathrm{BZ}^2(Q(\frac{1}{2})) = \emptyset = Q(\frac{1}{2})_I$ (which implies $\mathrm{BZ}'^2(Q(\frac{1}{2})) = \mathrm{BZ}_+'^2(Q(\frac{1}{2})) = \emptyset$). However, since the run-time of BZ depends on the size of the system of inequalities describing P (which in this case is exponential in n), the relaxation generated by BZ^2 is not obviously tractable. In contrast, note that it is easy to find an efficient separation oracle for $Q(\alpha)$ (e.g. by observing that $x \in Q(\alpha)$ if and only if $\sum_{i=1}^n |x_i - \frac{1}{2}| \leq \frac{n}{2} - \alpha$), and thus one could optimize a linear function over, say, $\mathrm{SA}^k(Q(\alpha))$ in polynomial time for any $k = O(1)$. The reader may refer to Figure 1.2 for a complete classification of operators that depend on the algebraic description of the input set P, as opposed to those that only require a weak separation oracle.

As for the Las-rank of $Q(\frac{1}{2})$, it is shown to be 1 for $n = 2$ in [Lau03a], and 2 for $n = 4$ in [Che07]. While Las depends on the algebraic description of the initial relaxation just like the Bienstock–Zuckerberg operators, the following observation can significantly simplify the analysis of the Las-rank of $Q(\alpha)$.

Proposition 60. *Suppose n, k are fixed integers and $\alpha \in (0,1)$. Define the vector $w \in \mathbb{R}^{\mathcal{A}_n^+}$ where*

$$w[S|_1] = (n - |S| - 2\alpha)2^{-|S|-1}, \ \forall S \subseteq [n].$$

Then $\mathrm{Las}^k(Q(\alpha)) \neq \emptyset$ if and only if $M_k(w) \succeq 0$.

Proof. Suppose $\mathrm{Las}^k(Q(\alpha)) \neq \emptyset$, and let $Y' \in \tilde{\mathrm{Las}}^k(Q(\alpha))$. Notice that every automorphism for the unit hypercube is also an automorphism for $Q(\alpha)$. If we take these $2^n n!$ automorphisms and apply them onto Y' as outlined in the proof of Proposition 38, we obtain $2^n n!$ matrices in $\tilde{\mathrm{Las}}^k(Q(\alpha))$. Let $\bar{Y}$ be the average of these matrices. Then by the symmetry of $Q(\alpha)$, we know that $\bar{Y} = M_k(y)$, where $y \in \mathbb{R}^{\mathcal{A}_n^+}$,

$$y[S|_1] = 2^{-|S|-1}, \ \forall S \subseteq [n].$$

By the convexity of $\tilde{\mathrm{Las}}^k(Q(\alpha))$, $\bar{Y} \in \tilde{\mathrm{Las}}^k(Q(\alpha))$, and thus satisfies (Las 2) for all of the 2^n equalities defining $Q(\alpha)$. In fact, due to the entries of $\bar{Y}$, the matrix $\bar{Y}^j$ is the same for all 2^n inequalities describing $Q(\alpha)$. Thus, using the inequality $\sum_{i=1}^n x_i \leq n - \alpha$, we obtain that

$$\begin{aligned} \bar{Y}^j[S|_1, T|_1] &= (n-\alpha)Y[S|_1, T|_1] - \sum_{i=1}^n Y[(S \cup \{i\})|_1, (T \cup \{i\})|_1] \\ &= (n - |S \cup T| - 2\alpha)2^{-|S \cup T|-1} \\ &= M_k(w)[S|_1, T|_1] \end{aligned}$$

for all $S, T \subseteq [n], |S|, |T| \leq k$. Hence, we deduce that $\mathrm{Las}^k(Q(\alpha)) \neq \emptyset \Rightarrow M_k(w) \succeq 0$.

The converse can be proven by tracing the above argument backwards. In establishing the SA_+-rank lower bound of $Q(\alpha)$, we have shown that $\bar{Y} \succeq 0$. Again, the matrix $\bar{Y}^j$ is exactly $M_k(w)$ for all 2^n inequalities describing $Q(\alpha)$. Since $M_k(w) \succeq 0$ by assumption, $\bar{Y} \in \tilde{\mathrm{Las}}^k(Q(\alpha))$. Thus, we obtain that $\frac{1}{2}\bar{e} \in \mathrm{Las}^k(Q(\alpha))$, and so $\mathrm{Las}^k(Q(\alpha)) \neq \emptyset$. $\square$

Thus, computing the Las-rank of $Q(\alpha)$ reduces to finding the largest k where the matrix $M_k(w)$ defined in the statement of Proposition 60 is positive semidefinite (which would then imply that the Las-rank of $Q(\alpha)$ is $k+1$). Using that, we are able to show the following:

Theorem 61. *For every $n \geq 2$, there exists $\alpha \in (0,1)$ such that $Q(\alpha)$ has* Las-*rank n.*

Before we prove Theorem 61, we need some notation. Define the matrix $Z \in \mathbb{R}^{\mathcal{A}_n^+ \times \mathcal{A}_n^+}$ where

$$Z[S|_1, T|_1] = \begin{cases} 1 & \text{if } S \subseteq T; \\ 0 & \text{otherwise.} \end{cases}$$

Z is the *zeta matrix* of $\{1, \ldots, n\}$. It is well known that Z is invertible. For example, if we order the rows and columns of Z such that their size is nondecreasing, then Z is upper-triangular. In particular, with such a row and column ordering, both the first row of Z and the last column of Z are all ones.

Laurent [Lau03a] showed the following relation between zeta matrices and moment matrices:

Proposition 62 (Lemma 2 in [Lau03a])**.** *Suppose $y \in \mathbb{R}^{\mathcal{A}_n^+}$. Define $u \in \mathbb{R}^{\mathcal{A}_n^+}$ where*

$$u[S|_1] = \sum_{T \supseteq S} (-1)^{|T \setminus S|} y[T|_1].$$

Then $M_n(y) = Z \operatorname{Diag}(u) Z^\top$.

Recall that $\operatorname{Diag}(u)$ denotes the diagonal matrix U where $U[S|_1, S|_1) = u[S|_1]$ for all $S \subseteq [n]$. Now we are ready to prove Theorem 61.

Proof of Theorem 61. By Proposition 60, it suffices to prove that, for every n, there exists α such that $M_{n-1}(w) \succeq 0$. Since the entries of the matrix $M_{n-1}(w)$ depend continuously on α, our claim follows if we prove that $M_{n-1}(w) \succ 0$ when $\alpha = 0$.

Let Z' denote the symmetric minor of the zeta matrix Z with the last row and column (which corresponds to the set $[n]|_1$) removed. Then $Z = \begin{pmatrix} Z' & \bar{e} \\ 0 & 1 \end{pmatrix}$. Also, define $u \in \mathbb{R}^{\mathcal{A}_n^+}$ such that $u[S|_1] = \sum_{T \supseteq S} (-1)^{|T \setminus S|} w[T|_1]$. Then

$$\begin{aligned}
& u[S|_1] \\
= \; & \sum_{T \supseteq S} (-1)^{|T \setminus S|} (n - |T| - 2\alpha) 2^{-|T|-1} \\
= \; & 2^{-n} \sum_{i=0}^{n-|S|} (-1)^i \binom{n-|S|}{i} (n - |S| - i - 2\alpha) 2^{n-|S|-i-1} \\
= \; & 2^{-n} \sum_{i=0}^{n-|S|} (n-|S|) \left((-1)^i \binom{n-|S|-1}{i} 2^{n-|S|-i-1} \right) - \alpha \left((-1)^i \binom{n-|S|}{i} 2^{n-|S|-i} \right) \\
= \; & 2^{-n} (n - |S| - \alpha).
\end{aligned}$$

Therefore, when $\alpha = 0$, $u[S|_1] > 0$ for all $S \subset [n]$, and $u[[n]|_1] = 0$. Let $u' \in \mathbb{R}^{\mathcal{A}_{n-1}^+}$ be the vector u with the entry corresponding to $[n]|_1$ removed. Then $u' > 0$, and $\mathrm{Diag}(u) = \begin{pmatrix} \mathrm{Diag}(u') & 0 \\ 0 & 0 \end{pmatrix}$.

Hence, by Proposition 62,

$$\begin{aligned} M_n(w) &= Z\,\mathrm{Diag}(u) Z^\top \\ &= \begin{pmatrix} Z' & \bar{e} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Diag}(u') & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z'^\top & 0 \\ \bar{e}^\top & 1 \end{pmatrix} \\ &= \begin{pmatrix} Z'\,\mathrm{Diag}(u') Z'^\top & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Since $M_{n-1}(w)$ is the symmetric minor of $M_n(w)$ with the last row and column removed, we obtain that $M_{n-1}(w) = Z'\,\mathrm{Diag}(u') Z'^\top$. Now Z' is upper-triangular (and thus invertible), and $\mathrm{Diag}(u') \succ 0$ (since $u' > 0$). Thus, $M_{n-1}(w) \succ 0$, and we are finished. □

In fact, with a more careful analysis, one can find an interval of α's where $Q(\alpha)$ attains Las-rank n. The following lemma will be helpful:

Lemma 63. *Let $Z^{(n)}$ be the $2^n \times 2^n$ zeta matrix on $[n]$, and let $\beta := \frac{3-\sqrt{5}}{2}$. Then β^{n-2i} is an eigenvalue of $Z^{(n)}(Z^{(n)})^\top$ with multiplicity $\binom{n}{i}$, for every $i \in \{0, 1, \ldots, n\}$.*

Proof. For this proof, we use the following ordering of rows and columns of $Z^{(n)}$: Given distinct subsets $S, S' \subseteq [n]$, we place the S-column in Z to the right of S'-column if the element with the largest index in the symmetric difference of S and S' is in S. For example, the columns of $Z^{(3)}$ correspond to the subsets

$$\emptyset, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\},$$

if we scan from left to right. With this ordering, it is not hard to see that $Z^{(n)} = \begin{pmatrix} Z^{(n-1)} & Z^{(n-1)} \\ 0 & Z^{(n-1)} \end{pmatrix}$ for all $n \geq 1$. Moreover,

$$\begin{aligned} Z^{(n)}(Z^{(n)})^\top &= \begin{pmatrix} Z^{(n-1)} & Z^{(n-1)} \\ 0 & Z^{(n-1)} \end{pmatrix} \begin{pmatrix} (Z^{(n-1)})^\top & (Z^{(n-1)})^\top \\ (Z^{(n-1)})^\top & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2Z^{(n-1)}(Z^{(n-1)})^\top & Z^{(n-1)}(Z^{(n-1)})^\top \\ Z^{(n-1)}(Z^{(n-1)})^\top & Z^{(n-1)}(Z^{(n-1)})^\top \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \otimes Z^{(n-1)}(Z^{(n-1)})^\top, \end{aligned}$$

where $\otimes$ denotes the Kronecker product. In fact, since $Z^{(1)}(Z^{(1)})^\top = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, we obtain that

$$Z^{(n)}(Z^{(n)})^\top = \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \otimes \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}}_{n \text{ times}},$$

for every $n \geq 1$. Since the eigenvalues of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ are β and β^{-1}, and that the eigenvalues of the Kronecker product are the pairwise products of the eigenvalues of the two matrices, our claim follows. $\square$

In particular, since $\beta < 1$, we obtain that the smallest eigenvalue of $Z^{(n)}(Z^{(n)})^\top$ is β^n. Then we have the following:

Theorem 64. *Suppose $n \geq 2$, and*

$$0 < \alpha \leq \frac{5^n - 4^n}{5^n - 2^{n+1} + 1}.$$

Then $Q(\alpha)$ has Las-*rank n.*

Proof. We follow the proof of Theorem 61, while sharpening a few details. Again, if we define $u \in \mathbb{R}^{\mathcal{A}_n^+}$ such that $u[S|_1] = \sum_{T \supseteq S} (-1)^{|T \setminus S|} w[T|_1]$. Then

$$u[S|_1] = 2^{-n}(n - |S| - \alpha)$$

for all $S \subseteq [n]$. Now $u[S|_1] > 0$ for all $S \subset [n]$, and $u[[n]|_1] = -2^{-n}\alpha$. Let $u' \in \mathbb{R}^{\mathcal{A}_{n-1}^+}$ be the vector u with the entry corresponding to $[n]|_1$ removed. Then $\text{Diag}(u) = \begin{pmatrix} \text{Diag}(u') & 0 \\ 0 & -2^{-n}\alpha \end{pmatrix}$.

Hence, by Proposition 62,

$$\begin{aligned} M_n(w) &= Z \, \text{Diag}(u) Z^\top \\ &= \begin{pmatrix} Z' & \bar{e} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Diag}(u') & 0 \\ 0 & -2^{-n}\alpha \end{pmatrix} \begin{pmatrix} Z'^\top & 0 \\ \bar{e}^\top & 1 \end{pmatrix} \\ &= \begin{pmatrix} Z' \, \text{Diag}(u') Z'^\top - 2^{-n}\alpha \bar{e}\bar{e}^\top & -2^{-n}\alpha\bar{e} \\ -2^{-n}\alpha\bar{e}^\top & -2^{-n}\alpha \end{pmatrix}. \end{aligned}$$

Since $M_{n-1}(w)$ is the symmetric minor of $M_n(w)$ with the last row and column removed, we obtain that $M_{n-1}(w) = Z' \, \text{Diag}(u') Z'^\top - 2^{-n}\alpha\bar{e}\bar{e}^\top$.

Next, since $u' \geq 2^{-n}(1 - \alpha)\bar{e}$,

$$Z' \, \text{Diag}(u') Z'^\top \succeq 2^{-n}(1 - \alpha) Z' Z'^\top.$$

Thus, to establish $M_{n-1}(w) \succeq 0$, it suffices to show that

$$(1 - \alpha) Z' Z'^\top - \alpha \bar{e}\bar{e}^\top \succeq 0. \tag{6.1}$$

Since $(1-\alpha)Z'Z'^\top \succeq 0$ and all vectors orthogonal to $\bar{e}$ have eigenvalue 0 for $\bar{e}\bar{e}^\top$, we see that (6.1) holds if and only if

$$\begin{aligned} & (1-\alpha)\bar{e}^\top Z'Z'^\top \bar{e} - \alpha \bar{e}^\top(\bar{e}\bar{e}^\top)\bar{e} \geq 0 \\ \iff \quad & (1-\alpha)\left(\sum_{i=0}^{n-1}\binom{n}{i}(2^i)^2\right) - \alpha(2^n-1)^2 \geq 0 \\ \iff \quad & \alpha \leq \frac{5^n-4^n}{5^n-4^n+(2^n-1)^2} \\ \iff \quad & \alpha \leq \frac{5^n-4^n}{5^n-2^{n+1}+1}. \end{aligned}$$

Thus, our claim follows. □

Therefore, similar to the case for $P(\alpha)$, the Las-rank of $Q(\alpha)$ is n for sufficiently small α. Also, as with $P(\alpha)$, the Las-rank of $Q(\alpha)$ varies under the choice of α. For instance, Figure 6.3 illustrates the Las-rank for $Q\left(\frac{k}{1000}\right)$ for $k \in [500]$, for several values of n.

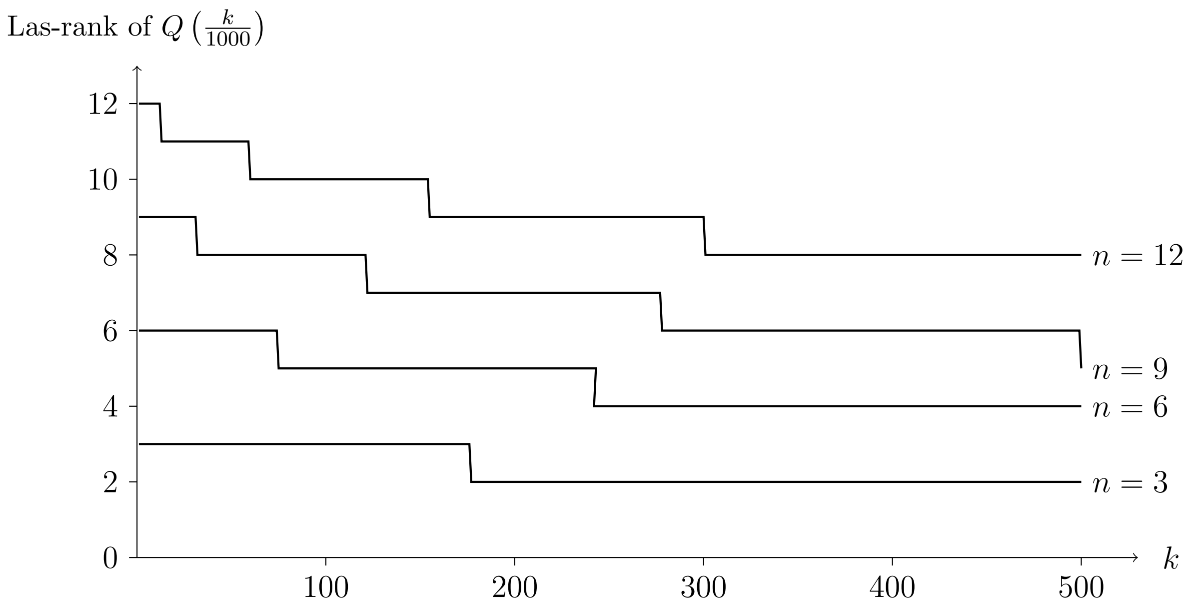

Figure 6.3: The Las-rank of $Q(\alpha)$ for varying values of α, for $n \in \{3, 6, 9, 12\}$.

The pattern is similar for all other values of n we were able to test — the Las-rank is around $\frac{n}{2}$ when $\alpha = \frac{1}{2}$, and slowly rises to n as α approaches 0.

Also, while the interval of α's where $Q(\alpha)$ has Las-rank n is probably wider than that shown in Theorem 64, computational results show that the size of this "window" of α's seem to shrink exponentially as n increases. Let $f(n)$ be the largest α where $Q(\alpha)$ has Las-rank n. We have computed $\log_2 f(n)$ to within two decimal place for $n \in \{2, 3, \ldots, 12\}$, as illustrated in Figure 6.4. Somewhat surprisingly, $\log_2(f(n))$ increases at an almost linear rate with respect to n, at least for the small values of n we have computed.

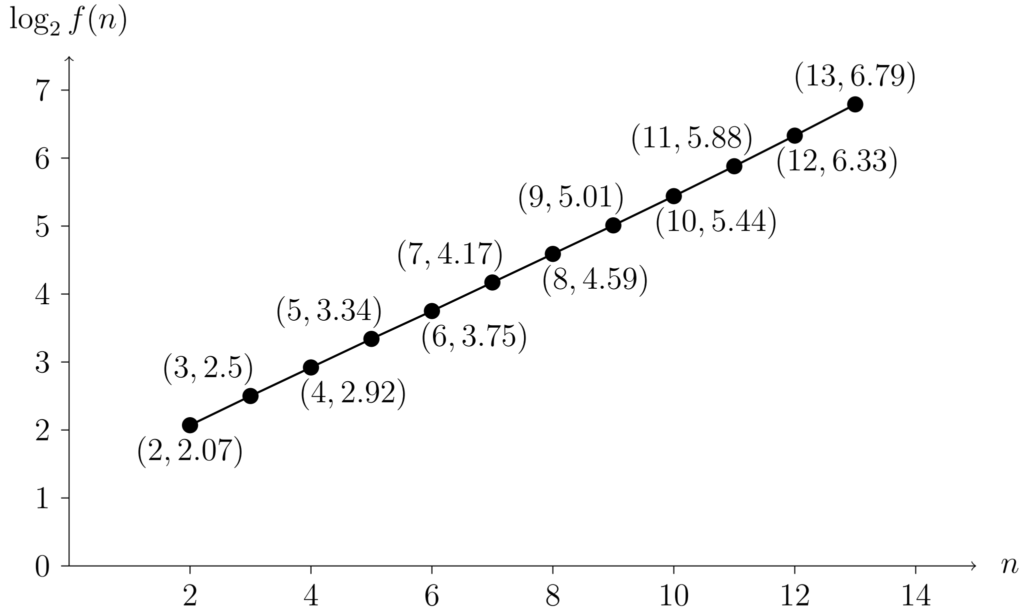

Figure 6.4: Computational results for $f(n) := \min\left\{\alpha : \text{Las}^{n-1}(Q(\alpha)) \neq \emptyset\right\}$.

In general, since all lift-and-project operators we have studied preserve containment, starting with a tighter initial relaxation might offer a lift-and-project operator a head start and yield stronger relaxations in fewer iterations. However, in the two examples we just saw, different lift-and-project operators utilized this head start in different ways. For operators such as LS_+, SA and SA_+ that use linear constraints to relate the entries of the matrices in the lifted space to P (e.g. imposing $\hat{x}(Ye_\alpha) \in K(P)$), the "head start" is preserved throughout the rounds in sort of a statical manner. For instance, we know from Corollary 59 that given α, α' where $0 < \alpha < \alpha' < \frac{1}{2}$,

$$\text{SA}_+^k(Q(\alpha)) = Q\left(\alpha + \frac{k}{2}\right) \supset Q\left(\alpha' + \frac{k}{2}\right) = \text{SA}_+^k(Q(\alpha')),$$

for all $k \in [n-1]$. However, they still converge to the integer hull in the same number of steps. On the other hand, the Las operator turns every inequality of in the description of the relaxation P into a semidefinite constraint, which have shown (at least on $P(\alpha)$ and $Q(\alpha)$) that it can compound with the improved initial relaxation to arrive at the integer hull in fewer iterations.

It would be interesting to better understand how the performance of Las depends on the system of inequalities (linear or polynomial) describing P, and whether it can benefit from some "preprocessing" of the set P, such as adding some selected Gomory–

Chvátal cuts, or including inequalities obtained from the refinement step in the Bienstock–Zuckerberg operators.

Chapter 7

On the Integrality Gaps of Lift-and-Project Relaxations

So far, we have been using the rank of a relaxation with respect to a lift-and-project operator as the measure of how far that relaxation is away from its integer hull. Another measure of the "tightness" of a relaxation that is commonly used and well-studied is its *integrality gap*. Let $P \subseteq [0,1]^n$ be a compact, convex set such that $P_I \neq \emptyset$, and suppose $c \in \mathbb{R}^n$. Then

$$\gamma_c(P) := \frac{\max\left\{c^\top x : x \in P\right\}}{\max\left\{c^\top x : x \in P_I\right\}}$$

is the *integrality gap* of P with respect to c. Obviously, if $P = P_I$, then $\gamma_c(P) = 1$ for all $c \in \mathbb{R}^n$. Otherwise, there exists some vector c such that $\gamma_c(P) > 1$. Of course, the minimization analog of the integrality gap can also be defined similarly, but we focus on the above version as we will mostly look at maximization problems.

Observe that, given two different relaxations of the same set of integer points such that one relaxation contains the other, then obviously the smaller set provides a better relaxation. This elementary fact is also reflected in the integrality gaps of these sets — given P, P' such that $P_I = P'_I$ and $P \subseteq P'$, then $\gamma_c(P) \leq \gamma_c(P')$ for every vector c. Thus, our analysis of relative strengths of lift-and-project operators in Chapter 3 immediately implies the following:

Corollary 65. *Suppose $P \subseteq [0,1]^n$, and two lift-and-project operators Γ_1, Γ_2 satisfy the conditions in either Theorem 11 or 12. Then*

$$\gamma_c(\Gamma_1(P)) \leq \gamma_c(\Gamma_2(P)),$$

for all $c \in \mathbb{R}^n$.

In general, the integrality gap provides a finer and more direction-specific measure of the tightness of a relaxation. For instance, in Chapter 6, we saw examples when adding positive semidefiniteness constraints does not provide a tighter relaxation. However, sometimes even when these constraints' inclusion produces a tighter set, they might still be deemed unhelpful if they trim off parts of the relaxations that are not in the direction of our interest. More precisely, given two relaxations P, P' such that $P_I = P'_I$ and $P \subset P'$, it is still possible that maximizing $c^\top x$ over P and P' yields the same optimal value for some vector c. This is equivalent to saying that P and P' have the same integrality gap with respect to c.

In this chapter, we first establish some tools that can help computing the integrality gap for general relaxations. We then apply these tools, and look at the integrality gaps for several lift-and-project relaxations. Finally, we will look at how the integrality gaps of various lift-and-project relaxations converge to 1. Many circumstantial pieces of evidence suggest that the convergence behaviours are quite different for operators that produce polyhedral versus semidefinite relaxations, and we discuss a few examples that highlight these tendencies.

7.1 Simplifying integrality gap computations by utilizing symmetries

Recall that in our discussion of constructing certificate matrices for $x \in \Gamma(P)$ in the presence of a permutation matrix Q such that $Q(P) = P$, we may assume that x inherits the symmetries of P and Q. Thus, the next result follows readily from Proposition 43.

Corollary 66. *Let $P \subseteq [0,1]^n$ be a compact, convex set, and Γ be a permutation-commutative lift-and-project operator such that $\Gamma(P)$ is also convex. Then, for every $c \in \mathbb{R}^n$, the integrality gap $\gamma_c(\Gamma(P))$ is attained by a vector $x \in \Gamma(P)$ with the property that $x_i = x_j$ whenever there exists a permutation matrix Q such that $Q(P) = P, Qc = c$ and $Qe_i = e_j$.*

In particular, in the case when P is symmetric and we are optimizing in the direction of $\bar{e}$, we have the following:

Corollary 67. *Let $P \subseteq [0,1]^n$ be a compact, convex set that is symmetric. If a lift-and-project operator Γ is permutation-commutative and $\Gamma(P)$ is convex, then*

- *$\gamma_{\bar{e}}(\Gamma(P))$ is attained by a multiple of $\bar{e}$;*
- *if $\ell\bar{e} \notin \Gamma(P)$ and $\ell_0\bar{e} \in \Gamma(P)$ for some $\ell_0 < \ell$, then*

$$\gamma_{\bar{e}}(\Gamma(P)) < \frac{\ell n}{\max\left\{\sum_{i=1}^{n} x_i : x \in P_I\right\}}.$$

Of course, an analog of Corollary 67 for $\gamma_{-\bar{e}}(\cdot)$ can be obtained by essentially the same observations. Thus, we see that in many cases, it suffices to check whether a certain multiple of $\bar{e}$ belongs to $\Gamma(P)$ to obtain a bound on $\gamma_{\bar{e}}(\Gamma(P))$. This structure, when present, can make the analysis a lot easier.

7.2 Obtaining integrality gap results from lower-bound results

Next, we show that many lower-bound results on lift-and-project relaxations readily lead to integrality gap results, simply by applying the above observations. First, we look at the fractional stable set polytope of complete graphs.

Proposition 68. *For all integers $n \geq 2, k \in \{0, 1, \ldots, n-2\}$ and lift-and-project operator $\Gamma \in \{\mathrm{LS}_0, \mathrm{LS}, \mathrm{SA}, \mathrm{SA}'\}$,*

$$\gamma_{\bar{e}}(\Gamma^k(FRAC(K_n)) = \frac{n}{k+2}.$$

Proof. Let P denote $FRAC(K_n)$. First, since

$$\mathrm{SA}'^k(P) \subseteq \mathrm{SA}^k(P) \subseteq \mathrm{LS}^k(P) \subseteq \mathrm{LS}_0^k(P)$$

for every k, it suffices to show that $\gamma_{\bar{e}}(\mathrm{SA}'^k(P) \geq \frac{n}{k+2}$ and $\gamma_{\bar{e}}(\mathrm{LS}_0^k(P) \leq \frac{n}{k+2}$. The former follows immediately from Proposition 17, as $\frac{1}{k+2}\bar{e} \in \mathrm{SA}'^k(P)$ achieves the claimed integrality gap. As for the latter, note that P is symmetric (in fact, $Q(P) = P$ for *every* permutation matrix Q), and LS_0 is permutation-commutative and produces convex relaxations. Thus, we may apply Corollary 67 and conclude that $\gamma_{\bar{e}}(\mathrm{LS}_0^k(P))$ is attained by a multiple of $\bar{e}$. Since it is easy to show that $\ell\bar{e} \notin \mathrm{LS}_0^k(P)$ for all $\ell > \frac{1}{k+2}$ (e.g. by induction on k), our result follows. $\square$

Note that in the proof of Theorem 18, we showed that

$$\mathrm{BZ}'^k(FRAC(K_n)) \supseteq \mathrm{SA}'^{2k+2}(FRAC(K_n)),$$

for every $n \geq 3$ and $k \geq 0$. We also obtained that the BZ'-rank of $FRAC(K_n)$ is at most $\left\lceil \frac{n+1}{2} \right\rceil$. Thus, we immediately obtain the following

Proposition 69. *For all integers $n \geq 3, k \in \left\{0, 1, \ldots, \left\lceil \frac{n}{2} \right\rceil - 2\right\}$,*

$$\gamma_{\bar{e}}(\mathrm{BZ}'^k(FRAC(K_n)) \geq \frac{n}{2k+4}.$$

Moreover, $\gamma_{\bar{e}}(\mathrm{BZ}'^k(FRAC(K_n)) = 1$ for all $k \geq \left\lceil \frac{n+1}{2} \right\rceil$.

Next, we turn to matching polytope. It was shown in [MS09] that $\gamma_{\bar{e}}(\mathrm{SA}^k(MT(K_{2n+1})))$ exhibits very interesting behaviour as k varies: It remains at $1 + \frac{1}{2n}$ for all $k \leq n-1$, then gradually decreases, and reaches 1 exactly when $k = 2n-1$. Relying on Stephen and Tunçel's results on $\mathrm{LS}_+^k(MT(K_{2n+1}))$ from [ST99], we show that the integrality gaps of the LS_+-relaxations behave quite differently.

Proposition 70. *Let $P = MT(K_{2n+1})$ for some integer $n \geq 1$. Then*

$$\gamma_{\bar{e}}(\mathrm{LS}_+^k(P)) = \begin{cases} 1 + \frac{1}{2n} & \text{if } k \leq n-1; \\ 1 & \text{if } k \geq n. \end{cases}$$

Proof. It was shown in [ST99] that $\mathrm{LS}_+^n(P) = P_I$, and thus the integrality gap is 1 for all $k \geq n$. On the other hand, they also showed that $\frac{1}{2n}\bar{e} \in \mathrm{LS}_+^{n-1}(P)$, and thus $\gamma_{\bar{e}}(\mathrm{LS}_+^{n-1}(P)) \geq 1 + \frac{1}{2n}$. It is not hard to show that $\ell\bar{e} \notin P$ for all $\ell > \frac{1}{2n}$. Since P is obviously symmetric, $\gamma_{\bar{e}}(P)$ is attained by a multiple of $\bar{e}$ and so we obtain that $\gamma_{\bar{e}}(P) \leq 1 + \frac{1}{2n}$. Then our result follows, as the integrality gap is monotonously decreasing with respect to k. □

We now look into the relaxations of the max-cut problem. Recall Lasserre's semidefinite relaxations (5.3) and Georgiou's lower-bound result (Theorem 27). While the relaxations (5.3) are technically not derived from any of the lift-and-project methods we have looked at, one can still exploit the symmetries in its objective function and feasible region and argue (along the lines of the proof of Corollary 45) that the maximum integrality gaps of (5.3) is achieved by a matrix M where $M[S_1|_1, T_1|_1] = M[S_2|_1, T_2|_1]$ whenever $|S_1 \cup T_1| = |S_2 \cup T_2|$. This tells us that the integrality gap of (5.3) with respect to $\bar{e}$, as first pointed out by Georgiou [Geo10], is

$$\frac{(2n+1)^2/4}{n(n+1)} = 1 + \frac{1}{4n(n+1)}.$$

Another relaxation of the max-cut problem that has been studied in the literature (and was referred to as the "edge model" in [Lau04]) is the following. Given a graph $G = (V, E)$, one can define its *cut polytope* to be

$$CUT(G) := \mathrm{conv}(\{\chi^{\delta(S)} : S \subseteq V\}).$$

Then a relaxation of $CUT(G)$ is the *metric polytope* of G,

$$MET(G) := \left\{ x \in [0,1]^E : 1 \leq \sum_{e \in C} x_e \leq |C| - 1, \forall \text{ odd cycles } C \subseteq E \right\}.$$

While not all integer points in $MET(G)$ are cuts, it is not hard to see that

$$\max \left\{ c^\top x : x \in MET(G) \cap \{0,1\}^E \right\} = \max \left\{ c^\top x : x \in CUT(G) \right\}$$

for all vectors $c \geq 0$. Thus, we can apply any of the lift-and-project methods to $MET(G)$ to approximate $CUT(G)$. Laurent [Lau04] showed that $\mathrm{Las}^k(MET(G))$ is a weaker relaxation than the k^{th} relaxation in (5.3).

Another similar relaxation for max-cut was proposed by Gouveia, Parrilo and Thomas in [GPT10], utilizing an observation that is akin to the notion of obstructions in the BZ operator: define $\mathcal{O}(G)$ to be the collection where $S \subseteq E$ is in $\mathcal{O}(G)$ if and only if S is the set of edges of an odd cycle in G. Then the size of the largest cut in G is equal to

$$\max \left\{ \sum_{e \in E} x_e : \prod_{e \in S} x_e = 0, \ \forall S \in \mathcal{O}(G) \right\}.$$

Using this, they defined the following hierarchy of relaxations:

$$\begin{aligned} \mathrm{TH}_k(CUT(G)) := \{x \in \mathbb{R}^E \quad : \quad & \exists y \in \mathbb{R}^{\mathcal{A}_{2k}^+}, y_{\mathcal{F}} = 1, y_{e|_1} = x_e \ \forall e \in E, \\ & y_{S|_1} = 0, \text{ if } S \supseteq T \text{ for some } T \in \mathcal{O}(G) \\ & M_k(y) \succeq 0\}. \end{aligned} \tag{7.1}$$

Then it is clear that $\mathrm{TH}_k(CUT(G)) \supseteq CUT(G)$ for all $k \geq 1$. This idea of describing a desired set of integer points by a list of "forbidden subsets" can also be applied to other combinatorial optimization problems. For instance, it was shown in [GPT10] that

$$\begin{aligned} \mathrm{TH}_k(STAB(G)) := \{x \in \mathbb{R}^V \quad : \quad & \exists y \in \mathbb{R}^{\mathcal{A}_{2k}^+}, y_{\mathcal{F}} = 1, y_{i|_1} = x_i \ \forall i \in V, \\ & y_{S|_1} = 0, \text{ if } S \supseteq \{i,j\} \text{ for some } \{i,j\} \in E \\ & M_k(y) \succeq 0\}. \end{aligned} \tag{7.2}$$

Notice that $\mathrm{TH}_1(STAB(G))$ is exactly the theta body of G, as defined by Lovász in [Lov79]. However, while $\mathrm{TH}_k(CUT(G))$ and $\mathrm{TH}_k(STAB(G))$ are intuitively simple to describe, they are rather weak relaxations. For instance, while the odd cycle inequalities are valid for simple linear relaxations such as $\mathrm{LS}_0(FRAC(G))$, they are generally not valid for $\mathrm{TH}_1(STAB(G))$ (although they are for $\mathrm{TH}_2(STAB(G))$, as shown in [GPT10]). Also, if G is an odd cycle of $2n+1$ vertices, then $\mathrm{TH}_n(CUT(G)) \neq CUT(G)$ [GPT10]. It would be interesting if there are other problems in which this theta body approach using obstruction sets can yield strong relaxations.

Finally, note that Lasserre's max-cut relaxations (5.3) can be interpreted as lifting from the vertex space and projecting into a space indexed by $V(G) \cup E(G)$. This idea of projecting a lifted space onto a set defined by the variables used in the objective function may also be extended to other problems. For instance, let ℓ be a sufficiently large constant. Then one can obtain an upper bound to the size of the largest stable set in a graph G by solving the following:

$$\begin{aligned} \max \quad & \sum_{i \in V(G)} y[i|_1] - \ell \left(\sum_{\{i,j\} \in E(G)} y[\{i,j\}\,|_1] \right) \\ \text{subject to} \quad & M_k(y) \succeq 0, y \in \mathbb{R}^{\mathcal{A}_{2k}^+}, y[\mathcal{F}] = 1. \end{aligned} \tag{7.3}$$

For instance, when $\ell = |V|$, then an integer solution would have a nonnegative objective value if and only if it corresponds to a stable set. Moreover, the feasible region of (7.3) is identical to that of (5.3), and insights obtained in one problem might be useful for the other. To go a bit further, one could theoretically combine the two ideas above, and (say) optimize the objective function in (7.3) over $\mathrm{TH}_k(STAB(G))$, or any other lift-and-project relaxations where the $y_{\{i,j\}_1}$ variables are present.

7.3 Integrality gaps of SA'_+-relaxations for matching

We now turn our attention to obtain a bound on the integrality gap of a specific family of relaxations. As seen in Chapter 5, sometimes we can use the symmetries of P to prove the existence of "nice" certificate matrices that have few parameters (e.g. Proposition 45). Herein, we give an application of this approach, and prove a result on the integrality gap for relaxations of the matching polytope produced by SA'_+, with respect to $\bar{e}$. Throughout this section, let G be K_{4n+1}. Since $\frac{1}{4n}\bar{e} \in MT(G)$ and the largest matching in G has size $2n$, we know that

$$\gamma_{\bar{e}}(MT(G)) \geq \frac{1}{4n}\binom{4n+1}{2}\frac{1}{2n} = \frac{4n+1}{4n}.$$

In fact, equality holds as it is not hard to see that $\ell\bar{e} \notin MT(G)$ for all $\ell > \frac{1}{4n}$. Here, we show that SA'_+ takes at most $n+1$ iterations to obtain improvement on this gap.

Theorem 71. *For all integers $n \geq 1$,*

$$\gamma_{\bar{e}}(\mathrm{SA}'^{n+1}_+(MT(G))) < \frac{4n+1}{4n}.$$

Before proving Theorem 71, we need some intermediate results. First, we describe another family of eigenvectors of $Y_{n,k,k}(m)$. We remark that these are generalizations of the second type of eigenvectors described after Proposition 35.

Proposition 72. *Define $x \in \mathbb{R}^{\mathcal{M}_{n,k}}$ where*

$$x[S] = \begin{cases} 1 & \text{if } S \text{ saturates vertex 1 but not 2;} \\ -1 & \text{if } S \text{ saturates vertex 2 but not 1;} \\ 0 & \text{otherwise.} \end{cases}$$

Then x is an eigenvector of $Y_{n,k,k}(m)$ with eigenvalue

$$\sum_{i=0}^{\min\left\{k-1, \left\lfloor \frac{n-2k-1}{2} \right\rfloor\right\}} |\mathcal{M}_{n-2k-1,i}| \left(m_{k+i} - (n-2k-1-2i)m_{k+i+1}\right).$$

Proof. First, we evaluate

$$\sum_{T \in \mathcal{M}_{n,k}} Y_{n,k,k}(m)[S,T]x[T]. \tag{7.4}$$

for the case when $x[S] = 0$. If S contains the edge $\{1,2\}$, then it is easy to see that there are no matchings T such that $S \cup T$ is a matching, and $x[T] \neq 0$. Thus, (7.4) evaluates to 0. Next, suppose S saturates neither vertex 1 nor vertex 2. Let T be a matching such that $x[T] = 1$ and $Y[S,T] \neq 0$. Then T contains an edge $\{1,i\}$ for some vertex i. Now observe that for the matching

$$T' := (T \setminus \{\{1,i\}\}) \cup \{\{2,i\}\},$$

$x[T'] = -1$ and $Y[S,T'] = Y[S,T]$. Thus, their contributions to (7.4) cancel each other out, and the sum also vanishes in this case. Likewise, suppose S contains edges $\{1,i\}$ and $\{2,j\}$ for some vertices i,j. If T is a matching such that $x[T] = 1$ and $Y[S,T] \neq 0$ (so T must contain $\{1,i\}$), then

$$T' := (T \setminus \{\{1,i\}\}) \cup \{\{2,j\}\}$$

satisfies $x[T'] = -1$ and $Y[S,T'] = Y[S,T]$, so again the sum evaluates to 0.

Next, we turn to the case $x[S] = 1$. Consider the matchings T such that $x[T] = 1$, $S \cup T$ is a matching, and $|T \setminus S| = i$. This implies that both S, T contain the same edge that is incident with vertex 1. Also, $|T \setminus S| = i$ implies that $Y[S,T] = m_{k+i}$. Now the i edges in $S \setminus T$ cannot saturate vertices already saturated by S, or the vertex 2. Thus, there are $|\mathcal{M}_{n-2k-1,i}|$ such choices, and such matchings T contribute a total of $|\mathcal{M}_{n-2k-1,i}| m_{k+i}$ to (7.4).

Likewise, if we consider the matchings T such that $x[T] = -1$, $S \cup T$ is a matching, and $|T \setminus S| = i+1$, we obtain that these matchings contribute $-|\mathcal{M}_{n-2k-1,i}|(n-2k-1-2i)m_{k+i+1}$. Summing these contributions over all possible i leads to our result. □

We then need the next result, which relates the entries in certificate matrices that are of our interest.

Corollary 73. *Let $\mathcal{M} := \bigcup_{i=0}^{k} \mathcal{M}_{n,i}$. Define $Y \in \mathbb{R}^{\mathcal{M} \times \mathcal{M}}$ such that*

$$Y[S,T] = \begin{cases} (n-2i-1)!! & \text{if } S \cup T \in \mathcal{M}_{n,i},\ i \in \{0,1,\ldots,2k-1\} \\ \alpha & \text{if } S \cup T \in \mathcal{M}_{n,2k} \\ 0 & \text{otherwise.} \end{cases}$$

If $Y \succeq 0$, then $\alpha = (n-4k-1)!!$.

Proof. For convenience, let p_i denote $(n-2i-1)!!$ throughout this proof. Our goal then is to show that $\alpha = p_{2k}$.

First, if α was in fact p_k, then we know from Lemma 29 that,

$$\sum_{S \in \mathcal{M}_{n,i}, T \in \mathcal{M}_{n,j}} Y[S,T] = (n-1)!! \binom{n/2}{i} \binom{n/2}{j},$$

for all $i, j \leq k$. Notice that the α entries only appear in the rows and columns of Y indexed by matchings of size k. Thus, if $\alpha < p_{2k}$, then

$$\left(\sum_{S,T \in \mathcal{M}_{n,k-1}} Y[S,T]\right)\left(\sum_{S,T \in \mathcal{M}_{n,k}} Y[S,T]\right) < \left(\sum_{S \in \mathcal{M}_{n,k-1}, T \in \mathcal{M}_{n,k}} Y[S,T]\right)^2,$$

and $Y \not\succeq 0$.

On the other hand, suppose $\alpha > p_{2k}$. Let Y' be the symmetric minor of Y indexed by $\mathcal{M}_{n,k}$. Then we know from Proposition 72 that

$$|\mathcal{M}_{n-2k-1,k-1}| \left(p_{2k-1} - (n-4k+1)\alpha\right).$$

is an eigenvalue of Y'. Since $\alpha > p_{2k} = \frac{1}{n-4k+1} m_{2k-1}$, this eigenvalue is negative. Thus, we conclude that $\alpha = p_{2k}$, and we are finished. $\square$

We are now ready to prove Theorem 71.

Proof of Theorem 71. Suppose $x \in \mathrm{SA}_+'^{n+1}(MT(G))$ attains $\gamma_{\bar{e}}(\mathrm{SA}_+'^{n+1}(MT(G)))$, and let Y be the corresponding certificate matrix in $\tilde{\mathrm{SA}}_+'^{n+1}(MT(G))$. Notice that for every pair of matchings S, T of the same size, there exists a permutation matrix Q such that $Q(MT(G)) = MT(G)$, and $Q\chi^S = \chi^\top$. Thus, by Corollary 44, we may assume that there exists $m_0, m_1, \ldots, m_{2n} \in \mathbb{R}$ such that

$$Y[S|_1, S'|_1] = \begin{cases} m_i & \text{if } S \cup S' \text{ is matching of size } i; \\ 0 & \text{otherwise.} \end{cases}$$

We further assume $Y[S|_1, S'|_1] = 0$ whenever $S \cup S'$ is not a matching due to (SA_+' 4). We also know from (SA_+ 1) that $m_0 = 1$.

To prove our result, it suffices to show that $m_1 < \frac{1}{4n}$. Suppose for a contradiction that m_1 is in fact $\frac{1}{4n}$ (it could not be greater, as $\ell\bar{e} \notin MT(G)$, $\forall \ell > \frac{1}{4n}$). We show by induction that this implies $m_i = \frac{(4n-2i)!!}{(4n)!!}$ for all $i \in [2n]$. Suppose $i \leq n+1$ and $m_j = \frac{(4n-2i)!!}{(4n)!!}$ for all $j \leq i$. Pick any matching $S \in \mathcal{M}_{n,i}$ and consider the column $Ye_{S|_1}$. Take a vertex v that is not saturated by S. Applying the degree constraint of v on the column $Ye_{S|_1}$ yields that

$$\sum_{e \in \delta(v)} Y[S|_1, e|_1] \leq Y[S|_1, \mathcal{F}].$$

Notice that $Y[S|_1, e|_1]$ must be 0 if the other endpoint of e is saturated by S, and m_{i+1} otherwise. Since $Y[S|_1, \mathcal{F}] = m_i$, we obtain that $(4n-2i)m_{i+1} \leq m_i$.

Next, pick an edge in S and call it f. The same degree constraint on the column $Ye_{(S \setminus f)|_1 \cap f|_0}$ is

$$(4n+2-2i)m_i - (4n-2i)m_{i+1} \leq m_{i-1} - m_i.$$

Therefore, using the fact that $m_{i-1} = (4n+2-2i)m_i$, the above is equivalent to $(4n-2i)m_{i+1} \geq m_i$. Thus, we obtain that $m_{i+1} = \frac{(4n-2i)!!}{(4n)!!}, \forall i \leq n+1$.

Next, we show that $m_i = \frac{(4n-2i)!!}{(4n)!!}$ for $i \geq n+2$. If i is even, then we can apply Corollary 73 and deduce that $m_i = \frac{(4n-2i)!!}{(4n)!!}$. If i is odd, then fix an edge $e \in E(G)$, and let Y' be the symmetric minor of Y such that Y' only contains the rows and columns of whose indices (which are matchings) contain e. Then

$$Y' = \frac{1}{(4n-2)!!}\begin{pmatrix} Y_{4n-1,0,0} & Y_{4n-1,0,1} & \cdots & Y_{4n-1,0,(i-1)/2} & \cdots \\ Y_{4n-1,1,0} & Y_{4n-1,1,1} & \cdots & Y_{4n-1,1,(i-1)/2} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \cdots \\ Y_{4n-1,(i-1)/2,0} & Y_{4n-1,(i-1)/2,1} & \cdots & Y_{4n-1,(i-1)/2,(i-1)/2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Also notice that the $Y_{4n-1,i,j}$ portion of Y' comes from the submatrix $Y_{4n+1,i+1,j+1}$ in Y. Since $i-1$ is even, we may apply the argument above, and infer from Corollary 73 that $Y'[S,T] = \frac{(4n-2-2(i-1))!!}{(4n-2)!!}$ whenever $S \cup T \in \mathcal{M}_{n-2,i-1}$. This implies that $m_i = \frac{(4n-2i)!!}{(4n)!!}$.

Iteratively, we can show that $m_i = \frac{(4n-2i)!!}{(4n)!!}$ for all $i \leq 2n$. Since all the remaining entries are 0, we have reduced the task of showing $\bar{e}^\top x < \frac{4n+1}{2}$ for all $x \in \mathrm{SA}_+'^{n+1}(MT(G))$ to showing that this particular matrix Y is not positive semidefinite.

To complete our proof, notice that $Y'' := \frac{1}{(4n)!!}\begin{pmatrix} 1 & Y_{4n+1,0,n+1} \\ Y_{4n+1,n+1,0} & Y_{4n+1,n+1,n+1} \end{pmatrix}$ is a symmetric minor of Y. From the proof of Lemma 29, we see that the entries in $Y_{4n+1,0,n+1}$ and $Y_{4n+1,n+1,0}$ both sum up to $\binom{(4n+1)/2}{n+1}$, while the entries in $Y_{4n+1,n+1,n+1}$ sum up to less than $\binom{(4n+1)/2}{n+1}^2$. Thus, $Y'' \not\succeq 0$, and consequently $Y \not\succeq 0$. $\square$

Also, since the Las'^k operator dominates $\mathrm{SA}_+'^k$, Theorem 71 immediately implies the following:

Corollary 74. *For all integers $n \geq 1$,*

$$\gamma_{\bar{e}}(\mathrm{Las}'^{n+1}(MT(G))) < \frac{4n+1}{4n}.$$

7.4 Integrality gaps of polyhedral versus semidefinite lift-and-project relaxations

We conclude this chapter by noting the following interesting phenomenon on the integrality gaps of polyhedral versus semidefinite relaxations. In many well-studied examples, when a polyhedral lift-and-project operator is applied iteratively, the integrality gap of the relaxations decrease to 1 gradually. On the other hand, the integrality gaps obtained from applying a semidefinite operator to the same initial relaxation experience sudden jumps as the number of iterations crosses some threshold. For instance, if we let $P = FRAC(K_n)$, then we saw in Propositions 68 and 69 that all hierarchies of polyhedral lift-and-project relaxations we have looked at, the integrality gap starts at $\frac{n}{2}$, then gradually decreases, and reaches 1 after $\Omega(n)$ iterations. On the other hand, it takes semidefinite operators such as $\mathrm{LS}_+, \mathrm{SA}_+$ and Las exactly one iteration to reach the integer hull of P, and thus the corresponding integrality gaps for these operators would dive from $\frac{n}{2}$ to 1 in just one iteration.

Another example is $P = MT(K_{2n+1})$. We saw in Proposition 70 that the

$$\gamma_{\bar{e}}(\mathrm{LS}_+^k(FRAC(K_{2n+1}))) = 1 + \frac{1}{2n}$$

for every $k \in \{0, 1, \ldots, n-1\}$, then suddenly drops to 1 when $k = n$. In contrast, Mathieu and Sinclair [MS09] showed the following:

Theorem 75 (Theorem 1.2 in [MS09])**.** *For $P = MT(K_{2n+1})$,*

- *For every $k \in \{0, 1, \ldots, n-1\}$, $\gamma_{\bar{e}}(\mathrm{SA}^k(P)) = 1 + \frac{1}{2n}$.*
- *If $n \leq k \leq 2n - \omega(\sqrt{n})$, then*

$$1 + \frac{1}{2n} - o\left(\frac{1}{n}\right) \leq \gamma_{\bar{e}}(\mathrm{SA}^k(P)) \leq 1 + \frac{1}{2n}.$$

- *If $2n - o(\sqrt{n}) \leq k \leq 2n - 2$, then*

$$1 < \gamma_{\bar{e}}(\mathrm{SA}^k(P)) \leq 1 + o\left(\frac{1}{n}\right).$$

- *If $k \geq 2n - 1$, then $\gamma_{\bar{e}}(\mathrm{SA}^k(P)) = 1$.*

Thus, while the integrality gap of the SA-relaxations coincide with that of the LS_+-relaxations for the first $n-1$ iterations, it then drops from $1 + \frac{1}{2n}$ to 1 over $\Omega(n)$ iterations.

Another example of similar behaviours (although not as extreme as that described above) was given by Goemans and Tunçel in [GT01]:

Theorem 76 (Theorem 4.4 in [GT01])**.** *For the set*

$$P := \left\{ x \in [0,1]^{2n} : \sum_{i \in S} x_i \leq n, \ \forall S, |S| = n+1 \right\},$$

1. *the* LS*-rank of P is $2n - 2$;*

2. *the* LS_+*-rank of* P *is* n*;*
3. *for every* $k \leq n - \sqrt{2n} + \frac{3}{2}$*,*

$$\max\left\{\bar{e}^\top y : y \in \mathrm{LS}^k(P)\right\} = \max\left\{\bar{e}^\top y : y \in \mathrm{LS}_+^k(P)\right\}.$$

Thus, we see that the integrality gaps of $\mathrm{LS}^k(P)$ and $\mathrm{LS}_+^k(P)$ with respect to $\bar{e}$ would agree for $\Omega(n)$ rounds, and then $\gamma_{\bar{e}}(\mathrm{LS}_+^k(P))$ would suddenly decrease to 1 rather rapidly, while $\gamma_{\bar{e}}(\mathrm{LS}^k(P))$ takes another $O(n)$ rounds to tail off to 1. Somewhat different from the examples we have seen above, $\gamma_{\bar{e}}(\mathrm{LS}_+^k(P))$ does decrease very slowly before suddenly dropping off (see Figure 3 in [GT01]).

It would be interesting to find out whether the above examples are merely coincidences, or this trend extends to lift-and-project relaxations for other optimization problems. A better understanding on this front may offer clues in how semidefinite and polyhedral operators differ in generating cuts.

It should be noted that the number of inequalities imposed by most lift-and-project methods (including LS and LS_+) are superpolynomial in n after $\Omega(\log(n))$ rounds. Thus it is possible that there is a threshold in the number of iterations which, after crossing, the increase in the number of new inequalities generated is so overwhelming that we would quickly converge to the integer hull.

Chapter 8

Conclusions and Future Research Directions

In this thesis, we looked at various lift-and-project operators and their relaxations for a number of combinatorial optimization problems. Through our work, we have now a much better understanding of the strongest existing operators (such as BZ' and BZ'_+), and have exposed some of their limitations by studying the characteristics of these operators (such as identifying the variables in their lifted spaces that are unhelpful). We analyzed the role of positive semidefiniteness plays in lift-and-project relaxations, and showed instances where these constraints are guaranteed to generate strong cuts (Chapter 4), and where they do not contribute (Chapter 6). We added to existing evidence that polyhedral lift-and-project methods can perform poorly on the stable set relaxations, and both polyhedral and nonpolyhedral lift-and-project methods can perform poorly on the matching relaxations. Most importantly, we presented and discussed many different tools for analyzing these lift-and-project relaxations. Through the new operators we defined, dominance and restricted reverse dominance relations we established, and frameworks we presented that simplify the construction and verification of certificate matrices, we have brought the forefront of the research on lift-and-project methods to the strongest operators, and have made future analyses of existing and new operators simpler, more systematic, and more transparent.

In this final chapter, we first use the techniques we developed in previous chapters to relate the $\mathrm{SA}'_+, \mathrm{BZ}'_+$ and Las'-rank of the matching relaxations of odd cliques. We then revisit the connections between matchings and integer partitions. We provide evidence that they seem to run much deeper than were shown in Chapter 5, and point out their implications on the relaxations of the matching polytope. Finally, we finish with some concluding remarks and discuss some possible future research directions.

8.1 Relating the SA'_+-, BZ'_+-and Las'-relaxations for matching

As we saw in Chapter 3, many lift-and-project operators perform poorly on $MT(G)$ when G is an odd clique. To summarize, $MT(K_{2n+1})$ is known to have LS_+-rank n [ST99], BCC-rank n^2 [ABN04], and SA-rank $2n-1$ [MS09]. We showed in this thesis that its BZ'-rank is at least $\left\lceil \sqrt{2n} - \frac{3}{2} \right\rceil$ (Theorem 15).

Next, we show that using the results we established earlier, we can relate the performances of the strongest lift-and-project operators with positive semidefiniteness, such as $\mathrm{SA}'_+, \mathrm{BZ}'_+$ and Las', on the matching relaxations. First, we have the following:

Theorem 77. *Let the SA'_+-rank and the BZ'_+-rank of $MT(K_{2n+1})$ be k and ℓ, respectively. Then*

$$\ell \geq \sqrt{2k-1} - 1.$$

Proof. We have shown in the proof of Theorem 15 that, for every $\ell \geq 1$, all tiers generated by BZ'^{ℓ} of size greater than $\frac{\ell(\ell+1)}{2} + \ell = \frac{(\ell+1)(\ell+2)}{2}$ are P-useless. Since BZ'^{ℓ}_+ generates exactly the same tiers, the observation applies here as well. Therefore, we may apply Theorem 14 and deduce that

$$\mathrm{BZ}'^{\ell}_+(P) \supseteq \mathrm{SA}'^{(\ell+1)(\ell+2)/2}_+(P),$$

for every $\ell \geq 1$. Observe that

$$\frac{(\ell+1)(\ell+2)}{2} \leq k-1 \iff \ell \leq \sqrt{2k - \frac{7}{4}} - \frac{3}{2}.$$

Thus, if $MT(G)$ has SA'_+-rank k, then $\mathrm{SA}'^{k-1}_+(P) \neq P_I$, and $\mathrm{BZ}'^{\ell}_+(P) \neq P_I$ for all $\ell \leq \sqrt{2k - \frac{7}{4}} - \frac{3}{2}$. Thus, we obtain that the BZ'_+-rank of $MT(G)$ is at least $\sqrt{2k - \frac{7}{4}} - \frac{1}{2} \leq \sqrt{2k-1} - 1$, as desired. $\square$

Of course, if we know that the SA'_+-rank of $MT(K_{2n+1})$ is k, then the BZ'_+-rank must be no more than k, since BZ'_+ dominates SA'_+. Thus, determining the SA'_+-rank in this case would give us both an upper bound and a lower bound of the BZ'_+-rank of the set.

Next, we show a similar result that relates the Las' and SA'_+ relaxations of $MT(K_{2n+1})$.

Proposition 78. *Suppose $G = K_{2n+1}$. Then*

$$\frac{1}{2n}\bar{e} \in \mathrm{SA}'^{k}_+(MT(G)) \Rightarrow \frac{1}{2n}\bar{e} \in \mathrm{Las}'^{k-1}(MT(G)),$$

for every integer $k \geq 2$.

Proof. Let $P := MT(G)$, and suppose $\frac{1}{2n}\bar{e} \in \mathrm{SA}_{+}'^{k}(P)$. By Theorem 71, we know that this implies $k \leq \frac{n}{2}$. Also, it follows from Corollary 73 that $\frac{1}{2n}\bar{e}$ has a unique matrix $Y \in \tilde{\mathrm{SA}}_{+}'^{k}(P)$. We show that $Y \in \tilde{\mathrm{Las}}_{+}'^{k-1}(P)$.

First, the fact that $Y \in \tilde{\mathrm{SA}}_{+}'^{k}(P)$ implies most of the conditions for Las'. It only remains to check $Y^j \succeq 0$ and $Y^j \geq 0$ for each inequality $j \in V(G)$ (since each inequality describing $MT(G)$ corresponds to a vertex in G). By the symmetry of G and $MT(G)$, the Y^j's are identical, and we only have to verify the conditions on Y^j for one particular j.

It turns out that Y^j is the matrix of all zeros. Let $\delta(j)$ denote the set of $2n$ edges that are incident with vertex j. Suppose $S, S' \subseteq E(G)$ and $|S|, |S'| \leq k$. We know that

$$Y^j[S|_1, S'|_1] = Y[S|_1, S'|_1] - \sum_{e \in \delta(j)} Y[(S \cup \{e\})|_1, (S' \cup \{e\})|_1].$$

If $Y[S|_1, S'|_1] = 0$, that means that $S \cup S'$ is not a matching of G, and thus $Y[(S \cup \{e\})|_1, (S' \cup \{e\})_1] = 0,\ \forall e \in \delta(j)$. Therefore, $Y^j[S|_1, S'|_1] = 0$ in this case.

Now suppose $S \cup S'$ is a matching of G. There are two cases: either $S \cup S'$ contains an edge incident with j, or it does not. First assume it does, and let e' be this edge. Notice that for any $e \in \delta(j), S \cup S' \cup \{e\}$ is a matching only when $e = e'$. Therefore,

$$Y^j[S|_1, S'|_1] = Y[S|_1, S'|_1] - Y[(S \cup \{e'\})|_1, (S' \cup \{e'\})|_1] = 0,$$

where the last equality follows from ($\mathrm{Las}'\,5$) (since $e' \in S \cup S'$).

For the other case, let $\ell := |S \cup S'|$. Then for any $e \in \delta(j)$, $S \cup S' \cup \{e\}$ is a matching if and only if e joins j to one of the $2n - 2\ell$ vertices not saturated by $S \cup S'$. Since $\ell \leq |S| + |S'| \leq 2(k-1) < n$, such an edge exists and let e' be one of them. Then from the construction of Y, we know that $Y[S|_1, S'|_1] = \frac{(2n-2\ell)!!}{(2n)!!}$ and $Y[(S \cup \{e'\})|_1, (S' \cup \{e'\})|_1] = \frac{(2n-2\ell-2)!!}{(2n)!!}$. Hence,

$$Y^j[S|_1, S'|_1] = \frac{(2n-2\ell)!!}{(2n)!!} - (2n-2\ell)\left(\frac{(2n-2\ell-2)!!}{(2n)!!}\right) = 0.$$

Thus, the symmetric minor of Y^j indexed by sets in $\mathcal{A}_{k-1}^{+}$ is the matrix of all zeros. Since Y^j satisfies (OMC) by construction and $\mathcal{A}_{k-1}^{+}$ generates $\mathcal{A}_{k-1}$, we obtain that the entire Y^j matrix is zero, and therefore, Y^j is trivially positive semidefinite and nonnegative. Hence, the claimed lower bound is established. $\square$

Since $\frac{1}{2n}\bar{e} \notin MT(K_{2n+1})_I$, Proposition 78 immediately implies the following:

Theorem 79. *Define*

$$\ell := \max\left\{k : \frac{1}{2n}\bar{e} \in \mathrm{SA}_{+}'^{k}(MT(K_{2n+1}))\right\}.$$

Then the Las*-rank of* $MT(K_{2n+1})$ *is at least* $\ell - 1$.

Theorems 77 and 79, together with what we showed in the proof of Theorem 15 and several other reverse dominance results that hold in general, gives us many handles on how the rank of $MT(K_{2n+1})$ for different operators relate to each other. Using these relations, establishing the rank of $MT(K_{2n+1})$ for an operator in Figure 8.1 leads to implications on the performances of all other operators in the chart.

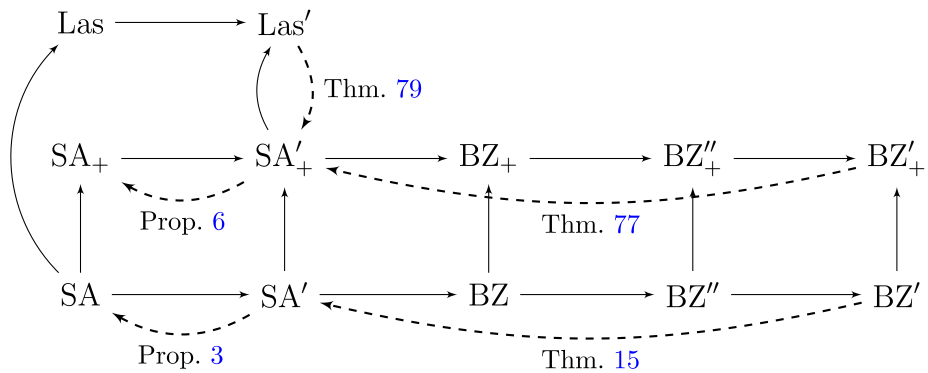

Figure 8.1: Relating the rank of $MT(K_{2n+1})$ for various operators.

8.2 More connections between matchings and integer partitions

Recall that in Chapter 5, we defined a family of vectors using integer partitions, and proved (Theorem 36) that those correspond to partitions with no more than one part of size greater than one are eigenvectors of $Y_{n,k,k}(m)$.

We believe that vectors corresponding to other partitions are also eigenvectors of $Y_{n,k,k}(m)$. In fact, we also believe that the multiplicities of these conjectured eigenvectors are related to the number of standard Young tableaux of certain shapes. Given any partition λ of an integer k, a *standard Young tableaux* of shape λ is an assignment of integers 1 to k to the k boxes of the Young diagram of shape λ, with each integer appearing exactly once, such that each row is increasing from left to right, and each column is increasing from top to bottom. For example, Figure 8.2 lists the nine standard Young tableaux of shape $\lambda = (4, 2)$:

We let $t(\lambda)$ denote the number of standard Young tableaux of shape λ. One way to compute $t(\lambda)$ is the following. For each box (i, j) in the Young diagram of λ, we let $\text{hook}(i, j)$ denote the *hook length* of (i, j), which is defined to be the number of boxes to the right and directly below (i, j), including the box (i, j). For instance, if we consider the Young diagram for $\lambda = (4, 4, 3, 1)$, and fill in each box with its hook length, then we obtain the Young diagram in Figure 8.3.

$$\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4&6\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\\hline 4&5\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&6\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&2&4&6\\\hline 3&5\\\cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&2&5&6\\\hline 3&4\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2&6\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&3&4&6\\\hline 2&5\\\cline{1-2}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4\\\cline{1-2}\end{array}$$

Figure 8.2: The nine standard Young tableaux of shape $(4, 2)$.

$$\begin{array}{|c|c|c|c|}\hline 7&5&4&2\\\hline 6&4&3&1\\\hline 4&2&1\\\cline{1-3} 1\\\cline{1-1}\end{array}$$

Figure 8.3: A Young diagram where each box is labelled by its hook length.

It is well known that, for a partition λ of k,

$$t(\lambda) = \frac{k!}{\prod_{(i,j):j \leq \lambda_i} \mathrm{hook}(i,j)}.$$

Thus, from the above, we see that the number of standard Young tableaux of shape $(4, 4, 3, 1)$ is

$$t(4, 4, 3, 1) = \frac{12!}{7 \cdot 5 \cdot 4 \cdot 2 \cdot 6 \cdot 4 \cdot 3 \cdot 1 \cdot 4 \cdot 2 \cdot 1 \cdot 1} = 2970.$$

Also, let $|\lambda|$ denote the size of a partition λ. Then our conjecture about these eigenvectors can be summarized as follows:

Conjecture 80. *Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be a partition of size k, and suppose $n \geq 2k + 2\lambda_1$. Then for every set of $2k + 2\lambda_1$ vertices S as labelled as in (5.7), z^S is an eigenvector of $Y_{n,k,k}(m)$ with multiplicity*

$$t\left(n - 2|\lambda|, 2\lambda_1, 2\lambda_2, \ldots, 2\lambda_\ell\right),$$

for every vector $m \in \mathbb{R}^{k+1}$.

The eigenvectors described in Conjecture 80 can be integral in establishing $Y_{n,k,k} \succeq 0$, as we believe that they are exactly the eigenvectors with nonzero eigenvalues of the matrix

$$Y_{n,k,k} - \tilde{L}_{n,i-1}^{\top} Y_{n,k-1,k-1} \tilde{L}_{n,i-1},$$

where $\tilde{L}_{n,i-1}$ is any matrix that satisfies $Y_{n,k-1,k-1}\tilde{L}_{n,i-1} = Y_{n,k-1,k}$. However, computations indicate that while L matrices are simple, $\tilde{L}$ matrices are hard to capture in general in this case. An observation that might be helpful is the following. Recall that the zeta matrix on the set $\{1, \ldots, n\}$ is the matrix $Z \in \mathbb{R}^{\mathcal{A}_n^+ \times \mathcal{A}_n^+}$ where

$$Z[S|_1, T|_1] = \begin{cases} 1 & \text{if } S \subseteq T; \\ 0 & \text{otherwise.} \end{cases}$$

It is well known that Z is invertible, and

$$Z^{-1}[S|_1, T|_1] = \begin{cases} -1^{|T \setminus S|} & \text{if } S \subseteq T; \\ 0 & \text{otherwise.} \end{cases}$$

Z^{-1} is also known as the *Möbuis matrix*. Since the $L_{n,i,j}$ matrices (defined immediately before Lemma 30) are submatrices of the zeta matrix, it is likely that the $\tilde{L}$ matrices are in some ways related the Möbuis matrix.

Next, we narrow our focus onto the matrices $Y_{n,k,k}$. First, we believe the following is true:

Conjecture 81. *For every $n \geq 1$, $Y_{4n+1,n,n} \succeq 0$.*

We saw in Section 5.4.2 that $Y_{4n+1,n,n} \succeq 0$ for all $n \leq 4$. In addition to being positive semidefinite, we believe we can say a lot more about these matrices. We let (n, λ)-vectors denote the vectors described in Conjecture 80 that are based on a graph with n vertices and the partition λ (so entries of (n, λ)-vectors are indexed by elements of $\mathcal{M}_{n,|\lambda|}$). Also, for every integer $k \geq 1$, we let $\mathcal{P}(k)$ denote the set of all partitions of size up to and including k. Then we have the following:

Conjecture 82. *Let n, k be integers where $4n \geq k$. Then for every $\lambda = (\lambda_1, \ldots, \lambda_\ell) \in \mathcal{P}(k)$, $Y_{n,k,k}$ has an eigenspace of dimension*

$$t(n - 2|\lambda|, 2\lambda_1, \ldots, 2\lambda_\ell),$$

which are spanned by the vectors

$$\left\{ Y_{n,k,|\lambda|} x : x \text{ is an } (n, \lambda)\text{-vector} \right\}.$$

Furthermore, the above sets account for all eigenvectors of $Y_{n,k,k}$ with nonzero eigenvalue.

Due to computational limitations, we could only completely verify Conjecture 82 up to $n = 13$ and $k = 3$. The following table shows all eigenvalues of $Y_{13,3,3}$ and their multiplicities.

For larger values of n, k, Theorem 36 does provide some eigenvectors for $Y_{n,k,k}$. Also, (n, λ)-vectors are verified (by hand) to be eigenvectors of $Y_{n,k,k}(m)$ for all λ where $|\lambda| \leq 6$.

If Conjecture 81 is true, then we can use Theorem 28 to show that $\frac{1}{4n}\bar{e}$ is contained in $\mathrm{SA}_+'^n(MT(K_{4n+1}))$, and it would follow that $MT(K_{4n+1})$ has SA_+'-rank at least n. This would imply that the BZ_+'-rank of $MT(K_{4n+1})$ is at least roughly $\sqrt{2n}$ (Theorem 77), and the Las'-rank is at least $n - 1$ (Theorem 79).

While our initial motivation of studying the $Y_{n,k,k}$ matrices was to establish their positive semidefiniteness and obtain a lower-bound result for the SA_+-rank, we are excited to have uncovered a wealth of combinatorial connections between matchings and integer partitions. It would be fascinating if future work allowed us to understand these connections in a more unifying way.

Eigenvalue	Multiplicity	Corresponding λ in $\mathcal{P}_3$
1287	$1 = \mathrm{t}(13)$	$\emptyset$
657	$65 = \mathrm{t}(11{,}2)$	$\{1\}$
287	$429 = \mathrm{t}(9{,}4)$	$\{2\}$
272	$936 = \mathrm{t}(9{,}2{,}2)$	$\{1,1\}$
105	$429 = \mathrm{t}(7{,}6)$	$\{3\}$
80	$6006 = \mathrm{t}(7{,}4{,}2)$	$\{2,1\}$
72	$4004 = \mathrm{t}(7{,}2{,}2{,}2)$	$\{1,1,1\}$
0	13870	
Total	25740	

Table 8.1: Eigenvalues and multiplicities of $Y_{13,3,3}$.

8.3 Final remarks

In this final section, we briefly remind the reader some of the main ideas and results we have seen throughout this thesis, and discuss several future research directions.

- We proposed many new, strong lift-and-project operators such as SA'_+ and BZ'_+, and showed how they relate to the existing ones, both through dominance and (under certain assumptions) reverse dominance relationships (Figure 1.3).
- We developed the notions of admissible operators, measure consistency of matrices, and P-useless variables. We used them to relate the strengths of different operators (Theorems 13 and 14), which allowed us to give the first known bad instances for the operator BZ (Theorems 15 and 18).
- We provided overall performance guarantee of a lift-and-project operator in the presence of ℓ-establishing variables and positive semidefiniteness constraints, and used this framework to prove some upper-bound results (e.g. Theorems 21 and 22).
- We presented ideas (such as using linear dependencies, connecting with combinatorial objects, and utilizing symmetries and commutative maps) to simplify the construction and verification of certificate matrices in lifted spaces.
- We characterized some sets where SA_+, a strong operator that imposes positive semidefiniteness constraints in its lifted spaces, does not perform better than polyhedral operators such as LS_0 (Theorem 53). We also looked at several examples whose SA_+- and Las-ranks are as high as the dimension of the relaxations.
- We showed that, often times, lower-bound results for lift-and-project relaxations readily lead to integrality gap results on these relaxations, and streamlined such derivations.

On the other hand, there are still a lot in the area to be uncovered, as there are still many optimization problems whose lift-and-project relaxations (especially those of the stronger operators such as Las and BZ_+) are not well-understood. For instance, it would be interesting to see if we can apply the new tools and techniques we have developed to problems such as the travelling salesman problem and vertex cover. Non-approximability results on these problems based on, say, the BZ_+ operator would be a significant improvement over our current knowledge of their hardness with respect to lift-and-project methods.

Lift-and-project relaxations have also been shown recently to be closely related with the extension complexity of sets. As mentioned in Chapter 1, Chan et. al. [CLRS13] proved that inapproximability results for the Sherali–Adams lift-and-project relaxations of approximate constraint satisfaction problems can be extended to lower-bound results on the extension complexity of the max-cut and max 3-sat polytopes. Thus, it is natural to ask if other lift-and-project relaxations also admit such potentials, and more importantly if hardness results on semidefinite representations can be obtained by this approach.

While one of the advantages of lift-and-project operators is that they are systematic and can be applied to any relaxation without additional problem-specific observations, there might be value in making these operators adapt to their initial relaxations in some way. Bienstock and Zuckerberg [BZ04] devised the first operators that generate different variables for different relaxations (or even different algebraic descriptions of the same relaxation — see Proposition 87 for an example). They showed that this flexibility can be very useful in attacking relaxations of some set covering problems, and perhaps tight relaxations for other hard problems can be found similarly by building a lift-and-project operator with suitable adaptations.

Looking further ahead, while we currently do have rather strong operators such as SA_+, Las, and BZ_+, we are interested in pushing the limits of lift-and-project methods, and build the strongest possible tightening operator that can arise in this approach. A sufficiently strong operator might provide a breakthrough in combinatorial optimization and approximation algorithms. For example, it is conceivable that such an operator may yield a $2 - \Theta(1)$ approximation algorithm for vertex cover. On the other hand, if this cannot be achieved by the strongest possible operator, then we obtain a hardness result that banishes all hopes of finding a $(2 - \Theta(1))$-approximation algorithm for this problem using the lift-and-project approach.

APPENDICES

Appendix A

The Original $\mathrm{BZ}, \mathrm{BZ}_+$ Operators

We now state the original BZ operator in our unifying language, and show that it is refined by BZ$'$.

A.1 Details of the original $\mathrm{BZ}, \mathrm{BZ}_+$ operators

The refinement step of BZ^k coincides with BZ'^k — both operators derive k-small obstructions from the linear inequalities describing P, and use them to construct $\mathcal{O}_k(P)$. Then BZ^k defines its set of walls to be

$$\mathcal{W}_k := \left\{ \bigcup_{i,j \in [\ell], i \neq j} (O_i \cap O_j) : O_1, \dots O_\ell \in \mathcal{O}_k, \ell \leq k+1 \right\}.$$

Note that unlike for BZ'^k, BZ^k does not guarantee that the singleton sets are walls, and we will see that this could make a difference in performance. As for the tiers, BZ^k defines them to be the sets of indices that can be written as the union of up to k walls in $\mathcal{W}_k$. Thus, BZ^k only generates a polynomial size subset of the tiers used in BZ'^k. Then the lifting step of BZ^k (and BZ_+^k) can be described as follows:

1. Define $\mathcal{A}'$ to be the set consisting of the following:
   - $\mathcal{F}$ and $i|_1, i|_0, \; \forall i \in [n]$.
   - Suppose $S := \bigcup_{i=1}^{\ell} W_i$ is a tier. Then we do the following:
     - For each ℓ-tuple of sets, $(T_1, \dots, T_\ell)$ such that $T_i \subseteq W_i, \; \forall i \in [\ell]$ and $\sum_{i=1}^{\ell} |T_i| \leq k$, include the set

$$\left(\bigcup_{i=1}^{\ell} W_i \setminus T_i \right) \Bigg|_1 \cap \left(\bigcup_{i=1}^{\ell} T_i \right) \Bigg|_0 . \tag{A.1}$$

If $\sum_{i=1}^{\ell} |T_i| = k$ and $T_\ell \subset W_\ell$, then include the set

$$\left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i\right)\Bigg|_1 \cap \left(\bigcup_{i=1}^{\ell-1} T_i\right)\Bigg|_0 \cap W_\ell|_{<|W_\ell|-|T_\ell|}. \tag{A.2}$$

2. Let $\tilde{\mathrm{BZ}}^k(P)$ denote the set of matrices $Y \in \mathbb{S}^{\mathcal{A}'}$ that satisfy all of the following conditions:

(BZ 1) $Y[\mathcal{F}, \mathcal{F}] = 1$.

(BZ 2) For any column x of the matrix Y,

(i) $0 \leq x_\alpha \leq x_{\mathcal{F}}$, for all $\alpha \in \mathcal{A}'$.

(ii) $\hat{x}(x) \in K(\mathcal{O}_k(P))$.

(iii) $x_{i|_1} + x_{i|_0} = x_{\mathcal{F}}$, for every $i \in [n]$.

(iv) For each $\alpha \in \mathcal{A}'$ in the form of $S|_1 \cap T|_0$, impose the inequalities

$$\begin{aligned} x_{i|_1} &\geq x_\alpha, \quad \forall i \in S; && \text{(A.3)}\\ x_{i|_0} &\geq x_\alpha, \quad \forall i \in T; && \text{(A.4)}\\ \sum_{i\in S} x_{i|_1} + \sum_{i \in T} x_{i|_0} - x_\alpha &\leq (|S| + |T| - 1)x_{\mathcal{F}}. && \text{(A.5)} \end{aligned}$$

(v) For each $\alpha \in \mathcal{A}'$ in the form $S|_1 \cap T|_0 \cap U|_{<r}$, impose the inequalities

$$\begin{aligned} x_{i|_1} &\geq x_\alpha, \quad \forall i \in S; && \text{(A.6)}\\ x_{i|_0} &\geq x_\alpha, \quad \forall i \in T; && \text{(A.7)}\\ \sum_{i\in U} x_{i|_0} &\geq (|U| - (r-1))x_\alpha. && \text{(A.8)} \end{aligned}$$

(vi) For each variable in the form (A.1), if $|W_\ell| + \sum_{i=1}^{\ell-1} |T_i| \leq k$, impose

$$\begin{aligned} &\sum_{U \subseteq W_\ell} x_{\left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i\right)|_1 \cap \left(\bigcup_{i=1}^{\ell-1} T_i\right)|_0 \cap (W_\ell \setminus U)|_1 \cap U|_0} \\ = \; & x_{\left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i\right)|_1 \cap \left(\bigcup_{i=1}^{\ell-1} T_i\right)|_0}. \end{aligned} \tag{A.9}$$

Otherwise, define $r := k - (\sum_{i=1}^{\ell-1} |T_i|)$, and impose

$$\begin{aligned} &\sum_{U \subseteq W_\ell, |U| \leq r} x_{\left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i\right)|_1 \cap \left(\bigcup_{i=1}^{\ell-1} T_i\right)|_0 \cap (W_\ell \setminus U)|_1 \cap U|_0} \\ + \; & x_{\left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i\right)|_1 \cap \left(\bigcup_{i=1}^{\ell-1} T_i\right)|_0 \cap W_\ell|_{<|W_\ell|-r}} \\ = \; & x_{\left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i\right)|_1 \cap \left(\bigcup_{i=1}^{\ell-1} T_i\right)|_0}. \end{aligned} \tag{A.10}$$

(BZ 3) For all $\alpha, \beta \in \mathcal{A}'$ such that $\alpha \cap \beta = \emptyset$, or $\alpha \cap \beta$ is contained in $O|_1$ for some k-small obstruction $O \in \mathcal{O}_k$, $Y[\alpha, \beta] = 0$.

(BZ 4) For all $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathcal{A}'$ such that $\alpha_1 \cap \beta_1 = \alpha_2 \cap \beta_2$, $Y[\alpha_1, \beta_1] = Y[\alpha_2, \beta_2]$.

3. Define
$$\mathrm{BZ}^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{BZ}}^k(P), \hat{x}(Ye_{\mathcal{F}}) = \hat{x} \right\},$$
and
$$\mathrm{BZ}_+^k(P) := \left\{ x \in \mathbb{R}^n : \exists Y \in \tilde{\mathrm{BZ}}_+^k(P), \hat{x}(Ye_{\mathcal{F}}) = \hat{x} \right\},$$
where $\tilde{\mathrm{BZ}}_+^k(P) := \tilde{\mathrm{BZ}}^k(P) \cap \mathbb{S}_+^{\mathcal{A}'}$.

In [BZ04], BZ was defined so that the first relaxation in the hierarchy is $\mathrm{BZ}^2(P)$, with $\mathrm{BZ}^{n+1}(P)$ being the n^{th} relaxation that is guaranteed to be P_I. We have modified their definitions and presented their operators such that the relaxations are instead $\mathrm{BZ}^1(P), \ldots, \mathrm{BZ}^n(P)$, to align them with the other named operators mentioned in this manuscript.

Next, we show that BZ' and BZ'_+ indeed dominate their original counterparts.

Proposition 83. *For every polytope $P \subseteq [0,1]^n$ and integer $k \geq 1$, $\mathrm{BZ}'^k(P) \subseteq \mathrm{BZ}^k(P)$ and $\mathrm{BZ}_+'^k(P) \subseteq \mathrm{BZ}_+^k(P)$.*

Proof. It is apparent that every variable generated by BZ^k is also generated by BZ'^k. The only nontrivial case is when BZ^k generates a variable in the form
$$\left. \left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i \right) \right|_1 \cap \left. \left(\bigcup_{i=1}^{\ell-1} T_i \right) \right|_0 \cap W_\ell|_{<|W_\ell|-|T_\ell|} \tag{A.11}$$
such that W_ℓ is not disjoint from $\bigcup_{i=1}^{\ell-1} W_i$. In this case if we define $W' := W_\ell \setminus \bigcup_{i=1}^{\ell-1} W_i$, then the above is equivalent to $\emptyset$ if $|W'| \leq |T_\ell|$, and
$$\left. \left(\bigcup_{i=1}^{\ell-1} W_i \setminus T_i \right) \right|_1 \cap \left. \left(\bigcup_{i=1}^{\ell-1} T_i \right) \right|_0 \cap W'|_{<|W'|-|T_\ell|}$$
otherwise, which we know is generated by BZ'^k.

Also, the condition (BZ′ 3) is more easily triggered than (BZ 3), and thus BZ' forces more variables to be zero and is more restrictive. It is also not hard to see that the constraints (2.7)–(2.14) imply their corresponding counterparts (A.3)–(A.10) in BZ. Hence, we have $\tilde{\mathrm{BZ}}'^k(P) \subseteq \tilde{\mathrm{BZ}}^k(P)$, and it follows readily that $\mathrm{BZ}'^k(P) \subseteq \mathrm{BZ}^k(P)$ and $\mathrm{BZ}_+'^k(P) \subseteq \mathrm{BZ}_+^k(P)$. ☐

As Bienstock and Zuckerberg proved in [BZ04], the original BZ operator can efficiently solve many set covering type problems which require exponential effort to solve by previously known operators such as SA. However, since BZ^k does not ensure that it generates walls of small sizes, its tiers (which are unions of walls) could all be large, and the lifted set of variables $\mathcal{A}'$ does not necessarily contain $\mathcal{A}_k$ as in BZ'^k. In fact, in some cases, BZ^k performs no better than one round of LS.

Proposition 84. *Let p, q be positive integers such that $1 \leq q < p$, and let*

$$P := \left\{ x \in [0,1]^p : \sum_{i=1}^{p} x_1 \leq q + \frac{1}{2} \right\}.$$

If $(k+1)(k+2) \leq p - q$ and $k+1 \leq q$, then $\mathrm{BZ}^k(P) = \mathrm{LS}(P)$ and $\mathrm{BZ}_+^k(P) = \mathrm{LS}_+(P)$.

Proof. Since $q + \frac{1}{2} > k+1$, there are no k-small obstructions of size $k+1$ or less. Thus, $S \subseteq [n]$ is a k-small obstruction if and only if $|S| \geq p - (k+1)$, which implies that every wall (and hence, every tier) has size at least $p - (k+1)^2$. If $p - (k+1)^2 - (k+1) \geq q$, then we see that every tier is P-useless. The only remaining variables that are not useless are $\mathcal{F}, i|_1$ and $i|_0$ for all $i \in [n]$. Thus, $\mathrm{BZ}^k(P) = \mathrm{LS}(\mathcal{O}_k(P))$ and $\mathrm{BZ}_+^k(P) = \mathrm{LS}_+(\mathcal{O}_k(P))$.

Furthermore, $\mathcal{O}_k(P) = P$ whenever $k+1 \leq p-q$, which is implied by $(k+1)(k+2) \leq p - q$. Thus, our claim follows. □

Since $\mathrm{LS}(P) \subset P$ whenever $P \neq P_I$, the above implies that one can construct examples in which $\mathrm{LS}^2(P) \subset \mathrm{BZ}^k(P)$ for arbitrarily large k. On the other hand, it is easy to obtain a lift-and-project operator that has the unique strength of BZ, while also refining the earlier operators (for instance, by simply taking $\Gamma^k(P) = \mathrm{SA}^k(P) \cap \mathrm{BZ}^k(P)$).

A.2 Strong and tractable — the $\mathrm{BZ}'', \mathrm{BZ}''_+$ operators

We can take this one step further. Recall that BZ' generates exponentially many variables in its lifted space, and thus does not admit a straightforward polynomial-time implementation. However, the number of variables generated becomes polynomial in n if we instead use the original BZ's rule of generating tiers (i.e., defining S to be a tier if it is a union of up to k walls). Let BZ'' denote this new operator. Then BZ'' is just like the original BZ, except it has polynomially more variables, always ensures the singleton sets are walls, and imposes the condition ($\mathrm{BZ}'\,3$) instead of the weaker ($\mathrm{BZ}\,3$). Also, just like ($\mathrm{SA}'\,4$) and ($\mathrm{SA}'_+\,2$), the condition ($\mathrm{BZ}'\,3$) can be efficiently verified, given we have an efficient separation oracle for P, and the condition is only checked polynomially many times. Replacing ($\mathrm{BZ}\,3$) with ($\mathrm{BZ}'\,3$) boasts the advantage of eliminating the operator's dependence on the set of obstructions in the lifting step, and allows us state the operator as a two-step process. Thus. if $k = O(1)$ and we have a compact description of P, then $\mathrm{BZ}''^k(P)$ is tractable. It is also not hard to see that BZ'' dominates both SA' and BZ. Moreover, the following is true:

Proposition 85. *The BZ''-rank of P is at most $\left\lceil \frac{n+1}{2} \right\rceil$, for all $P \subseteq [0,1]^n$.*

Proof. Let $Y \in \tilde{\mathrm{BZ}}''^k(P)$ such that $k \geq \frac{n+1}{2}$. We show that $\hat{x}(Ye_{\mathcal{F}}) \in K(P_I)$. Notice that BZ''^k generates $S := [k]$ is a tier (derived from k singleton-set walls), and we know by (2.9) and the symmetry of Y that

$$Ye_{\mathcal{F}} = \sum_{T \subseteq S} Ye_{T|_1 \cap (S \setminus T)|_0}. \tag{A.12}$$

In the remainder of this proof, we let Y_T denote $Ye_{T|_1 \cap (S \setminus T)|_0}$ to reduce cluttering. Note that since $|S| = k$, BZ''^k does generate the variable $T|_1 \cap (S \setminus T)|_0$ for all $T \subseteq S$, and so Y_T is well defined.

Next, we prove that $\hat{x}(Y_T) \in K(P_I)$ for every $T \subseteq S$. Then by (A.12), it follows that $\hat{x}(Ye_{\mathcal{F}}) \in K(P_I)$. For convenience, we let $\bar{S}$ denote $[n] \setminus S$. Notice that

$$(Y_T)_{\mathcal{F}} = \sum_{S' \subseteq \bar{S}} (Y_T)_{S'|_1 \cap (\bar{S} \setminus S')|_0} \tag{A.13}$$

by (2.9). Also, since $k \geq \frac{n+1}{2}$, $|\bar{S}| = n - k \leq k - 1$. Hence, $\{j\} \cup \bar{S}$ is a tier for all $j \in [n]$, and

$$(Y_T)_{j|_1} = \sum_{S' \subseteq \bar{S}} (Y_T)_{(j \cup S')|_1 \cap (\bar{S} \setminus S')|_0}, \quad \forall j \in [n]. \tag{A.14}$$

Next, for all $T' \subseteq \bar{S}$, we define $Y_{T,T'} \in \mathbb{R}^{n+1}$ such that

$$(Y_{T,T'})_i = \begin{cases} (Y_T)_{T'|_1 \cap (\bar{S} \setminus T')|_0} & \text{if } i = 0 \text{ or } i \in T \cup T'; \\ 0 & \text{otherwise.} \end{cases}$$

From (A.13), (A.14), and the construction of $Y_{T,T'}$, we obtain that

$$\hat{x}(Y_T) = \sum_{T' \subseteq \bar{S}} Y_{T,T'}, \ \forall T \subseteq S.$$

Thus, it suffices to show that $Y_{T,T'} \in K(P_I)$, $\forall T \subseteq S, T' \subseteq \bar{S}$. This is obviously true if $(Y_{T,T'})_0 = 0$. If $(Y_{T,T'})_0 > 0$, then by (BZ′ 3) we know that $(T \cup T')|_1 \cap ([n] \setminus (T \cup T'))|_0 \cap P \neq \emptyset$. Since $Y_{T,T'} = \begin{pmatrix} (Y_{T,T'})_0 \\ (Y_{T,T'})_0 \chi^{T \cup T'} \end{pmatrix}$, it follows that $Y_{T,T'} \in K(P_I)$, completing the proof. $\square$

Likewise, we can define BZ''_+ to be the positive semidefinite counterpart of BZ'', and obtain a tractable operator that dominates both SA'_+ and BZ_+. Therefore, it follows that the BZ''_+-rank of any $P \subseteq [0,1]^n$ is also at most $\left\lceil \frac{n+1}{2} \right\rceil$. Moreover, observe that the essential ingredients used in the above proof are the presence of the variables in $\mathcal{A}_{\lceil n+1/2 \rceil}$ in the lifted space and the condition (BZ′ 3), which also applies for the SA'^k_+ relaxation for any $k \geq \frac{n+1}{2}$. Thus, the above proof can be slightly modified to show that the SA'_+-rank of any polytope contained in $[0,1]^n$ is at most $\left\lceil \frac{n+1}{2} \right\rceil$. In contrast, we have seen in Corollary 59 an example in which the SA_+-rank is n.

While we do not have an example of a set whose BZ-rank exceeds $\left\lceil \frac{n+1}{2} \right\rceil$, we do have an instance in which BZ'' outperforms BZ.

Proposition 86. *Let* $P := \left\{x \in [0,1]^7 : \sum_{i=1}^{7} 2x_i \leq 7\right\}$. *Then*

$$y := (0.76, 0.76, 0.76, 0.3, 0.3, 0.3, 0.3)^\top \in \mathrm{BZ}(P) \setminus \mathrm{BZ}''(P).$$

Proof. First, it is easy to see that $P_I = \left\{x \in [0,1]^7 : \sum_{i=1}^{7} x_i \leq 3\right\}$, and $\mathcal{O}_1(P) = P$. It can also be checked that $y \in \mathrm{BZ}(P)$. We next show that BZ'' cuts off y. First, the 1-small obstructions of P is the collection of subsets of $[7]$ of size at least 5, and it is not hard to see that $\mathcal{O}_1(P) = P$.

Since each wall is an intersection of up to two obstructions, every subset of $[7]$ of size between 3 and 5 is a wall. These sets are also exactly the tiers, as every tier is consisted of one wall in BZ''. Suppose for a contradiction that there exists a certificate matrix $Y \in \tilde{\mathrm{BZ}}''(P)$ for y. Consider the tier $S := \{1,2,3\}$. By (2.14), we know that

$$Ye_{\mathcal{F}} = Ye_{S|_1} + \sum_{i \in S} Ye_{(S\setminus\{i\})|_1 \cap i|_0} + Ye_{S|_{<2}}. \tag{A.15}$$

Since $\hat{x}(Ye_\alpha) \in K(\mathcal{O}_1(P)) = K(P)$ for all variables $\alpha \in \mathcal{A}'$, we know from (A.15) we can write $\hat{x}(Ye_{\mathcal{F}})$ as $z + w$, where $z := \hat{x}(Ye_{S|_1})$, and $w \in K(P)$.

Now, applying (2.10) of $S|_1$ on the column $Ye_{\mathcal{F}}$, we obtain that

$$Y[1|_1, \mathcal{F}] + Y[2|_1, \mathcal{F}] + Y[3|_1, \mathcal{F}] - Y[S|_1, \mathcal{F}] \leq (|S|-1)Y[\mathcal{F}, \mathcal{F}].$$

Hence, $z_0 = Y[\mathcal{F}, S|_1] = Y[S|_1, \mathcal{F}] \geq 3(0.76) - 2 = 0.28$, and $w_0 = 1 - z_0 \geq 0.72$. We also know that $\sum_{i=1}^{7} w_i \leq \frac{7}{2} w_0$ (as $w \in K(P)$).

For $j \in \{4,5,6,7\}$, since $j|_1 \cap S|_1 \cap P = \emptyset$, our strengthened rule (BZ' 3) requires that $Y[j|_1, S|_1] = 0$ (this is what sets BZ'' apart from BZ in this example). Therefore, we have

$$\sum_{i=1}^{7} z_i = \sum_{i=1}^{7} Y[i|_1, S|_1] \leq 3Y[\mathcal{F}, S|_1] = 3z_0.$$

Thus, the inequality

$$\sum_{i=1}^{7} x_i = \sum_{i=1}^{7} (z_i + w_i) \leq 3z_0 + \frac{7}{2} w_0 \leq 3(0.28) + \frac{7}{2}(0.72) = 3.36$$

is valid for $\mathrm{BZ}''(P)$. However, $\sum_{i=1}^{7} y_i = 3.48$, which implies that $y \notin \mathrm{BZ}''(P)$. □

Next, we remark that, in general, adding redundant inequalities to the system $Ax \leq b$ could generate more obstructions and walls, and thus can improve the performance of BZ (and its variants). An example of this phenomenon is the following:

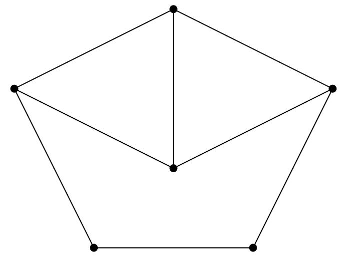

Figure A.1: A graph for which BZ performs better on $FRAC(G)$ with a redundant inequality.

Proposition 87. *Let G be the graph in Figure A.1. Further let P be the set defined by the facets of $FRAC(G)$ and P' be the system P with the additional (redundant) inequality*

$$\sum_{i \in V} x_i \leq 3.$$

Then

$$\mathrm{BZ}'_+(P) \supset \mathrm{BZ}(P') = P_I.$$

Proof. For the first claim, notice that the obstructions generated by BZ'_+ are exactly the edge sets, so $\mathcal{O}_k(P) = (P)$. This also implies that all walls and tiers have size 1, so

$$\mathrm{BZ}'_+(P) = \mathrm{LS}_+(\mathcal{O}_k(P)) = \mathrm{LS}_+(P) \neq P_I,$$

as it is shown in [LT03] that P has LS_+-rank 2.

For the second claim, notice that with the additional inequality in P', all sets of size at least 4 are 1-small obstructions, and thus all sets of size 2 are walls (and hence tiers). In this case, $\mathrm{BZ}(P') \subseteq \mathrm{SA}^2(P') = P_I$. $\square$

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